VISUALIZING GROVER'S SEARCH ALGORITHM

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1. Oracle operators

If we want an operator U_x with the following behavior

$$U_x|x\rangle = -|x\rangle$$
$$U_x|x_\perp\rangle = |x_\perp\rangle \tag{1}$$

where all $|x_\perp\rangle$ are orthogonal to $|x\rangle$, then the operator takes the form

$$U_x = I - 2|x\rangle\langle x| \tag{2}$$

Let's demonstrate this with a couple of examples.

$$\begin{aligned} U_x|x\rangle &= (I - 2|x\rangle\langle x|)|x\rangle \\ &= |x\rangle - 2|x\rangle\langle x|x\rangle \\ &= |x\rangle - 2|x\rangle \\ &= -|x\rangle \end{aligned} \tag{3}$$

because $\langle x|x\rangle = 1$.

$$\begin{aligned} U_x|x_\perp\rangle &= (I - 2|x\rangle\langle x|)|x_\perp\rangle \\ &= |x_\perp\rangle - 2|x\rangle\langle x|x_\perp\rangle \\ &= |x_\perp\rangle - 2\langle x|x_\perp\rangle|x_\perp\rangle \\ &= |x_\perp\rangle \end{aligned} \tag{4}$$

because $\langle x|x_\perp\rangle = 0$.

Now let's look at the result of acting on an arbitrary state $|\alpha\rangle$, that is not completely orthogonal to $|x\rangle$.

$$\begin{aligned} U_x|\alpha\rangle &= (I - 2|x\rangle\langle x|)|\alpha\rangle \\ &= |\alpha\rangle - 2|x\rangle\langle x|\alpha\rangle \\ &= (1 - 2\langle x|\alpha\rangle)|\alpha\rangle \end{aligned} \tag{5}$$

In words, the result is the original state vector minus twice the overlap or projection of $|\alpha\rangle$ on $|x\rangle$. We can draw this out geometrically.

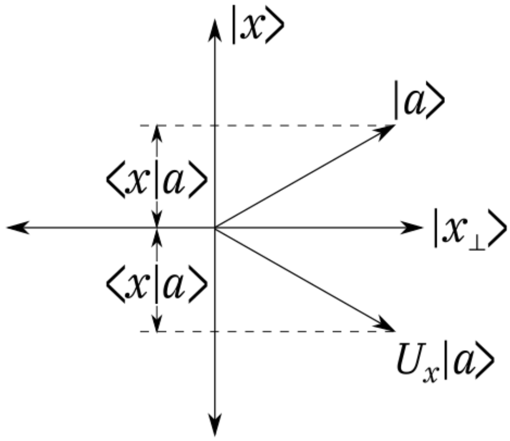

FIGURE 1. Geometric demonstration that the operator U_x has the effect of reflecting $|\alpha\rangle$ across $|x_\perp\rangle$, the vector orthogonal to $|x\rangle$.

2. Symmetries of a single qubit system

A single qubit's state $|s\rangle$

$$|s\rangle = (a + ib)|0\rangle + (c + id)|1\rangle \tag{6}$$

written in terms of the 4 real numbers a, b, c, d can be fully described by 3 real numbers, since the state must be normalized,

$$\langle s|s\rangle = a^2 + b^2 + c^2 + d^2 = 1, \tag{7}$$

which makes one of the real numbers dependent on the other three.

A single qubit's rotational symmetries can be described by the matrices

$$\begin{pmatrix} a + ib & -(c - id) \\ c + id & -(a + ib) \end{pmatrix} \tag{8}$$

where a, b, c, d are real numbers. The normalization condition on the state of the qubit requires that the determinant of this matrix be 1.

Another way to describe a single qubit's rotational symmetries is by the quaternion:

$$a + bi + cj + dk \tag{9}$$

where the real numbers a, b, c, d of the matrix and quaternion are equal respectively. The determinant of the matrix is the norm of the corresponding quaternion.

Since the matrix has determinant 1 as a consequence of the normalization condition of the qubit's state, the corresponding quaternion has norm 1.

3. Symmetries of a two qubit system

A system of two qubits can be described by $2 \times 3 = 6$ real numbers. The system transforms as the tensor product of two independent matrices that each describe a single qubit, as explained above.

$$\left(\begin{array}{cc|cc} ia & -\bar{z} & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ z & -ia & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}$}} & ib & -\bar{w} \\ & & w & -ib \end{array}\right) \tag{10}$$

where a, b are real numbers and z, w are complex numbers. The numbers a, z, b, w are made up of 6 independent real numbers. Each of the two block matrices can be related to independent quaternions.

The dynamics of a two qubit system can be thought of geometrically in 4D euclidean space where each of the two quaternions is acted upon independently by operators associated with each spin respectively.