Mathematics 4N Pablo Díaz Viñambres

Exercise 8

$\boxed{1-}$ Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \quad -5 \le x \le 5, \quad t \ge 0$$

$$u(-5,t) = u(5,t) = 0, \quad t \ge 0$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad -5 \le x \le 5$$

a) Find the exact solutions of the wave equation for the sets of initial conditions

- $f(x) = \cos\left(\frac{\pi x}{2}\right), \quad g(x) = 0$

We can use the d'Alembert formula:

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(s)\,ds\right]$$

Since $c^2 = 4 \Rightarrow c = 2$ and $g(x) = 0$ we have

$$u(x,t) = \frac{1}{2}\left[f(x+2t) + f(x-2t)\right] =$$

$$= \frac{1}{2}\cos\left(\frac{\pi(x+2t)}{2}\right) + \cos\left(\frac{\pi(x-2t)}{2}\right)$$

- $f(x) = 0 \;;\; g(x) = \pi\cos\left(\frac{\pi x}{2}\right)$

We apply the same formula and get

$$u(x,t) = \frac{1}{2}\cdot\frac{1}{2}\int_{x-2t}^{x+2t} \pi\cos\left(\frac{\pi s}{2}\right) ds =$$

$$= \frac{\pi}{4}\int_{x-2t}^{x+2t} \cos\left(\frac{\pi s}{2}\right) ds = \frac{\pi}{4}\cdot\frac{2}{\pi}\,\mathrm{sen}\left(\frac{\pi r}{2}\right)\Bigg|_{x-2t}^{x+2t} =$$

$$= \frac{1}{2}\left(\mathrm{sen}\left(\frac{\pi(x+2t)}{2}\right) - \mathrm{sen}\left(\frac{\pi(x-2t)}{2}\right)\right)$$

b) Set up a FD scheme, for using central differences.

Using the central derivative formula

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u(x, t+\Delta t) - 2u(x,t) + u(x,t+\Delta t)}{\Delta t^2}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{\Delta x^2}$$

and defining U_i^n as the approximation of $u(x_i, t_n)$, where $x_i = -5 + i\Delta x$, $t_n = n\Delta t$

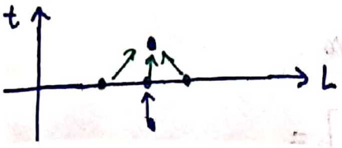
we get

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \underset{\text{C.D.}}{\approx} \frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\Delta t^2} = c^2 \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2}$$

$$\Rightarrow U_i^{n+1} = -U_i^{n-1} + 2(1-\alpha^2)U_i^n + \alpha^2(U_{i-1}^n + U_{i+1}^n)$$

where $\alpha := \frac{c\Delta t}{\Delta x} = \frac{2\Delta t}{\Delta x}$, $n > 0$, $1 \le i \le \frac{10}{\Delta x} = m$

For $n = 0$ (the first row) we must use the IC, since the points U_i^{-1} are ghost points



First, notice that the points in green are $u(x,0)$ given by the IC $u(x,0) = f(x)$, now since

$\frac{\partial u}{\partial t}\Big|_{t=0} = g(x)$ we can use central differences again and get

$$\frac{U_i^1 - U_i^{-1}}{2\Delta t} \approx \frac{\partial u(x_i, t)}{\partial t}\Big|_{t=0} = g(x_i) \Rightarrow$$

$$\Rightarrow U_i^{-1} = U_i^1 - 2\Delta t\, g(x_i)$$

So we get

$$U_i^1 = -U_i^{-1} + 2(1-\alpha^2)U_i^0 + \alpha^2(U_{i-1}^0 + U_{i+1}^0) =$$

$$= -U_i^1 + 2\Delta t\, g(x_i) + 2(1-\alpha^2) f(x_i) + \alpha^2(f(x_{i-1}) + f(x_{i+1})),$$

$1 \le i \le 10/\Delta x = m$

for the first row.

Note that $U_0^n = U_m^n = 0$ for all n because of the BCs $u(-5,t) = u(5,t) = 0$

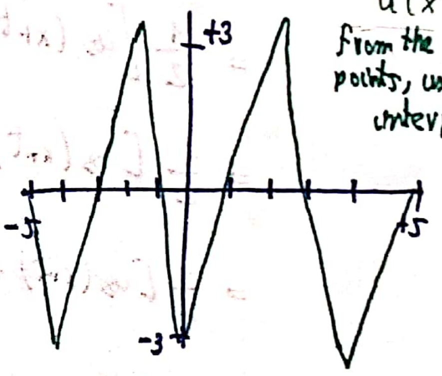
- Using $f(x) = \cos\left(\frac{\pi x}{2}\right)$, $g(x) = 0$, $\overset{\Delta x}{\overset{\shortparallel}{h}} = 1$ and $\overset{\Delta t}{\overset{\shortparallel}{k}} = 0.5$ use the formula to find approx. for $u(x_i, 0.5)$, $i = 1 \dots 9$.

$u(x_i, 0.5) \approx U_i^1$ (first row), then

$$u(x_i, 0.5) \approx \underbrace{0.5\,g(x_i)}_{0} + (1 - 2^2)\, f(x_i) + \frac{2^2}{2}\left(f(x_{i-1}) + f(x_{i+1})\right) =$$

$$= -3\cos\left(\frac{\pi x_i}{2}\right) + 2\cos\left(\frac{\pi x_{i-1}}{2}\right) + 2\cos\left(\frac{\pi x_{i+1}}{2}\right), \quad i = 2 \dots 8^{*}$$

x_i	$\cos\left(\frac{\pi x_i}{2}\right)$	$U_i^1 \approx u(x_i, 0.5)$
-4	1	-3
-3	0	0
-2	-1	3
-1	0	0
0	1	-3
1	0	0
2	-1	3
3	0	0
4	1	-3



approximation of $u(x, 0.5)$ from the given points, using linear interpolation

* for $i = 1$ we don't have the $2\cos\left(\frac{\pi x_{i-1}}{2}\right)$ term and for $i = 9$ we don't have $2\cos\left(\frac{\pi x_{i+1}}{2}\right)$, because of the BCs

$\boxed{2-}$ Use D'Alembert method to find the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } t>0 \text{ and all } x \in \mathbb{R} \text{ with}$$

$$\underbrace{u(x,0) = \cos(x)}_{f(x)} \quad , \quad \frac{\partial u(x,0)}{\partial t} = xe^{-x^2} := g(x)$$

We apply d'Alembert solution and get:

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(s)\,ds \right] \quad (c=1)$$

$$= \frac{1}{2}\left[f(x+t) + f(x-t) + \int_{x-t}^{x+t} g(s)\,ds \right] =$$

$$= \frac{1}{2}\left[\cos(x+t) + \cos(x-t) + \int_{x-t}^{x+t} s e^{-s^2}\,ds \right] =$$

$$= \frac{1}{2}\left[\cos(x+t) + \cos(x-t) \right] + \frac{1}{2}\left(-\frac{1}{2}\right) e^{-s^2} \Big|_{x-t}^{x+t} =$$

$$= \frac{1}{2}\left[\cos(x+t) + \cos(x-t) \right] - \frac{1}{4}\left[e^{-(x+t)^2} - e^{-(x-t)^2} \right]$$

$\boxed{3}$ Solve the differential equation

$$\begin{cases} \frac{\partial u}{\partial t} - 2\frac{\partial^2 u}{\partial x^2} = 0 \\ u(0,t) = u(\pi, t) = 0 \quad \leftarrow \text{Dirichlet BCs} \\ u(x,0) = x(x-\pi) \end{cases}$$

for $x \in [0,\pi]$ and $t > 0$

We use separation of variables assuming $u(x,t) = F(x)\,G(t)$:

$$\frac{\partial}{\partial t}\big(F(x)\,G(t)\big) = 2\,\frac{\partial^2\big(F(x)\,G(t)\big)}{\partial x^2} \Rightarrow$$

$$\Rightarrow F(x)\,\dot{G}(t) = 2\,F''(x)\,G(t) \Rightarrow$$

$$\Rightarrow \frac{\dot{G}(t)}{2\,G(t)} = \frac{F''(x)}{F(x)} = k \ (\text{constant})$$

and putting $k = -\lambda^2$, $\lambda \in \mathbb{R}$ we get

$$\begin{cases} \dot{G}(t) = -\lambda^2 2\, G(t) \overset{\text{solve}}{\underset{\text{ODE}}{\Rightarrow}} G(t) = C\,e^{-\lambda^2 2t} \\ F''(x) = -\lambda^2 F(x) \overset{\text{solve}}{\underset{\text{ODE}}{\Rightarrow}} F(x) = A\cos\lambda x + B\sin\lambda x \end{cases}$$

Now we apply the BCs:

$$u(0,t) = u(\pi,t) = 0$$

$$u(0,t) = F(0)\,G(t) = 0 \Rightarrow A \cdot C\,\underbrace{e^{-\lambda^2 2t}}_{\neq 0} = 0 \Rightarrow AC = 0$$

but if $C = 0$ then $u \equiv 0$, therefore $\underline{A = 0}$

$$u(\pi,t) = F(\pi)\,G(t) = 0 \Rightarrow \underbrace{BC}_{\neq 0 \text{ (would lead to } u \equiv 0)} e^{-\lambda^2 2t}\sin(\lambda\pi) = 0 \Rightarrow \sin(\lambda\pi) = 0 \Rightarrow$$

$$\Rightarrow \lambda_n = n, \ n \in \mathbb{Z}$$

therefore

$$u_n(x,t) = \underbrace{BC}_{A_n}\sin(\lambda_n x)\,e^{-2\lambda^2 t} = A_n\,e^{-2\lambda^2 t}\sin(\lambda_n x)$$

Then the general solution is the superposition $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

Now applying the IC: $u(x,0) = f(x)$

$$A_0 + \sum_{n=1}^{\infty} A_n\,\cancel{e^0}\sin(nx) = f(x) = x(x-\pi)$$

Since $f(x) = x(x-\pi)$ is continuous and has a bounded derivative in $[0,\pi]$, we can compute its odd extension:

$$f(x) = \sum_{n=1}^{+\infty} b_n \sin(nx)$$, by comparison $A_0 = 0$ and $A_n = b_n$ in the IC equation.

Therefore, $$\boxed{u(x,t) = \sum_{n=1}^{+\infty} b_n e^{-2n^2 t} \sin(nx)}$$, where

$$b_n = \frac{2}{\pi} \int_0^{\pi} x(x-\pi) \sin(nx)\, dx = \frac{2}{\pi} \frac{\pi n \sin(\pi n) + 2\cos(\pi n) - 2}{n^3}$$

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