Exercise Sheet 11

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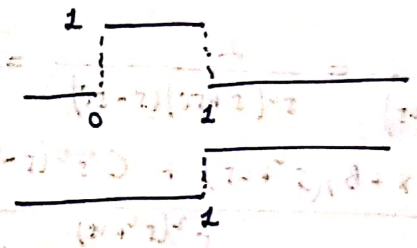
1-

Linear ramp:
$$r(t) = \begin{cases} t & \text{if } t \leq 1 \\ 1 & \text{if } t > 1 \end{cases}$$

a) Rewrite $r(t)$ using shifted Heaviside functions

If we consider
$$u_1(t) = u(t) - u(t-1)$$
$$u_2(t) = u(t-1)$$



we can write $r(t)$ as
$$r(t) = \big(u(t) - u(t-1)\big)\, t + u(t-1) =$$
$$= t\,u(t) + (1-t)\,u(t-1)$$

if we only consider $t \in [0, +\infty)$, then
$$r(t) = \underbrace{\big(1 - u(t-1)\big)}_{\text{ramp } 0 \text{ to } 1} t + \underbrace{u(t-1)}_{\text{constant}} = \big(f_2(t) - f_3(t)\big) f_1(t) + f_3(t) =$$
$$= \boxed{t + (1-t)\,u(t-1)}$$

c) Solve the ODE $y'' + 25y = v(t)$ with zero initial conditions
$y(0) = y'(0) = 0$

Applying the Laplace transform we get

$$\mathcal{L}[y'' + 25y] = \mathcal{L}[v(t)] \Rightarrow s^2\mathcal{L}[y] + 25\,\mathcal{L}[y] = \frac{1 - e^{-s}}{s^2} \Rightarrow$$

$$\Rightarrow \mathcal{L}[y] = \frac{1 - e^{-s}}{s^2 \cdot (s^2+25)} = \frac{1}{s^2(s^2+25)} - e^{-s}\frac{1}{s^2(s^2+25)}$$

$$\frac{1}{s^2(s^2+25)} = \frac{1}{s^2(s+5i)(s-5i)} = \frac{As+B}{s^2} + \frac{C}{s+5i} + \frac{D}{s-5i} =$$

$$= \frac{(As+B)(s^2+25) + Cs^2(s-5i) + Ds^2(s+5i)}{s^2(s^2+25)} = \frac{As^3 + 25As + Bs^2 + 25B + Cs^3 - 5iCs^2 + Ds^3 + 5iDs^2}{s^2(s^2+25)}$$

$$\Rightarrow \begin{cases} A + C + D = 0 \Rightarrow C + D = 1/25 \\ B - 5iC + 5iD = 0 \Rightarrow D - C = -1/25 \cdot (1/5i) \\ 25A + 25B = 0 \Rightarrow A = -1/25 \\ 25B = 1 \Rightarrow B = 1/25 \end{cases}$$

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b)

$$\mathcal{L}[v(t)] = \int_0^{+\infty} e^{-st}\left(t + (1-t)u(t-1)\right)dt =$$

$$= \mathcal{L}[t] - \left(\int_0^{+\infty} e^{-st}(t-1)\,u(t-1)\,dt\right) = \quad \mathcal{L}[u(t-a)f(t-a)] = e^{-as}\mathcal{L}[f(t)]$$

Second shift theorem

$$= \frac{1}{s^2} - e^{-s}\frac{1}{s^2} = \frac{1-e^{-s}}{s^2}$$

$$\Rightarrow \begin{cases} A = -1/25, \quad B = 1/25 \\ 2D = 1/25 - 1/25\,(1/5i) \Rightarrow D = \frac{1}{50}\left(1 - \frac{1}{5i}\right) = \frac{1}{50}\left(\frac{5i-1}{5i}\right) \cdot \left(\frac{i}{i}\right) = \\ \qquad = \frac{1}{50}\left(\frac{+5+i}{+5}\right) = \frac{1}{250}(5+i) \\ C = \frac{1}{25} - \frac{1}{250}(5+i) = \\ \quad = \frac{1}{250}(10-5-i) = \frac{1}{250}(5-i) \end{cases}$$

then $\frac{1}{s^2(s^2+25)} = \frac{1}{25}\left(\frac{-s+1}{s^2}\right) + \frac{1}{250}\left(\frac{5-i}{s+5i}\right) + \frac{1}{250}\left(\frac{5+i}{s-5i}\right)$

2- Dirac delta $\delta(t-a)$, $a \geq 0$, $\mathcal{L}[\delta(t-a)] = e^{-as}$

a) Solve the ODE $y'' + y = \delta(t)$ with zero ICs and show it has the same solution as $y'' + y = 0$, $y(0)=0$, $y'(0)=1$

Dirac δ: $\mathcal{L}[y'' + y] = \mathcal{L}[\delta(t)] \Leftrightarrow s^2\mathcal{L}[y] + \mathcal{L}[y] = 1 \Leftrightarrow$

$\Leftrightarrow \mathcal{L}[y] = \frac{1}{s^2+1} \Rightarrow y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin(t)$

Initial velocity: $\mathcal{L}[y''+y] = 0 \Leftrightarrow s^2\mathcal{L}[y] - sf(0) - f'(0) + \mathcal{L}[y] \Leftrightarrow$

$\Leftrightarrow s^2\mathcal{L}[y] + \mathcal{L}[y] = 1$, the same expression that we reached above

Therefore, the application of the Dirac delta impulse to a second-order system is equivalent to entering a initial velocity

b) Solve the ODE $y''' + 8y = 0$, $y(0) = y'(0) = 0$ and $y''(0) = 1$

$\mathcal{L}[y''' + 8y] = 0 \Rightarrow s^3\mathcal{L}[y] - 1 + 8\mathcal{L}[y] = 0 \Rightarrow$

$\Rightarrow \mathcal{L}[y] = \frac{1}{s^3+2^3} = \frac{1}{(s+2)(s^2-2s+4)} = \frac{A}{s+2} + \frac{Bs+C}{s^2-2s+4} =$

$= \frac{As^2 - 2As + 4A + Bs^2 + 2Bs + Cs + 2C}{(s+2)(s^2-2s+4)} \Rightarrow \begin{cases} A + B = 0 \\ -2A + 2B + C = 0 \Rightarrow 4B + C = 0 \\ 4A + 2C = 1 \Rightarrow -4B + 2C = 1 \end{cases}$

$3C = 1$

$C = 1/3$

$4B + 1/3 = 0 \Rightarrow B = -\frac{1}{12}$

$A = \frac{1}{12}$

then $\mathcal{L}[y] = \frac{1}{12}\frac{1}{s+2} + \frac{1}{12}\frac{-s+4}{s^2-2s+4}$

$s^2 - 2s + 4 = (s^2 - 2s + 1) + 3 = (s-1)^2 + 3$

$$\Rightarrow \mathcal{L}^{-1}[\mathcal{L}[y]] = \frac{1}{12}\left(\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{s-1}{(s-1)^2+3}\right] + \mathcal{L}\left[\frac{5}{(s-1)^2+3}\right]\right)$$

use First Shift theorem

$$\Rightarrow y(t) = \frac{1}{12}\left(e^{-2t} - e^{t}\cos(\sqrt{3}\,t) + \frac{5}{\sqrt{3}}e^{t}\sin(\sqrt{3}\,t)\right) := f_b(t)$$

c) Solve the ODE $y''' + 8y = \delta(t-1)$ with $y(0) = y'(0) = y''(0) = 0$

$\mathcal{L}[y''' + 8y] = \mathcal{L}[\delta(t-1)] \Rightarrow s^3\mathcal{L}[y] + 8\mathcal{L}[y] = e^{-s} \Rightarrow$

$$\Rightarrow \mathcal{L}[y] = e^{-s}\underbrace{\frac{1}{s^3+8}}_{\text{found inv. LT at b)}} \xRightarrow{\text{Second Shift theorem}} y(t) = u(t-1)\,f_b(t-1)$$

$\boxed{3-}$ Convolution $f*g := \int_0^t f(\tau)\,g(t-\tau)\,d\tau = \int_0^t g(\tau) f(t-\tau)\,d\tau$

a) Compute $e^{at} * \sin\omega t$ with a, ω real constants

We know that $\mathcal{L}[e^{at}] = \frac{1}{s-a}$, $\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$

using the Convolution Theorem we know that

$\mathcal{L}[f*g] = \mathcal{L}[f]\cdot\mathcal{L}[g] \Rightarrow \mathcal{L}[e^{at} * \sin\omega t] =$

$$= \frac{\omega}{(s^2+\omega^2)(s-a)} = \frac{\omega}{(s-\omega i)(s+\omega i)(s-a)} = \frac{A}{(s-\omega i)} + \frac{B}{(s+\omega i)} + \frac{C}{(s-a)}$$

$$= \frac{As^2 - aAs + \omega i As - \omega i a A + Bs^2 - aBs - \omega i Bs + a\omega i B + Cs^2 + \omega^2 C}{\ldots}$$

$$\begin{cases} A + B + C = 0 \Rightarrow C = -(A+B) \\ (\omega i - a)A - (\omega i + a)B = 0 \\ -\omega i a A + \omega i a B + \omega^2 C = \omega \Rightarrow ia(B-A) + \omega C = 1 \Rightarrow \end{cases}$$

$\Rightarrow ia(B-A) \quad -\omega(A+B) = 1 \Rightarrow$

$\Rightarrow$

b) $t * t^n$, $\mathcal{L}[t] = \frac{1}{s^2}$, $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

$$\mathcal{L}[t * t^n] = \frac{1}{s^2} \cdot \frac{n!}{s^{n+1}} = \frac{n!}{s^{n+3}} = \frac{(n+2)!}{s^{(n+2)+1}} \cdot \frac{1}{(n+1)(n+2)} \Rightarrow$$

$$\Rightarrow t * t^n = \frac{1}{(n+1)(n+2)} t^{n+2}$$

c) Using convolution, find the inverse Laplace Transform of $F(s) = \frac{s}{(s^2+1)^2}$

The convolution theorem also states $\mathcal{L}^{-1}[F \cdot G] = \mathcal{L}^{-1}[F] * \mathcal{L}^{-1}[G]$

$$F(s) = \frac{s}{(s^2+1)^2} = \frac{s}{s^2+1} \cdot \frac{1}{s^2+1} \text{ , then}$$

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] * \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] =$$

$$= \cos(t) * \sin(t)$$

$\boxed{4-}$ Solve the integral - differential equations using the Laplace transform:

a) $L i'(t) + R i(t) + \frac{1}{C} \int_0^t i(\tau) d\tau = \delta(t)$

$f(t) = 0.1 i'(t) + 11 i(t) + 100 \int_0^t i(\tau) d\tau = \delta(t)$, $i(0) = 0$

$$\mathcal{L}[f(t)] = \mathcal{L}[\delta(t)] \Rightarrow 0.1 s \mathcal{L}[i] + 11 \mathcal{L}[i] + 100 \cdot \frac{\mathcal{L}[i]}{s} = 1 \Rightarrow$$

$$\Rightarrow \mathcal{L}[i] \left(0.1 s + 11 + \frac{100}{s}\right) = 1 \Rightarrow$$

$$\Rightarrow \mathcal{L}[i] = \frac{s}{0.1 s^2 + 11 s + 100} = \frac{s}{0.1(s+10)(s+100)} = \frac{s}{0.1} \cdot \left(\frac{A}{s+100} + \frac{B}{s+10}\right)$$

$$= \frac{s}{0.1} \cdot \left(\frac{As + 10A + Bs + 100B}{(s+10)(s+100)}\right) \Rightarrow \begin{cases} 10A + 100B = 0 \Rightarrow \cancel{-10B + 100B = 0 \Rightarrow B =} \\ A + B = 1 \Rightarrow A = 1 - B \end{cases}$$

$10 - 10B + 100B = 0 \Rightarrow$

$1 - B = \frac{10}{9}$

$\Rightarrow -10 = 90B \Rightarrow B = \frac{-1}{9}$

$$\Rightarrow \mathcal{L}[i] = \frac{s}{0.1} \, 105 \left(\frac{10/9}{s+100} - \frac{1/9}{s+10} \right) =$$

$$= \frac{100}{9} \frac{s}{s+100} - \frac{10}{9} \frac{s}{s+10} =$$

$$= \frac{100}{9}\left(1 - \frac{100}{s+100}\right) - \frac{10}{9}\left(1 - \frac{10}{s+10}\right)$$

$$= \frac{100}{9} - \frac{1000}{9} \frac{20}{s+(20)^2} - \frac{10}{9} + \frac{10\sqrt{10}}{9} \frac{\sqrt{10}}{s+(\sqrt{10})^2} \Rightarrow$$

$$\Rightarrow i(t) = \frac{100}{9}\delta(t) - \frac{10}{9}\delta(t) - \frac{1000}{9}\sin(20t) + \frac{10\sqrt{10}}{9}\sin(\sqrt{10}\,t)$$

b) $y(t) + 4t * y(t) = 2t$

$$\mathcal{L}[y(t) + 4t * y(t)] = \mathcal{L}[2t] \Rightarrow \mathcal{L}[y(t)] + 4\mathcal{L}[t * y(t)] = 2\mathcal{L}[t] \Rightarrow$$

$$\Rightarrow \mathcal{L}[y(t)] + 4\mathcal{L}[t]\,\mathcal{L}[y(t)] = 2\mathcal{L}[t] \Rightarrow Y(s) + 4\,Y(s)\cdot\frac{1}{s^2} = \frac{2}{s^2} \Rightarrow$$

Convolution theorem

$$\Rightarrow Y(s)\left(1 + \frac{4}{s^2}\right) = \frac{2}{s^2} \Rightarrow Y(s) = \frac{2}{s^2+4} \Rightarrow$$

$$\Rightarrow \boxed{y(t) = \sin(2t)}$$

c) $y(t) + \cosh(t) * y(t) = t + e^t$

$$\mathcal{L}[y(t)] + \mathcal{L}[\cosh(t) * y(t)] = \mathcal{L}[t + e^t] \Rightarrow$$

$$\Rightarrow Y(s) + \frac{s}{s^2-1} Y(s) = \frac{1}{s^2} + \frac{1}{s-1} \Rightarrow$$

$$\Rightarrow Y(s)\underbrace{\left(1 + \frac{s}{s^2-1}\right)}_{\frac{s^2-1+s}{s^2-1}} = \frac{s-1+s^2}{s^2(s-1)} \Rightarrow Y(s) = \frac{\frac{s-1+s^2}{s^2(s-1)}}{\frac{s^2-1+s}{s^2-1}} =$$

$$= \frac{\cancel{(s^2+s-1)}(s+1)\cancel{(s-1)}}{\cancel{(s^2+s-1)}\,s^2\cancel{(s-1)}} = \frac{s+1}{s^2} = \frac{1}{s^2} + \frac{1}{s} \Rightarrow$$

$$\Rightarrow \boxed{y(t) = t + 1.}$$