Pricing Continuously Funded Everlasting Options

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ABSTRACT
In this paper, we discuss the everlasting options of a special funding style - continuous funding, which is especially useful for DeFi scenarios. We show that there is an analytical formula for the pricing of such everlasting options.

The recent "Everlasting Options" paper by White, D. *et al.*[*1*] has introduced a new type of financial derivative, *Everlasting Options*, inspired by the funding-fee-based paradigm introduced by BitMEX for the perpetual futures[*2*]. Perpetual futures have become extremely popular since their inception and brought to the industry a new category of derivatives, i.e. the derivatives with positions maintained by regularly paying funding fees. Everlasting Options adopt this paradigm to options markets and have the potential to avoid the rolling issues and largely reduce the degree of liquidity fragmentation[*1*].

Everlasting Options work similarly to Perpetual Futures: a long position is maintained by paying funding fees to a short position. However, in the case of everlasting options, funding fee is charged as $(MARK - PAYOFF)$. Considering that theoretically $MARK$ should always be higher than $PAYOFF$, the funding fee should always be positive (i.e. long positions always pay short positions). For options, $(MARK - PAYOFF)$ has a specific financial significance - time value. Therefore, the mechanism of maintaining an everlasting option position has a very obvious financial significance too: option buyers pay the time values of the options associated with specific funding periods.

By means of the no-arbitrage argument, [*1*] has proved a pricing framework for funding-fee-based perpetual derivatives:

$$E = \frac{1}{F}\left[\frac{F}{F+1}P_{t_1} + \left(\frac{F}{F+1}\right)^2 P_{t_2} + ...\right]$$

where F is the payment frequency and P_{t_i} is the price of the regular option with expiration of t_i (REGPRICE of t_i).

We are especially interested in adopting this pricing method in DeFi scenarios, where funding is usually accrued on a per-block basis. This corresponds to very large F. Mathematically, the cases of large F are more convenient to be treated as $F \to \infty$. When F converges to infinity, this leads to a special funding style - continuous funding. That is, the funding fee that one long contract should pay one short contract is accrued continuously. This is similar to how interest is accrued for continuously compounded interest rate.

Let's denote the funding period as T (e.g. 1 day) and payment interval as $\Delta t = T/F$ (e.g. 1 hour if $F = 24$). Then we can rewrite the formula above. Note that $t_i = i\Delta t = iT/F$ and $i = Ft_i/T$, we have

$$E = \Delta t \left[\left(\frac{1}{1+1/F} \right)^{Ft_1/T} P_{t_1} + \left(\frac{1}{1+1/F} \right)^{Ft_2/T} P_{t_2} + ... \right]$$

When $F \to \infty$, $\Delta t \to 0$, we have

$$\left(\frac{1}{1+1/F} \right)^{F} \to e$$

And the summation converges to an integral

$$E = \int_0^\infty \frac{1}{T} e^{-t/T} P_t dt$$

Specifically, for everlasting call and put options, we have

$$\begin{aligned} C^{ever} &= \int_0^\infty \frac{1}{T} e^{-t/T} C(t) dt \\ P^{ever} &= \int_0^\infty \frac{1}{T} e^{-t/T} P(t) dt \end{aligned}$$

If we price $C(t)$ and $P(t)$ by the classic Black-Scholes pricing formula [3] with interest rate ignored:

$$\begin{aligned} C(t) &= N\left(\frac{\ln(S/K)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t} \right) S - N\left(\frac{\ln(S/K)}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t} \right) K \\ P(t) &= -N\left(-\frac{\ln(S/K)}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t} \right) S + N\left(-\frac{\ln(S/K)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t} \right) K \end{aligned}$$

where

$$
\begin{aligned}
N &= \text{CDF of the normal distribution} \\
S &= \text{spot price of the underlyer} \\
K &= \text{strike} \\
\sigma &= \text{volatility of } S
\end{aligned}
$$

then we can prove the following pricing formula for the continuously-funded everlasting options.

Pricing Formula

Let's divide C^{ever} and P^{ever} into intrinsic value and time value:

$$
\begin{aligned}
C^{ever} &= \max(S-K,0) + TimeValue_{call} \\
P^{ever} &= \max(K-S,0) + TimeValue_{put}
\end{aligned}
$$

We can prove that the call and put options at the same strike have the same time value $TimeValue_{call} = TimeValue_{put} = V$, given by

$$
V = \begin{cases} \frac{K}{u}\left(\frac{S}{K}\right)^{-\frac{u-1}{2}}, & \text{if } S \geqslant K \\ \frac{K}{u}\left(\frac{S}{K}\right)^{\frac{u+1}{2}}, & \text{if } S < K \end{cases}
$$

where

$$
u = \sqrt{1 + \frac{8}{\sigma^2 T}}
$$

The details of derivations are laid out in the appendix.

References

(1) White, Dave.; Bankman-Fried, Sam (2021). *"Everlasting Options"*. https://www.paradigm.xyz/2021/05/everlasting-options/
(2) *"Perpetual Contracts Guide"*. https://www.bitmex.com/app/perpetualContractsGuide
(3) Black, Fischer; Scholes, Myron (1973). *"The Pricing of Options and Corporate Liabilities"*. Journal of Political Economy. 81 (3): 637–654.

Appendix A. Details of Pricing Formula Derivations

Starting from the Black-Scholes pricing formula with interest rate ignored, we have

$$
\begin{aligned}
C(t) &= N\left(\frac{\ln(S/K)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\right) S - N\left(\frac{\ln(S/K)}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t}\right) K \\
P(t) &= -N\left(-\frac{\ln(S/K)}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t}\right) S + N\left(-\frac{\ln(S/K)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\right) K
\end{aligned}
$$

where

$$
\begin{aligned}
N &= \text{CDF of the normal distribution} \\
S &= \text{spot price of the underlyer} \\
K &= \text{strike} \\
\sigma &= \text{volatility of } S
\end{aligned}
$$

Let $\tau = \frac{t}{T}$, $a = \frac{ln(S/K)}{\sigma\sqrt{T}}$, $b = \frac{1}{2}\sigma\sqrt{T}$, then we can rewrite the Black-Scholes pricing formula for call options as

$$
C(t) = C(T\tau) = N\left(\frac{a}{\sqrt{\tau}} + b\sqrt{\tau}\right) S - N\left(\frac{a}{\sqrt{\tau}} + b\sqrt{\tau}\right) K
$$

and

$$
C^{ever} = \int_0^\infty e^{-\tau} C(T\tau) d\tau
$$

Let's rescale and still use the letter t in the place of τ, we have:

$$
\begin{aligned}
C^{ever} &= \frac{S}{\sqrt{2\pi}} \int_0^\infty e^{-t} \left[\int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}} dy\right] dt - \frac{K}{\sqrt{2\pi}} \int_0^\infty e^{-t} \left[\int_{-\infty}^{\frac{a}{\sqrt{t}}-b\sqrt{t}} e^{-\frac{y^2}{2}} dy\right] dt \\
&= SI_1 - KI_2
\end{aligned}
$$

with

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t} \left[\int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}} dy\right] dt \\
&= -\frac{1}{\sqrt{2\pi}} \int_0^\infty d(e^{-t})[...] \\
&= -\frac{1}{\sqrt{2\pi}} e^{-t}[...]|_0^\infty + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t} d[...]
\end{aligned}
$$

The first part of I_1:

$$-\frac{1}{\sqrt{2\pi}}e^{-t}[...]|_0^\infty = \frac{1}{\sqrt{2\pi}}e^{-t}\left[\int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}}dy\right]_{t=0} - \frac{1}{\sqrt{2\pi}}e^{-t}\left[\int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}}dy\right]_{t=\infty}$$

within which the second item always converges to 0:

$$e^{-t}\left[\int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}}dy\right]_{t=\infty} = 0$$

If $S > K, a > 0$, when $t \to 0$, $\frac{a}{\sqrt{t}} + b\sqrt{t} \to \infty$, we have:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}}dy \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}dy = 1$$

If $S < K, a < 0$, when $t \to 0$, $\frac{a}{\sqrt{t}} + b\sqrt{t} \to -\infty$, we have:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}}dy \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\infty} e^{-\frac{y^2}{2}}dy = 0$$

Second part of I_1

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t}d[...] &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t}e^{-\frac{1}{2}\left(\frac{a}{\sqrt{t}}+b\sqrt{t}\right)^2}\frac{d}{dt}\left(\frac{a}{\sqrt{t}}+b\sqrt{t}\right)dt \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}\left(\frac{a}{\sqrt{t}}+b\sqrt{t}\right)^2-t}\left(-\frac{a}{2}t^{-3/2}+\frac{b}{2}t^{-1/2}\right)dt\end{aligned}$$

let $\sqrt{t} = x$

$$\begin{aligned}&= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}\left(\frac{a}{x}+bx\right)^2-x^2}\left(-\frac{a}{2}x^{-3}+\frac{b}{2}x^{-1}\right)2xdx \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}\left(\frac{a}{x}+bx\right)^2-x^2}\left(-ax^{-2}+b\right)dx = A\end{aligned}$$

Similarly,

$$\begin{aligned}I_2 &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t}\left[\int_{-\infty}^{\frac{a}{\sqrt{t}}-b\sqrt{t}} e^{-\frac{y^2}{2}}dy\right]dt \\ &= -\frac{1}{\sqrt{2\pi}}\int_0^\infty d\left(e^{-t}\right)[...] = -\frac{1}{\sqrt{2\pi}}e^{-t}[...]|_0^\infty + \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t}d[...]\end{aligned}$$

The first part of I_2:

$$-\frac{1}{\sqrt{2\pi}}e^{-t}[...]|_0^\infty = \frac{1}{\sqrt{2\pi}}e^{-t}\left[\int_{-\infty}^{\frac{a}{\sqrt{t}}-b\sqrt{t}} e^{-\frac{y^2}{2}}dy\right]_{t=0} - \frac{1}{\sqrt{2\pi}}e^{-t}\left[\int_{-\infty}^{\frac{a}{\sqrt{t}}-b\sqrt{t}} e^{-\frac{y^2}{2}}dy\right]_{t=\infty}$$

within which the second item always converges to 0:

$$\frac{1}{\sqrt{2\pi}}e^{-t}\left[\int_{-\infty}^{\frac{a}{\sqrt{t}}-b\sqrt{t}} e^{-\frac{y^2}{2}}dy\right]_{t=\infty} = 0$$

If $S > K, a > 0$, when $t \to 0$, $\frac{a}{\sqrt{t}} - b\sqrt{t} \to \infty$, we have:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{a}{\sqrt{t}}-b\sqrt{t}} e^{-\frac{y^2}{2}}dy \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}dy = 1$$

If $S < K, a < 0$, when $t \to 0$, $\frac{a}{\sqrt{t}} - b\sqrt{t} \to -\infty$, we have:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}}dy \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\infty} e^{-\frac{y^2}{2}}dy = 0$$

Second part of I_2:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t}d[...] &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t}e^{-\frac{1}{2}\left(\frac{a}{\sqrt{t}}-b\sqrt{t}\right)^2}\frac{d}{dt}\left(\frac{a}{\sqrt{t}}-b\sqrt{t}\right)dt \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}\left(\frac{a}{x}-bx\right)^2-x^2}\left(-ax^{-2}-b\right)dx = B\end{aligned}$$

Then we have:

$$C^{ever} = \begin{cases} S(1+A) - K(1+B), & \text{if } S > K \\ SA - KB, & \text{if } S < K \end{cases}$$

where

$$A = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}\left(\frac{a}{x}+bx\right)^2-x^2}\left(-ax^{-2}+b\right)dx$$

$$B = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}\left(\frac{a}{x}-bx\right)^2-x^2}\left(-ax^{-2}-b\right)dx$$

Or we can rewrite C^{ever} as the following, which is naturally divided into the intrinsic value (the first part) and the time value (the second part):

$$C^{ever} = \max(S-K,0) + (SA - KB)$$

With a similar derivation, we have the following for everlasting put options:

$$P^{ever} = \max(K-S,0) + (SA - KB)$$

Note that the everlasting call and put options have the same time value:

$$V = SA - KB$$

Now let's derive A and B.

$$\begin{aligned}
\sqrt{2\pi}A &= \int_0^{\infty} e^{-\frac{1}{2}(\frac{a}{x}+bx)^2 - x^2}(-ax^{-2}+b)dx \\
&= e^{-ab}\int_0^{\infty} e^{-\frac{1}{2}(\frac{a}{x^2}+b_1x^2)}(-ax^{-2}+b)dx \\
&= e^{-ab}\int_0^{\infty} e^{-\frac{1}{2}(\frac{a}{x^2}+b_1x^2)}\left(ad\left(\frac{1}{x}\right)+bdx\right) \\
&= e^{-ab}\int_0^{\infty} e^{-\frac{1}{2}\left[\left(\frac{ab_1}{y}\right)^2+y^2)\right]}\left(ab_1d\frac{1}{y}+\frac{b}{b_1}dy\right) \\
&= e^{-ab}\int_0^{\infty} e^{-\frac{1}{2}\left[\left(\frac{c}{y}\right)^2+y^2)\right]}\left(d\left(\frac{c}{y}\right)+\frac{b}{b_1}dy\right)
\end{aligned}$$

where

$$b_1 = \sqrt{b^2+1}, y = b_1x, c = ab_1$$

Let $f(c) = \int_0^{\infty} e^{-\frac{1}{2}\left[\frac{c^2}{y^2}+y^2\right]}dy$. When $a>0, c>0$, it's easy to see:

$$f(c) = \int_0^{\infty} e^{-\frac{1}{2}\left[\frac{c^2}{y^2}+y^2\right]}dy = \int_{\frac{c}{y}\to\infty}^{\frac{c}{y}\to 0} e^{-\frac{1}{2}\left[\frac{c^2}{y^2}+y^2\right]}d\left(\frac{c}{y}\right) = -\int_0^{\infty} e^{-\frac{1}{2}\left[\frac{c^2}{y^2}+y^2\right]}d\left(\frac{c}{y}\right)$$

Therefore we can calcualte $f(c)$ as follows.

$$
\begin{aligned}
f(c) &= \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\left[\frac{c^2}{y^2}+y^2\right]} d\left(y-\frac{c}{y}\right) \\
&= \frac{e^{-c}}{2}\int_0^\infty e^{-\frac{1}{2}\left(y-\frac{c}{y}\right)^2} d\left(y-\frac{c}{y}\right) \\
&= \frac{e^{-c}}{2}\int_{-\infty}^\infty e^{-\frac{1}{2}z^2} dz \\
&= e^{-c}\sqrt{\frac{\pi}{2}}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
A &= \frac{1}{\sqrt{2\pi}} e^{-ab}\left[-f(c)+\frac{b}{b_1}f(c)\right] \\
&= -\frac{1}{2}e^{-a(b+b1)}\left(1-\frac{b}{b_1}\right) \\
&= -\frac{1}{2}e^{-w(1+u)}\left(1-\frac{1}{u}\right) \\
&= -\frac{1}{2}\left(\frac{S}{K}\right)^{-\frac{u+1}{2}}\left(1-\frac{1}{u}\right)
\end{aligned}
$$

where

$$
w = ab = \ln{(S/K)}/2, u = b_1/b = \sqrt{1+\frac{8}{\sigma^2 T}}
$$

Let $A = g(a,b)$, we have

$$
\begin{aligned}
B = g(a,-b) = -\frac{1}{2}e^{-a(-b+b1)}\left(1+\frac{b}{b_1}\right) = -\frac{1}{2}e^{w(1-u)}\left(1+\frac{1}{u}\right) \\
= -\frac{1}{2}\left(\frac{S}{K}\right)^{\frac{1-u}{2}}\left(1+\frac{1}{u}\right)
\end{aligned}
$$

$$
V = SA - KB = \frac{K}{u}\left(\frac{S}{K}\right)^{-\frac{u-1}{2}}
$$

Similarly, when $a<0, c<0$, we have

$$
f(c) = \int_0^\infty e^{-\frac{1}{2}\left[\frac{c^2}{y^2}+y^2\right]} dy = e^c\sqrt{\frac{\pi}{2}}
$$

$$A = \frac{1}{\sqrt{2\pi}} c^{-ab} \left(1 + \frac{b}{b_1}\right) f(c) = \frac{1}{2} e^{a(b_1 - b)} \left(1 + \frac{b}{b_1}\right) = \frac{1}{2} e^{w(u-1)} \left(1 + \frac{1}{u}\right)$$

$$B = \frac{1}{\sqrt{2\pi}} c^{-ab} \left(1 + \frac{b}{b_1}\right) f(c) = \frac{1}{2} e^{a(b_1 + b)} \left(1 - \frac{b}{b_1}\right) = \frac{1}{2} e^{w(u+1)} \left(1 - \frac{1}{u}\right)$$

$$V = SA - KB = \frac{K}{u} \left(\frac{S}{K}\right)^{\frac{u+1}{2}}$$

The case of $a = 0$ needs to be treated separately:

$$\begin{aligned} A &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}(b+2)^2 x^2} b dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2} b_1^2 x^2} \frac{b}{b_1} d(b_1 x) \\ &= \frac{1}{2u} \end{aligned}$$

$$B = -\frac{1}{2u}$$

$$V = SA - KB = \frac{K}{u}$$

Note that V is continuous at $a = 0$, i.e. the point of ATM.