VISUALIZING GROVER'S SEARCH ALGORITHM

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1. How Grover's Works

If we want an operator U_x with the following behavior

$$\begin{aligned} U_x|x\rangle &= -|x\rangle \\ U_x|x_\perp\rangle &= |x_\perp\rangle \end{aligned} \tag{1}$$

where all $|x_\perp\rangle$ are orthogonal to $|x\rangle$, then the operator takes the form

$$U_x = I - 2|x\rangle\langle x| \tag{2}$$

Let's demonstrate this with a couple of examples.

$$\begin{aligned} U_x|x\rangle &= (I - 2|x\rangle\langle x|)|x\rangle \\ &= |x\rangle - 2|x\rangle\langle x|x\rangle \\ &= |x\rangle - 2|x\rangle \\ &= -|x\rangle \end{aligned} \tag{3}$$

because $\langle x|x\rangle = 1$.

$$\begin{aligned} U_x|x_\perp\rangle &= (I - 2|x\rangle\langle x|)|x_\perp\rangle \\ &= |x_\perp\rangle - 2|x\rangle\langle x|x_\perp\rangle \\ &= |x_\perp\rangle - 2\langle x|x_\perp\rangle|x_\perp\rangle \\ &= |x_\perp\rangle \end{aligned} \tag{4}$$

because $\langle x|x_\perp\rangle = 0$.

Now let's look at the result of acting on an arbitrary state $|\alpha\rangle$, that is not completely orthogonal to $|x\rangle$.

$$\begin{aligned} U_x|\alpha\rangle &= (I - 2|x\rangle\langle x|)|\alpha\rangle \\ &= |\alpha\rangle - 2|x\rangle\langle x|\alpha\rangle \\ &= (1 - 2\langle x|\alpha\rangle)|\alpha\rangle \end{aligned} \tag{5}$$

In words, the result is the original state vector minus twice the overlap or projection of $|\alpha\rangle$ on $|x\rangle$. We can draw this out geometrically.

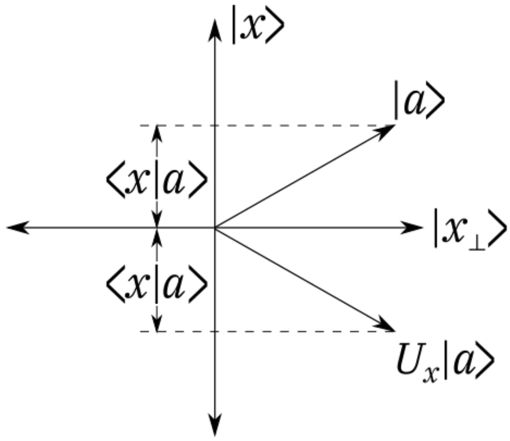

FIGURE 1. Geometric demonstration that the operator U_x has the effect of reflecting $|\alpha\rangle$ across $|x_\perp\rangle$, the vector orthogonal to $|x\rangle$.