Assignment 5

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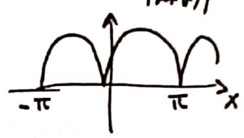
$\boxed{1-}$ Determine the coefficients of the trigonometric Fourier series for:

a) $f(x) = \cos^2 x = \dfrac{1+\cos 2x}{2} = \dfrac{1}{2} + \dfrac{1}{2}\cos 2x$, then

$a_0 = a_2 = \dfrac{1}{2}$ and the rest of the coefficients are zero.

b) $f(x) = |\sin(x)|$, even function, we only need to compute a_n terms:

period $= \pi$ $\Rightarrow$ $\left(L = \frac{\pi}{2}\right)$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)|\,dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\,dx = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} |\sin(x)| \cos(2nx)\,dx =$$

$$= \frac{4}{\pi}\int_0^{\pi} \sin(x)\cos(2nx)\,dx = \frac{4}{\pi}\int_0^{\pi} \frac{\sin((2n+1)x) + \sin((1-2n)x)}{2}\,dx =$$

$$= \frac{-2}{\pi}\left[+\frac{1}{2n+1}\cos((2n+1)x) + \frac{1}{1-2n}\cos((1-2n)x)\right]_0^{\pi} =$$

$\in\{1,3,\dots\} \Rightarrow \cos = -1$

$$= \cancel{+\frac{3}{(1-2n)^2}} - \frac{2}{1-4n^2}\Big[(1-2n)\cos((2n+1)\pi) - (1-2n)\underbrace{\cos(0)}_{1}$$
$$+ (1+2n)\cos((1-2n)\pi) - (1+2n)\underbrace{\cos(0)}_{1}\Big] =$$

$\in\{-1,-3,\dots\} \Downarrow \cos = -1$

$$= -\frac{2}{1-4n^2}\left(-2(1-2n) - 2(1+2n)\right) =$$

$$= 4$$

c) $f(x) = x^2 - x$ for $-1 \le x \le 1$, with period $T = 2L = 2 \Rightarrow L = 1$

$$a_0 = \frac{1}{2}\int_{-1}^{1} x^2 - x\,dx = \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{-1}^{1} = \frac{1}{2}\Big(\underbrace{-\frac{1}{6} - \frac{1}{3} - \frac{1}{2}}_{-1}\Big) =$$
$$= \cancel{1}\ -\frac{1}{2}$$

$$a_n = \int_{-1}^{1} (x^2 - x)\cos(\pi n x)\,dx = \underbrace{\sin(\pi n x)}_{0 \text{ for } x = 1,\,-1 \text{ and every } n}\frac{1}{\pi n}(x^2 - x)\Big|_{-1}^{1}$$

$$- \int_{-1}^{1}(2x-1)\sin(\pi n x)\cdot\frac{1}{\pi n}\,dx =$$

$$= \frac{-1}{\pi n}\int_{-1}^{1}(2x-1)\sin(\pi n x)\,dx = +\frac{1}{(\pi n)^2}\left(\cos(\pi n x)(2x-1)\Big|_{-1}^{1} - 2\int_{-1}^{1}\cos \pi n x\,dx\right) =$$

$$= \frac{1}{(\pi n)^2} \left(2\cos(\pi n) - 3\cos(-\pi n) - 2\underbrace{\cancel{\frac{\sin(\pi n)}{\pi n}}}_{0} + 2\underbrace{\cancel{\frac{\sin(-\pi n)}{\pi n}}}_{0} \right) =$$

$$= \frac{1}{(\pi n)^2} \left(\cos(\pi n) - 3\cos(+\pi n) \right) = \frac{1}{(\pi n)^2} \left(-2\cos(\pi n) \right) =$$

$$= \begin{cases} \frac{-2}{(\pi n)^2} & \text{if } n \text{ is even} \\ \frac{2}{(\pi n)^2} & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \int_{-1}^{1} (x^2 - x)\sin(\pi n x)\,dx = \frac{1}{\pi n}\left(-(x^2 - x)\cos(\pi n x)\Big|_{-1}^{1} + \int_{-1}^{1} (2x-1)\cos(\pi n x)\,dx \right)$$

$$= \frac{1}{(\pi n)} \left(2\cos(-\pi n) + \int_{-1}^{1} (2x-1)\cos(\pi n x)\,dx \right) =$$

$$= \frac{2\cos(\pi n)}{(\pi n)} + \frac{1}{(\pi n)^2} \left(\underbrace{\cancel{\sin(\pi n x)}}_{0 \text{ for all } n \text{ when } x = \pm 1} (2x-1)\Big|_{-1}^{1} - \int_{-1}^{1} 2\sin(\pi n x)\,dx \right) =$$

$$= \frac{2\cos(\pi n)}{(\pi n)} + \frac{2}{(\pi n)^3} \left(\underbrace{\cos(\pi n) - \cos(-\pi n)}_{0} \right) = \frac{2\cos(\pi n)}{(\pi n)} = \begin{cases} \frac{2}{(\pi n)} & \text{if } n \text{ is even} \\ \frac{-2}{(\pi n)} & \text{if } n \text{ is odd} \end{cases}$$

we get $a_n = -b_n$ for all $n > 0$!

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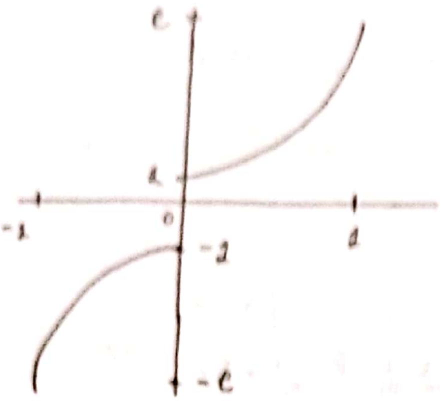
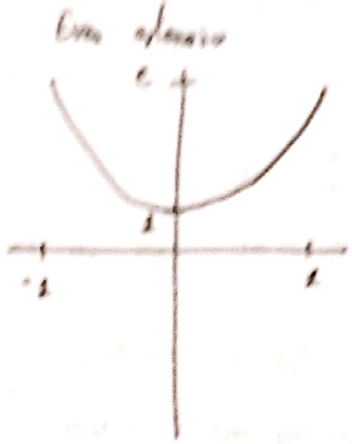
a) $f(x) = e^x$, $0 \le x \le 1$. Sketch both odd and even extensions on $[-1,1]$.

Odd extension



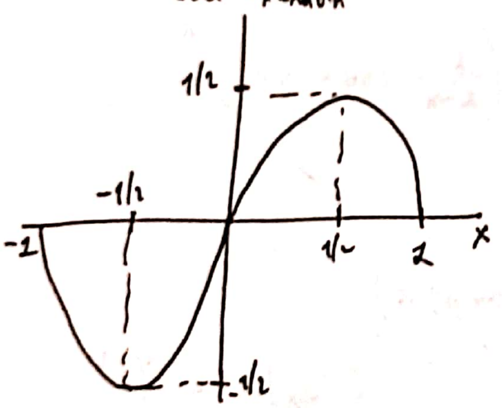
Even extension



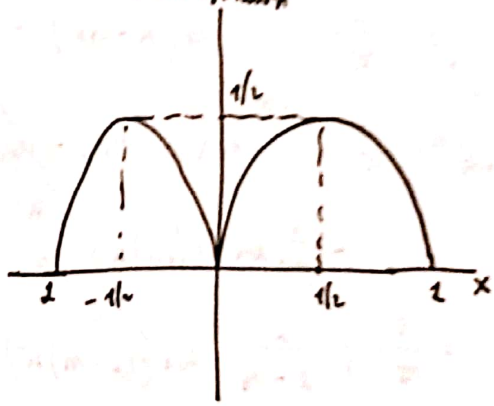
b) The even extension will yield better results since it is continuous on the interval, while the odd extension has a discontinuity at $x=0$ that will cause instability near that point when approximating.

c) $f(x) = x - x^2$, $0 \le x \le 1$

Odd extension



Even extension



Both are continuous, but the odd extension has a derivative at $x=0$:

$f'(x) = 1-2x$, $\quad f_o(-x) = -f(x) = x^2 - x \Rightarrow f_o(x) = x^2+x$

$f_o'(x) = 1+2x$

and $\nexists$ since $f_o'(0) = f'(0) = 1 \Rightarrow f_o^*$, the odd ext., has a derivative at that point

However, the even function doesn't:

$f_e^*(-x) = f_e(x) = x^2 - x \Rightarrow f_e'(x) = 2x - 1$

and $f'(0) = 1 \neq -1 = f_e'(x) \Rightarrow f_e^*$, the even ext, doesn't have a derivative at that point

3- Determine the coefficients of the complex Fourier series for:

a) $f(x) = \sin^2 x = \frac{1-\cos 2x}{2}$, where $a_0 = \frac{1}{2}$, $a_1 = -\frac{1}{2}$ then

$L = \frac{\pi}{2}$

$\sin\left(\frac{n\pi x}{L}\right) = \sin\left(\frac{n\pi x}{\pi/2}\right) = \sin(2nx)$

cos " cos "

$$c_n = \begin{cases} a_n & \text{for } n=0 \\ \frac{a_n - ib_n}{2} & \text{for } n>0 \\ \frac{a_n + ib_{-n}}{2} & \text{for } n<0 \end{cases} = \begin{cases} 1/2 & \text{for } n=0 \\ -1/4 & \text{for } n>0 \\ -1/4 & \text{for } n<0 \end{cases}$$

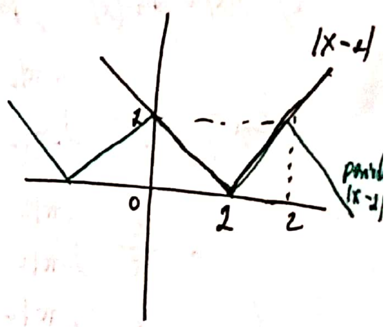
b) $f(x) = |x-1|$ for $x \in [-1,1]$ with period $T = 2L = 2 \Rightarrow L = 1$

$$\cancel{c_n = \frac{1}{2}\int_{-1}^{1} |x-1|\, d}$$

$$c_n = \frac{1}{2}\int_{-1}^{1} f(x)\, e^{-in\pi x}\, dx =$$

f(x) is even

$$= \int_{0}^{1} (1-x)\, e^{-in\pi x}\, dx =$$



$$= \frac{1}{-in\pi}\left[(1-x)\, e^{-in\pi x}\Big|_0^1 + \int_0^1 e^{-in\pi x}\, dx\right] =$$

$$= -\frac{1}{in\pi}\left[-e^0 - \frac{1}{in\pi}\left[e^{-in\pi} - e^0\right]\right] =$$

$$= \frac{1}{in\pi}\left(1 + \frac{1}{in\pi}\left(e^{-in\pi} - 1\right)\right) =$$

$$= \frac{1}{in\pi}\cdot\frac{i}{i} - \frac{1}{(n\pi)^2}\left(e^{-in\pi} - 1\right) = -\frac{i}{(n\pi)^2} - \frac{1}{(n\pi)^2}\left(\cos(-n\pi) + i\cancel{\sin(-n\pi)}\right) -$$

0

$$= -\frac{i}{(n\pi)^2} - \frac{\cos(n\pi) - 1}{(n\pi)^2} = \frac{1}{(n\pi)^2}\left(-\cos(n\pi) - 1 - i\right)$$

if n is even: $c_n = \frac{1}{n\pi^2}(-1 - i)$

if n is odd: $c_n = \frac{1}{n\pi^2}(0 - i)$

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$\boxed{4-}$

a) $f(x) = e^x$, $T = 2\pi$ Compute the error E_N for $N = 1, 4, 8$

~~$N = 2$~~ $\cdot$ ~~$\int_{-\pi}^{\pi} |f(x)|^2 dx = \int_{-\pi}^{\pi} e^{2x} dx = \frac{e^{2\pi} - e^{-2\pi}}{2} = \|f\|^2_{-\pi}^{\pi}$~~

~~$|c_1|^2 + |c_2|^2 = \left|\frac{e^{-\pi} - e^{\pi}}{2\pi(1+1^2)}(1+i)\right|^2 + \left|\frac{e^{+\pi} - e^{-\pi}}{2\pi(1+2^2)}(1+2i)\right|^2$~~

~~In general, if i is even~~

~~$|c_i|^2 + |c_{i+1}|^2 = e$~~

$$|c_i|^2 = \left(\frac{e^{-\pi} - e^{\pi}}{2\pi(1+n^2)}\right)^2 |1+in|^2 = (1+n)^2 \left(\frac{e^{-2\pi} + e^{2\pi} - 2}{4\pi^2 (1+n^2)^2}\right) \; ??$$

Using Parseval Identity:
$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \int_{-\pi}^{\pi} e^{2x} dx = \frac{e^{2\pi} - e^{-2\pi}}{2}$$

$$\parallel$$

$$2\pi \sum_{-\infty}^{\infty} |c_i|^2$$

$\boxed{4-}$ a) $f(x) = e^x$, $T = 2\pi$. Compute the error E_N for $N = 2, 4, 8$

$N=2$: $$\int_{-\pi}^{\pi} |f(x)|^2 dx = \int_{-\pi}^{\pi} e^{2x} dx = \frac{1}{2} e^{2x}\Big|_{-\pi}^{\pi} =$$

$$= \frac{e^{2\pi} - e^{-2\pi}}{2}$$

And we have $|c_1|^2 + |c_2|^2 = \left| \frac{e^{-\pi} - e^{\pi}}{2\pi(1+1^2)} (1+i) \right|^2 +$

$$\left| \frac{e^{\pi} - e^{-\pi}}{2\pi(1+2^2)} (1+2i) \right|^2 = \left(\frac{e^{-\pi} - e^{\pi}}{4\pi}\right) \underbrace{|1+i|^2}_{2} + \frac{e^{\pi} - e^{-\pi}}{10\pi} \underbrace{|1+2i|^2}_{5} =$$

$$\cancel{= \left(\frac{e^{-\pi} - e^{\pi}}{4\pi}\right)^2 + \left(\frac{e^{\pi} - e^{-\pi}}{10\pi}\right)^2 =}$$

multiply by conj, root and square ...

$$= \frac{e^{-2\pi} - e^{2\pi}}{2\pi} + \frac{e^{2\pi} - e^{-2\pi}}{2\pi} = \frac{e^{-2\pi} - e^{2\pi}}{\pi}$$

and $$E_N = \left| \frac{e^{2\pi} - e^{-2\pi}}{2} - \frac{e^{-2\pi} - e^{2\pi}}{\pi} \right| = \frac{e^{2\pi}(\pi+2) - e^{-2\pi}(\pi+2)}{2\pi}$$

$N=4$: $$|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 = \frac{(e^{-2\pi} - e^{2\pi})}{\pi} + \frac{e^{-\pi} - e^{\pi}}{20\pi} \underbrace{|1+3i|^2}_{20} + \frac{e^{\pi} - e^{-\pi}}{34\pi} |1$$

$$= \_\_\_$$