

VISUALIZING GROVER'S SEARCH ALGORITHM

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1. HOW GROVER'S WORKS

If we want an operator U_x with the following behavior

$$(1) \quad \begin{aligned} U_x|x\rangle &= -|x\rangle \\ U_x|y\rangle &= |y\rangle \end{aligned}$$

where all $|y\rangle$ are orthogonal to $|x\rangle$, then the operator takes the form

$$(2) \quad U_x = I - 2|x\rangle\langle x|$$

Let's demonstrate this with a couple of examples.

$$(3) \quad \begin{aligned} U_x|x\rangle &= (I - 2|x\rangle\langle x|)|x\rangle \\ &= |x\rangle - 2|x\rangle\langle x|x\rangle \\ &= |x\rangle - 2|x\rangle \\ &= -|x\rangle \end{aligned}$$

because $\langle x|x\rangle = 1$.

$$(4) \quad \begin{aligned} U_x|y\rangle &= (I - 2|x\rangle\langle x|)|y\rangle \\ &= |y\rangle - 2|x\rangle\langle x|y\rangle \\ &= |y\rangle - 2\langle x|y\rangle|y\rangle \\ &= |y\rangle \end{aligned}$$

because $\langle x|y\rangle = 0$.

Now let's look at the result of acting on an arbitrary state $|\alpha\rangle$, that is not completely orthogonal to $|x\rangle$.

$$(5) \quad \begin{aligned} U_x|\alpha\rangle &= (I - 2|x\rangle\langle x|)|\alpha\rangle \\ &= |\alpha\rangle - 2|x\rangle\langle x|\alpha\rangle \\ &= (1 - 2\langle x|\alpha\rangle)|\alpha\rangle \end{aligned}$$

In words, the result is the original state vector minus twice the overlap or projection of $|\alpha\rangle$ on $|x\rangle$. We can draw this out geometrically.

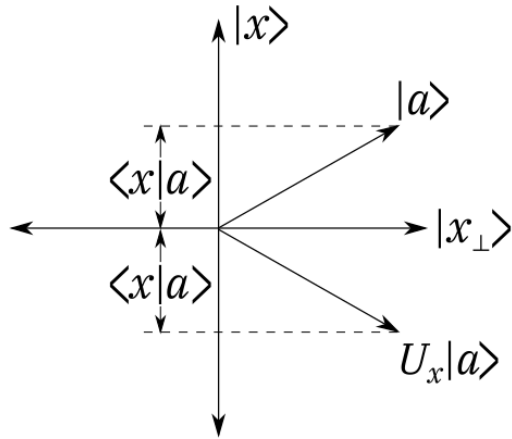


FIGURE 1. Geometric demonstration that the operator U_x has the effect of reflecting $|a\rangle$ across $|x_\perp\rangle$, the vector orthogonal to $|x\rangle$.