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Mathematics 4N : Assignment 2

$\boxed{2.-}$ Lagrange interpolation. Cardinal functions: $l_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$

a) Compute the cardinal functions for $x_0 = 0, x_1 = 1, x_2 = 4$

$$l_0(x) = \frac{x-1}{0-1} \cdot \frac{x-4}{0-4} = \frac{+(x-1)(x-4)}{4}$$

$$l_1(x) = \frac{x-0}{1-0} \cdot \frac{x-4}{1-4} = \frac{-x(x-4)}{3}$$

$$l_4(x) = \frac{x-0}{4-0} \cdot \frac{x-1}{4-1} = \frac{x(x-1)}{12}$$

b) Verify $\begin{cases} l_i(x_i) = 1 & \forall i = 0, \dots, n \\ l_j(x_j) = 0 & \forall i,j = 0, \dots, n \text{ with } i \neq j \end{cases}$

$$l_0(0) = \frac{(0-1)(0-4)}{4} = 1, \quad l_0(1) = \frac{(1-1)(1-4)}{4} = 0, \quad l_0(4) = \frac{(4-1)(4-4)}{4} = 0$$

$$l_1(0) = 0, \quad l_1(1) = \frac{-1(1-4)}{3} = 1, \quad l_1(4) = 0$$

$$l_4(0) = 0, \quad l_4(1) = 0, \quad l_4(4) = \frac{4(4-1)}{12} = 1$$

c) Given $(x_i, y_i)_{i=1}^{n}$, show that the interpolation polynomial is given by

$$p_n(x) = \sum_{i=0}^{n} y_i \, l_i(x)$$

- l_i is a polynomial of order n

PROOF

By definition, $l_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$, since $x - x_j \neq 0$ then l_i is clearly a polynomial of order n, since we are multiplying n terms of the form $x - a$.

- P_n is a polynomial of order n

PROOF

P_n is a polynomial of order n since it is a linear combination of polynomials of order n in the $\mathbb{P}_n$ space.

$i=0$

- $p_n(x_i) = y_i \quad \forall i = 0, \dots, n$

<u>PROOF</u>

$$p_n(x_i) = \sum_{j=0}^{n} y_j \, l_j(x_i) = y_i$$

$l_j(x_i) = 0$ if $j \neq i$
$l_j(x_i) = 1$ if $i = j$

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$\boxed{4-}$ Gauß - Legendre quadrature

$G(f)(-1, 1)$ with 3 nodes on $[-1,1]$ given by

$$x_0 = -\sqrt{\tfrac{3}{5}} \qquad x_1 = 0 \qquad x_2 = +\sqrt{\tfrac{3}{5}}$$
$$\omega_0 = \tfrac{5}{9} \qquad \omega_1 = \tfrac{8}{9} \qquad \omega_2 = \tfrac{5}{9}$$

a) Transfer to an arbitrary interval (a,b) to obtain an approximation

$$G(f)(a,b) \approx \int_a^b f(x)\,dx$$

$[-1,1] \longrightarrow [a,b]$

We can transfer the integral using the transformation $t \longmapsto \frac{b-a}{2}t + \frac{b+a}{2} = x$,

with $\frac{dx}{dt} = \frac{b-a}{2}dt$

Which gives us the approximation:

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)dt$$

$$\approx \frac{b-a}{2}\left[\frac{5}{9} f\left(\frac{b-a}{2}\cdot\left(-\frac{\sqrt{3}}{5}\right) + \frac{b+a}{2}\right) + \frac{8}{9} f\left(\frac{b+a}{2}\right) + \frac{5}{9} f\left(\frac{b-a}{2}\cdot\frac{\sqrt{3}}{5} + \frac{b+a}{2}\right)\right]$$

$$= \frac{b-a}{18}\left[5 f\left(\frac{a(1+\sqrt{3}) + b(1-\sqrt{3})}{2}\right) + 8 f\left(\frac{a+b}{2}\right) + 5 f\left(\frac{a(1-\sqrt{3}) + b(1+\sqrt{3})}{2}\right)\right]$$

b) What error estimate would you expect for the composite Gauß-Legendre rule G_m?

If we compare this formula to the ones found for the Simpsons Rule, midpoint Rule and Trapezoidal Rule, we can find a pattern:

SIMPLE ERROR

$$\int_a^b f(x)\,dx - Q(a,b) = \frac{(b-a)^{m+1}}{n} f^{(m)}(\xi) \text{ for some } \xi \in (a,b)$$

COMPOSITE ERROR

$$\int_a^b f(x)\,dx - Q_m(a,b) = \frac{(b-a)\,h^{m}}{n} f^{(m)}(\xi) \text{ for some } \xi \in (a,b)$$

Therefore, we can expect an error of

$$E(a,b) = \int_a^b f(x)dx - G_m(f)(a,b) = \frac{(b-a)h^6}{2016000} f^{(6)}(\xi), \ \xi \in (a,b)$$

c) Assume that (a,b) is small enough so $f^{(6)}$ is almost constant on (a,b).
Derive error estimates for E_1 and E_2 only based on the values of G_1 and G_2.

If we let $H = b-a$, then

$$I(a,b) - G_1(a,b) \approx CH^7$$
$$I(a,b) - G_2(a,b) \approx 2C\left(\frac{H}{2}\right)^7$$

therefore

$$G_2(a,b) - G_1(a,b) \approx CH^7\left(1-\frac{1}{2^6}\right) \Rightarrow CH^7 \approx \frac{64}{63}\left(G_2(a,b) - G_1(a,b)\right)$$

which gives us

$$E_1(a,b) = I(a,b) - G_1(a,b) \approx \boxed{\frac{64}{63}\left(S_2(a,b) - S_1(a,b)\right)}$$
$$E_2(a,b) = I(a,b) - G_2(a,b) \approx \frac{64}{63} \cdot \frac{1}{64}\left(S_2(a,b) - S_1(a,b)\right) =$$
$$= \boxed{\frac{1}{63}\left(S_2(a,b) - S_1(a,b)\right)}$$

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