$\boxed{\text{Exercise Sheet 10}}$

$\boxed{1-}$ Compute the Laplace transform of the following functions.

a) $f(t) = 5 + t^3 - 4t^6$

$$\mathcal{L}[f(t)] = 5\mathcal{L}[1] + \mathcal{L}[t^3] - 4\mathcal{L}[t^6] =$$

linearity

$$= \frac{5}{s} + \frac{6}{s^4} - \frac{2880}{s^7}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

b) $f(t) = te^{2t}$

$$\mathcal{L}[f(t)] = \mathcal{L}[e^{2t}.\, t] = F'(s-2) = \frac{1}{(t-2)^2}$$

where $f'(t) = t$; $F'(t) = \mathcal{L}[t] = \frac{1}{t^2}$

by the first shift theorem

c) $f(t) = e^{-t}\cos(5t)$

$$\mathcal{L}[f(t)] = \mathcal{L}[e^{-t}\cos(5t)] = \mathcal{L}[\cos(5t')] = \frac{t'}{(t')^2+25} = \frac{t+1}{(t+1)^2+25}$$

$t' = t+1$, ~~thus~~

first shift theorem

$\boxed{2-}$ Find the inverse Laplace transform of the following functions

INV. TRANSFORM $\boxed{\mathcal{L}^{-1}[\mathcal{L}[f(t)]] = f(t)}$

a) $F(s) = -\frac{4}{s^2} + \frac{3}{s^5} = -4\frac{1!}{s^{1+1}} + \frac{3}{4!}\frac{4!}{s^{4+1}} \Rightarrow$

$$\Rightarrow \mathcal{L}^{-1}[F(s)] = -4t + \frac{3}{4!}t^4 = -4t + \frac{1}{8}t^4$$

linearity and

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

b) $$F(s) = \frac{s+13}{s^2-6s+6} = \frac{s+13}{(s-3)^2-3} = \frac{s}{(s-3)^2-3} + \frac{13}{(s-3)^2-3} =$$

$$= \frac{s-3}{(s-3)^2-(\sqrt{3})^2} + \frac{16\sqrt{3}}{3}\,\frac{\sqrt{3}}{(s-3)^2-(\sqrt{3})^2}$$

Then $$\mathcal{L}^{-1}[F(s)] \underset{\text{linearity}}{=} \mathcal{L}^{-1}\left[\frac{s-3}{(s-3)^2-(\sqrt{3})^2}\right] + \frac{16\sqrt{3}}{3}\,\mathcal{L}^{-1}\left[\frac{\sqrt{3}}{(s-3)^2-(\sqrt{3})^2}\right] =$$

$$= e^{3t}\left[\cosh(\sqrt{3}\,t) + \frac{16\sqrt{3}}{3}\sinh(\sqrt{3}\,t)\right]$$

1st Shift theorem: $\mathcal{L}^{-1}[F(s-a)] = e^{at}f(t)$

$\mathcal{L}[\sinh(at)] = \frac{a}{s^2-a^2}$, $\mathcal{L}[\cosh(at)] = \frac{s}{s^2-a^2}$

c) $F(s) = \frac{1}{(s-1)(s^2+1)}$, $\frac{1}{(s-1)(s^2+1)} \underset{\text{part. frac. decomposition}}{=} \frac{A(s^2+1) + (Bs+C)(s-1)}{(s-1)(s^2+1)} =$

$$= \frac{As^2+A+Bs^2-Bs+Cs-C}{(s-1)(s^2+1)}$$

$$\begin{array}{l|l} \cdot (s)^2 & A+B=0 \Rightarrow 2+C+C=0 \Rightarrow C=-1 \\ \cdot (s)^1 & C-B=0 \Rightarrow B=C \Rightarrow B=-1 \\ \cdot (s)^0 & A-C=2 \Rightarrow A=2+C=1 \end{array}$$

then $F(s) = \frac{1}{s-1} - \frac{s+1}{s^2+1} = \frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}$

and $\mathcal{L}^{-1}[F(s)] = \underset{(A)}{e^t} - \underset{(B)}{\cos(t)} - \underset{(C)}{\sin(t)}$

(A) 1st shift theorem $\mathcal{L}^{-1}[F(s-a)] = e^{at} f(t)$

(A) $\mathcal{L}[1] = \frac{1}{s}$

(C) $\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}$

(B) $\mathcal{L}[\cos(at)] = \frac{s}{s^2+a^2}$

linearity

$\boxed{3-}$ Decide if the statements are true or false.

a) f, g have Laplace transform $\Rightarrow \mathcal{L}(f-g) = \mathcal{L}(f) - \mathcal{L}(g)$

This is <u>true</u> by linearity of the integral: $\begin{cases} \int f+g \, dx = \int f \, dx + \int g \, dx \\ \int \lambda f \, dx = \lambda \int f \, dx \end{cases}$ (works for definite and indefinite integrals)

$$\mathcal{L}(f-g) = \int_0^{+\infty} e^{-st}(f(t)-g(t))\,dt = \int_0 e^{-st} f(t) - e^{-st} g(t)\,dt =$$

$$= \int_0^{+\infty} e^{-st} f(t)\,dt - \int_0^{+\infty} e^{-st} g(t)\,dt = \mathcal{L}(f) - \mathcal{L}(g) \blacksquare$$

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b) If f and g have a Laplace transform $\Rightarrow \mathcal{L}(f\cdot g) = \mathcal{L}(f)\cdot\mathcal{L}(g)$

False, a counterexample is

$f(t) = t$, $g(t) = e^{at}$. with $\mathcal{L}(f) = \frac{1}{s^2}$, $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

but $\mathcal{L}(f\cdot g) = \mathcal{L}(te^{at}) \underset{\uparrow}{=} \frac{1}{(s-a)^2} \neq \frac{1}{s^2}\cdot\frac{1}{(s-a)} = \mathcal{L}(f)\cdot\mathcal{L}(g)$

1st shift theorem

c) If $0 \le f(t)$ $\forall t \geq 0 \Rightarrow \mathcal{L}(f)(s) \geq 0$ $\forall s \,/\, \exists \mathcal{L}(f)(s)$.

TRUE, since the integral satisfies the property

$f(x) \geq 0 \Rightarrow \int_a^b f(x)\,dx \geq 0$

$\forall x \in [a,b]$

b can be $\to +\infty$ for our case

then $\mathcal{L}(f)(s) = \int_0^{+\infty} \underbrace{e^{-st} f(t)}_{\geq 0 \text{ since } e^{-st} > 0 \text{ and } f(t) \geq 0}\,dt \geq 0$ $\quad \forall s \,/\, \exists \mathcal{L}(f)(s)$

d) If $0 \le f(t) \le 1$ $\forall t \geq 0 \Rightarrow \exists \mathcal{L}(f)(s)$ $\forall s > 0$

FALSE, a counterexample can be given using the Dirichlet function

$$f(t) = \begin{cases} 0 & \text{if } t \in \mathbb{Q} \\ 1 & \text{if } t \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

that is not piecewise continuous nor integrable, so the integral $\mathcal{L}[f(t)](s) = \int_0^{+\infty} e^{-st} f(t)\,dt$ doesn't exist

4- Use the Laplace transform to solve the following IVPs:

a) $y'' + y' - 6y = 0$, $y(0) = 1$, $y'(0) = 1$

We apply the Laplace transform

$$\mathcal{L}[y'' + y' - 6y] = 0 \Rightarrow s^2 Y(s) - s\,y(0) - y'(0) + s\,Y(s) - y(0) - 6Y(s) = 0$$

$$\Rightarrow (s^2 + s - 6)\,Y(s) - s - 2 = 0 \Rightarrow Y(s) = \frac{s+2}{s^2+s-6} = \frac{s+2}{(s+3)(s-2)} =$$

$$= \frac{A(s+3) + B(s-2)}{(s+3)(s-2)} = \frac{4}{5}\,\frac{1}{s-2} + \frac{1}{5}\,\frac{1}{s+3}$$

$$\begin{array}{l|l} (s)^1 & A + B = 1 \Rightarrow A = 1 - B \Rightarrow A = 4/5 \\ (s)^0 & 3A - 2B = 2 \Rightarrow 3 - 3B - 2B = 2 \Rightarrow 5B = 1 \Rightarrow B = 1/5 \end{array}$$

and

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{4}{5}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] =$$

$$= \frac{4}{5}\,e^{2t} + \frac{1}{5}\,e^{-3t}$$

b) $y''' + y' = 1$, $y(0) = 1$, $y'(0) = -1$, $y''(0) = -1$

We apply the Laplace transform

$$\mathcal{L}[y''' + y'] = \mathcal{L}[1] \Leftrightarrow s^3 Y(s) - s^2 y(0) - s\,y'(0) - \cancel{y''(0)} + s\,Y(s) - \cancel{y(0)} = \frac{1}{s}$$

$$\Leftrightarrow (s^3 + s)\,Y(s) - s^2 + s = \frac{1}{s} \Leftrightarrow$$

$$\Leftrightarrow Y(s) = \frac{s^2 - s}{s^2(s^2+1)} = \frac{s-1}{s(s^2+1)} =$$

$$= \frac{A}{s} + \frac{Bx + C}{s^2+1} = -\frac{1}{s} + \frac{s+1}{s^2+1} =$$

$$\begin{array}{l|l} (s)^2 & A + B = 0 \Rightarrow B = 1 \\ (s)^1 & C = 1 \\ (s)^0 & A = -1 \end{array}$$

$$= -\frac{1}{s} + \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

then $y(t) = \mathcal{L}^{-1}\left[-\frac{1}{s} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right] = -1 + \sin(t) + \cos(t)$

c) $y'' + 5y' + 6y = 0$, $y(0) = -2$, $y'(0) = 1$

We apply the Laplace transform:

$\mathcal{L}[y'' + 5y' + 6y] = 0 \Leftrightarrow s^2Y(s) - sy(0) - y'(0) + 5sY(s) - 5y(0) + 6Y(s) = 0$

$\Leftrightarrow (s^2+5s+6)Y(s) + 2s - 1 + 10 = 0 \Leftrightarrow$

$\Leftrightarrow Y(s) = \frac{-2s-9}{s^2+5s+6} = \frac{A}{s+2} + \frac{B}{s+3} =$

$= \frac{-5}{s+2} + \frac{3}{s+3}$

$(s)^1 \;\; A + B = -2 \Rightarrow A = -2 - B \Rightarrow A = -5$

$(s)^0 \;\; 3A + 2B = -9 \Rightarrow -6 - 3B + 2B = -9 \Rightarrow +B = +3$

and then

$y(t) = \mathcal{L}^{-1}\left[-\frac{5}{s+2} + \frac{3}{s+3}\right] = -5e^{-2t} + 3e^{-3t}$