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FINM 33000: Homework 6 Solutions

Problem 1

Part (a)

Let's first table out the variables we have from this. The number of shares we purchase 1 year from now is:

$$N = \frac{3000}{min(30, 0.75S_T)}$$

Our profit/share from this is:

$$P = \big(S_T - min(30{,}0.75S_T)\big)$$

Hence, we can see that our net profit a year from now is:

$$\begin{aligned} PN &= \big(S_T - min(30{,}0.75S_T)\big)\frac{3000}{min(30, 0.75S_T)} \\ &= 3000\left(\frac{S_T}{min(30, 0.75S_T)} - 1\right) \\ &= 3000\left(\frac{4}{3}max\left(1, \frac{S_T}{40}\right) - 1\right) \\ &= 3000\left(\frac{1}{3} + \frac{1}{30}max(0, S_T - 40)\right) \\ &= 1000 + 100\,max(0, S_T - 40) \end{aligned}$$

Essentially, our net cashflow at time t is a combination of 1000 dollars plus 100 calls with strike 40. Further solving:

$$\begin{aligned} Plan &= 1000e^{-0.05} + 100C(40) \\ &= 1000e^{-0.05} + 100\big(S_0N(d_1) - Ke^{-0.05}N(d_2)\big) \\ &= 951.23 + 100(4.48) \\ &= 1399.23 \end{aligned}$$

Part (b)

Since we are essentially long 100 units of strike 40 calls in the plan, what we can do is short 100 units of strike 40 calls with time to maturity of 1 year to do a perfect hedge. Hence, our combined plan will have a time-1 value of 1000 regardless of S_T's price. This will have no need for rebalancing.

Part (c)

Since we have no options to use to hedge, we would instead need to look at the delta of the option. We need to net the delta of our portfolio to zero to perfectly hedge.

As of time-0, our delta for each option:

$$N(d_1) = 0.63$$

Hence the number of shares we need to hedge is approximate 63 shares, i.e. we should short 63 shares at time-0 to hedge our portfolio. However, we always need to be dynamically rebalance as our delta, $N(d_1)$, changes in this case. Hence, we want to always be short $\frac{dC}{dS}$ shares.

Problem 2
Part (a)
Let x_t be the number of units in our bank account. Then our portfolio is:
$$L_t = \beta \frac{L_t}{S_t} S_t + x_t e^{rt}$$
$$L_t(1-\beta) = x_t e^{rt}$$
$$x_t = \frac{L_t(1-\beta)}{e^{rt}}$$
Hence we are always short some value since $\beta > 1$.

Part (b)
Since the trade is self-financing, and since r is constant:
$$dL_t = \beta \frac{L_t}{S_t} dS_t$$
i.e. the change of the portfolio value is equal to change in price of the stock multiplied by the number of those stocks held.
Taking that, and we can see that:
$$dL_t = \beta \frac{L_t}{S_t}(\mu S_t dt + \sigma S_t dW_t)$$
$$dL_t = \beta L_t(\mu dt + \sigma dW_t)$$

Hence, we can see that it is a geometric Brownian motion with its drift being $\beta\mu$ and its volatility being $\beta\sigma$.

Problem 3
Part (a)
$$C_t + Ke^{-r(T-t)} = P_t + S_t$$

Part (b)
Using put call parity:
$$C_t + Ke^{-r(T-t)} = P_t + S_t$$
$$S_t N(d_1) - Ke^{-r(T-t)} N(d_2) + Ke^{-r(T-t)} = P_t + S_t$$
$$P_t = S_t(N(d_1) - 1) + Ke^{-r(T-t)}\big(1 - N(d_2)\big)$$
$$P_t = Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1)$$

Part (c)
$$\frac{dP_t}{dS_t} = -N(-d_1)$$

Part (d)
$$\frac{d^2 P_t}{dS_t^2} = \frac{d\frac{dP_t}{dS_t}}{dS_t} = \frac{d\big(-N(-d_1)\big)}{dS_t} = \frac{d\big(-N(-d_1)\big)}{d(-d_1)} \frac{d(-d_1)}{dS_t} = \frac{N'(d_1)}{S_t \sigma \sqrt{T-t}}$$
Note that $-N(-d_1) = N(d_1)$.