Exercise #11

31. October 2022

Problem 1. (Ramp input)

A standard input function in the analysis of dynamic systems is the *linear ramp*, which can be seen as a "smoother alternative" to the Heaviside input. The unit ramp increases linearly from zero (at $t = 0$) to 1 (at time $t = a$), and then remains equal to 1 for $t > a$. In particular, for $a = 1$, we can write this function as

$$r(t) = \begin{cases} t & \text{if } t \leq 1\,, \\ 1 & \text{if } t > 1\,. \end{cases}$$

Instead of defining $r(t)$ *piecewise*, as done above, we can conveniently rewrite it in one single expression, with the help of a shifted Heaviside function

$$u(t-1) = \begin{cases} 0 & \text{if } t < 1\,, \\ 1 & \text{if } t > 1\,. \end{cases}$$

a) Remembering that unit steps can be used to "switch on/off" different parts of a function, rewrite $r(t)$ as *one single expression* containing only the functions $f_1(t) = t$, $f_2(t) = 1$ and $f_3(t) = u(t-1)$.

b) Using the expression obtained for item a), compute the Laplace transform of $r(t)$.

c) Using the Laplace transform, solve the ODE $y'' + 25y = r(t)$ with zero initial conditions.

d) Either sketch (by hand) or plot (using e.g. Python) the solution $y(t)$ from $t = 0$ to $t = 4$.

Problem 2. (Dirac input)

Consider the Dirac delta $\delta(t-a)$, with $a \geq 0$. In this exercise, you are asked to *use the Laplace transform* to solve various initial value problems, some of them containing Dirac inputs.

a) Solve the ODE $y'' + y = \delta(t)$, with zero initial conditions, and show that it has the same solution as the ODE $y'' + y = 0$, with $y(0) = 0$ and $y'(0) = 1$.
Remark: this result illustrates the important fact that, in a second-order system, the application of a Dirac impulse is equivalent to enforcing an "initial velocity".

b) Solve the ODE $y''' + 8y = 0$, with $y(0) = y'(0) = 0$ and $y''(0) = 1$.
Hint: remember the identity $s^3 + a^3 = (s+a)(s^2 - as + a^2)$.

c) Solve the ODE $y''' + 8y = \delta(t-1)$, with $y(0) = y'(0) = y''(0) = 0$.
Hint: see if you can reuse some of the computations done for item b).

Problem 3. (Convolution)

For two functions $f(t)$ and $g(t)$, consider the convolution defined as

$$f * g := \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau = \int_0^t g(\tau)f(t-\tau)\,\mathrm{d}\tau\,.$$

a) Compute $\mathrm{e}^{at} * \sin \omega t$, with a and ω being two (given) real constants.

b) Compute $t * t^n$, with n being a given natural number.

c) Using convolution, find the inverse Laplace transform of $F(s) = \frac{s}{(s^2+1)^2}$

Problem 4. (Integral equations)

Using the Laplace transform, solve the following integral(-differential) equations.

a) "Lightning-struck" RLC circuit:

$$Li'(t) + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau = \delta(t)\,,$$

with $L = 0.1$, $R = 11$, $C = 0.01$ and $i(0) = 0$.

b) $y(t) + 4t * y(t) = 2t.$

c) $y(t) + \cosh(t) * y(t) = t + e^t.$

The next exercises are optional and should not be handed in!

Problem 5. (Heaviside)

Below you will find four functions defined piecewise. Rewrite each of them *using only one expression* – with the aid of one or multiple **Heaviside** functions – and compute their Laplace transforms.

a)

$$f(t) = \begin{cases} -t^2 + 4t - 3 & \text{if } 1 < t < 3\,, \\ 0 & \text{otherwise .} \end{cases}$$

b)

$$f(t) = \begin{cases} t & \text{if } t \leq 1\,, \\ \cosh(t-1) & \text{if } t > 1\,. \end{cases}$$

c)

$$f(t) = \begin{cases} t & \text{if } t \leq 1\,, \\ \cosh(t-1) & \text{if } 1 < t < 2\,, \\ 0 & \text{if } t > 2\,. \end{cases}$$

d)

$$f(t) = \begin{cases} \sin \pi t & \text{if } t \leq 1\,, \\ 0 & \text{if } 1 < t < 2\,, \\ t & \text{if } t > 2\,. \end{cases}$$

– *Note:* If you feel the need to practice further with Laplace transforms, shifting theorems, convolution, etc, the exercises at the end of each section (Secs. 6.3–6.5) in the textbook should offer just the extra training you need.