Assignment 4 Pablo Díaz Viñambres

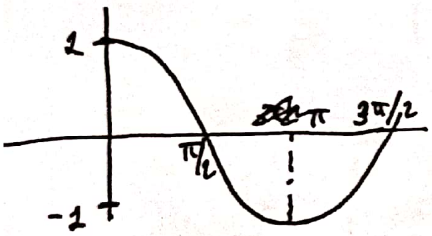
$\boxed{2-1}$ $f: \mathbb{R} \to \mathbb{R}$ is periodic if $\exists\, p>0 \,/\, f(x+p) = f(x) \;\; \forall x \in \mathbb{R}$.
The fundamental period of f is the smallest number that satisfies this.

a) Prove or give c.e.: Every periodic function has a fundamental period.

The statement is true for most periodic functions but if we take a constant function $f(x) = a$, $\forall x \in \mathbb{R}$. Then $f(x+p) = f(x) = a$ for all $p>0$, and therefore there isn't a smallest period, since the set $(0, +\infty)$ is open.

b) Find the fundamental period of the following functions:

• $f(x) = \cos x$

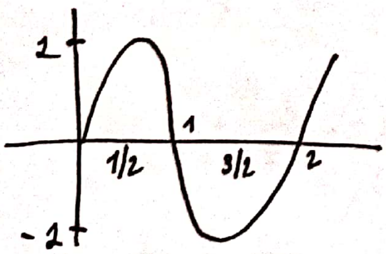


The cosine function satisfies $\cos(x) = \cos(x + 2k\pi)$ $\forall x \in \mathbb{R}, \forall k \in \mathbb{N}$

therefore, periods are ~~[illegible]~~ and $\{2\pi, 4\pi, \dots\}$

$p = 2\pi$ is the fundamental period.

~~4~~ • $f(x) = \sin(\pi x)$

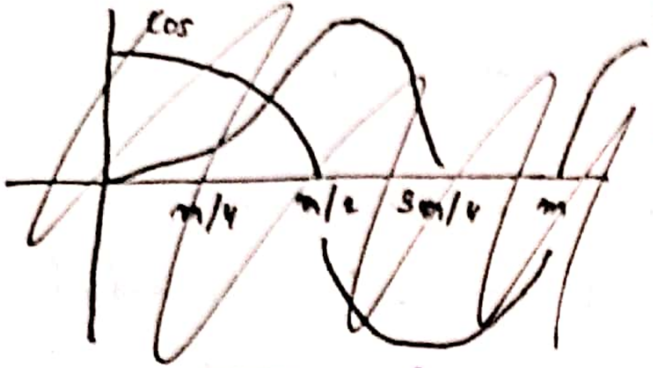


Since $\sin(\pi x) = \sin(\pi x + 2k\pi) = \sin(\pi(x+2k))$ $\forall x \in \mathbb{R}, \forall k \in \mathbb{N}$

then periods are $\{2, 4, 6, 8, \dots\}$ and $p = 2$ is the fundamental period.

Fo $\cdot\, f(x) = \cos\left(\frac{2\pi}{m}x\right) + \sin\left(\frac{2\pi}{n}x\right)$, $n, m \in \mathbb{N}$

a)



For $\cos\left(\frac{2\pi}{m}x\right)$, the periods will be $\{m, 2m, 3m, \dots\}$

since $\cos\left(\frac{2\pi}{m}x\right) = \cos\left(\frac{2\pi}{m}x + 2k\pi\right) =$

$= \cos\left(\frac{2\pi}{m}(x + mk)\right)$, therefore the fundamental period is m. Same goes for the $\sin\left(\frac{2\pi}{n}x\right)$.

Therefore, since $\cos\left(\frac{2\pi}{m}x\right) = \cos\left(\frac{2\pi}{m}(x+m)\right)$ and

$\sin\left(\frac{2\pi}{m}x\right) = \sin\left(\frac{2\pi}{m}(x+m)\right)$ then

$$f(x) = f(x+m)$$

and m is the fundamental period of f.

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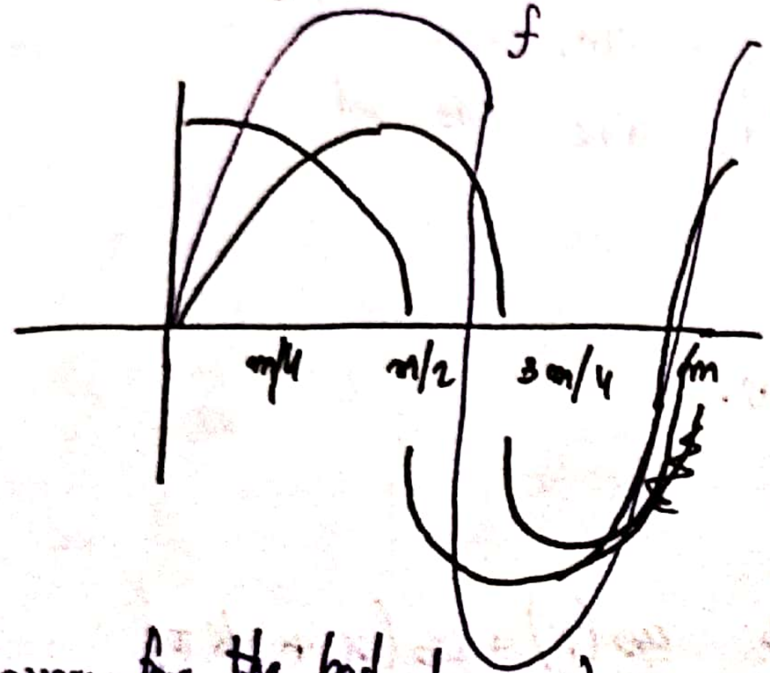


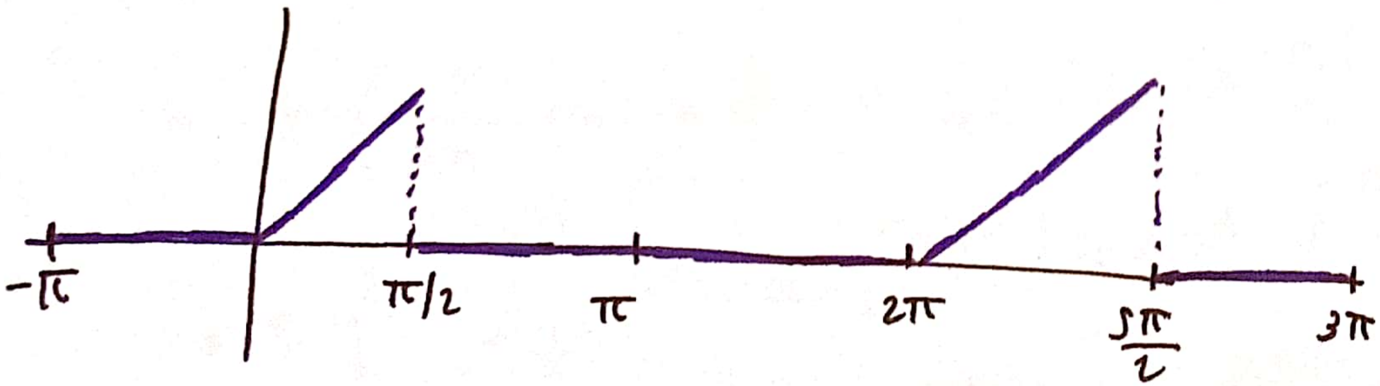
(sorry for the bad drawing)

$\boxed{3-}$

Sketch the following 2π-periodic functions over $-3\pi < x < 3\pi$ and find their Fourier series. In each case, plot the truncated series (Python)

a) $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \text{ or } \frac{\pi}{2} < x \le \pi \\ x & \text{if } 0 \le x \le \pi/2 \end{cases}$

$(L = \text{half-period} = \pi)$



Now, we find the Fourier Coefficients

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi/2} x\,dx = \frac{1}{2\pi}\left(\frac{1}{2}x^2\right)\Big|_0^{\pi/2} = \frac{1}{2\pi}\,\frac{\pi^2}{8} = \frac{\pi}{16}$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \frac{1}{\pi}\int_0^{\pi/2} x\cos(mx)\,dx =$$

$$= \frac{1}{\pi}\left[\frac{x}{m}\sin(mx)\Big|_0^{\pi/2} - \frac{1}{m}\int_0^{\pi/2}\sin(mx)\,dx\right] =$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2m}\sin\left(m\frac{\pi}{2}\right) + \frac{1}{m^2}\left(\cos\left(m\frac{\pi}{2}\right) - 1\right)\right]$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{1}{\pi}\int_0^{\pi/2} x\sin(mx)\,dx =$$

$$= \frac{1}{\pi}\left[\frac{-x}{m}\cos(mx)\Big|_0^{\pi/2} + \frac{1}{m}\int_0^{\pi/2}\cos(mx)\,dx\right] =$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2m}\cos\left(m\frac{\pi}{2}\right) + \frac{1}{m^2}\left(\sin\left(m\frac{\pi}{2}\right)\right)\right]$$

then

$$S_N(x) = \frac{\pi}{16} + \sum_{m=1}^{N} \frac{1}{\pi}\left(\frac{\pi}{2m}\sin\left(m\frac{\pi}{2}\right) + \frac{1}{m^2}\left(\cos\left(\frac{m\pi}{2}\right) - 1\right)\right)\cos(mx) + \underbrace{\frac{1}{\pi}\left(-\frac{\pi}{2m}\cos\left(m\frac{\pi}{2}\right) + \frac{1}{m^2}\sin\left(m\frac{\pi}{2}\right)\right)\sin(mx)}$$

if $m \bmod 4 = 1$, then $\sin\left(m\frac{\pi}{2}\right) = 1$, $\cos\left(m\frac{\pi}{2}\right) = 0$

$$\rightarrow \frac{1}{\pi}\left(\frac{\pi}{2m}\cdot 1 - \frac{1}{m^2}\right)\cdot\cos(mx) + \frac{1}{\pi}\left(+\frac{1}{m^2}\right)\sin(mx)$$

if $m \bmod 4 = 2$, then $\sin\left(m\frac{\pi}{2}\right) = 0$, $\cos\left(m\frac{\pi}{2}\right) = -1$

$$\rightarrow \frac{1}{\pi}\left(\frac{1}{m^2}(-2)\right)\cos(mx) + \frac{1}{\pi}\left(\frac{\pi}{2m}\right)\sin(mx)$$

if $m \bmod 4 = 3$, then $\sin\left(m\frac{\pi}{2}\right) = -1$, $\cos\left(m\frac{\pi}{2}\right) = 0$

$$\rightarrow \frac{1}{\pi}\left(\frac{-\pi}{2m} - \frac{1}{m^2}\right)\cos(mx) + \frac{1}{\pi}\left(-\frac{1}{m^2}\right)$$

if $m \bmod 4 = 0$, then $\sin\left(m\frac{\pi}{2}\right) = 0$, $\cos\left(m\frac{\pi}{2}\right) = 1$

$$\rightarrow \frac{1}{\pi}\left(-\frac{\pi}{2m}\right)\sin(mx)$$

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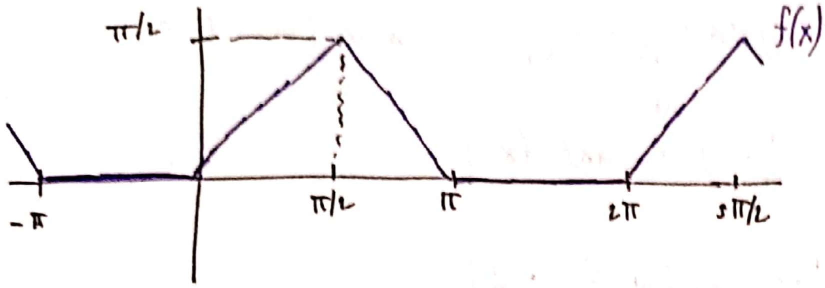
b) $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ x & \text{if } 0 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < \pi \end{cases}$



$L = \pi$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} \pi - x\,dx \right) =$$

$$= \frac{1}{2\pi}\left(\underset{\uparrow \text{ from a)}}{\frac{\pi^2}{8}} + \underset{\uparrow \text{ same area}}{\frac{\pi^2}{8}} \right) = \frac{\pi^{2}}{8}$$

$$f^*(x) = \begin{cases} 0 & \text{if } -\pi < x < \frac{\pi}{2} \text{ or } \frac{\pi}{2} \leq x \leq \pi \\ \frac{\pi}{2} + x & \text{if } -\frac{\pi}{2} < x < 0 \\ \frac{\pi}{2} - x & \text{if } 0 < x \leq \frac{\pi}{2} \end{cases}$$

$\cancel{a_n = \frac{1}{\pi} \int_{-\pi} f(x) \cos(nx)}$

Notice that $f^*(x) = f\left(x + \frac{\pi}{2}\right)$ is an even function. Therefore we can skip b_n coefficients (and a_0 will be the same). We can then compute the Fourier Series for $f'(x)$ and then translate it back:

$$a'_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f^*(x) \cos(nx)\,dx =$$

$$\cancel{= \frac{1}{\pi}\left[\int_{-\pi/2}^{0} \left(\frac{\pi}{2} + x\right) \cos(nx)\,dx + \int_0^{\pi/2} \left(\frac{\pi}{2} - x\right) \cos(nx)\,dx \right]}$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \left(\frac{\pi}{2} - x\right) \cos(nx)\,dx =$$

$$= \frac{2}{\pi} \cdot \left(\frac{\pi}{2} \int_0^{\pi/2} \cos(nx) - \int_0^{\pi/2} x \cos(nx)\,dx \right) =$$

$$= \frac{1}{n}\left| \sin\left(\frac{\pi}{2} n\right) - 1 \right| + \underbrace{\frac{2}{\pi} \int_0^{\pi/2} x \cos(nx)\,dx}_{\text{done in class}} =$$

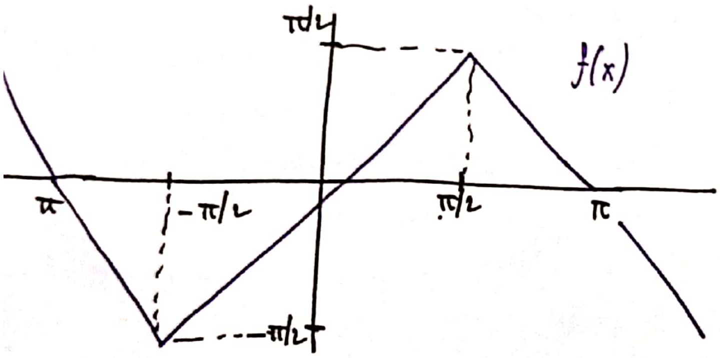
$$= \frac{1}{n}\left| \sin\left(\frac{\pi}{2} n\right) - 1 \right| + \frac{2[(-1)^n - 1]}{\pi n^2}$$

And then,

$$S'(x) = \frac{\pi^2}{8} + \sum_{n=1}^{N} \left(\cos\left(\frac{\pi}{2} n\right) - 1 + \frac{2\left[(-1)^n - 1\right]}{\pi n^2} \right) \cos(nx), \quad \text{and we have}$$

$$S(x) = S'\left(x - \frac{\pi}{2}\right) = \frac{\pi^2}{8} + \sum_{n=1}^{N} \left(\cos\left(\frac{\pi}{2} n\right) - 1 + \frac{2\left[(-1)^n - 1\right]}{\pi n^2} \right) \cos\left(n\left(x - \frac{\pi}{2}\right)\right)$$

c) $$f(x) = \begin{cases} -\pi - x & \text{if } -\pi < x < -\pi/2 \\ x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x \leq \pi/2 \end{cases}$$



- $f(x)$ is odd $\Rightarrow a_n = 0 \ \forall n \in \mathbb{N}$ (0 incl.)

Then, we only need to compute b_n coeffs:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \operatorname{sen}(nx)\, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \operatorname{sen}(nx)\, dx =$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} x \operatorname{sen}(nx)\, dx + \int_{\pi/2}^{\pi} (\pi - x) \operatorname{sen}(x)\, dx \right] =$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} x \operatorname{sen}(nx)\, dx - \int_{\pi/2}^{\pi} x \operatorname{sen}(nx)\, dx + \overbrace{\int_{\pi/2}^{\pi} \pi\, dx}^{\pi^2/2} \right] = (*)$$

since $I(x) = \int x \operatorname{sen}(nx)\, dx = -\cos(nx) \cdot \frac{1}{n} x - \frac{1}{n} \int \cos(nx)\, dx = -\frac{1}{n}\left(x \cos(nx) + \frac{1}{n} \operatorname{sen}(nx) \right)$

then

$$(*) = \frac{2}{\pi} \left(-I(\pi) + 2I(\pi/2) - I(0) + \pi^2/2 \right]$$

with

$$I(0) = -\frac{1}{n}\left(0 \cdot \cos\big(\underset{n\cdot 0}{\cancel{0x}}\big) + \frac{1}{n}\sin(n \cdot 0)\right) = 0$$

$$I(\pi/2) = -\frac{1}{n}\left(\frac{\pi}{2}\cos\left(n\frac{\pi}{2}\right) + \frac{1}{n}\sin\left(n\frac{\pi}{2}\right)\right)$$

$$I(\pi) = -\frac{1}{n}\left(\pi\cos(n\pi) + \frac{1}{n}\underset{0}{\cancel{\sin(n\pi)}}\right) = -\frac{\pi}{n}(-1)^n$$

and

$$S_N(x) = \frac{2}{\pi}\sum_{n=1}^{N}\left(\sin(nx)\Big/\frac{\pi^2}{2} + 2\left(-\frac{1}{n}\left(\frac{\pi}{2}\cos\left(n\frac{\pi}{2}\right) + \frac{1}{n}\sin\left(n\frac{\pi}{2}\right)\right)\right)\right.$$

$$\left.\left. + \frac{\pi}{n}(-1)^n\right)\right)$$