COMP0130 Robot Vision and Navigation

1D: Single-Epoch GNSS Positioning

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Session Objectives

Show how to compute a GNSS position solutions using measurements from a single point in time.

- An exact solution using 4 satellites only.
- A least-squares solution using more than 4 satellites.

Single-Epoch GNSS Positioning

Measurements Output (or Observables)

Pseudo-range

- Range (m) from satellite to user antenna plus clock offset ρ_a^s
- Obtained from code tracking

Pseudo-range rate

- Range rate (m/s) from satellite to user antenna plus clock drift $\dot{\rho}_a^s$

Doppler shift

- Doppler shift (Hz) = – pseudo-range rate * carrier frequency / speed of light $\Delta f_{ca,a}^s$

"Carrier phase" or Accumulated Delta Range (ADR)

- Current reference carrier phase
- Plus number of integer carrier cycles since start of tracking Φ_a^s
- Only available where carrier phase tracking is implemented
- Units may be metres, carrier cycles or radians

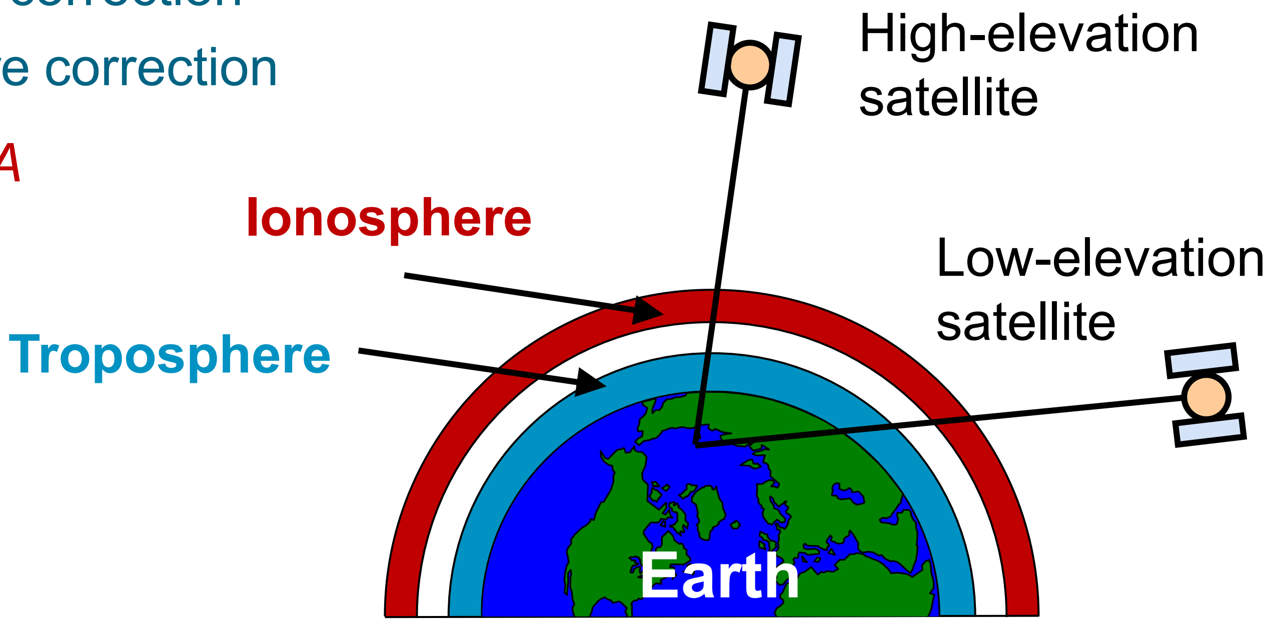
Single-Epoch GNSS Positioning

Measurement Correction

Several corrections are applied to the raw GNSS receiver measurements before using them to compute position:

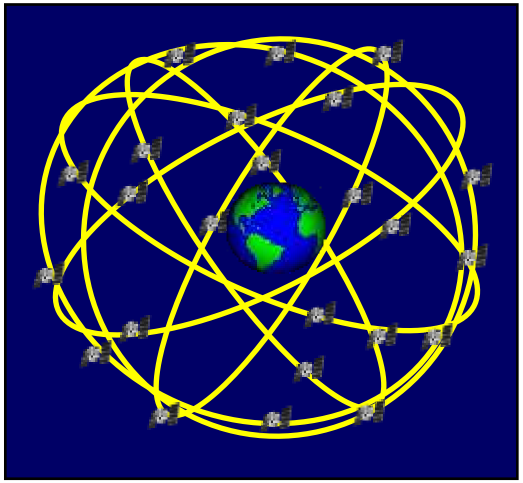
- Satellite clock correction
- Ionosphere correction
- Troposphere correction

See Lecture 2A



Single-Epoch GNSS Positioning

We Need to Know The Satellite Positions



GPS, Galileo and BeiDou satellites broadcast a series of parameters describing their orbits

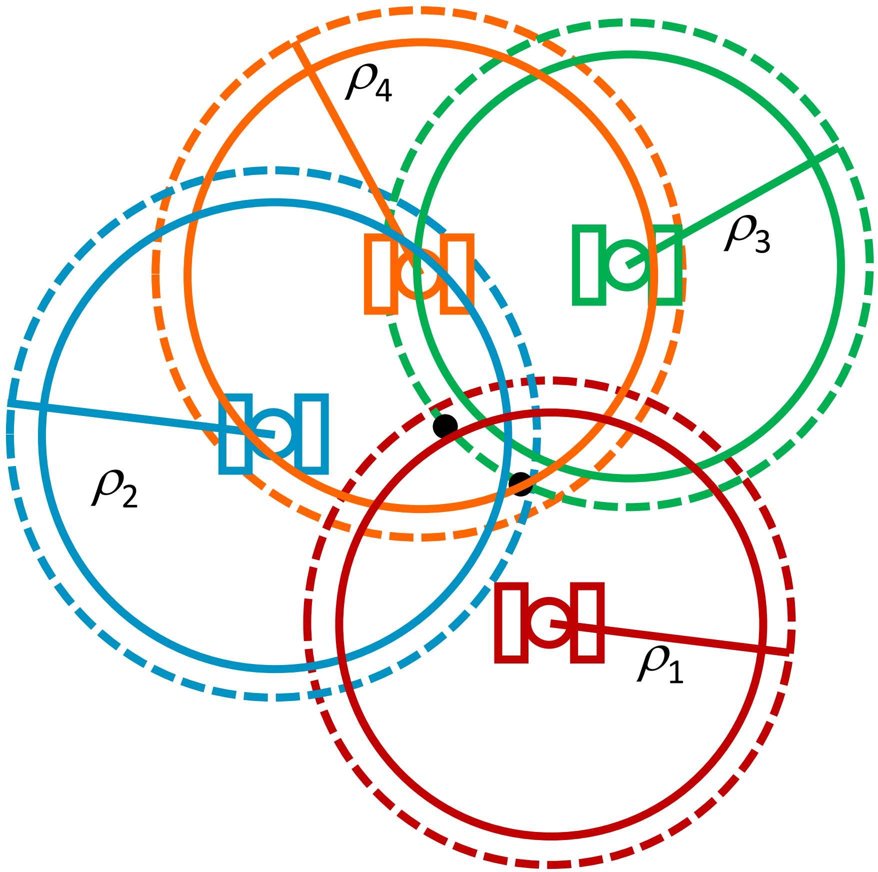
- Known as the **ephemeris**
- Repeated every 30 seconds (GPS C/A code)
- Updated every two hours
- *A mathematical model is used to calculate position and velocity using these parameters*

GLONASS satellites broadcast position, velocity and lunisolar acceleration at a reference time

- Repeated every 30 seconds; Updated every 30 minutes
- *A force model is used to calculate the current position*

GNSS processing software will calculate satellite positions

Positioning geometry in 3 dimensions



With GNSS *pseudo-*ranges, you need a 4th satellite to resolve the receiver clock error

Coordinate System

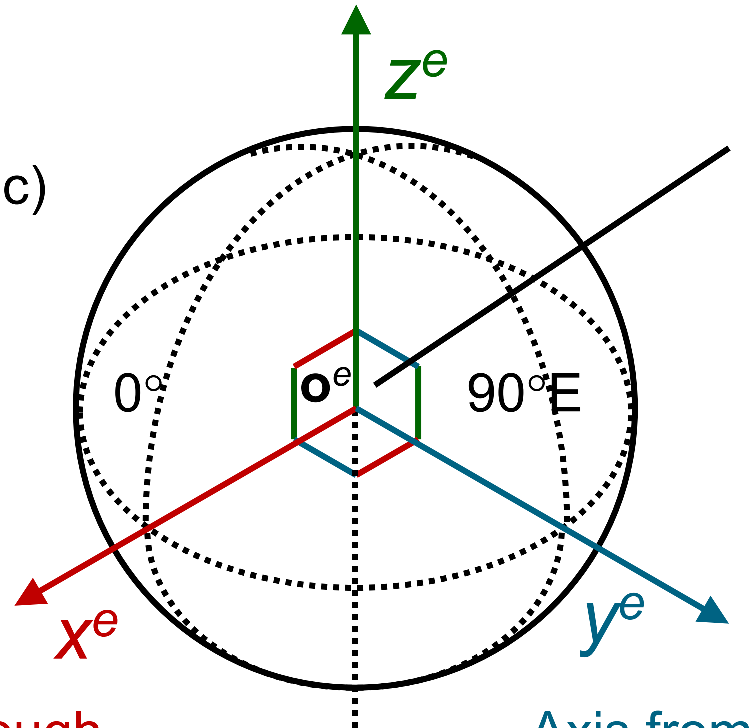
An Earth-centred Earth-fixed (ECEF) frame

with Cartesian position coordinates (i.e., geocentric)

is normally used

An alternative is Earth-centred inertial (ECI)

Earth rotation (spin) axis



Origin at the centre of the Earth

Axis from Centre through zero longitude meridian

Axis from Centre through 90° east meridian

Single-Epoch GNSS Positioning

Notation

$\hat{\mathbf{r}}^{e}_{ea}$

Estimated

Cartesian Position

Of User Antenna, a

With respect to origin of ECEF frame, e

Resolved about the ECEF frame, e, axes

$$\mathbf{r}^{e}_{ea} = \begin{pmatrix} x^{e}_{ea} \\ y^{e}_{ea} \\ z^{e}_{ea} \end{pmatrix}$$

$\mathbf{r}^{e}_{e1}, \mathbf{r}^{e}_{e2}, \mathbf{r}^{e}_{e3} \ldots$ **Positions of satellites 1, 2, 3 …**

$\tilde{\rho}^{1}_{a,C}, \tilde{\rho}^{2}_{a,C}, \tilde{\rho}^{3}_{a,C} \ldots$ **Measured pseudo-ranges from satellites 1, 2, 3 … to user antenna** (*corrections applied*)

$\delta\hat{\rho}^{a}_{c}$ **Estimated range error due to receiver clock offset**

Note: Pseudo-ranges or satellite positions must be compensated for the Sagnac effect

Single-Epoch GNSS Positioning

Ranging Geometry

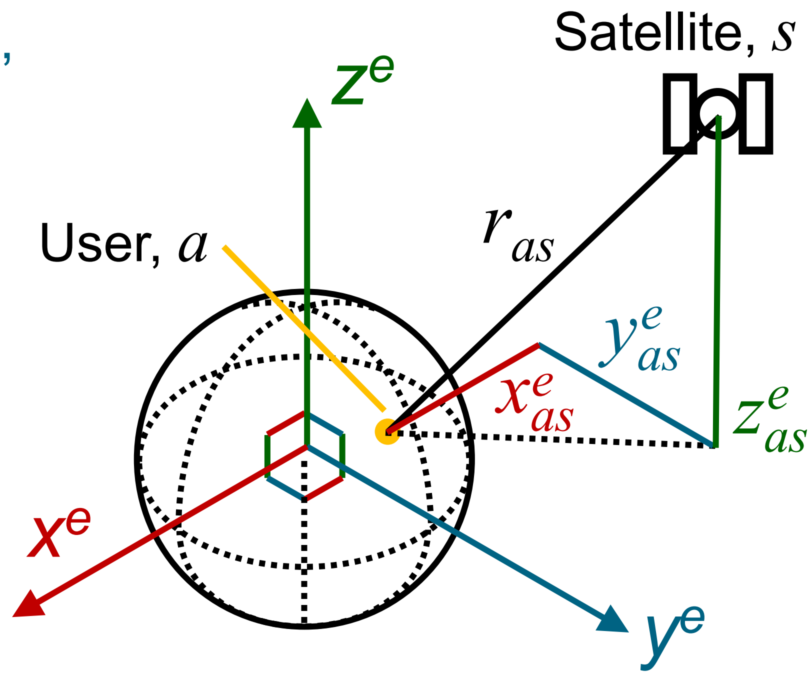
Applying Pythagorus' Theorem Twice,
The user – satellite distance is

$$r_{as}^{2} = x_{as}^{e\;2} + y_{as}^{e\;2} + z_{as}^{e\;2}$$

(User to satellite) =
(Earth to satellite) – (Earth to User)

$$\Rightarrow \mathbf{r}_{as}^{e} = \mathbf{r}_{es}^{e} - \mathbf{r}_{ea}^{e} \qquad y_{as}^{e} = y_{es}^{e} - y_{ea}^{e}$$

$$x_{as}^{e} = x_{es}^{e} - x_{ea}^{e} \qquad z_{as}^{e} = z_{es}^{e} - z_{ea}^{e}$$



$$\therefore\; r_{as}^{2} = \left(x_{es}^{e} - x_{ea}^{e}\right)^{2} + \left(y_{es}^{e} - y_{ea}^{e}\right)^{2} + \left(z_{es}^{e} - z_{ea}^{e}\right)^{2}$$

Pseudo-range = range + receiver clock offset
(*where satellite clock offset is corrected*) $\Rightarrow$

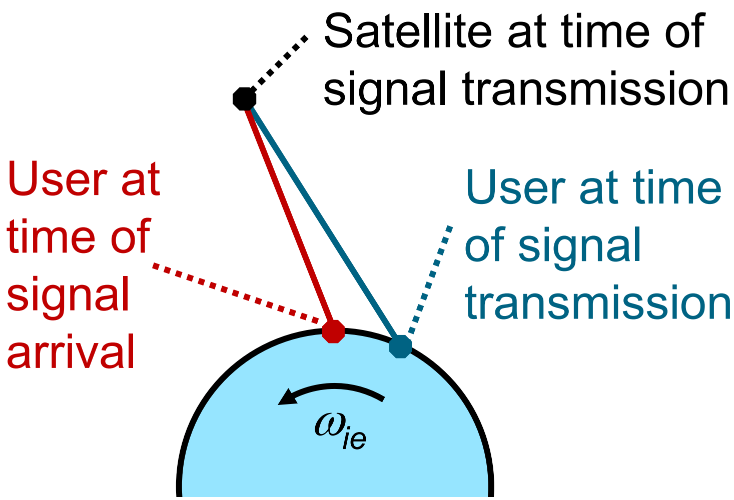
$$\rho_{a,C}^{s} = r_{as} + \delta\rho_{c}^{a}$$

Single-Epoch GNSS Positioning

Sagnac Effect

A change in the apparent speed of light due to coordinate frame rotation with respect to inertial space:

- It applies to position computation in any Earth-referenced frame
- Ignoring it results in position errors of up to 40m



A Sagnac correction is typically applied to the satellite positions:

- We replace $\mathbf{r}_{es}^{e}$ with $\mathbf{r}_{Is}^{I} = \mathbf{C}_e^I \mathbf{r}_{es}^{e}$

where

$$\mathbf{C}_e^I \approx \begin{pmatrix} 1 & \omega_{ie} r_{as}/c & 0 \\ -\omega_{ie} r_{as}/c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ω_{ie} is Earth rotation rate

c is the speed of light

Single-Epoch GNSS Positioning

The Positioning Equations

Range error due to receiver clock (unknown)

User antenna position (unknown)

$$\tilde{\rho}_{a,C}^{1} = \sqrt{\left(x_{I1}^{I} - \hat{x}_{ea}^{e+}\right)^2 + \left(y_{I1}^{I} - \hat{y}_{ea}^{e+}\right)^2 + \left(z_{I1}^{I} - \hat{z}_{ea}^{e+}\right)^2} + \delta\hat{\rho}_{c}^{a+}$$

$$\tilde{\rho}_{a,C}^{2} = \sqrt{\left(x_{I2}^{I} - \hat{x}_{ea}^{e+}\right)^2 + \left(y_{I2}^{I} - \hat{y}_{ea}^{e+}\right)^2 + \left(z_{I2}^{I} - \hat{z}_{ea}^{e+}\right)^2} + \delta\hat{\rho}_{c}^{a+}$$

$$\tilde{\rho}_{a,C}^{3} = \sqrt{\left(x_{I3}^{I} - \hat{x}_{ea}^{e+}\right)^2 + \left(y_{I3}^{I} - \hat{y}_{ea}^{e+}\right)^2 + \left(z_{I3}^{I} - \hat{z}_{ea}^{e+}\right)^2} + \delta\hat{\rho}_{c}^{a+}$$

$$\tilde{\rho}_{a,C}^{4} = \sqrt{\left(x_{I4}^{I} - \hat{x}_{ea}^{e+}\right)^2 + \left(y_{I4}^{I} - \hat{y}_{ea}^{e+}\right)^2 + \left(z_{I4}^{I} - \hat{z}_{ea}^{e+}\right)^2} + \delta\hat{\rho}_{c}^{a+}$$

Satellite positions (known and corrected for Sagnac effect)

Pseudo-range measurements (known and corrected for satellite clock, ionosphere and troposphere)

Single-Epoch GNSS Positioning

Positioning with 4 Satellites (1)

$$\tilde{\rho}_{a,C}^{1} = \sqrt{\left(x_{I1}^{I} - \hat{x}_{ea}^{e+}\right)^2 + \left(y_{I1}^{I} - \hat{y}_{ea}^{e+}\right)^2 + \left(z_{I1}^{I} - \hat{z}_{ea}^{e+}\right)^2} + \delta\hat{\rho}_{c}^{a}$$

$$\tilde{\rho}_{a,C}^{2} = \sqrt{\left(x_{I2}^{I} - \hat{x}_{ea}^{e+}\right)^2 + \left(y_{I2}^{I} - \hat{y}_{ea}^{e+}\right)^2 + \left(z_{I2}^{I} - \hat{z}_{ea}^{e+}\right)^2} + \delta\hat{\rho}_{c}^{a}$$

$$\tilde{\rho}_{a,C}^{3} = \sqrt{\left(x_{I3}^{I} - \hat{x}_{ea}^{e+}\right)^2 + \left(y_{I3}^{I} - \hat{y}_{ea}^{e+}\right)^2 + \left(z_{I3}^{I} - \hat{z}_{ea}^{e+}\right)^2} + \delta\hat{\rho}_{c}^{a}$$

$$\tilde{\rho}_{a,C}^{4} = \sqrt{\left(x_{I4}^{I} - \hat{x}_{ea}^{e+}\right)^2 + \left(y_{I4}^{I} - \hat{y}_{ea}^{e+}\right)^2 + \left(z_{I4}^{I} - \hat{z}_{ea}^{e+}\right)^2} + \delta\hat{\rho}_{c}^{a}$$

Problem – *Equations are highly nonlinear*

Solution – *Linearisation*

This requires a prediction of the user position and clock offset

$$\hat{x}_{ea}^{e-}, \hat{y}_{ea}^{e-}, \hat{z}_{ea}^{e-}, \delta\hat{\rho}_{c}^{a-}$$

The last known position is typically used, where available

Otherwise, the centre of the Earth can be used

Single-Epoch GNSS Positioning

Positioning with 4 Satellites (2)

Predict the ranges:

$$\hat{r}_{as}^{-} = \sqrt{\left[\hat{\mathbf{r}}_{Is}^{I} - \hat{\mathbf{r}}_{ea}^{e-}\right]^{\mathrm{T}}\left[\hat{\mathbf{r}}_{Is}^{I} - \hat{\mathbf{r}}_{ea}^{e-}\right]} = \sqrt{\left[\mathbf{C}_{e}^{I}\hat{\mathbf{r}}_{es}^{e} - \hat{\mathbf{r}}_{ea}^{e-}\right]^{\mathrm{T}}\left[\mathbf{C}_{e}^{I}\hat{\mathbf{r}}_{es}^{e} - \hat{\mathbf{r}}_{ea}^{e-}\right]}$$

$$= \sqrt{\left(\hat{x}_{Is}^{I} - \hat{x}_{ea}^{e-}\right)^{2} + \left(\hat{y}_{Is}^{I} - \hat{y}_{ea}^{e-}\right)^{2} + \left(\hat{z}_{Is}^{I} - \hat{z}_{ea}^{e-}\right)^{2}} \qquad s \in 1,2,3,4$$

$$\hat{\mathbf{r}}_{ea}^{e-} = \begin{pmatrix} \hat{x}_{ea}^{e-} \\ \hat{y}_{ea}^{e-} \\ \hat{z}_{ea}^{e-} \end{pmatrix}$$

Then linearise the positioning equations (1st order Taylor series):

$$\begin{pmatrix} \tilde{\rho}_{a,C}^{1} - \hat{r}_{a1}^{-} - \delta\hat{\rho}_{c}^{a-} \\ \tilde{\rho}_{a,C}^{2} - \hat{r}_{a2}^{-} - \delta\hat{\rho}_{c}^{a-} \\ \tilde{\rho}_{a,C}^{3} - \hat{r}_{a3}^{-} - \delta\hat{\rho}_{c}^{a-} \\ \tilde{\rho}_{a,C}^{4} - \hat{r}_{a4}^{-} - \delta\hat{\rho}_{c}^{a-} \end{pmatrix} \approx \mathbf{H} \begin{pmatrix} \hat{x}_{ea}^{e+} - \hat{x}_{ea}^{e-} \\ \hat{y}_{ea}^{e+} - \hat{y}_{ea}^{e-} \\ \hat{z}_{ea}^{e+} - \hat{z}_{ea}^{e-} \\ \delta\hat{\rho}_{c}^{a+} - \delta\hat{\rho}_{c}^{a-} \end{pmatrix}$$

$$\mathbf{H} = \left.\begin{pmatrix} \dfrac{\partial \rho_{a}^{1}}{\partial x_{ea}^{e}} & \dfrac{\partial \rho_{a}^{1}}{\partial y_{ea}^{e}} & \dfrac{\partial \rho_{a}^{1}}{\partial z_{ea}^{e}} & \dfrac{\partial \rho_{a}^{1}}{\partial \rho_{c}^{a}} \\ \dfrac{\partial \rho_{a}^{2}}{\partial x_{ea}^{e}} & \dfrac{\partial \rho_{a}^{2}}{\partial y_{ea}^{e}} & \dfrac{\partial \rho_{a}^{2}}{\partial z_{ea}^{e}} & \dfrac{\partial \rho_{a}^{2}}{\partial \rho_{c}^{a}} \\ \dfrac{\partial \rho_{a}^{3}}{\partial x_{ea}^{e}} & \dfrac{\partial \rho_{a}^{3}}{\partial y_{ea}^{e}} & \dfrac{\partial \rho_{a}^{3}}{\partial z_{ea}^{e}} & \dfrac{\partial \rho_{a}^{3}}{\partial \rho_{c}^{a}} \\ \dfrac{\partial \rho_{a}^{4}}{\partial x_{ea}^{e}} & \dfrac{\partial \rho_{a}^{4}}{\partial y_{ea}^{e}} & \dfrac{\partial \rho_{a}^{4}}{\partial z_{ea}^{e}} & \dfrac{\partial \rho_{a}^{4}}{\partial \rho_{c}^{a}} \end{pmatrix}\right|_{\mathbf{r}_{ea}^{e} = \hat{\mathbf{r}}_{ea}^{e-}}$$

H = Measurement Matrix

Single-Epoch GNSS Positioning

Positioning with 4 Satellites (3)

$$\rho_{a,C}^{s-} = \sqrt{\left(x_{Is}^{I} - x_{ea}^{e}\right)^2 + \left(y_{Is}^{I} - y_{ea}^{e}\right)^2 + \left(z_{Is}^{I} - z_{ea}^{e}\right)^2} + \delta\rho_c^{a-} \qquad s \in 1,2,3,4$$

Differentiating using the chain rule:

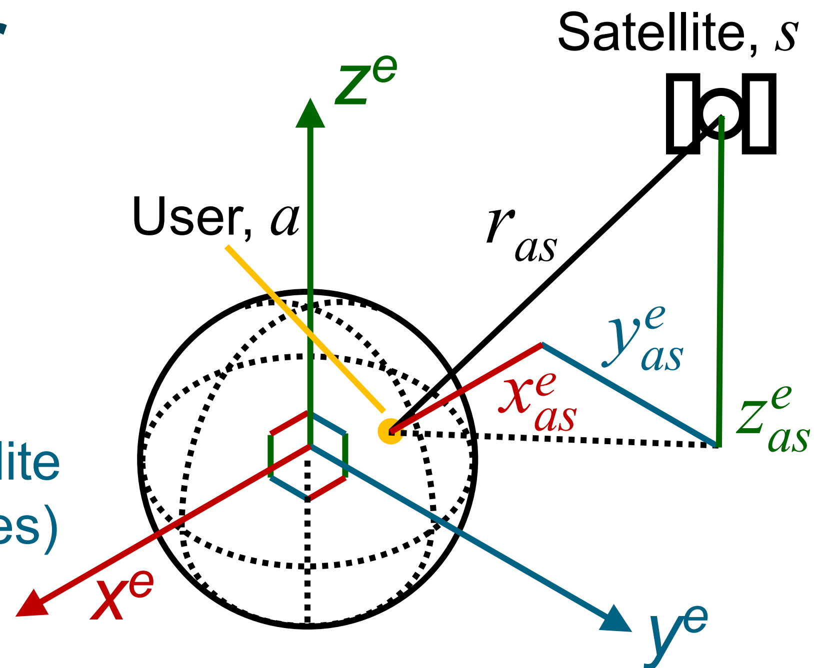
$$\mathbf{H} = \left.\begin{pmatrix} \frac{\partial\rho_a^1}{\partial x_{ea}^e} & \frac{\partial\rho_a^1}{\partial y_{ea}^e} & \frac{\partial\rho_a^1}{\partial z_{ea}^e} & \frac{\partial\rho_a^1}{\partial\rho_c^a} \\ \frac{\partial\rho_a^2}{\partial x_{ea}^e} & \frac{\partial\rho_a^2}{\partial y_{ea}^e} & \frac{\partial\rho_a^2}{\partial z_{ea}^e} & \frac{\partial\rho_a^2}{\partial\rho_c^a} \\ \frac{\partial\rho_a^3}{\partial x_{ea}^e} & \frac{\partial\rho_a^3}{\partial y_{ea}^e} & \frac{\partial\rho_a^3}{\partial z_{ea}^e} & \frac{\partial\rho_a^3}{\partial\rho_c^a} \\ \frac{\partial\rho_a^4}{\partial x_{ea}^e} & \frac{\partial\rho_a^4}{\partial y_{ea}^e} & \frac{\partial\rho_a^4}{\partial z_{ea}^e} & \frac{\partial\rho_a^4}{\partial\rho_c^a} \end{pmatrix}\right|_{\mathbf{r}_{ea}^e = \hat{\mathbf{r}}_{ea}^{e-}} = \begin{pmatrix} -\frac{x_{I1}^I - \hat{x}_{ea}^{e-}}{\hat{r}_{a1}^-} & -\frac{y_{I1}^I - \hat{y}_{ea}^{e-}}{\hat{r}_{a1}^-} & -\frac{z_{I1}^I - \hat{z}_{ea}^{e-}}{\hat{r}_{a1}^-} & 1 \\ -\frac{x_{I2}^I - \hat{x}_{ea}^{e-}}{\hat{r}_{a2}^-} & -\frac{y_{I2}^I - \hat{y}_{ea}^{e-}}{\hat{r}_{a2}^-} & -\frac{z_{I2}^I - \hat{z}_{ea}^{e-}}{\hat{r}_{a2}^-} & 1 \\ -\frac{x_{I3}^I - \hat{x}_{ea}^{e-}}{\hat{r}_{a3}^-} & -\frac{y_{I3}^I - \hat{y}_{ea}^{e-}}{\hat{r}_{a3}^-} & -\frac{z_{I3}^I - \hat{z}_{ea}^{e-}}{\hat{r}_{a3}^-} & 1 \\ -\frac{x_{I4}^I - \hat{x}_{ea}^{e-}}{\hat{r}_{a4}^-} & -\frac{y_{I4}^I - \hat{y}_{ea}^{e-}}{\hat{r}_{a4}^-} & -\frac{z_{I4}^I - \hat{z}_{ea}^{e-}}{\hat{r}_{a4}^-} & 1 \end{pmatrix}$$

where $\hat{r}_{as}^{-} = \sqrt{\left(x_{Is}^{I} - \hat{x}_{ea}^{e-}\right)^2 + \left(y_{Is}^{I} - \hat{y}_{ea}^{e-}\right)^2 + \left(z_{Is}^{I} - \hat{z}_{ea}^{e-}\right)^2} \qquad s \in 1,2,3,4$

Single-Epoch GNSS Positioning

Line of Sight Unit Vector

$$\mathbf{u}_{as}^{e} = \begin{pmatrix} \left(x_{Is}^{I} - x_{ea}^{e}\right)/r_{as} \\ \left(y_{Is}^{I} - y_{ea}^{e}\right)/r_{as} \\ \left(z_{Is}^{I} - z_{ea}^{e}\right)/r_{as} \end{pmatrix} \approx \begin{pmatrix} x_{as}^{e}/r_{as} \\ y_{as}^{e}/r_{as} \\ z_{as}^{e}/r_{as} \end{pmatrix}$$



This describes the direction of the satellite from the user (here, in ECEF coordinates)

For unit vectors, $\left|\mathbf{u}_{as}^{e}\right| = 1$

Thus, only two components are independent

It is calculated as a vector using

$$\mathbf{u}_{as}^{e} = \frac{\mathbf{C}_{e}^{I}\left(\hat{r}_{as}^{-}\right)\mathbf{r}_{es}^{e} - \hat{\mathbf{r}}_{ea}^{e-}}{\hat{r}_{as}^{-}} \qquad \hat{r}_{as}^{-} = \left|\mathbf{C}_{e}^{I}\left(\hat{r}_{as}^{-}\right)\mathbf{r}_{es}^{e} - \hat{\mathbf{r}}_{ea}^{e-}\right|$$

This has the same direction as the vector from the receiver to the satellite, but unit magnitude

Single-Epoch GNSS Positioning

4-Satellite Position Solution (1)

Solving the linearised positioning equations:

$$\begin{pmatrix} \tilde{\rho}^{1}_{a,C} - \hat{r}^{-}_{a1} - \delta\hat{\rho}^{a-}_{c} \\ \tilde{\rho}^{2}_{a,C} - \hat{r}^{-}_{a2} - \delta\hat{\rho}^{a-}_{c} \\ \tilde{\rho}^{3}_{a,C} - \hat{r}^{-}_{a3} - \delta\hat{\rho}^{a-}_{c} \\ \tilde{\rho}^{4}_{a,C} - \hat{r}^{-}_{a4} - \delta\hat{\rho}^{a-}_{c} \end{pmatrix} \approx \mathbf{H} \begin{pmatrix} \hat{x}^{e+}_{ea} - \hat{x}^{e-}_{ea} \\ \hat{y}^{e+}_{ea} - \hat{y}^{e-}_{ea} \\ \hat{z}^{e+}_{ea} - \hat{z}^{e-}_{ea} \\ \delta\hat{\rho}^{a+}_{c} - \delta\hat{\rho}^{a-}_{c} \end{pmatrix}$$

$$\mathbf{H} = \begin{pmatrix} -u^{e}_{a1,x} & -u^{e}_{a1,y} & -u^{e}_{a1,z} & 1 \\ -u^{e}_{a2,x} & -u^{e}_{a2,y} & -u^{e}_{a2,z} & 1 \\ -u^{e}_{a3,x} & -u^{e}_{a3,y} & -u^{e}_{a3,z} & 1 \\ -u^{e}_{a4,x} & -u^{e}_{a4,y} & -u^{e}_{a4,z} & 1 \end{pmatrix}$$

Solution is given by:

$$\mathbf{u}^{e-}_{as} = \frac{\mathbf{C}^{I}_{e}\mathbf{r}^{e}_{es} - \hat{\mathbf{r}}^{e-}_{ea}}{\hat{r}^{-}_{as}} \qquad s \in 1,2,3,4$$

$$\begin{pmatrix} \hat{x}^{e+}_{ea} \\ \hat{y}^{e+}_{ea} \\ \hat{z}^{e+}_{ea} \\ \delta\hat{\rho}^{a+}_{c} \end{pmatrix} \approx \begin{pmatrix} \hat{x}^{e-}_{ea} \\ \hat{y}^{e-}_{ea} \\ \hat{z}^{e-}_{ea} \\ \delta\hat{\rho}^{a-}_{c} \end{pmatrix} + \mathbf{H}^{-1} \begin{pmatrix} \tilde{\rho}^{1}_{a,C} - \hat{r}^{-}_{a1} - \delta\hat{\rho}^{a-}_{c} \\ \tilde{\rho}^{2}_{a,C} - \hat{r}^{-}_{a2} - \delta\hat{\rho}^{a-}_{c} \\ \tilde{\rho}^{3}_{a,C} - \hat{r}^{-}_{a3} - \delta\hat{\rho}^{a-}_{c} \\ \tilde{\rho}^{4}_{a,C} - \hat{r}^{-}_{a4} - \delta\hat{\rho}^{a-}_{c} \end{pmatrix}$$

Single-Epoch GNSS Positioning

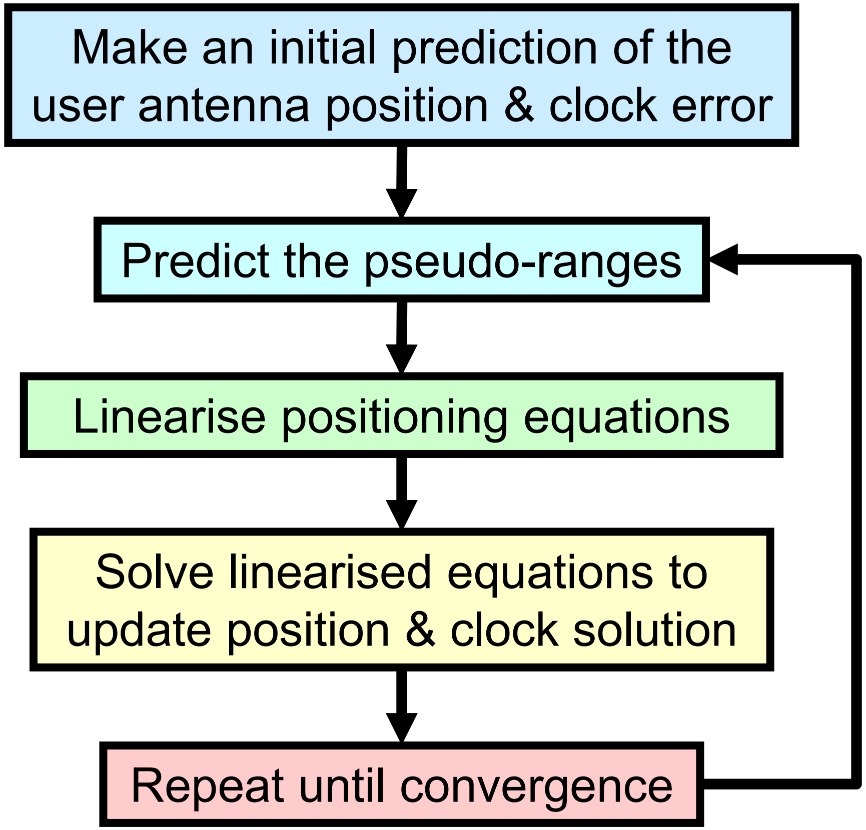
4-Satellite Position Solution (2)

Linearisation introduces errors

- The more accurate the predicted antenna position, the smaller the errors will be
- Large predicted clock errors do not affect the position solution

If the linearisation error is too large, the solution must be iterated:

- *New prediction = old solution*



See *RVN GNSS Positioning Example.xlsx* on Moodle

Single-Epoch GNSS Positioning

Position Solution with 5 or more Satellites

Using signals from more than 4 satellites usually improves accuracy

Linearised positioning equations:

$$\begin{pmatrix} \tilde{\rho}_a^1 - \hat{r}_{a1}^- - \delta\hat{\rho}_c^{a-} \\ \tilde{\rho}_a^2 - \hat{r}_{a2}^- - \delta\hat{\rho}_c^{a-} \\ \vdots \\ \tilde{\rho}_a^m - \hat{r}_{am}^- - \delta\hat{\rho}_c^{a-} \end{pmatrix} \approx \mathbf{H} \begin{pmatrix} \hat{x}_{ea}^{e+} - \hat{x}_{ea}^{e-} \\ \hat{y}_{ea}^{e+} - \hat{y}_{ea}^{e-} \\ \hat{z}_{ea}^{e+} - \hat{z}_{ea}^{e-} \\ \delta\hat{\rho}_c^{a+} - \delta\hat{\rho}_c^{a-} \end{pmatrix} \qquad \mathbf{H} = \begin{pmatrix} -u_{a1,x}^e & -u_{a1,y}^e & -u_{a1,z}^e & 1 \\ -u_{a2,x}^e & -u_{a2,y}^e & -u_{a2,z}^e & 1 \\ \vdots & \vdots & \vdots & \vdots \\ -u_{am,x}^e & -u_{am,y}^e & -u_{am,z}^e & 1 \end{pmatrix}$$

Solution is overdetermined where

number of measurements (m) > number of unknowns (4)

There is no exact solution

H is not square, so cannot be inverted

Least-squares estimation must be used

Single-Epoch GNSS Positioning

Least-Squares Position Determination

Expressing this as a nonlinear least-squares problem

$\mathbf{z} = \mathbf{h}(\mathbf{x})$, where $\mathbf{x} = \begin{pmatrix} x_{ea}^{e} \\ y_{ea}^{e} \\ z_{ea}^{e} \\ \delta\rho_{c}^{a} \end{pmatrix}$ $\mathbf{z} = \begin{pmatrix} \rho_{a,C}^{1} \\ \rho_{a,C}^{2} \\ \vdots \\ \rho_{a,C}^{m} \end{pmatrix}$

and

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}(x_{ea}^{e}, y_{ea}^{e}, z_{ea}^{e}, \delta\rho_{c}^{a}) = \begin{pmatrix} \sqrt{\left[\mathbf{C}_{e}^{I}\left(r_{a1}\right)\mathbf{r}_{e1}^{e} - \mathbf{r}_{ea}^{e}\right]^{\mathrm{T}}\left[\mathbf{C}_{e}^{I}\left(r_{a1}\right)\mathbf{r}_{e1}^{e} - \mathbf{r}_{ea}^{e}\right]} + \delta\rho_{c}^{a} \\ \sqrt{\left[\mathbf{C}_{e}^{I}\left(r_{a2}\right)\mathbf{r}_{e2}^{e} - \mathbf{r}_{ea}^{e}\right]^{\mathrm{T}}\left[\mathbf{C}_{e}^{I}\left(r_{a2}\right)\mathbf{r}_{e2}^{e} - \mathbf{r}_{ea}^{e}\right]} + \delta\rho_{c}^{a} \\ \vdots \\ \sqrt{\left[\mathbf{C}_{e}^{I}\left(r_{am}\right)\mathbf{r}_{em}^{e} - \mathbf{r}_{ea}^{e}\right]^{\mathrm{T}}\left[\mathbf{C}_{e}^{I}\left(r_{am}\right)\mathbf{r}_{em}^{e} - \mathbf{r}_{ea}^{e}\right]} + \delta\rho_{c}^{a} \end{pmatrix}$$

Single-Epoch GNSS Positioning

Least-Squares Position Solution (Unweighted)

The standard basic least-squares solution applies

With equal weighting:

$$\begin{aligned}\hat{\mathbf{x}}^{+} &= \hat{\mathbf{x}}^{-} + \delta\mathbf{x} \\ &\approx \hat{\mathbf{x}}^{-} + \left(\mathbf{H}^{\mathrm{T}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{b}\end{aligned}$$

where the measurement matrix is $\mathbf{H} = \dfrac{\partial \mathbf{h}}{\partial \mathbf{x}} = \begin{pmatrix} -u^{e}_{a1,x} & -u^{e}_{a1,y} & -u^{e}_{a1,z} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ -u^{e}_{am,x} & -u^{e}_{am,y} & -u^{e}_{am,z} & 1 \end{pmatrix}$

and the measurement innovation is

$$\mathbf{b} = \delta\mathbf{z}^{-} = \tilde{\mathbf{z}} - \mathbf{h}\left(\hat{\mathbf{x}}^{-}\right) = \begin{pmatrix} \tilde{\rho}^{1}_{a,C} \\ \vdots \\ \tilde{\rho}^{m}_{a,C} \end{pmatrix} - \begin{pmatrix} \hat{r}^{-}_{a1} + \delta\hat{\rho}^{a}_{c} \\ \vdots \\ \hat{r}^{-}_{am} + \delta\hat{\rho}^{a}_{c} \end{pmatrix}$$

$$\begin{pmatrix} \hat{r}^{-}_{a1} \\ \vdots \\ \hat{r}^{-}_{am} \end{pmatrix} = \begin{pmatrix} \sqrt{\left[\mathbf{C}^{I}_{e}\left(\hat{r}^{-}_{a1}\right)\hat{\mathbf{r}}^{e}_{e1} - \hat{\mathbf{r}}^{e-}_{ea}\right]^{\mathrm{T}}\left[\mathbf{C}^{I}_{e}\left(\hat{r}^{-}_{a1}\right)\hat{\mathbf{r}}^{e}_{e1} - \hat{\mathbf{r}}^{e-}_{ea}\right]} \\ \vdots \\ \sqrt{\left[\mathbf{C}^{I}_{e}\left(\hat{r}^{-}_{am}\right)\hat{\mathbf{r}}^{e}_{em} - \hat{\mathbf{r}}^{e-}_{ea}\right]^{\mathrm{T}}\left[\mathbf{C}^{I}_{e}\left(\hat{r}^{-}_{am}\right)\hat{\mathbf{r}}^{e}_{em} - \hat{\mathbf{r}}^{e-}_{ea}\right]} \end{pmatrix}$$

Single-Epoch GNSS Positioning

Least-Squares Position Solution (Weighted)

The standard basic least-squares solution applies

With unequal weighting:

$$\begin{aligned}\hat{\mathbf{x}}^{+} &= \hat{\mathbf{x}}^{-} + \delta\mathbf{x} \\ &\approx \hat{\mathbf{x}}^{-} + \left(\mathbf{H}^{\mathrm{T}}\mathbf{C}_{\mathbf{z}}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{C}_{\mathbf{z}}^{-1}\mathbf{b}\end{aligned}$$

where $\mathbf{b}$ and $\mathbf{H}$ are as for equal weighting and

$$\mathbf{C}_{\mathbf{z}} = \begin{pmatrix} \sigma_{z1}^{2} & 0 & \cdots & 0 \\ 0 & \sigma_{z2}^{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{zm}^{2} \end{pmatrix}$$

Measurement error SD

Zenith measurement error SD

$$\sigma_{zs} = \frac{\sigma_{z0}}{\sin\theta_{nu}^{as}}$$

Elevation angle

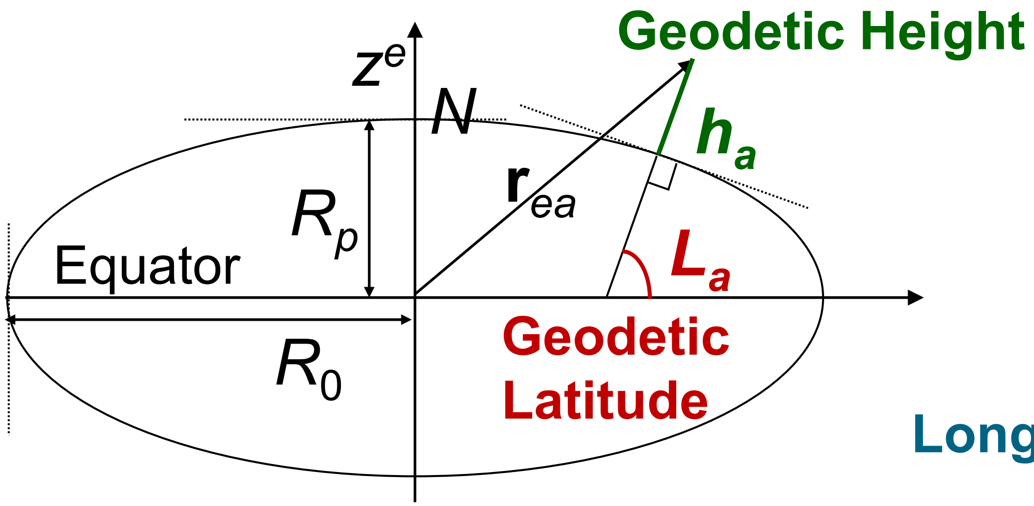
Other models may also be used

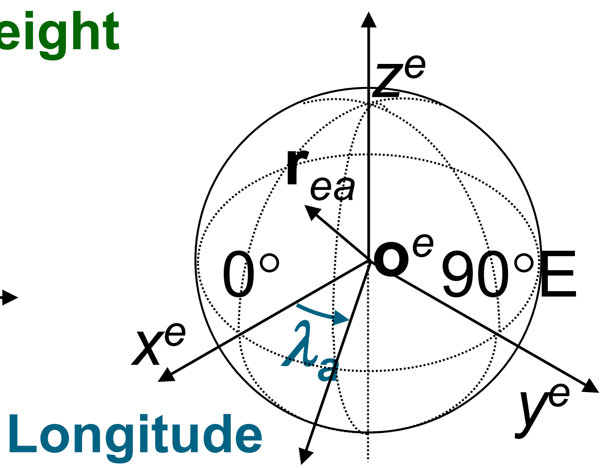
Single-Epoch GNSS Positioning

Latitude, longitude and height (1)

ECEF-referenced position has limited practical use

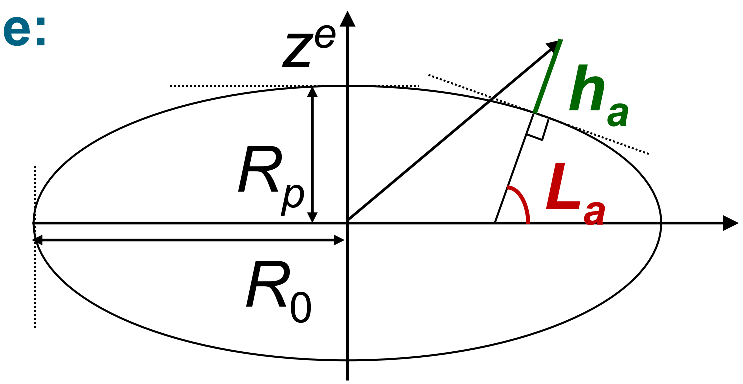
Latitude, longitude and height are more useful

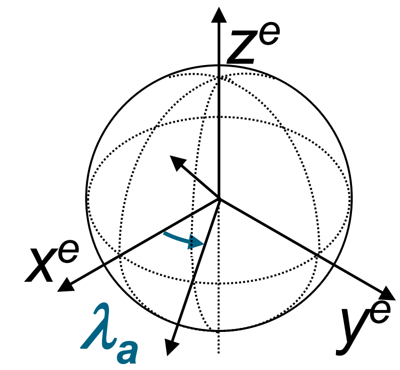




Latitude, longitude and height (2)

Conversion formulae:





$$\tan \lambda_a = \frac{y_{ea}^e}{x_{ea}^e}$$

$$\tan \zeta_a = \frac{z_{ea}^e}{\sqrt{1-e^2}\sqrt{x_{ea}^{e\ 2} + y_{ea}^{e\ 2}}}$$

$$R_E = \frac{R_0}{\sqrt{1-e^2 \sin^2 L_a}}$$

$$\tan L_a \approx \frac{z_{ea}^e\sqrt{1-e^2} + e^2 R_0 \sin^3 \zeta_a}{\sqrt{1-e^2}\left(\sqrt{x_{ea}^{e\ 2} + y_{ea}^{e\ 2}} - e^2 R_0 \cos^3 \zeta_a\right)}$$

$$h_a = \frac{\sqrt{x_{ea}^{e\ 2} + y_{ea}^{e\ 2}}}{\cos L_a} - R_E$$

WGS84 datum: R_0 = 6,378,137.0 m e = 0.0818191908425

Single-Epoch GNSS Positioning

Latitude, longitude and height (3)

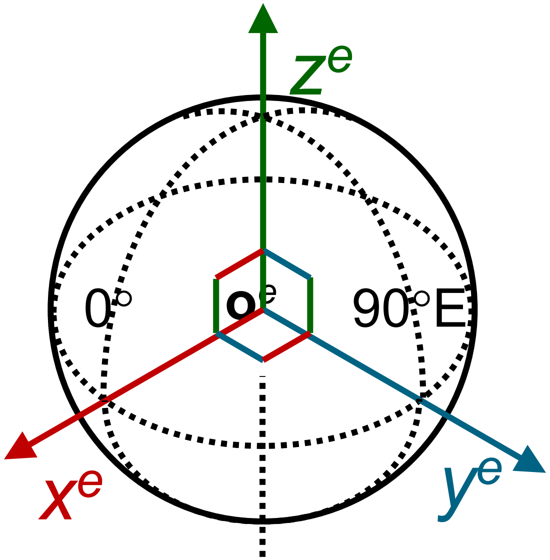
Converting back to Cartesian Position

$$x_{ea}^{e} = \left(R_E + h_a\right)\cos L_a \cos\lambda_a$$

$$y_{ea}^{e} = \left(R_E + h_a\right)\cos L_a \sin\lambda_a$$

$$z_{ea}^{e} = \left[\left(1 - e^2\right)R_E + h_a\right]\sin L_a$$

$$R_E = \frac{R_0}{\sqrt{1 - e^2 \sin^2 L_a}}$$



This is exact

WGS84 datum: R_0 = 6,378,137.0 m e = 0.0818191908425

Single-Epoch GNSS Positioning

Obtaining Velocity

Velocity obtained from differentiating the position solution is very noisy

A much better solution can be obtained from pseudo-range rate measurements $\dot{\rho}_a^s$

Pseudo-range rate = Range rate (m/s) from satellite to user antenna + receiver clock drift

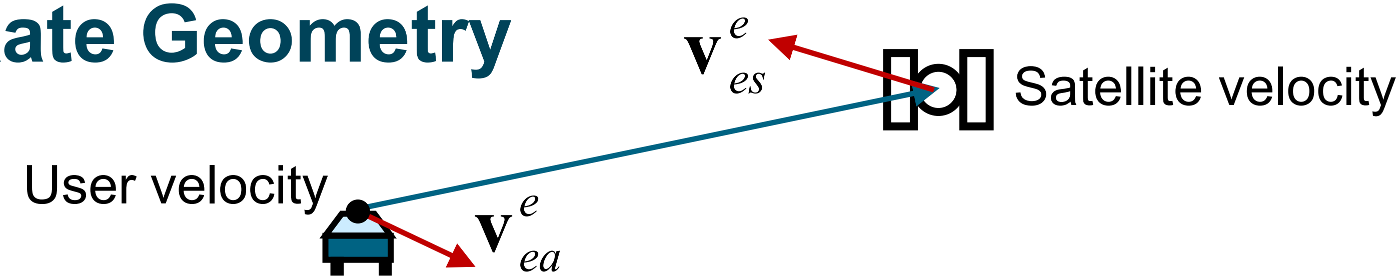
Pseudo-range rate = – Doppler shift (Hz) * speed of light / carrier frequency

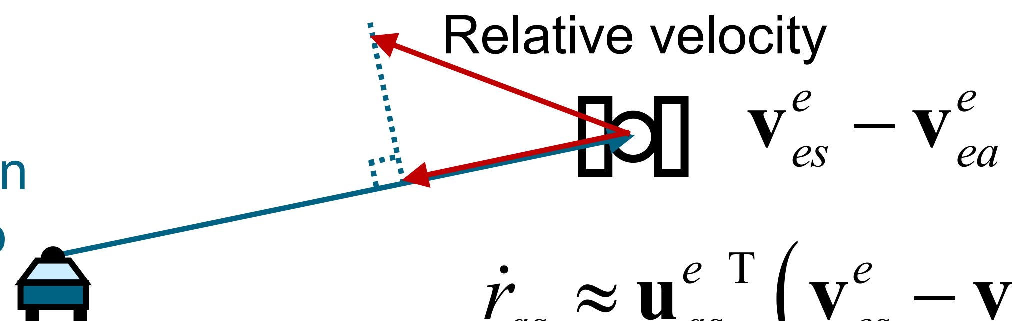
Doppler shift is obtained from carrier tracking

- Much smaller errors from noise and multipath interference than the code tracking used to measure pseudo-range

Single-Epoch GNSS Positioning

Range Rate Geometry





Range rate is the projection of the relative velocity onto the line of sight:

$$\dot{r}_{as} \approx \mathbf{u}_{as}^{e\,\mathrm{T}} \left(\mathbf{v}_{es}^{e} - \mathbf{v}_{ea}^{e} \right)$$

Accounting for the Sagnac effect…

$$\dot{r}_{as} = \hat{\mathbf{u}}_{as}^{e\,\mathrm{T}} \left[\mathbf{C}_{e}^{I} \left(\mathbf{v}_{es}^{e} + \mathbf{\Omega}_{ie}^{e} \mathbf{r}_{es}^{e} \right) - \left(\mathbf{v}_{ea}^{e} + \mathbf{\Omega}_{ie}^{e} \mathbf{r}_{ea}^{e} \right) \right]$$

Pseudo-range rate = range rate + receiver clock drift (*where satellite clock offset & drift are corrected*) :

$$\dot{\rho}_{a,C}^{s} = \dot{r}_{as} + \delta\dot{\rho}_{c}^{a}$$

$$\mathbf{\Omega}_{ie}^{e} = \begin{pmatrix} 0 & -\omega_{ie} & 0 \\ \omega_{ie} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Single-Epoch GNSS Positioning

Least-Squares Velocity Solution (Unweighted)

The standard equally-weighted basic least-squares solution applies:

$$\hat{\mathbf{x}}^{+} \approx \hat{\mathbf{x}}^{-} + \left(\mathbf{H}^{\mathrm{T}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{b}$$

where

$$\hat{\mathbf{x}}^{+} = \begin{pmatrix} \hat{v}^{e+}_{ea,x} \\ \hat{v}^{e+}_{ea,y} \\ \hat{v}^{e+}_{ea,z} \\ \delta\hat{\dot{\rho}}^{a+}_{c} \end{pmatrix} \quad \hat{\mathbf{x}}^{-} = \begin{pmatrix} \hat{v}^{e-}_{ea,x} \\ \hat{v}^{e-}_{ea,y} \\ \hat{v}^{e-}_{ea,z} \\ \delta\hat{\dot{\rho}}^{a-}_{c} \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} \tilde{\dot{\rho}}^{1}_{a,C} - \hat{\dot{\rho}}^{1-}_{a,C} \\ \tilde{\dot{\rho}}^{2}_{a,C} - \hat{\dot{\rho}}^{2-}_{a,C} \\ \vdots \\ \tilde{\dot{\rho}}^{m}_{a,C} - \hat{\dot{\rho}}^{m-}_{a,C} \end{pmatrix}$$

The measurement matrix is the same as for the position solution

$$\mathbf{H} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}} = \begin{pmatrix} -u^{e}_{a1,x} & -u^{e}_{a1,y} & -u^{e}_{a1,z} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ -u^{e}_{am,x} & -u^{e}_{am,y} & -u^{e}_{am,z} & 1 \end{pmatrix}$$