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FINM 33000: Homework 3 Solutions

Problem 1

(a) Firstly, given that we have the up, middle and down state, we know that:

$$p_u + p_m + p_d = 1$$

Also, using stock S:

$$\frac{1}{B_T}\big(p_u \cdot S_T(w_u) + p_m \cdot S_T(w_m) + p_d \cdot S_T(w_d)\big) = S_0$$
$$\frac{1}{1.2}(240p_u + 120p_m + 60p_d) = 145$$
$$4p_u + 2p_m + p_d = 2.9$$

Using option C:

$$\frac{1}{B_T}\big(p_u \cdot C_T(w_u) + p_m \cdot C_T(w_m) + p_d \cdot C_T(w_d)\big) = C_0$$
$$\frac{1}{1.2}(30p_m + 30p_d) = 10$$
$$p_m + p_d = 0.4$$

Using these 3 equations, we can see that individually for each asset we have more than one martingale measure possible. However, for the market of assets $\{B, S, C\}$, we can see that the set of probabilities that make the full set of assets martingale can only be restricted to one outcome, i.e. $\{p_u, p_m, p_d\} = \{0.6, 0.1, 0.3\}$. Hence, the market is complete.

(b) We need to make it such that:

$$\begin{pmatrix} X_T(w_u) \\ X_T(w_m) \\ X_T(w_d) \end{pmatrix} = \begin{pmatrix} 120 \\ 60 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.2 & 240 & 0 \\ 1.2 & 120 & 30 \\ 1.2 & 60 & 30 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Solving for $\{a, b, c\}$, we can see that:

$$\{a, b, c\} = \{-100, 1, 2\}$$

This means that we lend 100 units of B, long 1 unit of S, and long 2 units of C. That gives you the desired payoff at time T. Hence, yes, this can be done using a portfolio of B, S and C.

(c) Using the replication portfolio:

$$\begin{aligned} X_0 &= -100B_0 + S_0 + 2C_0 \\ &= -100 + 145 + 20 \\ &= 65 \end{aligned}$$

Using the pricing probabilities:

$$\begin{aligned} X_0 &= \frac{1}{1.2}(120p_u + 60p_m + (0)p_d) \\ &= \frac{1}{1.2}(72 + 6) \\ &= 65 \end{aligned}$$

Problem 2

(a) Firstly, given that we have the up, middle and down state, we know that:

$$p_u + p_m + p_d = 1$$

Also, using stock S:

$$\frac{1}{B_T}\big(p_u \cdot S_T(w_u) + p_m \cdot S_T(w_m) + p_d \cdot S_T(w_d)\big) = S_0$$

$$\frac{1}{1.2}(240p_u + 120p_m + 60p_d) = 145$$

$$4p_u + 2p_m + p_d = 2.9$$

Using option C:

$$\frac{1}{B_T}\big(p_u \cdot C_T(w_u) + p_m \cdot C_T(w_m) + p_d \cdot C_T(w_d)\big) = C_0$$

$$\frac{1}{1.2}(24p_m + 36p_d) = 11$$

$$2p_m + 3p_d = 1.1$$

This set of equations have no viable solution as it's determinant is zero. Hence, the market is incomplete.

(b) We need to make it such that:

$$\begin{pmatrix} X_T(w_u) \\ X_T(w_m) \\ X_T(w_d) \end{pmatrix} = \begin{pmatrix} 120 \\ 60 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.2 & 240 & 0 \\ 1.2 & 120 & 24 \\ 1.2 & 60 & 36 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Solving for $\{a, b, c\}$, we can see that there is no solution. Hence, it is not possible to replicate the payoff.

(c) Since there is no solution in (b), (c) cannot be solved.

Problem 3

Let's plot these payoffs out on a table. We can see instantly that, in terms of Profit and Loss(P/L), our end state is the following:

P/L	**White Sox win 0**	**White Sox win 1**	**White Sox win 2**	**White Sox win 3**	**White Sox win 4**
Cubs win 0					+1000
Cubs win 1					+1000
Cubs win 2					+1000
Cubs win 3					+1000
Cubs win 4	-1000	-1000	-1000	-1000	-

Looking backwards, this means that at any point such that the score is even (i.e. (0,0), (1,1), (2,2), (3,3)), you want the P/L to be exactly 0. Hence:

P/L	**White Sox win 0**	**White Sox win 1**	**White Sox win 2**	**White Sox win 3**	**White Sox win 4**
Cubs win 0	0				+1000
Cubs win 1		0			+1000
Cubs win 2			0		+1000
Cubs win 3				0	+1000
Cubs win 4	-1000	-1000	-1000	-1000	-

Working backwards, we can see that the P/L of the current state is just the combined P/L of the next two possible states divided by 2. For example, at (3,3), we want to place 1000 worth of bet to end up at either (3,4) or (4,3). Hence, filling up the table:

P/L	**White Sox win 0**	**White Sox win 1**	**White Sox win 2**	**White Sox win 3**	**White Sox win 4**
Cubs win 0	0	+312.5	+625	+875	+1000
Cubs win 1	-312.5	0	+375	+750	+1000
Cubs win 2	-625	-375	0	+500	+1000
Cubs win 3	-875	-750	-500	0	+1000
Cubs win 4	-1000	-1000	-1000	-1000	-

From this table, it becomes evident that the bet as of (0,0) is just 312.5. Given that our borrowing costs nothing, whenever we win or lose, we can "hold on to our losses" at zero costs. In this case as well. Our probability of 55% does not play a part in this table as it just decides whether we will win or lose with a likelier probability. It should not change the bet we make at each stage.