Pricing Continuously Funded Power Perpetuals

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ABSTRACT
In this paper, we discuss the power perpetuals of a special funding style - continuous funding, which is especially useful for DeFi scenarios. We show that there is an analytical formula for the pricing of such everlasting options.

The "Power Perpetuals" paper by White, D. *et al.*[1] has introduced another instance of the funding-fee-based perpetual derivatives, *Power Perpetuals*.

The no-arbitrage argument in [2] has generally proved the pricing framework for funding-fee-based perpetual derivatives:

$$P = \frac{1}{F}\left[\frac{F}{F+1}P_{t_1} + \left(\frac{F}{F+1}\right)^2 P_{t_2} + ...\right]$$

where F is the payment frequency and P_{t_i} is the price of the regular derivative of the type with expiration of t_i (REGPRICE of t_i).

For a continuously-funded funding-fee-based perpetual derivative, the corresponding portfolio is of a continuous spectrum of classical options. That is, a continuously-funded perpetual derivatives with funding period T is equivalent to a portfolio consisting of all "regular" (i.e. expiring) derivative of expiration $t \in (0, \infty)$ (and all the other parameters being the same) with an exponentially decaying weight density:

$$w(t) = \frac{1}{T}e^{-t/T}$$

It has already been proved that, under the Black-Scholes assumptions, the theoretical price of a power perp expiring at t is given by[1]:

$$P_t = S^p e^{t(p-1)(r+\frac{p}{2}\sigma^2)}$$

where p is the power, r is the risk-free interest rate, and σ is the volatility.

Let's denote $h = (p-1)(r + \frac{p}{2}\sigma^2)$, then $P_t = S^p e^{ht}$. It is very straightforward to derive the theoretical price of a continuously-funded power perp with funding period T as follows.

$$\begin{aligned} P &= \int_0^\infty P_t w(t) \\ &= \int_0^\infty S^p e^{ht} \frac{e^{-t/T}}{T} \\ &= \frac{S^p}{T} \int_0^\infty e^{-(\frac{1}{T}-h)t} \\ &= \frac{S^p}{1-hT} \end{aligned}$$

Please note the formula above only holds when $hT < 1$, otherwise the Gaussian integral does not converge.

If the actual mark price out of trading is at the theoretical price, then the funding fee to be paid by a long to a short for funding period T is:

$$F = P - S^p = S^p \frac{hT}{1-hT}$$

The Greeks

Given the pricing formula, it is easy to get the *Delta*, *Gamma* and *Vega* of time value:

$$\Delta = \frac{\partial P}{\partial S} = \frac{pS^{p-1}}{1-hT}$$

$$\Gamma = \frac{\partial^2 P}{\partial S^2} = \frac{p(p-1)S^{p-2}}{1-hT}$$

$$\nu = \frac{\partial V}{\partial h}\frac{\partial h}{\partial \sigma} = \frac{S^p p(p-1)T\sigma}{(1-hT)^2}$$

The case of $p = 2$ is especially interesting because the *Gamma* is no longer dependent on the underlying price, but only on the volatility:

$$\Gamma = \frac{2}{1-hT} = \frac{2}{1-(r+\frac{\sigma^2}{2})T}$$

References

(1) White, Dave; Robinson, Dan; Koticha, Zubin; Leone, Andrew; Gauba, Alexis; Krishnan, Aparna (2021). *"Power Perpetuals"*.
https://www.paradigm.xyz/2021/08/power-perpetuals/

(2) White, Dave; Bankman-Fried, Sam (2021). *"Everlasting Options"*.
https://www.paradigm.xyz/2021/05/everlasting-options/