Everlasting Options with Interest Rate

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ABSTRACT
In this paper, we discuss the general cases of everlasting options with non-zero interest-fee rates and calculate their theoretical prices.

1. Theoretical Mark Price

In the original paper[*1*], we discuss the everlasting options in a simple case – assuming a zero risk-free interest rate. In this paper, we discuss the more general cases with a non-zero interest rate. We skip all the discussions about the general concept of everlasting options, for which readers can refer to [*1*] and [*2*].

Following the same pricing rationale in [*1*], we can prove that the theoretical mark prices of everlasting options with a non-zero risk-free interest rate r are as follows:

$$C^{ever} = \begin{cases} SA - KB + (S - \frac{K}{1+rT}) & \text{if } S \geqslant K, \\ SA - KB & \text{if } S < K. \end{cases}$$

$$P^{ever} = \begin{cases} SA - KB & \text{if } S \geqslant K, \\ SA - KB - (S - \frac{K}{1+rT}) & \text{if } S < K. \end{cases}$$

where

$$A = \begin{cases} \frac{1}{2}\left(\frac{S}{K}\right)^{-\frac{1}{2}(1+u)p}\left(\frac{1}{u} - 1\right) & \text{if } S \geqslant K, \\ \frac{1}{2}\left(\frac{S}{K}\right)^{-\frac{1}{2}(1-u)p}\left(\frac{1}{u} + 1\right) & \text{if } S < K. \end{cases}$$

$$B = \begin{cases} \frac{1}{2(1+rT)}\left(\frac{S}{K}\right)^{\frac{1}{2}(1+w)q}\left(\frac{1}{w} - 1\right) & \text{if } S \geqslant K, \\ \frac{1}{2(1+rT)}\left(\frac{S}{K}\right)^{\frac{1}{2}(1-w)q}\left(\frac{1}{w} + 1\right) & \text{if } S < K. \end{cases}$$

where

$$
\begin{aligned}
u &= \frac{\sqrt{p^2 + \frac{8}{\sigma^2 T}}}{p} \\
w &= \frac{\sqrt{q^2 + \frac{8(1+rT)}{\sigma^2 T}}}{-q} \\
p &= 1 + \frac{2r}{\sigma^2} \\
q &= 1 - \frac{2r}{\sigma^2}
\end{aligned}
$$

Please refer to the appendix at the end of this paper for the detailed math of the derivation.

2. The Special Case of $r = 0$

When $r = 0$, we have $p = q = 1, u = -w$. It's easy to see that the formula above degenerates to the simple version in [*1*].

When $S \geqslant K$,

$$
\begin{aligned}
SA - KB &= \frac{1}{2} S \left(\frac{S}{K}\right)^{-\frac{1+u}{2}} \left(\frac{1}{u} - 1\right) - \frac{1}{2} K \left(\frac{S}{K}\right)^{\frac{1-u}{2}} \left(-\frac{1}{u} - 1\right) \\
&= \frac{K}{2} \left[\left(\frac{S}{K}\right)^{1-\frac{1+u}{2}} \left(\frac{1}{u} - 1\right) - \left(\frac{S}{K}\right)^{\frac{1-u}{2}} \left(-\frac{1}{u} - 1\right)\right] \\
&= \frac{K}{u} \left(\frac{S}{K}\right)^{\frac{1-u}{2}}
\end{aligned}
$$

When $S < K$,

$$
\begin{aligned}
SA - KB &= \frac{1}{2} S \left(\frac{S}{K}\right)^{-\frac{1-u}{2}} \left(\frac{1}{u} + 1\right) - \frac{1}{2} K \left(\frac{S}{K}\right)^{\frac{1+u}{2}} \left(-\frac{1}{u} + 1\right) \\
&= \frac{K}{2} \left[\left(\frac{S}{K}\right)^{1-\frac{1-u}{2}} \left(\frac{1}{u} + 1\right) - \left(\frac{S}{K}\right)^{\frac{1+u}{2}} \left(-\frac{1}{u} + 1\right)\right] \\
&= \frac{K}{u} \left(\frac{S}{K}\right)^{\frac{1+u}{2}}
\end{aligned}
$$

Therefore, when $r = 0$, $SA - KB$ is the same as the time value V in [*1*].

3. Funding

The funding fees charged in a funding period is based on the same formula as the zero-rate everlasting options:

$$\text{FundingFee} = \text{MarkPrice} - \text{PayOff}$$

However, with a non-zero risk-free interest rate r, the *pay-off* should be defined slightly differently: $\max(\frac{S-K}{1+rT}, 0)$ for calls, and $\max(\frac{K-S}{1+rT}, 0)$ for puts. For options that are traded at the theoretical mark price, the funding fees charged for a funding period T are as follows. (Please notice that these are theoretical values only, and in practices the mark prices are usually determined by the markets so might not necessarily be the same as these.)

$$\text{Funding}(C^{ever}) = \begin{cases} SA - KB + S\frac{rT}{1+rT} & \text{if } S \geqslant K, \\ SA - KB & \text{if } S < K, \end{cases}$$

$$\text{Funding}(P^{ever}) = \begin{cases} SA - KB & \text{if } S \geqslant K, \\ SA - KB - S\frac{rT}{1+rT} & \text{if } S < K, \end{cases}$$

It's easy to see such a setting of funding fee is consistent with the funding of perpetual futures. Consider a portfolio consisting of 1 long position in call and 1 short position in put at the same strike, i.e. the put-call parity portfolio $(C^{ever}, -P^{ever})$. With put-call parity, such a portfolio synthesizes a unit of perpetual futures and thus should have the same funding fees as the latter. The funding fee charged on this portfolio for a funding period is as follows. When $S \geqslant K$:

$$\begin{aligned} \text{Funding}(C^{ever}, -P^{ever}) &= \begin{cases} \left(SA - KB + S\frac{rT}{1+rT}\right) - (SA - KB) & \text{if } S \geqslant K, \\ (SA - KB) - \left(SA - KB - S\frac{rT}{1+rT}\right) & \text{if } S < K. \end{cases} \\ &= S\frac{rT}{1+rT} \qquad \forall S \end{aligned}$$

Since the result holds for both $S \geqslant K$ and $S < K$, this theoretical value of the funding fee always equals to that of perpetual futures[*3*]. This is important because an arbitrage would be created if these two were not the same.

4. The Greeks

Following the notation of [*1*], let's still denote $V = SA - KB$. However, please notice that V does no longer have the meaning of "time value". But we still have the relations that

$$\Delta_{call} = \begin{cases} \frac{\partial V}{\partial S} + 1 & \text{if } S > K, \\ \frac{\partial V}{\partial S} & \text{if } S < K. \end{cases}$$

$$\Delta_{put} = \begin{cases} \frac{\partial V}{\partial S} & \text{if } S > K, \\ \frac{\partial V}{\partial S} - 1 & \text{if } S < K. \end{cases}$$

and

$$\Gamma_{call} = \Gamma_{put} = \frac{\partial^2 V}{\partial S^2}$$

Let's calculate the derivatives of V which can easily give the Greeks of the options.

When $S > K$, we have $A < 0, B < 0$,

$$\ln(-A) = \ln\frac{1}{2} + \left(-\frac{1+u}{2}p\right)(\ln S - \ln K) + \ln\left(-\frac{1}{u} + 1\right)$$

$$\ln(-B) = \ln\frac{1}{2(1+rT)} + \left(\frac{1+w}{2}q\right)(\ln S - \ln K) + \ln\left(-\frac{1}{w} + 1\right)$$

When $S < K$, we have $A > 0, B > 0$,

$$\ln A = \ln\frac{1}{2} + \left(-\frac{1-u}{2}p\right)(\ln S - \ln K) + \ln\left(\frac{1}{u} + 1\right)$$

$$\ln B = \ln\frac{1}{2(1+rT)} + \left(\frac{1-w}{2}q\right)(\ln S - \ln K) + \ln\left(\frac{1}{w} + 1\right)$$

Therefore,

$$\frac{\partial A}{\partial S} = \begin{cases} \frac{A}{S} \cdot \left(-\frac{(1+u)p}{2}\right) & \text{if } S > K, \\ \frac{A}{S} \cdot \left(-\frac{(1-u)p}{2}\right) & \text{if } S < K. \end{cases}$$

$$\frac{\partial B}{\partial S} = \begin{cases} \frac{B}{S} \cdot \frac{(1+w)q}{2} & \text{if } S > K, \\ \frac{B}{S} \cdot \frac{(1-w)q}{2} & \text{if } S < K. \end{cases}$$

Therefore,

$$\frac{\partial V}{\partial S} = A + S\frac{\partial A}{\partial S} - K\frac{\partial B}{\partial S}$$
$$= \begin{cases} A \cdot \left(1 - \frac{(1+u)p}{2}\right) - B\frac{K}{S} \cdot \left(\frac{(1+w)q}{2}\right) & \text{if } S > K, \\ A \cdot \left(1 - \frac{(1-u)p}{2}\right) - B\frac{K}{S} \cdot \left(\frac{(1-w)q}{2}\right) & \text{if } S < K. \end{cases}$$

$$\frac{\partial^2 V}{\partial S^2} = \begin{cases} \frac{A}{S} \cdot \left[\frac{(1+u)p}{2} - 1\right] \frac{(1+u)p}{2} - \frac{BK}{S^2} \cdot \left[\frac{(1+w)q}{2} - 1\right] \frac{(1+w)q}{2} & \text{if } S > K, \\ \frac{A}{S} \cdot \left[\frac{(1-u)p}{2} - 1\right] \frac{(1-u)p}{2} - \frac{BK}{S^2} \cdot \left[\frac{(1-w)q}{2} - 1\right] \frac{(1-w)q}{2} & \text{if } S < K. \end{cases}$$

References

(1) 0xAlpha; Fang, Daniel; Chen, Richard (2021). *"The Exchange Protocol of Everlasting Options"*. https://github.com/deri-protocol/whitepaper/blob/master/deri_everlasting_options_whitepaper.pdf
(2) White, Dave.; Bankman-Fried, Sam (2021). *"Everlasting Options"*. https://www.paradigm.xyz/2021/05/everlasting-options/
(3) *"Perpetual Contracts Guide"*. https://www.bitmex.com/app/perpetualContractsGuide

Appendices

Appendix A. Details of Pricing Formula Derivations

Following the same pricing rationale in [*1*], let's start from the Black-Scholes pricing formula with interest rate:

$$C(t) = N\left(\frac{\ln(S/K)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t} + \frac{r}{\sigma}\sqrt{t}\right)S - N\left(\frac{\ln(S/K)}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t} + \frac{r}{\sigma}\sqrt{t}\right)Ke^{-rt}$$
$$P(t) = -N\left(-\frac{\ln(S/K)}{\sigma\sqrt{t}} - \frac{1}{2}\sigma\sqrt{t} - \frac{r}{\sigma}\sqrt{t}\right)S + N\left(-\frac{\ln(S/K)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t} - \frac{r}{\sigma}\sqrt{t}\right)Ke^{-rt}$$

where

$$\begin{aligned}
N &= \text{CDF of the normal distribution} \\
S &= \text{spot price of the underlyer} \\
K &= \text{strike} \\
\sigma &= \text{volatility of } S \\
r &= \text{risk-free interest rate}
\end{aligned}$$

Let

$$\begin{aligned}
\tau &= \frac{t}{T} \\
\alpha &= \frac{\ln(\frac{S}{K})}{\sigma\sqrt{T}} \\
\beta_1 &= \frac{1}{2}\sigma\sqrt{T} + \frac{r}{\sigma}\sqrt{t} = \frac{1}{2}\sigma\sqrt{T}(1 + \frac{2r}{\sigma^2}) = \frac{1}{2}\sigma\sqrt{T}p \\
\beta_2 &= -\frac{1}{2}\sigma\sqrt{T} + \frac{r}{\sigma}\sqrt{t} = -\frac{1}{2}\sigma\sqrt{T}(1 - \frac{2r}{\sigma^2}) = -\frac{1}{2}\sigma\sqrt{T}q \\
p &= 1 + \frac{2r}{\sigma^2} \\
q &= 1 - \frac{2r}{\sigma^2}
\end{aligned}$$

then we can rewrite the Black-Scholes pricing formula for call options as

$$C(t) = C(T\tau) = N\left(\frac{\alpha}{\sqrt{\tau}} + \beta_1\sqrt{\tau}\right)S - N\left(\frac{\alpha}{\sqrt{\tau}} + \beta_2\sqrt{\tau}\right)Ke^{-rt}$$

and

$$C^{ever} = \int_0^\infty e^{-\tau}C(T\tau)d\tau$$

Let's rescale and still use the letter t in the place of τ, we have:

$$
\begin{aligned}
C^{ever} &= \int_0^\infty e^{-t} C(Tt) dt \\
&= S \int_0^\infty e^{-t} N\left(\frac{\alpha}{\sqrt{t}} + \beta_1 \sqrt{t}\right) dt - K \int_0^\infty e^{-t} e^{-rTt} N\left(\frac{\alpha}{\sqrt{t}} + \beta_2 \sqrt{t}\right) dt \\
&= \frac{S}{\sqrt{2\pi}} \int_0^\infty e^{-t} \left[\int_{-\infty}^{\frac{\alpha}{\sqrt{t}} + \beta_1 \sqrt{t}} e^{-\frac{y^2}{2}} dy\right] dt - \frac{K}{\sqrt{2\pi}} \int_0^\infty e^{-(1+rT)t} \left[\int_{-\infty}^{\frac{\alpha}{\sqrt{t}} - \beta_2 \sqrt{t}} e^{-\frac{y^2}{2}} dy\right] dt \\
&= SI_1 - KI_2
\end{aligned}
$$

with

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t} \left[\int_{-\infty}^{\frac{\alpha}{\sqrt{t}} + \beta_1 \sqrt{t}} e^{-\frac{y^2}{2}} dy\right] dt \\
I_2 &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(1+rT)t} \left[\int_{-\infty}^{\frac{\alpha}{\sqrt{t}} + \beta_2 \sqrt{t}} e^{-\frac{y^2}{2}} dy\right] dt \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t} \left[\int_{-\infty}^{\frac{\alpha\sqrt{1+rT}}{\sqrt{t}} + \frac{\beta_2}{\sqrt{1+rT}} \sqrt{t}} e^{-\frac{y^2}{2}} dy\right] \frac{dt}{1+rT} \\
&= \frac{1}{(1+rT)} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t} \left[\int_{-\infty}^{\frac{\alpha'}{\sqrt{t}} + \beta_2' \sqrt{t}} e^{-\frac{y^2}{2}} dy\right] dt
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha' &= \sqrt{1+rT}\alpha = \sqrt{1+rT} \frac{\ln(\frac{S}{K})}{\sigma\sqrt{T}} \\
\beta_2' &= \frac{\beta_2}{\sqrt{1+rT}} = -\frac{1}{2}\sigma\sqrt{T} q \frac{1}{\sqrt{1+rT}}
\end{aligned}
$$

Notice that I_1 and I_2 contain the same integral:

$$
I(a, b) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t} \left[\int_{-\infty}^{\frac{a}{\sqrt{t}} + b\sqrt{t}} e^{-\frac{y^2}{2}} dy\right] dt
$$

with $(a = \alpha, b = \beta_1)$ and $(a = \alpha', b = \beta_2')$, respectively.

Let's now calculate this integral I.

$$
\begin{aligned}
I &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t} \left[\int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}} dy \right] dt \\
&= -\frac{1}{\sqrt{2\pi}} \int_0^\infty d(e^{-t}) \left[\int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}} dy \right] \\
&= -\frac{1}{\sqrt{2\pi}} e^{-t}[...]|_0^\infty + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t} d[...]
\end{aligned}
$$

The first part of I:

$$
-\frac{1}{\sqrt{2\pi}} e^{-t}[...]|_0^\infty = \frac{1}{\sqrt{2\pi}} e^{-t} \left[\int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}} dy \right]_{t=0} - \frac{1}{\sqrt{2\pi}} e^{-t} \left[\int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}} dy \right]_{t\to\infty}
$$

within which the second item always converges to 0:

$$
e^{-t} \left[\int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}} dy \right]_{t\to\infty} = 0
$$

If $a > 0$, when $t \to 0$, $\frac{a}{\sqrt{t}} + b\sqrt{t} \to \infty$, we have:

$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}} dy \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = 1
$$

If $a = 0$, when $t \to 0$, $\frac{a}{\sqrt{t}} + b\sqrt{t} \to 0$, we have:

$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}} dy \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{y^2}{2}} dy = \frac{1}{2}
$$

If $a < 0$, when $t \to 0$, $\frac{a}{\sqrt{t}} + b\sqrt{t} \to -\infty$, we have:

$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{y^2}{2}} dy \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\infty} e^{-\frac{y^2}{2}} dy = 0
$$

Let's denote the second part of I as M:

$$\begin{aligned} M &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t} d[...] \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t} e^{-\frac{1}{2}\left(\frac{a}{\sqrt{t}}+b\sqrt{t}\right)^2} \frac{d}{dt}\left(\frac{a}{\sqrt{t}} + b\sqrt{t}\right) dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2-\frac{1}{2}\left(\frac{a}{x}+bx\right)^2} d\left(\frac{a}{x} + bx\right) \end{aligned}$$

where $x = \sqrt{t}$.

$$\begin{aligned} M &= \frac{e^{-ab}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}\left(\frac{a^2}{x^2}+(b^2+2)x^2\right)} d\left(\frac{a}{x} + bx\right) \\ &= \frac{e^{-ab}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}\left(\frac{a^2}{x^2}+(b^2+2)x^2\right)} d\left(\frac{a}{x}\right) + \frac{e^{-ab}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}\left(\frac{a^2}{x^2}+(b^2+2)x^2\right)} bdx \\ &= M_1 + M_2 \end{aligned}$$

where

$$\begin{aligned} M_1 &= \frac{e^{-ab}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}\left(\frac{a^2}{x^2}+(b^2+2)x^2\right)} d\left(\frac{a}{x}\right) \\ M_2 &= \frac{e^{-ab}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}\left(\frac{a^2}{x^2}+(b^2+2)x^2\right)} bdx \end{aligned}$$

Let $b_1 = \sqrt{b^2+2}$, $c = ab_1 = a\sqrt{b^2+2}$, then we have

$$M_1 = \frac{e^{-ab}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}\left(\frac{c^2}{(b_1 x)^2}+(b_1 x)^2\right)} d\left(\frac{c}{b_1 x}\right)$$

Let $\frac{c}{b_1 x} = by$.

If $a > 0$, when $x \to 0$, $y \to \infty$; when $x \to \infty$, $y \to 0$ we have

$$\begin{aligned} M_1 &= \frac{e^{-ab}}{\sqrt{2\pi}} \int_\infty^0 e^{-\frac{1}{2}\left((by)^2+\frac{c^2}{(by)^2}\right)} d\left(by\right) \\ &= -\frac{e^{-ab}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}\left((by)^2+\frac{c^2}{(by)^2}\right)} d\left(by\right) \end{aligned}$$

Let $by = b_1 z$, then

$$\begin{aligned} M_1 &= -\frac{e^{-ab}}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}\left((b_1 z)^2 + \frac{a^2}{z^2}\right)} d\left(b_1 z\right) \\ &= -\frac{b_1}{b} M_2 \end{aligned}$$

So let's only calculate M_2.

$$\begin{aligned} M_2 &= \frac{e^{-ab}}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}\left(\frac{c^2}{(b_1 x)^2} + (b_1 x)^2\right)} d(b_1 x) \frac{b}{b_1} \\ &= \frac{e^{-ab}}{\sqrt{2\pi}} \frac{b}{b_1} \int_0^{\infty} e^{-\frac{1}{2}\left(\frac{c^2}{x^2} + x^2\right)} dx \\ &= \frac{e^{-ab}}{\sqrt{2\pi}} \frac{b}{b_1} \cdot L \end{aligned}$$

where

$$L = \int_0^{\infty} e^{-\frac{1}{2}\left(\frac{c^2}{x^2} + x^2\right)} dx$$

Let $x = \frac{c}{z}$. Since $c > 0$, when $x \to 0$, $z \to \infty$; when $x \to \infty$, $z \to 0$

$$\begin{aligned} L &= \int_{\infty}^{0} e^{-\frac{1}{2}\left(z^2 + \frac{c^2}{z^2}\right)} d(\frac{c}{z}) \\ &= \int_0^{\infty} e^{-\frac{1}{2}\left(z^2 + \frac{c^2}{z^2}\right)} d(-\frac{c}{z}) \\ &= \int_0^{\infty} e^{-\frac{1}{2}\left(x^2 + \frac{c^2}{x^2}\right)} d(-\frac{c}{x}) \end{aligned}$$

$$\begin{aligned} 2L &= \int_0^{\infty} e^{-\frac{1}{2}\left(x^2 + \frac{c^2}{x^2}\right)} d\left(x - \frac{c}{x}\right) \\ &= \int_0^{\infty} e^{-\frac{1}{2}\left(x - \frac{c}{x}\right)^2 - c} d\left(x - \frac{c}{x}\right) \\ &= e^{-c} \int_0^{\infty} e^{-\frac{1}{2}\left(x - \frac{c}{x}\right)^2} d\left(x - \frac{c}{x}\right) \end{aligned}$$

Let $y = x - \frac{c}{x}$. With $c > 0$, when $x \to 0$, $y \to -\infty$; when $x \to +\infty$, $y \to +\infty$, therefore

$$2L = e^{-c} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}y^2} dy = e^{-c}\sqrt{2\pi}$$

$$
\begin{aligned}
M_2 &= \frac{e^{-ab}}{\sqrt{2\pi}}\frac{b}{b_1} \cdot L \\
&= \frac{e^{-ab}}{\sqrt{2\pi}}\frac{b}{b_1} \cdot \frac{1}{2}e^{-c}\sqrt{2\pi} \\
&= \frac{1}{2}e^{-ab-c}\frac{b}{b_1} \\
&= \frac{1}{2}e^{-a(b+b_1)}\frac{b}{b_1} \\
&= \frac{1}{2}e^{-ab(1+\frac{\sqrt{b^2+2}}{b})}\frac{b}{\sqrt{b^2+2}}
\end{aligned}
$$

$$
\begin{aligned}
M_1 &= -\frac{b_1}{b}M_2 \\
&= -\frac{1}{2}e^{-ab(1+\frac{\sqrt{b^2+2}}{b})}
\end{aligned}
$$

$$
\begin{aligned}
M &= M_1 + M2 \\
&= \frac{1}{2}e^{-ab(1+\frac{\sqrt{b^2+2}}{b})}\left(\frac{b}{\sqrt{b^2+2}} - 1\right)
\end{aligned}
$$

If $a < 0$, when $x \to 0$, $y \to -\infty$; when $x \to \infty$, $y \to 0$. With a similar derivation, we have

$$
\begin{aligned}
M_1 &= \frac{e^{-ab}}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-\frac{1}{2}\left((by)^2+\frac{c^2}{(by)^2}\right)} d\,(by) \\
&= \frac{b_1}{b}M_2
\end{aligned}
$$

Now we only need to recalculate L with the condition $c < 0$. Let $x = \frac{c}{z}$. Since $c < 0$, when $x \to 0$, $z \to -\infty$; when $x \to \infty$, $z \to 0$

$$
\begin{aligned}
L &= \int_{-\infty}^{0} e^{-\frac{1}{2}\left(z^2+\frac{c^2}{z^2}\right)} d(\frac{c}{z}) \\
&= \int_{0}^{\infty} e^{-\frac{1}{2}\left(z^2+\frac{c^2}{z^2}\right)} d(\frac{c}{z}) \\
&= \int_{0}^{\infty} e^{-\frac{1}{2}\left(x^2+\frac{c^2}{x^2}\right)} d(\frac{c}{x})
\end{aligned}
$$

$$
\begin{aligned}
2L &= \int_0^\infty e^{-\frac{1}{2}\left(x^2+\frac{c^2}{x^2}\right)} d\left(x+\frac{c}{x}\right) \\
&= \int_0^\infty e^{-\frac{1}{2}\left(x+\frac{c}{x}\right)^2+c} d\left(x+\frac{c}{x}\right) \\
&= e^c \int_0^\infty e^{-\frac{1}{2}\left(x+\frac{c}{x}\right)^2} d\left(x+\frac{c}{x}\right)
\end{aligned}
$$

Let $y = x + \frac{c}{x}$. With $c < 0$, when $x \to 0$, $y \to -\infty$; when $x \to +\infty$, $y \to +\infty$, therefore

$$
2L = e^c \int_{-\infty}^{+\infty} e^{-\frac{1}{2}y^2} dy = e^c \sqrt{2\pi}
$$

$$
\begin{aligned}
M_2 &= \frac{e^{-ab}}{\sqrt{2\pi}} \frac{b}{b_1} \cdot L \\
&= \frac{e^{-ab}}{\sqrt{2\pi}} \frac{b}{b_1} \cdot \frac{1}{2} e^c \sqrt{2\pi} \\
&= \frac{1}{2} e^{-ab+c} \frac{b}{b_1} \\
&= \frac{1}{2} e^{a(-b+b_1)} \frac{b}{b_1} \\
&= \frac{1}{2} e^{ab(\frac{\sqrt{b^2+2}}{b}-1)} \frac{b}{\sqrt{b^2+2}}
\end{aligned}
$$

$$
\begin{aligned}
M_1 &= \frac{b_1}{b} M_2 \\
&= \frac{1}{2} e^{ab(\frac{\sqrt{b^2+2}}{b}-1)}
\end{aligned}
$$

$$
\begin{aligned}
M &= M_1 + M2 \\
&= \frac{1}{2} e^{ab(\frac{\sqrt{b^2+2}}{b}-1)} \left(\frac{b}{\sqrt{b^2+2}} + 1 \right)
\end{aligned}
$$

The case of $a = 0$ needs to be handled separately.

$$
\begin{aligned}
M &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2 - \frac{1}{2}\left(\frac{a}{x}+bx\right)^2} d\left(\frac{a}{x} + bx\right) \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}(b^2+2)x^2} d\left(bx\right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{b}{\sqrt{b^2+2}} \int_0^\infty e^{-\frac{1}{2}(\sqrt{b^2+2}x)^2} d\left(\sqrt{b^2+2}x\right) \\
&= \frac{b}{2\sqrt{b^2+2}}
\end{aligned}
$$

Thus $I = \frac{1}{2} + \frac{b}{2\sqrt{b^2+2}}$ at $a = 0$. Notice the value of I at $a = 0$ retains the continuity around this point, so we can combine the case of $a = 0$ together with $a > 0$. Please notice that this is just an arbitrary choice – it's the same if put it with the other side. Let's summarize the result of integral I

$$
I = \begin{cases} 1 + \frac{1}{2} e^{-ab(1+\frac{\sqrt{b^2+2}}{b})} \left(\frac{b}{\sqrt{b^2+2}} - 1\right) & \text{if } a \geqslant 0, \\ \frac{1}{2} e^{ab(\frac{\sqrt{b^2+2}}{b}-1)} \left(\frac{b}{\sqrt{b^2+2}} + 1\right) & \text{if } a < 0. \end{cases}
$$

Now we have calculated the integral I, we have the evaluation of I_1 and I_2:

$$
\begin{aligned}
I_1 &= I(a,b)|_{a=\alpha, b=\beta_1} \\
I_2 &= \frac{1}{1+rT} I(a,b)|_{a=\alpha', b=\beta_2'}
\end{aligned}
$$

Denote

$$
\begin{aligned}
u &= \frac{\sqrt{1 + \frac{2}{\beta_1^2}}}{\beta_1} = \frac{\sqrt{\sigma^2 T p^2 + 8}}{\sigma\sqrt{T} p} = \frac{\sqrt{p^2 + \frac{8}{\sigma^2 T}}}{p} \\
w &= \frac{\sqrt{1 + \frac{2}{\beta_2'^2}}}{\beta_2'} = \frac{\sqrt{\sigma^2 T q^2 + 8(1+rT)}}{-\sigma\sqrt{T} q} = \frac{\sqrt{q^2 + \frac{8(1+rT)}{\sigma^2 T}}}{-q}
\end{aligned}
$$

and notice $\alpha'\beta_2' = \alpha\beta_2$ rewrite I_1 and I_2 as:

$$
\begin{aligned}
I_1 &= \begin{cases} 1 + \frac{1}{2} e^{-\alpha\beta_1(1+u)} \left(\frac{1}{u} - 1\right) & \text{if } \alpha \geqslant 0, \\ \frac{1}{2} e^{\alpha\beta_1(-1+u)} \left(\frac{1}{u} + 1\right) & \text{if } \alpha < 0. \end{cases} \\
I_2 &= \begin{cases} \frac{1}{1+rT} + \frac{1}{2(1+rT)} e^{-\alpha\beta_2(1+w)} \left(\frac{1}{w} - 1\right) & \text{if } \alpha \geqslant 0, \\ \frac{1}{2(1+rT)} e^{\alpha\beta_2(-1+w)} \left(\frac{1}{w} + 1\right) & \text{if } \alpha < 0. \end{cases}
\end{aligned}
$$

Let's denote

$$A = \begin{cases} \frac{1}{2}e^{-\alpha\beta_1(1+u)}\left(\frac{1}{u}-1\right) & \text{if } \alpha \geqslant 0, \\ \frac{1}{2}e^{\alpha\beta_1(-1+u)}\left(\frac{1}{u}+1\right) & \text{if } \alpha < 0. \end{cases}$$

$$B = \begin{cases} \frac{1}{2(1+rT)}\frac{1}{2}e^{-\alpha\beta_2(1+w)}\left(\frac{1}{w}-1\right) & \text{if } \alpha \geqslant 0, \\ \frac{1}{2(1+rT)}\frac{1}{2}e^{\alpha\beta_2(-1+w)}\left(\frac{1}{w}+1\right) & \text{if } \alpha < 0. \end{cases}$$

Then we have:

$$\begin{aligned} C^{ever} &= SI_1 - KI_2 \\ &= \begin{cases} (S - \frac{K}{1+rT}) + SA - KB & \text{if } S \geqslant K, \\ SA - KB & \text{if } S < K. \end{cases} \end{aligned}$$

Notice

$$\alpha\beta_1 = \frac{\ln\frac{S}{K}}{\sigma\sqrt{T}}(\frac{1}{2}\sigma\sqrt{T} + \frac{r}{\sigma}\sqrt{T}) = \ln\frac{S}{K}\frac{1}{2}(1+\frac{2r}{\sigma^2}) = \ln\frac{S}{K}\frac{p}{2}$$

$$\alpha\beta_2 = \frac{\ln\frac{S}{K}}{\sigma\sqrt{T}}(-\frac{1}{2}\sigma\sqrt{T} + \frac{r}{\sigma}\sqrt{T}) = -\ln\frac{S}{K}\frac{1}{2}(1-\frac{2r}{\sigma^2}) = -\ln\frac{S}{K}\frac{q}{2}$$

When $\alpha \geqslant 0$,

$$\begin{aligned} A &= \frac{1}{2}\exp\left[-\ln\frac{S}{K}\frac{1}{2}(1+u)p\right]\left(\frac{1}{u}-1\right) \\ &= \frac{1}{2}\left(\frac{S}{K}\right)^{-\frac{1}{2}(1+u)p}\left(\frac{1}{u}-1\right) \end{aligned}$$

$$\begin{aligned} B &= \frac{1}{2(1+rT)}\exp\left[-\left(-\ln\frac{S}{K}\frac{q}{2}\right)(1+w)\right]\left(\frac{1}{w}-1\right) \\ &= \frac{1}{2(1+rT)}\left(\frac{S}{K}\right)^{\frac{1}{2}(1+w)q}\left(\frac{1}{w}-1\right) \end{aligned}$$

When $\alpha < 0$,

$$\begin{aligned}
A &= \frac{1}{2} \exp\left[\ln \frac{S}{K} \frac{1}{2}(-1+u)p\right] \left(\frac{1}{u}+1\right) \\
&= \frac{1}{2} \left(\frac{S}{K}\right)^{\frac{1}{2}(-1+u)p} \left(\frac{1}{u}+1\right) \\
&= \frac{1}{2} \left(\frac{S}{K}\right)^{-\frac{1}{2}(1-u)p} \left(\frac{1}{u}+1\right)
\end{aligned}$$

$$\begin{aligned}
B &= \frac{1}{2(1+rT)} \exp\left[\left(-\ln \frac{S}{K} \frac{q}{2}\right)(-1+w)\right] \left(\frac{1}{w}+1\right) \\
&= \frac{1}{2(1+rT)} \left(\frac{S}{K}\right)^{-\frac{1}{2}(-1+w)q} \left(\frac{1}{w}+1\right) \\
&= \frac{1}{2(1+rT)} \left(\frac{S}{K}\right)^{\frac{1}{2}(1-w)q} \left(\frac{1}{w}+1\right)
\end{aligned}$$

In summary,

$$C^{ever} = \begin{cases} (S - \frac{K}{1+rT}) + SA - KB & \text{if } S \geqslant K, \\ SA - KB & \text{if } S < K. \end{cases}$$

where

$$A = \begin{cases} \frac{1}{2} \left(\frac{S}{K}\right)^{-\frac{1}{2}(1+u)p} \left(\frac{1}{u}-1\right) & \text{if } S \geqslant K, \\ \frac{1}{2} \left(\frac{S}{K}\right)^{-\frac{1}{2}(1-u)p} \left(\frac{1}{u}+1\right) & \text{if } S < K. \end{cases}$$

$$B = \begin{cases} \frac{1}{2(1+rT)} \left(\frac{S}{K}\right)^{\frac{1}{2}(1+w)q} \left(\frac{1}{w}-1\right) & \text{if } S \geqslant K, \\ \frac{1}{2(1+rT)} \left(\frac{S}{K}\right)^{\frac{1}{2}(1-w)q} \left(\frac{1}{w}+1\right) & \text{if } S < K. \end{cases}$$

where

$$
\begin{aligned}
u &= \frac{\sqrt{p^2 + \frac{8}{\sigma^2 T}}}{p} \\
w &= \frac{\sqrt{q^2 + \frac{8(1+rT)}{\sigma^2 T}}}{-q} \\
p &= 1 + \frac{2r}{\sigma^2} \\
q &= 1 - \frac{2r}{\sigma^2}
\end{aligned}
$$

Let's now derive P^{ever}. With the put-call parity, we have $P(t) = C(t) - S + Ke^{-rt}$ holds for any $t > 0$, therefore

$$
\begin{aligned}
P^{ever} &= \int_0^\infty e^{-t} P(Tt)dt \\
&= \int_0^\infty e^{-t} C(Tt)dt - S\int_0^\infty e^{-t}dt + K\int_0^\infty e^{-t}e^{-rTt}dt \\
&= C^{ever} - (S - \frac{K}{1+rT})
\end{aligned}
$$

That is

$$
P^{ever} = \begin{cases} SA - KB & \text{if } S \geqslant K, \\ SA - KB - (S - \frac{K}{1+rT}) & \text{if } S < K. \end{cases}
$$