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FINM 33000: Homework 5 Solutions

Problem 1
Let W be a Brownian motion such that:

$$Z_t = exp(W_t^2 - 1), \qquad for\ t \geq 0$$

(a) Let $X_t = W_t$ and $dX_t = dW_t$. Let $f(x) = e^{x^2-1}$. Then, $f'(x) = 2xe^{x^2-1}$ and $f''(x) = 4x^2e^{x^2-1} + 2e^{x^2-1}$.
We can use Ito's to derive the following:

$$\begin{aligned} dZ_t = df(X_t) &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \\ &= 2X_te^{X_t^2-1}dX_t + \frac{1}{2}(4X_t^2e^{X_t^2-1} + 2e^{X_t^2-1})(dX_t)^2 \\ &= exp(W_t^2 - 1)(2W_t^2 + 1)\boldsymbol{dt} + 2W_t exp(W_t^2 - 1)\boldsymbol{dW_t} \end{aligned}$$

(b) Let $X_t = W_t^2 - 1$ and $dX_t = dt + 2W_tdW_t$. Let $f(x) = e^x$. Then, $f'(x) = f''(x) = e^x$.
We can then derive the following:

$$\begin{aligned} dZ_t = df(X_t) &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \\ &= e^{X_t}dX_t + \frac{1}{2}e^{X_t}(dX_t)^2 \\ &= e^{X_t}(dt + 2W_tdW_t) + \frac{1}{2}e^{X_t}(dt + 2W_tdW_t)^2 \\ &= e^{X_t}(dt + 2W_tdW_t) + \frac{1}{2}e^{X_t}((dt)^2 + 4W_t^2(dW_t)^2 + 4dtW_tdW_t) \\ &= exp(W_t^2 - 1)(2W_t^2 + 1)\boldsymbol{dt} + 2W_t exp(W_t^2 - 1)\boldsymbol{dW_t} \end{aligned}$$

(c) Drift term, $\mu = exp(W_t^2 - 1)(2W_t^2 + 1)$. Since $exp(W_t^2 - 1) > 0$ and $(2W_t^2 + 1) > 0$, Z_t is not martingale.

Problem 2
We are given:

$$\mathrm{d}X_t = \kappa(\theta - X_t)dt + \sigma dW_t$$

(a) Using $e^{\kappa t}x$, we can see that this is a variation for $f: \mathbb{R}^2 \to \mathbb{R}$, i.e. $f(X_t, Y_t)$ is an Ito process with $Y_t = t$:

$$\begin{aligned} d(e^{\kappa t}X_t) &= \frac{\delta f}{\delta x}dX_t + \frac{\delta f}{\delta t}dt + \frac{1}{2}\frac{\delta^2 f}{\delta x^2}(dX_t)^2 + \frac{1}{2}\frac{\delta^2 f}{\delta t^2}(dt)^2 + \frac{\delta^2 f}{\delta x \delta t}(dX_t)(dt) \\ &= e^{\kappa t}dX_t + \kappa e^{\kappa t}X_tdt + \frac{1}{2}(0)(dX_t)^2 + \frac{1}{2}\kappa^2e^{\kappa t}X_t(dt)^2 + \kappa e^{\kappa t}(dX_t)(dt) \\ &= e^{\kappa t}(\kappa(\theta - X_t)dt + \sigma dW_t) + \kappa e^{\kappa t}X_tdt \\ &= \kappa\theta e^{\kappa t}dt + e^{\kappa t}\sigma dW_t \end{aligned}$$

We can rewrite the following in integral expression, i.e.:

$$e^{\kappa t}X_t = X_0 + \int_0^t \kappa\theta e^{\kappa s}\,ds + \int_0^t e^{\kappa s}\,\sigma dW_t$$

Further simplifying:

$$X_t = e^{-\kappa t} X_0 + \int_0^t \kappa\theta e^{\kappa(s-t)}\, ds + \int_0^t e^{\kappa(s-t)}\, \sigma dW_t$$

Hence, given some time s and as of time T:

$$X_T = e^{-\kappa T} X_0 + \int_0^T \kappa\theta e^{\kappa(s-T)}\, ds + \int_0^T e^{\kappa(s-T)}\, \sigma dW_s$$

(b) Taking the last equation, and using our definition, we know that our equation's first two terms are the mean and the last term is variance.

Simplifying the second term:

$$\int_0^T \kappa\theta e^{\kappa(s-T)}\, ds = \left[\theta e^{\kappa(s-T)}\right]_0^T$$
$$= \theta(1 - e^{-\kappa T})$$

Using the additional information for part (b), term 3 has mean 0.

Hence:

$$E(X_T) = e^{-\kappa T} X_0 + \theta(1 - e^{-\kappa T})$$

Then:

$$Var(X_T) = Var\left(\int_0^T e^{\kappa(s-T)}\, \sigma dW_s\right)$$
$$= \int_0^T e^{2\kappa(s-T)}\, \sigma^2 ds$$
$$= \frac{\sigma^2}{2\kappa}\left[e^{2\kappa(s-T)}\right]_0^T$$
$$= \frac{\sigma^2}{2\kappa}(1 - e^{-\kappa T})$$

Problem 3

For each item:

(a) $\kappa = 8, \theta = 10$
(b) $\kappa = 3, \theta = 15$
(c) $\kappa = 8, \theta = 15$
(d) $\kappa = 3, \theta = 10$
(e) $\kappa = 3, \theta = 5$
(f) $\kappa = 8, \theta = 5$