COMP0130 Robot Vision and Navigation

3A: Motion Sensing, Dead Reckoning and Inertial Navigation

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Session Objectives

- Explain the main principles of motion sensing using a variety of technologies including
  - Accelerometers
  - Gyroscopes
  - Wheel speed sensors
  - Magnetometers
  - Barometers
  - Doppler radar and sonar

- Show how to compute a navigation solution using dead reckoning
- Show how to compute an inertial navigation solution
- Explain how navigation errors arise in these techniques

Contents

1. Introduction to Motion Sensing

Limitations of GNSS

Indoor and dense urban environments

- Buildings block, attenuate and reflect signals
- Less accurate in “urban canyons” and indoors
- No reception at all deep inside buildings

Tunnels, underground and underwater

- No signal

Deliberate jamming and spoofing

- Military applications
- Road user charging avoidance
- Disrupting stolen vehicle & employee tracking

Unintentional Interference

- Communications in bands adjacent to GNSS
- Military communications in the L5/E5 band

Other Absolute Positioning Technologies

Other Radio Positioning Techniques

- Ultrawideband (UWB) *Short range, high precision, often used indoors*
- VOR and DME *Backup system used by aircraft*

Radio Signals of Opportunity

- Wi-Fi and Bluetooth Low Energy
- Mobile phone signals
- Radio and TV broadcasts (FM, DAB, DVB-T etc)


Environmental Feature Matching

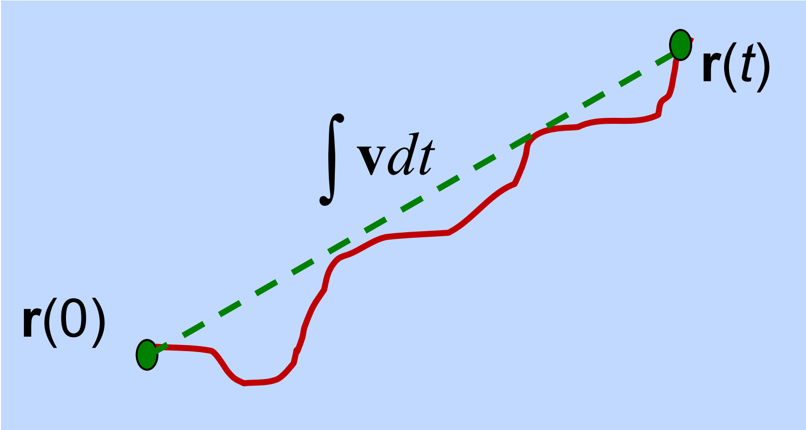
- Visual Features
- Road map matching
- Terrain referenced navigation
- Magnetic anomaly matching



All rely on availability of signals or recognisable features

1. Introduction to Motion Sensing

Dead Reckoning



Key benefit

- No external transmitters or landmarks needed

Key drawback

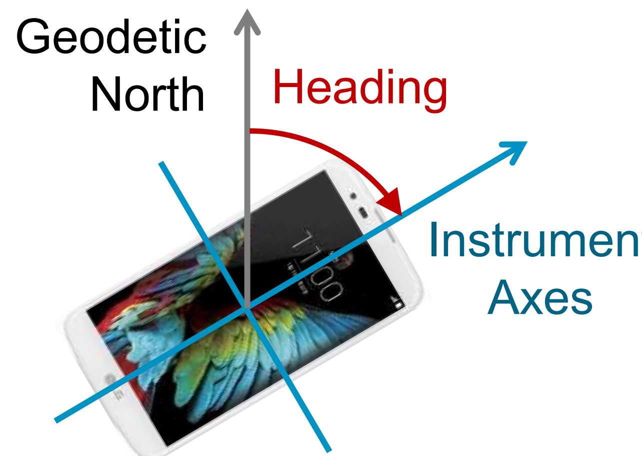
- Position error grows with time

- Measures velocity or change in position
- Integrating/summing gives the current position
- Measurements made in sensor body frame
- Need to know attitude/orientation to determine direction of motion with respect to the Earth

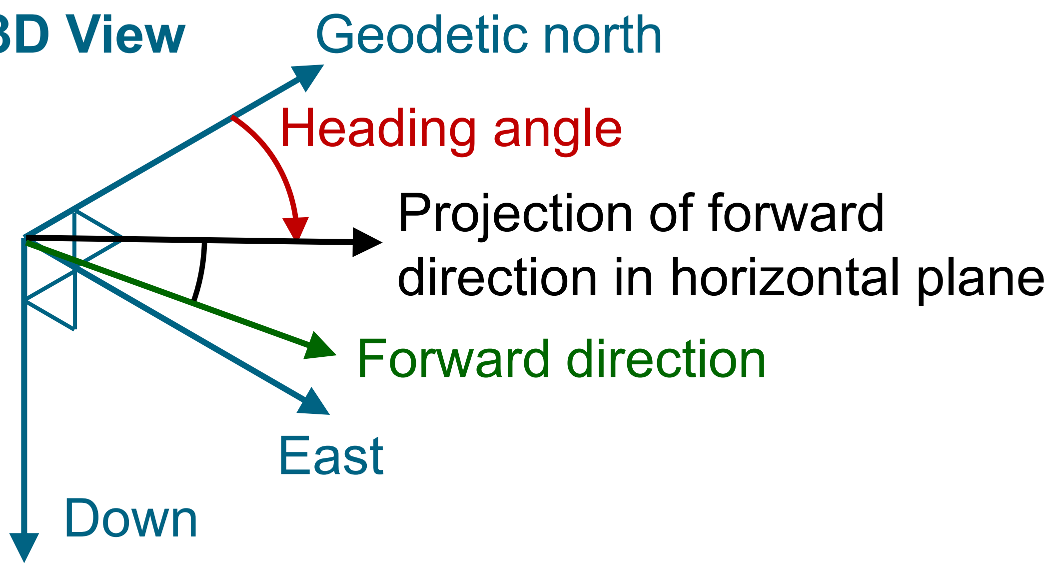
Heading Angle

Heading or **azimuth** is the clockwise angle in the horizontal plane from geodetic north to the forward axis of the instrument or vehicle.

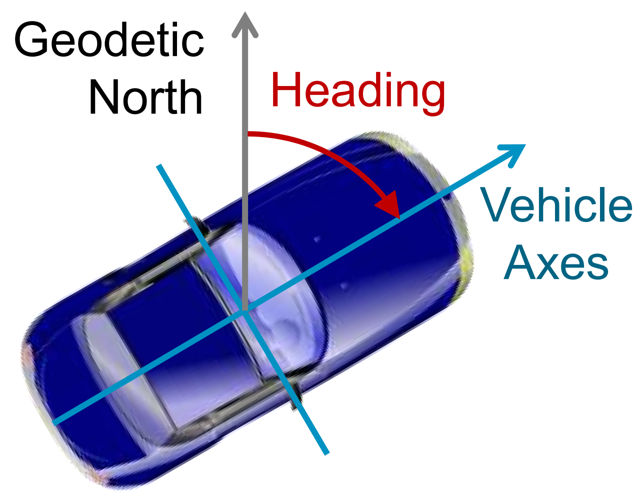
A change in heading is a **yaw** rotation.



3D View



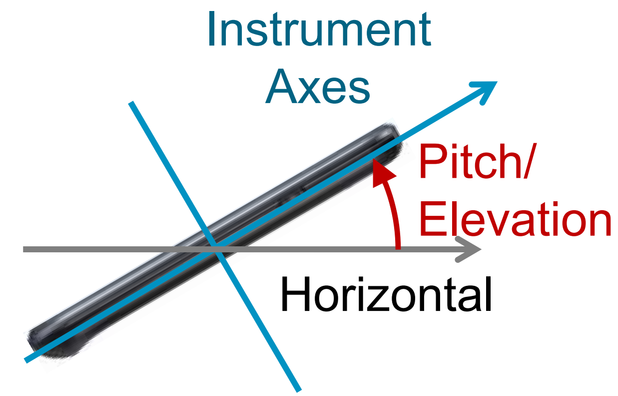
2D Views



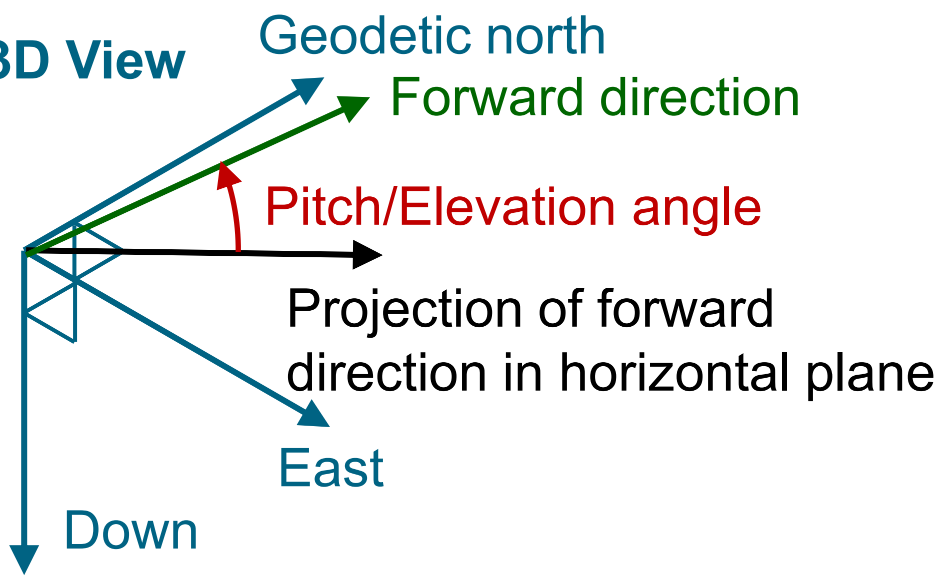
1. Introduction to Motion Sensing

Pitch/Elevation Angle

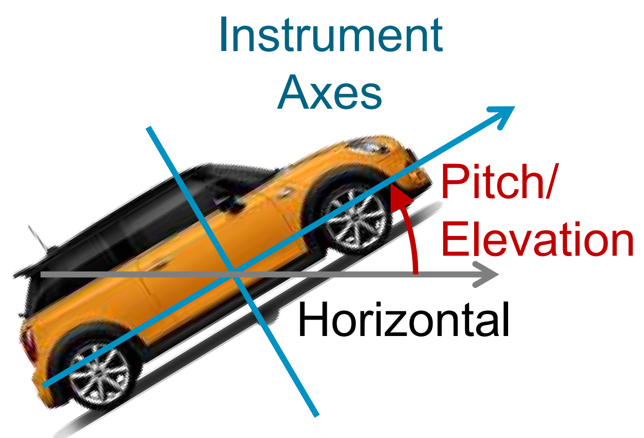
Pitch or **Elevation** is the angle of the forward axis of the instrument or vehicle above the horizontal plane.



3D View



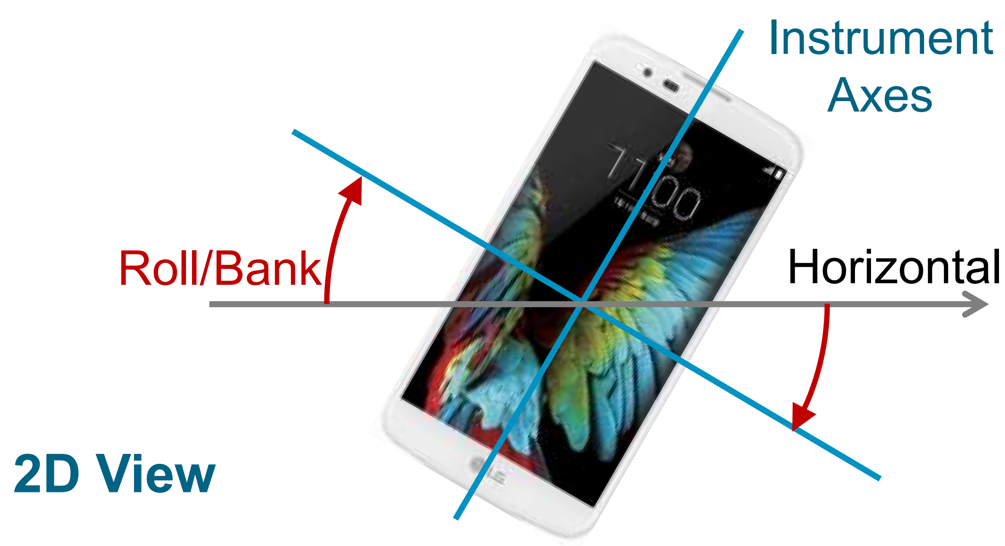
2D Views



1. Introduction to Motion Sensing

Roll/Bank Angle

Roll or **Bank** is the angle of the left-pointing axis of the instrument or vehicle above the horizontal plane and the right-pointing axis below the horizontal.



1. Introduction to Motion Sensing

Direction and Distance

Distance travelled

- Land: Wheel speed odometer
- Air: Doppler radar, LIDAR
- Water: Doppler sonar, Ship's log
- Pedestrian dead reckoning

Direction

- Aircraft requires 3D attitude (or orientation)
- Land and water vehicles can be assumed level (roll $\approx$ 0; pitch $\approx$ 0)
- Heading obtainable from
  - Magnetic compass
  - Trajectory (*from position fixing*)
  - Gyroscope (*after initialisation*)

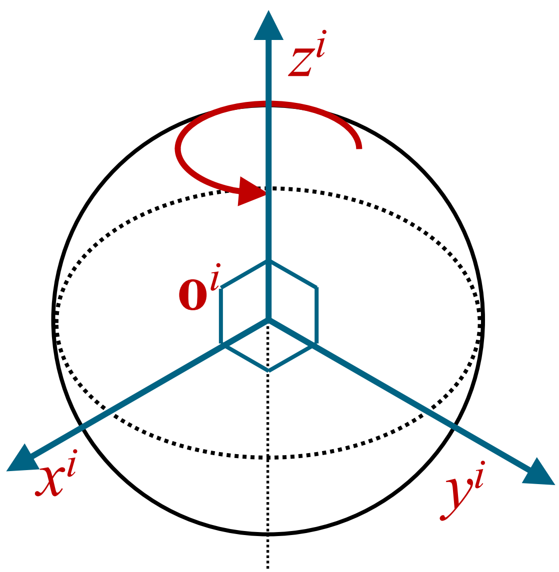
Inertial Navigation

Determines distance and direction (*after initialisation*)

Contents

2. Accelerometers

Earth-Centred Inertial (ECI) frame



Gyros and accelerometers measure motion with respect to an inertial frame

- An **inertial** co-ordinate frame does not accelerate or rotate with respect to the rest of the universe
- ECI frame approximates this
- ***z* axis** is Earth rotation axis
- ***x* and *y* axes** are fixed w.r.t. the stars

– they are fixed w.r.t. Earth in an Earth-centred Earth-fixed (ECEF) frame

- Symbol ***i***

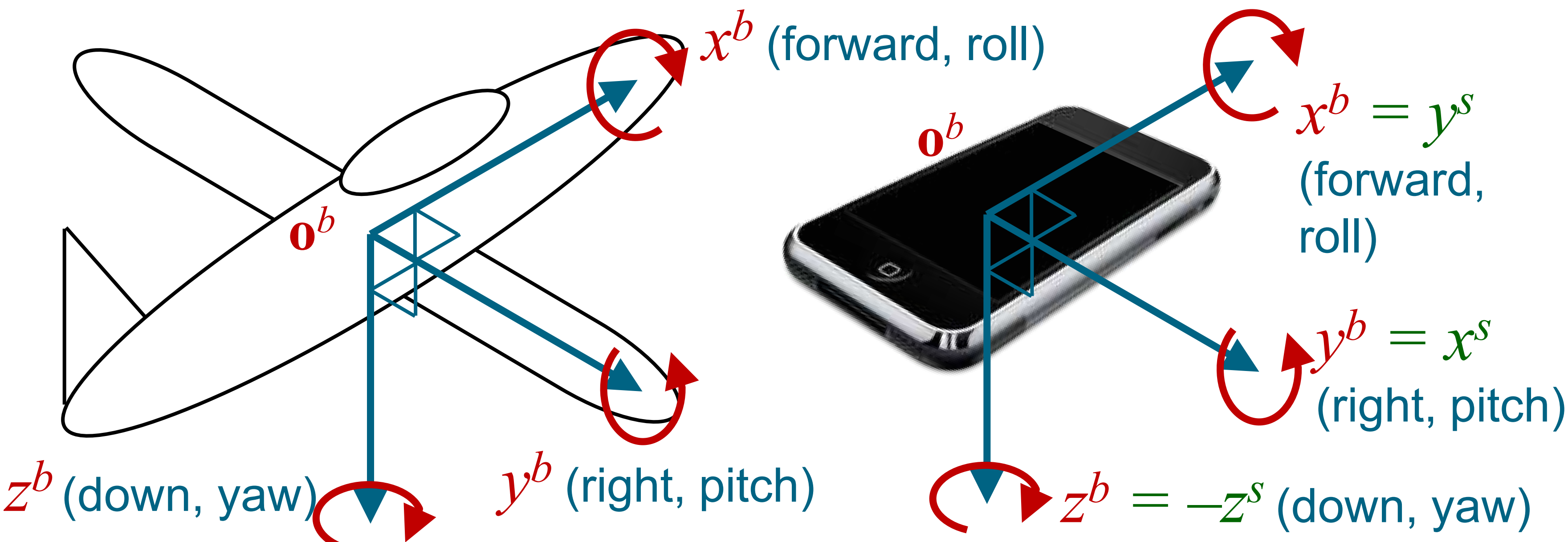
2. Accelerometers

Body frame

Gyros and accelerometers measure motion of a **body** with respect to an inertial frame.

A **body** co-ordinate frame (symbol ***b***) describes the object whose motion is measured.

Smartphone sensors (***s***) use a different convention



2. Accelerometers

Introducing Specific Force (1)

Specific force is the *nongravitational* force per unit mass on a body, ***b***, sensed w.r.t. an inertial frame, ***i***

- People and instruments sense specific force
- Gravitation (the mass-attraction force) is not sensed because it acts equally on all points, causing them to move together
- Other forces are sensed as they are transmitted from point to point
- The sensation of weight is caused by forces opposing gravitation

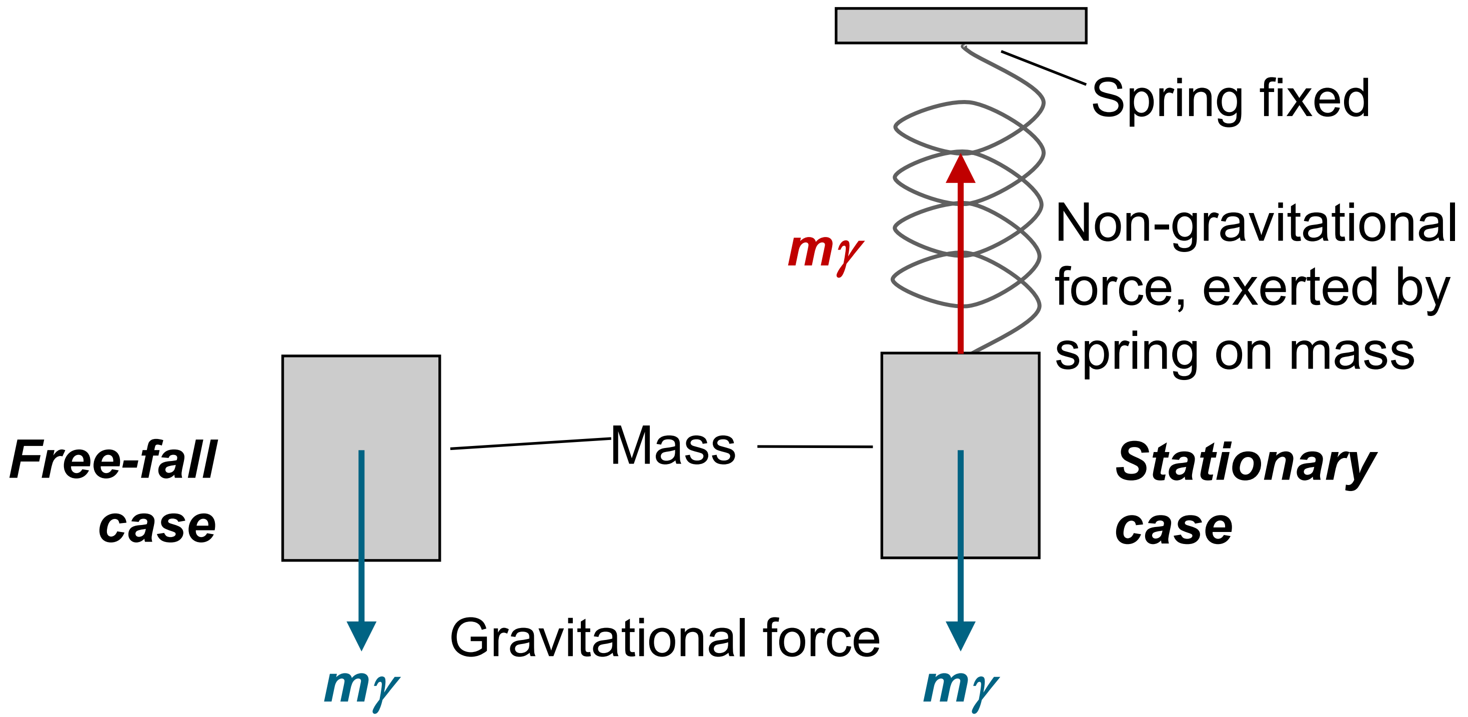
$$\mathbf{f}_{ib}^{\gamma} = \mathbf{a}_{ib}^{\gamma} - \boldsymbol{\gamma}_{ib}^{\gamma}$$

Specific force **Inertially-referenced acceleration** **Acceleration due to gravitational force**

γ is the resolving frame

2. Accelerometers

Introducing Specific Force (2)

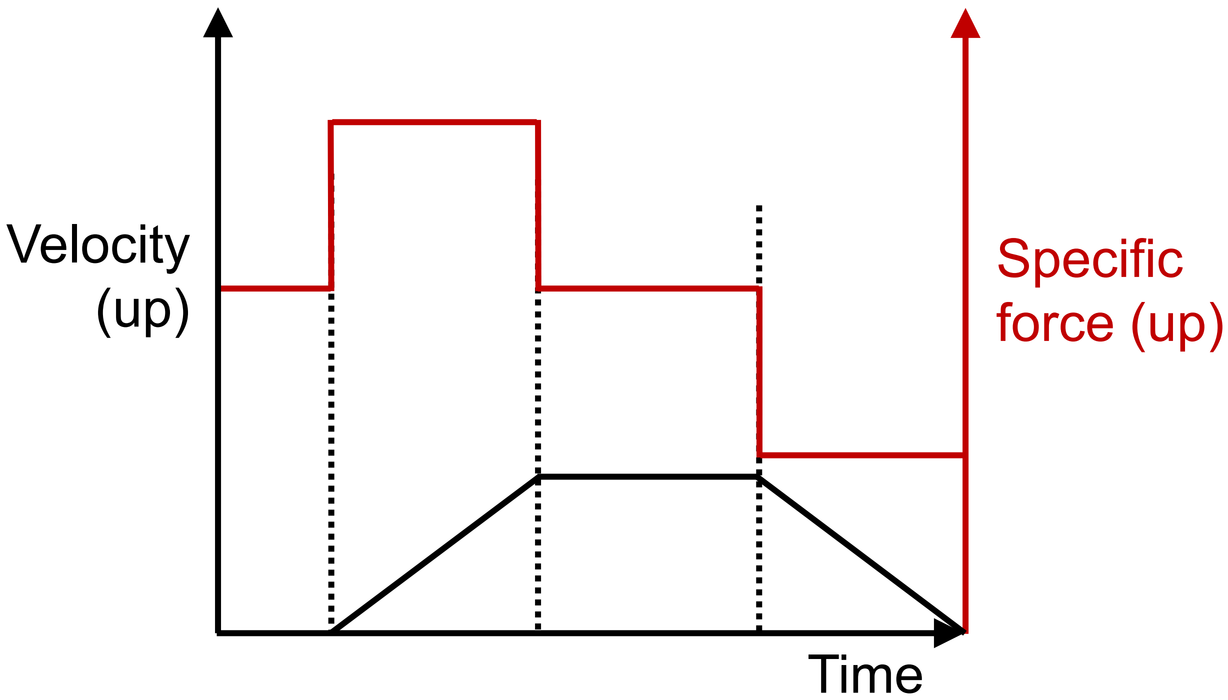


Free-fall case	Stationary case
Downwards acceleration = γ	**Downwards acceleration = 0**
Downwards specific force = 0	**Downwards specific force = $-\gamma$**

2. Accelerometers

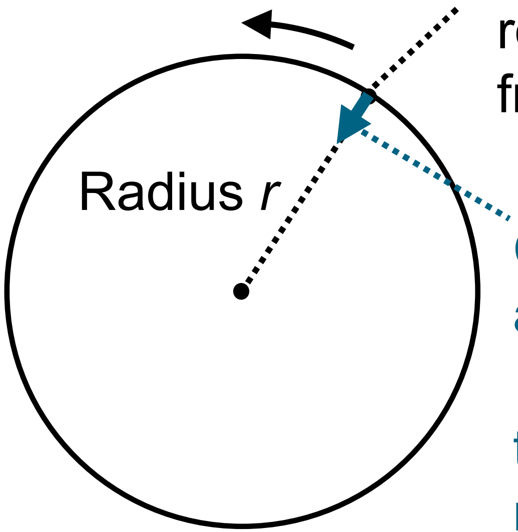
Introducing Specific Force (3)

Example: *a lift moving up*



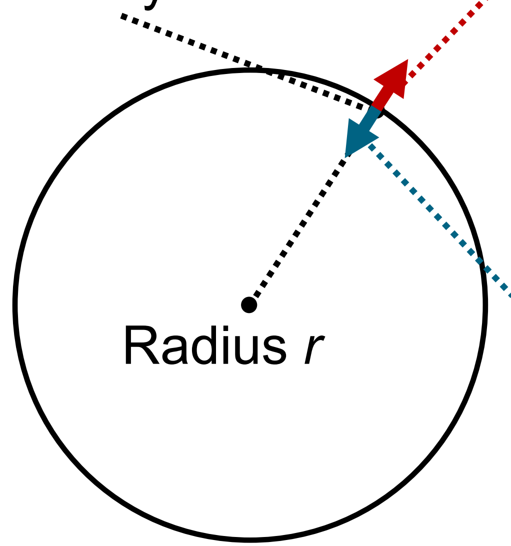
Gravity and Rotating Reference Frames (1)

Inertial frame perspective



Centripetal acceleration is a real force

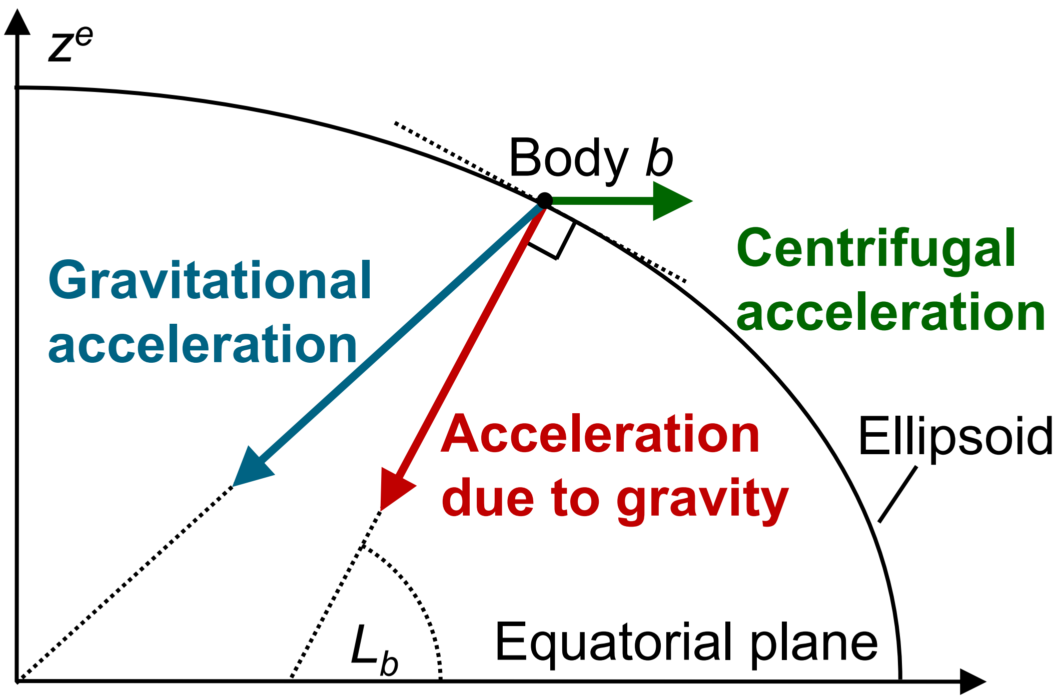
Rotating frame perspective



Centrifugal acceleration is a virtual force (or pseudo-force)

2. Accelerometers

Gravity and Rotating Reference Frames (2)



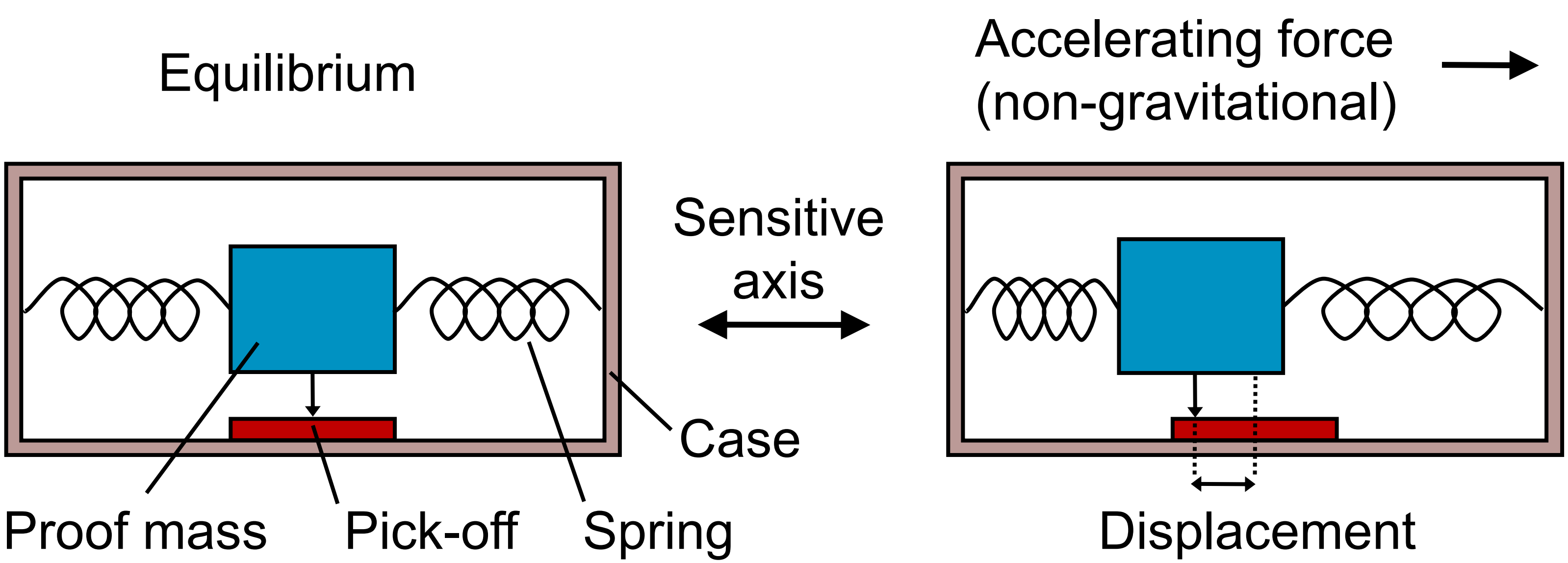
Specific force → $$\mathbf{f}_{ib}^{\gamma} = \mathbf{a}_{eb}^{\gamma} - \mathbf{g}^{\gamma}$$ ← Acceleration due to gravity

Earth-referenced acceleration

2. Accelerometers

Simple Accelerometer

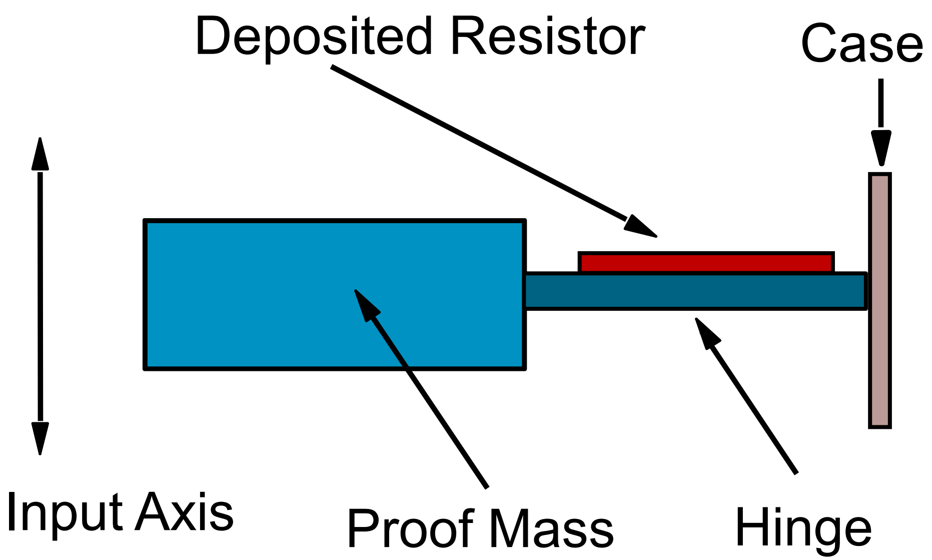
Accelerometers: measure the specific force of the device body frame with respect to an inertial frame, $\mathbf{f}_{ib}^{b}$



2. Accelerometers

Low-cost Pendulous Accelerometer

Strain gauge pick-up



Micro-electromechanical systems (MEMS) technology

- Enables mass production

- Small and light

- Low performance

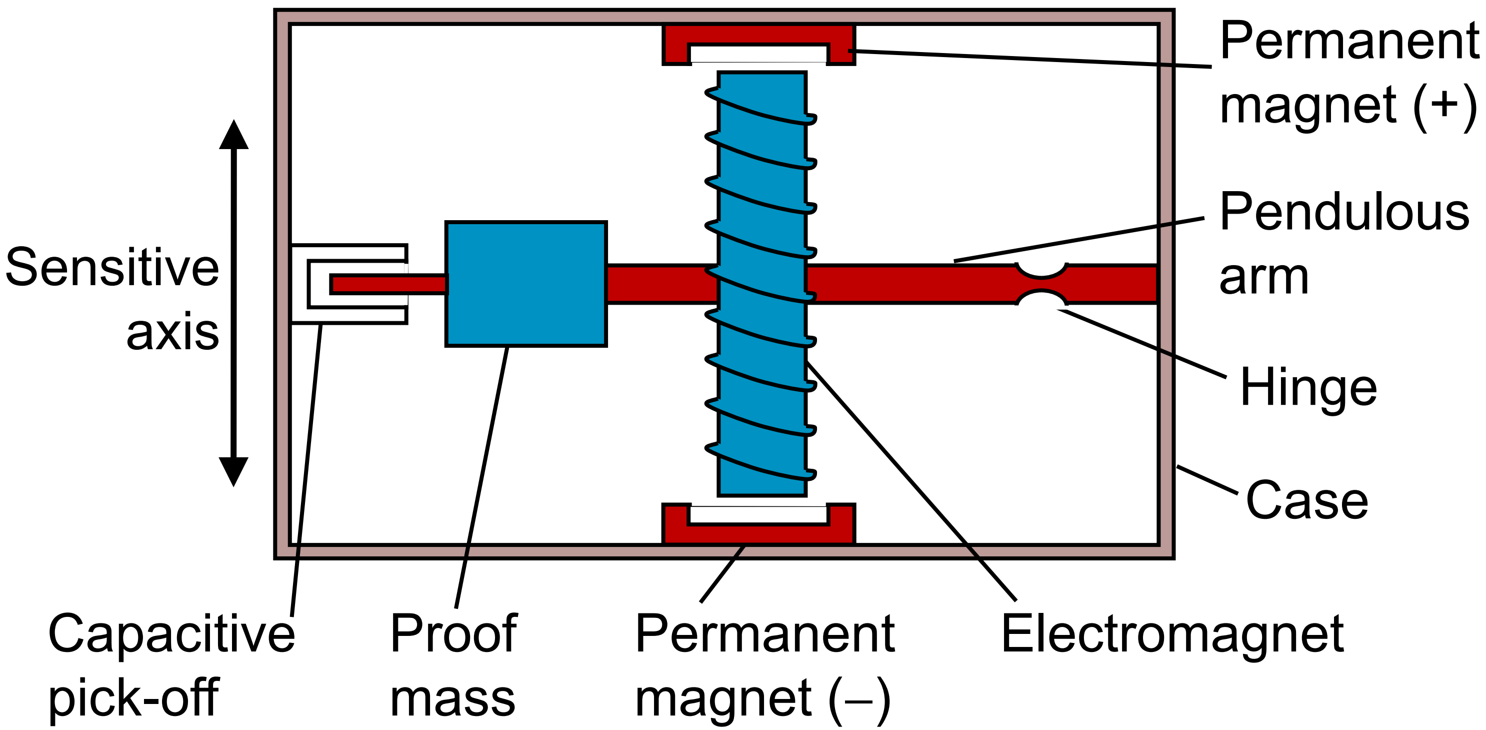
Can also have a capacitive pick-up

Hinge bends under force

- Resistor stretches and compresses, changing resistance
- Hinge also provides resistance, acting as a spring

2. Accelerometers

Force-feedback Pendulous Accelerometer



High performance but expensive

2. Accelerometers

Error Sources

Noise:

- Electrical noise, vibration resonance
- Integrating accelerometer random noise produces a velocity random walk
- Random walk error SDs vary as $\sqrt{\text{time}}$
- *Can not be calibrated*

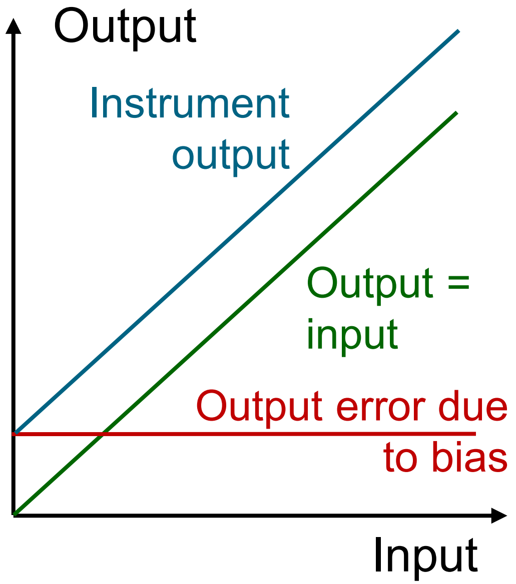
Systematic Errors:

- Constant, turn-on and slowly time-varying components
- All of these vary with temperature
- *Constant components can be calibrated in a laboratory*
- *Turn-on and slowly time-varying components can be calibrated using other sensors, including GNSS*

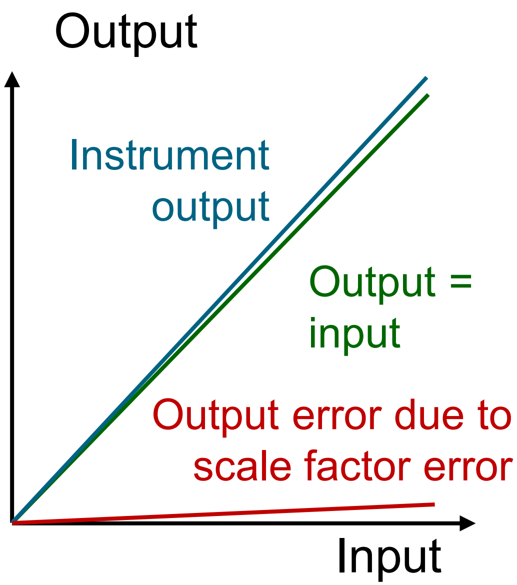
2. Accelerometers

Sensor Errors

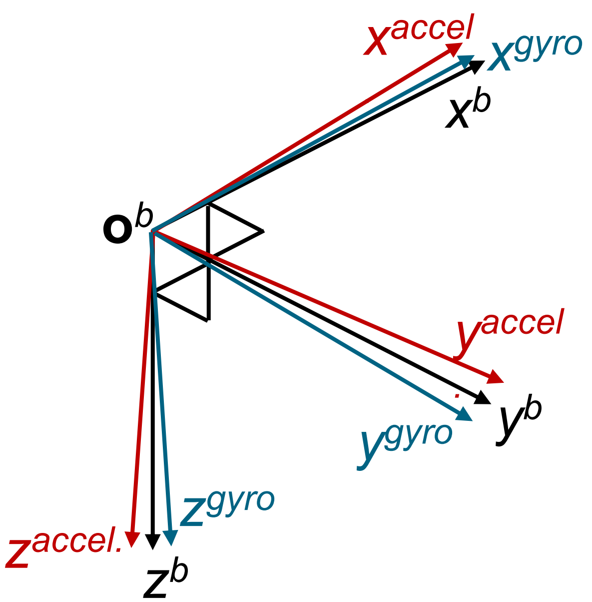
Bias



Scale factor error



Cross-coupling error



These systematic errors can be calibrated

- When the system is stationary, output = reaction to gravity + bias + noise
- Can also compare outputs with specific force determined by other means

Contents

3. Gyroscopes

Gyroscopes

Gyroscopes (gyros) measure the angular rate of a **body** with respect to an **inertial** frame

Optical:

- Ring Laser Gyro (RLG)
- Interferometric Fibre-optic Gyro (IFOG)
- Medium or High performance

Vibratory:

- MEMS – Low performance
- Or Quartz – Medium performance

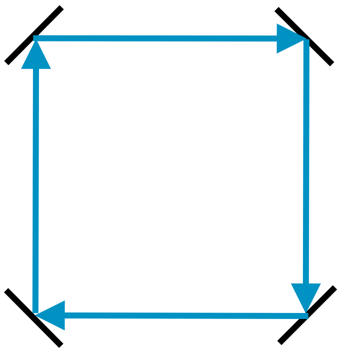
Spinning mass:

- Traditional technology, *now limited to specialist applications*
- Medium or High performance

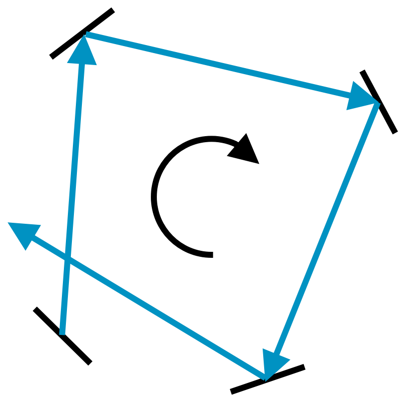
3. Gyroscopes

Optical gyros – The Sagnac Effect

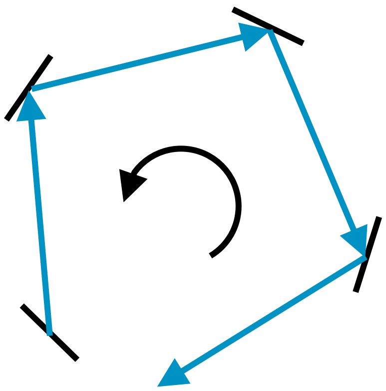
Four mirrors



No rotation



Rotation in same direction as light – path length increases

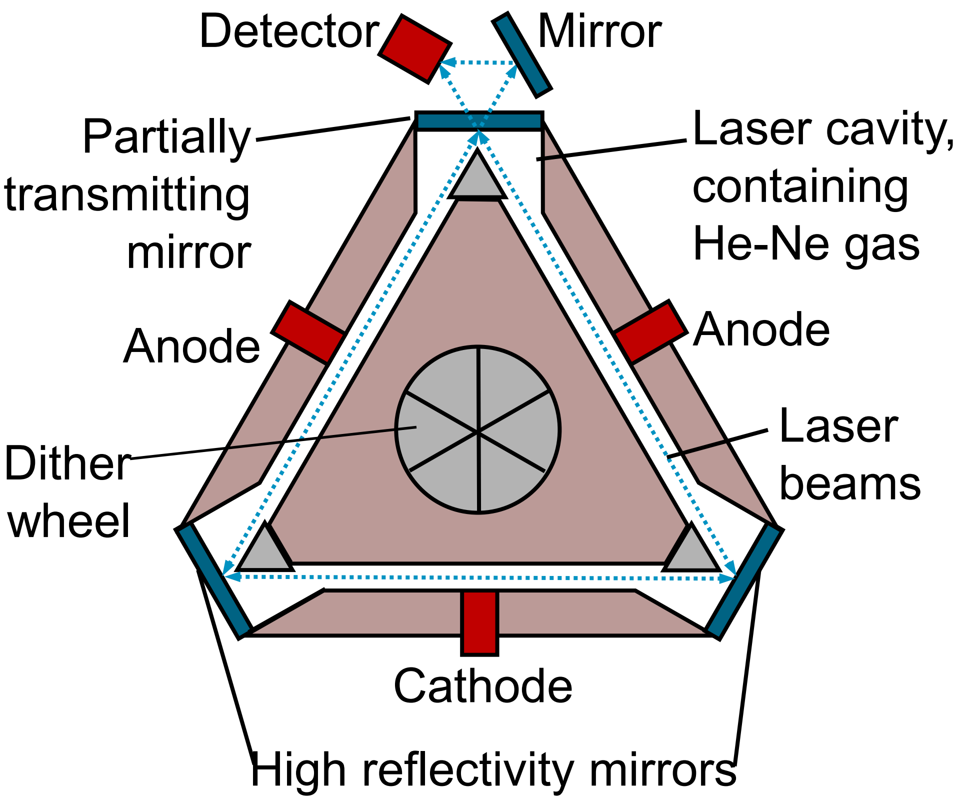


Rotation in opposite direction to light – path length decreases

Inertial frame perspective – speed of light is constant

3. Gyroscopes

Ring Laser Gyro (RLG)



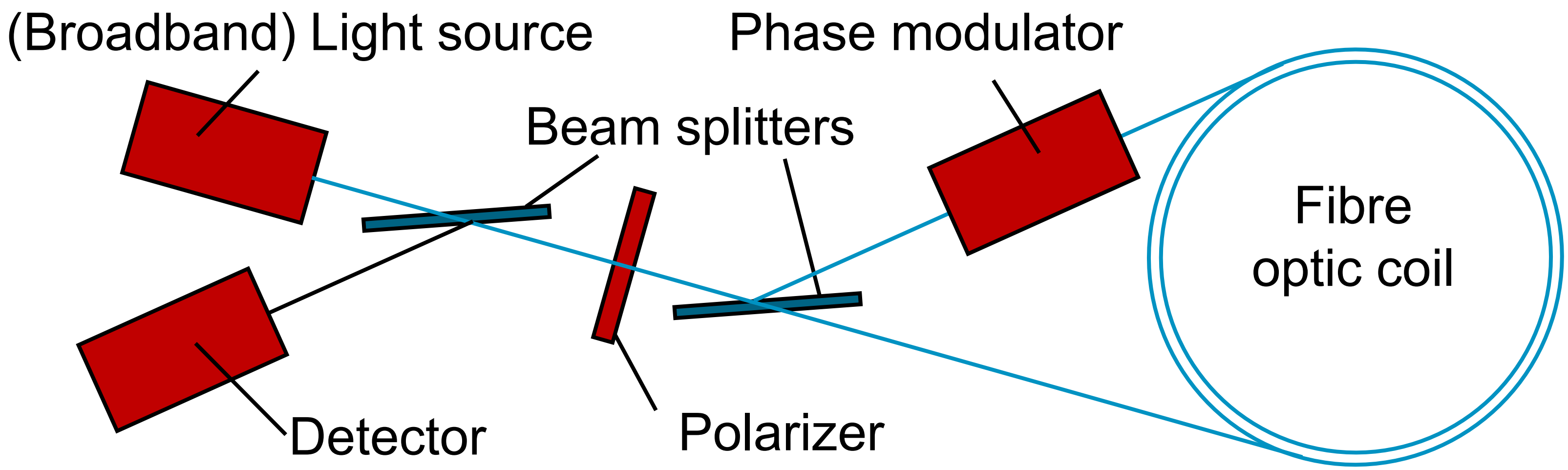
Two resonant lasing modes

- One in each direction
- These differ when laser is rotated
- Beat frequency proportional to angular rate
- Measured at detector

Honeywell GG1320
0.04 °/hr drift
8.6 cm diameter

3. Gyroscopes

Interferometric Fibre-optic Gyro (IFOG)

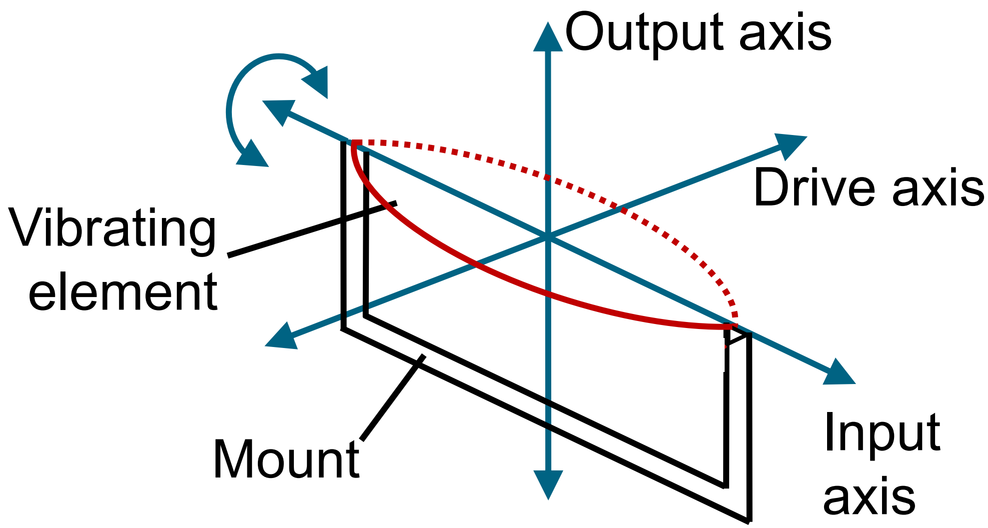


- Light sent in both directions around coil
- Phase difference between light output from the two directions is proportional to angular rate

LITEF LFK95
500 m coil
1 °/hr drift
8 cm
diameter

3. Gyroscopes

Vibratory Gyro



- Driven vibratory element
  - Can be string, beam, 2 beams, tuning fork, ring, cylinder, hemisphere
- Coriolis effect produces vibration ⊥ driven vibration & sensor rotation axis

3. Gyroscopes

Error Sources

Noise:

- Electrical noise, vibration resonance
- Integrating gyro random noise produces an attitude random walk
- Random walk error SDs vary as $\sqrt{\text{time}}$
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Systematic Errors:

- Constant, turn-on and slowly time-varying components
- All of these vary with temperature
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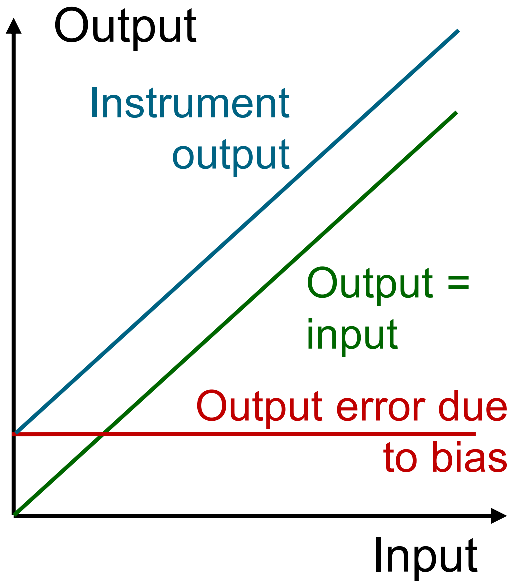
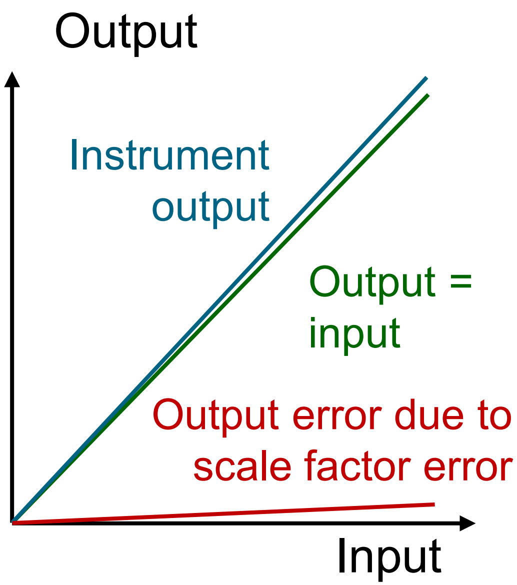
3. Gyroscopes

Systematic Errors

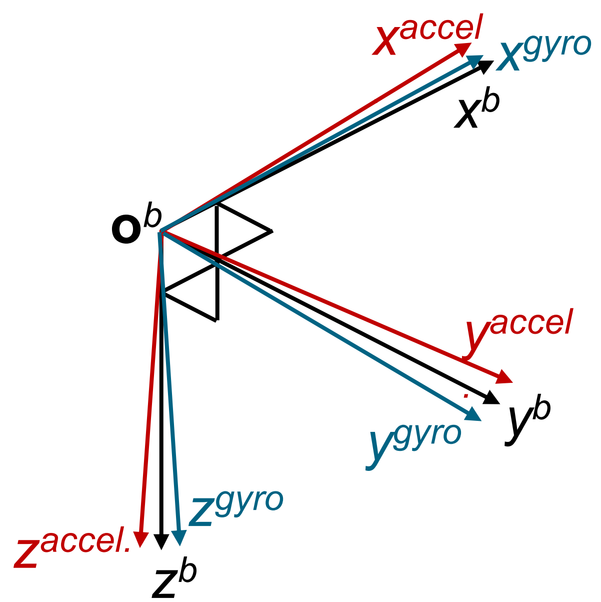
Bias



Scale factor error



Cross-coupling error



These systematic errors can be calibrated

- When the system is stationary, sensor output = bias + noise
- Can also compare Gyro outputs with rotations measured by other means

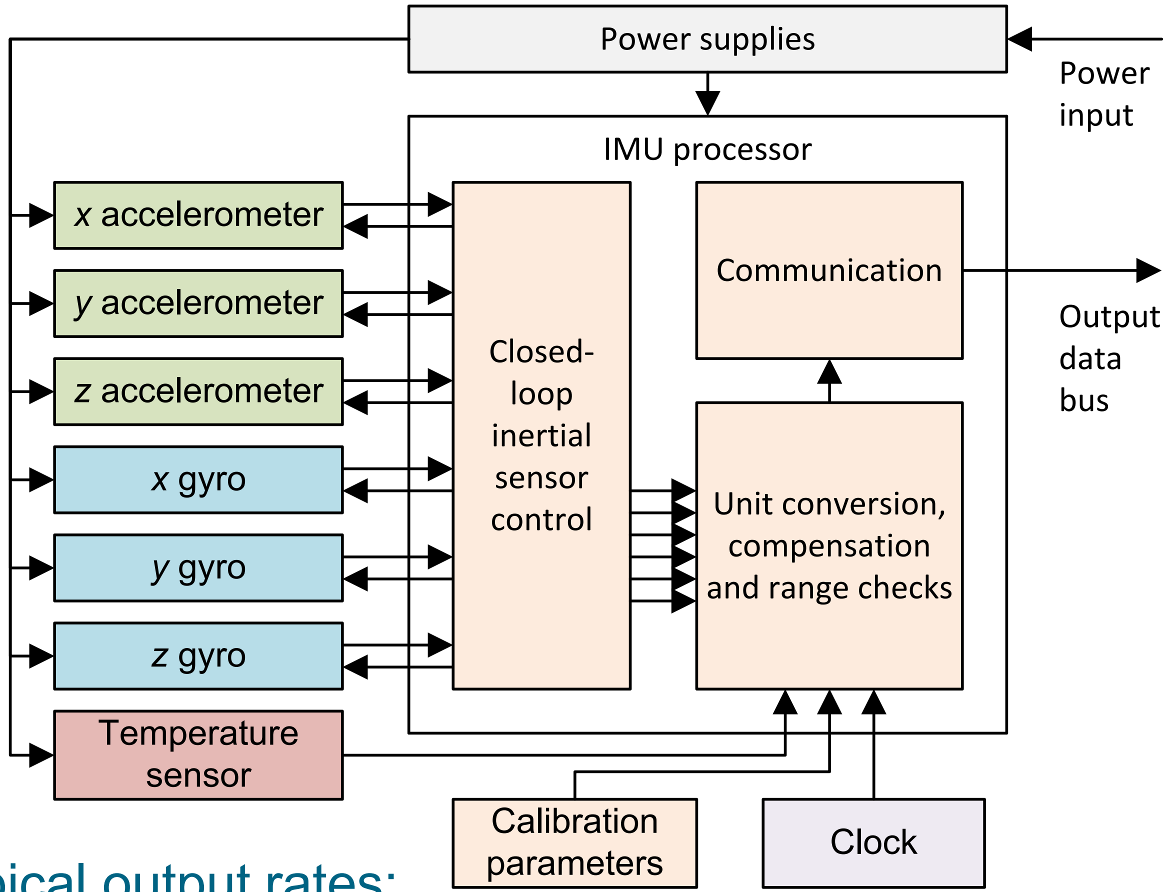
Contents

4. Inertial Measurement Units and their Applications

Inertial Measurement Unit (IMU)

Combines accelerometers with gyroscopes

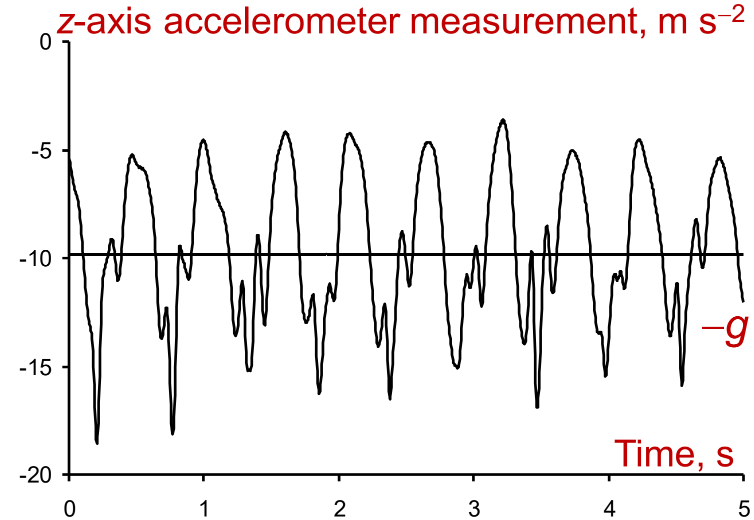
Image: University of Michigan

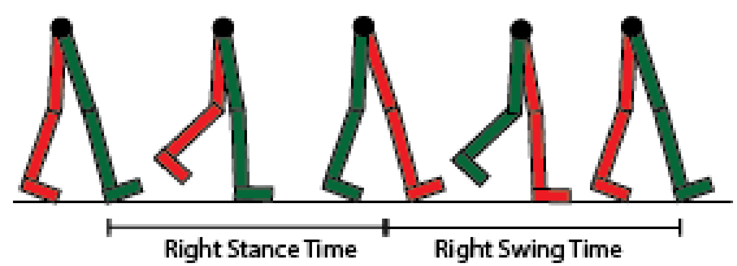


Typical output rates:
100 Hz – 1 kHz

Pedestrian Dead Reckoning (PDR) by Step Detection

- Sensors can be anywhere on the body
  Back pack, belt-mounted, hand-held
- Steps detected from vertical or resultant accelerometer signal
- Accelerometer variance and step frequency used to estimate step length
- Accuracy: 2–10% of distance travelled
- Largely insensitive to sensor quality
- Can distinguish between running, walking and other behaviour
- Best performance requires calibration to the individual user,
  e.g. through GNSS integration



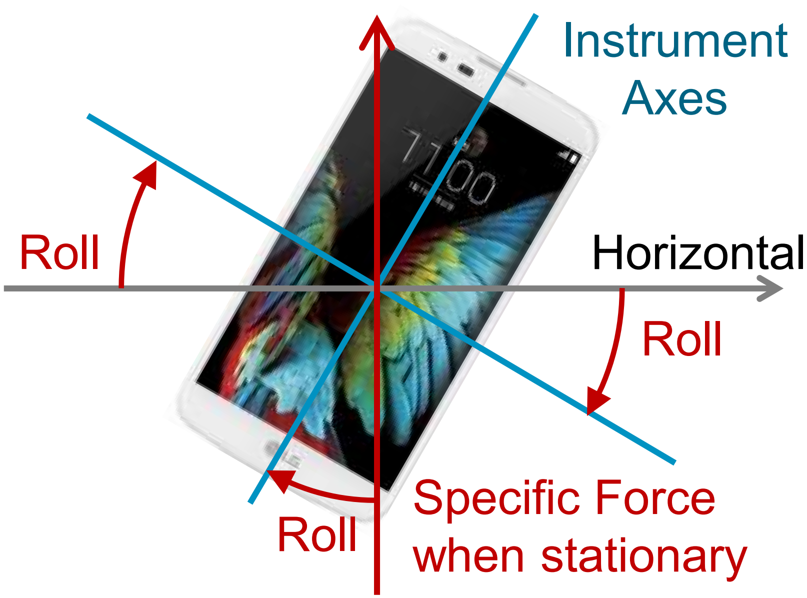


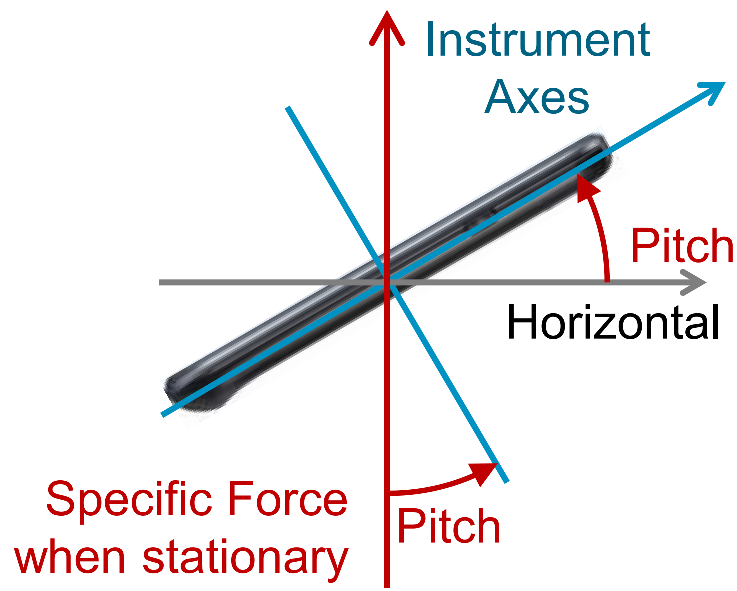
Device Orientation from Specific Force

Roll and **pitch** describes the orientation of a body with respect to **gravity**

When a device is stationary, **specific force** is equal and opposite to **acceleration due to gravity**

Measuring specific force enables these angles to be deduced





Orientation from Specific Force – “Levelling”

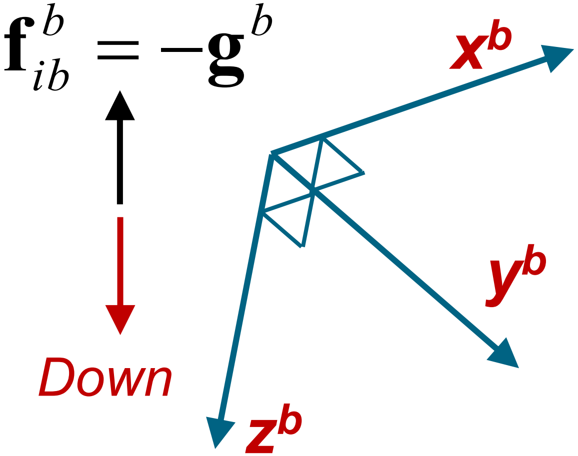
For a **stationary device**, the sensed specific force is due to **reaction to gravity** only

$$\boxed{\tilde{\mathbf{f}}_{ib}^{b}} = \tilde{\mathbf{C}}_{n}^{b} \begin{array}{cl} \left(\begin{array}{c} 0 \\ 0 \\ -\boxed{g(\hat{L}_b, \hat{h}_b)} \end{array}\right) & \begin{array}{l} \text{North} \\ \text{East} \\ \text{Down} \end{array} \end{array}$$

Specific force measured in body frame

Acceleration due to gravity

$$\mathbf{f}_{ib}^{b} = -\mathbf{g}^{b}$$

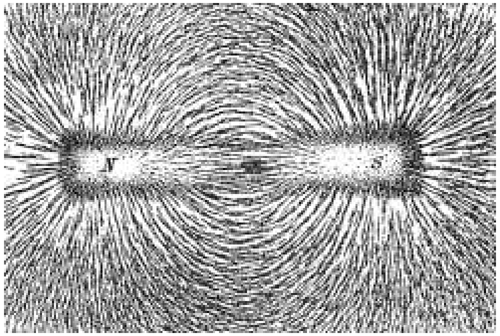


- Solving gives **pitch**, *θ*, and **roll**, *ϕ*, attitude, *but not heading*
- 10^{-3} g measurement accuracy → 1 mrad levelling accuracy
- **Levelling does not work when the device is accelerating**

Contents

5. Magnetometers and Barometers

3-Axis Magnetometer



- Three sensors measure magnetic flux density in three perpendicular directions
- A **compass** determines heading from the direction of the magnetic field projected onto the horizontal plane
- If the magnetometer is level…

Horizontal magnetometer measurements

Heading of device relative to direction of magnetic field

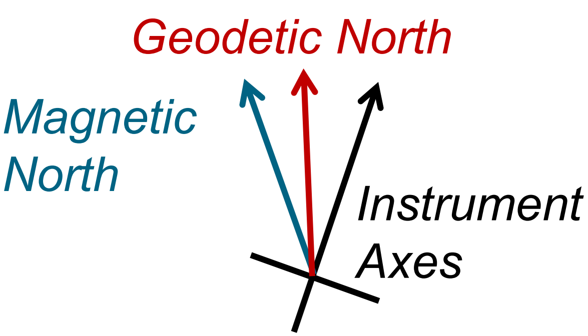
$$\boxed{\tilde{\psi}_{mb}} = \arctan_2\left(-\boxed{\tilde{m}_y, \tilde{m}_x}\right)$$

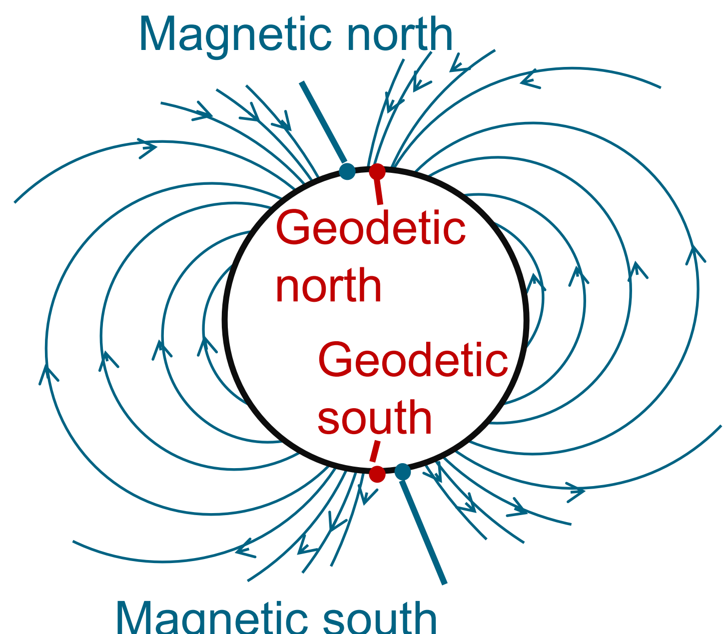
- Otherwise, we must mathematically rotate the sensor measurements
  - This needs a bank and elevation solution

5. Magnetometers and Barometers

Geodetic and Magnetic Heading

- The heading of the compass or 3-axis magnetometer with respect to magnetic north is measured
- A model of the Earth's magnetic field and the position solution is used to convert this to heading w.r.t. true north
- Model must be kept up to date as the Earth's magnetic field moves. An up-to-date model is accurate to ~0.5°



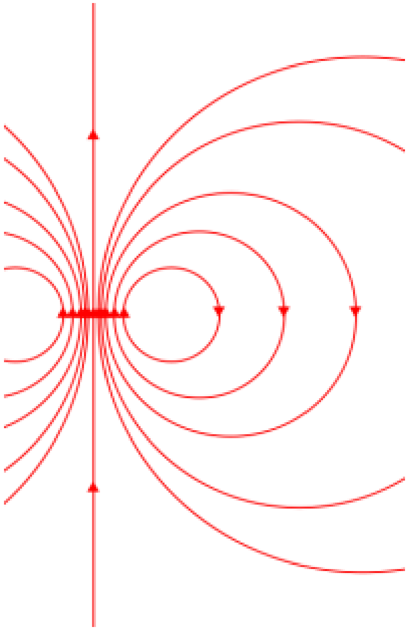


5. Magnetometers and Barometers

Errors due to Equipment Magnetism

Magnetic heading computation assumes only the Earth's magnetic field is measured

- Errors occur when this is not the case



Equipment magnetism errors

- Caused by the host equipment/ vehicle
- Hard-iron magnetism produces a constant field in the sensor body frame
  - Unless equipment is switched on or off
- Soft-iron magnetism distorts the Earth's field
- Both can be calibrated by "swinging"
  - Measure the magnetic field with different sensor orientations at the same location
  - Set calibration parameters to give a constant magnitude

5. Magnetometers and Barometers

Errors due to Magnetic Anomalies

Magnetic heading computation assumes only the Earth's magnetic field is measured

- Errors occur when this is not the case

Environmental magnetic anomalies

- Caused by other vehicles, bridges, lampposts, power lines *etc*
- Can be significant several metres away
- Mask and distort measurements of the Earth's field
- NOT practical to calibrate

Other ways of sensing heading are needed




Barometer

Measures **pressure**

Atmospheric pressure varies with

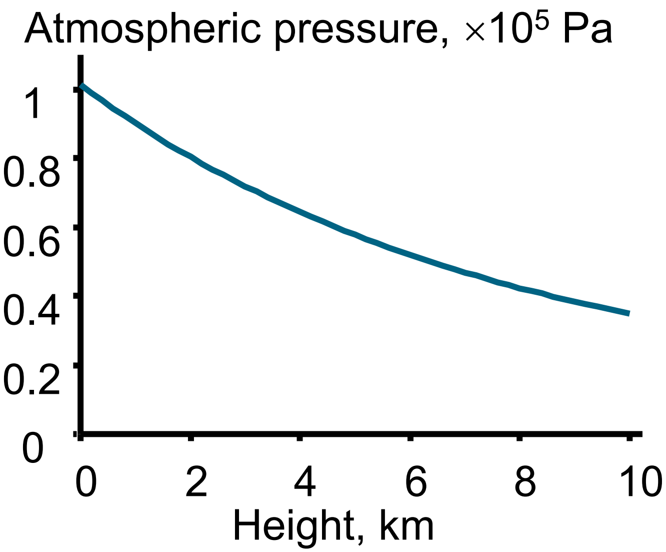
- Height
- The weather

Height can thus be inferred from a measurement of air pressure

But, the result depends on the weather

- Regular calibration is thus needed
- Or, only change in height is obtained

Aircraft use *flight levels* based on a standard barometric height model, not the true height



5. Magnetometers and Barometers

MEMS Inertial Measurement Units

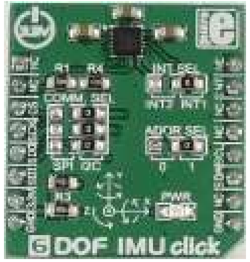
MEMS IMUs often incorporate additional sensors

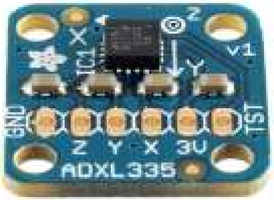
6 DOF IMU: 3 accelerometers + 3 gyros

9 DOF IMU: 3 accelerometers + 3 gyros
+ 3 magnetometers

10 DOF IMU: 3 accelerometers + 3 gyros
+ 3 magnetometers **+ 1 barometer**

Many smartphones have a 10 DOF IMU now





DOF = degrees of freedom

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6. Sensing Motion with respect to the Environment

Wheel Speed Odometry

A **wheel speed sensor** measures the rotation of a land vehicle's wheels:

Distance travelled = Number of rotations $\times$ Wheel radius $\times$ 2π

Speed = Distance travelled / Time taken

Wheel speed sensors are also used for anti-lock braking systems (ABS) and traction control

Wheel rotation is measured using a **rotary encoder**

- No significant error biases or scale factor errors
- Resolution of a few degrees (for vehicles)

6. Sensing Motion with respect to the Environment

Wheel Speed Odometry - Errors

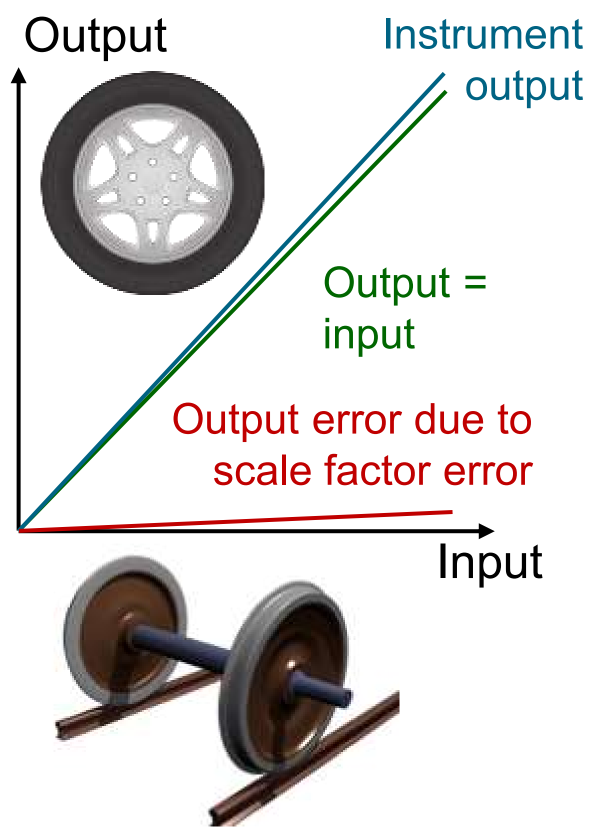
Errors arise from conversion of rotation into linear motion

This requires precise knowledge of wheel radius

- Road vehicle tyre radius varies due to wear, pressure and temperature introduces 1–3% **scale factor** errors
    - Can be calibrated through integration with GNSS (or other sensors)
- Train wheels are conical – radius varies

Wheel slip also occurs

- Driving wheels can rotate faster than the vehicle moves
- Non-driving wheels can rotate slower
- A big problem for trains due to lower friction



6. Sensing Motion with respect to the Environment

Wheel Speed Odometry - Turns

During turns, each wheel travels at a different speed

- Wheel speed sensors measure wheel speed not vehicle speed

Determining the correct vehicle speed requires

- Averaging speeds of left and right wheels, or
- Applying compensation based on the turn rate (e.g. from a gyro)

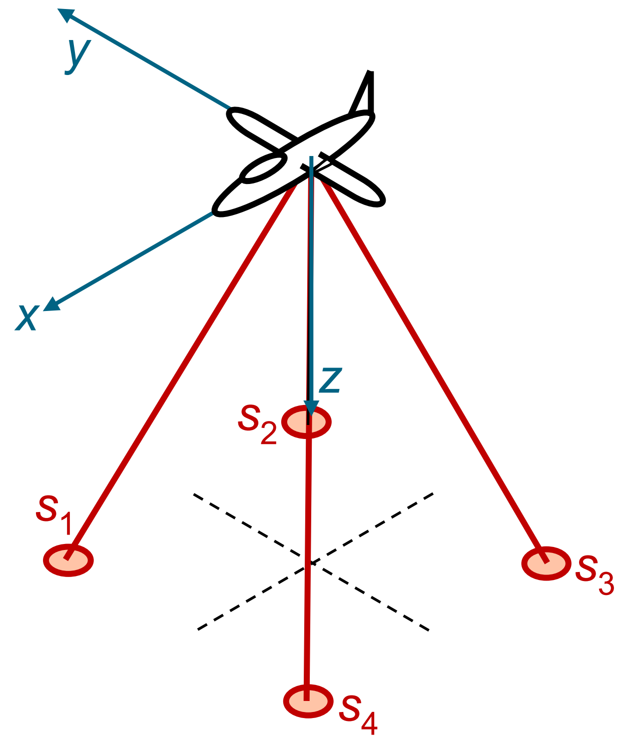
Turns can be measured by comparing rotation of different wheels

- Known as **Differential odometry**
- Noisy due to quantisation
- Can be biased by road camber

6. Sensing Motion with respect to the Environment

Doppler Sonar and Radar

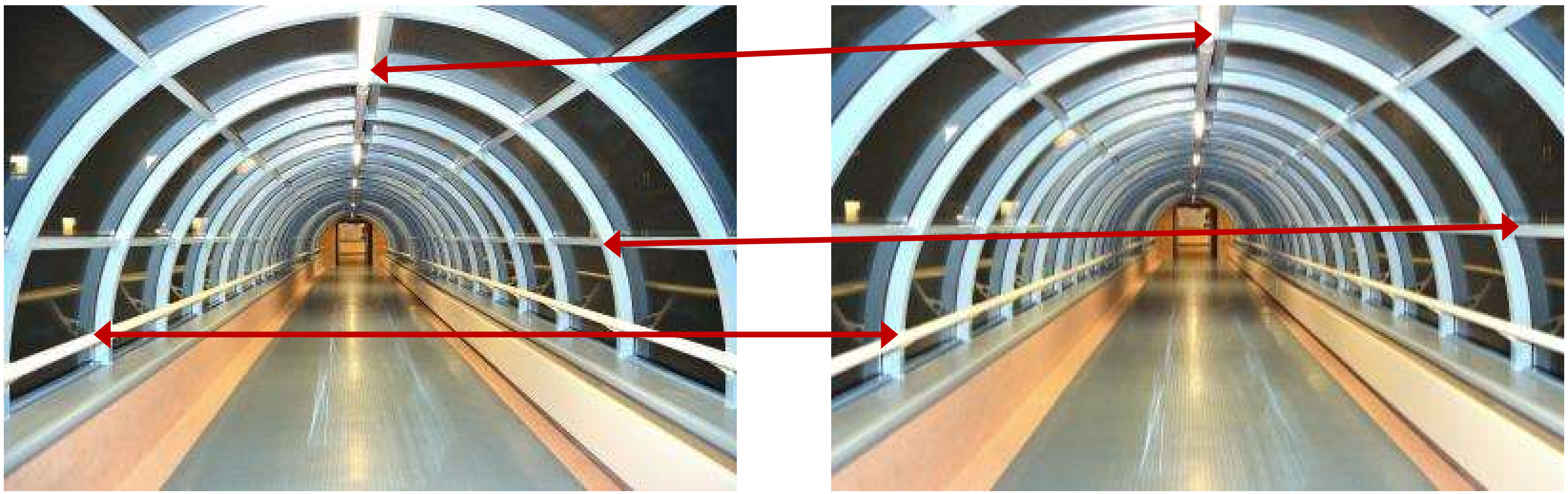
- Above ground, 3 or 4 radar beams are bounced off the ground
- On and underwater, 3 or 4 sonar beams are bounced off the seabed (or river bed)
- In both cases the signal Doppler shift is measured
- Doppler shift is proportional to the range rate along the beam
- Velocity (resolved about body axes) may be derived from 3 range rates



- A 4th range rate provides outlier detection
- For sonar, knowledge of the speed of sound is essential for best accuracy

6. Sensing Motion with respect to the Environment

Visual Odometry



Motion is inferred by comparing features across successive images

Vision-based navigation will be covered in detail later in the module

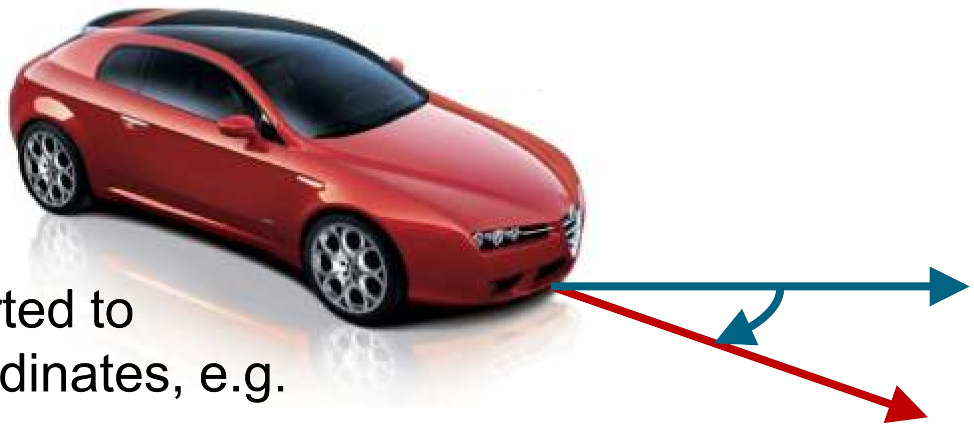
Contents

7. Dead Reckoning

Motion Within an Environment

Sensors measure motion in body axes:

- Forward-backward
- Left-right
- Vertical



Navigation and positioning require this motion to be converted to the external environment's coordinates, e.g.

- North-south
- East-west
- Vertical

This requires the sensor **orientation** – *as described previously*

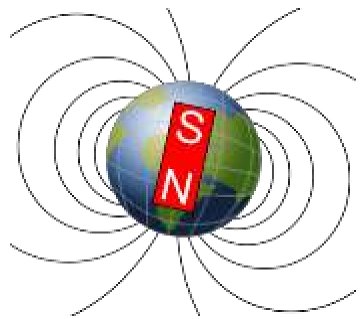
- Aircraft, spacecraft and underwater vehicles need **3D orientation**
- Land vehicles, ships and pedestrians need **heading**

7. Dead Reckoning

Heading Determination

Magnetic Compass

Heading from Earth's magnetic field

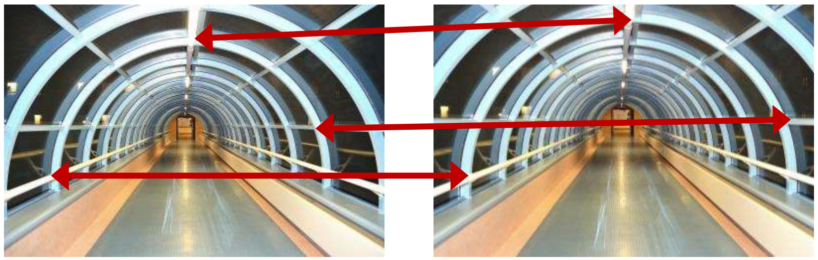


Gyroscope

Directly measures rotation

Rotation from optical flow

Compares successive images



Differential odometry

Compares rotation of different wheels to measure turning

Heading from Angular Rate

Gyroscopes, wheel-speed odometry and visual odometry all measure angular rate

Heading is the integral of angular rate in the horizontal plane:

$$\psi_{pb}(t) = \psi_{pb}(t_0) + \int_{t_0}^{t} \omega_{pb,z}^{b}(t')dt'$$

or

$$\psi_{pb,k} = \psi_{pb,k-1} + \omega_{pb,z,k}^{b}\left(t_k - t_{k-1}\right)$$

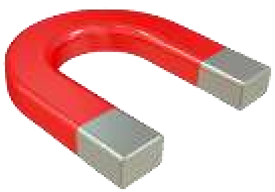
where $\omega_{pb,z,k}^{b}$ is averaged over the interval t_{k-1} to t_k.

An initial heading is required.

p denotes a planar frame

7. Dead Reckoning

Magnetic and Gyroscopic Heading



Magnetic heading

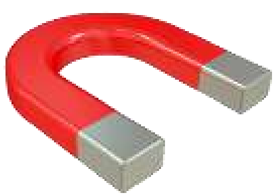
- Accuracy does not degrade over time
- Exhibits errors due to environmental magnetic anomalies

Gyroscopic heading

- High short-term stability
- Accuracy degrades over time
- Requires initialisation

Integrating magnetic and gyroscopic heading can combine the advantages of both types of sensor

Gyro-Smoothed Magnetic Heading (1)



Magnetometer → Smoothing algorithm ← Gyro

Magnetometer → Magnetic heading

Smoothing algorithm → Smoothed heading

Gyro → Rotation from gyro

Weighting

$$\boxed{\tilde{\psi}_{nb,S}(t)} = \boxed{W_M}\boxed{\tilde{\psi}_{nb,M}(t)} + \left(1 - W_M\right)\left[\boxed{\tilde{\psi}_{nb,S}(t-\tau)} + \boxed{\int_{t-\tau}^{t} \tilde{\omega}_{ib}\,dt}\right]$$

Gyro yaw rotation measurement

Smoothed heading

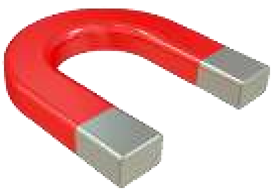
Magnetic heading

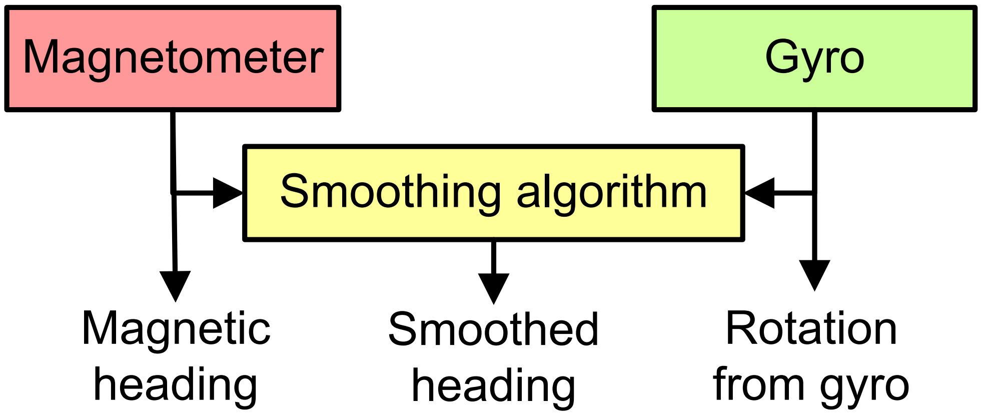
Old smoothed heading

- Basic low-pass filter for magnetometer and high-pass filter for gyro
- Reduces impact of gyro bias and magnetic anomalies

7. Dead Reckoning

Gyro-Smoothed Magnetic Heading (2)





A suitable weighting is

$$W_M = \frac{\boxed{\sigma_g}\,\boxed{\tau}}{\boxed{\sigma_m}}$$

Update interval

Gyro angular rate error standard deviation

Magnetic heading error standard deviation

- Units need to be consistent, i.e. degrees or radians
- W_M must be between 0 and 1

7. Dead Reckoning

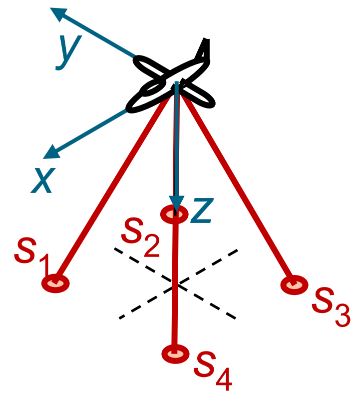
Distance Measurement

Wheel-Speed Odometry

Measures wheel rotation

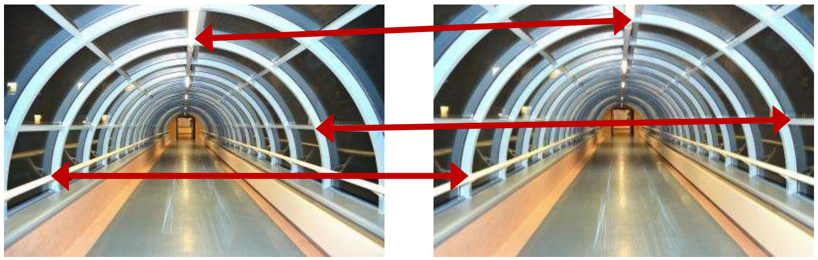
Doppler Radar and Sonar

Obtains velocity from Doppler shift



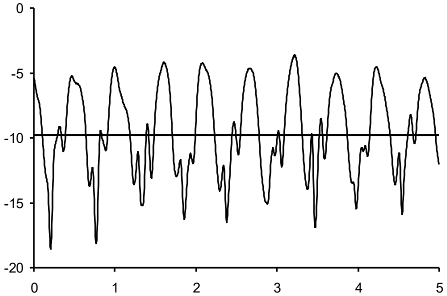
Visual Odometry

Compares successive images



Pedestrian Dead Reckoning

Detects and measures steps



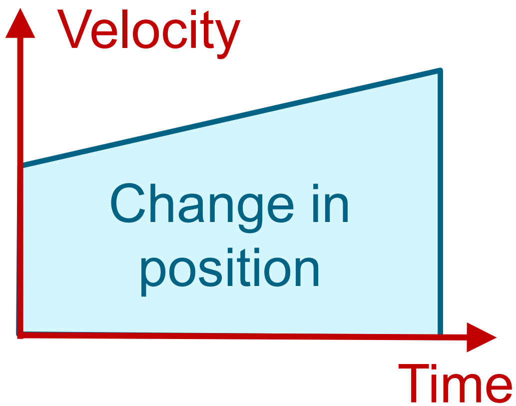
7. Dead Reckoning

DR Positioning in One Dimension

At time t_k, sensors output:

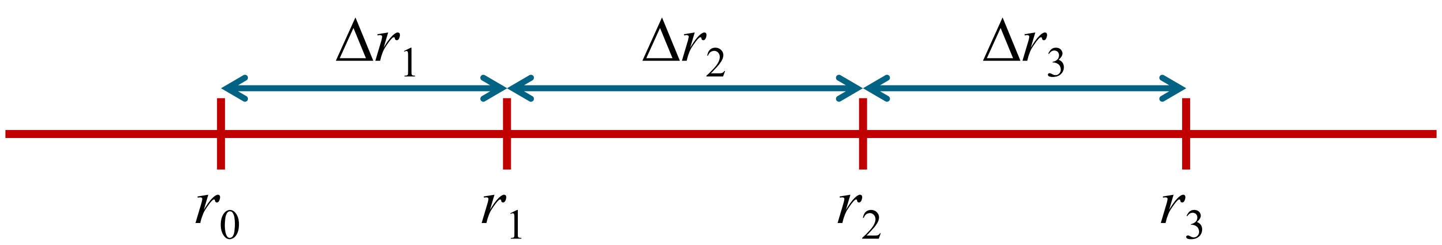
- Distance travelled, Δr_k, between times t_{k-1} and t_k
- Or, average speed, v_k, between times t_{k-1} and t_k

$$\Rightarrow \qquad \Delta r_k = v_k \left(t_k - t_{k-1} \right)$$



The position is then the sum of the initial position, r_0, and the distance measurements:

$$r_k = r_0 + \sum_{i=1}^{k} \Delta r_i$$

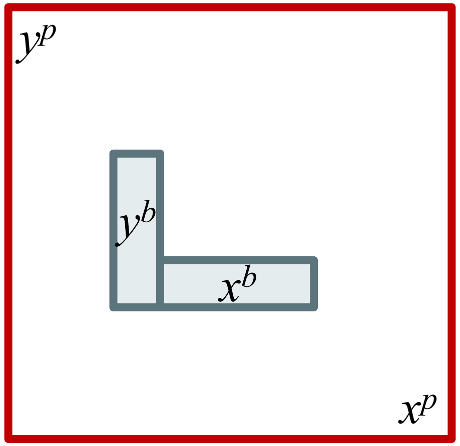


7. Dead Reckoning

DR Positioning in Two Dimension

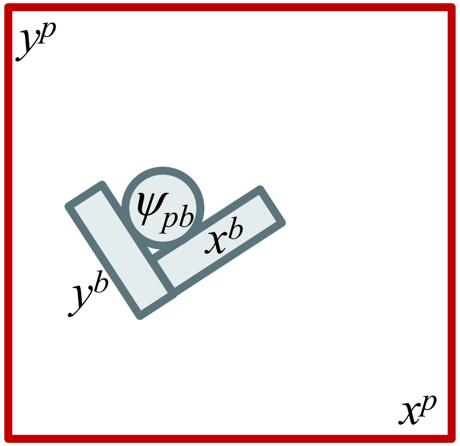
- Position is the integral of velocity

$$\begin{pmatrix} x^p_{pb}(t) \\ y^p_{pb}(t) \end{pmatrix} = \begin{pmatrix} x^p_{pb}(t_0) \\ y^p_{pb}(t_0) \end{pmatrix} + \int_{t_0}^{t} \begin{pmatrix} v^p_{pb,x}(t') \\ v^p_{pb,y}(t') \end{pmatrix} dt'$$



- The heading solution, ψ_{pb}, is used to transform the velocity measurements from body frame to environment frame axes

$$\begin{pmatrix} v^p_{pb,x}(t') \\ v^p_{pb,y}(t') \end{pmatrix} = \begin{pmatrix} \cos\psi_{pb}(t') & -\sin\psi_{pb}(t') \\ \sin\psi_{pb}(t') & \cos\psi_{pb}(t') \end{pmatrix} \begin{pmatrix} v^b_{pb,x}(t') \\ v^b_{pb,y}(t') \end{pmatrix}$$



p denotes a planar frame

7. Dead Reckoning

Dead Reckoning Error Sources

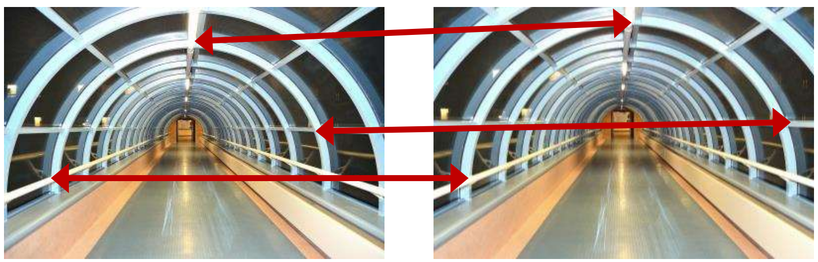
Error Source	Impact on Position Error
Distance/velocity measurement noise	Random walk error; standard deviation grows as √time
Distance/velocity measurement bias error	Error along direction travelled grows linearly with time
Distance/velocity measurement scale factor error	Error along direction travelled; grows in proportion to distance travelled
Heading measurement noise	Random walk error; standard deviation grows as √time
Heading measurement bias	Error perpendicular to direction travelled; grows in proportion to distance travelled

7. Dead Reckoning

Dead Reckoning in 3D

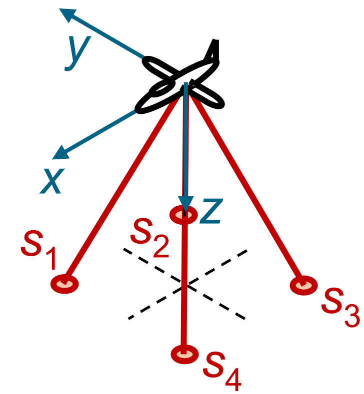
Visual Odometry

Compares successive images



Doppler Radar and Sonar

Obtains velocity
from Doppler shift



Attitude & Heading Reference

Uses inertial and magnetic sensors to maintain 3D orientation solution

Gives direction of velocity from Doppler radar/sonar or visual odometry measurements

Inertial Navigation

Integrates accelerometer and gyroscope measurements

Contents

8. Inertial Navigation

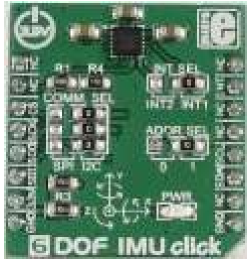
Inertial Sensor Fundamentals

Inertial sensors measure motion of a body with respect to inertial space, *not the surrounding environment*

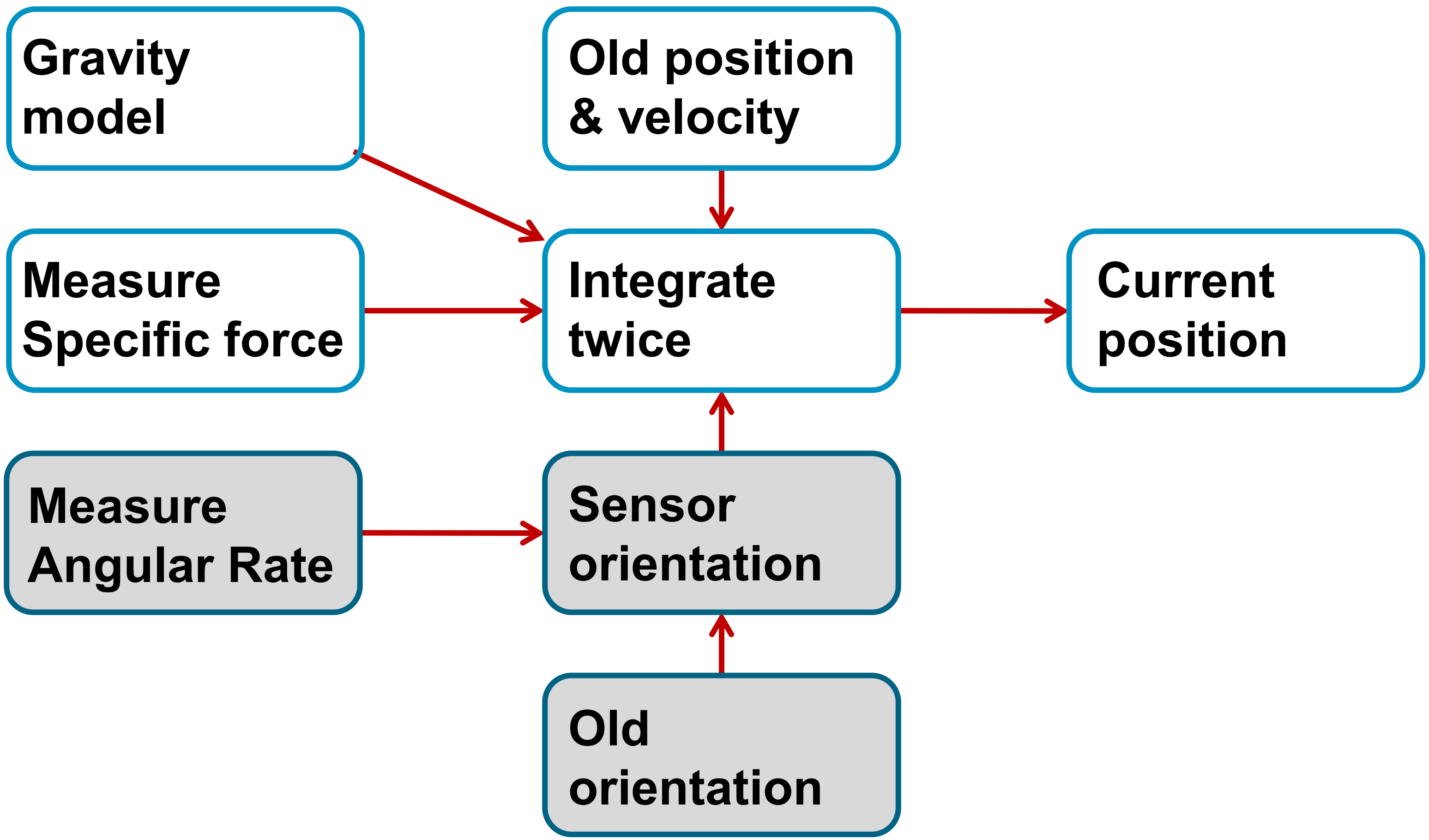
Gyroscopes (gyros) measure angular rate

Accelerometers measure specific force

Devices that measure the motion of a body *with respect to features in the environment* are **not** inertial sensors



Fundamentals of Inertial Navigation



8. Inertial Navigation

Applications

- Aircraft Navigation
  - Civil, military, general, helicopters, UAVs
- Motorsport and Train Navigation
- Remote and autonomous underwater vehicles
- Mobile Mapping and Surveying
- Military Ships and submarines
- Guided weapons

8. Inertial Navigation

One-Dimensional Inertial Navigation

- Velocity is the integral of acceleration

$$v_{pb}(t) = v_{pb}(t_0) + \int_{t_0}^{t} a_{pb}(t')dt'$$

- Position is the integral of velocity
  and the double integral of acceleration

$$\begin{aligned} r_{pb}(t) &= r_{pb}(t_0) + \int_{t_0}^{t} v_{pb}(t')dt' \\ &= r_{pb}(t_0) + (t - t_0)v_{pb}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t'} a_{pb}(t'')dt''dt' \end{aligned}$$

p denotes a planar frame

8. Inertial Navigation

Two-Dimensional Inertial Navigation (1)

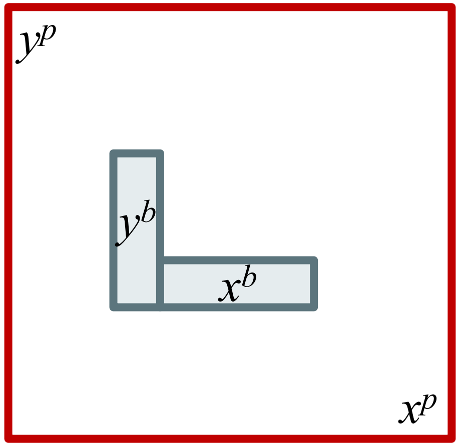
- Velocity is the integral of acceleration

$$\begin{pmatrix} v^p_{pb,x}(t) \\ v^p_{pb,y}(t) \end{pmatrix} = \begin{pmatrix} v^p_{pb,x}(t_0) \\ v^p_{pb,y}(t_0) \end{pmatrix} + \int_{t_0}^{t} \begin{pmatrix} a^p_{pb,x}(t') \\ a^p_{pb,y}(t') \end{pmatrix} dt'$$

- Position is the integral of velocity

$$\begin{pmatrix} x^p_{pb}(t) \\ y^p_{pb}(t) \end{pmatrix} = \begin{pmatrix} x^p_{pb}(t_0) \\ y^p_{pb}(t_0) \end{pmatrix} + \int_{t_0}^{t} \begin{pmatrix} v^p_{pb,x}(t') \\ v^p_{pb,y}(t') \end{pmatrix} dt'$$

Horizontal plane



p denotes a planar frame

Two-Dimensional Inertial Navigation (2)

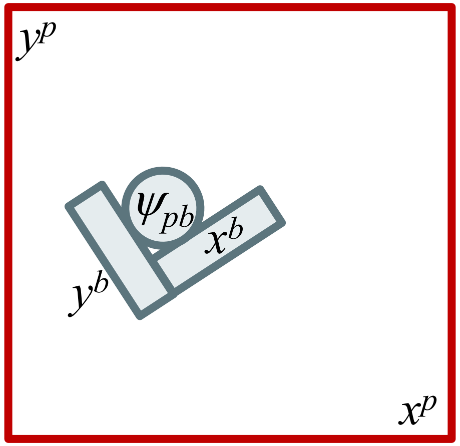
- But what about the sensor orientation?

$$\begin{pmatrix} a^p_{pb,x}(t') \\ a^p_{pb,y}(t') \end{pmatrix} = \begin{pmatrix} \cos\psi_{pb}(t') & -\sin\psi_{pb}(t') \\ \sin\psi_{pb}(t') & \cos\psi_{pb}(t') \end{pmatrix} \begin{pmatrix} a^b_{pb,x}(t') \\ a^b_{pb,y}(t') \end{pmatrix}$$

- Heading is the integral of angular rate

$$\psi_{pb}(t) = \psi_{pb}(t_0) + \int_{t_0}^{t} \omega^b_{pb,z}(t')\,dt'$$

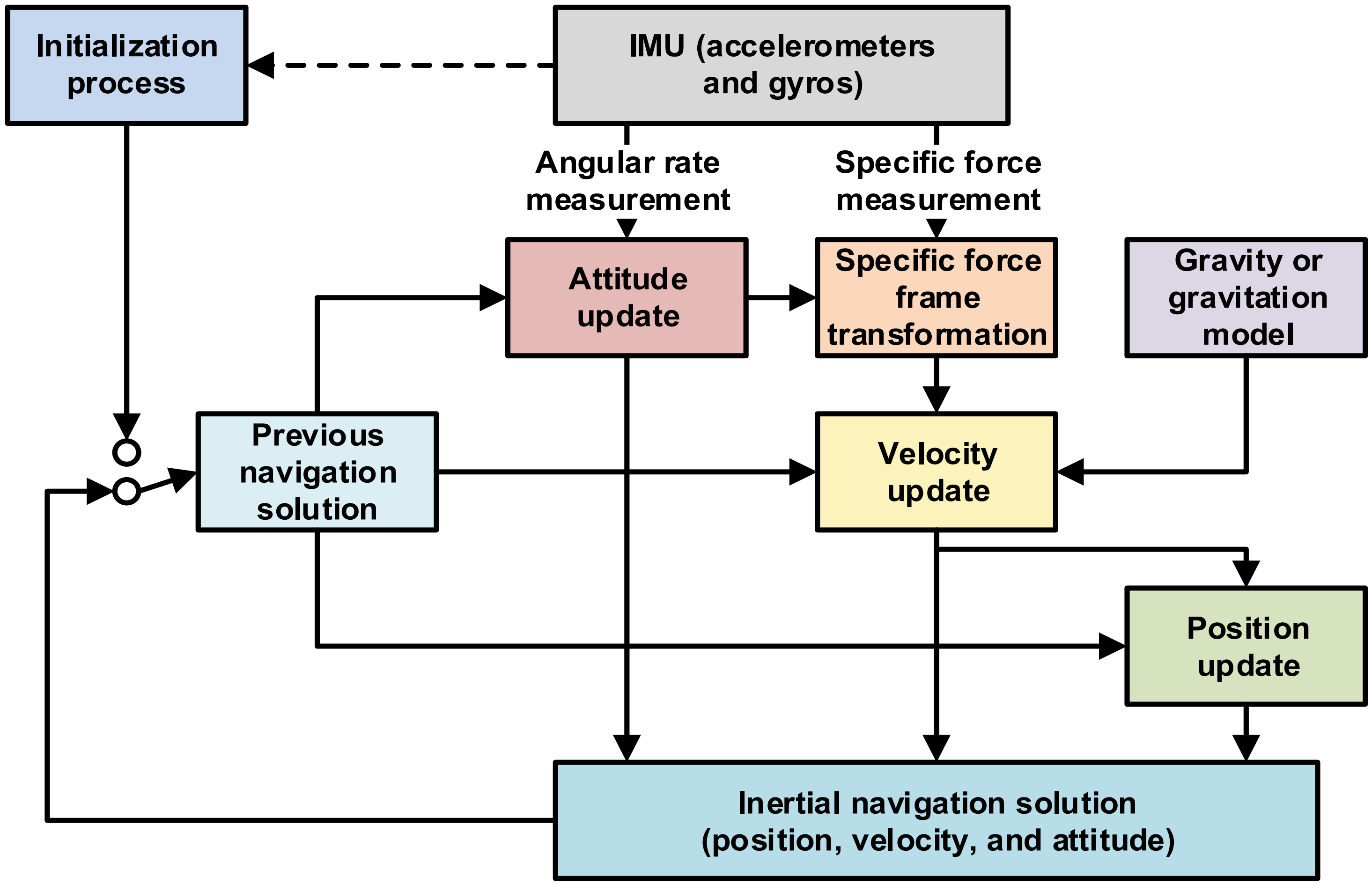
Horizontal plane



p denotes a planar frame

8. Inertial Navigation

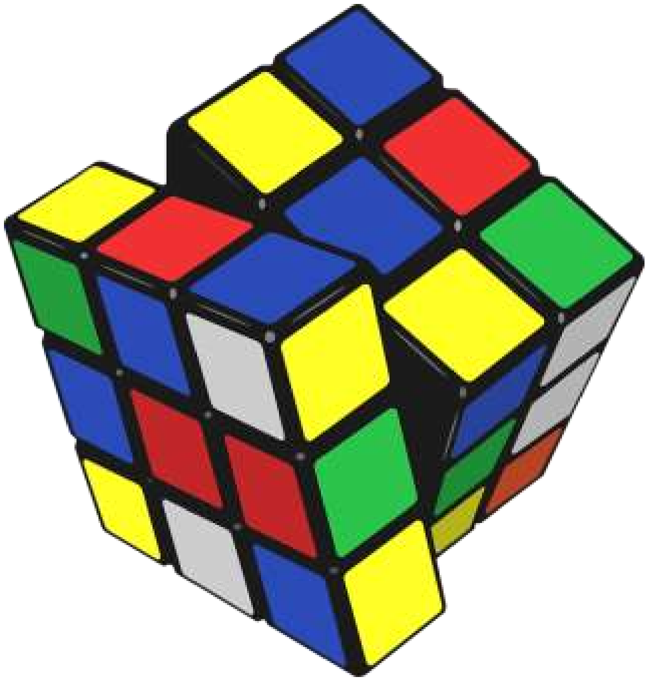
3D Inertial Navigation Processing



8. Inertial Navigation

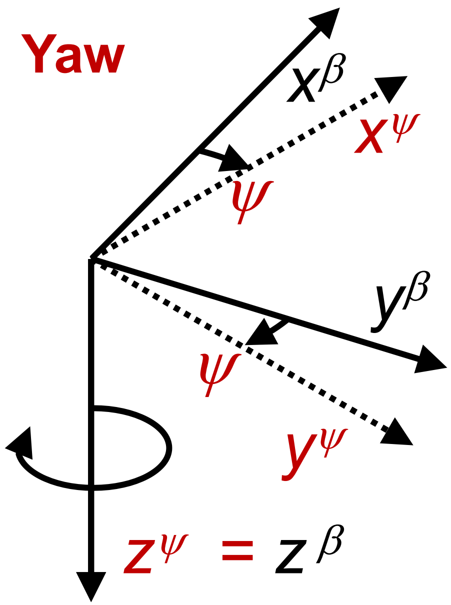
3D Attitude/Orientation Representations

- **Euler angles**
- **Coordinate transformation matrix** (rotation matrix)
- Quaternions *– see books*
- Rotation vector *– see books*



8. Inertial Navigation

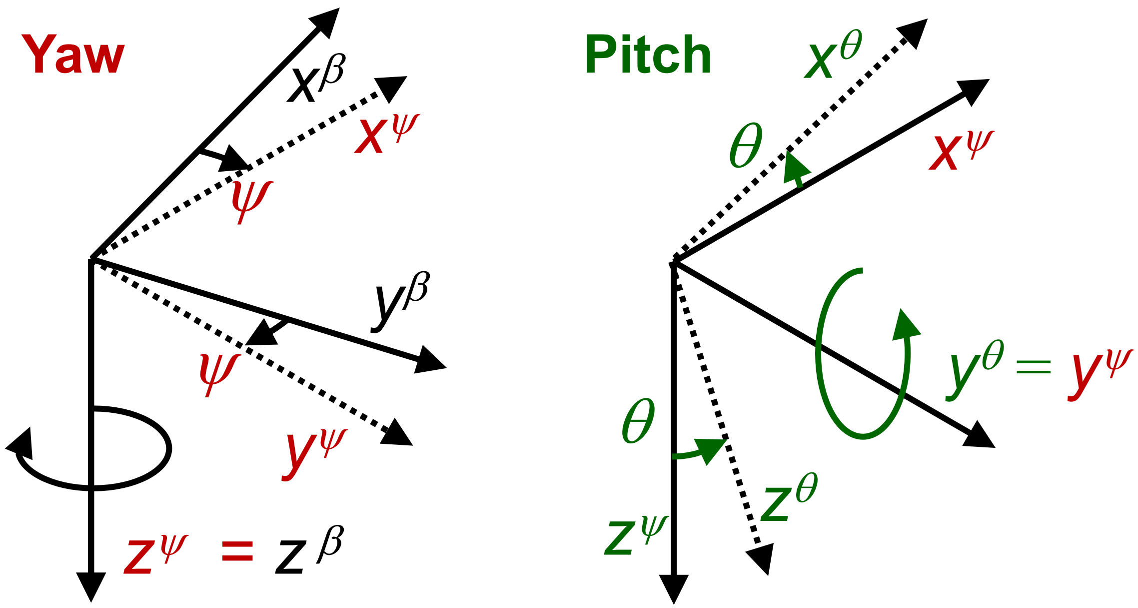
Euler Attitude (1)



Euler angles are the most intuitive way of describing attitude

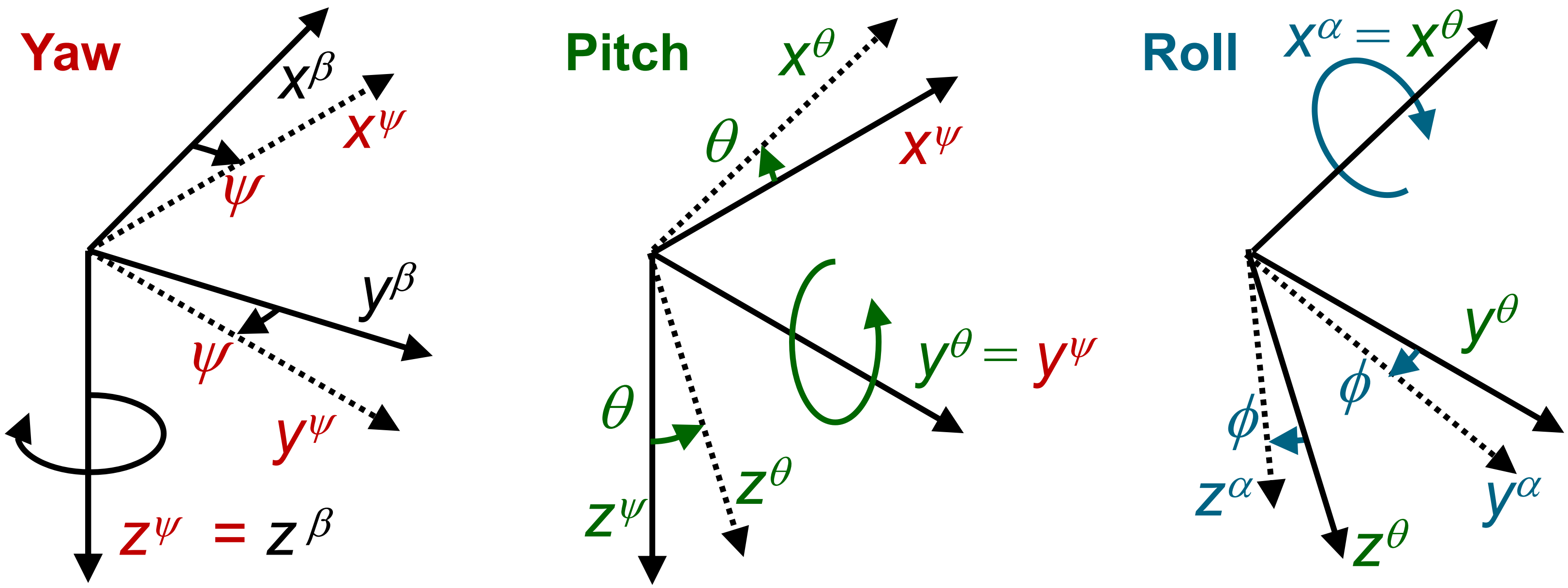
8. Inertial Navigation

Euler Attitude (2)



Euler angles are the most intuitive way of describing attitude

Euler Attitude (3)



Euler angles are the most intuitive way of describing attitude

But they are very difficult to manipulate mathematically

8. Inertial Navigation

Coordinate Transformation Matrix (1)

A co-ordinate transformation matrix (or direction cosine matrix), **C**, performs 2 functions:

- Representing an attitude i.e. the α coordinate frame with respect to the β frame
- Transformation of a vector **x** from representation in one frame to representation in another

$$\mathbf{C}_{\alpha}^{\beta} = \begin{bmatrix} \alpha_x \to \beta_x & \alpha_y \to \beta_x & \alpha_z \to \beta_x \\ \alpha_x \to \beta_y & \alpha_y \to \beta_y & \alpha_z \to \beta_y \\ \alpha_x \to \beta_z & \alpha_y \to \beta_z & \alpha_z \to \beta_z \end{bmatrix}$$

$$\mathbf{x}_{\delta\gamma}^{\beta} = \mathbf{C}_{\alpha}^{\beta} \mathbf{x}_{\delta\gamma}^{\alpha}$$

It is also known as the rotation matrix, **R**

8. Inertial Navigation

Coordinate Transformation Matrix (2)

To reverse a rotation or co-ordinate transformation, use the transpose of the matrix

$$\mathbf{C}_\alpha^\beta = \left(\mathbf{C}_\beta^\alpha\right)^\mathrm{T}$$

To perform successive transformations or rotations, multiply the matrices

- *The order of multiplication is critical*
- *A mathematical property of matrix multiplication*
- *Also a physical property of rotations*

$$\mathbf{C}_\alpha^\gamma = \mathbf{C}_\beta^\gamma \mathbf{C}_\alpha^\beta$$

$$\mathbf{C}_\alpha^\gamma \neq \mathbf{C}_\alpha^\beta \mathbf{C}_\beta^\gamma$$

Co-ordinate transformation matrices are orthonormal matrices

- *Only 3 components are independent*

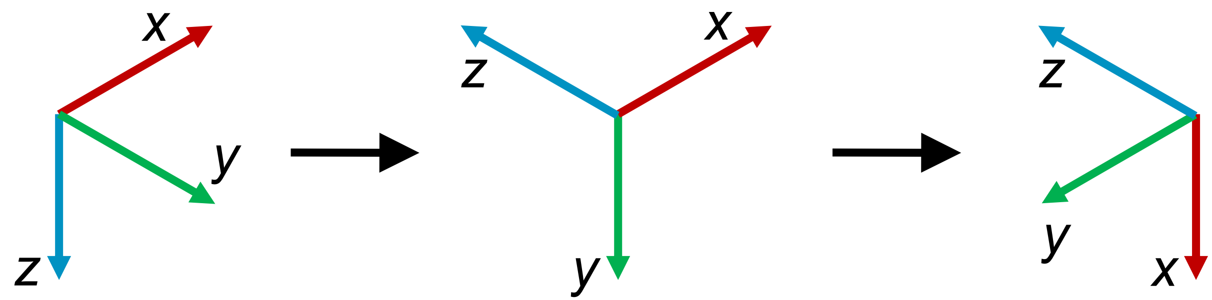
$$\mathbf{C}_\alpha^\beta \mathbf{C}_\beta^\alpha = \mathbf{I}_3$$

8. Inertial Navigation

Order of Rotation

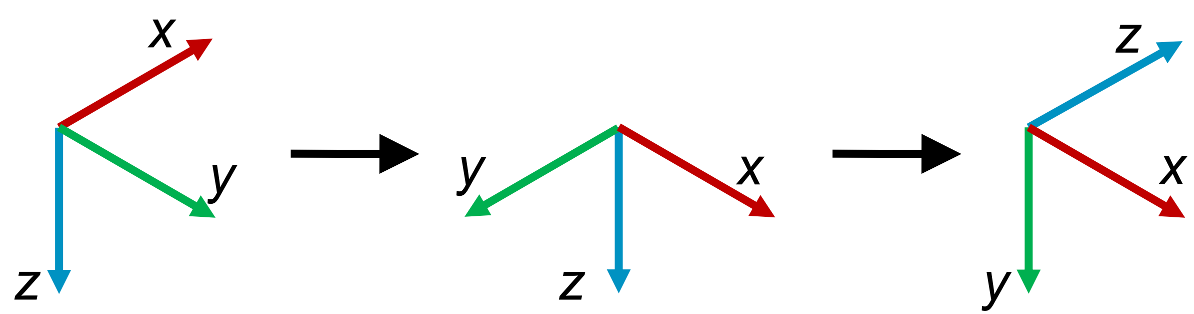
Performing the same rotations in different orders leads to different outcomes

"Noncommutativity"



Rotate +90° about body *x* axis

Rotate +90° about body *z* axis



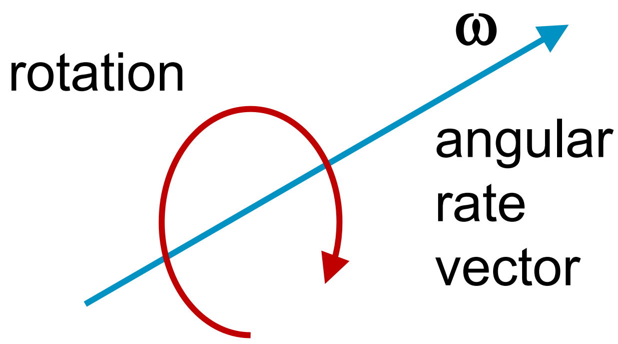
Rotate +90° about body *z* axis

Rotate +90° about body *x* axis

Angular Rate Vector

$$\boldsymbol{\omega}_{\beta\alpha}^{\gamma}$$

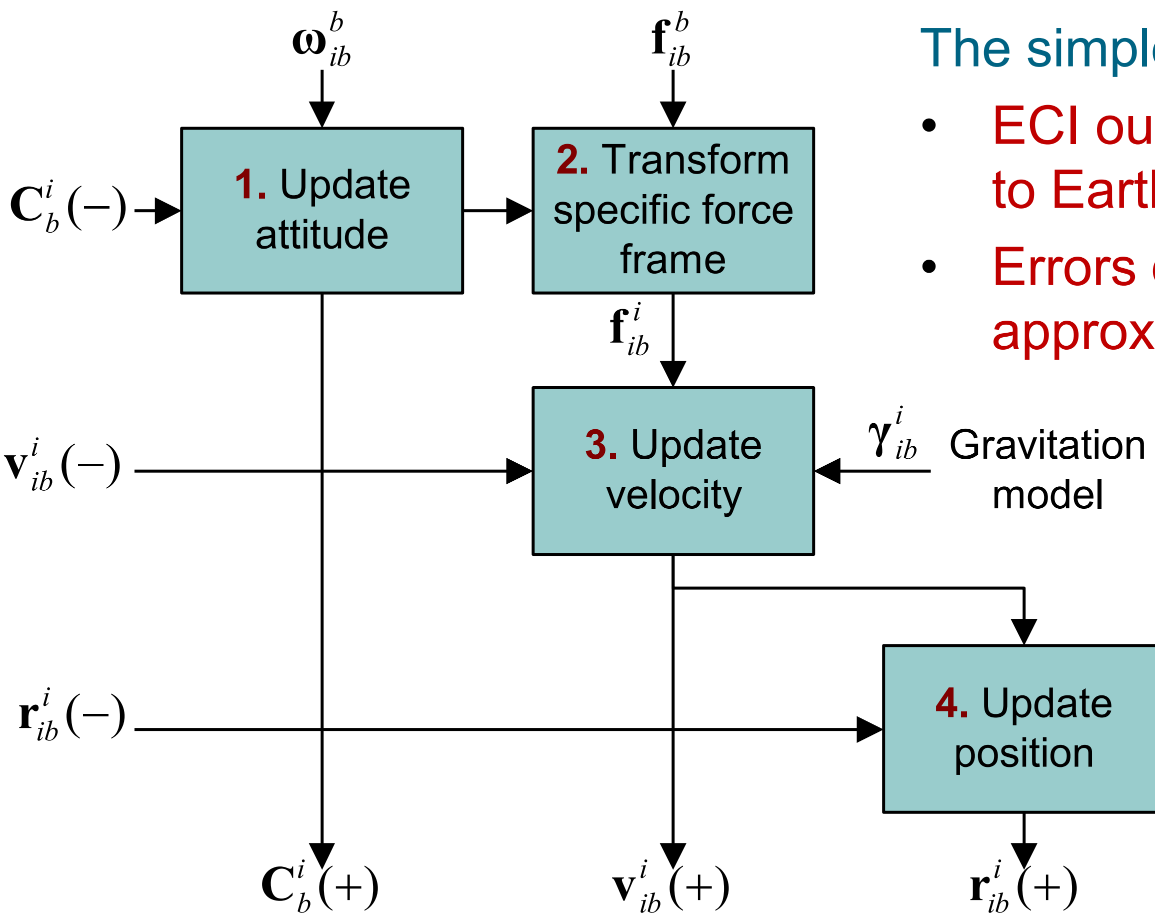
The **angular rate** vector is the rate of rotation of the α frame axes with respect to the β frame axes, resolved about the γ frame axes



Example: Gyros measure the angular rate of the IMU body frame, ***b***, with respect to inertial space, ***i***, resolved about body-frame axes

$$\boldsymbol{\omega}_{ib}^{b}$$

First-order ECI-frame Navigation Equations



The simplest implementation

- ECI outputs must be transformed to Earth-referenced for use
- Errors due to first-order approximation

8. Inertial Navigation

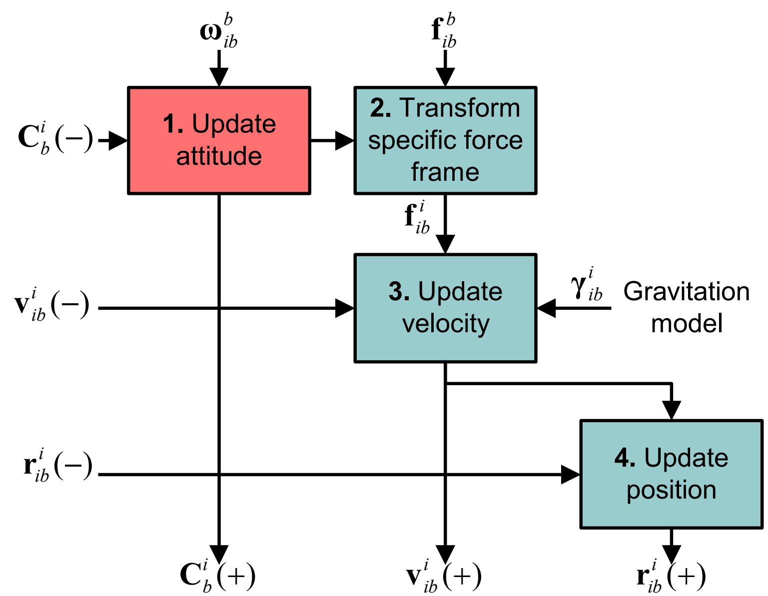
ECI-frame Step 1: Update attitude

Rotation of body frame

Body to inertial frame coordinate transformation matrix

$$\boxed{\mathbf{C}_b^i(+)} = \boxed{\mathbf{C}_b^i(-)}\boxed{\mathbf{C}_{b+}^{b-}}$$

After Before



Simplest form is 1st order

$$\mathbf{C}_{b+}^{b-} \approx \mathbf{I}_3 + \boldsymbol{\Omega}_{ib}^b \boxed{\tau_i}$$

Time interval

where

$$\boldsymbol{\Omega}_{ib}^b = \left[\boldsymbol{\omega}_{ib}^b \wedge\right] = \begin{pmatrix} 0 & -\omega_{ib,z}^b & \omega_{ib,y}^b \\ \omega_{ib,z}^b & 0 & -\omega_{ib,x}^b \\ -\omega_{ib,y}^b & \omega_{ib,x}^b & 0 \end{pmatrix}$$

is a skew-symmetric matrix

$$\left[\mathbf{a}\wedge\right]\mathbf{b} \equiv \mathbf{a}\wedge\mathbf{b} \equiv \mathbf{a}\times\mathbf{b}$$

The vector (cross product), expressed in matrix form

8. Inertial Navigation

ECI-frame Step 2: Transform specific force

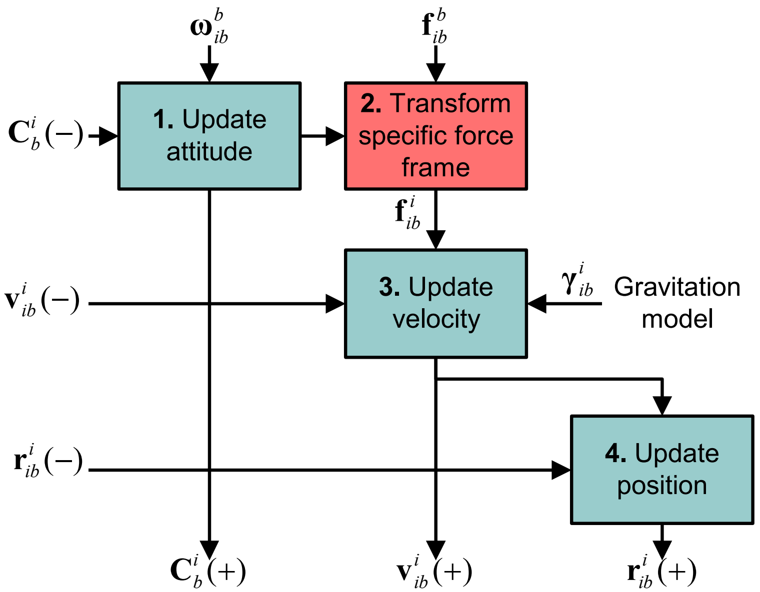
Specific force resolving axes are transformed from body to inertial by applying the **C** matrix

$$\mathbf{f}_{ib}^{i} = \mathbf{C}_{b}^{i}\mathbf{f}_{ib}^{b}$$

Measurement is averaged over time ***t* – *τ*** $t - \tau_i$ to t and **C** is available at both times. Thus...

$$\mathbf{f}_{ib}^{i} \approx \tfrac{1}{2}\left(\mathbf{C}_{b}^{i}(-) + \mathbf{C}_{b}^{i}(+)\right)\mathbf{f}_{ib}^{b}$$

Note the mean of the 2 attitudes is not exactly the mean of the 2 **C** matrices



8. Inertial Navigation

ECI-frame Step 3: Update velocity

Acceleration is the specific force + the gravitational acceleration

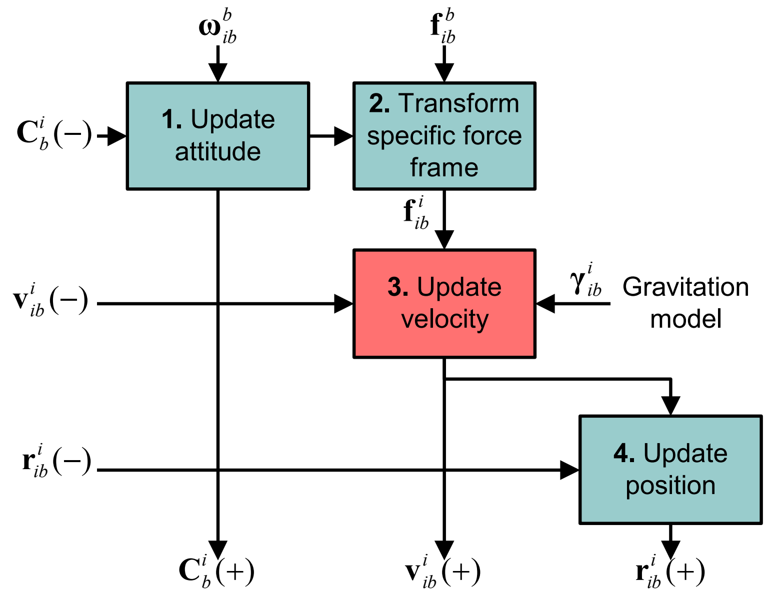
$$\mathbf{a}^{i}_{ib} = \mathbf{f}^{i}_{ib} + \boldsymbol{\gamma}^{i}_{ib}(L_b, h_b)$$

Where the resolving axes and reference frame are the same…

$$\dot{\mathbf{v}}^{i}_{ib} = \frac{d\mathbf{v}^{i}_{ib}}{dt} = \mathbf{a}^{i}_{ib}$$

Where variations in the acceleration over the update interval are not known…

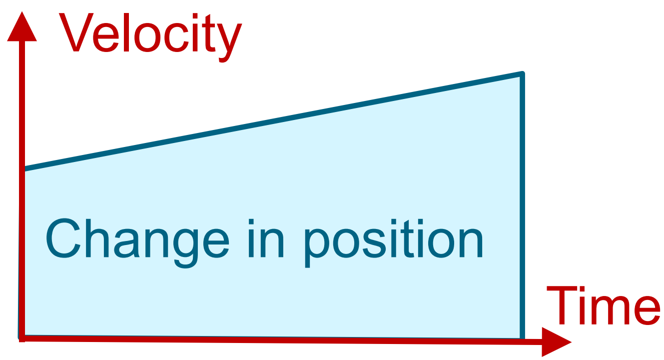
$$\mathbf{v}^{i}_{ib}(+) = \mathbf{v}^{i}_{ib}(-) + \tau_i \mathbf{a}^{i}_{ib}$$

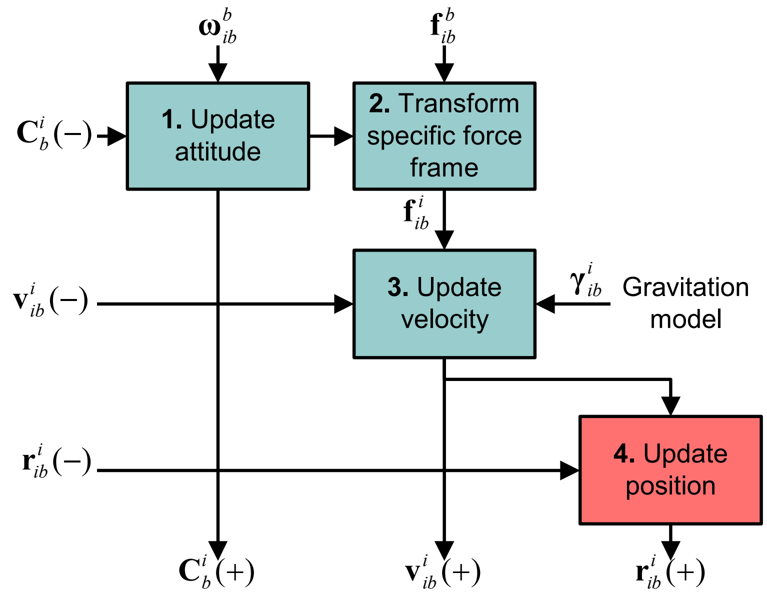


ECI-frame Step 4: Update position

Where the resolving axes and reference frame are the same…

$$\dot{\mathbf{r}}^{i}_{ib} = \mathbf{v}^{i}_{ib}$$





Velocity is known at the start and finish, so is modelled as a linear function of time:

$$\mathbf{r}^{i}_{ib}(+) = \mathbf{r}^{i}_{ib}(-) + \frac{\tau_i}{2}\left(\mathbf{v}^{i}_{ib}(-) + \mathbf{v}^{i}_{ib}(+)\right)$$

Alternative Implementations

Precision instead of first-order approximation

Higher **accuracy** – particularly at lower update rates

Increased complexity

ECEF-frame and NED-frame instead of ECI-frame

Navigation solution is **already Earth-referenced** – no conversion needed

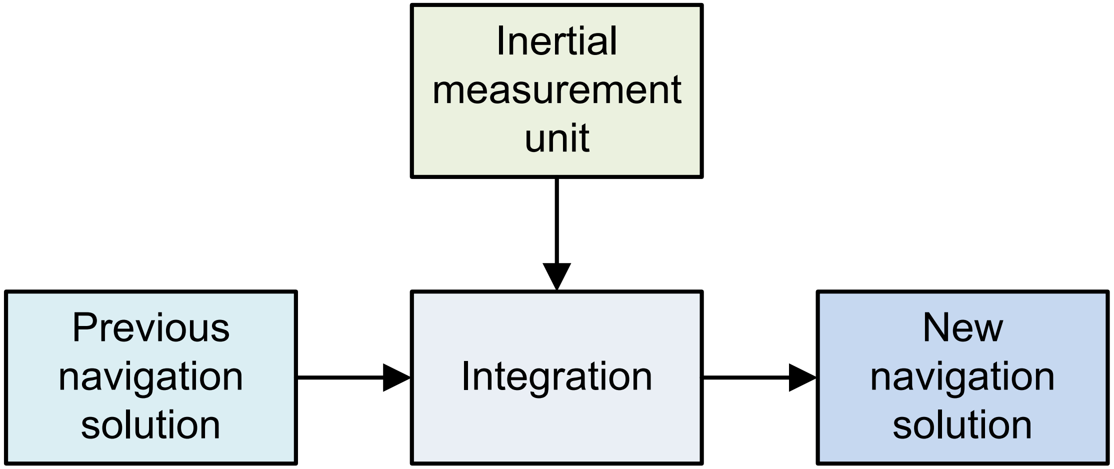
Increased complexity – extra terms required for:

- Impact of Earth rotation on attitude update
- Centrifugal force of Earth's rotation: *Gravitation* → *Gravity*
- Coriolis acceleration
- Rotation of NED axes w.r.t. Earth as the device moves (NED-frame only)

See textbooks for more

Initialisation and Alignment

Before an INS can be used to provide a navigation solution, that navigation solution must be **initialised**



8. Inertial Navigation

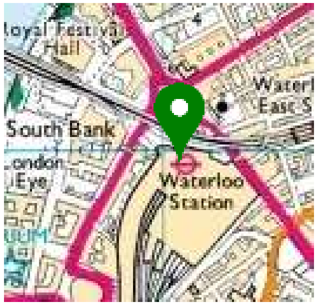
Position and Velocity Initialisation

INS position & velocity must be initialised from an external source

Methods include:

- Placing the INS at a known point to initialise position
- Maintaining the INS stationary with respect to the Earth enables the Earth-referenced velocity to be initialised at zero
- Using position and/or velocity from another technology, e.g. GNSS
- Using position and velocity from another INS

In each case the lever arm from the INS to the known position must be measured. Attitude may be needed to convert from the body frame and angular rate is needed to calculate the lever arm velocity



8. Inertial Navigation

Attitude Initialisation (1)

Attitude may be initialised in a number of ways

- Gyrocompassing: Measuring Earth rotation (high-grade gyros only)
- Accelerometer levelling
- Bearings from known objects
- Magnetic compass
- From another INS
- Multi-antenna GNSS

8. Inertial Navigation

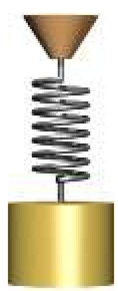
Attitude Initialisation (2)

Attitude may be initialised in a number of ways

Most INS applications require attitude to 1 mrad or better and most initialisation techniques do not meet this

$\therefore$ Initialisation is typically followed by fine alignment

- Attitude errors are inferred from the growth in velocity errors
- INS/GNSS integration; transfer alignment; quasi-stationary alignment (*see textbooks*)



Inertial Error Propagation

Where an INS is the sole navigation sensor

- Accelerometer and gyro errors will integrate up to give position, velocity & attitude errors that grow with time
- There will be cumulative errors due to navigation equations approximations
- Initialisation errors propagate through the navigation equations

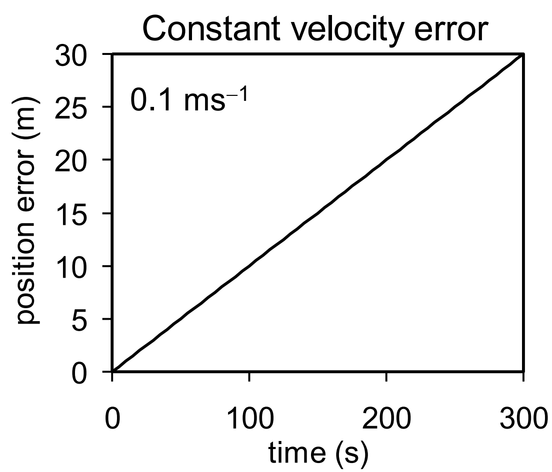
Accurate determination of INS error propagation is complex

- Full simulation is required to study it properly
- Some simple examples follow

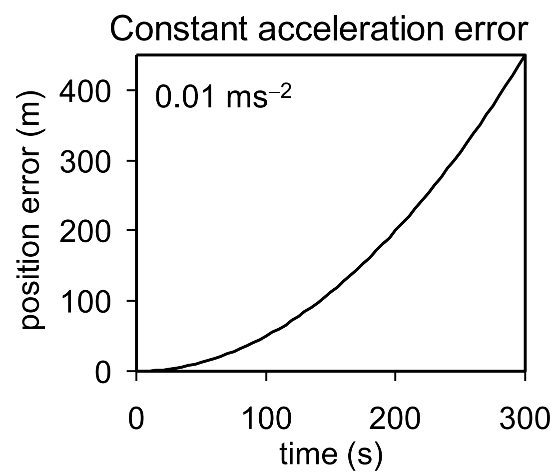
8. Inertial Navigation

Short-term Straight-line Error Propagation

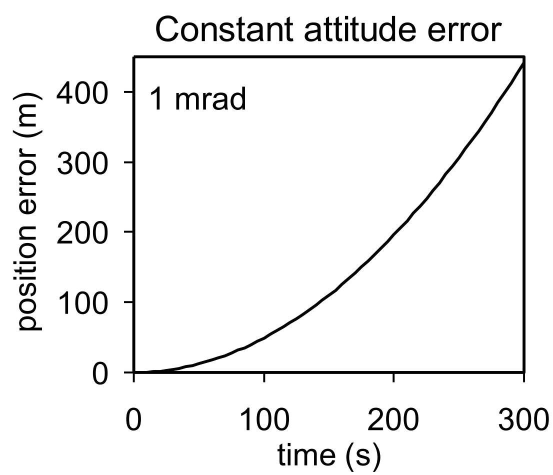
Linear



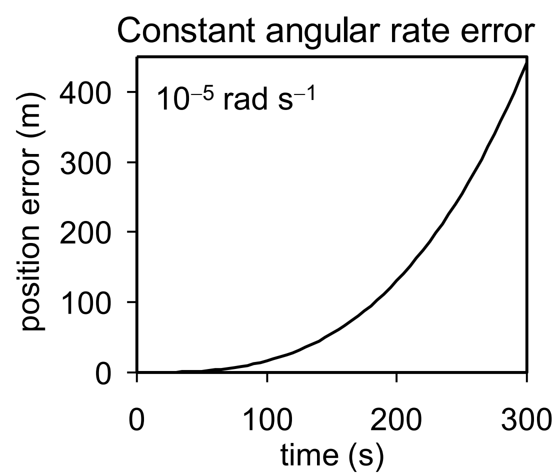
Quadratic



Quadratic



Cubic



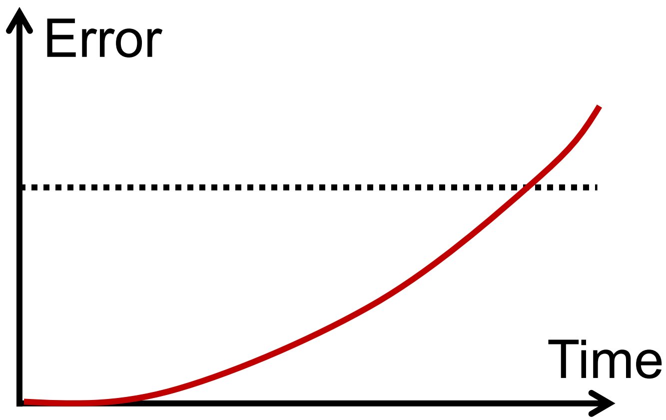
Inertial Navigation – Pros and cons

- High bandwidth (≥ 50 Hz) navigation solution
- Low short term noise
- Provides attitude, angular rate and acceleration as well as position and velocity
- Position accuracy degrades over time
- For high performance (1.5 km drift in 1 hour):
  - Cost is high (>£ 50 000)
  - Size, mass and power consumption are high
- Small, lightweight, low-cost inertial sensors are available:
  - Navigation performance orders of magnitude poorer
  - Only useful for seconds or minutes

Accurate and Reliable Navigation

Inertial Navigation and other Dead Reckoning techniques:

- Position error grows with time
- At some point, it will exceed the performance specification



Absolute Positioning (e.g. GNSS)**:**

- Only works when a number of sufficient good radio signals are received *or* distinct and identifiable environmental features observed
- This can never be guaranteed 100% of the time

Multisensor Integrated Navigation

- Continuously available navigation solution
- Position error is normally small