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FINM 33000: Homework 1 Solutions

Problem 1

(a) Given two assets B and B^*, with $B_0 = B_0^* = 1$ and $B_T = e^{rT}$, $B_T^* = e^{r^*T}$ where $r < r^*$, we can short B and long B^* (or borrow using B and take that money and place it in B^*).
Our initial value, V_0, is:
$$(-B_0, +B_0^*) \Longrightarrow V_0 = 0$$
Our value at time T, V_T, is:
$$V_T = e^{r^*T} - e^{rT}$$
We can see that for any $T > 0$, $V_T > 0$. Based on this, our construction is a type 1 arbitrage.

(b) Given 3 available assets:
A: Discount bond Z, with $Z_0 = 0.9$
B: Stock S with $S_0 = 100.0$
C: European call on S with strike 110, expiry T, where $C_0 = 0.5$
What we can do essentially is short a unit of stock, buy some unit of discount bond with par value 100, and also buy one unit of call, i.e. $(A, -B, C)$. In terms of valuation, for our initial value, V_0, we get:
$$\left(+\frac{199}{180}A, -B, +C\right) \Longrightarrow V_0 = 0$$
Our value at time T, V_T, is the following:
For $S_T \geq 110$:
$$V_T = \frac{199}{180} \cdot 100 - S_T + S_T - 110 = \frac{5}{9}$$
For $S_T < 110$:
$$V_T = \frac{199}{180} \cdot 100 - S_T + 0 > \frac{5}{9}$$
This is a type 1 arbitrage.
Note: If we are unable to buy our discount bond in odd lots, we would need to adjust our numbers accordingly such that all these number of units above are all whole lots.

(c) Given 3 available assets:
S: Stock S with $S_0 = 100.0$
G: Contract G with payoff $min(S_T, 110)$ with $G_0 = 85$
C: European call on S with strike 110, expiry T, where $C_0 = 20$
Based on the payoff, we can see that G is essentially just $S - C$. However, their prices do not match up even though they arrive at the same payoff.
Our initial value, V_0,:
$$(S, -G, -C) \Longrightarrow V_0 = -5$$
Our value at time T, V_T, is:
$$V_T = -S_T + min(S_T, 110) + min(0, S_T - 110) = 0$$
Hence, this is a Type 2 arbitrage. Since Type 2 arbitrage is available, Type 1 arbitrage must be available as well.

(d) Given 4 different assets, $Z_0, C(20.0), C(22.5), C(25.0)$, we can construct a portfolio that shorts 2 units of $C(20.0)$, longs 1 unit of $C(22.5)$, and longs 1 unit of $C(25.0)$. To balance that one, we will also buy 7.5 units of Z_0 amounting to 6.75 of initial cash outflow. This means that:

Our initial value, V_0, is:
$$\left(+7.5Z_0, -2C(20.0), +C(22.5), +C(25.0)\right) \Rightarrow V_0 = -12.8 + 3.1 + 1 - 6.75 = -1.95$$

Based on this, we have a couple of payoffs at V_T.
When $S_T < 20$, we just get the bond returns, i.e.:
$$V_T = 7.5$$
When $20 \leq S_T < 22.5$, we lose money from the short bonds, but still stay positive from the bond returns:
$$V_T = 2 \cdot (20 - S_T) + 7.5 > 2.5$$
When $22.5 \leq S_T < 25$, we still are positive because our bond covers any max loss from the options while $C(22.5)$ kicks in:
$$V_T = -5 + (22.5 - S_T) + 7.5 > 0$$
Only in our worst-case scenario, where $S_T \geq 25$, would we break-even when $C(25.0)$ kicks in, i.e.:
$$V_T = 0$$
Hence, since $V_T > 0 \;\forall\; S_T$, this is a Type 2 arbitrage.

(e) Assuming that $S_T > 0$ and $P(S_T \neq 100) > 0$, and that we have 3 assets: X, Y and Z. We can see that just buying 50 units of X and also buying one unit of yields you a final payoff that is always positive.
Our initial value, V_0, is:
$$(+50X, +Y) \Rightarrow V_0 = 0$$
For our final value, our payoff is:
$$V_T = -100\, log \frac{S_T}{100} + S_T - 100$$
Hence, $V_T \geq 0 \;\forall\; S_T$. This is a Type 1 arbitrage.

Problem 2

(a) If Trump only wins if all 3 of the states wins, we have a clear arbitrage that we can get by long one unit of US.Biden and going short one unit of PA.Biden. These yields $V_0 = 0.01$. In the case where Biden wins, if Biden also wins PA, we get $V_T = 0$. However, if Biden wins but loses PA, we get $V_T = 1$. Hence:
$$V_T \geq 0$$
In the case where Trump wins:
$$V_T = 0$$
This is a Type 2 arbitrage.

(b) In this case, we can take in borrow 1 unit of bank account and then long 1 contracts US.Biden and PA.Trump respectively. This yields These yields $V_0 = -0.01$. At time T, if Trump wins, that means $V_T = 0$ and if Trump loses $V_T \geq 0$.
This is a Type 2 Arbitrage.