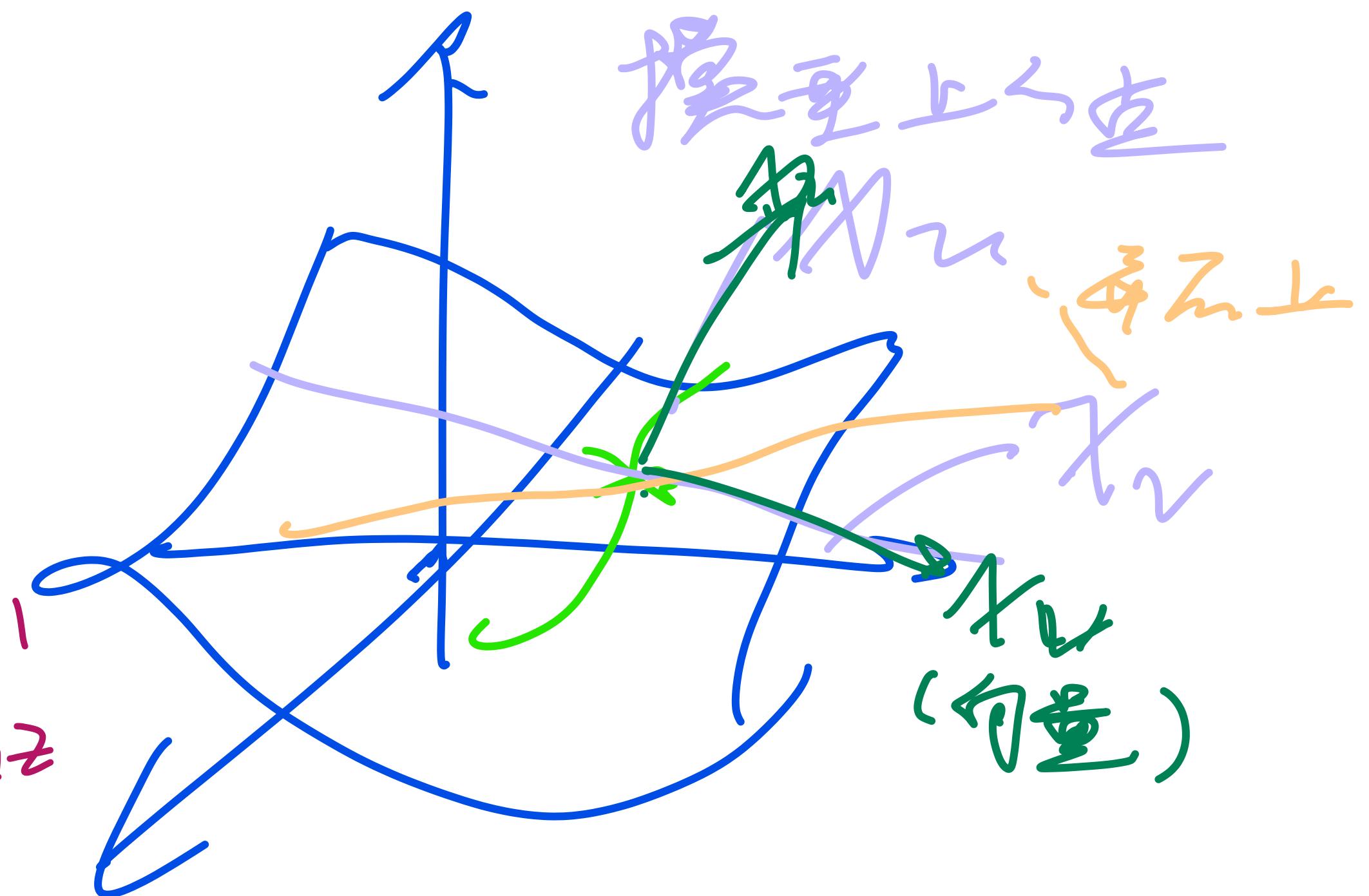


surface

$\begin{matrix} X \\ \text{vec for} \\ \text{变换} - M_{12} \end{matrix}$



$u, v \rightarrow x(u, v), y(u, v), z(u, v)$

Q. 程也从(经过点). 那么它立了  $d$  的 tangent 线一定  
在  $X_u, X_v$  define 在曲面上. (那么  $\pi(u, v) = \frac{X_u \wedge X_v}{\|X_u \wedge X_v\|}$ )

$$|a|=1$$

$$|a(t)|=1$$

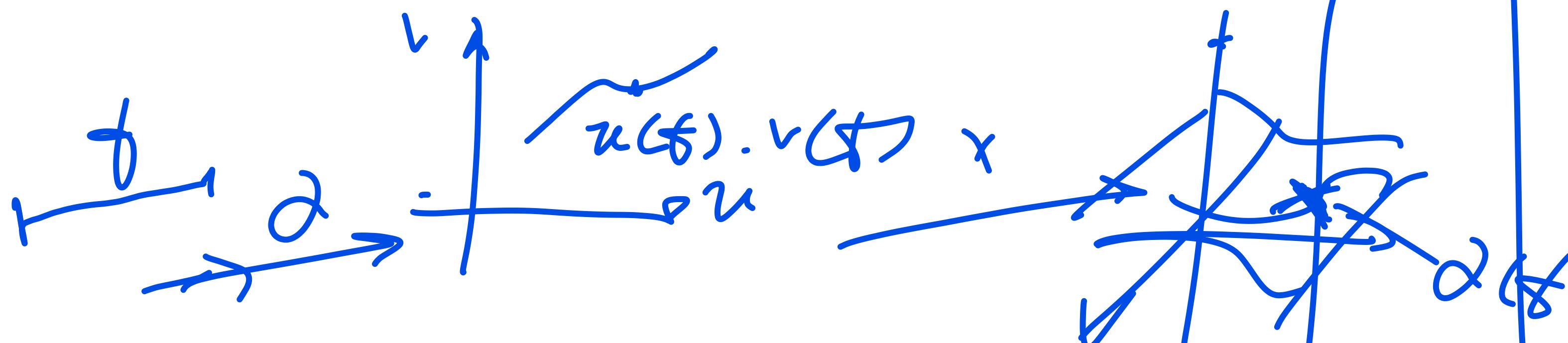
$$\Rightarrow a'(t) \cdot a(t) = 0$$

det. p war +  $\langle a, b \rangle$

$$\langle a, b \rangle^2 + \|a \wedge b\|^2$$

$$= (|a||b| \cos \theta)^2 + (\|a\| \|b\| \sin \theta)^2 = (|a||b|)^2$$

Btr,



$$\partial(t) = X(u(t), v(t))$$

$$\omega'(t) = X_u \cdot u' + V' X_v \quad (\text{chain rule})$$

$$\begin{aligned} |\omega'(t)|^2 &= \\ &< X_u u' + X_v v', X_u u' + X_v v' \rangle \\ &= \underbrace{\langle X_u, X_u \rangle}_{2\langle X_u, X_v \rangle u' v' + \underbrace{\langle X_v, X_v \rangle}_{\text{fundamental form}}} (u')^2 + \end{aligned}$$

~~first fundamental form~~  
 $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$

(如果  $X_u, X_v$  是平行的  $\rightarrow E=G=0$ ,  $F=0$ )



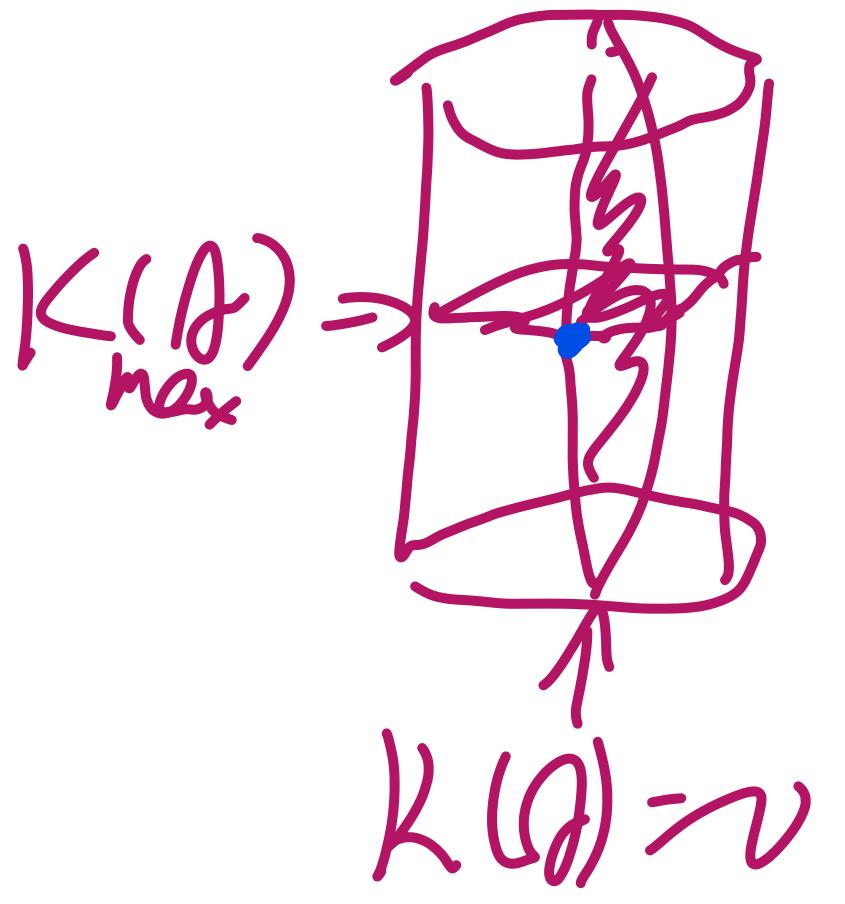
$\delta \sim 1$

( $\text{gap}^2$ )  
curvature

$K(\delta) = f(K_1, K_2, \delta)$

enforce  $\nabla$  equation

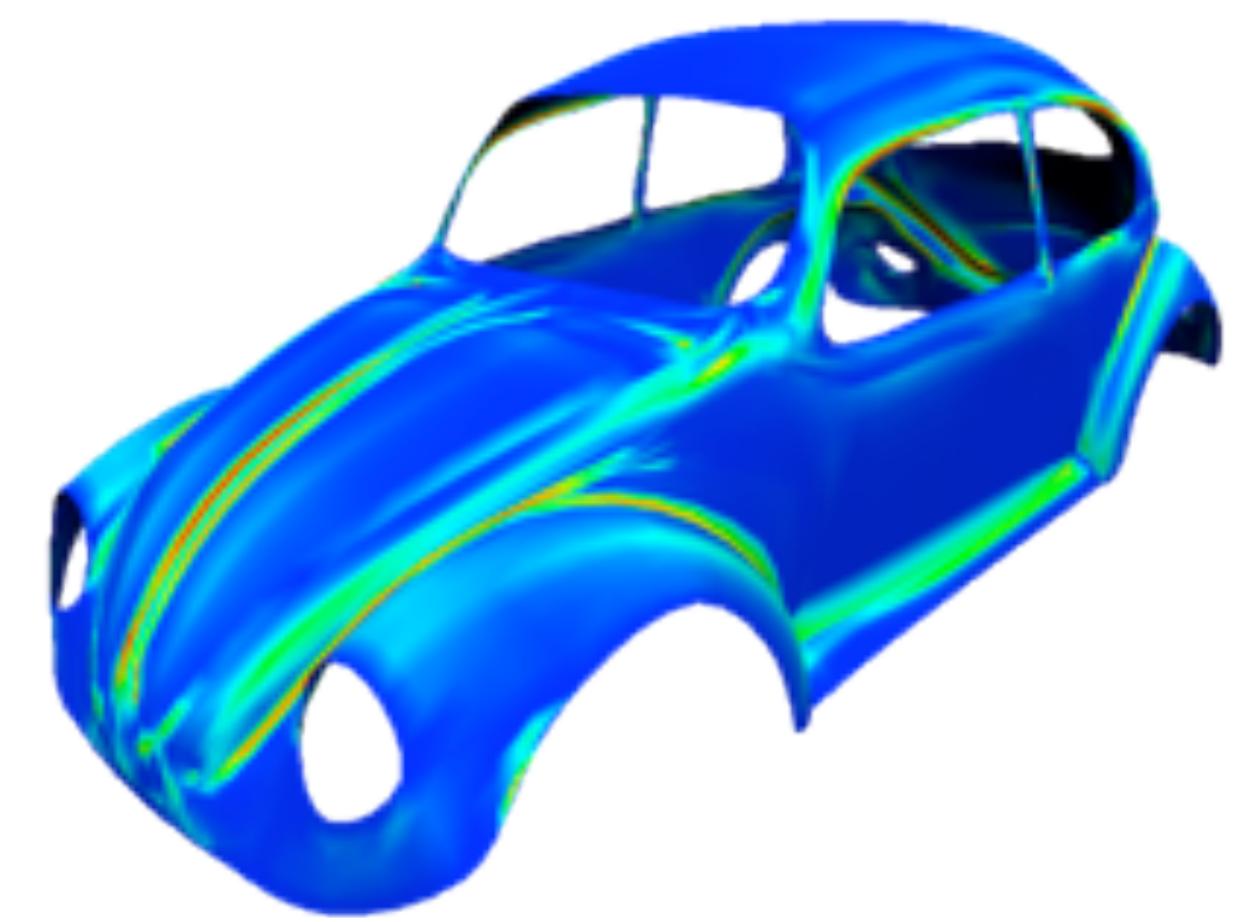
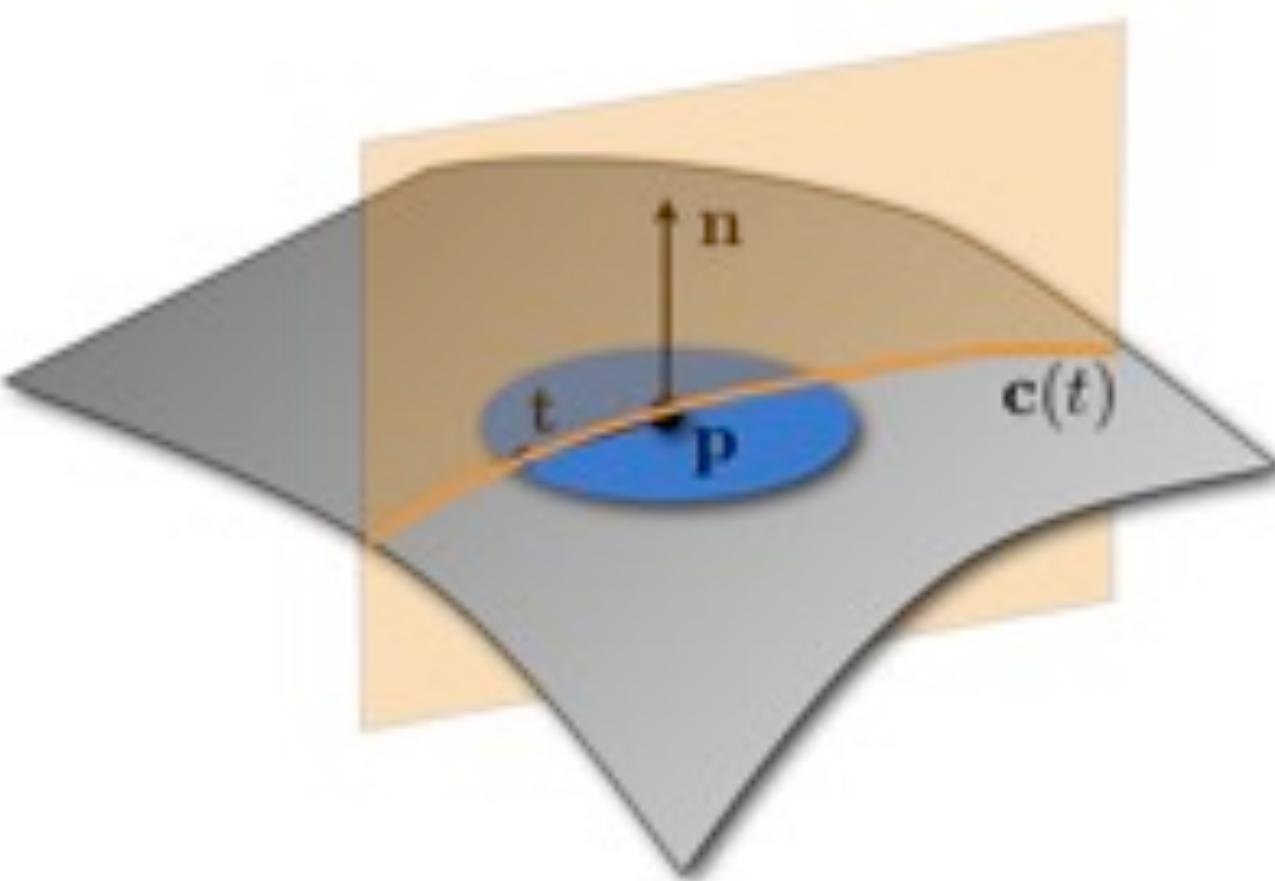
$$\int K d\mu d\nu = f(x)$$







# Differential Geometry



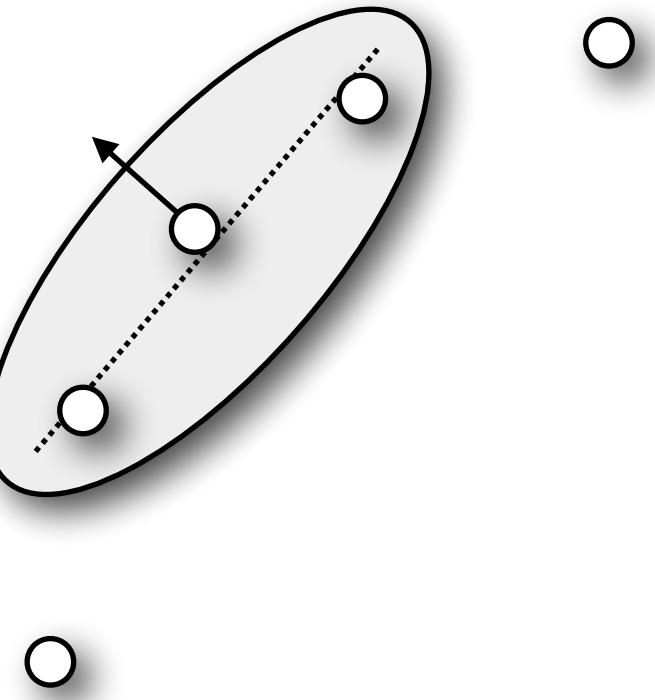
# Last Time



- Explicit Reconstruction
  - Fast Marching Method
- Implicit Reconstruction
  - SDF from point clouds
  - (SDF from range scans)
  - Fast Marching Cubes

# Normal Estimation

- Find normal  $\mathbf{n}_i$  for each sample point  $\mathbf{p}_i$ 
  1. Examine local neighborhood for each point
    - Set of  $k$  nearest neighbors
  2. Compute best approximating tangent plane
    - Covariance analysis
  3. Determine normal orientation
    - MST propagation



# Implicit Reconstruction

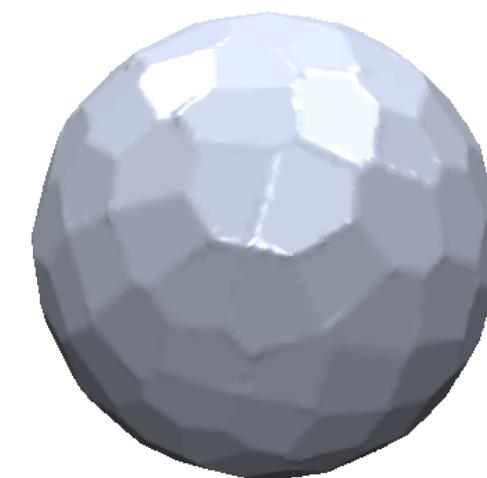
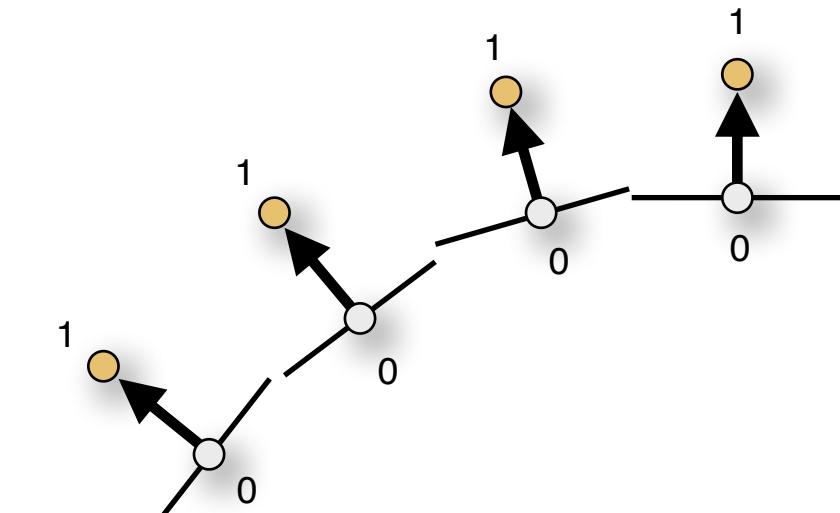
- Scattered data interpolation problem

- On-surface constraints  $\text{dist}(\mathbf{p}_i) = 0$
- Off-surface constraints  $\text{dist}(\mathbf{p}_i + \mathbf{n}_i) = 1$

- Radial basis functions (RBFs)

$$\text{dist}(\mathbf{x}) = \sum_i w_i \cdot \varphi(\|\mathbf{x} - \mathbf{c}_i\|)$$

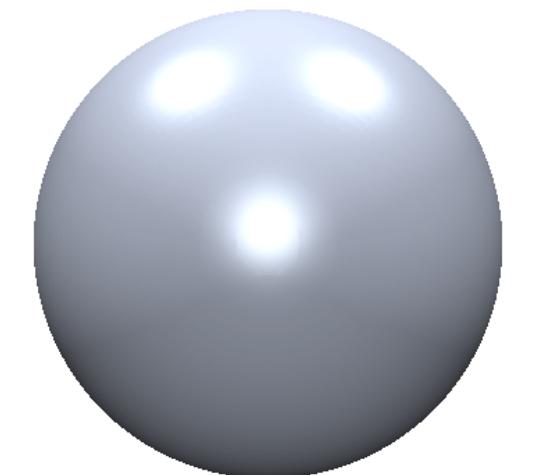
- Solve symmetric linear system for weights  $w_i$



Hoppe '92



Compact RBF Wendland  $C^2$

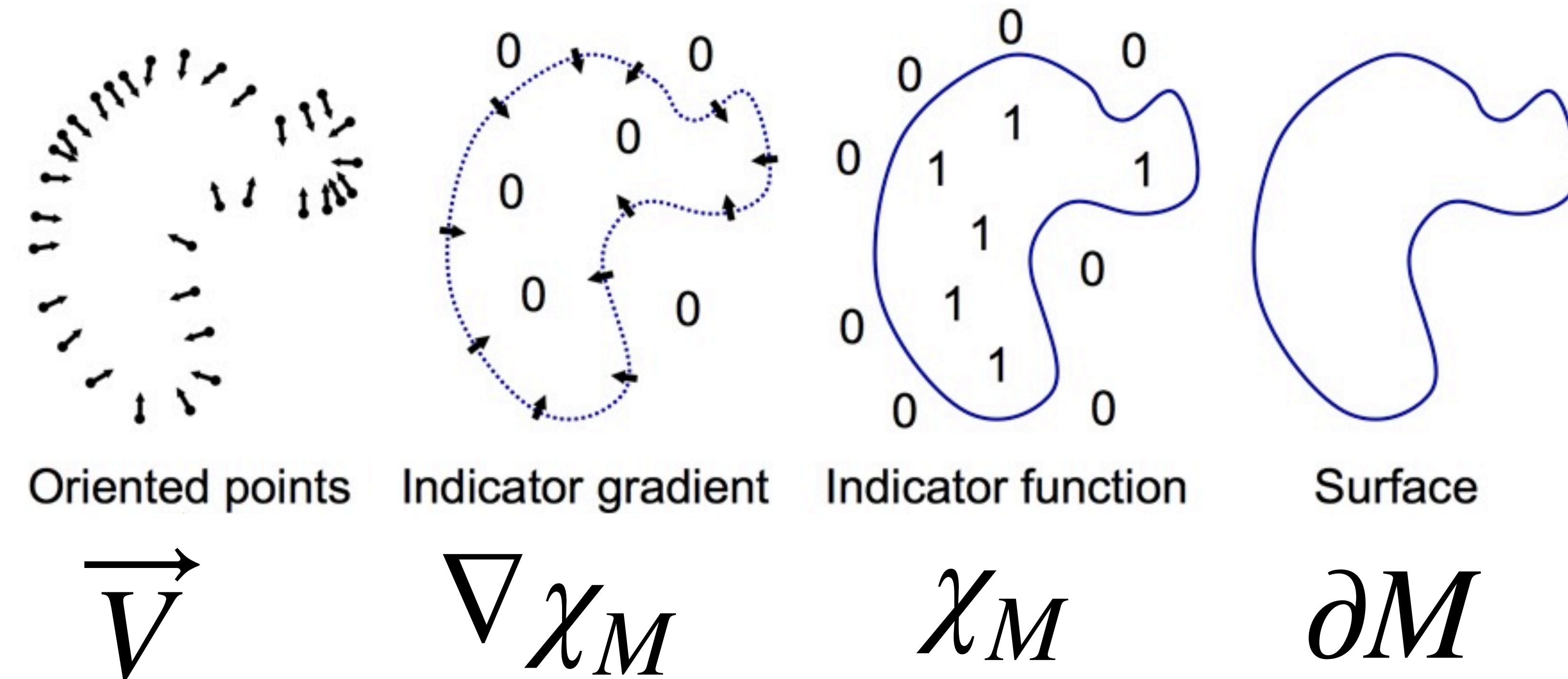


Global RBF Triharmonic

# Poisson Surface Reconstruction



- relationship between normal field and gradient of indicator function



# Outline



- **Differential Geometry**
- Discrete Differential Geometry
- Mesh Quality Measures

# Motivation



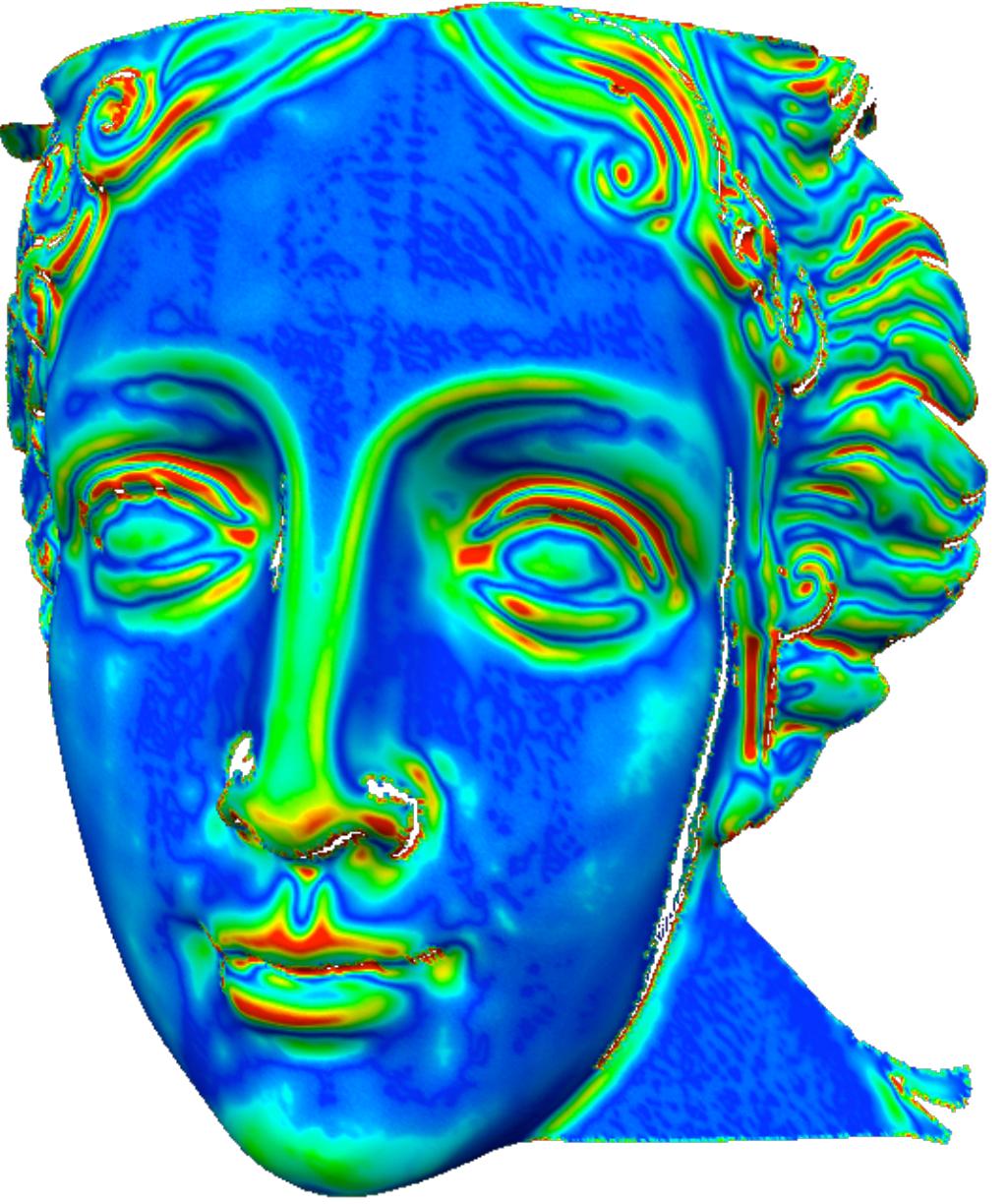
# Motivation



- We need differential geometry to compute

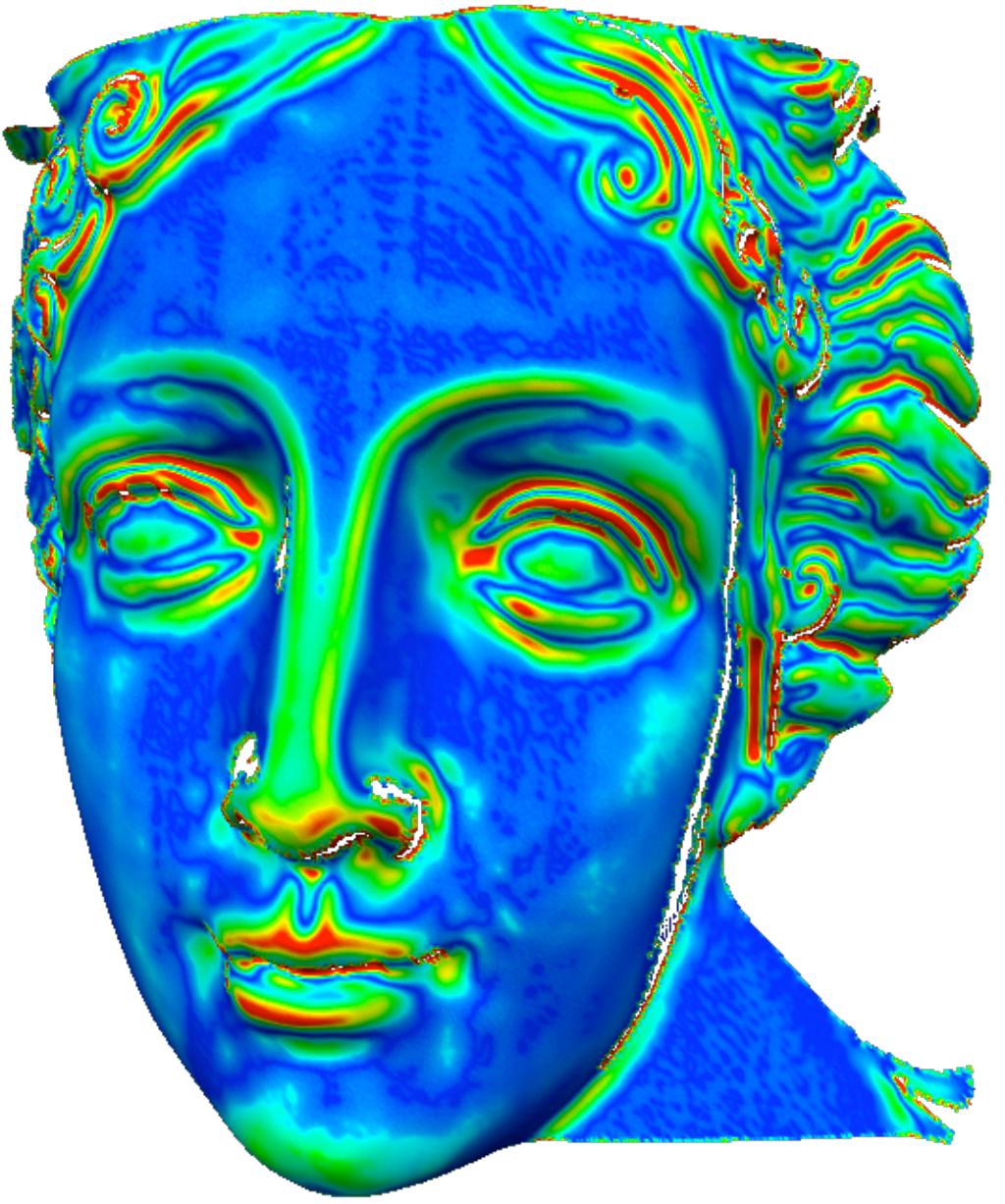
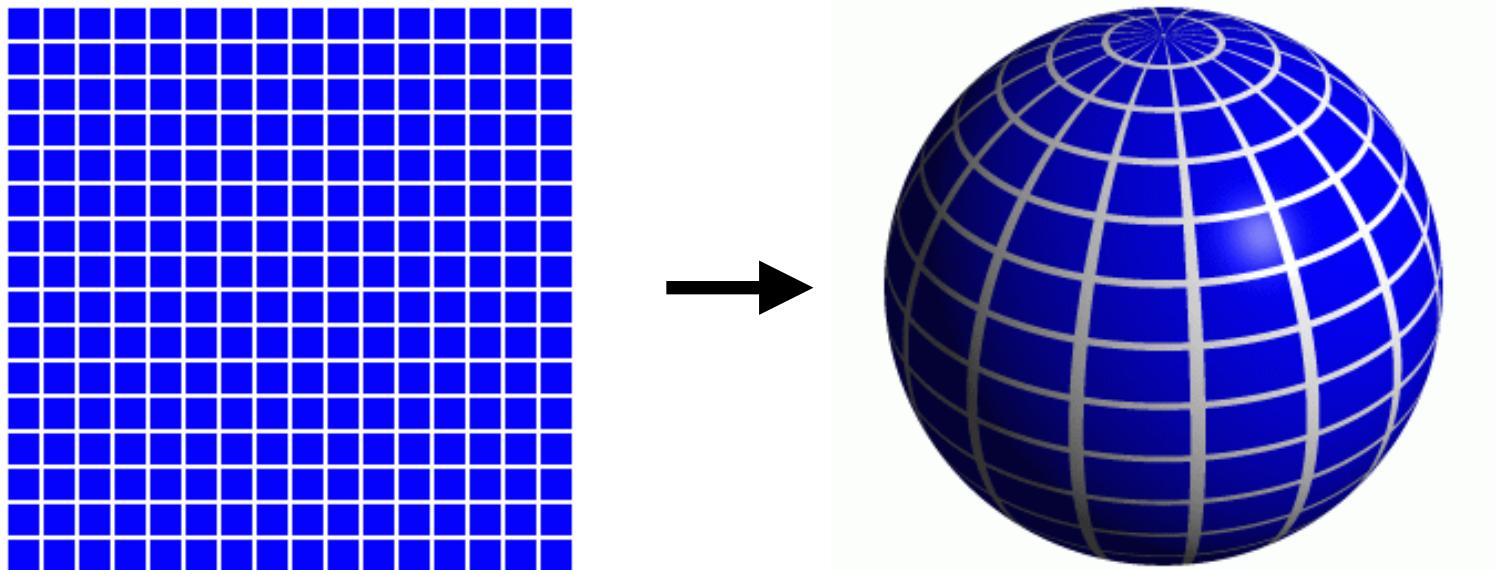
# Motivation

- We need differential geometry to compute
  - surface curvature



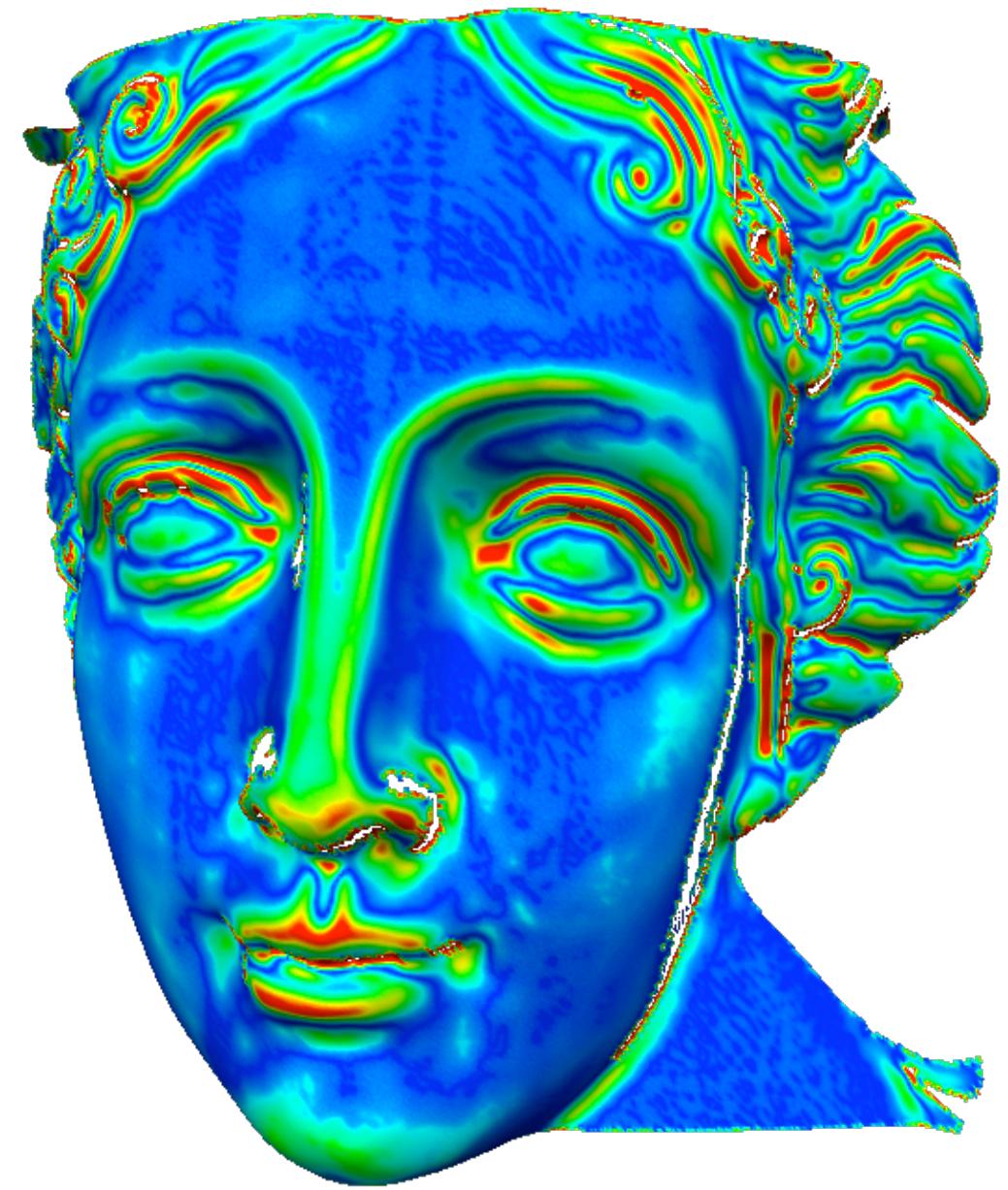
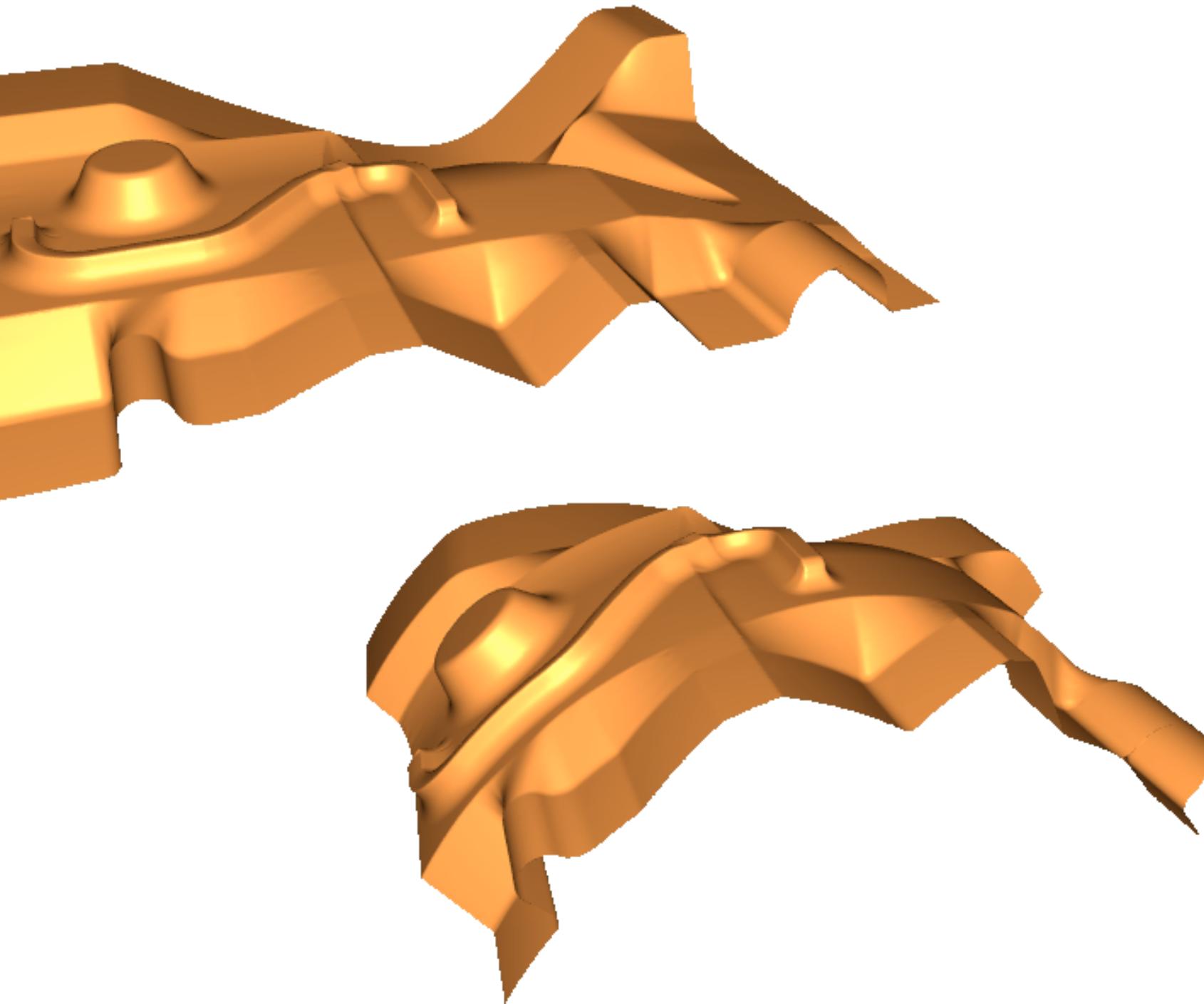
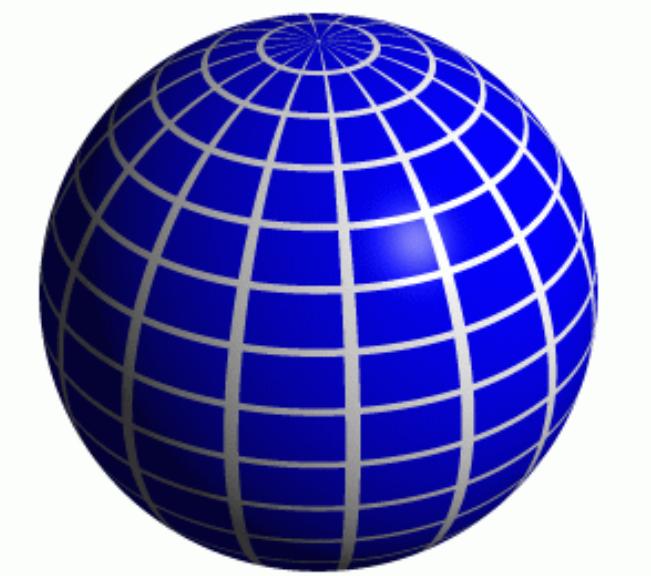
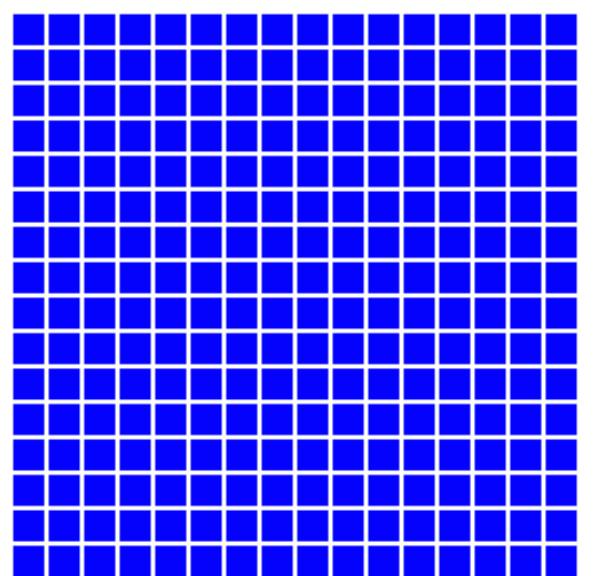
# Motivation

- We need differential geometry to compute
  - surface curvature
  - parameterization distortion



# Motivation

- We need differential geometry to compute
  - surface curvature
  - parameterization distortion
  - deformation energies



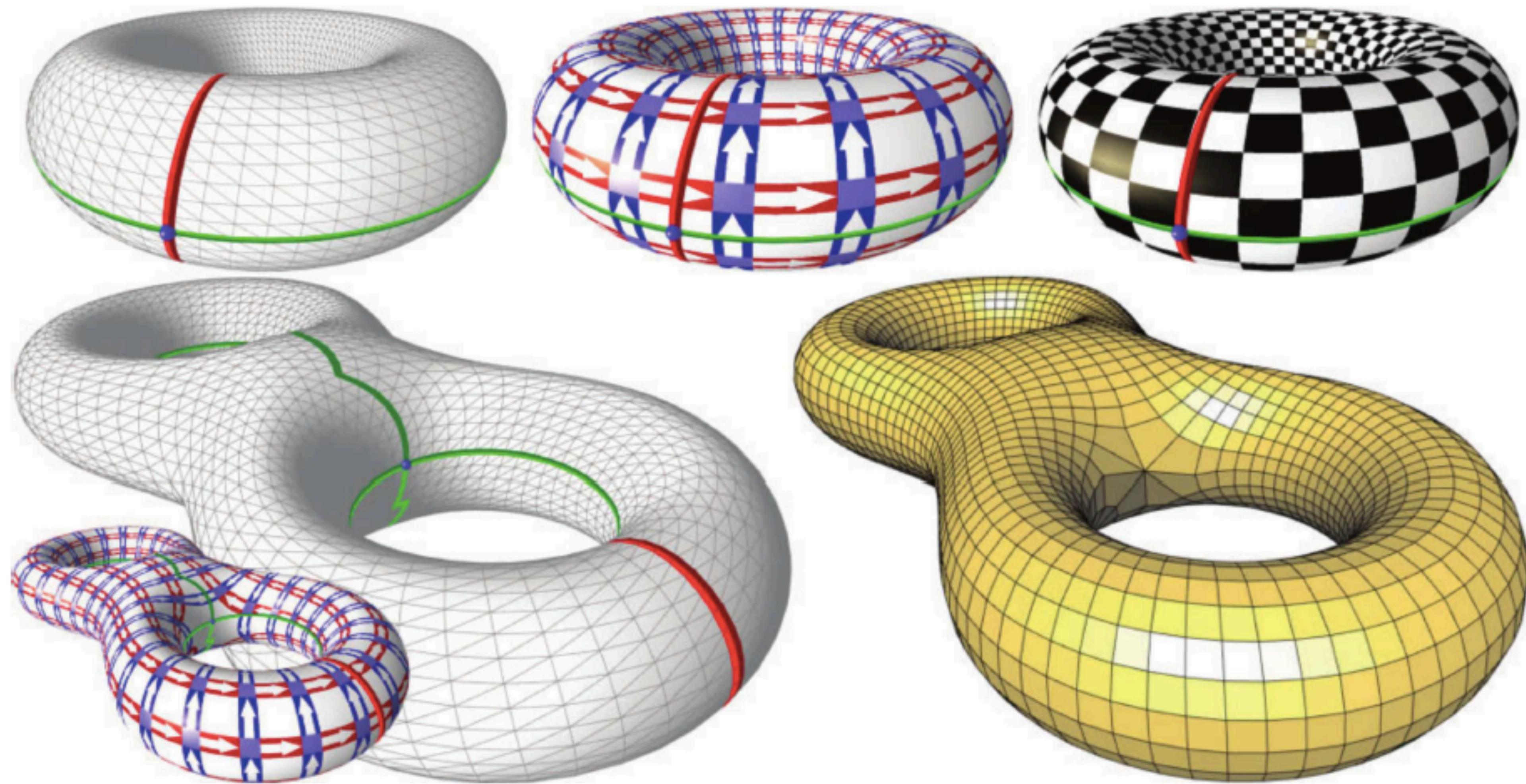
# Texture Mapping



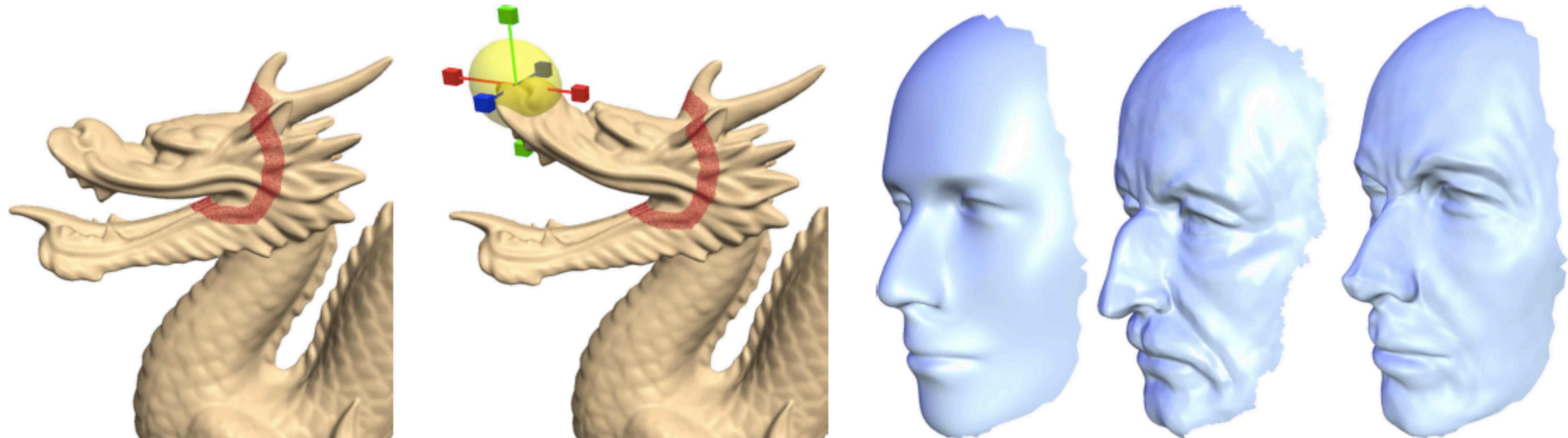
# Texture Mapping



# Parameterization + Meshing



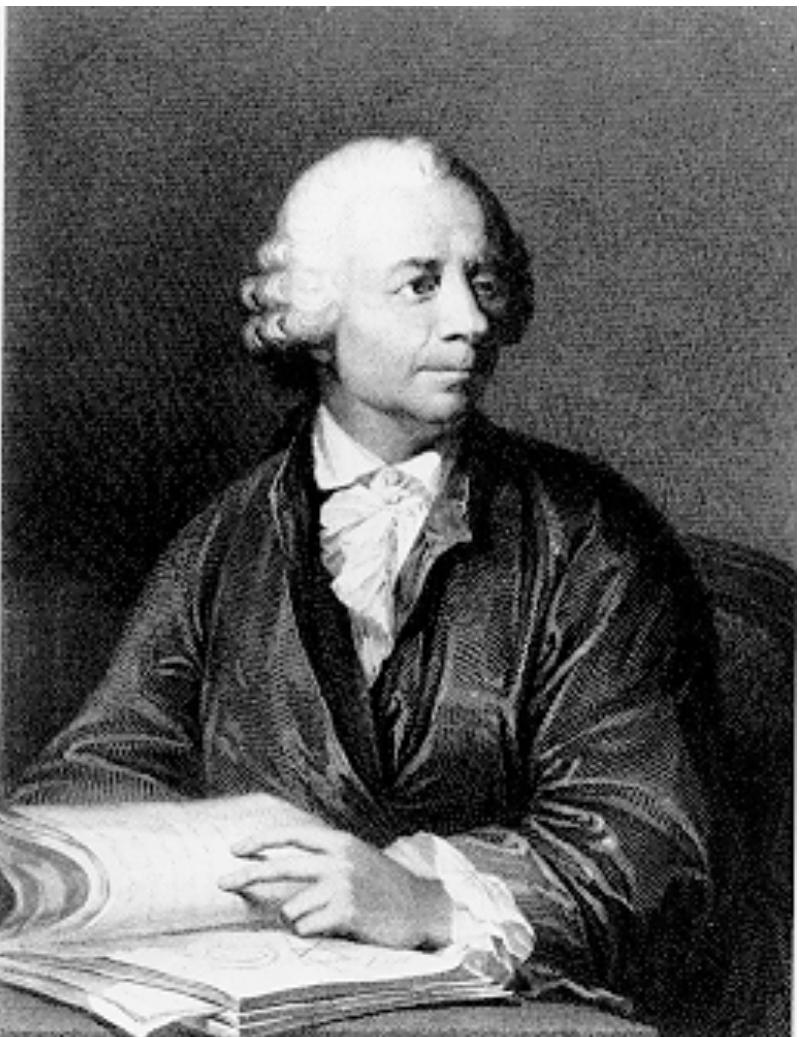
# Surface Deformation



# Differential Geometry



- Manfredo P. do Carmo: *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976
- Andrew Pressley: *Elementary Differential Geometry*, Springer, 2010



Leonard Euler (1707 - 1783)



Carl Friedrich Gauss (1777 - 1855)

# Overview — Curves



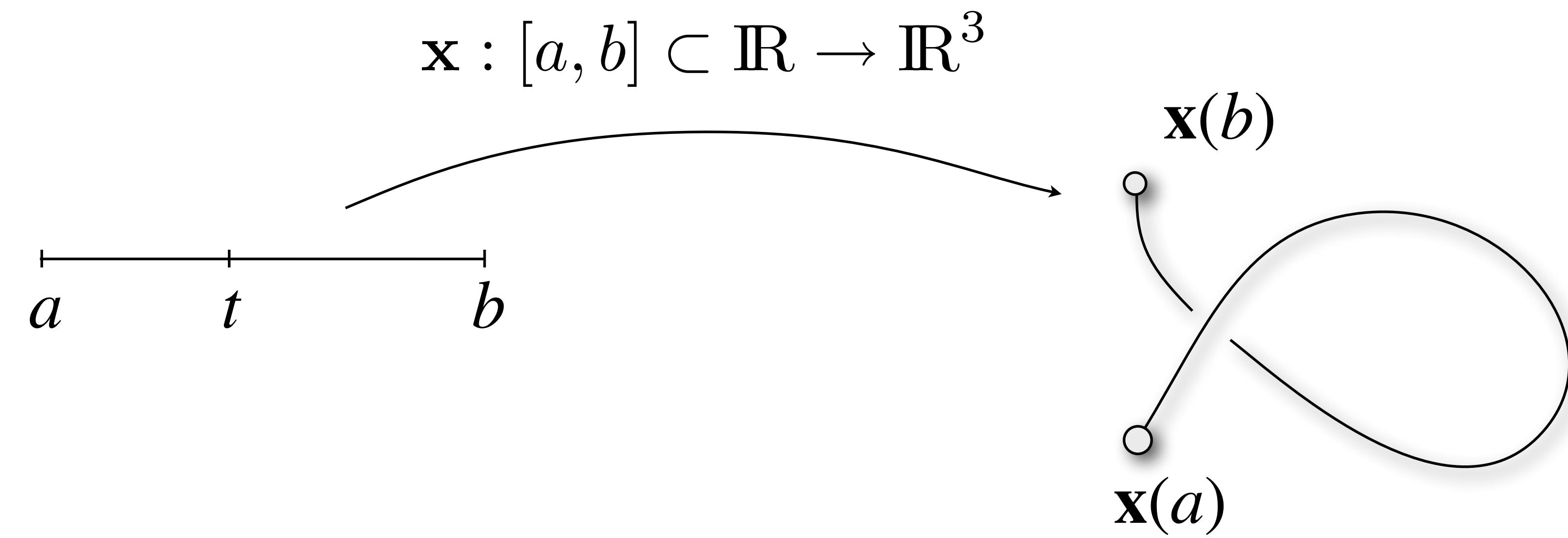
- **Parameterized** curves + **trace** of a curve
- **Ambiguity** due to parameterization
- Instantaneous **velocity** and **acceleration**
- constant **speed** versus constant **velocity**
- measuring curve length
- constant speed: velocity orthogonal to acceleration
- no self-intersection; regular parameterization

# Overview (cont.)



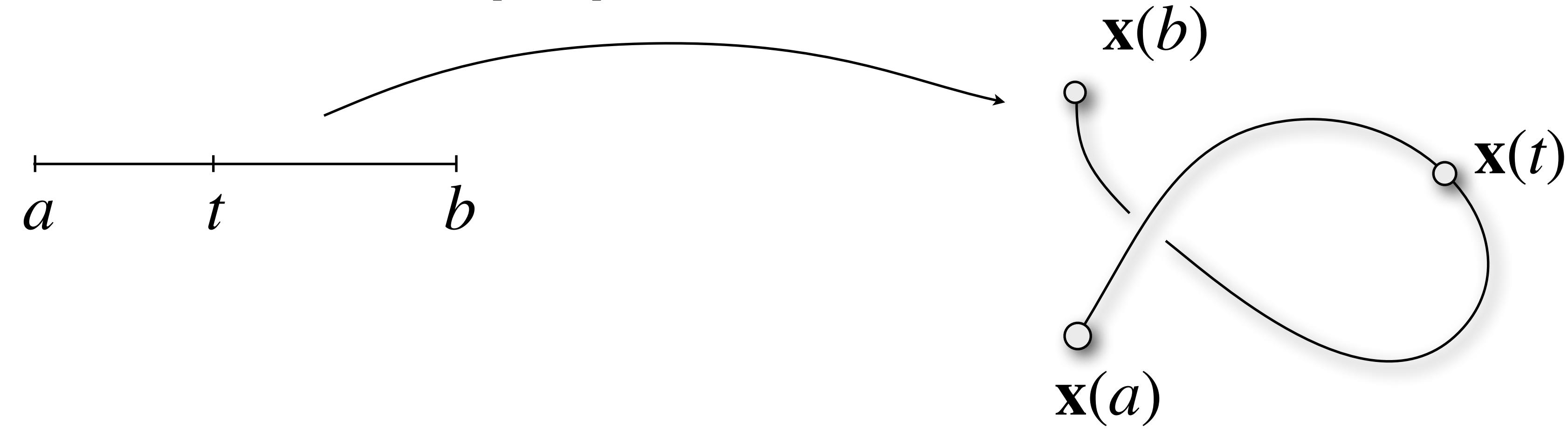
- radius of curvature
- Frenet frame; osculating plane
- tangent, normal, binormal
- — — — end curves

# Parametric Curves



# Parametric Curves

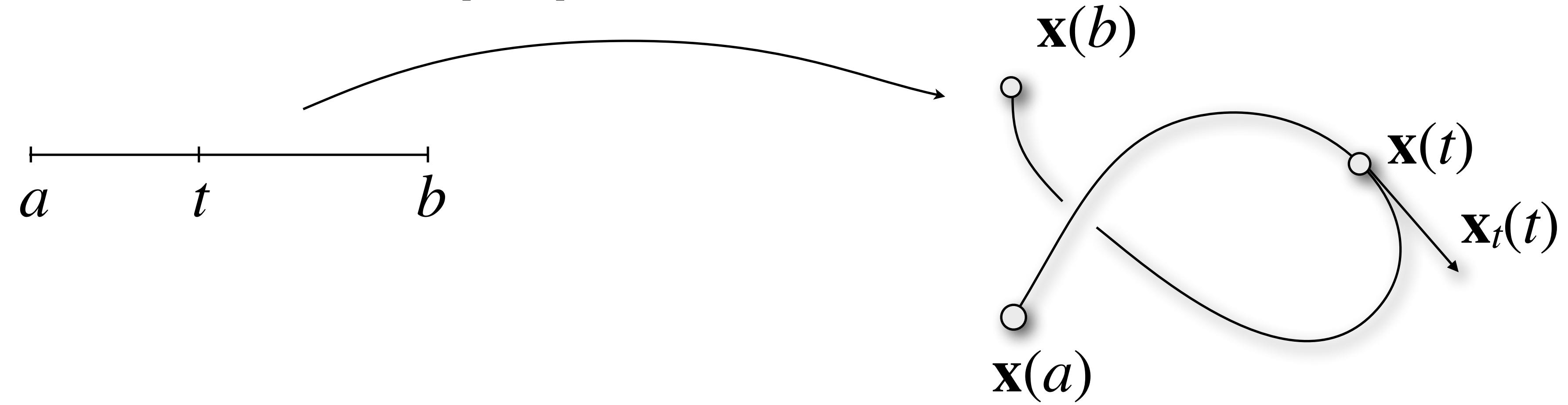
$$\mathbf{x} : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^3$$



$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

# Parametric Curves

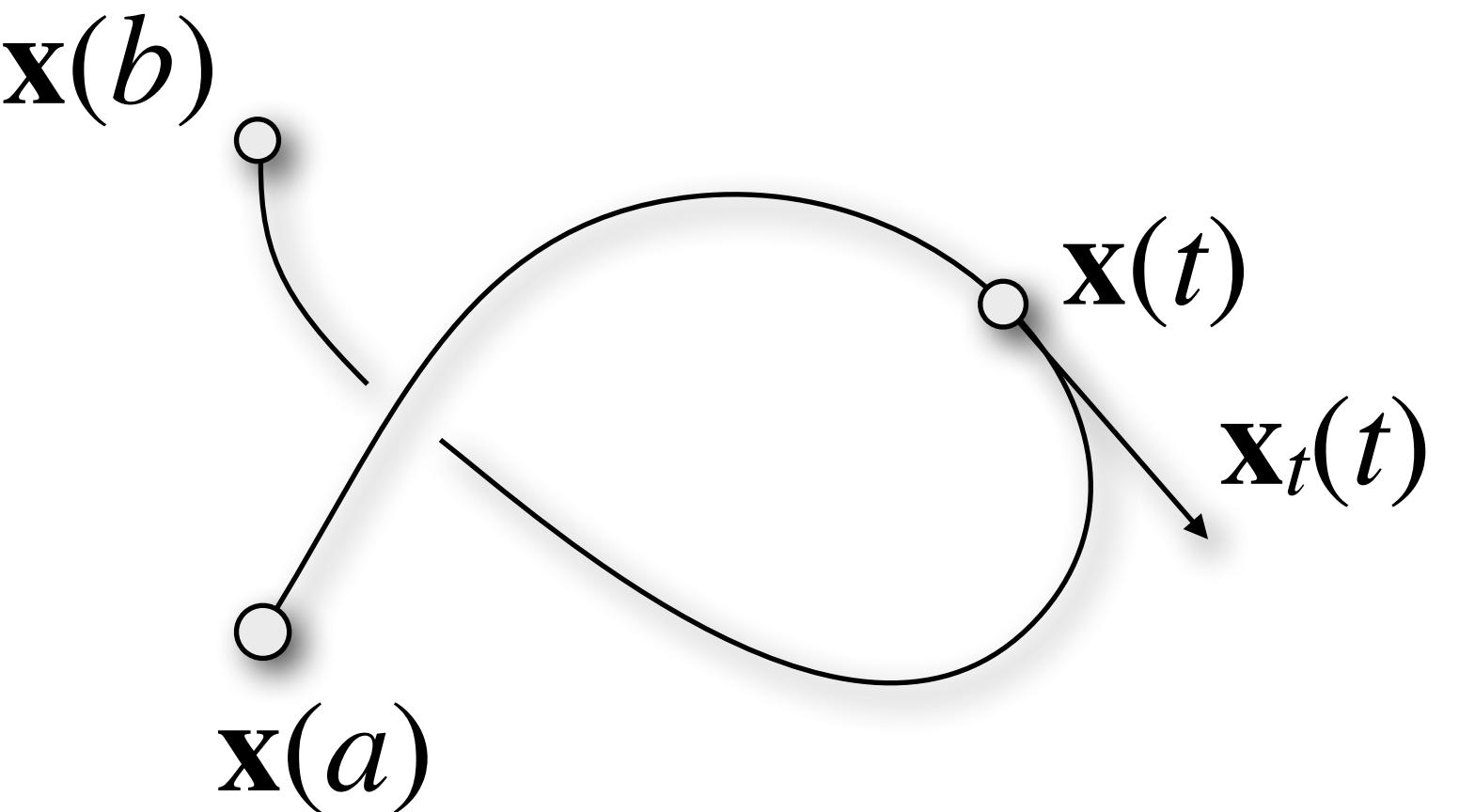
$$\mathbf{x} : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^3$$



$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \quad \mathbf{x}_t(t) := \frac{d\mathbf{x}(t)}{dt} = \begin{pmatrix} dx(t)/dt \\ dy(t)/dt \\ dz(t)/dt \end{pmatrix}$$

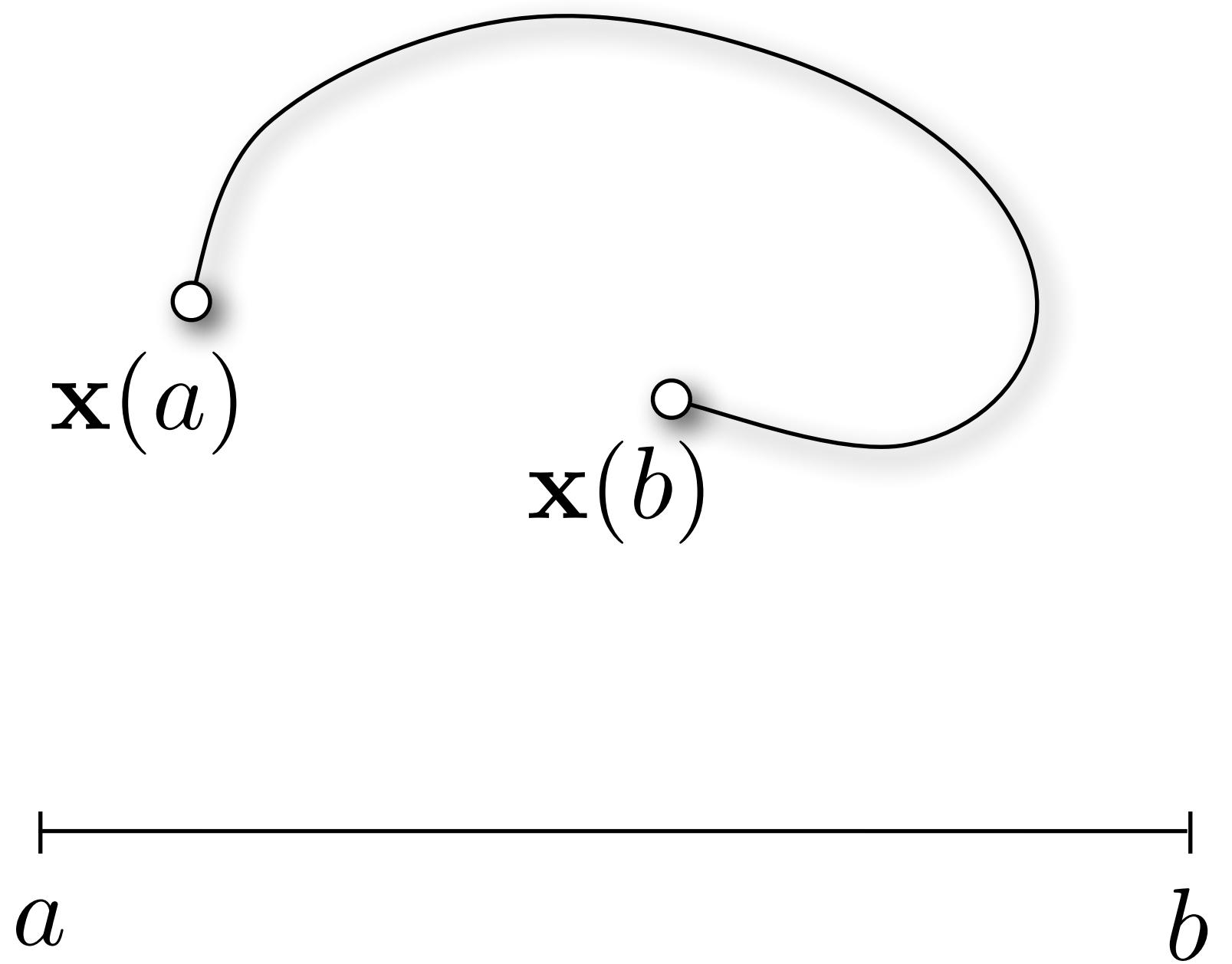
# Parametric Curves

- A parametric curve  $\mathbf{x}(t)$  is
  - *simple*  $\mathbf{x}(t)$  is injective (no self-intersections)
  - *differentiable*  $\mathbf{x}_t(t)$  is defined for all  $t \in [a, b]$
  - *regular*  $\mathbf{x}_t(t) \neq 0$  for all  $t \in [a, b]$



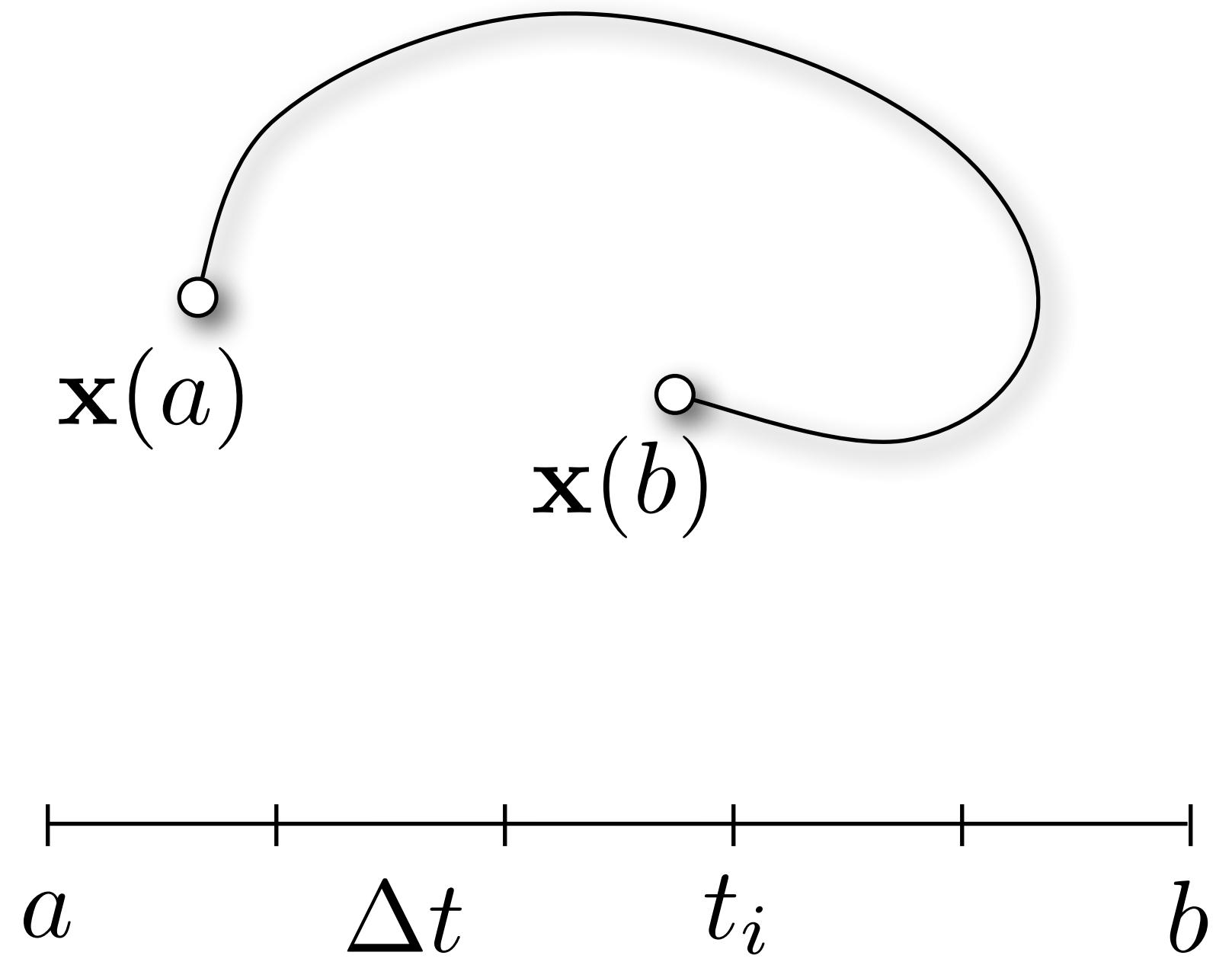
# Length of a Curve

- Let  $t_i = a + i\Delta t$  and  $\mathbf{x}_i = \mathbf{x}(t_i)$



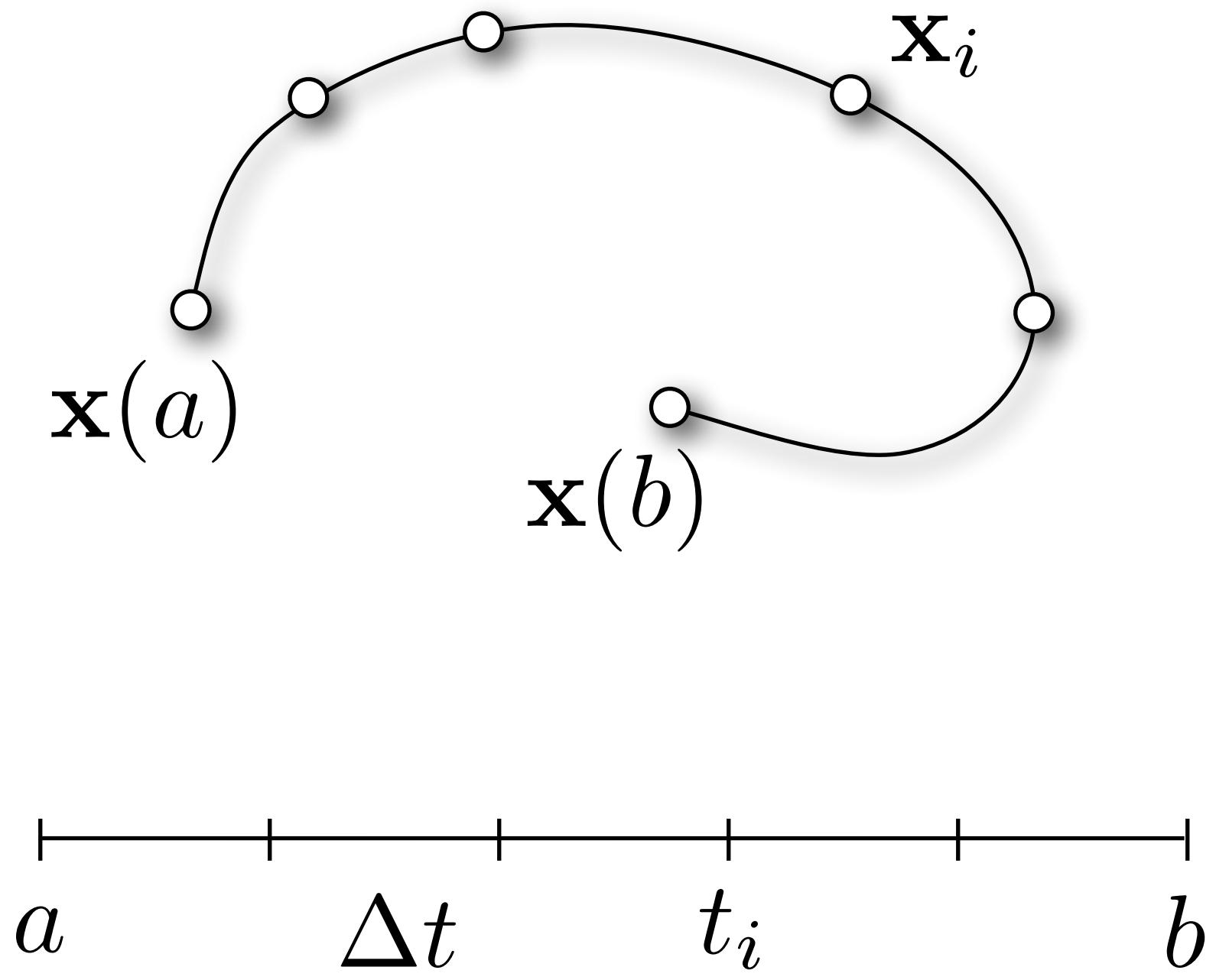
# Length of a Curve

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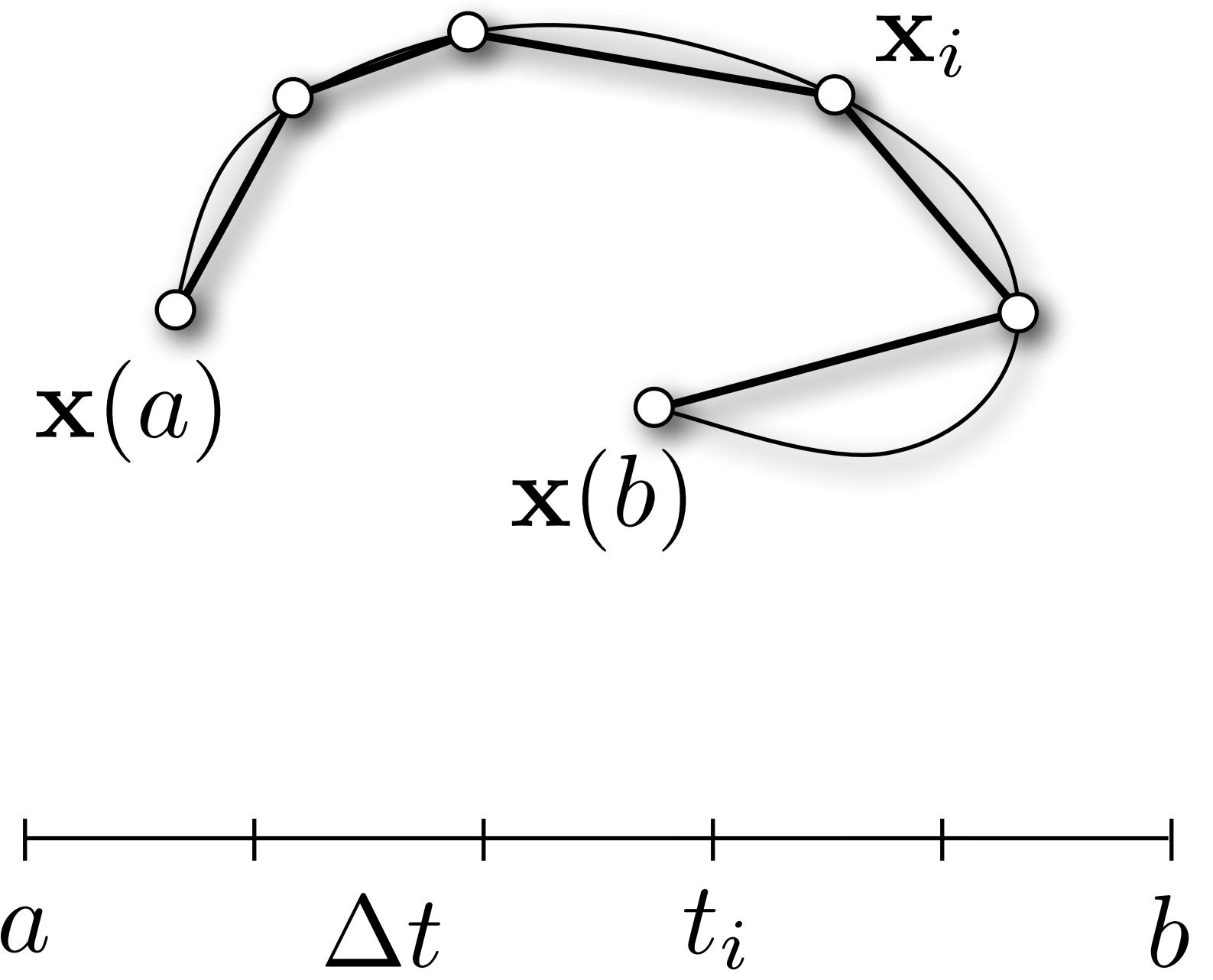
# Length of a Curve

- Let  $t_i = a + i\Delta t$  and  $\mathbf{x}_i = \mathbf{x}(t_i)$



# Length of a Curve

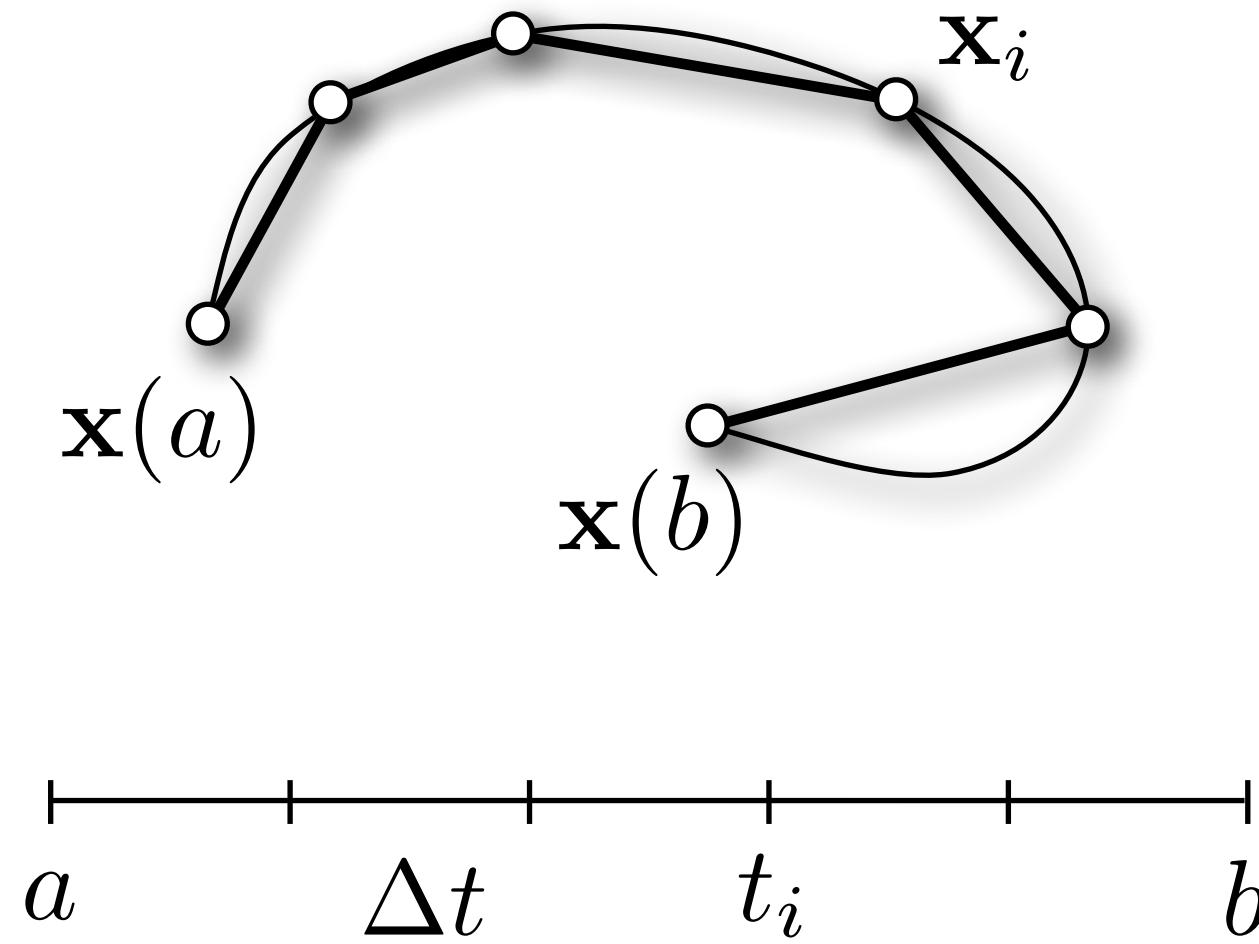
- Let  $t_i = a + i\Delta t$  and  $\mathbf{x}_i = \mathbf{x}(t_i)$



# Length of a Curve

- Polyline *chord length*

$$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \left\| \frac{\Delta \mathbf{x}_i}{\Delta t} \right\| \Delta t, \quad \Delta \mathbf{x}_i := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$$



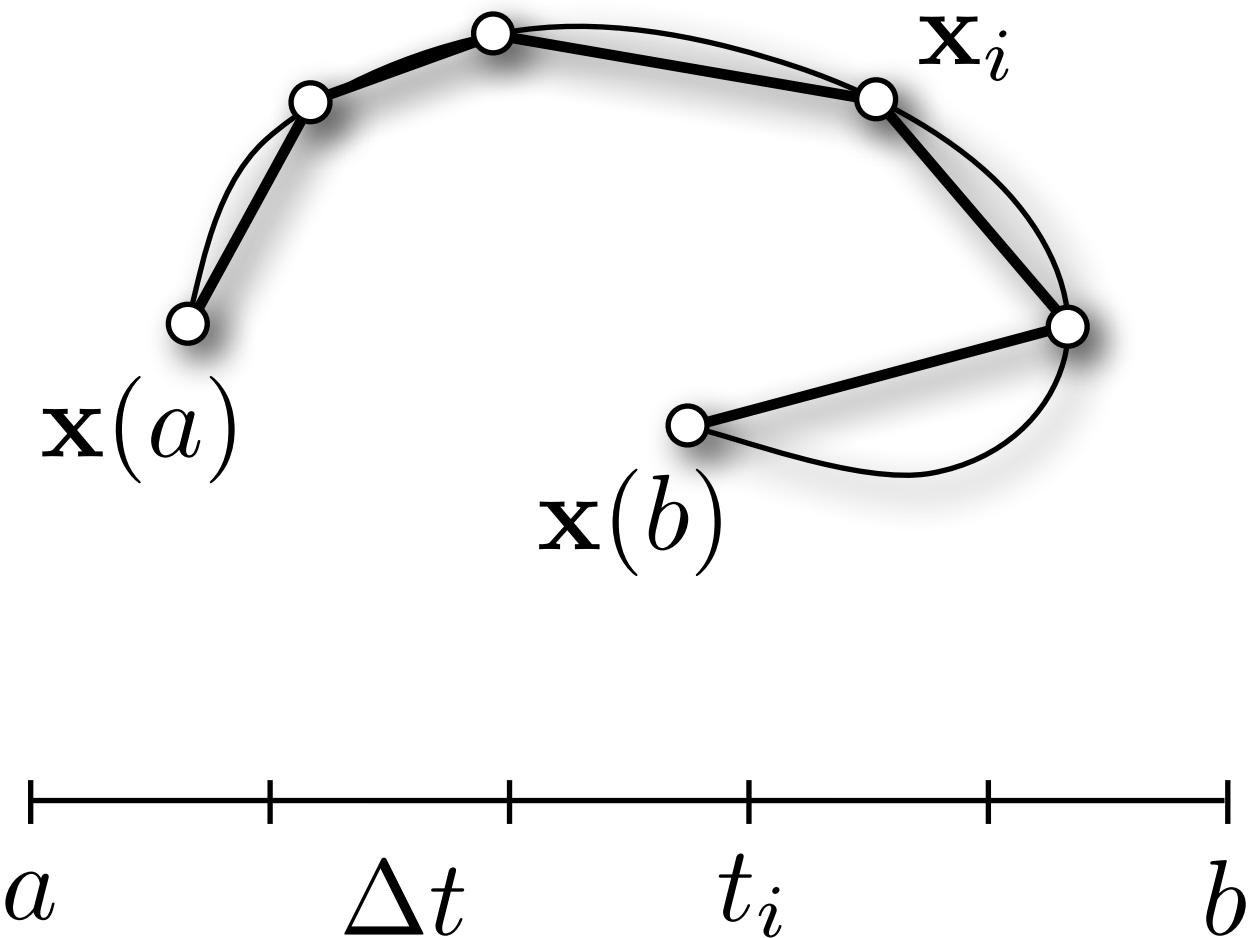
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- Curve *arc length* ( $\Delta t \rightarrow 0$ )

$$s = s(t) = \int_a^t \|\mathbf{x}_t\| dt$$



# Re-Parameterization



# Re-Parameterization



- Mapping of parameter domain

$$u : [a, b] \rightarrow [c, d]$$

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$$u : [a, b] \rightarrow [c, d]$$

- Re-parameterization w.r.t.  $u(t)$

$$[c, d] \rightarrow \mathbb{R}^3, \quad t \mapsto \mathbf{x}(u(t))$$

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$$u : [a, b] \rightarrow [c, d]$$

- Re-parameterization w.r.t.  $u(t)$

$$[c, d] \rightarrow \mathbb{R}^3, \quad t \mapsto \mathbf{x}(u(t))$$

- Derivative (chain rule)

$$\frac{d\mathbf{x}(u(t))}{dt} = \frac{d\mathbf{x}}{du} \frac{du}{dt} = \mathbf{x}_u(u(t)) u_t(t)$$

# Re-Parameterization



- Example

$$\mathbf{f} : \left[0, \frac{1}{2}\right] \rightarrow \mathbb{R}^2 , \quad t \mapsto (4t, 2t)$$

$$\phi : \left[0, \frac{1}{2}\right] \rightarrow [0, 1] , \quad t \mapsto 2t$$

$$\mathbf{g} : [0, 1] \rightarrow \mathbb{R}^2 , \quad t \mapsto (2t, t)$$

# Re-Parameterization



- Example

$$\mathbf{f} : \left[0, \frac{1}{2}\right] \rightarrow \mathbb{R}^2 , \quad t \mapsto (4t, 2t)$$

$$\phi : \left[0, \frac{1}{2}\right] \rightarrow [0, 1] , \quad t \mapsto 2t$$

$$\mathbf{g} : [0, 1] \rightarrow \mathbb{R}^2 , \quad t \mapsto (2t, t)$$

$$\Rightarrow \mathbf{g}(\phi(t)) = \mathbf{f}(t)$$

# Arc Length Parameterization



# Arc Length Parameterization



- Mapping of parameter domain:

$$t \mapsto s(t) = \int_a^t \|\mathbf{x}_t\| dt$$

# Arc Length Parameterization



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- Parameter  $s$  for  $\mathbf{x}(s)$  equals length from  $\mathbf{x}(a)$  to  $\mathbf{x}(s)$

$$\mathbf{x}(s) = \mathbf{x}(s(t)) \quad ds = \|\mathbf{x}_t\| dt$$

# Arc Length Parameterization



- Mapping of parameter domain:

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- Parameter  $s$  for  $\mathbf{x}(s)$  equals length from  $\mathbf{x}(a)$  to  $\mathbf{x}(s)$

$$\mathbf{x}(s) = \mathbf{x}(s(t)) \quad ds = \|\mathbf{x}_t\| dt$$

- Special properties of resulting curve

$$\|\mathbf{x}_s(s)\| = 1, \quad \mathbf{x}_s(s) \cdot \mathbf{x}_{ss}(s) = 0$$

# The Frenet Frame



- Taylor expansion

$$\mathbf{x}(t + h) = \mathbf{x}(t) + \mathbf{x}_t(t) h + \frac{1}{2}\mathbf{x}_{tt}(t) h^2 + \frac{1}{6}\mathbf{x}_{ttt}(t) h^3 + \dots$$

- Define local frame  $(\mathbf{t}, \mathbf{n}, \mathbf{b})$  (*Frenet frame*)

$$\mathbf{t} = \frac{\mathbf{x}_t}{\|\mathbf{x}_t\|}$$

tangent

$$\mathbf{n} = \mathbf{b} \times \mathbf{t}$$

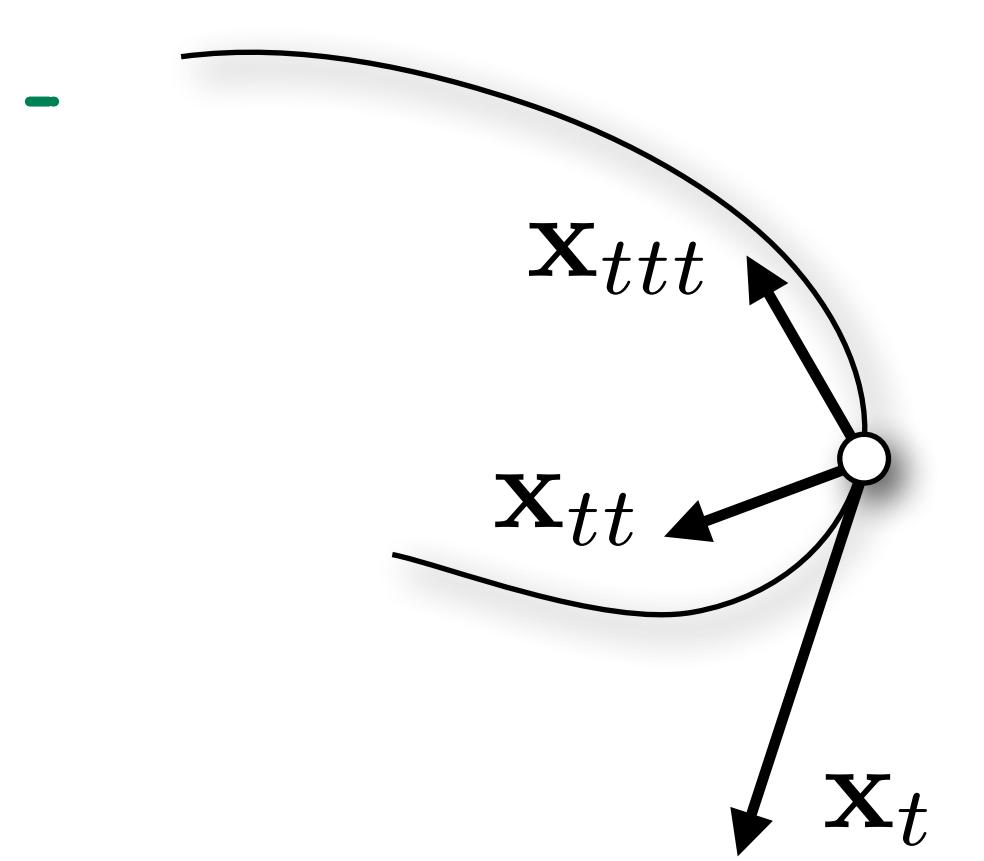
principal normal

$$\mathbf{b} = \frac{\mathbf{x}_t \times \mathbf{x}_{tt}}{\|\mathbf{x}_t \times \mathbf{x}_{tt}\|}$$

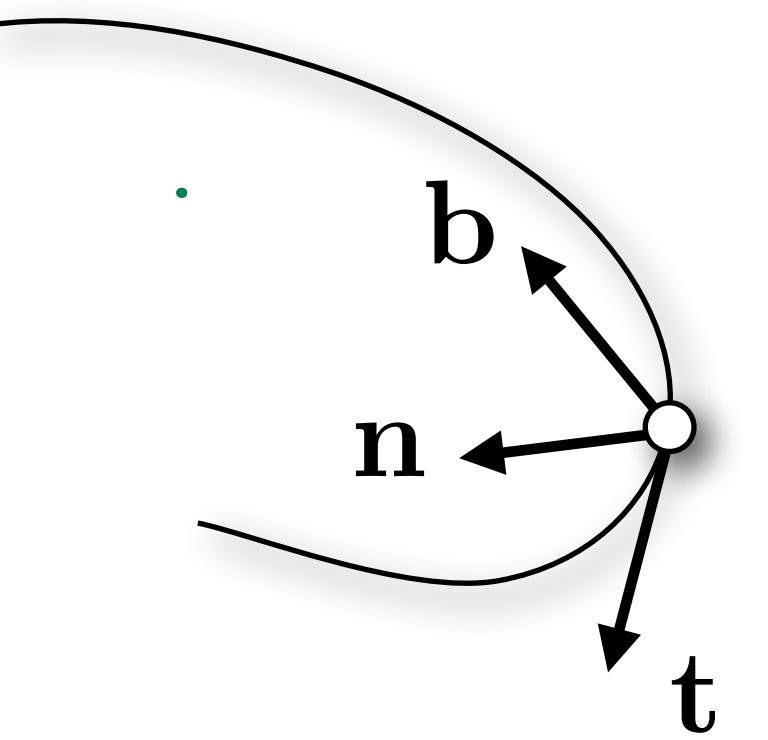
binormal

# The Frenet Frame

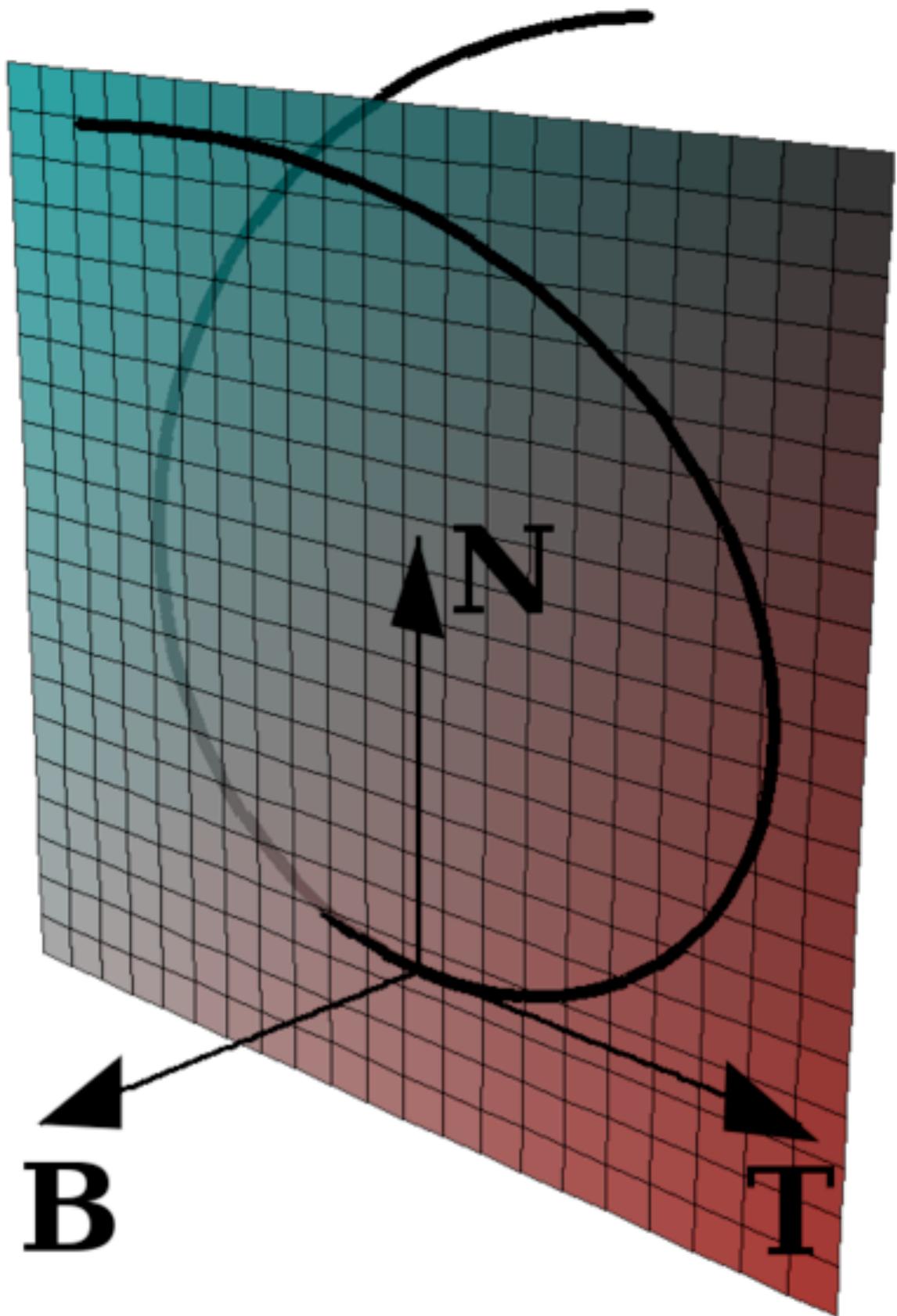
- Orthonormalization of local frame



local affine frame



Frenet frame



# The Frenet Frame



# The Frenet Frame



- *Frenet-Serret*: Derivatives w.r.t. arc length  $s$

$$\mathbf{t}_s = +\kappa \mathbf{n}$$

$$\mathbf{n}_s = -\kappa \mathbf{t} + \tau \mathbf{b}$$

$$\mathbf{b}_s = -\tau \mathbf{n}$$

# The Frenet Frame



- *Frenet-Serret*: Derivatives w.r.t. arc length  $s$

$$\begin{aligned}\mathbf{t}_s &= +\kappa \mathbf{n} \\ \mathbf{n}_s &= -\kappa \mathbf{t} \quad +\tau \mathbf{b} \\ \mathbf{b}_s &= -\tau \mathbf{n}\end{aligned}$$

- Curvature (*deviation from straight line*)

$$\kappa = \|\mathbf{x}_{ss}\|$$

# The Frenet Frame



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$$\begin{aligned}\mathbf{t}_s &= +\kappa \mathbf{n} \\ \mathbf{n}_s &= -\kappa \mathbf{t} \quad +\tau \mathbf{b} \\ \mathbf{b}_s &= -\tau \mathbf{n}\end{aligned}$$

- Curvature (*deviation from straight line*)

$$\kappa = \|\mathbf{x}_{ss}\|$$

- Torsion (*deviation from planarity*)

$$\tau = \frac{1}{\kappa^2} \det([\mathbf{x}_s, \mathbf{x}_{ss}, \mathbf{x}_{sss}])$$

# Curvature and Torsion



- Planes defined by  $x$  and two vectors:
  - *osculating plane*: vectors  $t$  and  $n$
  - *normal plane*: vectors  $n$  and  $b$
  - *rectifying plane*: vectors  $t$  and  $b$
- Osculating circle
  - second order contact with curve
  - center  $c = x + (1/\kappa)n$
  - radius  $1/\kappa$

# Curvature and Torsion



# Curvature and Torsion



- Curvature: Deviation from straight line

# Curvature and Torsion



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- Torsion: Deviation from planarity

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- Curvature: Deviation from straight line
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- Independent of parameterization

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- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
- Independent of parameterization
  - intrinsic properties of the curve

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- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
- Independent of parameterization
  - intrinsic properties of the curve
- Euclidean invariants

# Curvature and Torsion



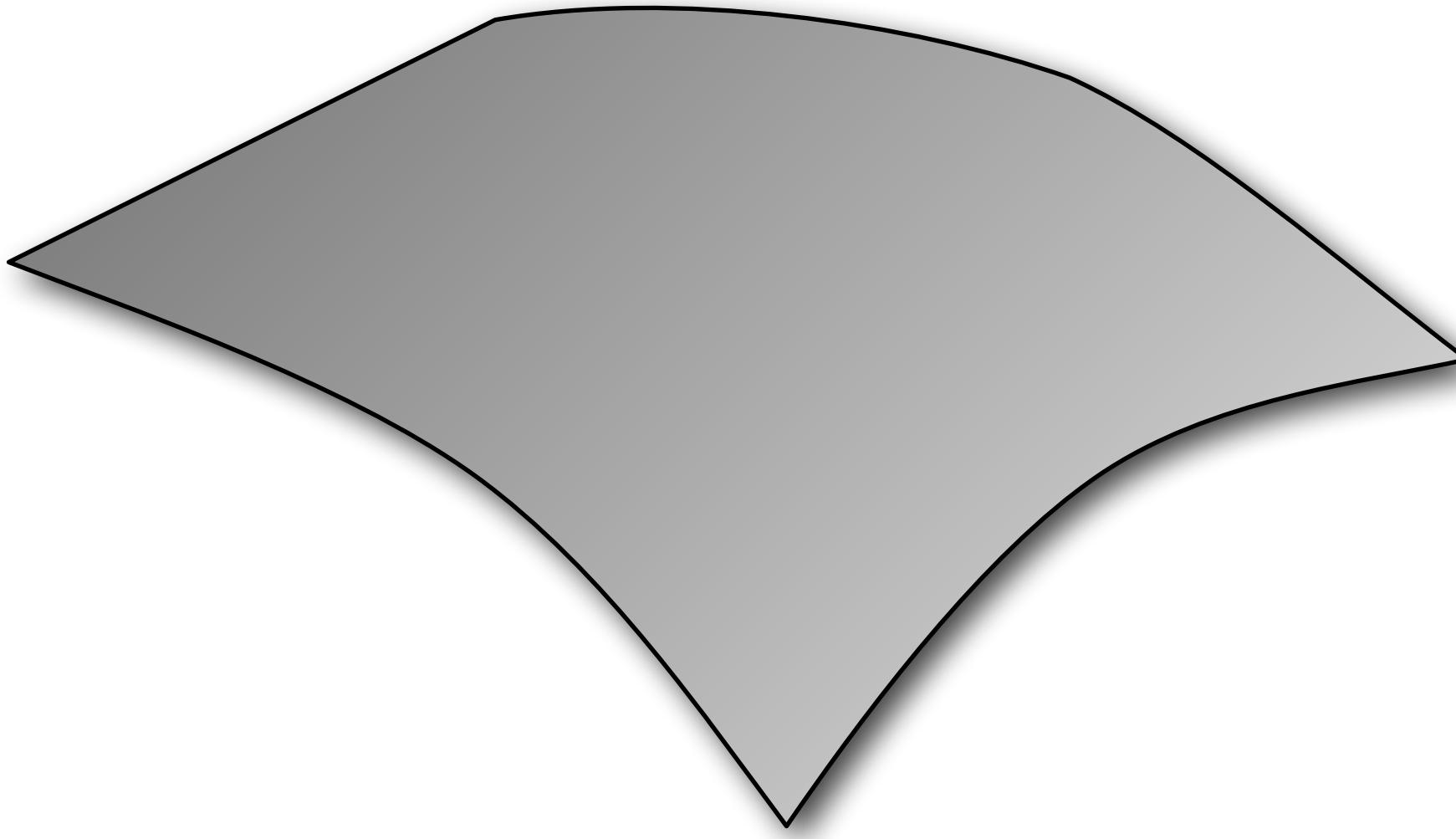
- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
- Independent of parameterization
  - intrinsic properties of the curve
- Euclidean invariants
  - invariant under rigid motion

# Curvature and Torsion



- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
- Independent of parameterization
  - intrinsic properties of the curve
- Euclidean invariants
  - invariant under rigid motion
- Define curve uniquely up to rigid motion

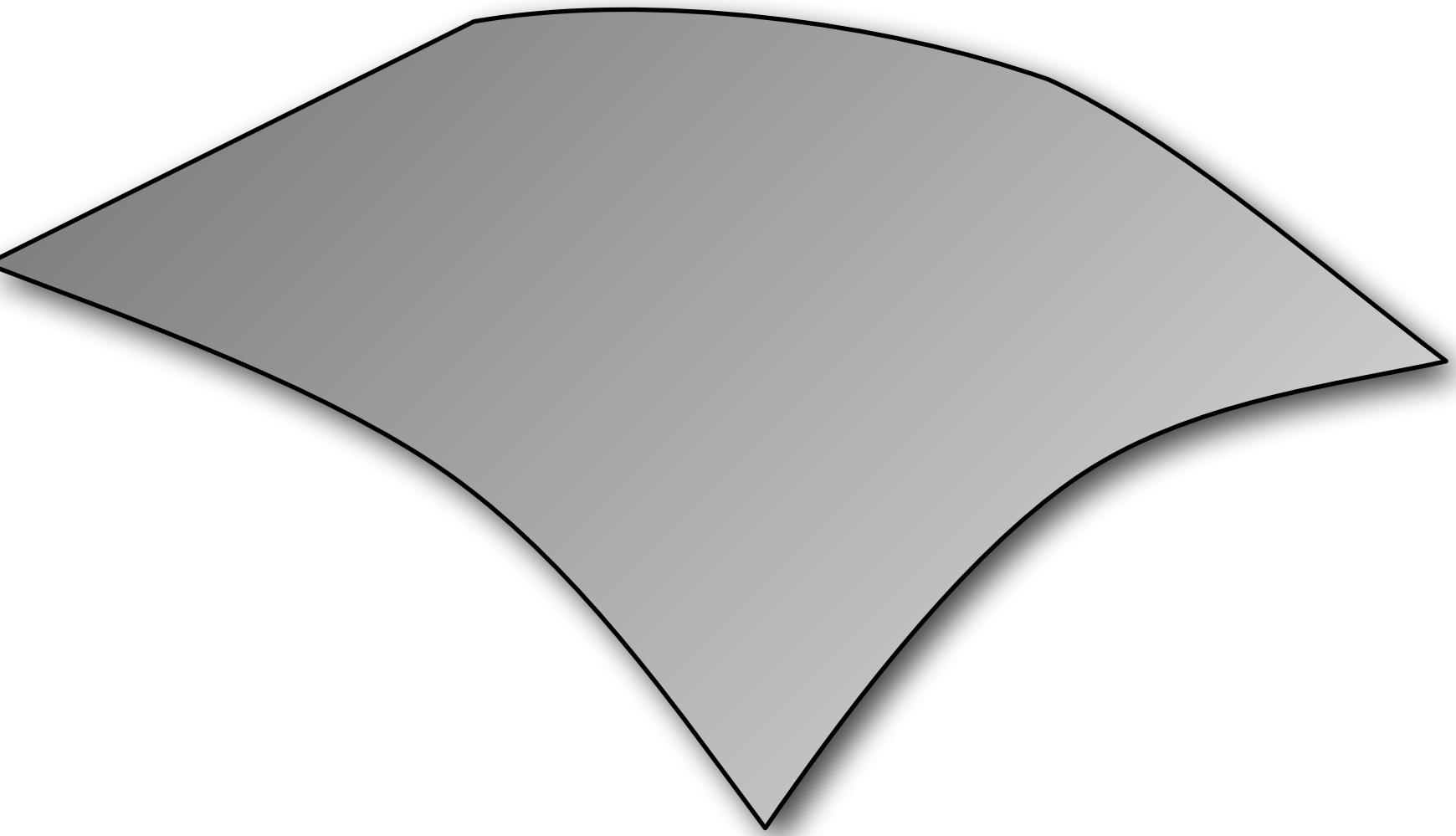
# Parametric Surfaces



# Parametric Surfaces

- Continuous surface

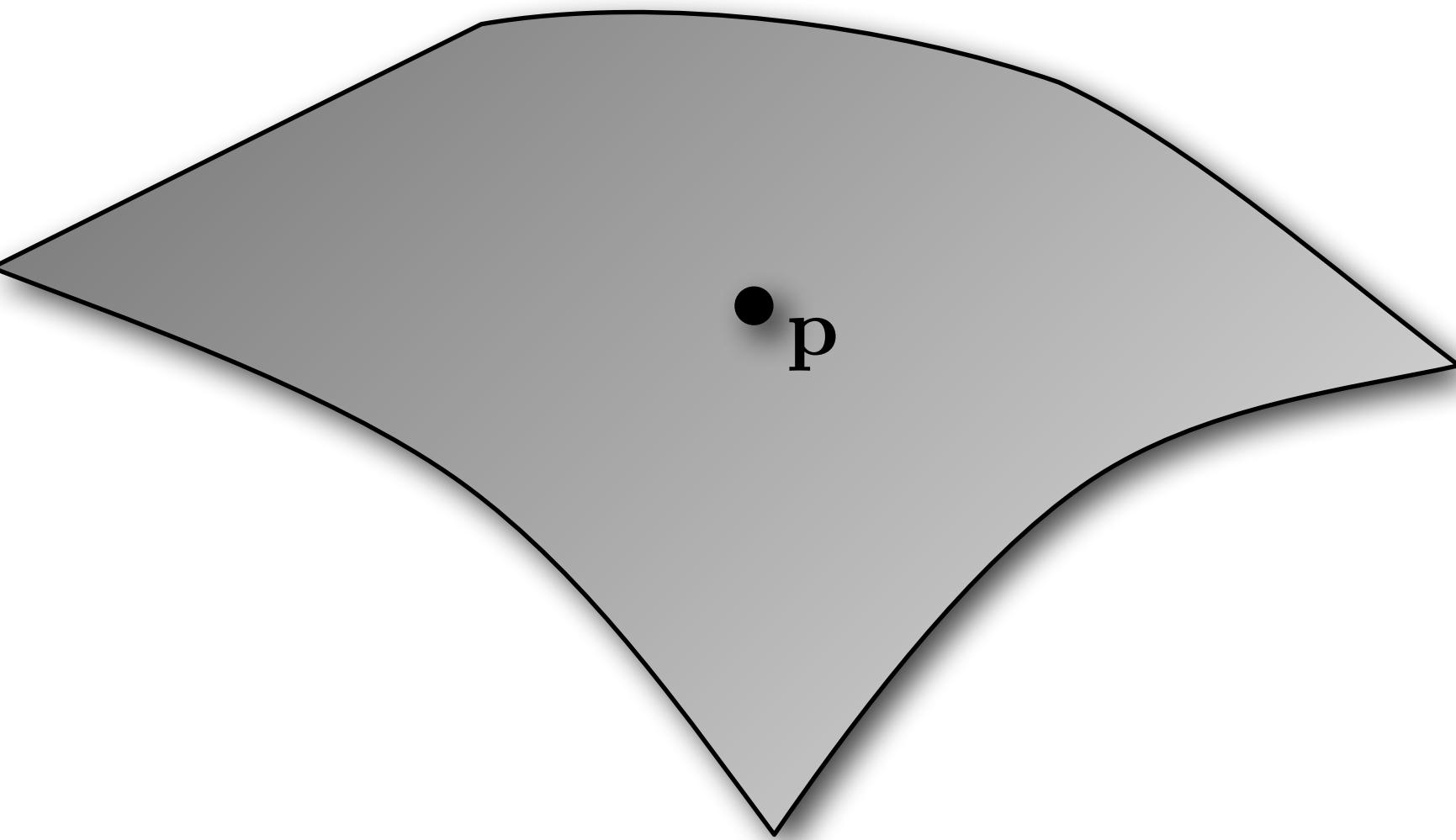
$$\mathbf{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$



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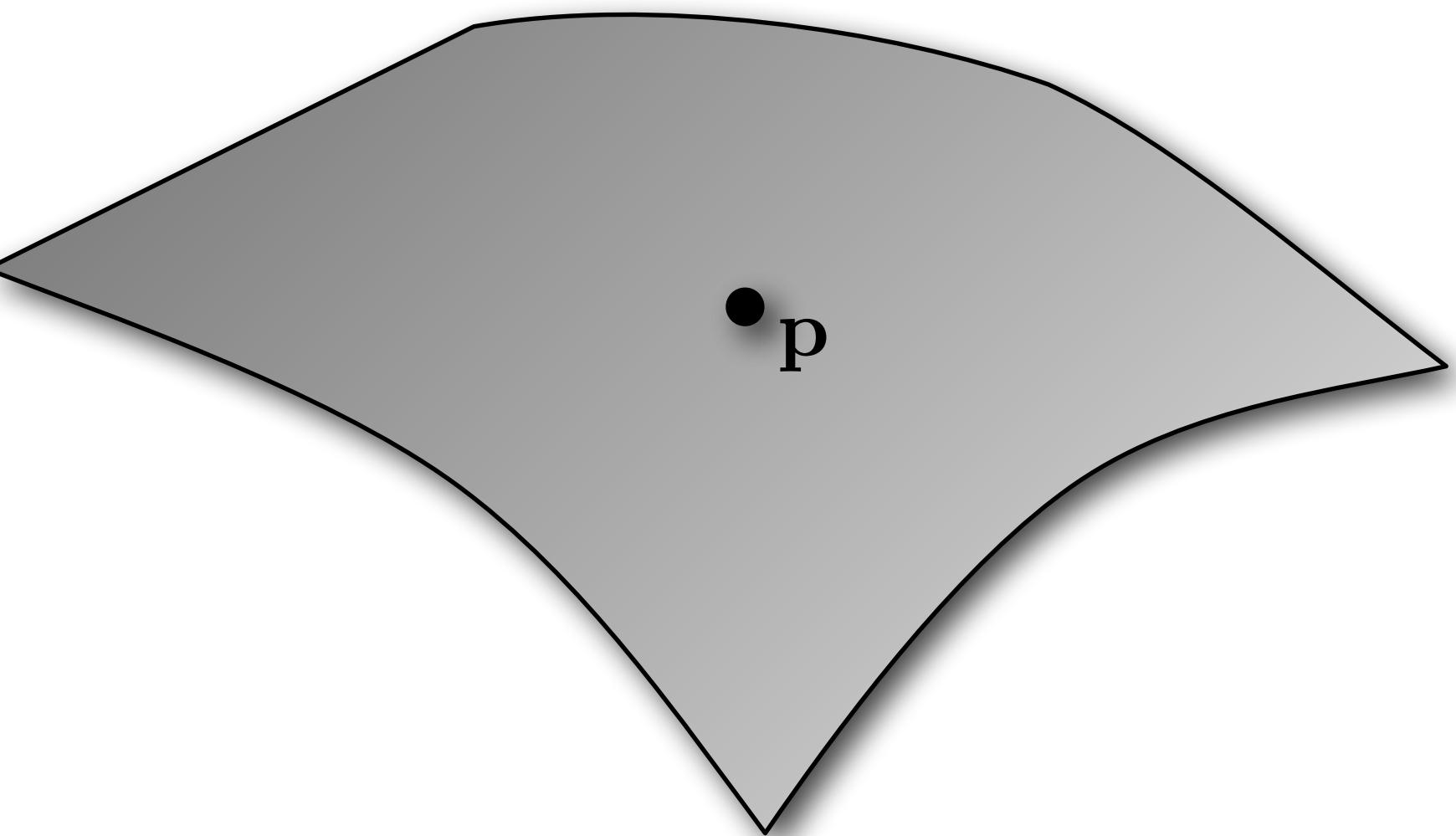
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$$\mathbf{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

- Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$



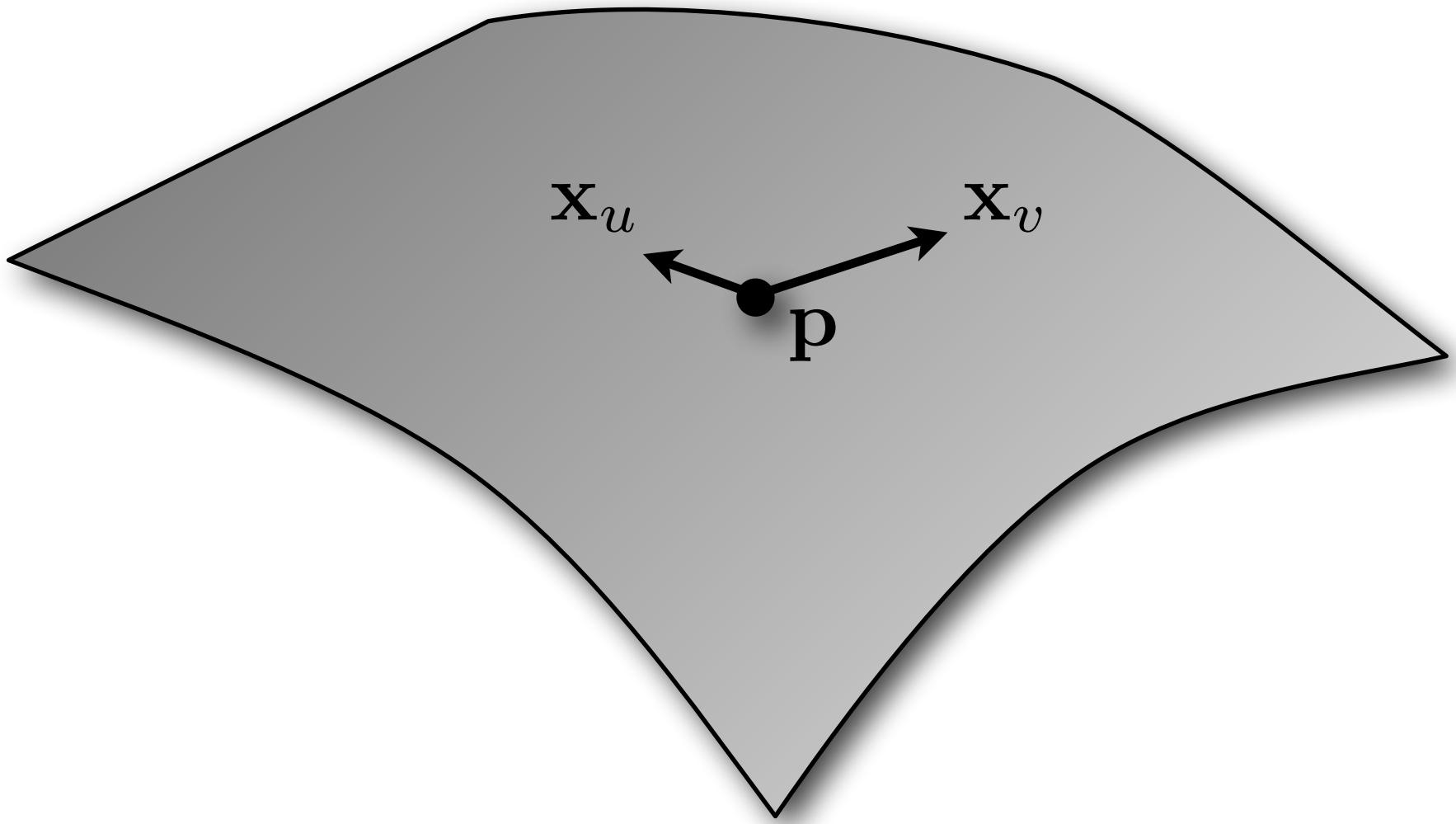
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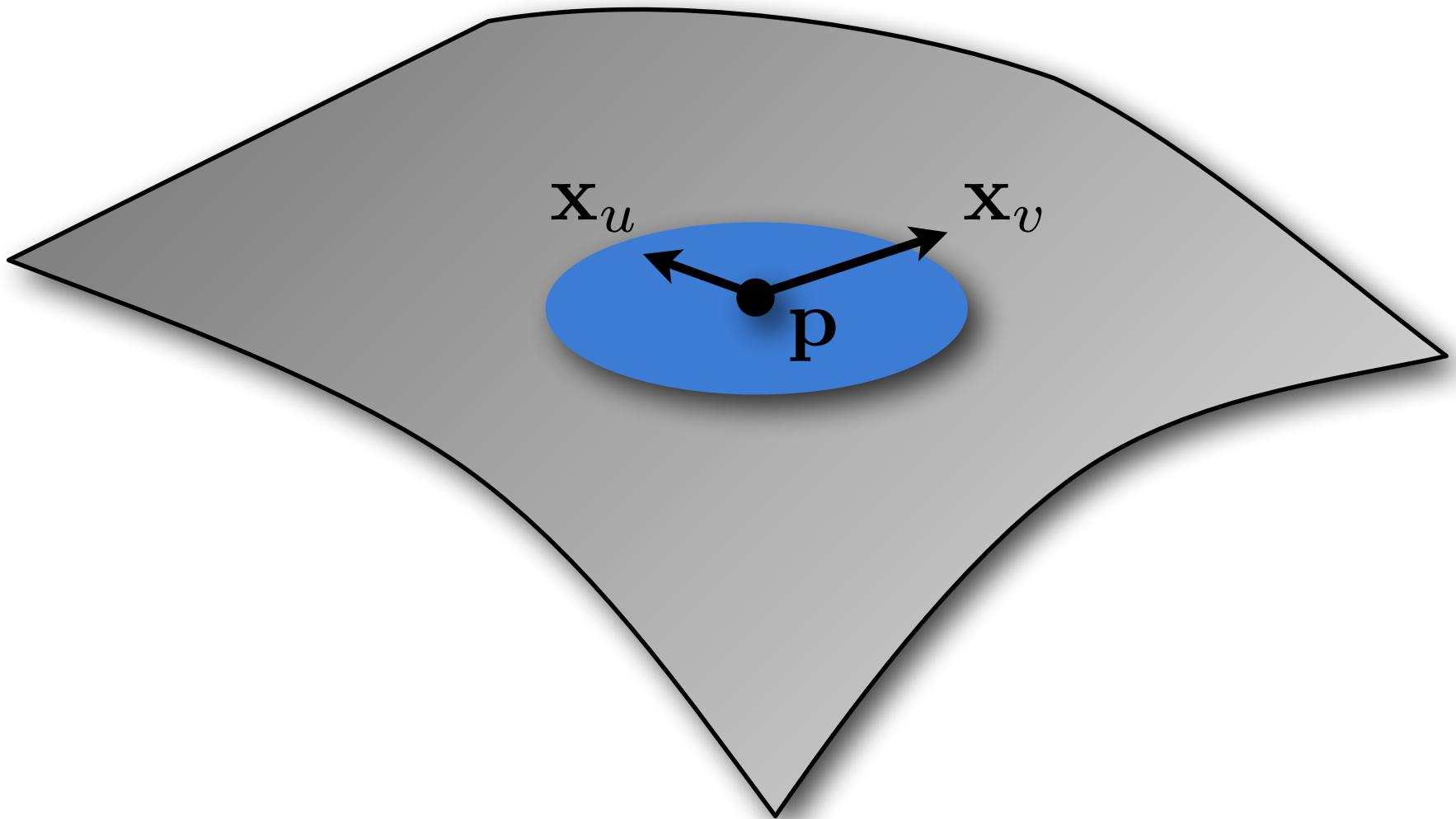
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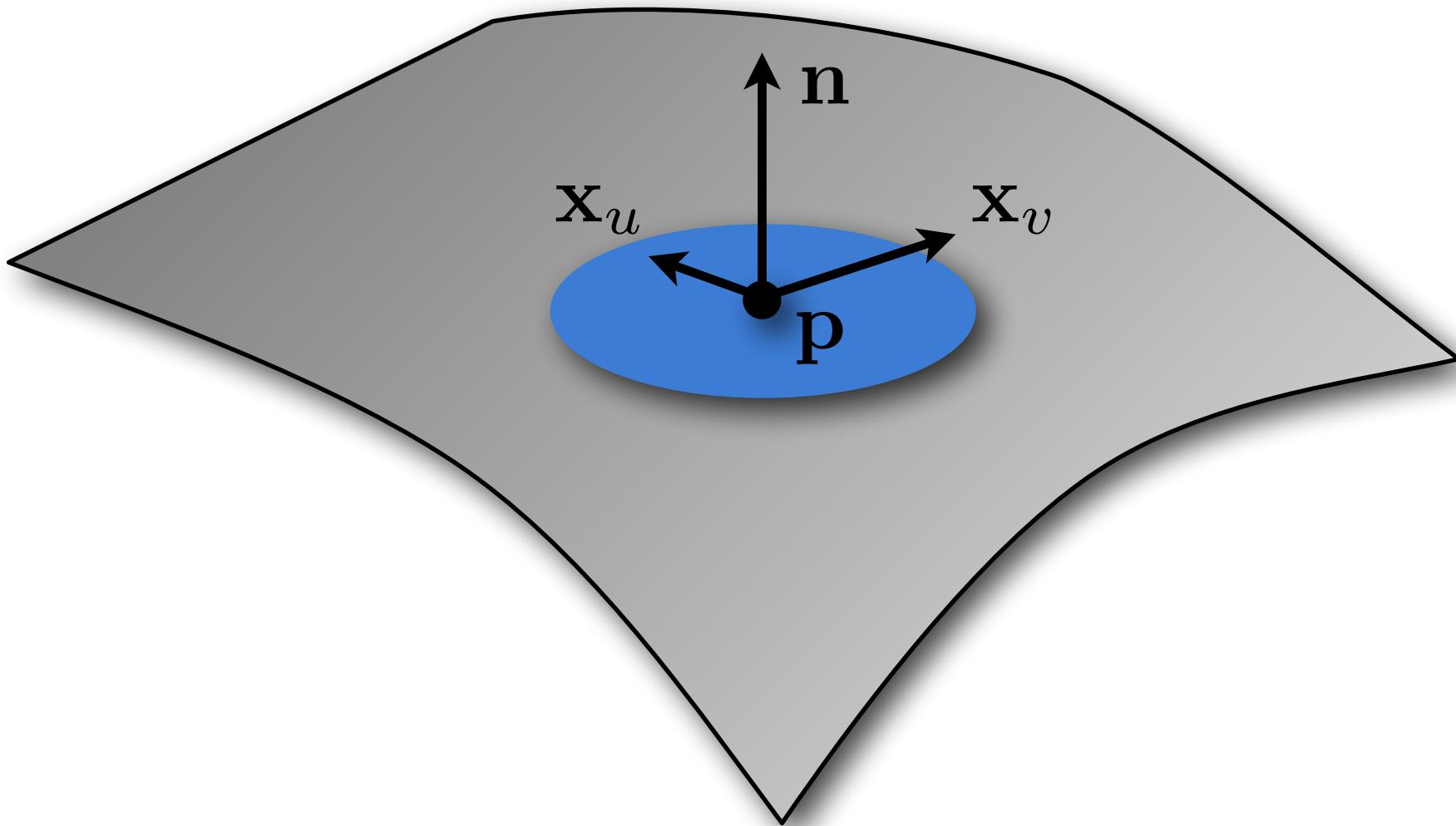
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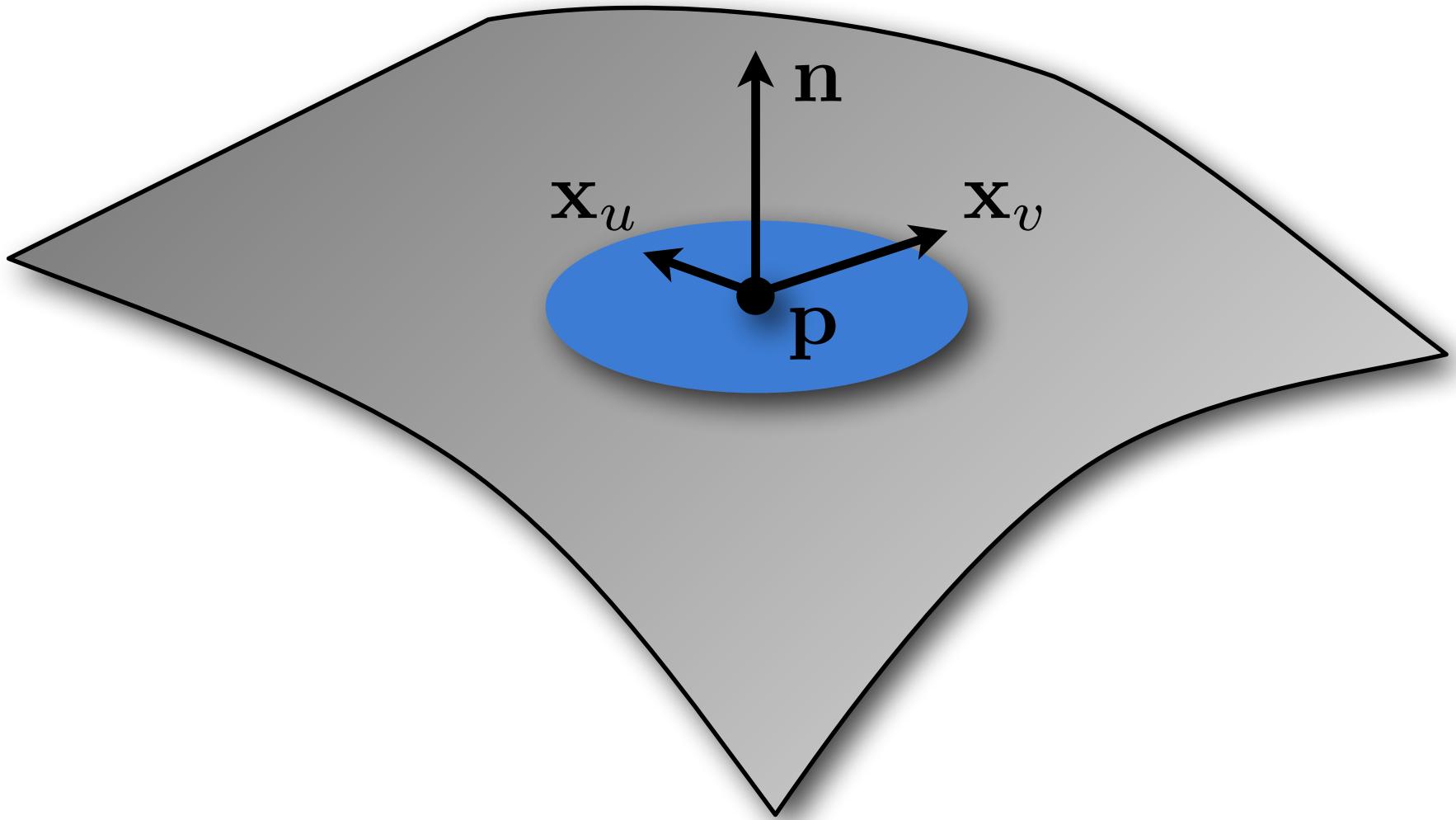
$$\mathbf{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

- Normal vector

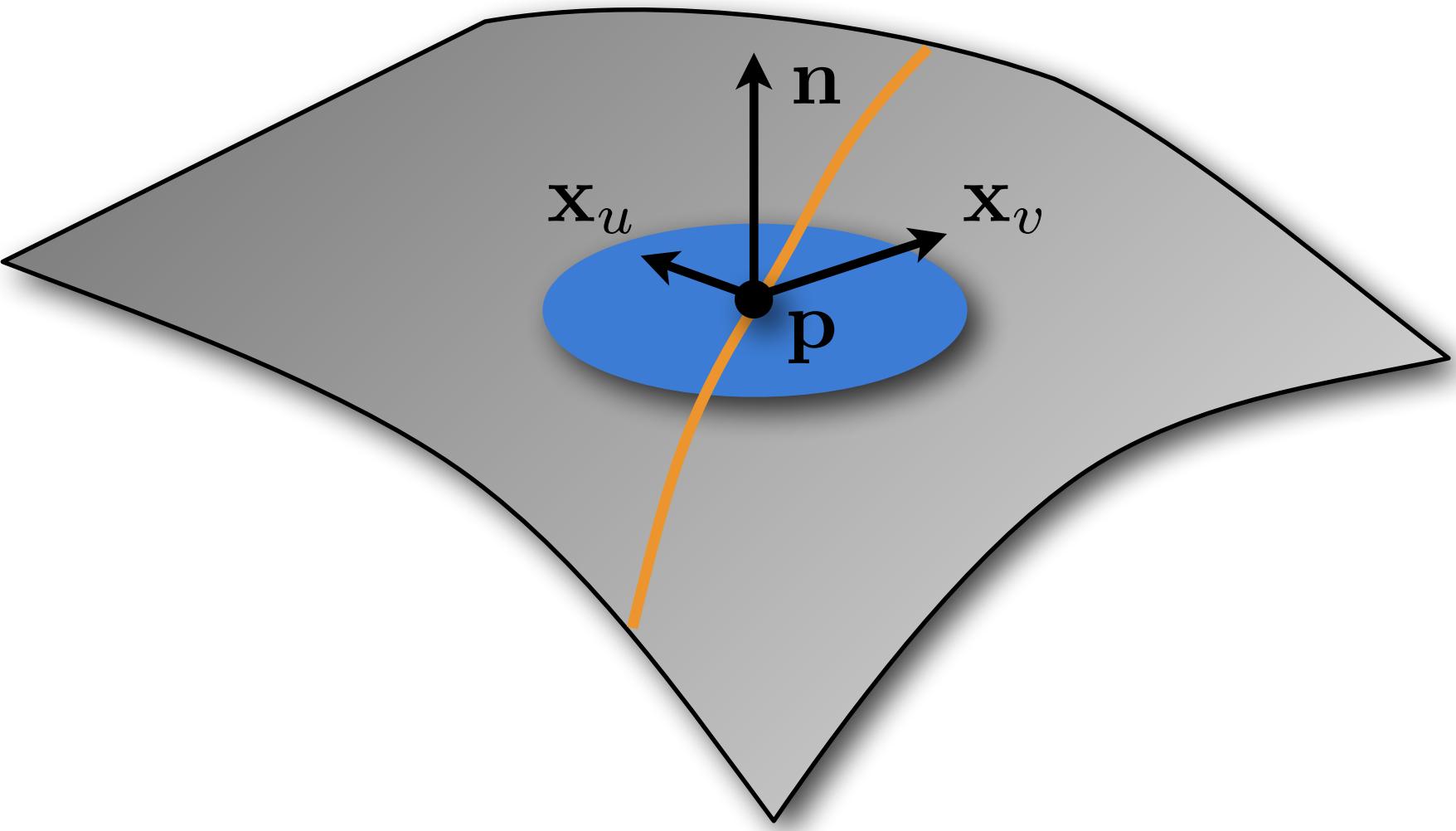
$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

- Assume *regular* parameterization

$$\mathbf{x}_u \times \mathbf{x}_v \neq 0$$

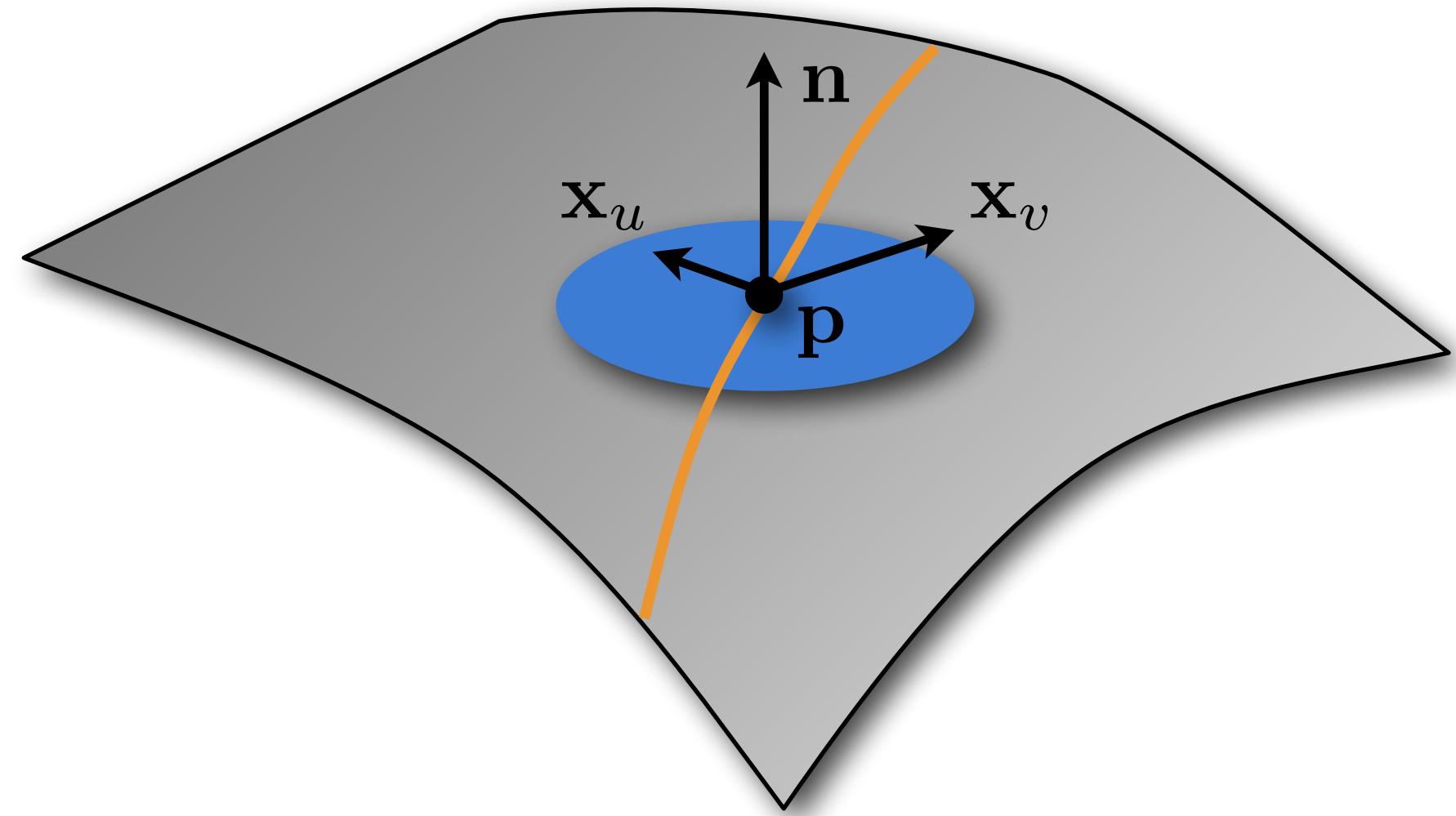


# Angles on Surface



# Angles on Surface

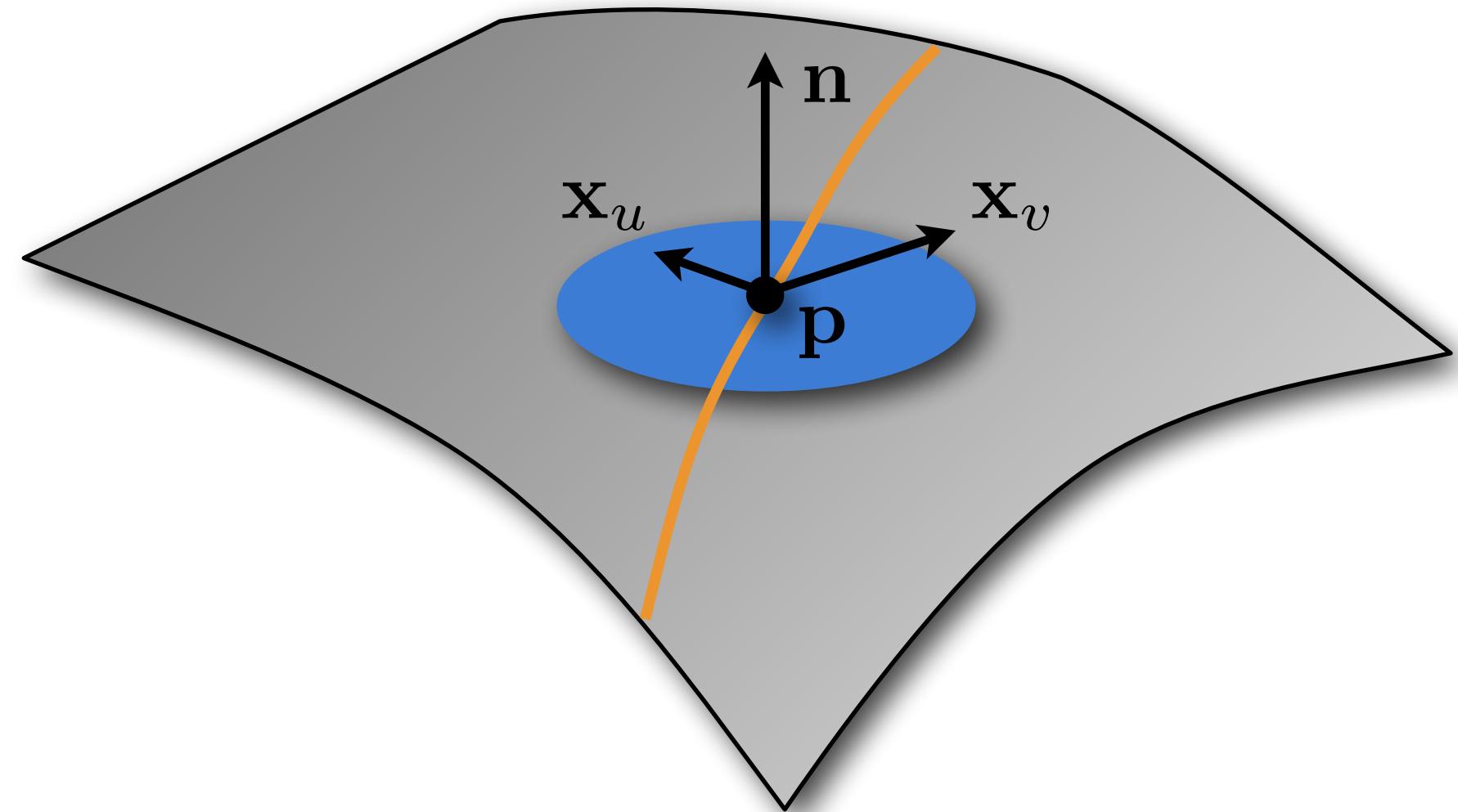
- Curve  $[u(t), v(t)]$  in  $uv$ -plane defines curve on the surface  $\mathbf{x}(u, v)$



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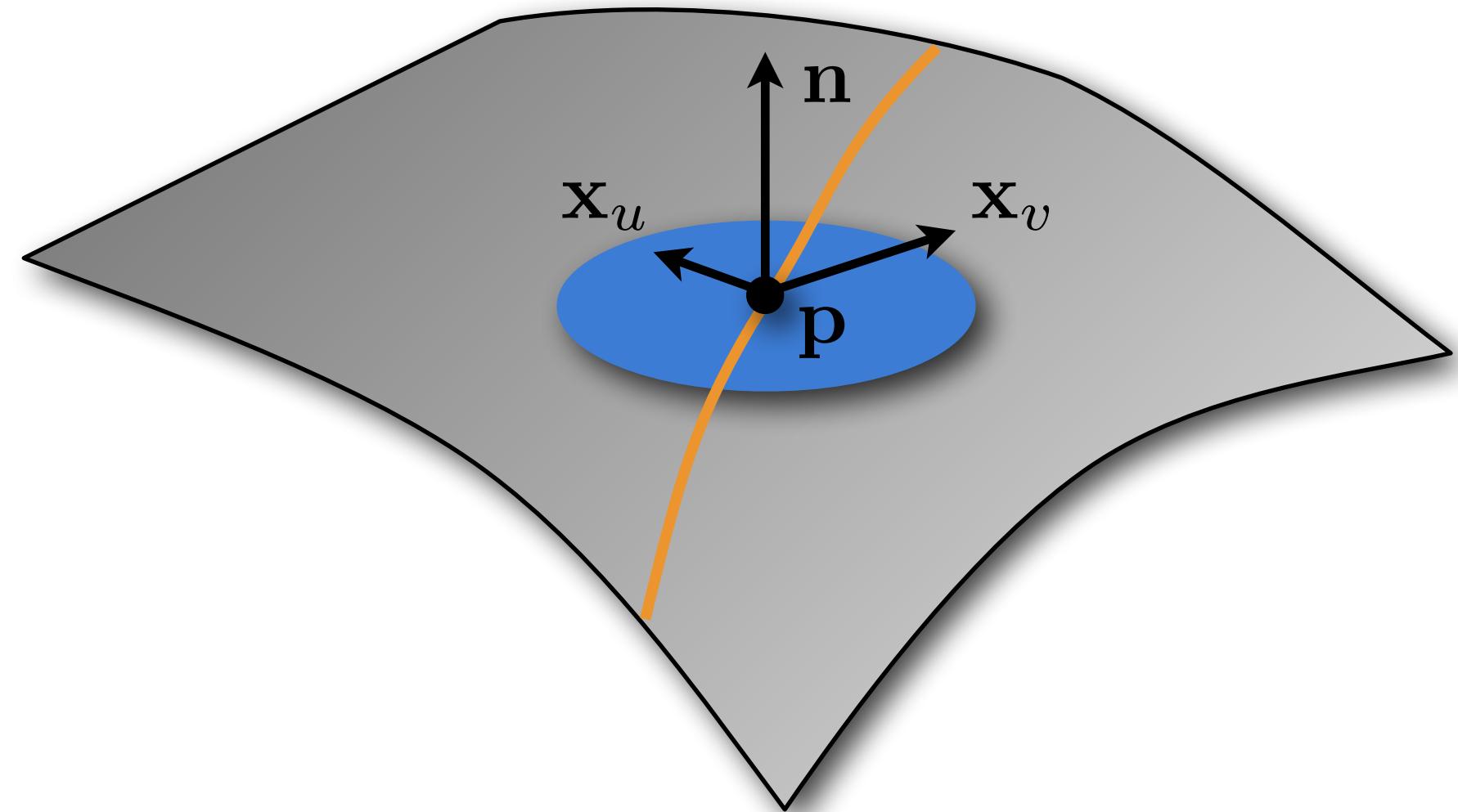


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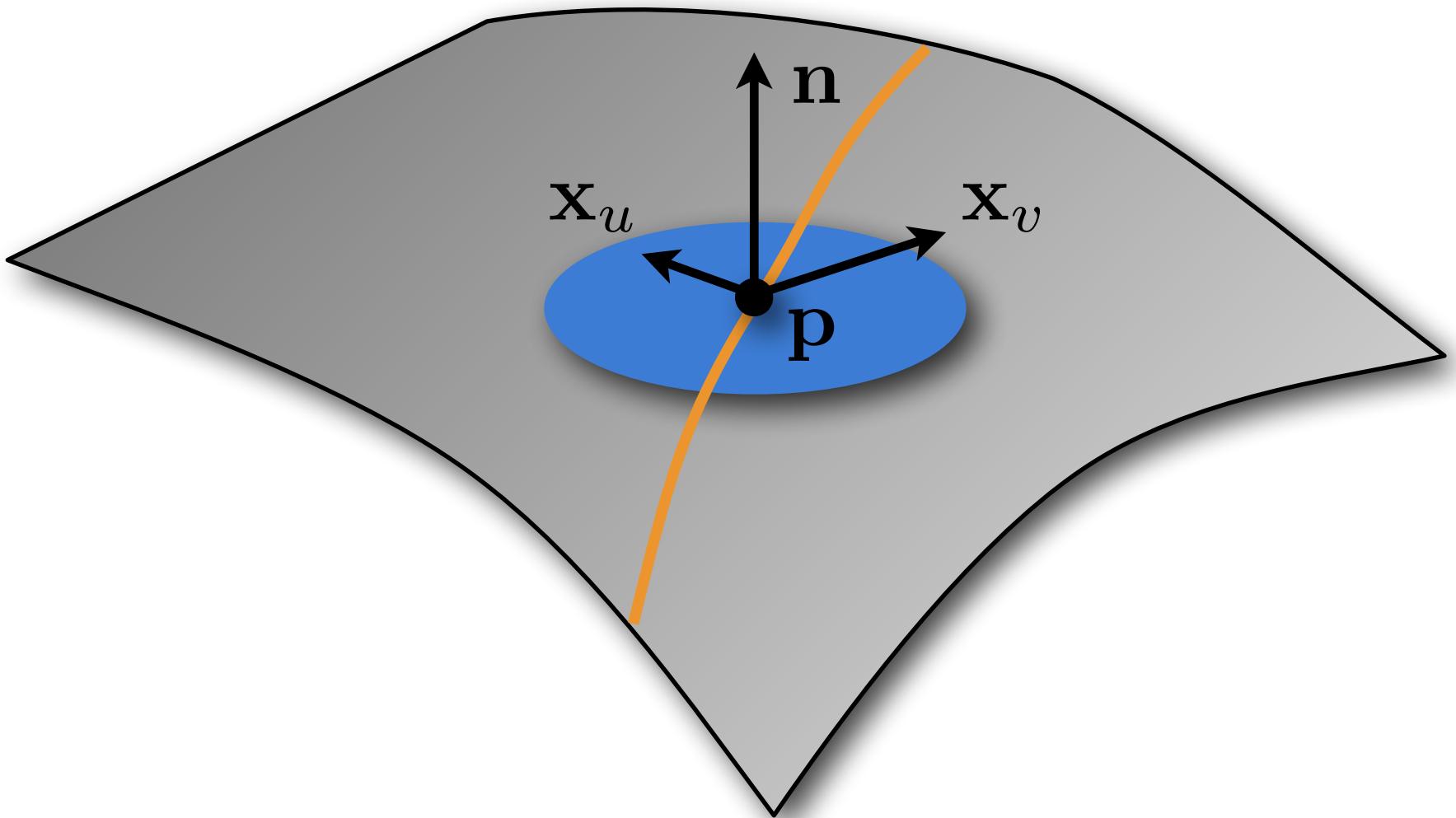


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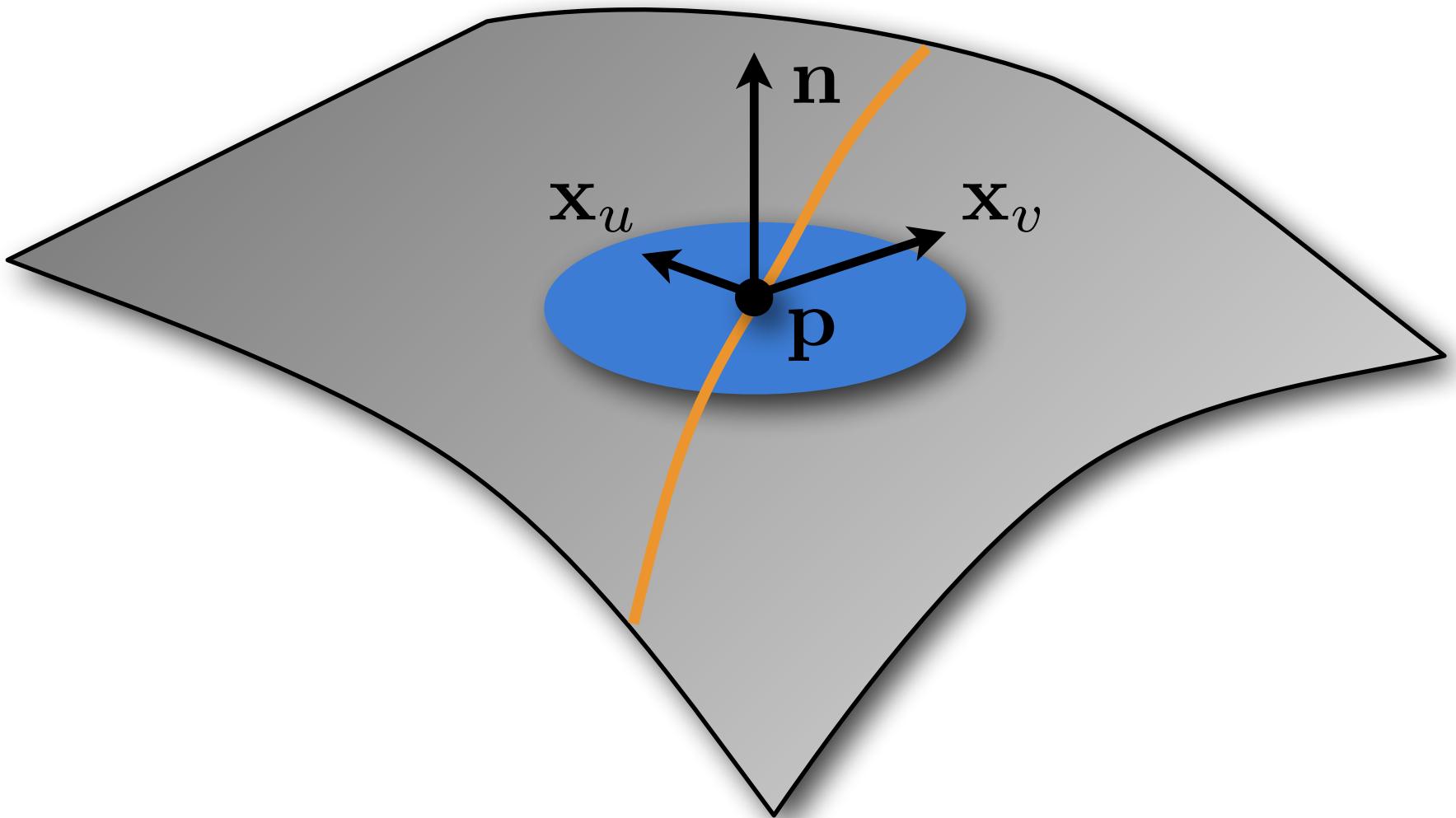


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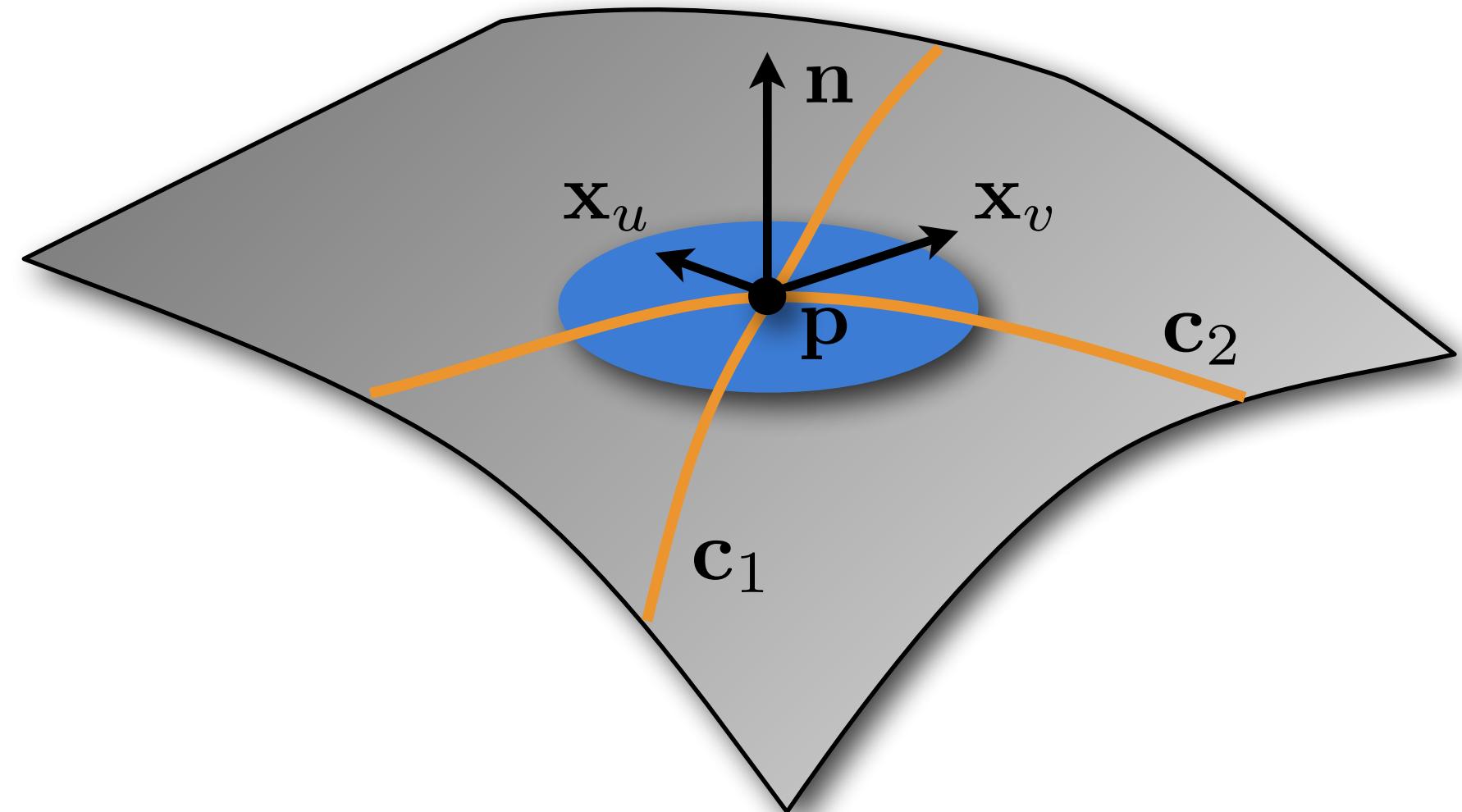
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$$\mathbf{t}_i = \alpha_i \mathbf{x}_u + \beta_i \mathbf{x}_v$$

- Compute inner product

$$\mathbf{t}_1^T \mathbf{t}_2 = \cos \theta \|\mathbf{t}_1\| \|\mathbf{t}_2\|$$



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# First Fundamental Form



- **First fundamental form**

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

- Defines inner product on tangent space

$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

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- First fundamental form I allows to measure  
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$$\begin{aligned} ds^2 &= \langle (du, dv), (du, dv) \rangle \\ &= Edu^2 + 2Fdudv + Gdv^2 \end{aligned}$$

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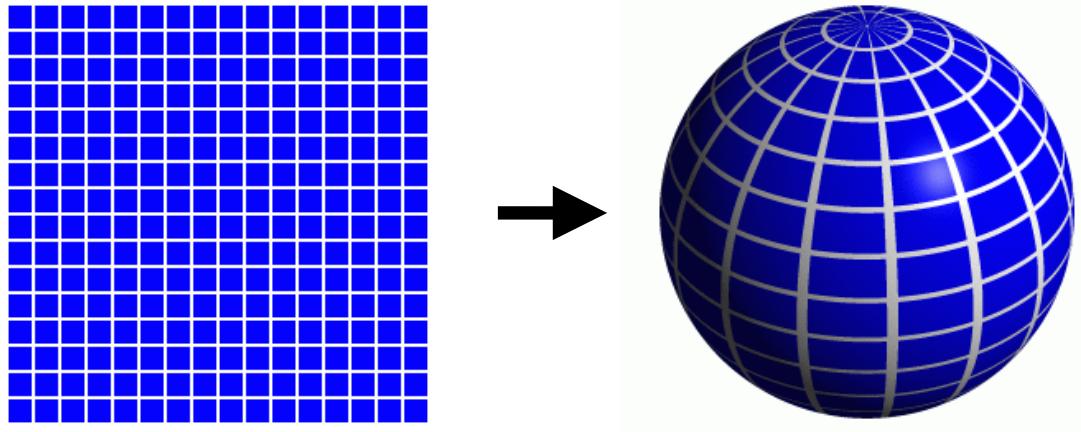
- Area

$$\begin{aligned} dA &= \|\mathbf{x}_u \times \mathbf{x}_v\| du dv \\ &= \sqrt{\mathbf{x}_u^T \mathbf{x}_u \cdot \mathbf{x}_v^T \mathbf{x}_v - (\mathbf{x}_u^T \mathbf{x}_v)^2} du dv \\ &= \sqrt{EG - F^2} du dv \end{aligned}$$

# Example: Sphere

- Spherical parameterization

$$\mathbf{x}(u, v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}, \quad (u, v) \in [0, 2\pi) \times [0, \pi)$$

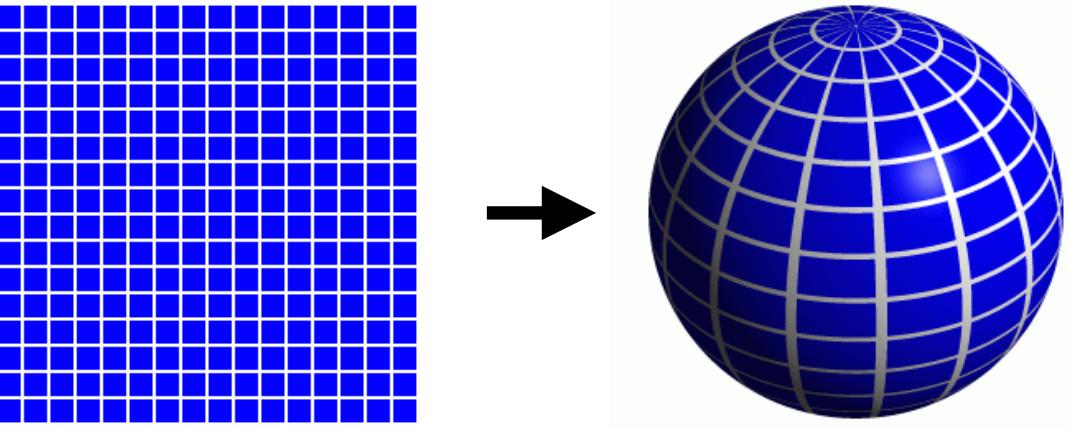


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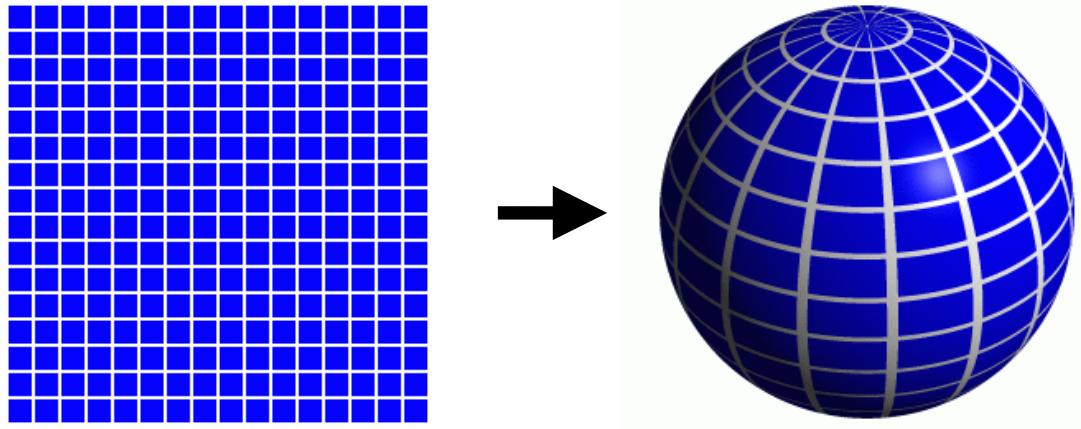
- Tangent vectors



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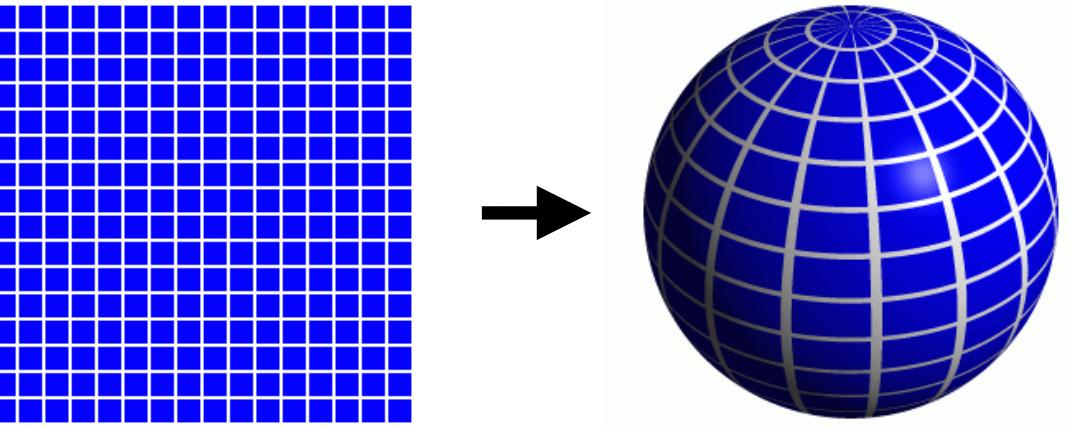
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$$\mathbf{x}_u(u, v) = \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix}$$

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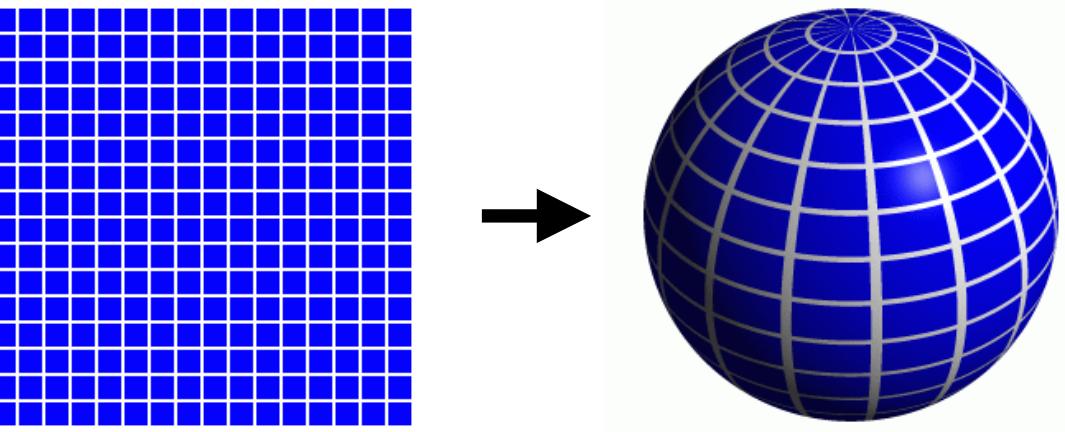
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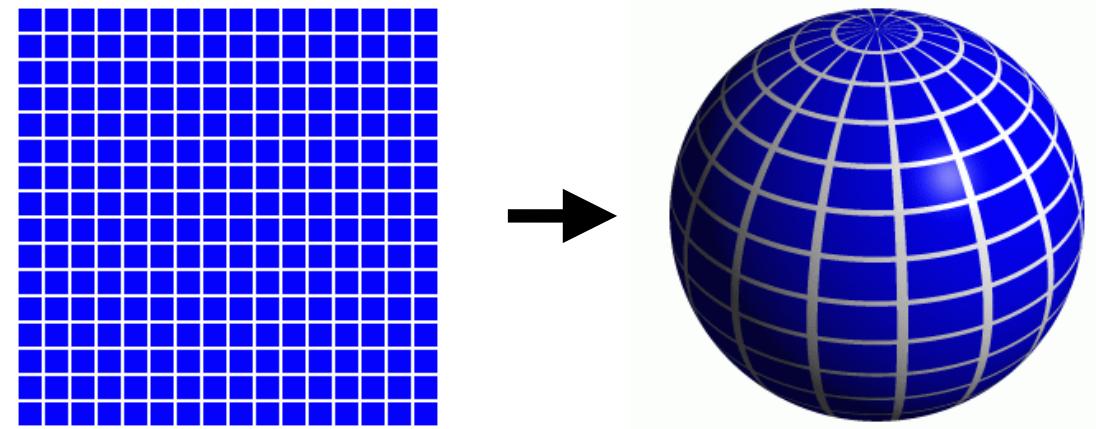
- First fundamental form

$$\mathbf{I} = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

# Example: Sphere



- Length of equator  $\mathbf{x}(t, \pi / 2)$

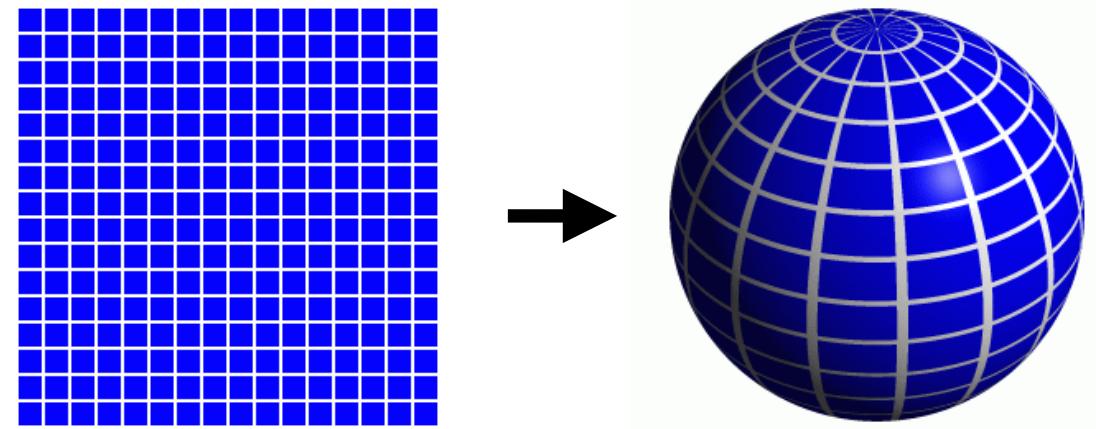


$$\int_0^{2\pi} 1 \, ds = \int_0^{2\pi} \sqrt{E (u_t)^2 + 2Fu_tv_t + G (v_t)^2} \, dt$$

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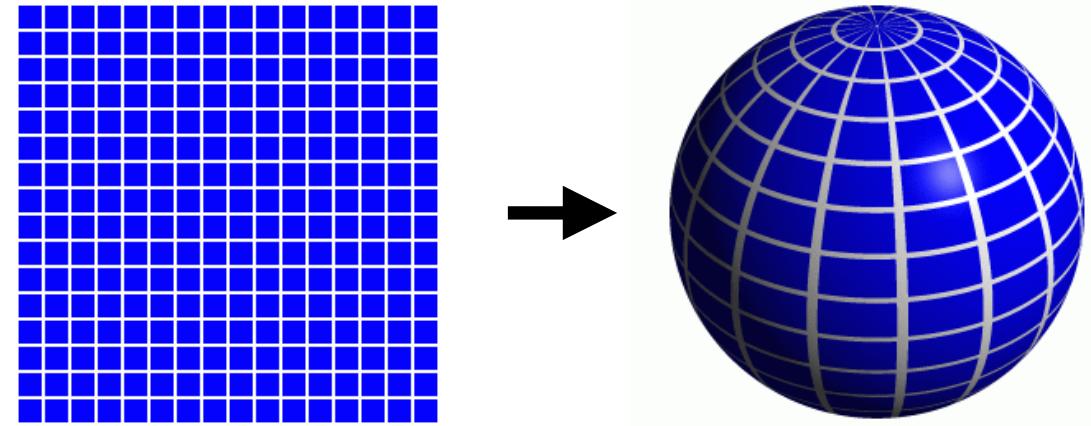
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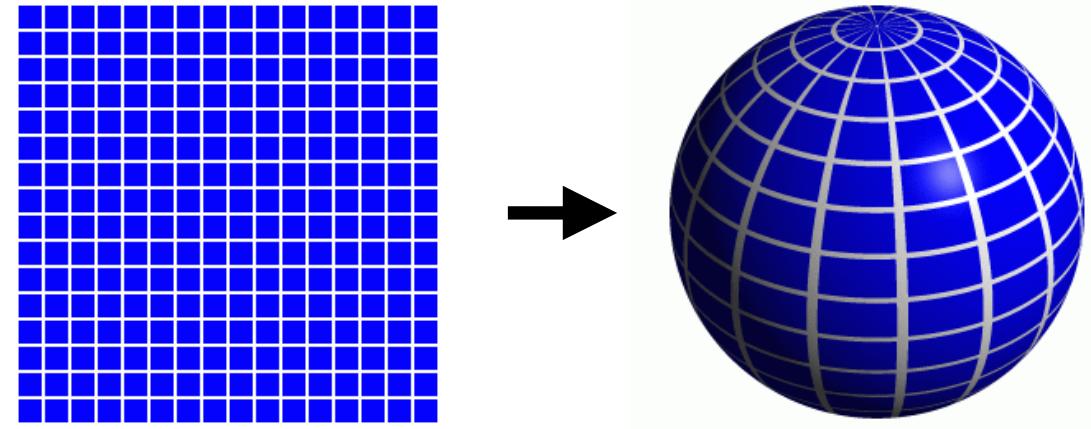
$$\begin{aligned} \int_0^{2\pi} 1 \, ds &= \int_0^{2\pi} \sqrt{E(u_t)^2 + 2Fu_tv_t + G(v_t)^2} \, dt \\ &= \int_0^{2\pi} \sin v \, dt \\ &= 2\pi \sin v = 2\pi \end{aligned}$$

(ranging  $v \in [0, \pi]$ )

# Example: Sphere



- Area of a sphere

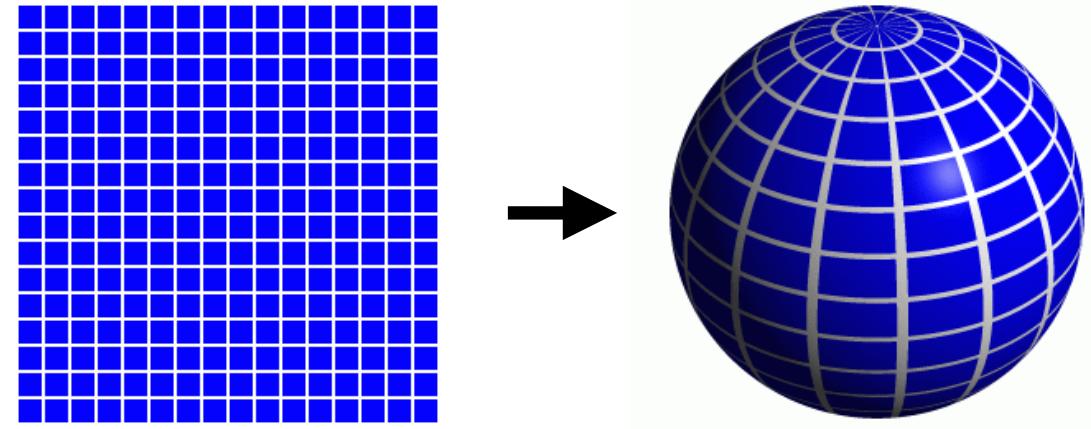


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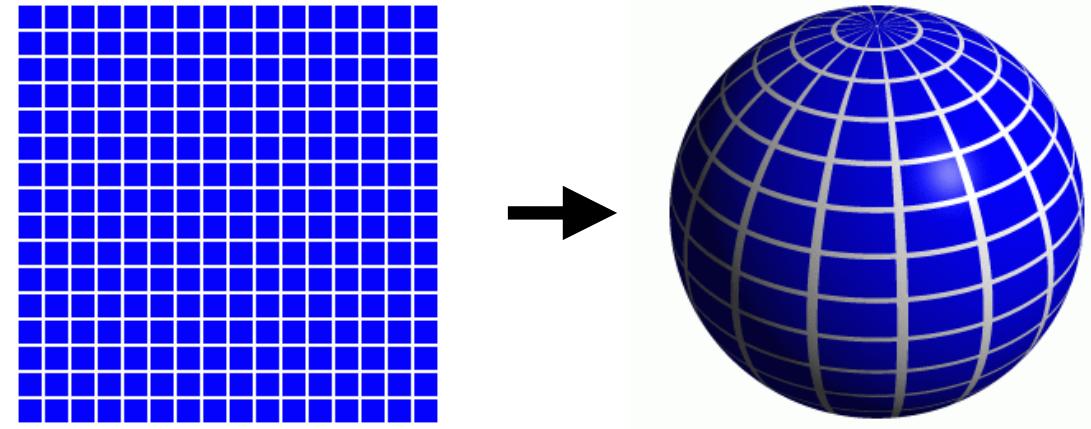


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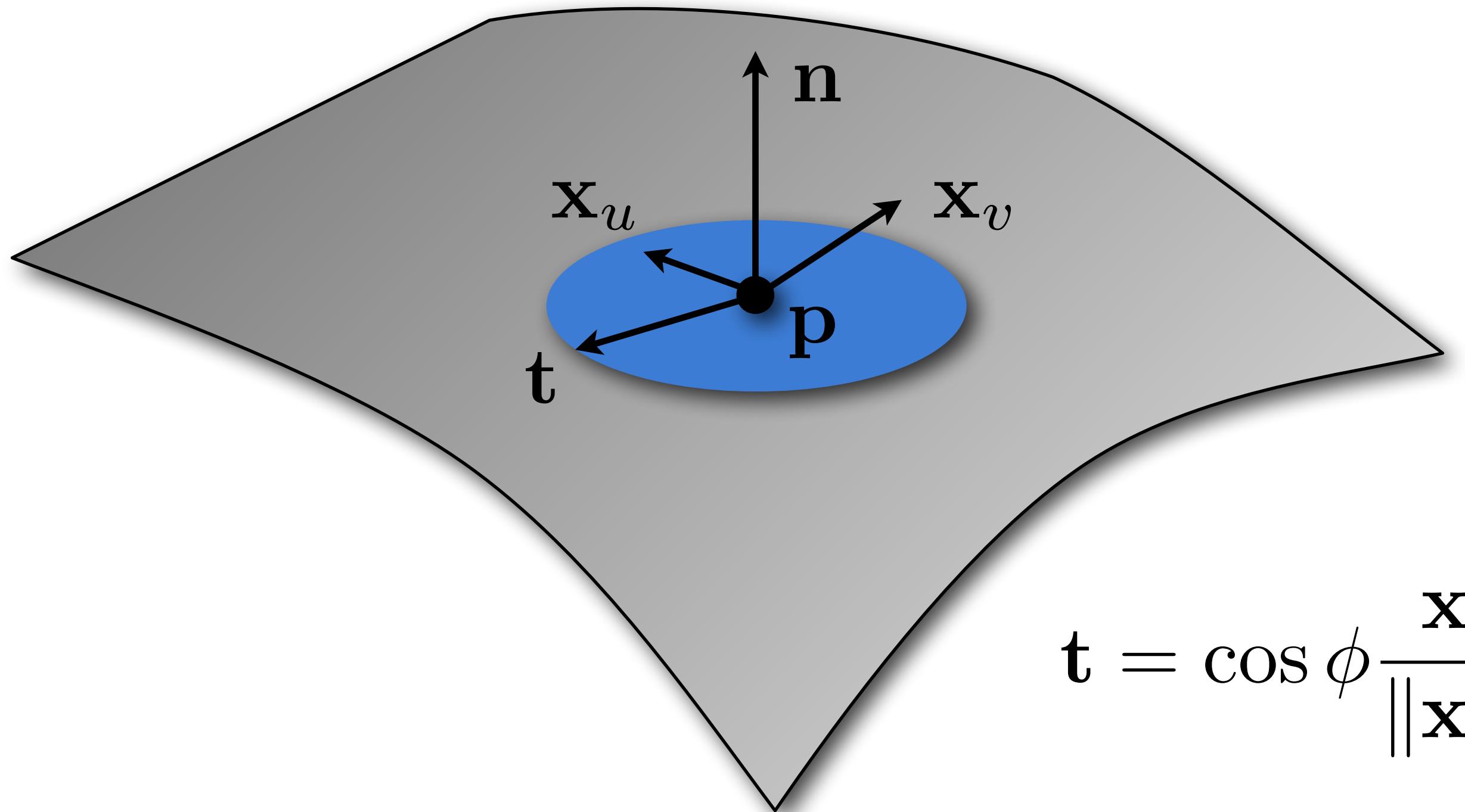
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# Normal Curvature

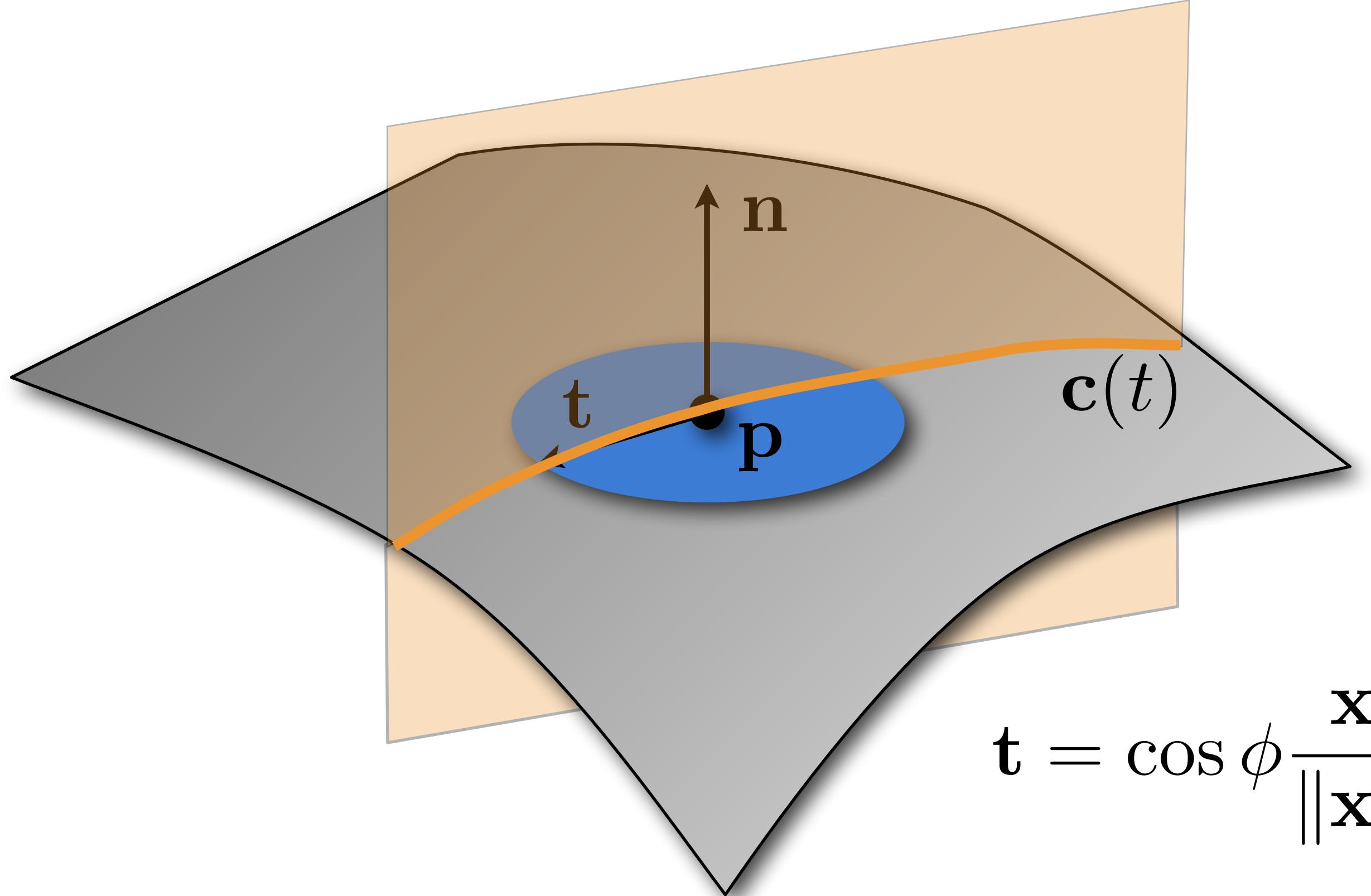
- Tangent vector  $t$ ...



$$\mathbf{t} = \cos \phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin \phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

# Normal Curvature

- .. defines intersection plane, yielding curve  $c(t)$



$$\mathbf{t} = \cos \phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin \phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

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**normal curvature** can be computed as

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \text{II} \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \mathbf{I} \bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2} \quad \begin{aligned} \mathbf{t} &= a\mathbf{x}_u + b\mathbf{x}_v \\ \bar{\mathbf{t}} &= (a, b) \end{aligned}$$

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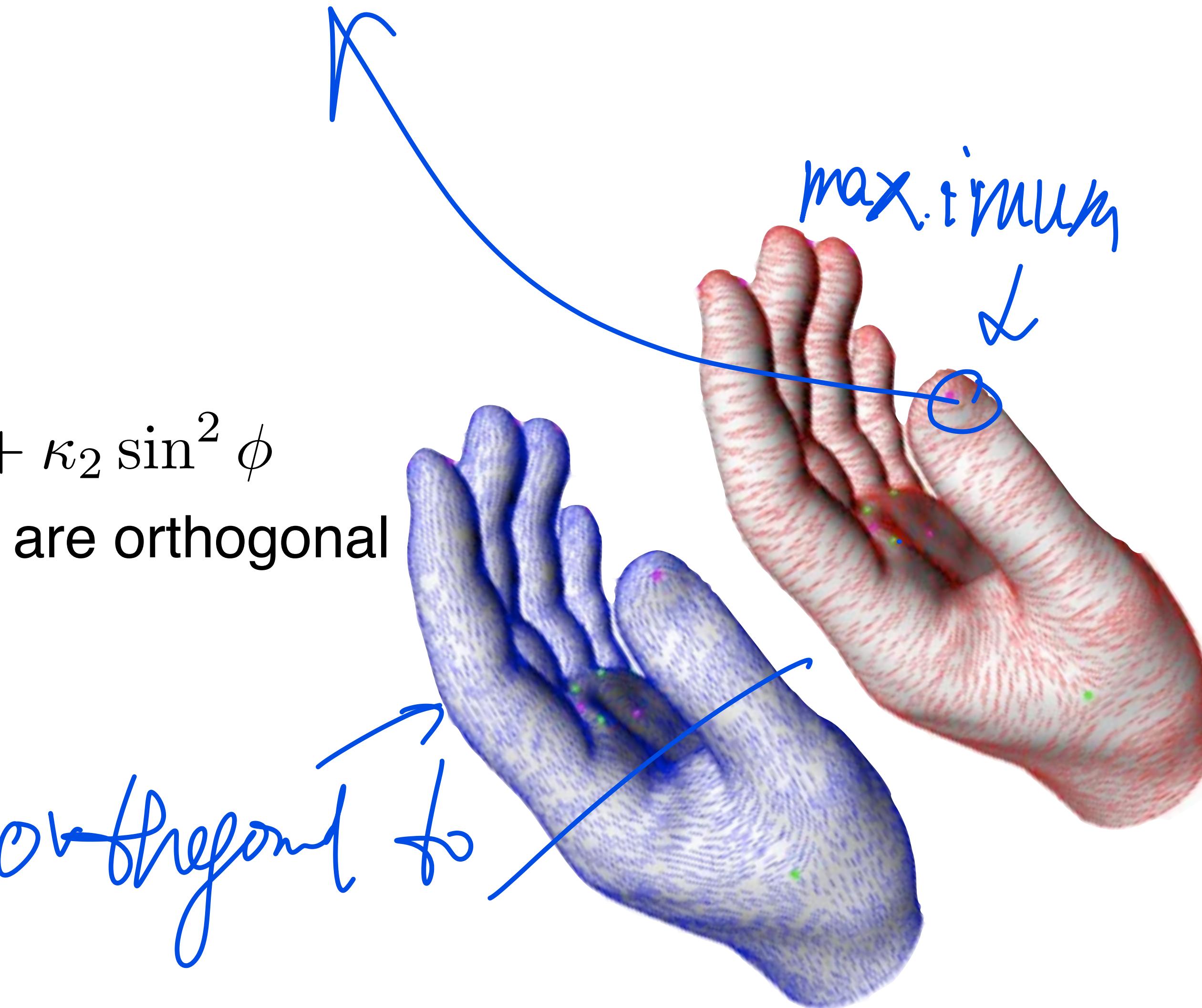
- Minimum curvature

$$\kappa_2 = \min_{\phi} \kappa_n(\phi)$$

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- Special curvatures

- Mean curvature

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

~~Gaussian curvature~~

$$K = \kappa_1 \cdot \kappa_2$$

# Curvature of Surfaces



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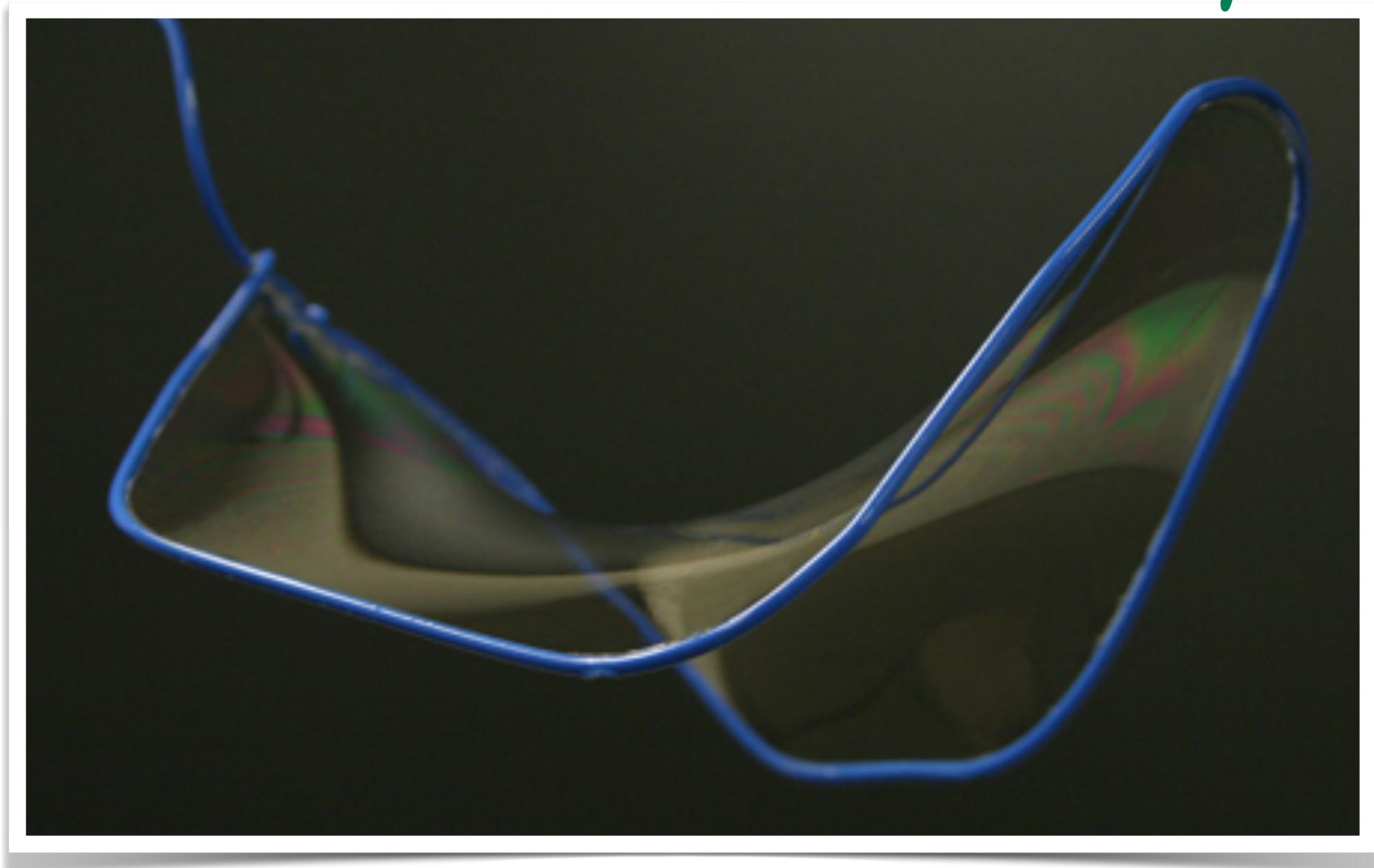
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(~~isoparametric~~ or ↴)



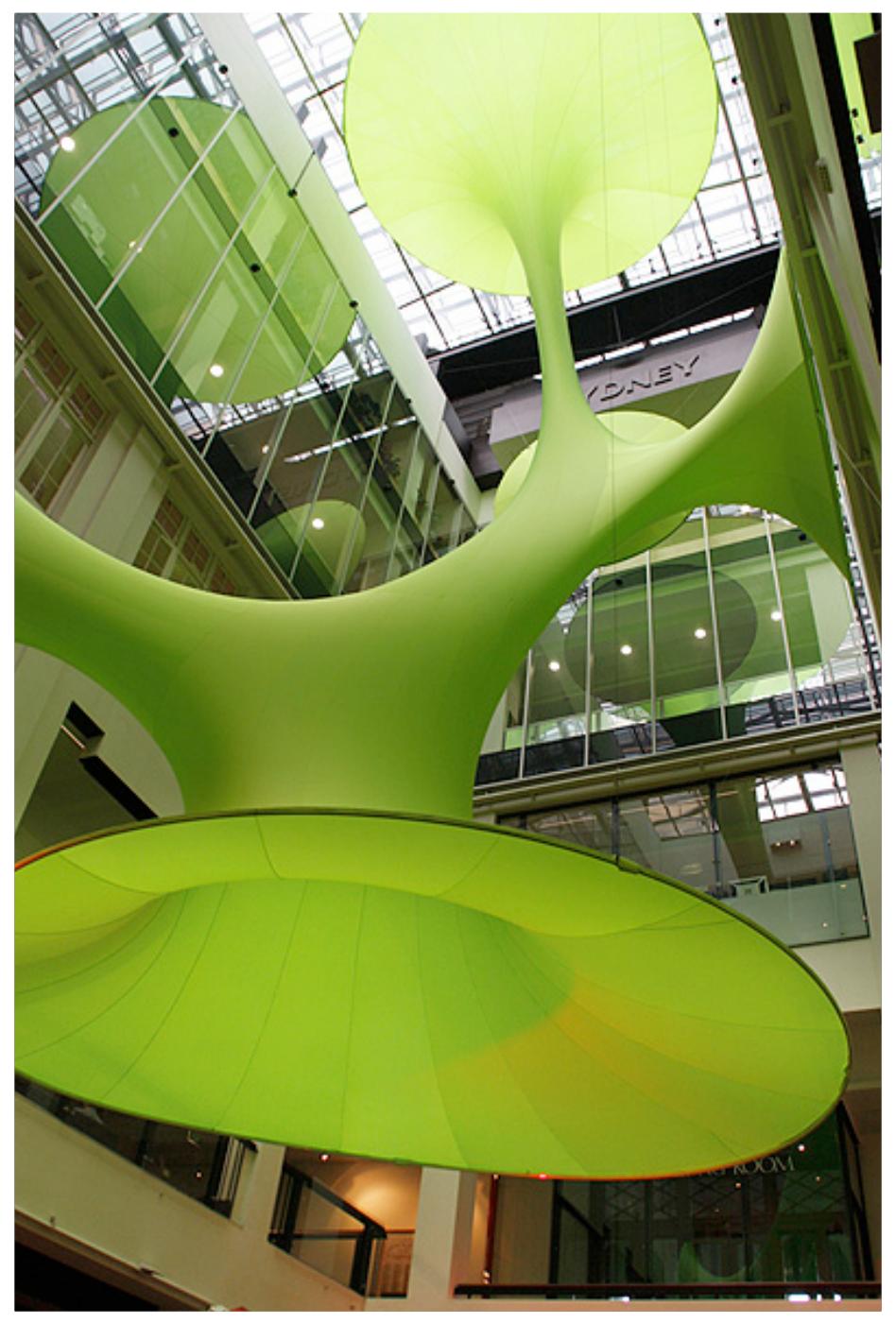
soap films

# Curvature of Surfaces

- Mean curvature
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Green Void, Sydney  
Architects: LAVA

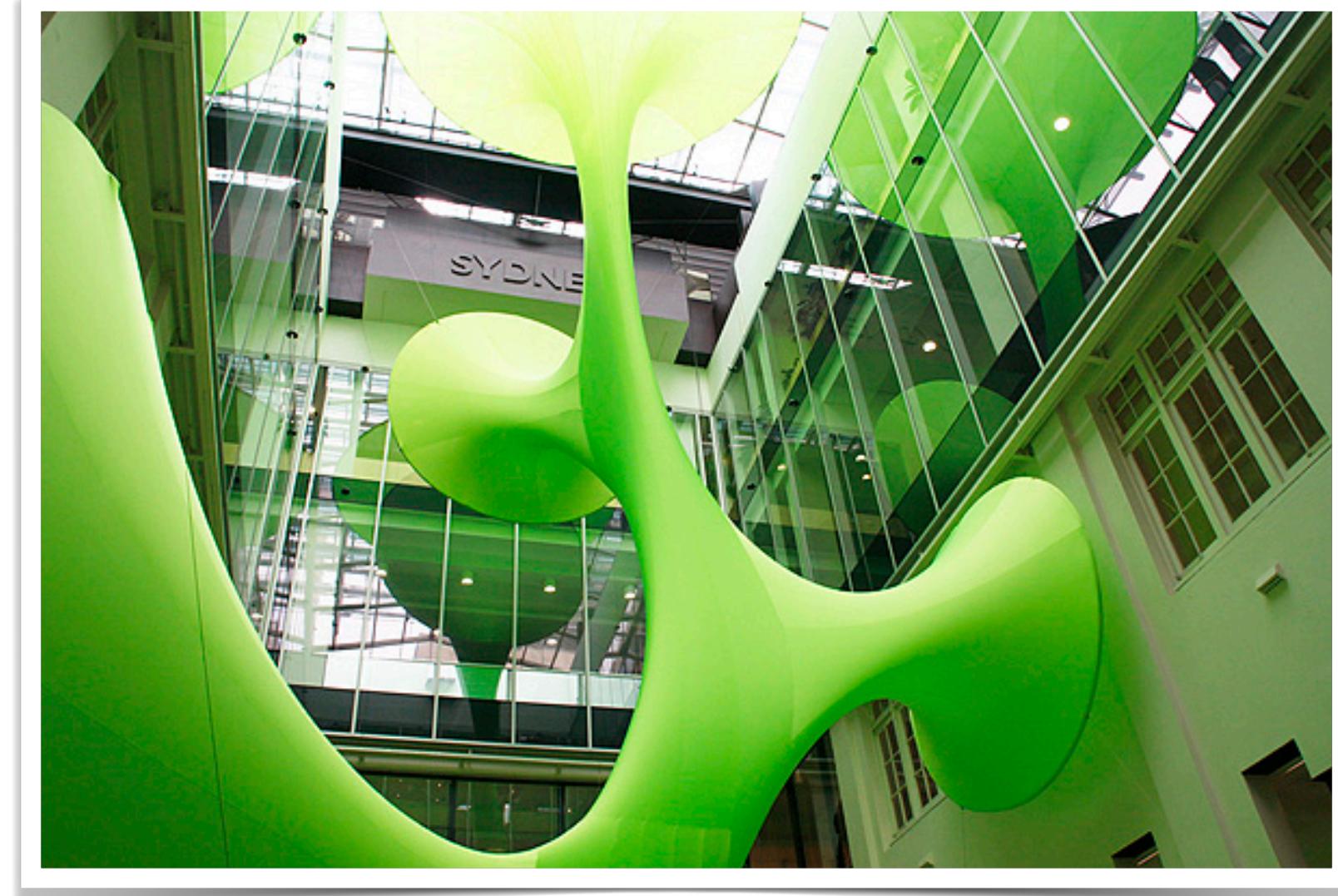


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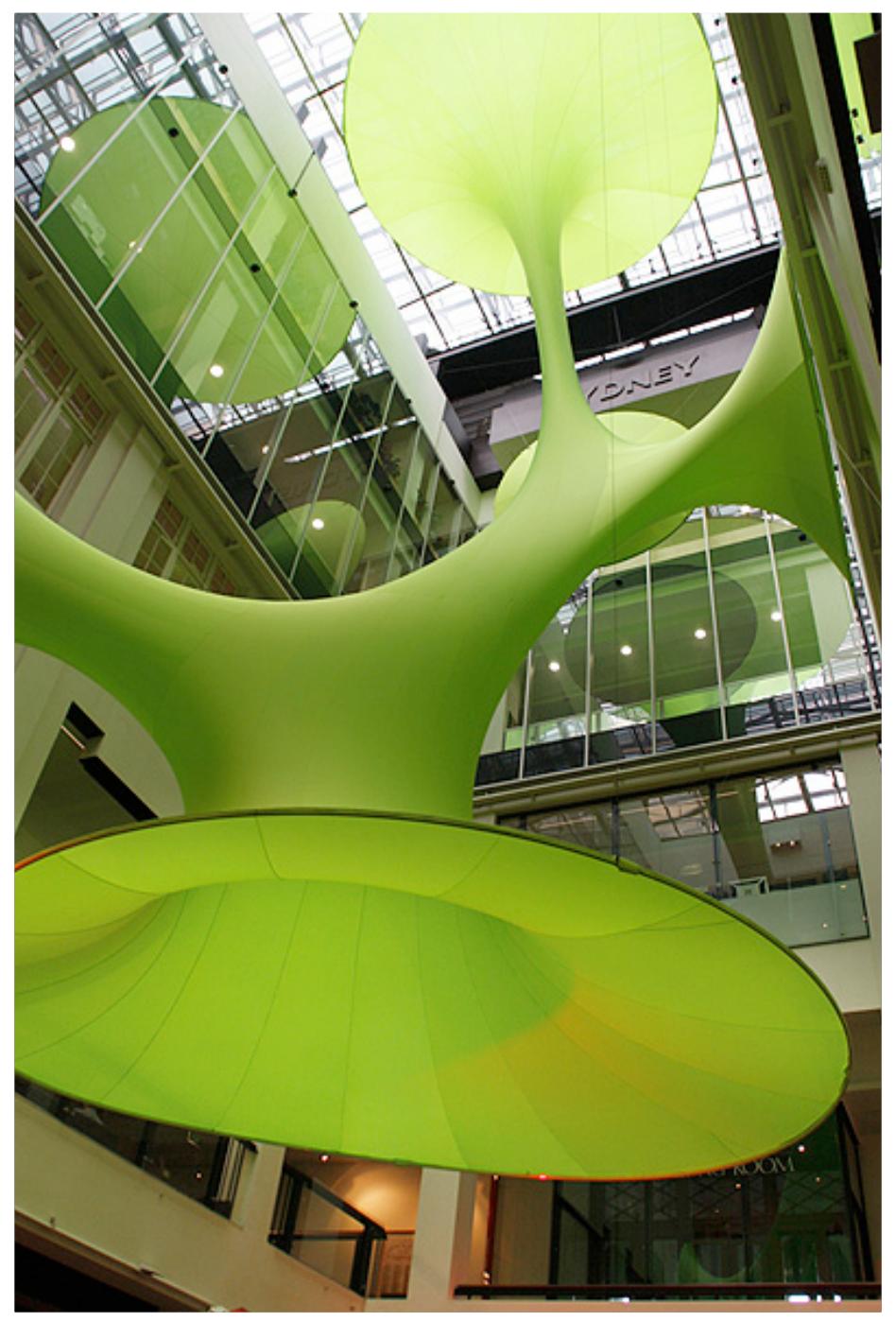
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# Curvature of Surfaces



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Disney Concert Hall, L.A.  
Architects: Gehry Partners



Timber Fabric  
IBOIS, EPFL

# Intrinsic Geometry



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$$K = \lim_{r \rightarrow 0} \frac{6\pi r - 3C(r)}{\pi r^3}$$

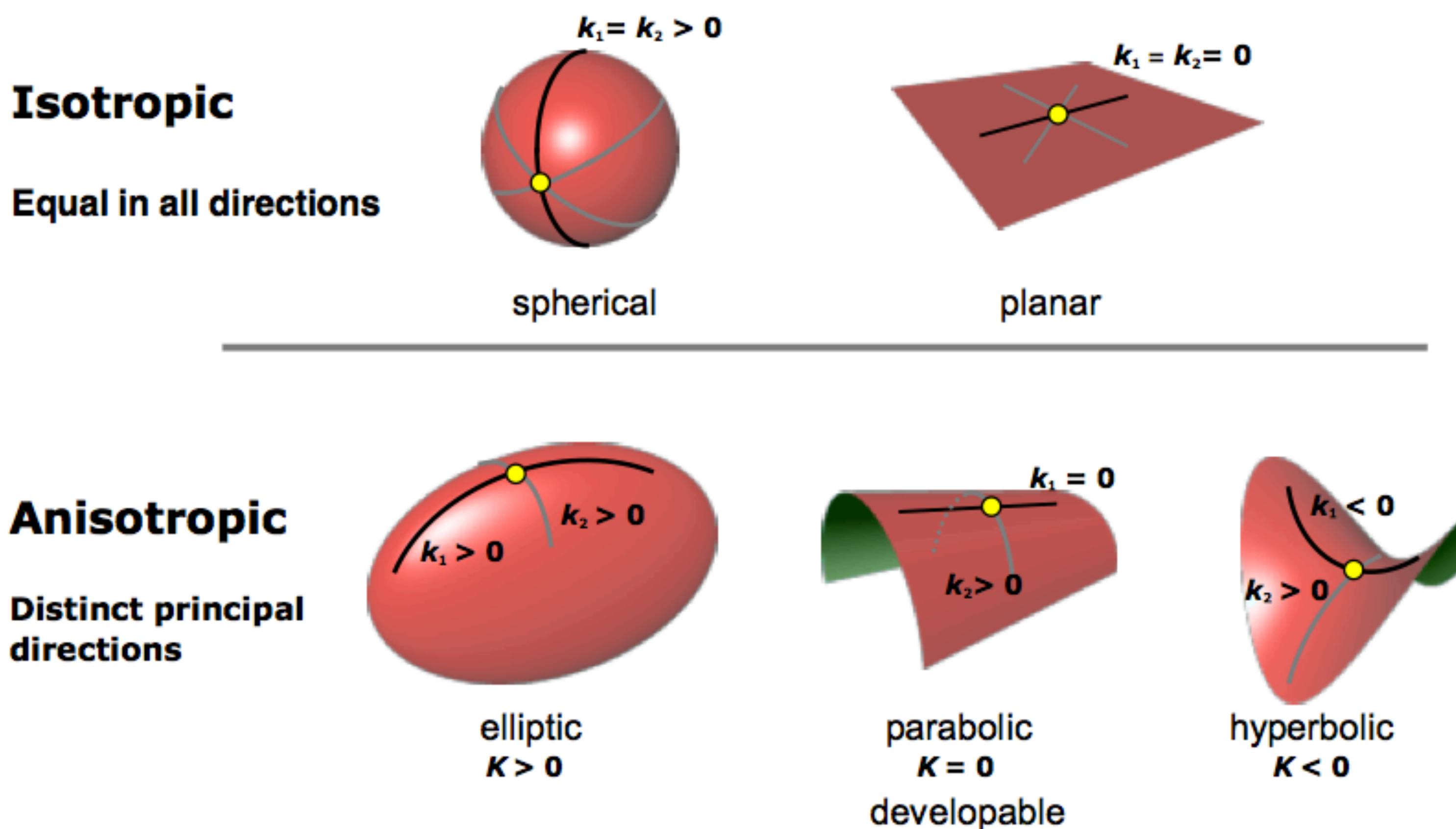
# Classification



- A point  $x$  on the surface is called
  - *elliptic*, if  $K > 0$
  - *hyperbolic*, if  $K < 0$
  - *parabolic*, if  $K = 0$
  - *umbilic*, if  $\kappa_1 = \kappa_2$

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# Gauss-Bonnet Theorem



- For **any** closed manifold surface with Euler characteristic  $\chi = 2 - 2g$

$$\int K = 2\pi\chi$$

# Gauss-Bonnet Theorem



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$$\int K = 2\pi\chi$$

$$\int K(\text{Hand}) = \int K(\text{Cow}) = \int K(\text{Sphere}) = 4\pi$$

# Gauss-Bonnet Theorem



# Gauss-Bonnet Theorem



- Sphere

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$$\kappa_1 = \kappa_2 = 1/r$$

# Gauss-Bonnet Theorem



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$$\kappa_1 = \kappa_2 = 1/r$$

$$K = \kappa_1 \kappa_2 = 1/r^2$$

# Gauss-Bonnet Theorem



- Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

$$K = \kappa_1 \kappa_2 = 1/r^2$$

$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$

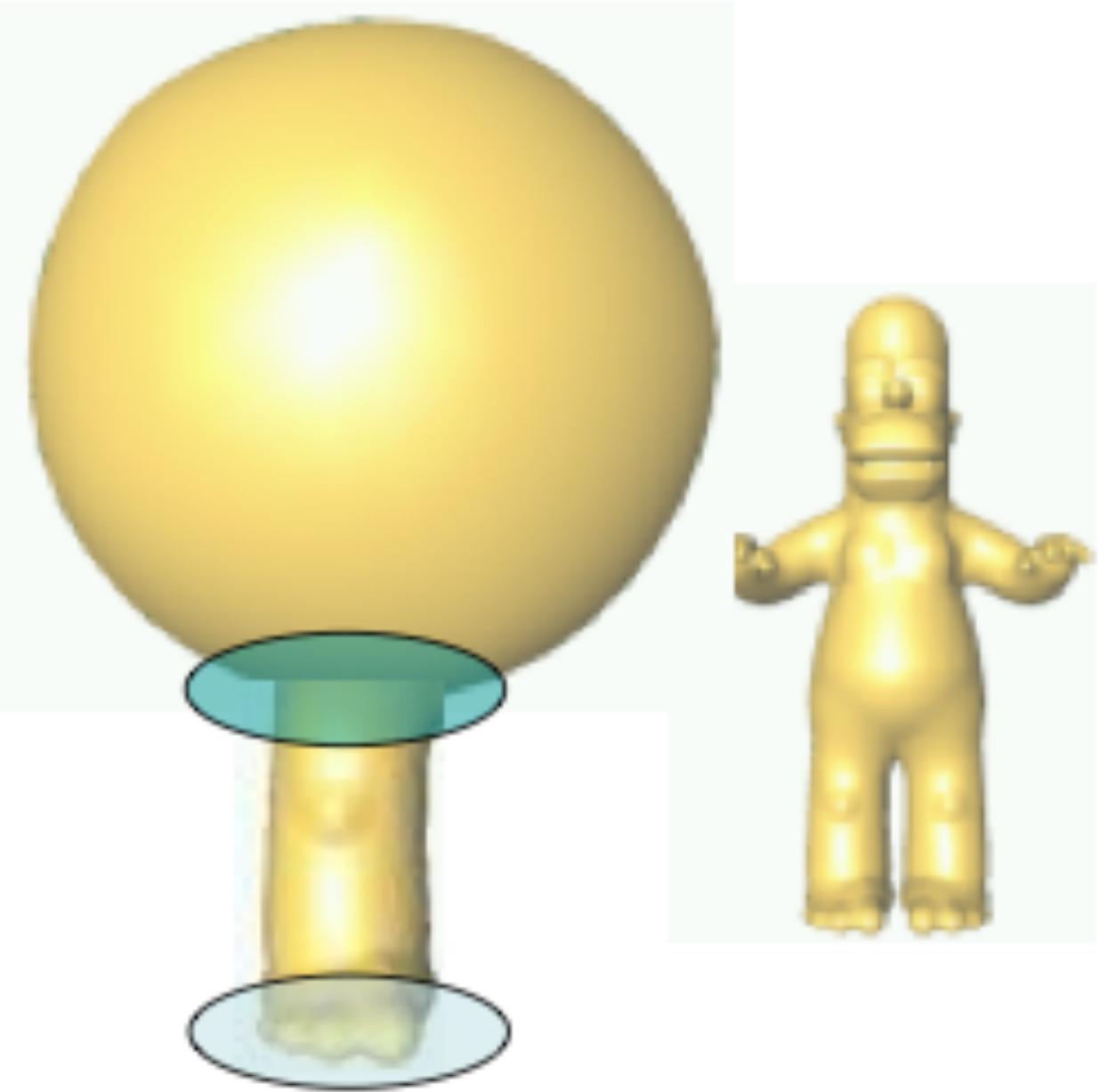
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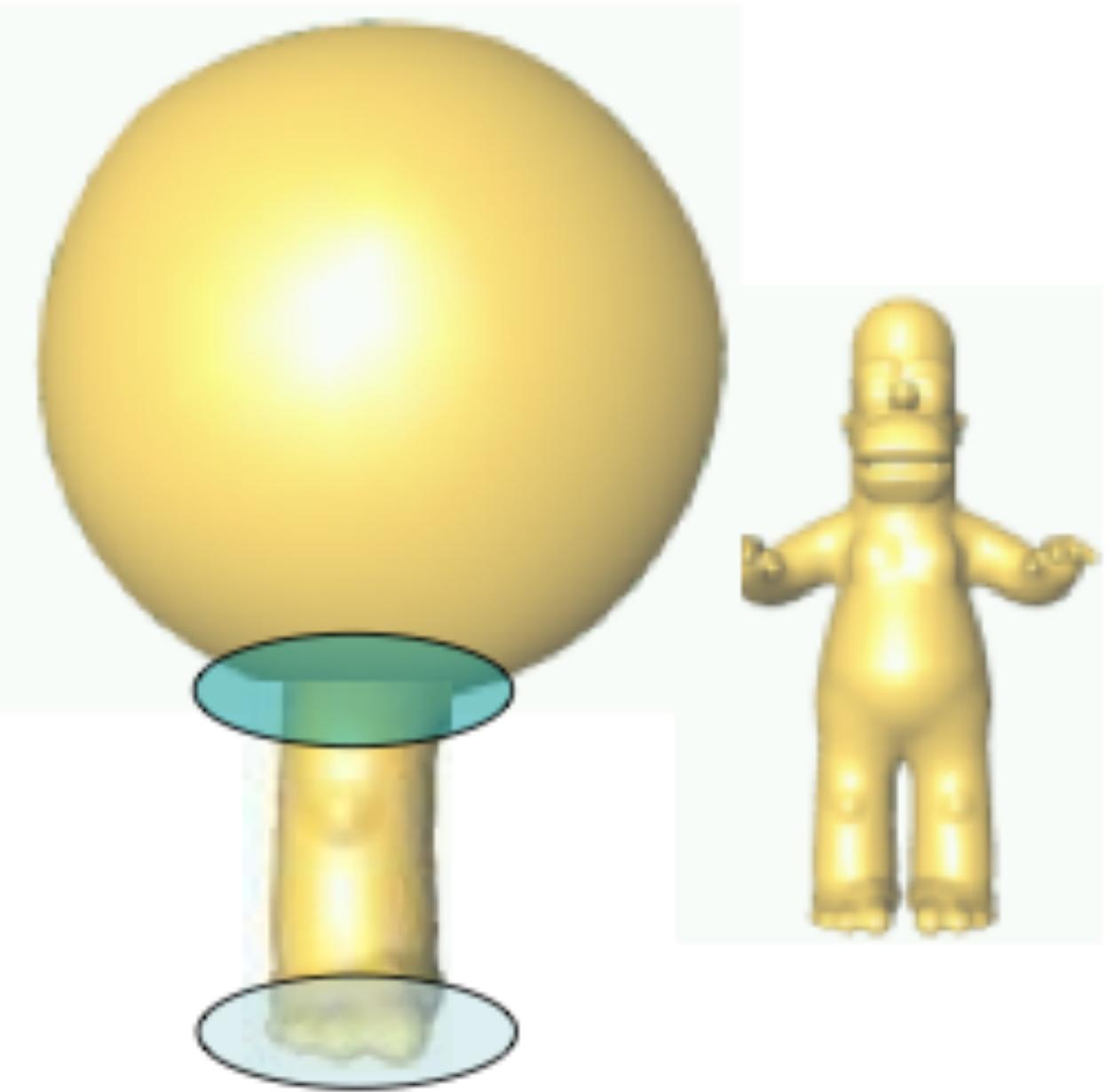
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$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$



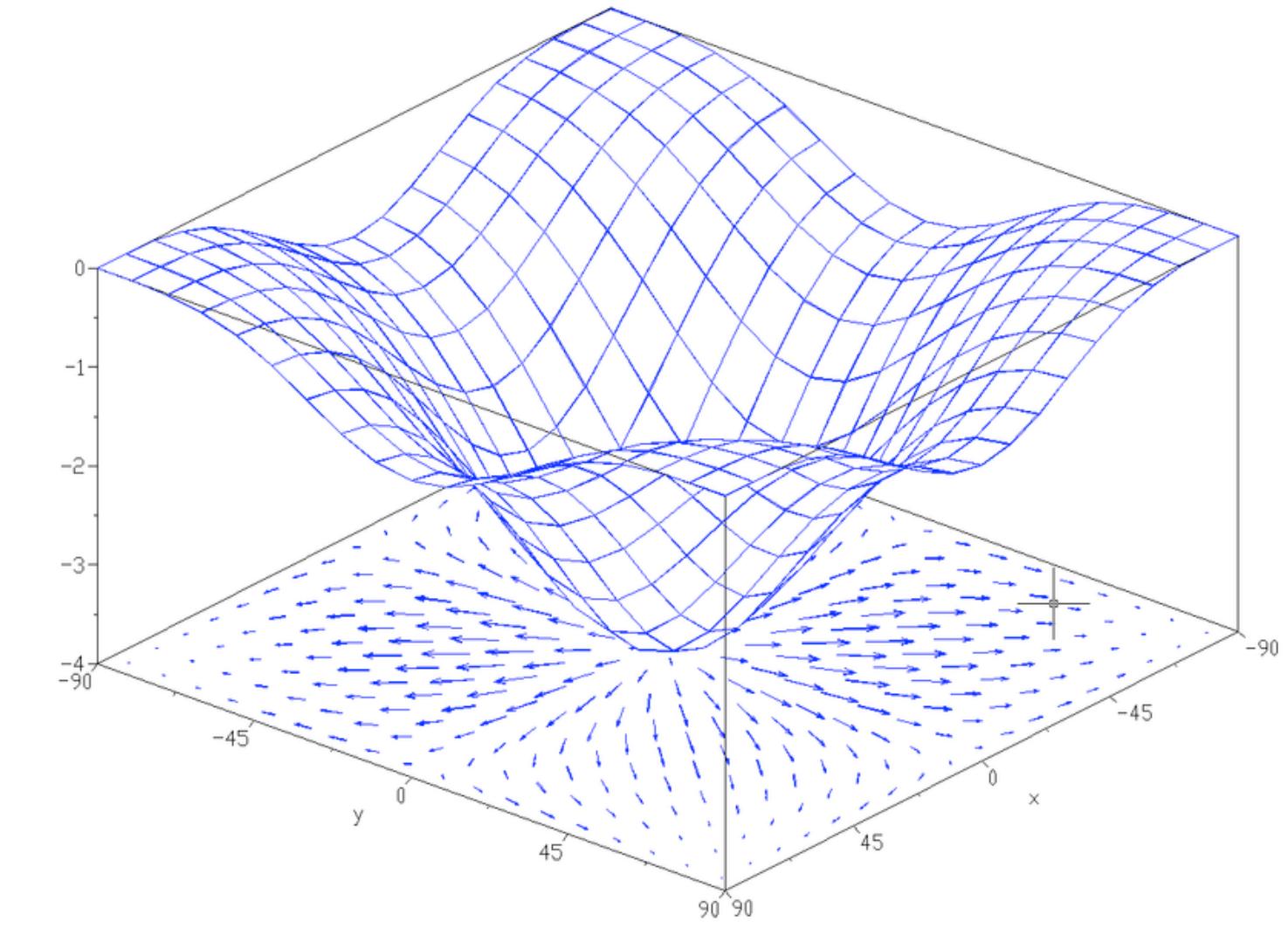
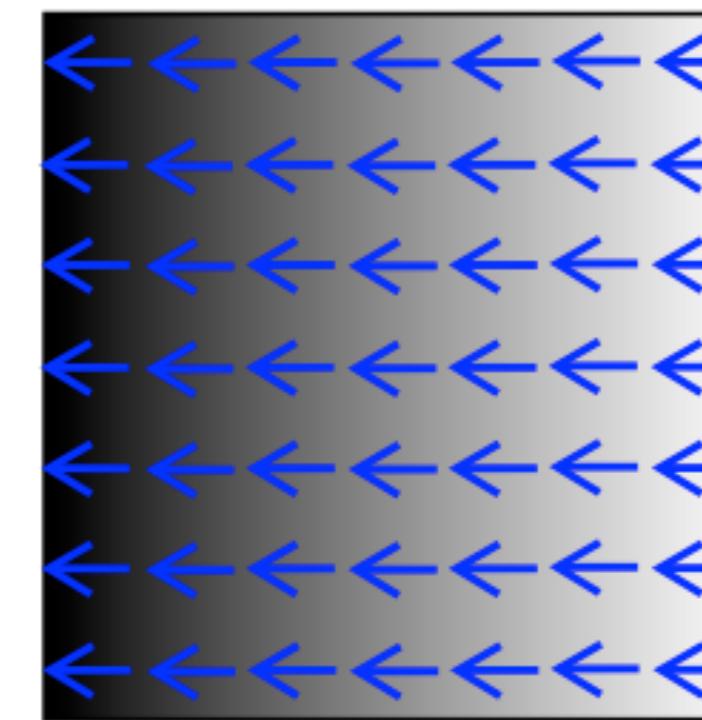
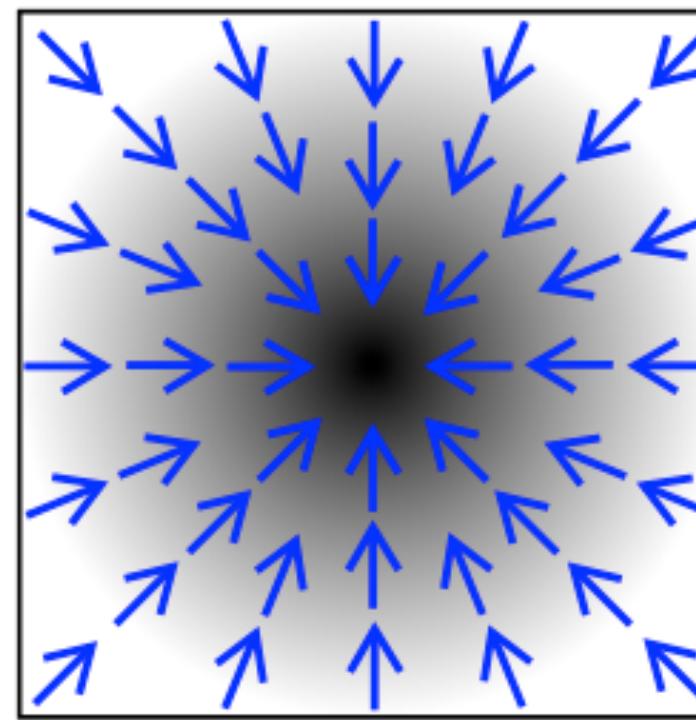
– when sphere is deformed new positive and negative curvature cancel out!

# Differential Operators

- Gradient

$$\nabla f := \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

- points in the direction of steepest ascent



# Differential Operators



- Divergence

$$\operatorname{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$

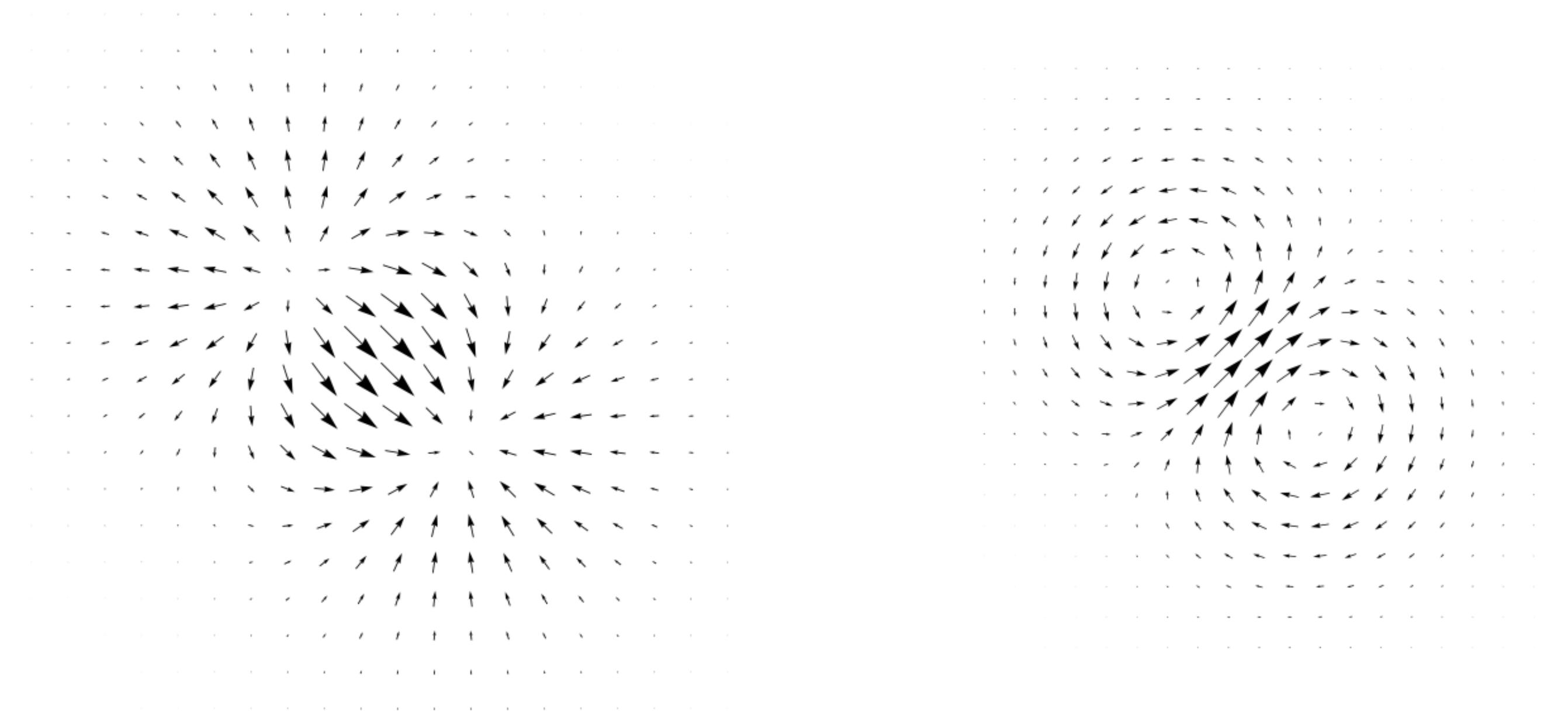
- volume density of outward flux of vector field
- magnitude of source or sink at given point
- Example: Incompressible fluid
  - velocity field is divergence-free

# Differential Operators



- Divergence

$$\operatorname{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$

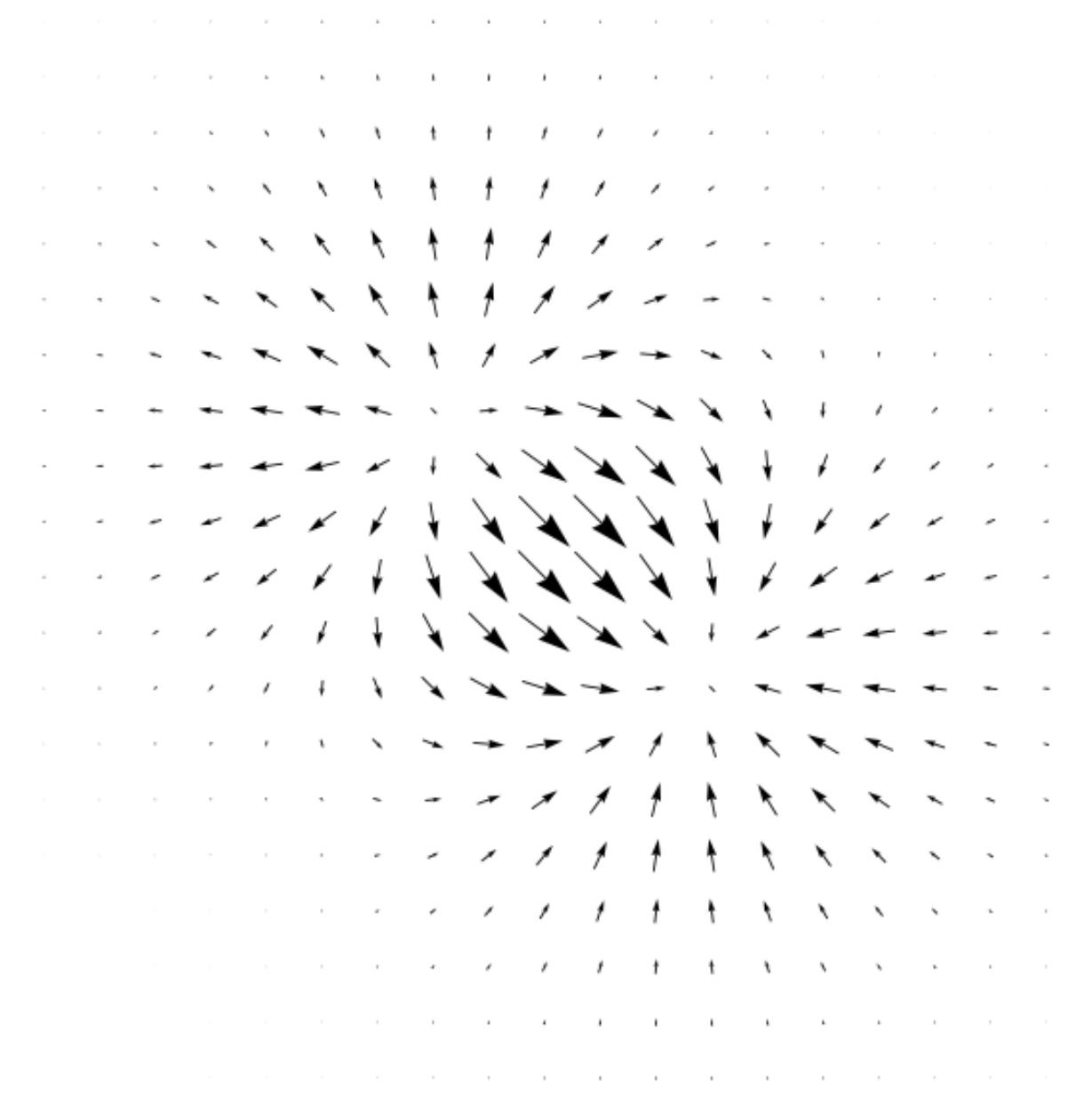


# Differential Operators

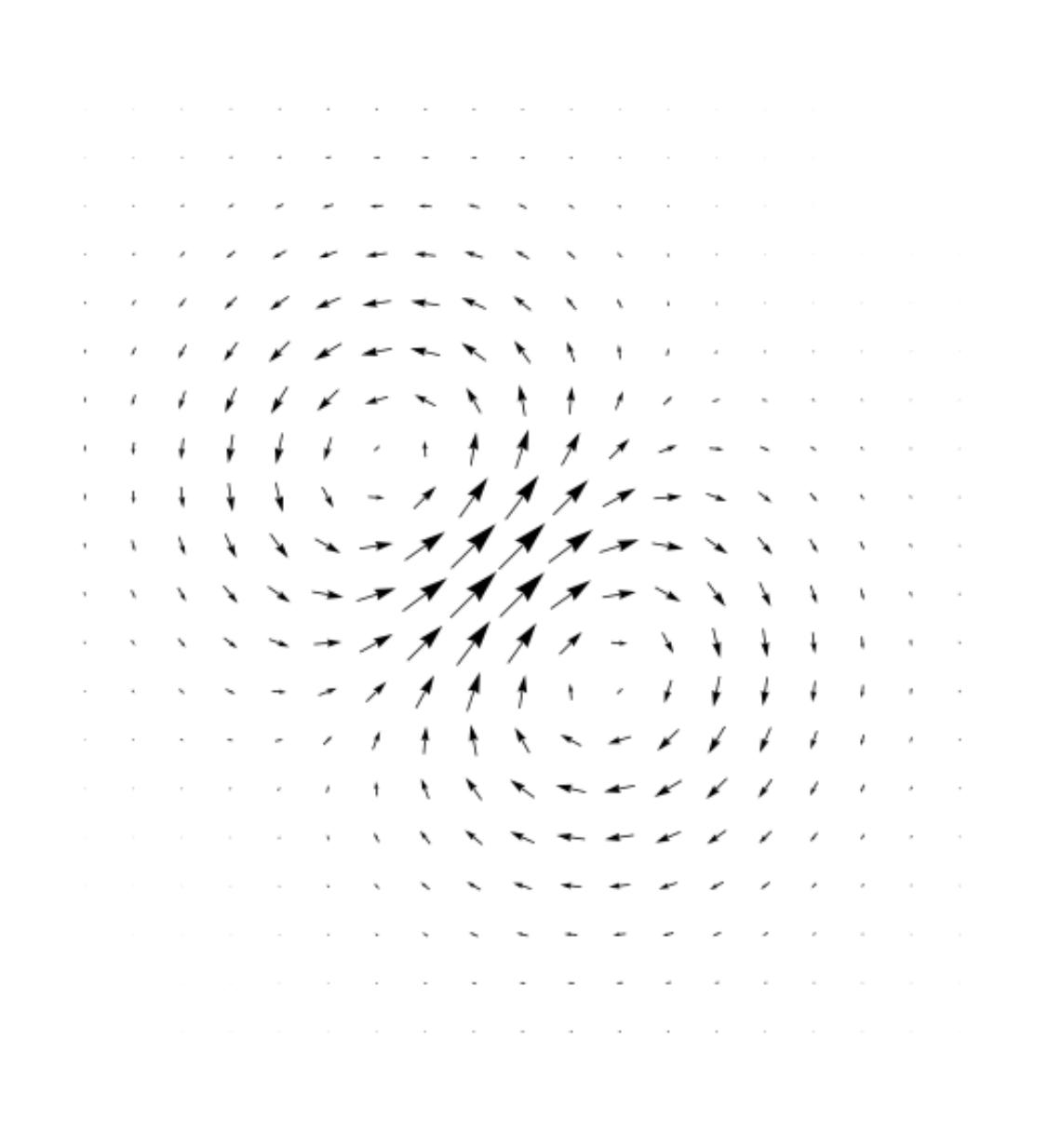


- Divergence

$$\operatorname{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$



high divergence



low divergence

# Literature



- G. Farin: ***Curves and Surfaces for CAGD***, Morgan Kaufmann, 2001.
- M. Do Carmo: ***Differential Geometry of Curves and Surfaces***, Prentice Hall, 1976.
- A. Pressley: ***Elementary Differential Geometry***, Springer, 2010