Exercise Sheet 6 Pablo Dolor Viviandes

Problem 1 : Find the Fourier transform of the following functions $f: \mathbb{R} \to \mathbb{R}$

a) $f(x) = e^{-|x|}$

We apply the formula $\cancel{*}\cancel{*}$ $\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{x} e^{-i\omega x}\,dx =$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|\cancel{x}-i\omega x}\,dx =$$

$$= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{0} e^{x-i\omega x}\,dx + \int_{0}^{+\infty} e^{-x-i\omega x}\,dx \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{1}{1-i\omega} e^{x(1-i\omega)}\Big|_{-\infty}^{0} + \frac{1}{-1-\omega} e^{x(-1-i\omega)}\Big|_{0}^{+\infty} \right) =$$

$$\overset{\cancel{*}}{=} \frac{1}{\sqrt{2\pi}} \left( \frac{1}{1-i\omega} + \frac{1}{1+i\omega} \right) = \frac{1}{\sqrt{2\pi}} \left( \frac{1+i\omega+1-i\omega}{1+\omega^2} \right) =$$

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$\cancel{*}$ $\lim_{x\to-\infty} \frac{1}{1-i\omega} e^{x(1-i\omega)} \cancel{(\ldots)} = \lim_{x\to-\infty} \frac{1}{1-i\omega} \underbrace{e^{x}}_{\to 0} \cdot \underbrace{e^{i(-x\omega)}}_{\text{bounded}} = 0$

$\lim_{x\to+\infty} \frac{1}{-1-i\omega} e^{-x(+1+i\omega)} \cancel{(\ldots)} = \lim_{x\to\infty} \frac{-1}{1+i\omega} e^{-x} \cdot e^{i(-x\omega)} = 0$

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$$= \frac{2}{\sqrt{2\pi}(1+\omega^2)}$$

$\cancel{*}\cancel{*}$ First, check if FT exists:

$$\int_{-\infty}^{+\infty} |f(x)|\,dx = \int_{-\infty}^{\infty} e^{-|x|}\,dx \underset{\uparrow \text{ even}}{=} 2\int_{0}^{+\infty} e^{-x}\,dx = 2 \cdot \overset{0 \,(x\to+\infty)}{-e^{-x}}\Big|_{0}^{+\infty} =$$

$= 2 \cdot (-1) = -2$

then $f(x) = e^{-|x|}$ is absolutely integrable and the FT exists.

b) $f(x) = x^2 e^{-x^2}$

Since $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^2 e^{-x^2} = 0$ then we can apply the FT derivative formula: $\mathcal{F}(f') = i\omega \mathcal{F}(f)$, we apply it two times since

$f'(x) = 2xe^{-x^2} + \frac{x^2}{-2x} e^{-x^2} = e^{-x^2}\left(2x - \frac{x}{2}\right) = \frac{3x}{2} e^{-x^2}$ also satisfies $\lim_{x\to\infty} f'(x) = 0$. Then $f''(x) = \frac{3}{2} e^{-x^2} + \frac{3\cancel{x}}{2} \cdot \frac{1}{-2\cancel{x}} e^{-x^2} =$

$= e^{-x^2}\left(\frac{3}{2} - \frac{3}{4}\right) = \frac{3}{4} e^{-x^2}$ and

$$\mathcal{F}(f'') = i\omega(i\omega\,\mathcal{F}(f)) = -\omega^2 \mathcal{F}(f) \Rightarrow$$

$$\Rightarrow \mathcal{F}(f) = -\frac{\mathcal{F}(f'')}{\omega^2}$$

since $\mathcal{F}(f'') \underset{\text{linearity}}{=} \frac{3}{4} \mathcal{F}(e^{-x^2}) \underset{\text{Kreyszig table}}{=} \frac{3}{4} \frac{1}{\sqrt{2}} e^{-\omega^2/4}$

then $\mathcal{F}(f) = -\frac{3e^{-\omega^2/4}}{4\sqrt{2}\cdot\omega^2}$

Problem 2 :: We define $f, g : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = e^{-x^2}, \quad g(x) = e^{-2x^2}$$

compute the convolution $(f * g)$ using the FT. First, let's compute its FT

$$\cancel{(f*g) = \mathcal{F}} \quad \mathcal{F}(f*g) = \sqrt{2\pi}\; \mathcal{F}(f) \cdot \mathcal{F}(g) =$$

$$\underset{\text{table}}{=} \sqrt{2\pi}\; \frac{1}{\sqrt{2}} e^{-\omega^2/4} \cdot \frac{1}{\sqrt{4}} e^{-\omega^2/8} =$$

$$= \frac{\sqrt{2\pi}\; e^{-3\omega^2/8}}{2\sqrt{2}} = \frac{\sqrt{\pi}}{2} e^{-3\omega^2/8}$$

and we now find the inverse fourier transform to get the original function :

$$\mathcal{F}^{-1}(\mathcal{F}(f*g)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\sqrt{\pi}}{2} e^{-3\omega^2/8} e^{i\omega x}\, dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\sqrt{\pi}}{2} e^{\frac{-3\omega^2}{8} \cdot i\omega x}$$

then we have $\mathcal{F}(f*g) = \frac{\sqrt{\pi}}{2} e^{-3\omega^2/8} = \frac{\sqrt{\pi}}{2} e^{-\omega^2/(4\cdot(2/3))} =$

$$= \frac{\sqrt{\pi}\cdot\sqrt{4/3}}{2} \cdot \frac{1}{\sqrt{4/3}} e^{-\omega^2/(4\cdot(2/3))} \underset{\text{table}}{=}$$

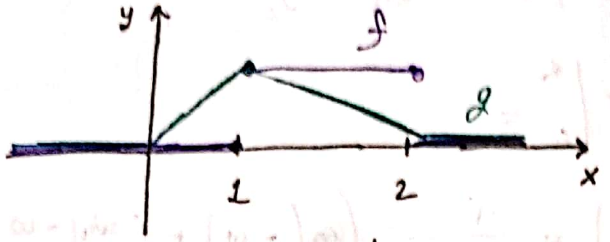
$$= \frac{\sqrt{4\pi}}{2\sqrt{3}} \cdot \mathcal{F}\left(e^{-2/3\,\omega^2}\right) \underset{\text{linearity}}{=}$$

$$= \mathcal{F}\left(\cancel{\frac{4\sqrt{\pi}}{}}\sqrt{\frac{\pi}{3}}\, e^{-2/3\,\omega^2}\right) \Rightarrow$$

$$\Rightarrow \underline{(f*g) = \sqrt{\frac{\pi}{3}}\, e^{-2/3\,\omega^2}}$$

Problem 3: We define the functions $f$ and $g$ as

$$f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0 & \text{else} \end{cases} \qquad g(x) = \begin{cases} x & \text{if } 0<x<1 \\ 2-x & \text{if } 1<x<2 \\ 0 & \text{else} \end{cases}$$




a) Use the definition of convolution to ~~find~~ show $f*f=g$

$$(f*f)(x) = \int_{-\infty}^{+\infty} f(y)\,f(x-y)\,dy = \int_0^1 f(y)\,f(x-y)\,dy =$$

$f(y)\,f(x-y)\ \boxed{=0}$ if $\boxed{y \notin (0,1)}$ or $x-y \notin (0,1)$

$f(y)\,f(x-y) = 1$ if $y \in (0,1)$ and $\underline{x-y \in (0,1)}$

$0<x-y<1 \Rightarrow -x<-y<1-x \Rightarrow$

$\Rightarrow \begin{cases} \boxed{x-1<y<x} \\ \boxed{0<y<1} \end{cases}$

$\cancel{= \int_{x-1}^{x} dy}$

$$= \int_{\max\{0,\,x-1\}}^{\min\{1,\,x\}} dy = \begin{cases} 1 & \text{if } 1<x \text{ and } 0>x-1 \Rightarrow 1>x \quad \underline{\underline{\text{NO}}} \\ x & \text{if } 1>x \text{ and } 0>x-1 \Rightarrow 1>x \\ 1-(x-1) & \text{if } 1<x \text{ and } 0<x-1 \Rightarrow 1<x \\ x-(x-1) & \text{if } 1>x \text{ and } 0<x-1 \Rightarrow 1<x \quad \underline{\underline{\text{NO}}} \end{cases}$$

$$\Rightarrow \begin{cases} x & \text{if } 1>x \\ 2-x & \text{if } 1<x \end{cases}$$

but the integral is 0 if $x \notin (0,2)$ since $f(x-y)=0$ when $y \in (0,1)$, then

$$(f*f)(x) = \begin{cases} x & \text{if } 0<x<1 \\ 2-x & \text{if } 1<x<2 \\ 0 & \text{else} \end{cases} = g(x)$$

b) Compute $\mathcal{F}(f)$

$$\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-i\omega x} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{i\omega} \, e^{-i\omega x} \Big|_0^1 =$$

$$\overset{\cdot\frac{i}{i}}{=} \frac{+i}{+\omega\sqrt{2\pi}} \left(e^{-i\omega} - 1\right) = \frac{i}{\omega\sqrt{2\pi}} \left(\cos(-\omega) + i\sin(-\omega) - 1\right) =$$

$$= \frac{i\left(\cos(\omega) - 1\right) - \sin(-\omega)}{\omega\sqrt{2\pi}} = \frac{\sin(\omega) + i\left(\cos(\omega) - 1\right)}{\omega\sqrt{2\pi}}$$

c) Compute $\mathcal{F}(g)$

$$\mathcal{F}(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x)\, e^{-i\omega x}\, dx =$$

$$= \frac{1}{\sqrt{2\pi}} \left( \int_0^1 x e^{-i\omega x} dx + \int_1^2 (2-x) e^{-i\omega x} dx \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \left( \int_0^1 x e^{-i\omega x} dx - \int_1^2 x e^{-i\omega x} dx + 2\int_1^2 e^{-i\omega x} dx \right) =$$

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$$\int x e^{-i\omega x} dx = x e^{-i\omega x} \cdot \frac{1}{-i\omega} - \int e^{-i\omega x} dx =$$

$$= \frac{-1}{i\omega} \left[ x e^{-i\omega x} + \frac{1}{i\omega} e^{-i\omega x} \right] =$$

$$= \frac{+i}{\omega} \left[ x e^{-i\omega x} + \frac{i}{\omega} e^{-i\omega x} \right] =$$

$$= e^{-i\omega x} \left[ \frac{i}{\omega} \cancel{x} \frac{-1}{\omega^2} \right]$$

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$$= \frac{1}{\sqrt{2\pi}} \left( \left| e^{-i\omega x} \left( \frac{i}{\omega} - \frac{1}{\omega^2} \right) \right|_0^1 - \left| e^{-i\omega x} \left( \frac{1}{\omega} - \frac{1}{\omega^2} \right) \right|_1^2 + 2\, \frac{i}{\omega}\, e^{-i\omega x} \Big|_1^2 \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left( -e^{-i2x} \left( \frac{i}{\omega} - \frac{1}{\omega^2} \right) + 2e^{-ix} \left( \frac{i}{\omega} - \frac{1}{\omega^2} \right) - \left( \frac{i}{\omega} - \frac{1}{\omega^2} \right) + 2\, \frac{i}{\omega} \left( e^{-i2x} - e^{ix} \right) \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left[ e^{-i2x} \left( \frac{1}{\omega^2} + \frac{i}{\omega} \right) + e^{-ix} \left( \frac{-2}{\omega^2} \right) - \left( \frac{i}{\omega} - \frac{1}{\omega^2} \right) \right] = \ldots$$

$$\dots = \frac{-1+2e^{i\omega}-e^{-2i\omega}}{\sqrt{2\pi}\ \omega^2}$$

Used table
have to finish,
not sure of calculations
were correct

Problem 4:

a) Compute the Fourier matrices $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$

In general form $(\mathcal{F}_N)_{nk} = w^{nk}$, with $w = e^{\frac{-2\pi}{N} i}$

$$\mathcal{F}_2 = \begin{pmatrix} 1 & 1 \\ 1 & w \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & e^{-\pi i} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathcal{F}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w^4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{-4\pi i/3} \\ 1 & e^{-4\pi i/3} & e^{-8\pi i/3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2}-i\frac{\sqrt{3}}{2} & -\frac{1}{2}+\frac{\sqrt{3}}{2}i \\ 1 & -\frac{1}{2}+\frac{\sqrt{3}}{2}i & \frac{1}{2}-i\frac{\sqrt{3}}{2} \end{pmatrix}$$

$$e^{-\frac{2\pi i}{3}} = e^{i\left(-\frac{2}{3}\pi\right)} = \cos\left(-\frac{2}{3}\pi\right) + i\sin\left(-\frac{2}{3}\pi\right) =$$

$$= \cos\left(\frac{2}{3}\pi\right) - i\sin\left(\frac{2}{3}\pi\right) =$$

$$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$e^{-\frac{4\pi i}{3}} = \cos\left(\frac{4}{3}\pi\right) - i\sin\left(\frac{4}{3}\pi\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$e^{-8\pi i/3} = \cos\left(\frac{8}{3}\pi\right) - i\sin\left(\frac{8}{3}\pi\right) = \cos\left(\frac{2}{3}\pi\right) - i\sin\left(\frac{2}{3}\pi\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\mathcal{F}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$$w^{1+4k} = w = e^{-\frac{\pi i}{2}} = -i$$
$$w^{2+4k} = w^2 = e^{-\pi i} = -1 \qquad \forall k \in \mathbb{N}$$
$$w^{3+4k} = w^3 = e^{-3\pi i/2} = i$$
$$w^{4k} = w^4 = e^{-2\pi i} = 1$$

b) Compute by hand the discrete FT of $f = (1, 2, 3, 1)$

$$\hat{f}_4 = \mathcal{F}_4 f \Rightarrow \hat{f}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ -2-i \\ 1 \\ -2+i \end{pmatrix}$$

c) The DFT of $f \in \mathbb{R}^{11}$ is $\hat{f} = (0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0)$

Express the entries $f_n$ of $f$ in terms of sine and cosine functions.

$$\hat{f} = \mathcal{F}_{11} f \Rightarrow f = \mathcal{F}_{11}^{-1} \hat{f} = \frac{1}{11} \overline{\mathcal{F}_{11}} \hat{f} =$$

$$(\mathcal{F}_{11})_{nk} = e^{\frac{-2\pi i}{11} \cdot nk} = \cos\left(\frac{-2\pi n k}{11}\right) + i \sin\left(\frac{-2\pi n k}{11}\right) \quad (*)$$

$$= \frac{1}{11} \left( \cos\left(\frac{-2\pi n k}{11}\right) - i \sin\left(\frac{-2\pi n k}{11}\right) \right)_{nk} \hat{f} =$$

apply $(*)$ and conjugate

$$= \frac{1}{11} \left( \cos\left(\frac{2\pi n k}{11}\right) + i \sin\left(\frac{2\pi n k}{11}\right) \right)_{nk} \hat{f} \Rightarrow$$

$\sin(-\alpha) = -\sin\alpha$

$\cos(-\alpha) = \cos\alpha$

$\hat{f} = (0,0,1,0,0,0,0,0,0,1,0)^T$: $k=2$, $k=9$

$$\Rightarrow f_n = \frac{1}{11} \left( \left( \cos\left(\frac{2\pi n \cdot 2}{11}\right) + \cos\left(\frac{2\pi n \cdot 9}{11}\right) \right) + i \left( \sin\left(\frac{2\pi n \cdot 2}{11}\right) + \sin\left(\frac{2\pi n \cdot 9}{11}\right) \right) \right)$$