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FINM 33000: Homework 2 Solutions

Problem 1

(a) Let our binary call payout at time T be denoted by C_T. We can see that:

$$C_T = \begin{cases} 1, \forall\, S_T \geq K \\ 0, \forall\, S_T < K \end{cases}$$

We can see that a payoff that sub replicates our 2.5 units of a binary option is a call spread such that buy a $C(22.5)$ and sell a $C(25)$.The 2.5 units of a binary option is consequently also super replicated by a call spread such that you buy $C(20)$ and sell $C(22.5)$.

Hence we can see that:

$$C(22.5) - C(25) \leq 2.5C_T \leq C(20) - C(22.5)$$

Simplifying this, we get:

$$\frac{4.15 - 2.6}{2.5} \leq C_0 \leq \frac{6.15 - 4.15}{2.5}$$
$$0.62 \leq C_0 \leq 0.8$$

(b) For our binary put option, at time T, we denote payout to be P_T. We can see that:

$$P_T = \begin{cases} 0, \forall\, S_T \geq K \\ 1, \forall\, S_T < K \end{cases}$$

Once again, similar to the logic above, we can super replicate our binary put by a call spread such that you sell a $C(22.5)$ and buy a $C(25)$ to build a "sell call spread". Similarly, we can sub replicate it by a "sell call spread" such that you sell a $C(20)$ and buy a $C(22.5)$. Now you realise here on a cashflow perspective that you would end up negative at time T if any of the call spreads are "in the money". Hence, what we want to do is to buy 2.5 units of a zero coupon bond. Hence:

$$2.5Z - \big(C(20) - C(22.5)\big) \leq 2.5P_T \leq 2.5Z - \big(C(22.5) - C(25)\big)$$

Simplifying:

$$0.95 - \frac{6.15 - 4.15}{2.5} \leq P_0 \leq 0.95 - \frac{4.15 - 2.6}{2.5}$$
$$0.15 \leq P_0 \leq 0.33$$

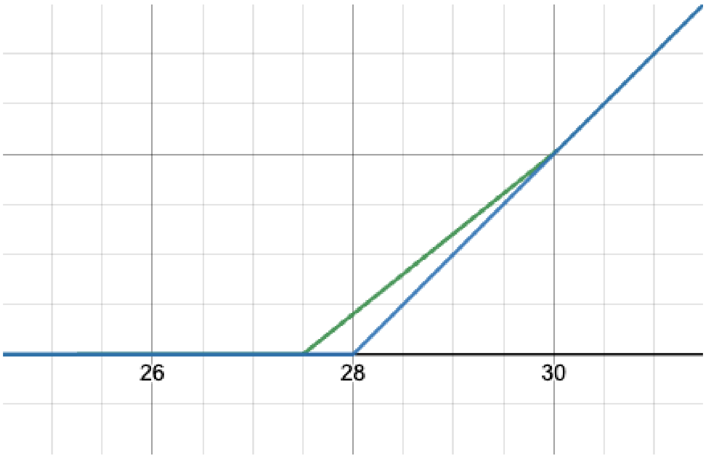
(c) For a contract to pay $max(2.5, S_T - 22.5)$, is essentially the same as a structure combining an option $C(25)$ and a discount bond returning 2.5. Hence, if we let bond payout at time T be Z_T per unit, we can say that:

$$max(2.5, S_T - 22.5) = 2.5Z_T + C_T(25)$$

Hence, the time-0 price of the contract is the time-0 price of 2.5 units of a discount bond and $C_0(25)$, i.e.:

$$\begin{aligned} 2.5Z_0 + C_0(25) &= 2.5 \cdot 0.95 + 2.6 \\ &= 5.45 \end{aligned}$$

(d) The upper bound of a 28 strike option can be done via replication using 0.8 units of a 27.5 strike option and 0.2 units of a 30 strike option. These give the best combinations to closely super replicate the option such that the payoff looks something as such:



Hence the upper bound of our call option is just:

$$\begin{aligned} C_0(28) &= 0.8C_0(27.5) + 0.2C_0(30) \\ &= 0.8 \cdot 1.5 + 0.2 \cdot 0.8 \\ &= 1.36 \end{aligned}$$

Problem 2

(a) For any $s > K_*$, we can see that:

$$\int_0^{K_*} f''(K)(K-s)^+ \, dK = 0$$

Then, following that, using the same logic where $s > K_*$, we can eliminate the $+$ limitation and substitute the ∞ in the integral given such that the equation is still equation, i.e.:

$$\int_{K_*}^{\infty} f''(K)(s-K)^+ \, dK = \int_{K_*}^{s} f''(K)(s-K) \, dK$$

Let us do integration by parts for this, setting $u = (s-K)$ and $dv = f''(K)dK$ such that we have $du = -dK$ and $v = \mathrm{f}'(\mathrm{K})$. Using these, we can see that:

$$\begin{aligned} \int_{K_*}^{s} f''(K)(s-K) \, dK &= [uv]_{K_*}^{s} - \int_{K_*}^{s} v \, du \\ &= [(s-K)f'(K)]_{K_*}^{s} + \int_{K_*}^{s} f'(K) \, dK \\ &= [(s-s)f'(s) - (s-K_*)f'(K_*)] + f(s) - f(K_*) \\ &= -(s-K_*)f'(K_*) + f(s) - f(K_*) \end{aligned}$$

Now, trying a new constraint such that $K_* > s$:

$$\int_{K_*}^{\infty} f''(K)(s-K)^+ \, dK = 0$$

Following on that constraint:

$$\int_0^{K_*} f''(K)(K-s)^+ \, dK = \int_s^{K_*} f''(K)(K-s) \, dK$$

Let us do integration by parts for this, setting $u = (K-s)$ and $dv = f''(K)dK$ such that we have $du = dK$ and $v = f'(K)$. Using these, we can see that:

$$\int_s^{K_*} f''(K)(K-s) \, dK = [uv]_s^{K_*} - \int_s^{K_*} v \, du$$

$$
\begin{aligned}
&= [(K-s)f'(K)]_s^{K_*} - \int_s^{K_*} f'(K)\, dK \\
&= [(K_* - s)f'(K_*) - (s-s)f'(s)] - f(K_*) + f(s) \\
&= -(s-K_*)f'(K_*) - f(K_*) + f(s)
\end{aligned}
$$

Now, in conclusion, for $s > K_*$:

$$
\begin{aligned}
f(s) &= f(K_*) - f'(K_*)(s-K_*) - (s-K_*)f'(K_*) + f(s) - f(K_*) \\
&= f(s)
\end{aligned}
$$

Similarly for $K_* > s$:

$$
\begin{aligned}
f(s) &= f(K_*) - f'(K_*)(s-K_*) - (s-K_*)f'(K_*) - f(K_*) + f(s) \\
&= f(s)
\end{aligned}
$$

Hence, the equation has been proven to hold for all cases.

(b) In this case, the put payoff is the part of the equation where:

$$\int_0^{K_*} f''(K)(K-s)^+\, dK$$

Approximating the integral as a Riemann Sum, substituting the formula for log payoff, and with the information that you have that we use puts and calls with strikes at positive integer multiples of 5 (e.g. 1950, 1955, 1960), you can see that:

$$
\begin{aligned}
\int_0^{K_*} f''(K)(K-s)^+\, dK &= \sum_{i=1}^{n} f''(K)(K-s)^+ \Delta K_i \\
&= \sum_{i=1}^{n} \left(-2log(K_*)\right)''(K-s)^+ \Delta K_i \\
&= \sum_{i=1}^{n} \frac{2}{K_*^2}(K-s)^+ \Delta K_i \\
&= \sum_{i=1}^{n} \frac{2}{K_*^2}(K-s)^+ 5
\end{aligned}
$$

Hence, the number of puts are just the co-efficient of each term in the summation, i.e. for 1950 strike puts, it is :

$$5 \cdot \frac{2}{K_*^2} = 5 \cdot \frac{2}{1950^2} = 2.63 \times 10^{-6}$$