Pricing Continuously Funded Power Perpetuals

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ABSTRACT
In this paper, we discuss the power perpetuals of a special funding style - continuous funding, which is especially useful for DeFi scenarios. We show that there is an analytical pricing formula for such power perpetuals.

Power Perpetuals are introduced by White, D. *et al.*[*1*] as yet another instance of the funding-fee-based perpetual derivatives.

The no-arbitrage argument in [*2*] has generally proved the pricing framework for funding-fee-based perpetual derivatives:

$$P = \frac{1}{F}\left[\frac{F}{F+1}P_{t_1} + \left(\frac{F}{F+1}\right)^2 P_{t_2} + ...\right]$$

where F is the payment frequency and P_{t_i} is the price of the regular derivative of the type with expiration of t_i (REGPRICE of t_i).

For a continuously-funded funding-fee-based perpetual derivative, the corresponding portfolio is of a continuous spectrum of classical options. That is, a continuously-funded perpetual derivatives with funding period T is equivalent to a portfolio consisting of all "regular" (i.e. expiring) derivative of expiration $t \in (0, \infty)$ (and all the other parameters being the same) with an exponentially decaying weight density:

$$w(t) = \frac{1}{T}e^{-t/T}$$

It has already been proved that, under the Black-Scholes assumptions, the theoretical price of a power perp expiring at t is given by[*1*]:

$$P_t = S^p e^{t(p-1)(r+\frac{p}{2}\sigma^2)}$$

where p is the power, r is the risk-free interest rate, and σ is the volatility.

Let's denote $h = (p-1)(r + \frac{p}{2}\sigma^2)$, then $P_t = S^p e^{ht}$. It is very straightforward to derive the theoretical price of a continuously-funded power perp with funding period T as follows.

$$\begin{aligned}
P &= \int_0^\infty P_t w(t) \\
&= \int_0^\infty S^p e^{ht} \frac{e^{-t/T}}{T} \\
&= \frac{S^p}{T} \int_0^\infty e^{-(\frac{1}{T}-h)t} \\
&= \frac{S^p}{1-hT}
\end{aligned}$$

Please note the formula above only holds when $hT < 1$, otherwise the Gaussian integral does not converge.

If the actual mark price out of trading is at the theoretical price, then the funding fee to be paid by a long to a short for funding period T is:

$$F = P - S^p = S^p \frac{hT}{1-hT}$$

The Greeks

Given the pricing formula, it is easy to get the *Delta*, *Gamma* and *Vega* of a power perpetual:

$$\Delta = \frac{\partial P}{\partial S} = \frac{pS^{p-1}}{1-hT} = \frac{pP}{S}$$

$$\Gamma = \frac{\partial^2 P}{\partial S^2} = \frac{p(p-1)S^{p-2}}{1-hT} = \frac{p(p-1)P}{S^2}$$

$$\nu = \frac{\partial V}{\partial h}\frac{\partial h}{\partial \sigma} = \frac{S^p p(p-1)T\sigma}{(1-hT)^2}$$

The case of $p = 2$ is especially interesting because the *Gamma* is no longer dependent on the underlying price, but only on the volatility:

$$\Gamma = \frac{2}{1-hT} = \frac{2}{1-(r+\frac{\sigma^2}{2})T}$$

Based on the Greeks, we can get an idea about the change of the derivative price δP caused by a small change of underlying price δS from the second-order Taylor expansion:

$$\delta P \approx \Delta \cdot \delta S + \frac{1}{2}\Gamma \cdot \delta S^2$$

$$\frac{\delta P}{P} \approx p\left(\frac{\delta S}{S}\right) + \frac{p(p-1)}{2}\left(\frac{\delta S}{S}\right)^2$$

References

(1) White, Dave; Robinson, Dan; Koticha, Zubin; Leone, Andrew; Gauba, Alexis; Krishnan, Aparna (2021). *"Power Perpetuals"*. https://www.paradigm.xyz/2021/08/power-perpetuals/

(2) White, Dave; Bankman-Fried, Sam (2021). *"Everlasting Options"*. https://www.paradigm.xyz/2021/05/everlasting-options/