Exercise Sheet 7 — Pablo Díaz Viñambres

## Problem 1:

Riccati equation: $\left.\begin{array}{l} v'(t) - v^2(t) = 1 \\ v(0) = 0 \end{array}\right\}$, can be transformed into

a linear, hom. ODE with $v(t) = \dfrac{-u'(t)}{u(t)}$ and $u(0) = 1$:

$$\left\{\begin{array}{ll} u''(t) + u(t) = 0 & (3) \\ u'(0) = 0 & (4) \\ u(0) = 1 & (5) \end{array}\right.$$

a) Explain why the Ricc. eq. doesn't admit superposition on the original form

For an ODE to be admit superposition, it must be both linear and homogeneous, but since the term $-v^2(t)$ cannot be written as a linear combination of $1, v,$ and $v^{(k)}$, then it is not linear and doesn't admit superposition.

b) $u_1(t), u_2(t)$ satisfy (3) and (4). $u_1(0) = 3$ and $u_2(0) = 2$. Explain why $u_1(t) + u_2(t)$ doesn't satisfy IVP while $u_1(t) - u_2(t)$ does.

Both the sum and difference satisfy (3) and (4), since $(u_1+u_2)'' = u_1'' + u_2''$, $(u_1 - u_2)'' = u_1'' - u_2''$, etc

but the condition (5) is only satisfied for $(u_1 - u_2)(t) = u_1(t) - u_2(t) = 1$ $(t=0)$

## Problem 2:

| PDE | Linear? | Homogeneous? |
|---|---|---|
| | Can be written as a linear combination of 1, u, and partial derivatives of u ⇕ | All terms depend on u and its partial derivatives, and vanish for $u \equiv 0$ ⇕ |
| Burger's equation | No, $u \frac{\partial u}{\partial x}$ depends on u | Yes |
| Laplace equation in polar coordinates | Yes | Yes |
| Poisson equation | Yes | No, +2 doesn't depend on u |
| Convection-reaction equation | No, $u^n$ is not linear | Yes |
| One-dimensional heat equation | Yes | No, $\frac{1}{1+t^2}$ doesn't depend on u |
| Bi-harmonic equation | Yes | Yes |

## Problem 3:

Wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{FL}{m} \frac{\partial^2 u}{\partial x^2}$. We grab it at points $\frac{L}{4}$ and $\frac{3L}{4}$ until it reaches height H and release it $\left( \frac{\partial u}{\partial t}\Big|_{t=0} \equiv 0 \right)$. String fixed
(trapezoidal shape)
at both ends: $u(0,t) = u(L,t) = 0$.

a) Determine the wave speed c in terms of the given constants F, L and m.

The general wave equation is written as $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, therefore $c = \sqrt{\frac{FL}{m}}$ is the wave speed.

c) Find a general solution for $u(x,t)$ fulfilling the BCs.

(we will use compact notation $F(x)=F$, $G(t)=G$)

BCs $\begin{cases} u(0,t)=0 \\ u(L,t)=0 \end{cases}$ We try $u(x,t)=F(x)G(t) \Rightarrow$

$$\Rightarrow \frac{\partial}{\partial t^2} FG = c^2 \frac{\partial}{\partial x^2} FG \Rightarrow F\frac{d^2G}{dt^2} = c^2 G \frac{d^2F}{dt}$$

Assuming $FG \neq 0$, divide by $FG$, then $\underbrace{\frac{1}{c^2}\frac{\ddot{G}}{G}}_{g(t)} = \underbrace{\frac{F''}{F}}_{f(x)}$ and

$g(t)=f(x) \Leftrightarrow g \equiv f$, then $\begin{cases} \frac{\ddot{G}}{c^2G} = k \\ \frac{F''}{F} = k \end{cases} \Rightarrow \begin{cases} \ddot{G} - kc^2G = 0 \\ F'' - kF = 0 \end{cases}$

And we look at the possibilities for $k$:

- $k=0$

$\begin{cases} \ddot{G}=0 \\ F''=0 \end{cases} \Rightarrow \begin{cases} G = at+b \\ F = \tilde{a}x+\tilde{b} \end{cases} \Rightarrow u(x,t)=(at+b)(\tilde{a}t+\tilde{b})$ that can only satisfy BCs if $\tilde{a}=\tilde{b}=0$, which contradicts ICs

- $k>0 \Rightarrow k=\mu^2$, $\mu \in \mathbb{R}^*$

$\begin{cases} \ddot{G} = (\mu c)^2 G \\ F'' = \mu^2 F \end{cases} \Rightarrow \begin{cases} G = Ae^{-\mu ct} + Be^{\mu ct} \\ F = Ce^{-\mu x} + De^{\mu x} \end{cases} \Rightarrow \mu(x,t) = (Ae^{-\mu ct}+Be^{\mu ct})(Ce^{-\mu x}+De^{\mu x})$

enforcing the BCs:

$$\begin{cases} u(0,t)=0 \\ u(L,t)=0 \end{cases} \Leftrightarrow \begin{cases} F(0)=0 \\ F(L)=0 \end{cases} \Rightarrow \begin{cases} Ce^{-0}+De^{0}=0 \\ Ce^{-\mu L}+De^{\mu L}=0 \end{cases} \Rightarrow \begin{cases} C+D=0 \\ C+\cancel{C}^{2\mu L}=0 \end{cases} \Rightarrow \begin{cases} C=0 \\ D=0 \end{cases}$$

and $u(x,t)=0$, which contradicts ICs.

- $K<0$, $K=-\lambda^2$, $\lambda\in\mathbb{R}^*$

$$\Rightarrow \begin{cases} \ddot{G}+c^2\lambda^2 G=0 \\ F+\lambda^2 F=0 \end{cases} \Rightarrow \begin{cases} G=A\cos(c\lambda t)+B\sin(c\lambda t) \\ F=C\cos(\lambda x)+D\sin(\lambda x) \end{cases}$$

enforcing BCs:

- $u(0,t)=0 \Leftrightarrow F(0)=0 \Rightarrow C\cos 0+D\sin 0=0 \Rightarrow C=0 \Rightarrow F=D\sin(\lambda x)$
- $u(L,t)=0 \Leftrightarrow F(L)=0 \Rightarrow D\sin\lambda L=0 \Rightarrow D=0$ NO, would get $u\equiv 0$

$$\Rightarrow \cancel{D}\sin\lambda L=0 \Leftrightarrow \lambda L=n\pi \Leftrightarrow$$

$$\Leftrightarrow \lambda_n=\frac{n\pi}{L}, \quad n\in\mathbb{N}$$

Then $u_n(x,t)=F_n(x)\,G_n(t)$ for each $\lambda_n$

and $u_n(x,t)=D\sin(\lambda_n x)[A\cos(\underbrace{c\lambda_n}_{\omega_n}t)+B\sin(\underbrace{c\lambda_n}_{\omega_n}t)]=$

$$=\boxed{[A_n\cos(\omega_n t)+B_n\sin(\omega_n t)]\sin(\lambda_n x)} \quad \omega_n=c\lambda_n,\ \lambda_n=\frac{n\pi}{L},\ n\in\mathbb{N}$$

d) Now consider $L=1$ and apply the ICs to find the solution $u(x,t)$ of the IVP.

We have $$\frac{\partial u_n}{\partial t} = \left[-\omega_n A_n \sin(\omega_n t) + \omega_n B_n \cos(\omega_n t)\right] \sin(\lambda_n x) = 0$$

IC $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

$$\Rightarrow \left.\frac{\partial u_n}{\partial t}\right|_{t=0} = \omega_n \left[A_n \cdot 0 + B_n \cdot 1\right] \sin(\lambda_n x) = 0 \Rightarrow B_n = 0$$

and the other IC gives us:

$$u_n(x,0) = f(x) \Leftrightarrow A_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$ which is not possible in general,

but since ~~u(x,t)~~ the wave eq. is linear and homogeneous then ~~we can define~~

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$ ~~would~~ will also be a solution.

Therefore, by comparing with the Fourier coefficients of $f$, we arrive at

$$u(x,t) = \sum_{n=1}^{+\infty} b_n \cos\left(\frac{n\pi c}{L} t\right) \sin\left(\frac{n\pi x}{L}\right), \text{ where } b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

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plugging in $L=2$ and solving the $b_n$ integral, we get

$$b_n = \frac{1}{\pi^2 n^2}\, 2 \left(4H \sin\left(\frac{\pi n}{4}\right) + 2\pi H n t \sin\left(\frac{n\pi}{4}\right)\sin\left(\frac{n\pi}{2}\right) + 4H \sin\left(\frac{3\pi n}{4}\right) - 4H\sin(n\pi) + \pi(-H)\, n \cos\left(\frac{\pi n}{4}\right) + \pi H n \cos\left(\frac{3\pi n}{4}\right)\right)$$

and

$$u(x,t) = \frac{2}{\pi^2}\sum_{n=1}^{+\infty}\left[\frac{1}{n^2}\left(4H\sin\left(\frac{\pi n}{4}\right) + 2\pi H n \sin\left(\frac{\pi n}{4}\right)\sin\left(\frac{\pi n}{2}\right) + 4H\sin\left(\frac{3\pi n}{4}\right) - 4H\sin(\pi n) - \pi H n\cos\left(\frac{\pi n}{4}\right) + \pi H n \cos\left(\frac{3\pi n}{4}\right)\right)\cos\left(\pi n\sqrt{\frac{F}{m}}\,t\right)\sin\left(\frac{n\pi x}{L}\right)\right]$$