Assignment Three

November 24, 2019

MCSC 6020G
Fall 2019
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Question One:

Analysis of matrix $T_N \in \mathbb{R}^{\mathrm{nxn}}$,

$$T_N = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

a) Find L_N such that $L_N L_N^T = T_N$

Note: All matrices in $\mathbb{R}^{\mathrm{nxn}}$ and no indices out of bounds.

First using cholesky decomposition, $T_N = PDP^T$,

$$
\begin{align}
P &= \{p_{i,j}\}_{\forall i,j} \tag{1}\\
p_{i,j} &= \begin{cases} 1 & i = j \\ \dfrac{1}{i+1} - 1 & j = i-1 \\ 0 & o/w \end{cases} \tag{2}
\end{align}
$$

$$
P^T = \{p_{j,i}\}_{\forall i,j} \tag{3}
$$

$$
\begin{align}
D &= \{d_{i,j}\}_{\forall i,j} \tag{4}\\
d_{i,j} &= \begin{cases} 1 + \dfrac{1}{i} & i = j \\ 0 & o/w \end{cases} \tag{5}
\end{align}
$$

Next split the diagonal matrix D and distribute among P and P^T,

$$
D' = \{d'_{i,j} = \sqrt{d_{i,j}}\}_{\forall i,j} \tag{6}
$$

$$
\begin{align}
PDP^T &= PD'D'P^T \tag{7}\\
&= \left(PD'\right)\left(D'P^T\right) \tag{8}
\end{align}
$$

$$
\begin{align}
L_N &= PD' \tag{9}\\
l_{i,j} &= \begin{cases} \sqrt{1 + \dfrac{1}{i}} & i = j \\ \left(\dfrac{1}{i} - 1\right)\sqrt{1 + \dfrac{1}{i-1}} & j = i-1 \\ 0 & o/w \end{cases} \tag{10}
\end{align}
$$

$$
L_N^T = \{l_{j,i}\}_{\forall i,j} \tag{11}
$$

$$
\boldsymbol{L_N L_N^T = T_N}
$$

Note: The equations were developed and verified experimentally using Python. The code can be found in the file <choleskyDecomp.py>.

(b) All eigenvectors and eigenvalues of T_N.

Through eigendecomposition:

$$\boldsymbol{T_N = Q\Lambda Q^T}$$

The diagonal matrix of eigenvalues[1]:

$$\Lambda = \{\lambda_{i,j}\}_{\forall i,j} \tag{1}$$

$$\lambda_{i,j} = \begin{cases} 4\left(sin\left[\frac{\pi i}{2(n+1)}\right]\right)^2 & i = j \\ 0 & o/w \end{cases} \tag*{^{1}(2)}$$

The column matrix of eigenvectors[1]:

$$Q = \{q_{i,j}\}_{\forall i,j} \tag{3}$$

$$q_{i,j} = \left\{\sqrt{\frac{2}{n+1}} sin\left(\frac{\pi ij}{n+1}\right)\right\}_{\forall i,j} \tag*{^{1}(4)}$$

$$Q^T = \{q_{j,i}\}_{\forall i,j} \tag{5}$$

Note: The equations were found in the textbook, Matrix Computations by Van Loan and verified experimentally using Python. The code can be found in the file <eigDecomp.py>.

[1] Matrix Computations, Van Loan, p 229