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FINM 33000: Homework 4 Solutions

Problem 1

(a) The following argument tries to prove that the no-arbitrage time-0 price of the call is zero. The argument presupposes that the following perfectly replicates the call on S_t with strike 105, and that hence the call price at time zero is zero:

$$\Theta_t^{rep} := \begin{cases} (0\ shares\ of\ asset, 0\ bonds), & if\ S_t \leq K \\ (1\ shares\ of\ asset, -K\ bonds), & if\ S_t > K \end{cases}$$

However, this proof is flawed

1. Logical Thought Process:
   Given there we are starting at $S_0 = 100$, we can see a non-zero probability that $S_{12} \geq 105$. This means that the price of the option cannot be zero, as there is a non-zero dollar value that has to be assigned to a non-zero probability of being above 105 at time $T = 12$. This means that the presupposed option price of zero dollars represents zero initial investment, no risk of loss, and some chance of gain. A "type 1" arbitrage. Hence, given that this is a multi-period scenario where we can reach at or above stock price 105 if given enough time, this "proof" is flawed.

2. Mathematical Proof:
   Given $V_t = \{C_t(105)\}$:
   $$P(V_{12} \geq 0) = 1$$
   Consequently, as well:
   $$P(V_{12} > 0) > 0$$
   If, in the event, $V_0 = 0$, we have a clear "type 1" arbitrage. Hence, this price cannot be a no-arbitrage price

(b) Firstly, we have to understand what our risk-neutral probability is, i.e. we want to see how our stock price moves and what our "Q" measure is, let's look at S_1:
$$S_0 = q(S_0 + 1) + (1 - q)(S_0 - 1)$$
Hence, we can clearly see that:
$$q = \frac{1}{2}$$
Furthermore, we can extrapolate that, given all the stock price movements are $+1$ or -1 of the prior time period's stock price, the risk-neutral probability for any "branch" is the same. Using this idea, we can now price our call.
Given that our call only yields a return that is ≥ 0 when $S_{12} \geq 105$, we are essentially looking for all permutations where we yield at or above five +1 moves, i.e.:
$$C_0(105) = \frac{1}{2^{12}} \sum_{n=5}^{12} \binom{12}{n} (n - 5)$$
$$= \frac{1}{2^{12}} \left(\binom{12}{6} + 2\binom{12}{7} + \cdots + \binom{12}{12} 7 \right)$$

Problem 2

(a) Let's assume our lower and upper bound to be $(2.00, 3.00)$ respectively. Given that our asset price is a bounded martingale, we can see that:

$$M_0 = 16, \mathrm{E}M_T = M_0 = 100p$$

Hence:

$$p = 0.16$$

(b) Given that $M_t = A^{S_t}$, we want to solve for:

$$\mathrm{E}(M_{t+1} - M_t) = 0$$

Firstly, we can see that:

$$M_{t+1} - M_t = A^{S_{t+1}} - A^{S_t}$$

Taking this into our equation, and taking the idea where we have information up till time t, and taking our measurements in numbers of cents:

$$\begin{aligned}
\mathrm{E}(M_{t+1} - M_t) = E(A^{S_{t+1}} - A^{S_t}) &= 0 \\
uA^{S_t+1} + (1-u)A^{S_t-1} - A^{S_t} &= 0 \\
uA^1 + (1-u)A^{-1} &= 1 \\
uA^2 - A + (1-u) &= 0 \\
\big(uA - (1-u)\big)(A-1) &= 0 \\
A &= \frac{1-u}{u}
\end{aligned}$$

(c) Given that:

$$E(M_T) = E(M_0) = A^{216}$$

Then:

$$\begin{aligned}
P\{S_t = 200\} \cdot A^{200} + P\{S_t = 300\} \cdot A^{300} &= A^{216} \\
(1 - P\{S_t = 300\}) \cdot A^{200} + P\{S_t = 300\} \cdot A^{300} &= A^{216} \\
P\{S_t = 300\} &= \frac{A^{216} - A^{200}}{A^{300} - A^{200}}
\end{aligned}$$