PABLO DÍAZ VIÑAMBRES

Mathematics 4N ex. sheet 1

$\boxed{1-}$ a) Compute all Taylor polynomials of $f(x) = x^4 + 2x^3 + x^2 + 5$ around $x_0 = 1$

A Taylor polynomial of degree n at x_0 is given by $P_n(x) = \sum_{k=0}^{n} \frac{f^{k)}(x_0)}{k!}(x-x_0)^k$

Since
$$f(1) = 9$$
$$f'(1) = 4x^3 + 6x^2 + 2x \Big]_{x=1} = 12$$
$$f''(1) = 12x^2 + 12x + 2 \Big]_{x=1} = 26$$
$$f'''(1) = 24x + 12 \Big]_{x=1} = 36$$
$$f^{IV)}(1) = 24$$
$$f^{V)}(1) = f^{k)}(0) = 0 \quad \forall k \geq 5$$

Then

$$9 + 12(x-1) + \frac{26}{2}(x-1)^2 + \frac{36}{6}(x-1)^3 + \frac{24}{24}(x-1)$$

$$= \underbrace{\underbrace{\underbrace{\underbrace{\underbrace{9}_{P_0(1)} + 12(x-1)}_{P_1(1)} + 13(x-1)^2}_{P_2(1)} + 6(x-1)^3}_{P_3(1)} + (x-1)^4}_{P_4(1) = P_k(1) \ \forall k \geq 4}$$

are the Taylor polynomials of the desired function. We could simplify P_4 and find that $P_4 = f$, which is true since the Taylor expansion of a polynomial is itself.

b) Compute the Taylor series of $g(x) = \ln(1+x)$ around $x_0 = 0$

A Taylor series can be seen as the infinite Taylor polynomial, we can define it at x_0

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{k)}(x_0)}{k!} (x - x_0)^k$$ and for $x_0 = 0$ we get

$$M(x) = \sum_{k=0}^{\infty} \frac{f^{k)}(0)}{k!} x^k$$, also called the McLaurin series.

Computing the derivatives: $g(0) = 0$

$$g'(x) = \frac{1}{1+x} \quad , \quad g'(0) = 1$$

$$g''(x) = \frac{-1}{(1+x)^2} \quad , \quad g''(0) = -1$$

$$g'''(x) = \frac{2}{(1+x)^3} \quad , \quad g'''(0) = 2$$

$$g^{IV)}(x) = \frac{-6}{(1+x)^4} \quad , \quad g^{IV)}(0) = -6$$

and we can easily prove that $g^{k)}(x) = \frac{(k-1)!}{(1+x)^k} \cdot (-1)^{k-1}$, by induction:

$\underline{n=1}$ $$g'(x) = \frac{(1-1)!}{(1+x)^1} \cdot (-1)^{1-1} = \frac{1}{1+x}$$

assume true for $n = k-1$, then for $\underline{n = k}$

$$g^{k)}(x) = \left(g^{k-1)}(x)\right)' = \left(\frac{(k-2)!}{(1+x)^{k-1}} (-1)^{k-2}\right)' = \frac{(k-1)!}{(1+x)^k} (-1)^{k-1}$$

Therefore $g^{k)}(0) = (k-1)! \, (-1)^{k-1}$ and we obtain

$$M(x) = \sum_{k=\cancel{0}\,1}^{\infty} \frac{(k-1)!\,(-1)^{k-1}}{k!} x^k = \sum_{k=\cancel{0}\,1}^{\infty} \frac{(-1)^{k-1}}{k} x^k =$$

$g(0)=0$ so we start at $k=1$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 \dots$$

PABLO DÍAZ VIÑAMBRES

$\boxed{2-}$ Let $u'(x) = \underbrace{\dfrac{3u(x) - 4u(x-h) + u(x-2h)}{2h}}_{u^*(x)} + e(h)$

a) Find an expression for the error $e(h)$

From the Taylor series:

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2!}u''(x) - \frac{h^3}{3!}u'''(x) + \dots \quad (1)$$

$$u(x-2h) = u(x) - 2hu'(x) + \frac{4h^2}{2!}u''(x) - \frac{8h^3}{3!}u'''(x) + \underbrace{\dots}_{\text{lower higher order terms, omitted}} \quad (2)$$

$\cancel{-4\cdot(1) + (2) \Rightarrow 4u(x-2h) - 4u(x-h) + u(x-2h)}$

$$4\cdot(1) - (2) \iff 4u(x-h) - u(x-2h) = 3u(x) - 2hu'(x) + h^2u''(x) + \frac{4}{3!}h^3u'''(x)$$

$$\iff \frac{3u(x) - 4u(x-h) + u(x-2h)}{2h} = u'(x) - \frac{h^2}{2h}u''(x) + \frac{4}{9}\frac{h^3}{2h}u'''(x)$$

$$\iff u^*(x) - u'(x) = -\cancel{\frac{h^2}{2h}}\frac{h}{2}u''(x) + \frac{4h^2}{98}u'''(x) \dots$$

Then $e(h) = -\dfrac{h}{2}u''(x) + \dfrac{4h^2}{18}u'''(x) \underbrace{\dots}_{}$

some less important terms, the convergence is quadratic since $e(h) \approx C h^2$

b) Now $u(x) = x\cos(x)$. Find approx to $u'(x)$ at $x = \pi/2$, $h = 0.2$

Substituting into the formula we get

$$u'(\pi/2) \approx \frac{\cancel{\frac{\pi}{2}3\cos(\frac{\pi}{2})} - 4(\frac{\pi}{2} - 0.2)\cos(\frac{\pi}{2} - 0.2) + \cos(\frac{\pi}{2} - 0.2)\cdot(\frac{\pi}{2} - 0.2)}{0.2} =$$

$$= \underline{\underline{-1.57501}}$$

the exact result is $u'(x) = \cos(x) - x\,\text{sen}(x) \mapsto u'(\frac{\pi}{2}) = -\frac{\pi}{2} = -1.570796321\dots$

Therefore $|e(h)| \approx 0.0042 < 10^{-2}$

PABLO DÍAZ VIÑAMBRES

$\boxed{3-}$ Find formula of the form $u''(x) \approx \frac{1}{h^2}\left(a_1 u(x) + a_2 u(x-h) + a_3 u(x - \frac{1}{2}h)\right)$

$$u''(x_c) = (u'(x_c))' \approx \frac{u'(x) - u'(x - {}^{1/2}h)}{(1/2)h} \underset{\substack{\uparrow \\ \text{use formula} \\ u'(x) \approx \frac{u(x) - u(x - 1/2 h)}{(1/2)h}}}{\approx} \frac{\frac{u(x) - u(x - 1/2 h)}{(1/2)h} - \frac{u(x - 1/2 h) - u(x-h)}{(1/2)h}}{(1/2)h} =$$

$$= \frac{4u(x) - 8u(x - 1/2 h) + 4u(x-h)}{h^2}$$, therefore $\underline{a_1 = 4,\ a_2 = 4,\ a_3 = -8}$

Since

$$u(x-h) = u(x) - h u'(x) + \frac{h^2}{2!} u''(x) - \frac{h^3}{3!} u'''(x) + \dots \quad \cdot 4$$

$$u(x - 1/2 h) = u(x) - \frac{1}{2} h u'(x) + \frac{h^2}{8} u''(x) - \frac{h^3}{8 \cdot 3!} u'''(x) + \dots \quad \cdot (-8)$$

then

$$\Rightarrow -8u(x - 1/2 h) + 4u(x-h) = -4u(x) - h^2 u''(x) + 2h^2 u''(x) + \frac{h^3}{3!} u'''(x) - \frac{3h^3}{3!} u'''(x) + \dots$$

$$\Rightarrow |u^*(x) - u'(x)| = \Big| \underbrace{-\frac{1}{3} h u'''(x)}_{\approx e(x) \approx C h} + \underset{\substack{\uparrow \\ \text{other higher order terms}}}{\dots} \Big|$$, and the method converges

~~quadratically~~ linearly

PABLO DIAZ ...

PABLO DÍAZ VIÑAMBRES

$\boxed{4-}$ Given the BVP: $u_{xx} + 2u_x + \pi^2 u = \cos(\pi x) - \pi(x+1)\sin(\pi x)$, $0 < x < 2$

$u(0) = 0$
$u(2) = 1$

a) Verify the exact solution is $u(x) = \frac{x}{2}\cos(\pi x)$

$$u(0) = 0$$
$$u(2) = \cos(2\pi) = 1$$
$$u_x = \frac{\cos(\pi x)}{2} - \frac{x\pi}{2}\operatorname{sen}(\pi x)$$
$$u_{xx} = -\frac{\operatorname{sen}(\pi x)\pi}{2} - \frac{\pi}{2}\operatorname{sen}(\pi x) - \frac{x\pi^2}{2}\cos(\pi x) =$$
$$= -\pi\operatorname{sen}(\pi x) - \frac{\pi^2}{2}x\cos(\pi x)$$

substituting:

$$-\pi\operatorname{sen}(\pi x) - \frac{\pi^2}{2}x\cos(\pi x) + 2\left(\frac{\cos(\pi x)}{2} - \frac{x\pi}{2}\operatorname{sen}(\pi x)\right) + \pi^2\left(\frac{x}{2}\cos\left(\frac{\pi}{x}\right)\right)$$
$$= (-\pi - x\pi)\operatorname{sen}(\pi x) + \left(\frac{-\pi^2}{2}x + 1 + \frac{\pi^2}{2}x\cos\left(\frac{\pi}{x}\right)\right) =$$
$$= \cos(\pi x) - \pi(x+1)\operatorname{sen}(\pi x)$$

b) Set up a finite difference scheme for this problem, using central differences. Use $\Delta x = \frac{2}{N}$.
Let $x_i = i\Delta x$, $i = 0, 1, \dots, N$

We use the central formulas
$$u'(x_i) \approx \frac{u(x_i+h) - u(x_i-h)}{2h} \mapsto \frac{u_{i+1} - u_{i-1}}{\cancel{h}(2/N)^2}$$
$$u''(x_i) \approx \frac{u(x_i+h) - 2u(x_i) + u(x_i-h)}{h^2} \mapsto \frac{u_{i+1} - 2u_i + u_{i-1}}{(2/N)^2}$$

~~then, since u(0)=0 and u(2)=1 we~~

PABLO DÍAZ VIÑANDRES

b) Set up a finite difference scheme for this problem, using central differences. Use $\Delta x = \frac{2}{N}$

Let $x_i = i\,\Delta x, \quad i = 0, 1, \dots, N$

We will use the central formulas: $u'(x_i) \approx \dfrac{u(x_i+h) - u(x_i-h)}{2h} = \dfrac{(u_{i+1} - u_{i-1})N}{4}$

$$u''(x_i) \approx \frac{u(x_i+h) - 2u(x_i) + u(x_i-h)}{h^2} = \frac{(u_{i+1} - 2u_i + u_{i-1})N^2}{4}$$

Substituting gives us:

$$\frac{N^2}{4}\left(u_{i+1} - 2u_i + u_{i-1}\right) + \frac{N}{2}\left(u_{i+1} - u_{i-1}\right) + \pi^2 u_i = \cos(\pi x_i) - \pi(x_i+1)\sin(\pi x_i) \Leftrightarrow$$

$$\Leftrightarrow u_{i+1}\left(\frac{N^2}{4} + \frac{N}{2}\right) + u_i\left(\frac{-N^2}{2} + \pi^2\right) + u_{i-1}\left(\frac{N^2}{4} - \frac{N}{2}\right) = \cos(\pi x_i) - \pi(x_i+1)\sin(\pi x_i)$$

Since $u_0 = 0$, $u_N = 1$ we arrive at a linear system

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{N^2}{4} - \frac{N}{2} & -\frac{N^2}{2} + \pi^2 & \frac{N^2}{4} + \frac{N}{2} & \dots & 0 \\ 0 & \frac{N^2}{4} - \frac{N}{2} & -\frac{N^2}{2} + \pi^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & & 1 \end{pmatrix}_{N \times N} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} 0 \\ \cos(\pi \cancel{u_1}^{2/N}) - \pi(\cancel{u_1}^{2/N} + 1)\sin(\pi \cancel{u_1}^{2/N}) \\ \cos(\pi \cancel{u_2}_{4/N}) - \pi(\cancel{u_2}_{4/N} + 1)\sin(\pi \cancel{u_2}_{4/N}) \\ \vdots \\ 1 \end{pmatrix}$$

that we can solve for u_i.

PABLO DÍAZ VIÑAMBRES

c) Now $N=4$. We arrange and solve the system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & \pi^2-8 & 6 & 0 & 0 \\ 0 & 2 & \pi^2-8 & 6 & 0 \\ 0 & 0 & 2 & \pi^2-8 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \cos\left(\frac{\pi}{2}\right) - \frac{3\pi}{2}\sin\left(\frac{\pi}{2}\right) \\ \cos(\pi) - \cancel{2\pi \sin(\pi)} \\ \cos\left(\frac{3\pi}{2}\right) - \frac{5\pi}{2}\sin\left(\frac{3\pi}{2}\right) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{3\pi}{2} \\ 1 \\ \frac{5\pi}{2} \\ 1 \end{pmatrix}$$

solving this we get

$u_1 = -2.4938$
$u_2 = -0.0083$
$u_3 = 1.0005$

(the results are very off, something must have gone wrong in the ~~[illegible]~~ solving)

NOT THE CASE, the approximation is just bad here for $N=4$ (Python gives the same result)

Then we have an error of

$e(1/2) = |u(1/2) - u_1| = +2.4938$
$e(1) = |u(1) - u_2| = 0.4917$
$e(3/2) = |u(3/2) - u_3| = 1.0005$

, ~~max~~ ~~e(...)~~
$e(2/2) = \max|u(x_i) - u_i| = \underline{2.49}$

d) The code was modified and run for $N = 10, 20, 40$ and we got the following errors

N	e(2/N)
10	1.760
20	0.2513
40	0.005664

We can deduce that the order is superlineal, probably quadratic from the nature of the finite difference methods for the second derivative.