Exercise 9 Pablo Díaz Viñambres

$\boxed{1-}$ Heat equation $u_t = u_{xx}$, $0<x<1$ and $t>0$

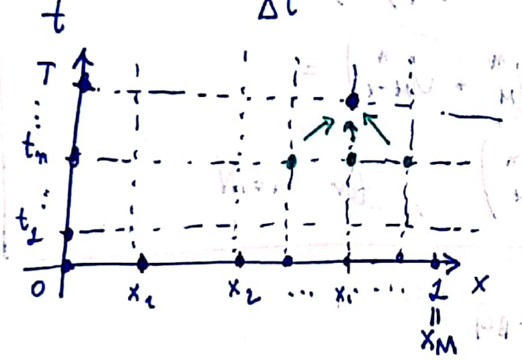
with IC $u(x,0) = f(x)$, $0<x<1$

and mixed BCs $\begin{cases} \partial_x u(0,t) = a\,(u(0,t) - g(t)) & a>0,\ g: \mathbb{R}^+ \to \mathbb{R},\ t>0 \\ \partial_x u(1,t) = 0 \ , & t>0 \end{cases}$

a) Set up an explicit FD scheme for the solution.

Plugging in the forward formula for u_t and the central one for u_{xx} we get the finite difference formula:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2} \Rightarrow$$



$$\Rightarrow U_i^{n+1} = U_i^n + \alpha\,(U_{i+1}^n - 2U_i^n + U_{i-1}^n) \quad (*)$$

$$\alpha := \frac{\Delta t c^2}{\Delta x}$$

The initial condition gives us $U_i^0 = f(x_i)$, now we need to get enough information from ~~4~~ the BCs to complete the stencil at the edges:

• $x=0 \longmapsto$ BC : $\dfrac{\partial u(0,t)}{\partial x} = a\,u(0,t) - a\,g(t)$ (Robin BC)

Applying central difference on $\dfrac{\partial u(0,t)}{\partial x}$ we get

$$\frac{\partial u(0,t)}{\partial x} = a\,u(0,t) - a\,g(t) \xmapsto[CD]{} \frac{U_1^n - U_{-1}^n}{2\Delta x} = a\,U_0^n - a\,g(t_n) \Rightarrow$$

$$\Rightarrow U_{-1}^n = -2a\,\Delta x\,(U_0^n - g(t_n)) + U_1^n$$

which gives us the ability to substitute the general stencil formula $(*)$ at $i=0$, if we already know U_0^n and U_1^n, which can be available from before.

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• $x = 1 \longmapsto$ BC: $\partial_x(1,t) = 0$ (Neumann BC)

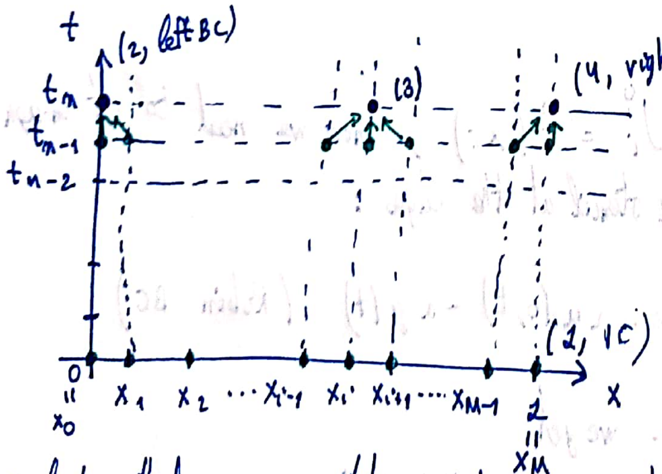
Applying central difference for $\partial_x(1,t)$ once again:

$$\partial_x(1,t) = 0 \xmapsto{CD} \frac{U_{M+1}^n - U_{M-1}^n}{2\Delta x} = 0 \Rightarrow U_{M+1}^n = U_{M-1}^n$$

and now we can substitute $(*)$ at $i = M$ ($x_M = 1$)

Putting it all together, we get

$$\boxed{\begin{aligned}
&(1)\ U_i^0 = f(x_i), \quad \text{for } i = 0, \dots, M\\
&(2)\ U_0^{n+1} = U_0^n + \alpha\left(U_1^n - 2U_0^n - 2a\Delta x\,(U_0^n - g(t_n)) + U_1^n\right)\\
&\qquad = U_0^n + 2\alpha\left(U_1^n - (1 + a\Delta x)\,U_0^n + a\Delta x\, g(t_n)\right), \text{ for } n \in \mathbb{N}\\
&(3)\ U_i^{n+1} = U_i^n + \alpha\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right), \quad \text{for } i = 1, \dots, M-1,\ n \in \mathbb{N}\\
&(4)\ U_M^{n+1} = U_M^n + \alpha\left(U_{M-1}^n - 2U_M^n + U_{M-1}^n\right) =\\
&\qquad = U_M^n + 2\alpha\left(U_{M-1}^n - U_M^n\right), \quad \text{for } n \in \mathbb{N}
\end{aligned}}$$



Explicit method map, computations can be done row by row

b) Set up an implicit FD scheme for the solution

We now apply the forward and central formulas, but we use the central one on t_{n+1} instead of t_n:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 x}{\partial x^2} \longmapsto \frac{U_i^{n+1} - U_i^{n}}{\Delta t} = \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{\Delta x^2} \Rightarrow$$

$$\Rightarrow -\alpha U_{i-1}^{n+1} + (1-2\alpha) U_i^{n+1} - \alpha U_{i+1}^{n+1} = U_i^n, \quad \text{for } i = 1, \dots, M-1 \ (*)$$

Again, we must apply the IC giving us $U_i^0 = f(x_i)$ and the BCs to avoid the ghost points U_{-1}^n and U_{M+1}^n:
(the deduction is equivalent to (a))

- $x = 0$:

$$U_{-1}^{n+1} = -2a\Delta x \left(U_0^{n+1} - g(t_n) \right) + U_1^{n+1}$$

- $x = 1$:

$$U_{M+1}^{n+1} = U_{M-1}^{n+1}$$

Therefore, we can apply the formula $(*)$ at $i = 0$:

$$+\alpha \left(a\Delta x \left(U_0^{n+1} - g(t_n) \right) \right) - \alpha U_1^{n+1} + (1-2\alpha) U_0^{n+1} - \alpha U_1^{n+1} = U_0^n$$

$$(\alpha a \Delta x + 1 - 2\alpha) U_0^{n+1} - 2\alpha U_1^{n+1} - \alpha a \Delta x\, g(t_n) = U_0^n$$

and $i = M$:

$$-\alpha U_{M-1}^{n+1} + (1-2\alpha) U_M^{n+1} - \alpha U_{M-1}^{n+1} = U_M^n$$

$$-2\alpha U_{M-1}^{n+1} + (1-2\alpha) U_M^{n+1} = U_M^n$$

and we get

left BC

$$\underbrace{\begin{pmatrix} \boxed{\alpha a \Delta x + 1 - 2\alpha} & \boxed{-2\alpha} & 0 & \cdots & 0 & 0 \\ \boxed{-\alpha} & \boxed{1-2\alpha} & \boxed{-\alpha} & \cdots & 0 & 0 \\ 0 & -\alpha & 1-2\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots 1-2\alpha & & -\alpha \\ 0 & 0 & 0 & \cdots & \boxed{-2\alpha} & \boxed{1-2\alpha} \end{pmatrix}}_{:= C \in \mathcal{M}_{(M+1)\times(M+1)}} \begin{pmatrix} U_0^{n+1} \\ U_1^{n+1} \\ \vdots \\ U_{M-1}^{n+1} \\ U_M^{n+1} \end{pmatrix} = \begin{pmatrix} U_0^n \\ U_1^n \\ \vdots \\ U_{M-1}^n \\ U_M^n \end{pmatrix} + \underbrace{\begin{pmatrix} \boxed{\alpha a \Delta x\, g(t_n)} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}}_{:= \vec{b} \in \mathcal{M}_{(M+1)\times 1}}$$

regular formula

right BC

left BC

Our final equations are

$$\boxed{\begin{aligned} \vec{U^0} &= \left(f(x_i)\right)_{(M+1)\times 1}, \quad \text{for } i=0,\dots,M \\ C\vec{U^{n+1}} &= \vec{U^n} + \vec{b}, \quad \text{for } n \in \mathbb{N} \end{aligned}}$$

where $(f(x_i))$ is the column vector with all the values of f at the points $x_0,\dots,x_M$ and C and $\vec{b}$ are where previously defined.

c) $a = \frac{\pi}{4}$, $g(t) = 0 \quad \forall t>0$ and

$$f(x) = \cos\left(\frac{\pi}{4}x\right) + \sin\left(\frac{\pi}{4}x\right)$$

Verify $u(x,t) = \left(\cos\left(\frac{\pi}{4}x\right) + \sin\left(\frac{\pi}{4}x\right)\right)e^{-\frac{\pi^2}{16}t}$ is a solution of the PDE.

1 - Satisfies IC:

$$u(x,0) = \cos\left(\frac{\pi}{4}x\right) + \sin\left(\frac{\pi}{4}x\right) e^0 = f(x) \quad \forall x \in [0,1] \checkmark$$

2 - Satisfies left IC:

$$\frac{\partial u}{\partial x} = \left(-\frac{\pi}{4}\sin\left(\frac{\pi}{4}x\right) + \frac{\pi}{4}\cos\left(\frac{\pi}{4}x\right)\right)e^{-\frac{\pi^2}{16}t}$$

$$\frac{\partial u(0,t)}{\partial x} = \underbrace{\frac{\pi}{4}}_{a}\underbrace{e^{-\frac{\pi^2}{16}t}}_{u(0,t)} = a\left(u(0,t) - g(t)\right) \quad \forall t \geq 0 \checkmark$$

3 - Satisfies right IC:

$$\frac{\partial u(1,t)}{\partial x} = \left(-\frac{\pi}{4}\frac{\sqrt{2}}{2} + \frac{\pi}{4}\frac{\sqrt{2}}{2}\right)e^{-\frac{\pi^2}{16}t} = 0 \quad \forall t \geq 0 \checkmark$$

4 - Satisfies heat equation:

$$\frac{\partial^2 u}{\partial x^2} = \left(-\frac{\pi^2}{16}\cos\left(\frac{\pi}{4}x\right) - \frac{\pi^2}{16}\sin\left(\frac{\pi}{4}x\right)\right)e^{-\frac{\pi^2}{16}t}$$

$$\parallel \qquad , \forall t \geq 0, \forall x \in [0,1]$$

$$\frac{\partial u}{\partial t} = -\frac{\pi^2}{16}\left(\cos\left(\frac{\pi}{4}x\right) + \sin\left(\frac{\pi}{4}x\right)\right)e^{-\frac{\pi^2}{16}t} \checkmark$$

$\boxed{2-}$ Use the Fourier Transform to find the solution to the PDE

$$u_t = t u_{xx}, \quad x \in \mathbb{R} \text{ and } t>0$$

with the IC

$$u(x,0) = e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$$

We apply the Fourier transform to the ODE and get
(x variable)

$$\mathcal{F}_x\left(\frac{\partial u}{\partial t}\right) = \mathcal{F}_x\left(t\frac{\partial^2 u}{\partial x^2}\right) \Leftrightarrow \frac{\partial}{\partial t}\left[\mathcal{F}_x(u)\right] = t(i\omega)^2 \mathcal{F}_x(u) \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial}{\partial t}\hat{u}(\omega,t) = -t\omega^2 \hat{u}(\omega,t) \underset{\text{Solve ODE}}{\Rightarrow} \hat{u}(\omega,t) = e^{-\frac{1}{2}t^2\omega^2}\cdot \phi$$

The initial condition gives us $u(x,0) = \underset{f(x)}{\underbrace{e^{-x^2/2}}}$, then

$$\mathcal{F}_x(u(x,0)) = \mathcal{F}_x(f(x)) \Rightarrow \hat{u}(\omega,0) = \hat{f}(\omega) \underset{e^{-1/2\cdot 0^2\cdot\omega^2}\cdot\phi = \phi}{\Rightarrow} \phi(\omega) = \hat{f}(\omega)$$

Then the solution on the freq. domain is given by

$$\hat{u}(\omega,t) = \hat{f}(\omega)e^{-1/2\cdot t^2\omega^2}$$

Now,

$$u(x,t) = \mathcal{F}^{-1}(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{f}(\omega)\, e^{-1/2 t^2\omega^2} e^{i\omega x}\,dx =$$

$$= \int_{-\infty}^{+\infty} \hat{f}(\omega)\,\hat{g}(\omega)\,e^{i\omega x}\,dx, \text{ where } \hat{g}(\omega) = \frac{e^{-1/2 t^2\omega^2}}{\sqrt{2\pi}}$$

Since $\mathcal{F}(f*g) = \mathcal{F}(f)\,\mathcal{F}(g)\sqrt{2\pi} \Rightarrow (f*g)(x) = \sqrt{2\pi}\,\mathcal{F}^{-1}(\hat{f}\cdot\hat{g}) =$

$= \int_{-\infty}^{+\infty} \hat{f}\,\hat{g}\,e^{i\omega x}\,d\omega$ then

$$u(x,t) = (f*g), \text{ where } g(x,t) = \mathcal{F}_x^{-1}\left(\frac{e^{\overset{-k=-\frac{1}{2}t^2}{\overbrace{-1/2 t^2}}\omega^2}}{\sqrt{2\pi}}\right) \underset{\text{Gaussian IT}}{=}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{2\left(\frac{1}{2}t^2\right)}}\, e^{-\frac{x^2}{4\left(\frac{1}{2}t^2\right)}} = \frac{1}{t\sqrt{2\pi}}\, e^{-\frac{x^2}{2t^2}}$$

Finally, $u(x,t) = \left(e^{-x^2/2} * \dfrac{e^{\frac{-x^2}{2}\cdot t^2}}{t\sqrt{2\pi}} \right)$

$\boxed{3-}$ Use the Fourier Transform to find the solution to the telegraph equation

$$u_{tt} + 2u_t + u = u_{xx} \quad , \quad x \in \mathbb{R} \, , \, t > 0$$

with ICs $\left.\begin{array}{l} u(x,0) = \text{sinc}(x) \\ \partial_t u(x,0) = -\text{sinc}(x) \end{array}\right\}$ for $x \in \mathbb{R}$, $\text{sinc}(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

We first apply the FT:

$$\mathcal{F}_x(u_{tt} + 2u_t + u) = \mathcal{F}_x(u_{xx}) \Rightarrow$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\hat{u}(\omega,t) + 2\frac{\partial}{\partial t}\hat{u}(\omega,t) + \hat{u}(\omega,t) = (i\omega)^2\hat{u}(\omega,t)$$

$$\Rightarrow \frac{\partial}{\partial t^2}\hat{u}(\omega,t) + 2\frac{\partial}{\partial t}\hat{u}(\omega,t) = -(1+\omega^2)\hat{u}(\omega,t)$$

~~If we define $\hat{g}(\omega,t) = \frac{\partial}{\partial t}\hat{u}(\omega,t)$ this become~~

$$\Rightarrow \frac{\partial}{\partial t^2}\hat{u}(\omega,t) + 2\frac{\partial}{\partial t}\hat{u}(\omega,t) + (1+\omega^2)\hat{u}(\omega,t) = 0$$, we will solve this ODE:

Characteristic equation roots:

$$\lambda^2 + 2\lambda + (1+\omega^2) = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 4(1+\omega^2)}}{2} = -1 \pm \sqrt{1 - 1 - \omega^2} =$$

$$= -1 \pm \omega i$$

Then the solution base of the higher order ODEs:

$$\{e^{-t}\cos(\omega t), e^{-t}\sin(\omega t)\}$$, then

$$\hat{u}(\omega,t) = Ae^{-t}\cos(\omega t) + Be^{-t}\sin(\omega t)$$ (the solution was checked at the end of the pdf)

We have the initial conditions:

$u(x,0) = \text{sinc}(x) \Rightarrow \hat{u}(\omega,0) = \mathcal{F}_x(\text{sinc}(x)) = \widehat{\text{sinc}}(\omega)$

then $Ae^{-0}\cos(0) + Be^{-0}\sin(0) = \widehat{\text{sinc}}(\omega) \Rightarrow A = \widehat{\text{sinc}}(\omega)$

$\partial_t u(x,0) = -\text{sinc}(x) \Rightarrow \partial_t\hat{u}(\omega,0) = -\widehat{\text{sinc}}(\omega) \Rightarrow$

$$\Rightarrow (-A\omega - B)\, e^{-0} \sin(0) + (-A + B\omega)\, e^{-0} \cos(0) = -\widehat{\text{sinc}}(\omega) \Rightarrow$$

$$\Rightarrow -\widehat{\text{sinc}}(\omega) + B\omega = -\widehat{\text{sinc}}(\omega) \Rightarrow B = 0$$

then the solution to the PDE in the duplace domain is

$$\hat{u}(\omega, t) = -\widehat{\text{sinc}}(\omega)\, e^{-t} \cos(\omega t)$$

and

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} -\widehat{\text{sinc}}(\omega)\, e^{-t} \cos(\omega t)\, e^{i\omega x}\, dx =$$

$$= \int_0^{+\infty} \widehat{\text{sinc}}(\omega) \left(-\frac{e^{-t} \cos(\omega t)}{\sqrt{2\pi}} \right) e^{i\omega x}\, dx$$

$$= (\text{sinc}(x) * g) \quad \text{where } g = \mathcal{F}_x^{-1}\left(-\frac{e^{-t}\cos(\omega t)}{\sqrt{2\pi}} \right) =$$

$$= -\frac{e^{-t}}{\sqrt{2\pi}} \mathcal{F}_x^{-1}(\cos(\omega t)) =$$

$$= -\frac{e^{-t}}{\sqrt{2\pi}}$$

Table lookup

$e^{-t}\cos(\omega t) = e\cos \qquad e^{-t}\sin(\omega t) = e\sin$

$\hat{u}: Ae^{-t}\cos(\omega t) + Be^{-t}\sin(\omega t)$

$\frac{\partial \hat{u}}{\partial t}: -Ae^{-t}\cos(\omega t) - \omega A e^{-t}\sin(\omega t) - Be^{-t}\sin(\omega t) + B\omega e^{-t}\cos(\omega t) =$

$= (-A\omega - B)\, e\sin + (-A + B\omega)\, e\cos$

$\frac{\partial^2 \hat{u}}{\partial t^2}: = (A\omega + B)\, e\sin + (-A\omega^2 - B\omega)\, e\cos + (A - B\omega)\, e\cos + (A\omega - B\omega^2)\, e\sin =$

$= (-B\omega^2 + 2A\omega + B)\, e\sin + (-A\omega^2 - 2B\omega + A)\, e\cos$

$e\sin$:

$\underbrace{-B\omega^2 + 2A\omega + B}_{\partial^2} + 2\underbrace{(-A\omega - B)}_{\partial} + \underbrace{(1+\omega^2)B}_{\hat{u}} =$

$= \cancel{-B\omega^2} + \cancel{2A\omega}^{+B} \cancel{-2A\omega} \cancel{-2B} + \cancel{B} + \cancel{B\omega^2} = \cancel{B}\ 0$

$e\cos$:

$\underbrace{-A\omega^2 - 2B\omega + A}_{\partial^2} + 2\underbrace{(-A + B\omega)}_{\partial} + \underbrace{(1+\omega^2)A}_{\hat{u}} =$

$= \cancel{-A\omega^2} \cancel{-2B\omega} \cancel{+A} \cancel{-2A} \cancel{+2B\omega} \cancel{+A} + \cancel{A\omega^2} = 0$