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Mathematics 4N - Assignment 3

$\boxed{1-}$ Rounding errors. Estimate mesh size h for which roundoff and approximation errors become comparable.

Since the convergence is quadratic, we have

$$\left| \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} - u''(x) \right| \approx C h^2$$

but in reality $\Delta \bar{u} = u(x-h) - u(x)$ (or $\Delta \overset{+}{u} = u(x+h) - u(x)$) is computed with a computer accuracy error ε_m: $\tilde{\Delta}_u = \Delta_u \pm \varepsilon_m$. Therefore

$$\frac{\tilde{\Delta}_u^- + \tilde{\Delta}_u^+}{h^2} = \frac{\Delta_u^- + \Delta_u^+ \pm 2\varepsilon_m}{h^2} = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} \pm \frac{2\varepsilon_m}{h^2} \Rightarrow$$

$$\Rightarrow \frac{\tilde{\Delta}_u^- + \tilde{\Delta}_u^+}{h^2} \approx u''(x) + C h^2 \pm \frac{2\varepsilon_m}{h^2} \Rightarrow \frac{\tilde{\Delta}_u^- + \tilde{\Delta}_u^+}{h^2} - u''(x) \approx \underbrace{C h^2}_{\text{method's error}} + \underbrace{\frac{2\varepsilon_m}{h^2}}_{\text{truncation error}}$$

Then we can find when

$$|C h^2| \approx \left| \frac{2\varepsilon_m}{h^2} \right| \Rightarrow h^4 \approx \frac{2\varepsilon_m}{|C|} \Rightarrow h \approx \sqrt[4]{\frac{2\varepsilon_m}{C}}$$

, since we are only estimating the order of magnitude, we take $\cancel{h \approx \sqrt[4]{\varepsilon_m}}$

$$h \in O\left(\sqrt[4]{\varepsilon_m}\right) = O\left(\sqrt[4]{10^{-16}}\right) = \underline{O\left(10^{-4}\right)}$$

$\boxed{3 -}$ Fixed - point method.

$$f(x) = 2x^3 - x^2 + 2x - 1 = 0 \Rightarrow x = g(x) \begin{cases} g_1(x) := \dfrac{-2x^3 + x^2 + 1}{2} = x \\ g_2(x) := \sqrt[3]{\dfrac{x^2 - 2x + 1}{2}} = x \end{cases}$$

a) Find the expression for g_1' and g_2' and select one of them.

$$g_1'(x) = \frac{-6x^2 + 2x}{2} = -3x^2 + x$$

$$g_2'(x) = \frac{\dfrac{2x - 2}{2}}{\left(\dfrac{x^2 - 2x + 1}{2}\right)^{2/3}} = \frac{x - 1}{\left(\dfrac{x^2 - 2x + 1}{2}\right)^{2/3}}$$

It's better to select $\underline{g_1}$ as our function for many reasons: it's derivative it's much faster to compute (~~square~~ roots are very slow operations), it's also continuous and a polynomial and it's much easier to check if it's bounded on an interval.

$\boxed{4-}$ Fluid dynamics problem, we got the equation $x + 2.93 \ln(2.32x / R) = 0$ and want to find it's roots. Set $R = 5000$ and use fixed-point with

$$x^{(k+1)} = g(x^{(k)}) \quad , \quad g(x) := -2.93 \ln(2.32x / 5000)$$

a) Compute g' and determine whether $g(x)$, $x>0$ is increasing, decreasing or non-monotone

$$g'(x) = \frac{-2.93 \cdot \cancel{2.32/5000}}{\cancel{2.32/5000}\,x} = -\cancel{2.930}\,2.93 \cdot \frac{1}{x} < 0 \text{ if } x > 0 \Rightarrow$$

$\Rightarrow g(x)$ is decreasing in $(0, +\infty)$

b) ~~Since~~ Compute maximum and minimum values of $g(x)$ in $[e, e^3]$

Since $g(x)$ is decreasing on that interval, then

$$\max_{x \in [e, e^3]} g(x) = g(e) \approx 13.97$$

$$\min_{x \in [e, e^3]} g(x) = g(e^3) \approx 20.12$$

c) Determine the maximum value L of $|g'(x)|$ in the interval $[e, e^3]$.

Since $g''(x) = \frac{1.93}{x^2} > 0$ and $g'(x) < 0$ in that interval, then g' is an increasing negative function, therefore $\max_{x \in [e, e^3]} |g'(x)| = \left| \min_{x \in [e,e^3]} g'(x) \right| =$

$$|g(e)| \approx \underline{\underline{0.71 := L}}$$

d) For $x^{(0)} = e^2$, perform the first fixed-point iter.

$$x^{(0)} = e^2$$
$$x^{(1)} = g(x^{(0)}) = g(e^2) \approx 12.04$$

e) Based on the values of $x^{(0)}$, $x^{(1)}$ and L, find an upper bound for reaching a tolerance of 10^{-3}.

Since the problem satisfies the method constraints on $[e, e^3]$: $\begin{cases} g([e, e^3]) \subset [e, e^3] \\ |g'(x)| \leq L < 1 \ \forall x \in [e, e^3] \end{cases}$

as seen on b) and c), then we can use our a-priori estimate error formula for finding that upper bound:

$$e'_{k+1} \leq \frac{L^{k+1}}{1-L} |x^{(1)} - x^{(0)}| \Rightarrow 10^{-3} < \frac{0.71^{k+1}}{1-0.71} |12.04 - e^2|$$

$$\Rightarrow 10^{-3} < \frac{0.71^{k+1}}{0.29} 4.65 \Rightarrow 6.23 \cdot 10^{-5} < 0.71^{k+1} \Rightarrow$$

$$\Rightarrow \frac{\ln(6.23 \cdot 10^{-5})}{\ln(0.71)} - 1 < k \Rightarrow \underline{\underline{27.27 < k}}$$

So we will need at most 28 iterations to reach that tolerance (but the method usually converges faster, this is a generous upper bound).

$\boxed{5-}$ We are solving the equation $\cos\theta - \frac{K_\theta}{WL}\theta = 0$ from the structural system analysis. Consider $L = 1$ m, $K = 2$ N/m, $K_\theta = 3$ Nm/rad and $W = 4$ N.

a) Using the simplification $\cos\theta \approx 1$, find θ.

$$1 - \frac{3\ \cancel{N}m/\text{rad}}{4\cancel{N}\cdot 1\cancel{m}}\theta = 0 \Rightarrow \underline{\underline{\theta = \frac{4}{3}\ \text{rad}}}$$

b) Using $x^{(0)} = \frac{4}{3}$, compute by hand the first iteration of Newton's method.

~~First, we need to compute the derivative find a function g(x) such that f(x) = ...~~

First, we need to find the derivative of $f(\theta) = \cos\theta - \frac{K_\theta}{WL}\theta =$

$$= \cos\theta - \frac{3}{4}\theta$$

$f'(\theta) = -\operatorname{sen}\theta - \frac{3}{4}$, now let's apply the iterative formula:

$$x^{(1)} = x^{(0)} - \frac{f(x^{(0)})}{f'(x^{(0)})} = \frac{4}{3} - \frac{\cos\frac{4}{3} - 1}{-\operatorname{sen}\frac{4}{3} - \frac{3}{4}} \approx \underline{\underline{1.33}}$$