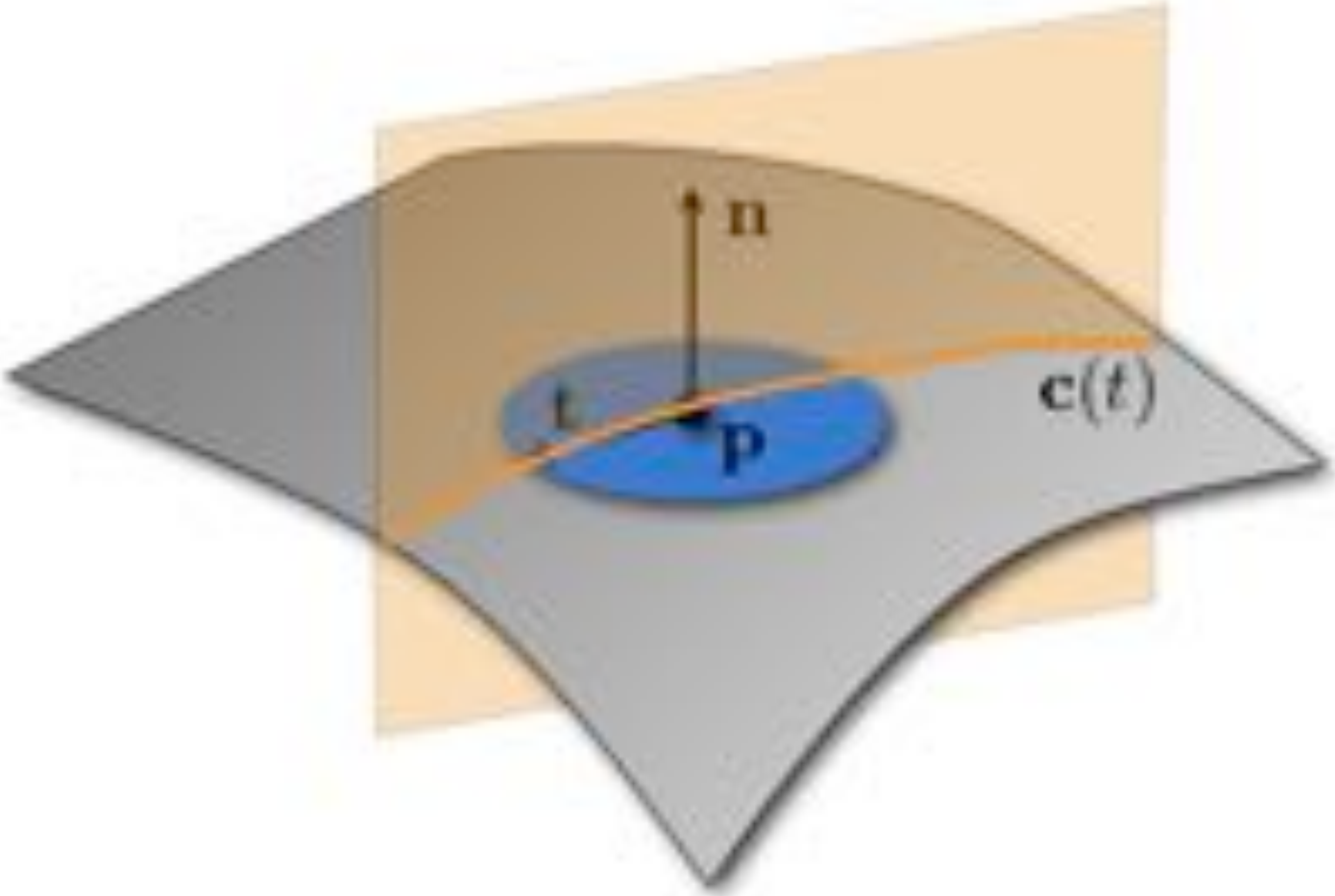
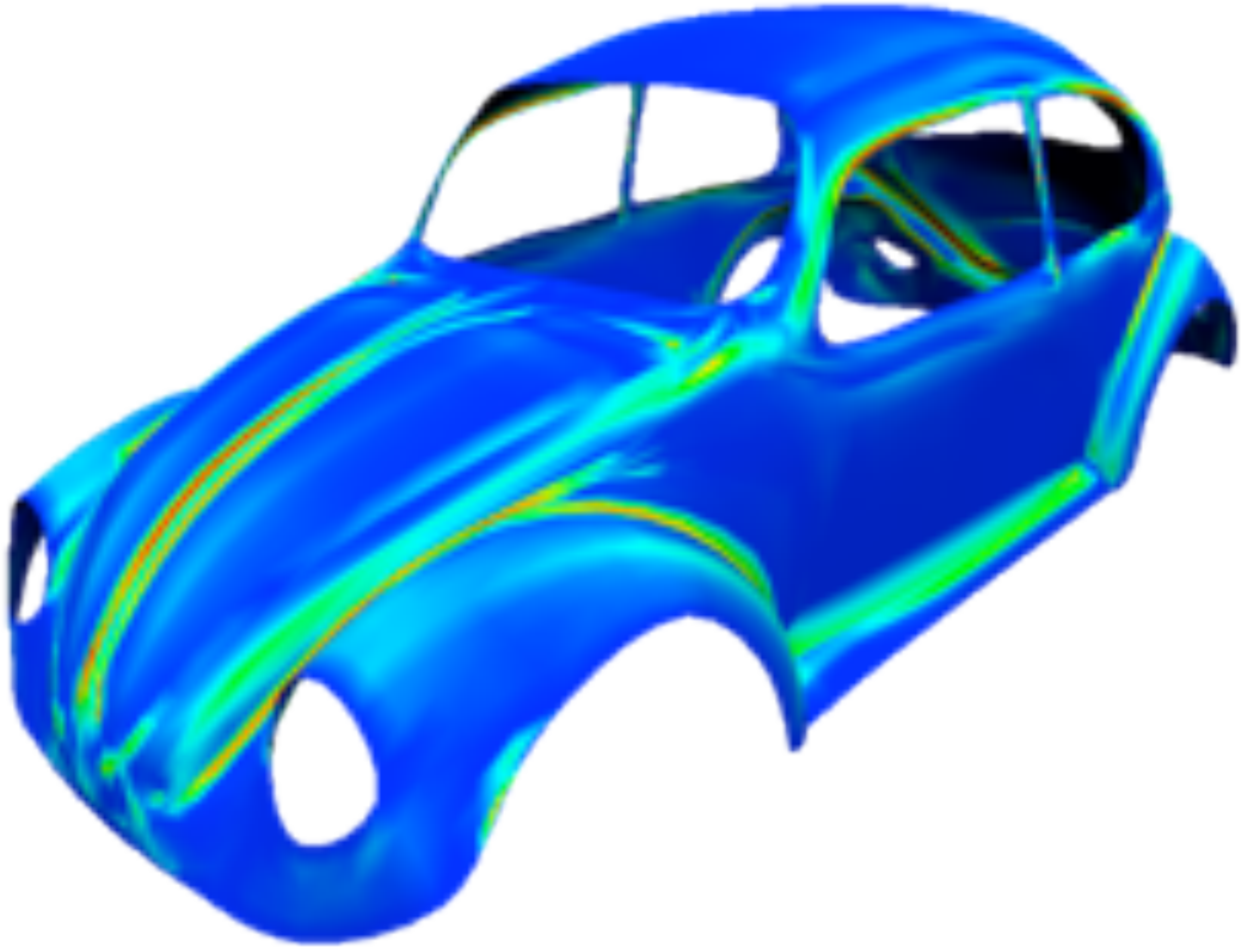
Differential Geometry





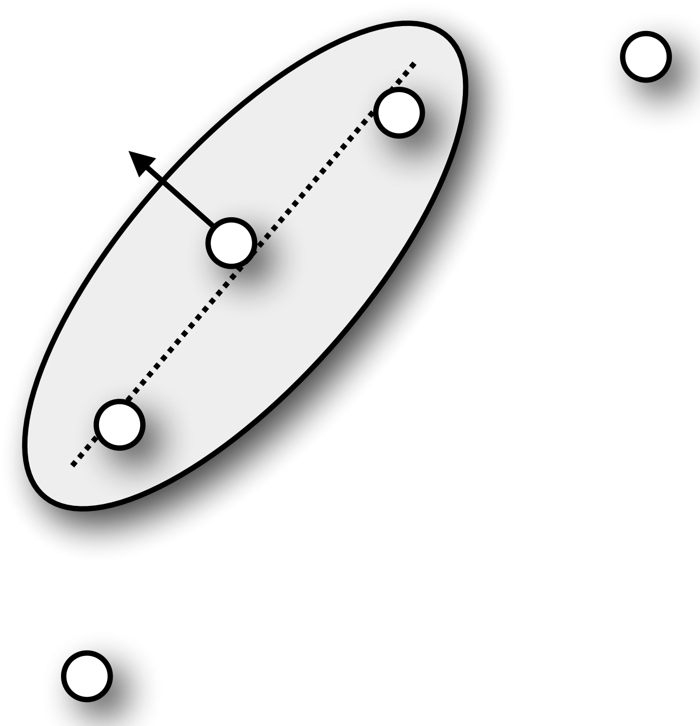
Last Time

- *Explicit Reconstruction*
  - Fast Marching Method
- *Implicit Reconstruction*
  - SDF from point clouds
  - (SDF from range scans)
  - Fast Marching Cubes

Normal Estimation

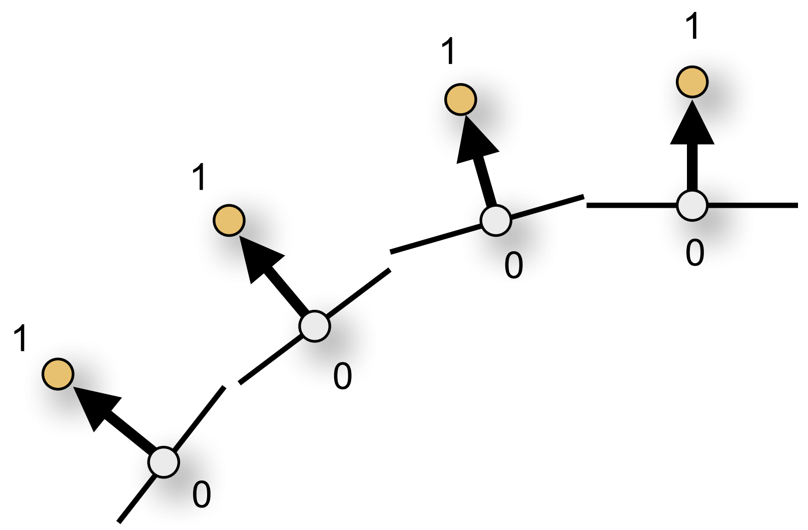
– Find normal $\mathbf{n}_i$ for each sample point $\mathbf{p}_i$

1. Examine local neighborhood for each point
   – Set of *k* nearest neighbors
2. Compute best approximating tangent plane
   – Covariance analysis
3. Determine normal orientation
   – MST propagation



Implicit Reconstruction

- Scattered data interpolation problem
  - On-surface constraints dist($\mathbf{p}_i$) = 0
  - Off-surface constraints dist($\mathbf{p}_i$ + $\mathbf{n}_i$) = 1



- Radial basis functions (RBFs)

$$\text{dist}(\mathbf{x}) = \sum_i w_i \cdot \varphi(\|\mathbf{x} - \mathbf{c}_i\|)$$

  - Solve symmetric linear system for weights w_i

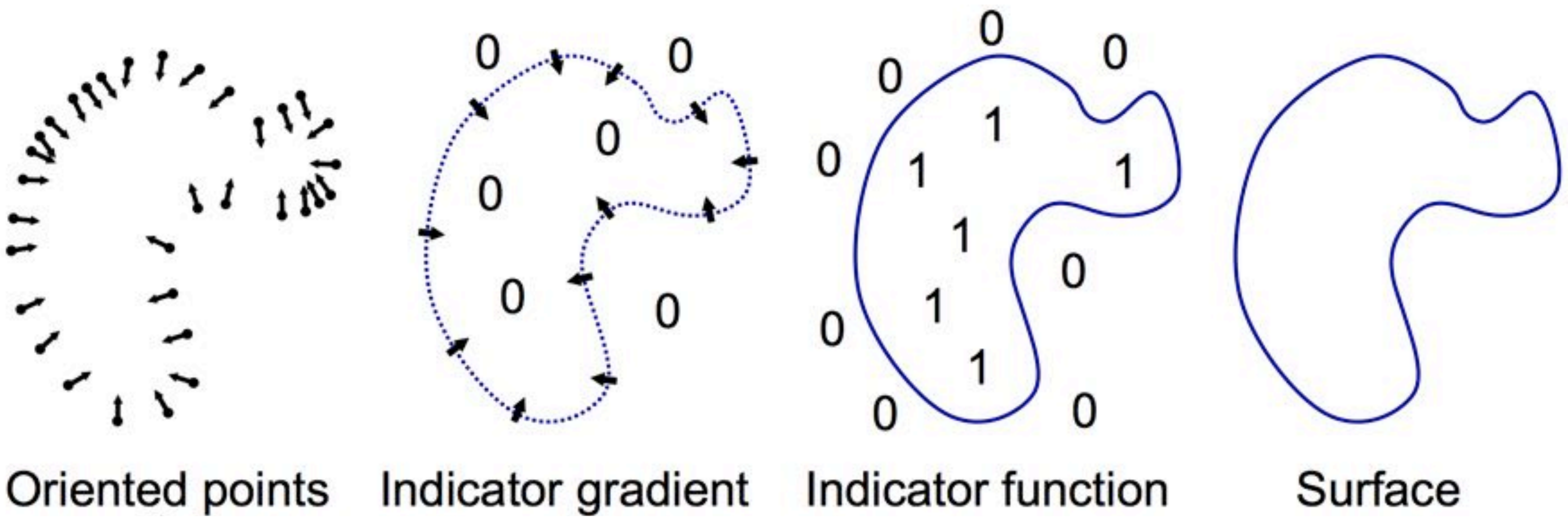
Hoppe '92

Compact RBF Wendland C^2

Global RBF Triharmonic

Poisson Surface Reconstruction

– relationship between normal field and gradient of indicator function



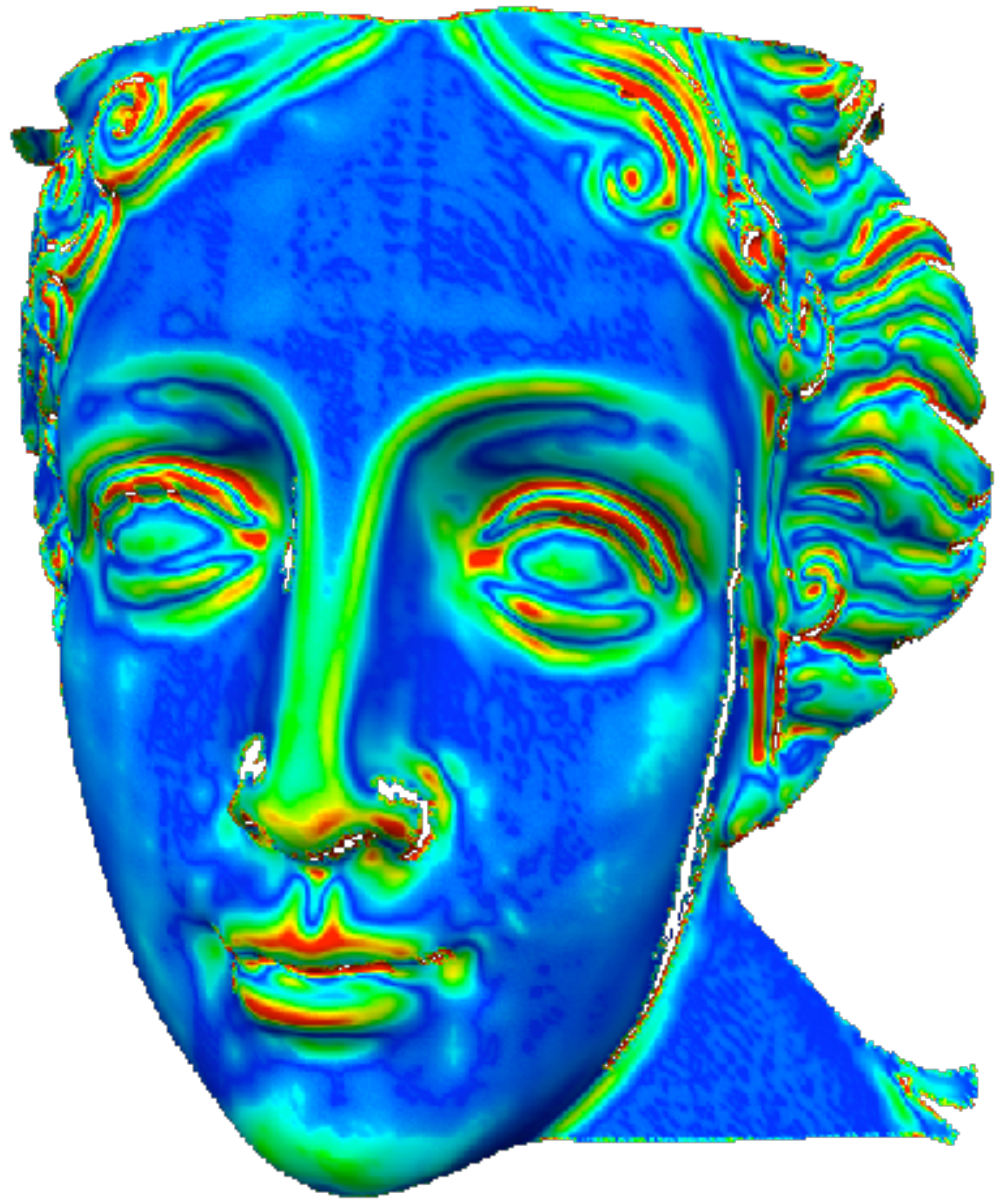
$\vec{V}$ $\nabla \chi_M$ χ_M ∂M

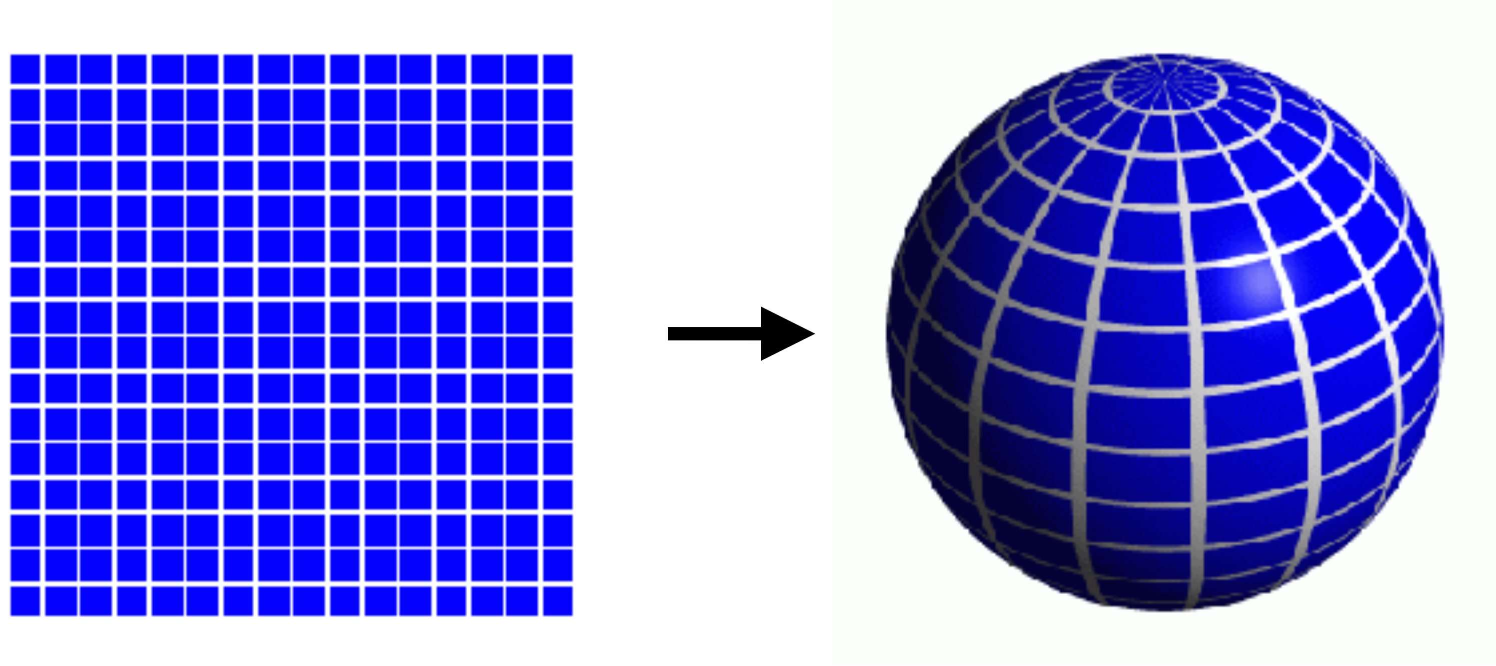
Outline

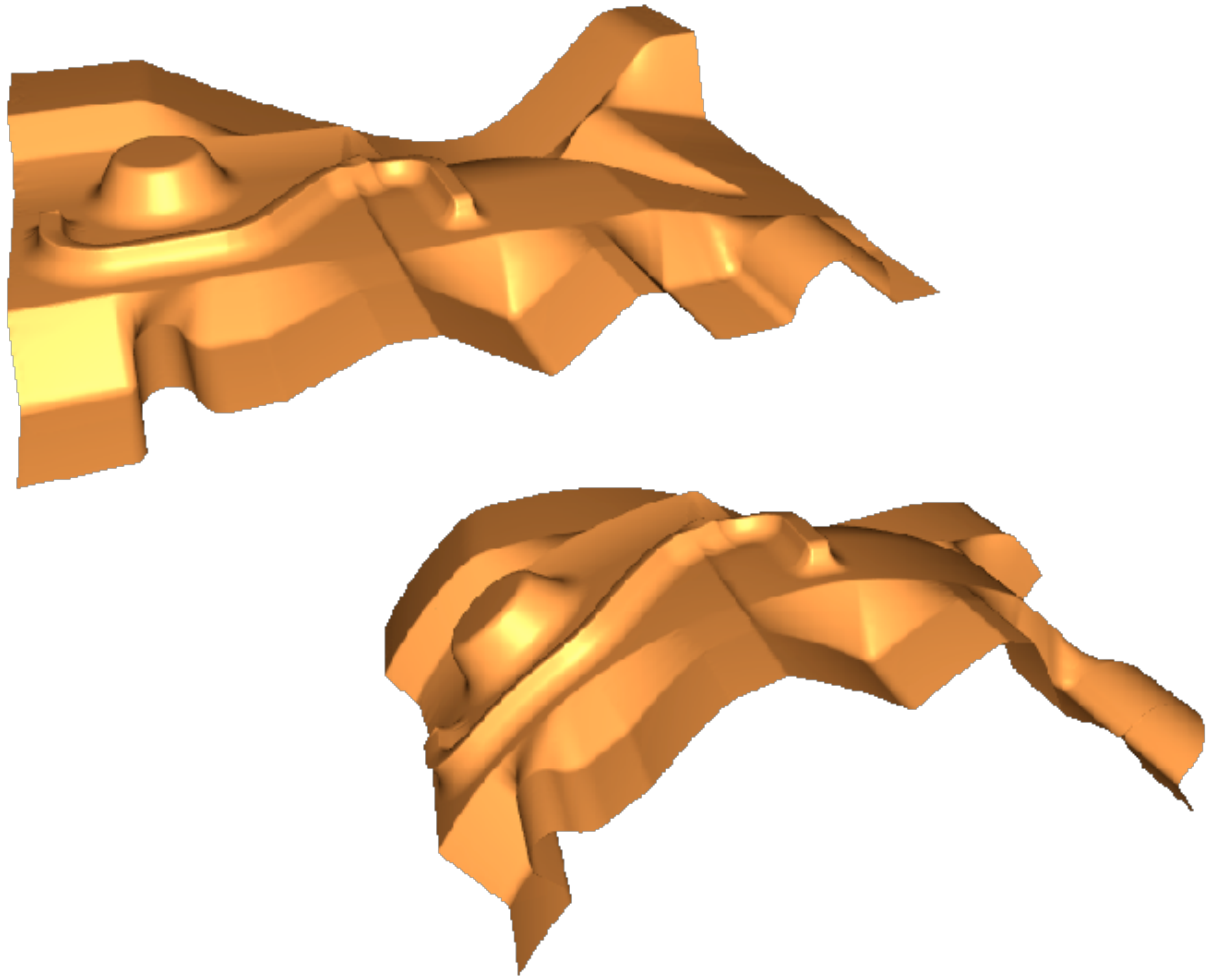
- **Differential Geometry**
- Discrete Differential Geometry
- Mesh Quality Measures

Motivation

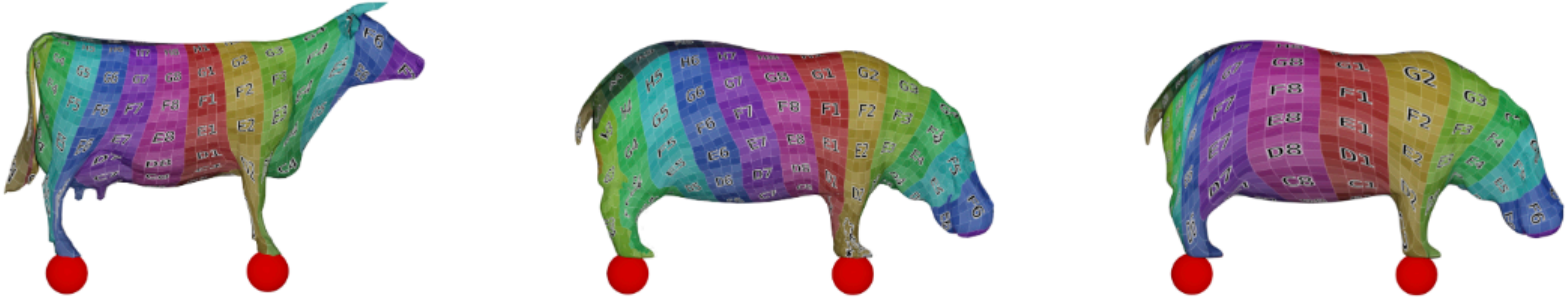
- We need differential geometry to compute
  - surface curvature
  - parameterization distortion
  - deformation energies



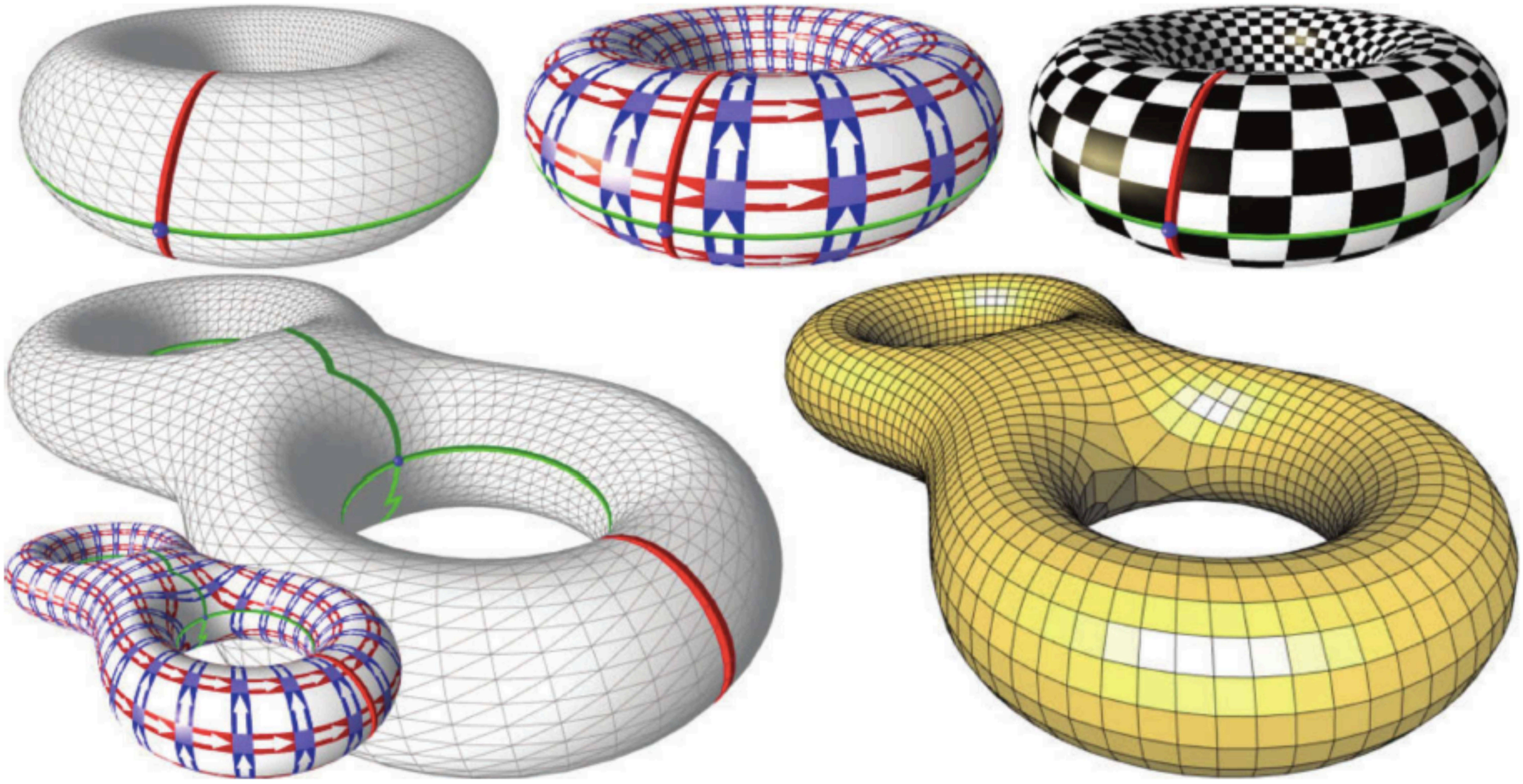




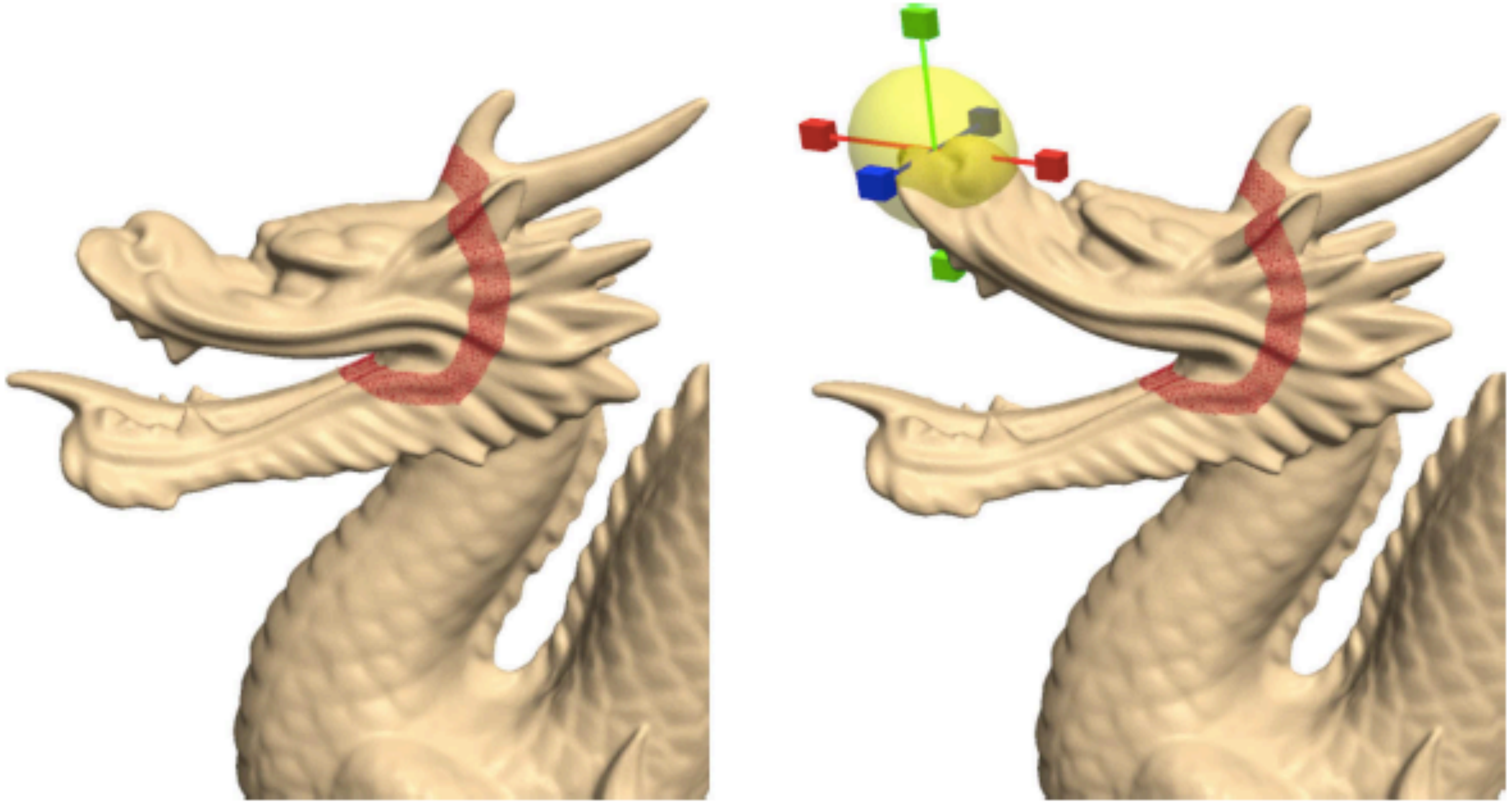
Texture Mapping

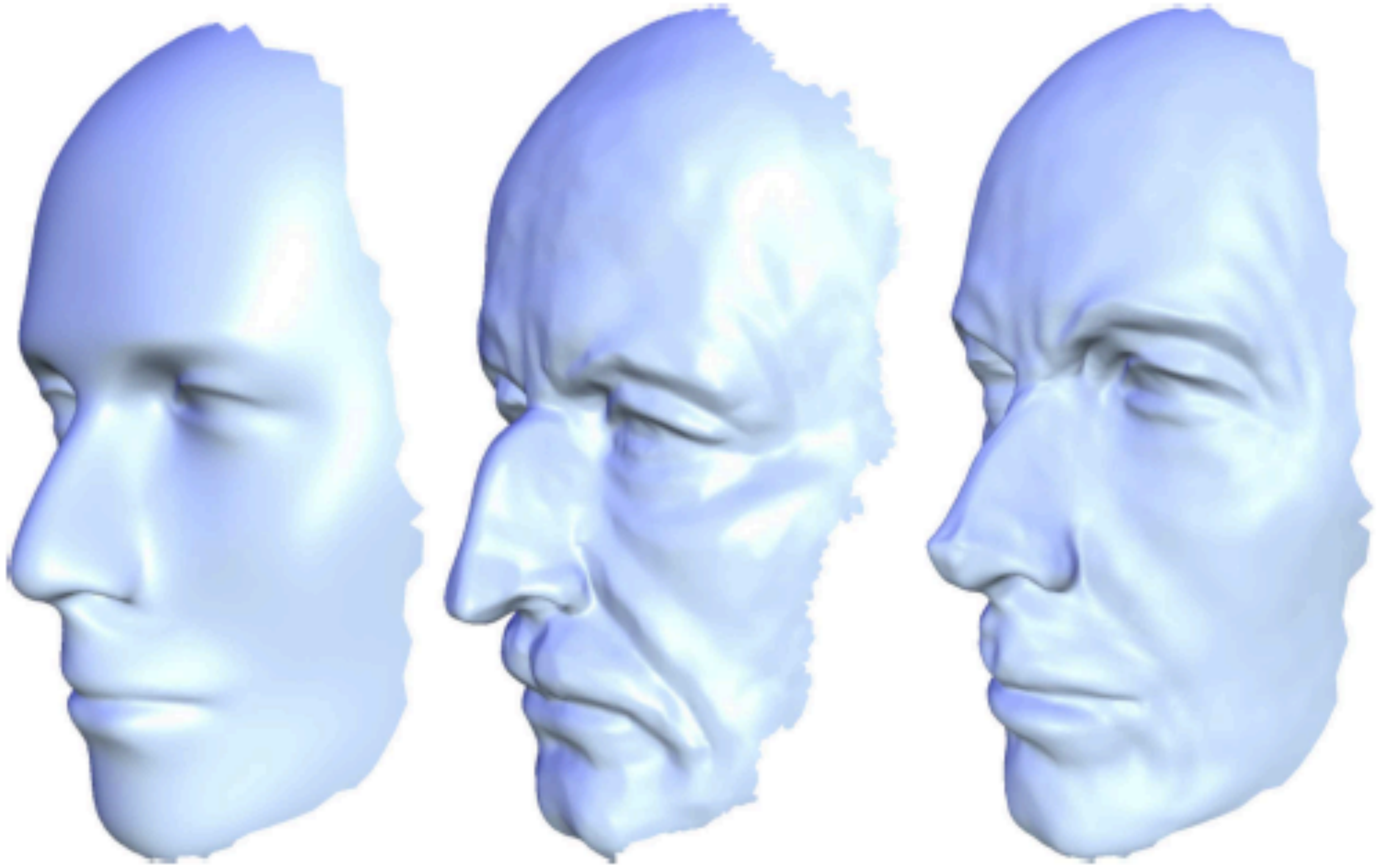


Parameterization + Meshing



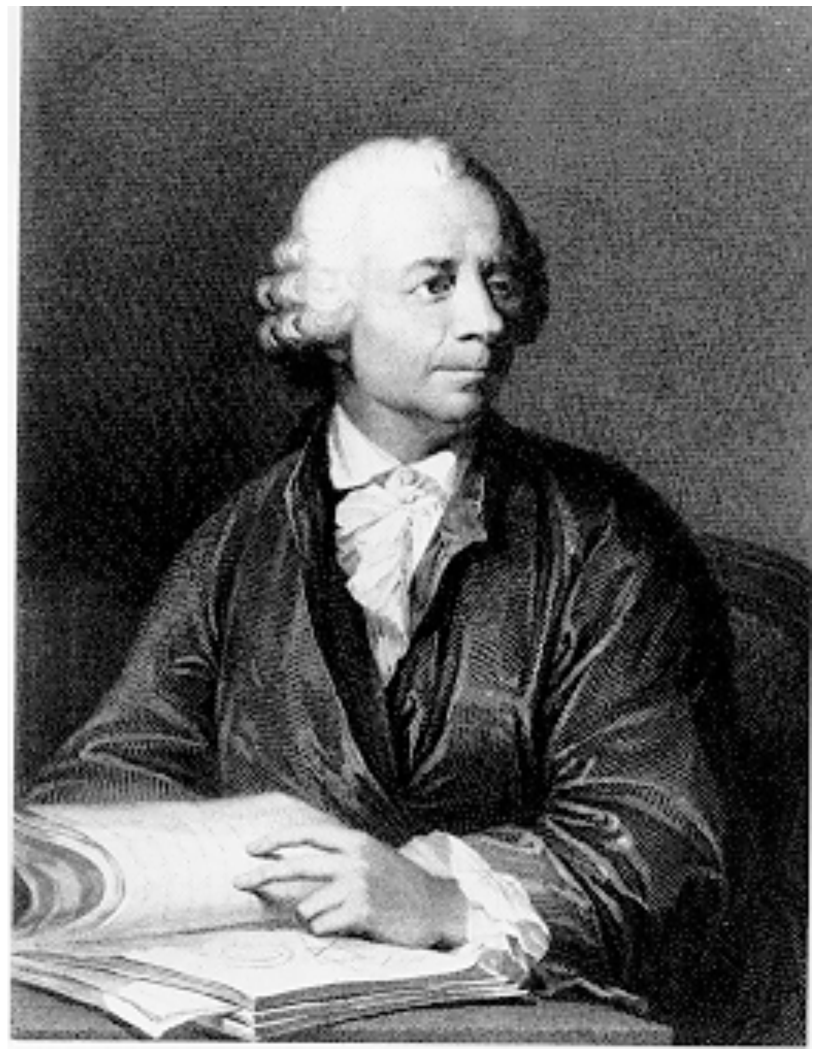
Surface Deformation





Differential Geometry

- Manfredo P. do Carmo: *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976; Chapters 1-4
- Andrew Pressley: *Elementary Differential Geometry*, Springer, 2010



Leonard Euler (1707 - 1783)

Carl Friedrich Gauss (1777 - 1855)

Overview — Curves

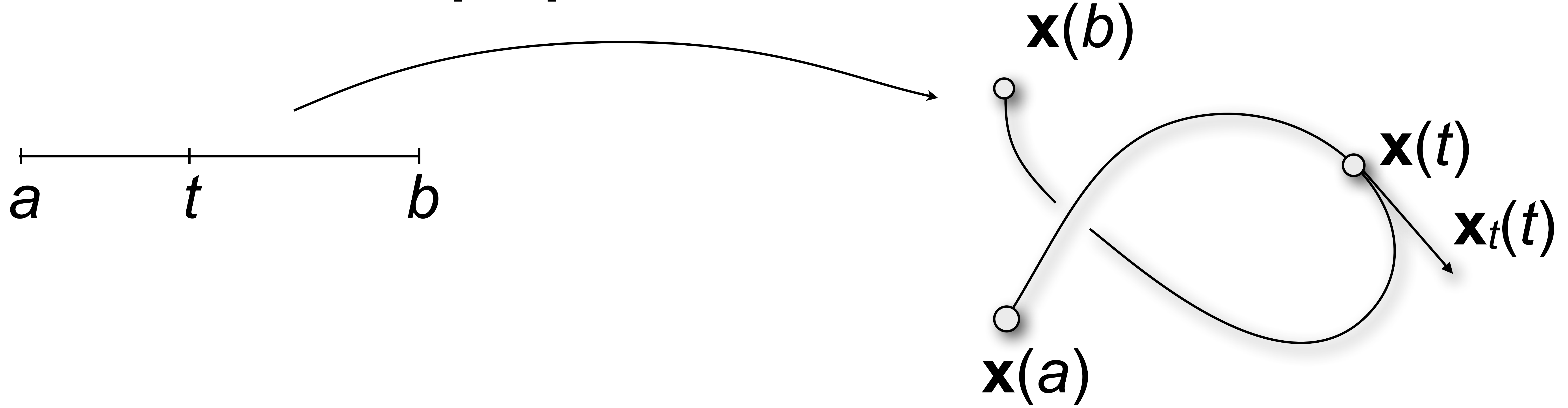
- **Parameterized** curves $\mathbf{x}(t)$ + **trace** of a curve
- **Ambiguity** due to parameterization; arc length parameterization $\alpha(s)$
- Instantaneous **velocity** $\alpha'(s)$ and **acceleration** $\alpha''(s)$
- constant **speed** versus constant **velocity**
- measuring curve length $\int_{t_0}^{t_1} ||\alpha'(t)|| dt$
- constant speed: velocity orthogonal to acceleration
- no self-intersection; regular parameterization

Overview (cont.)

- radius of curvature $\alpha''(s) = \kappa(s)\mathbf{n}(s)$
- Frenet frame $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$; osculating plane $\{\mathbf{t}(s), \mathbf{n}(s)\}$
- tangent, normal, binormal $\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s)$
- torsion $\tau(s)$
- Fundamental theorem of curves (we can recover curves uniquely, up to rigid transfers, from $\{\kappa(s), \tau(s)\}$
- ——- end curves

Parametric Curves

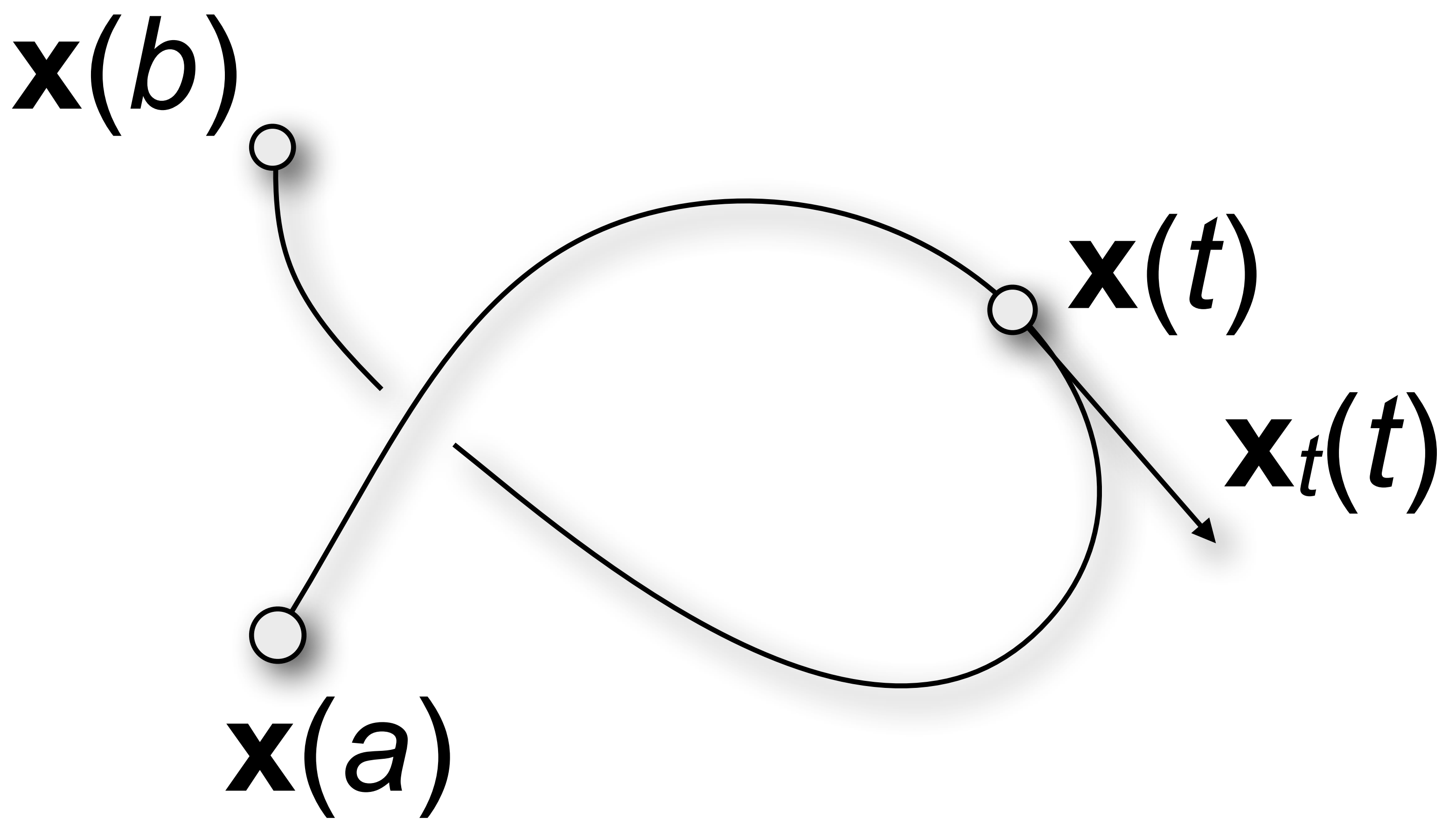
$$\mathbf{x} : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^3$$



$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad \mathbf{x}_t(t) := \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \begin{pmatrix} \mathrm{d}x(t)\,/\mathrm{d}t \\ \mathrm{d}y(t)\,/\mathrm{d}t \\ \mathrm{d}z(t)\,/\mathrm{d}t \end{pmatrix}$$

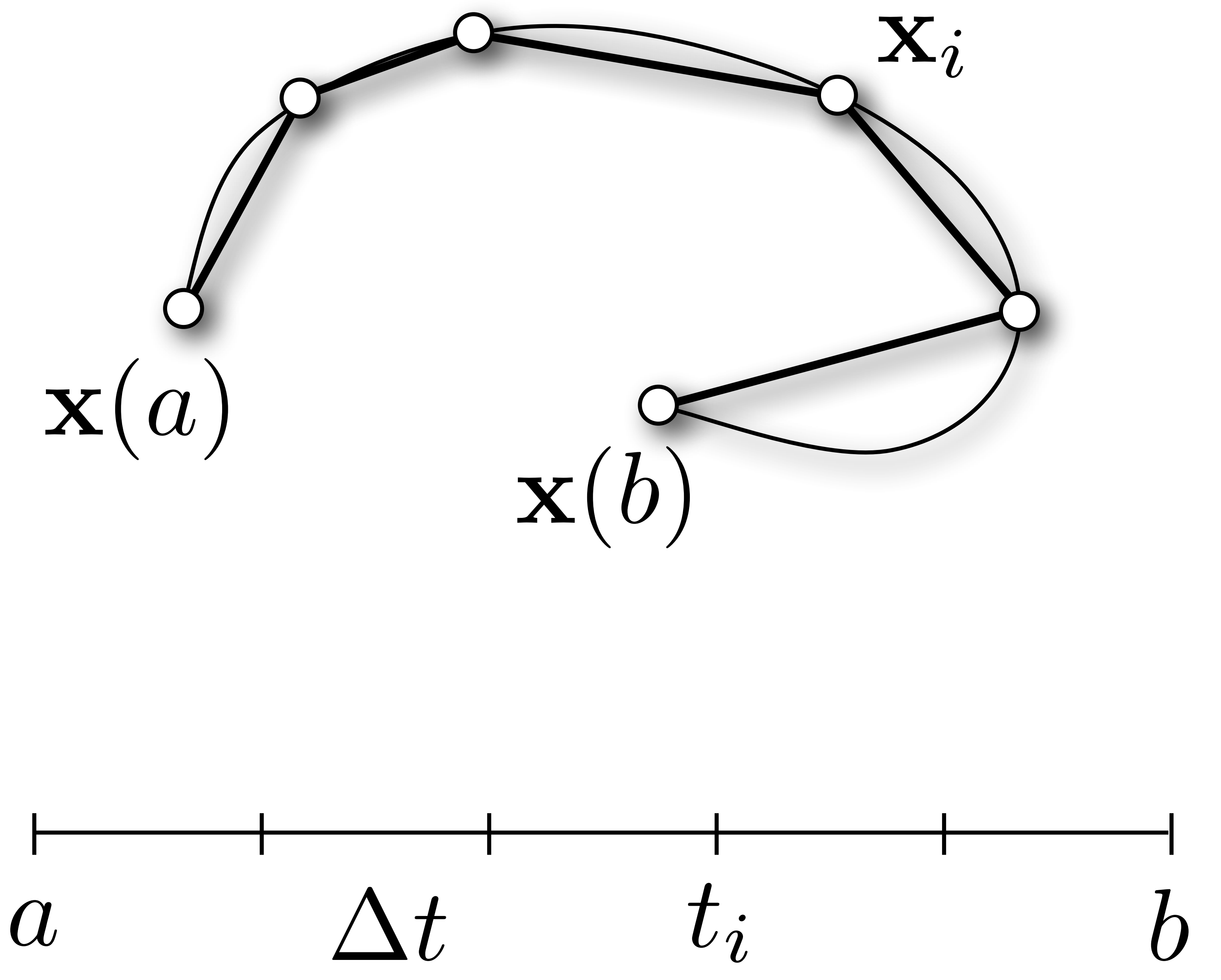
Parametric Curves

- A parametric curve $\mathbf{x}(t)$ is
  - *simple* $\mathbf{x}(t)$ is injective (no self-intersections)
  - *differentiable* $\mathbf{x}'(t) = \dfrac{d\mathbf{x}(t)}{dt}$ is defined for all $t \in [a, b]$
  - *regular* $\mathbf{x}'(t) \neq \mathbf{0}$ for all $t \in [a, b]$



Length of a Curve

- Let $t_i = a + i\Delta t$ and $\mathbf{x}_i = \mathbf{x}(t_i)$



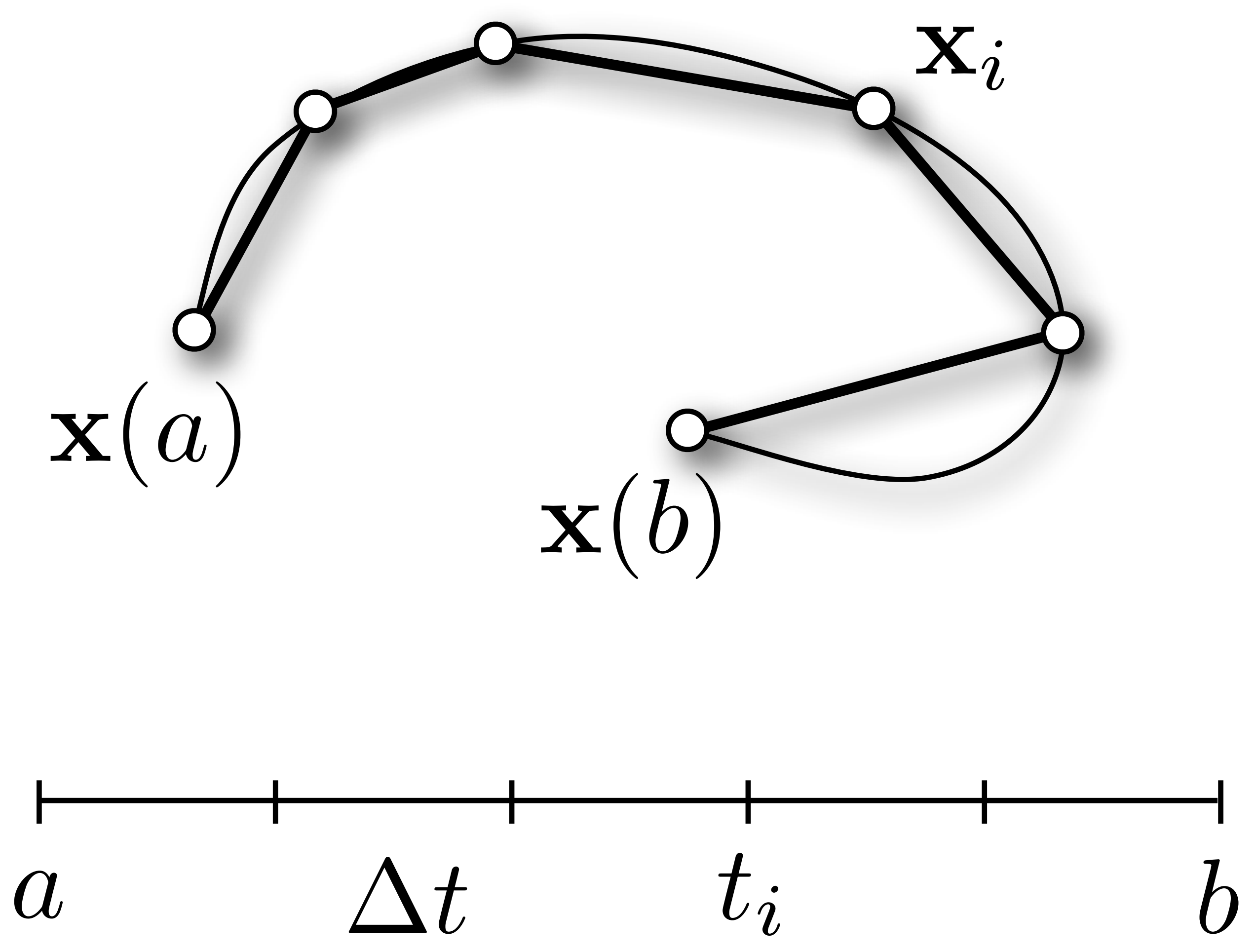
Length of a Curve

- Polyline *chord length*

$$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \left\| \frac{\Delta \mathbf{x}_i}{\Delta t} \right\| \Delta t \,, \quad \Delta \mathbf{x}_i := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$$

- Curve *arc length* $(\Delta t \rightarrow 0\,)$

$$s = s(t) = \int_a^t \|\mathbf{x}_t\| \,\mathrm{d}t$$



Re-Parameterization

- Mapping of parameter domain

$$u : [a, b] \to [c, d]$$

- Re-parameterization w.r.t. *u(t)*

$$[c, d] \to \mathrm{I\!R}^3 , \quad t \mapsto \mathbf{x}(u(t))$$

- Derivative (chain rule)

$$\frac{\mathrm{d}\mathbf{x}(u(t))}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t} = \mathbf{x}_u(u(t))\ u_t(t)$$

Re-Parameterization

- Example

$$\mathbf{f} : \left[0, \frac{1}{2}\right] \to \mathbb{R}^2 \quad , \quad t \mapsto (4t, 2t)$$

$$\phi : \left[0, \frac{1}{2}\right] \to [0, 1] \quad , \quad t \mapsto 2t$$

$$\mathbf{g} : [0, 1] \to \mathbb{R}^2 \quad , \quad t \mapsto (2t, t)$$

$$\Rightarrow \quad \mathbf{g}(\phi(t)) \; = \; \mathbf{f}(t)$$

Arc Length Parameterization

- Mapping of parameter domain:

$$t \;\mapsto\; s(t) = \int_a^t \|\mathbf{x}_t\| \, \mathrm{d}t$$

- Parameter *s* for **x**(*s*) equals length from **x**(*a*) to **x**(*s*)

$$\mathbf{x}(s) \;=\; \mathbf{x}(s(t)) \qquad\qquad \mathrm{d}s \;=\; \|\mathbf{x}_t\| \, \mathrm{d}t$$

- Special properties of resulting curve

$$\|\mathbf{x}_s(s)\| = 1 \,, \quad \mathbf{x}_s(s) \cdot \mathbf{x}_{ss}(s) = 0$$

The Frenet Frame

- Taylor expansion

$$\mathbf{x}(t+h) \;=\; \mathbf{x}(t) \;+\; \mathbf{x}_t(t)\,h \;+\; \frac{1}{2}\mathbf{x}_{tt}(t)\,h^2 \;+\; \frac{1}{6}\mathbf{x}_{ttt}(t)\,h^3 \;+\; \ldots$$

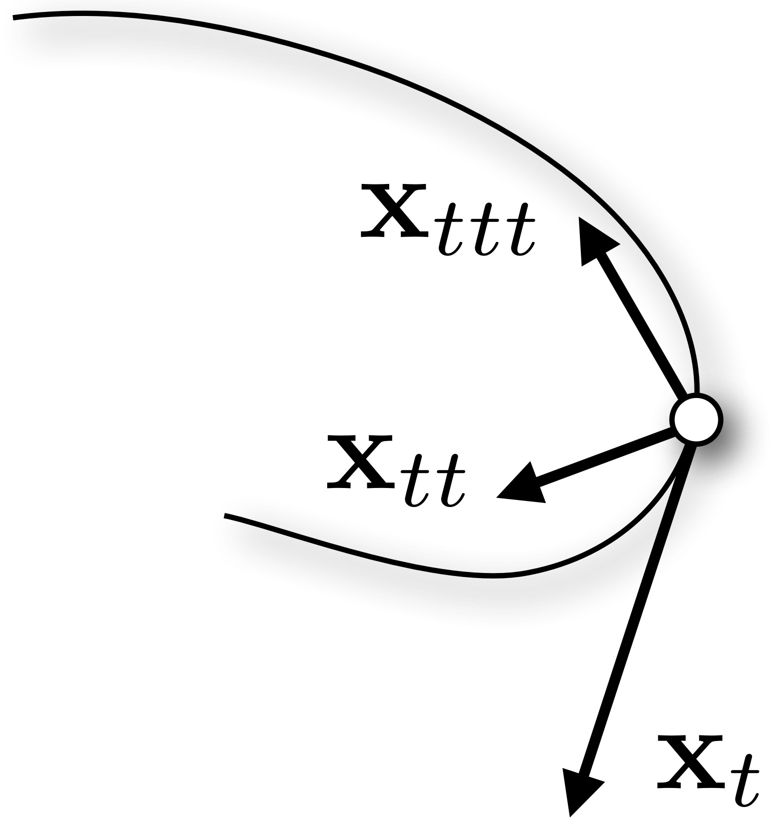
- Define local frame (**t**, **n**, **b**) (*Frenet frame*)

$$\mathbf{t} \;=\; \frac{\mathbf{x}_t}{\|\mathbf{x}_t\|} \qquad \mathbf{n} = \frac{\mathbf{x}_{tt}}{\|\mathbf{x}_{tt}\|} \qquad \mathbf{b} = \mathbf{t} \times \mathbf{n}$$

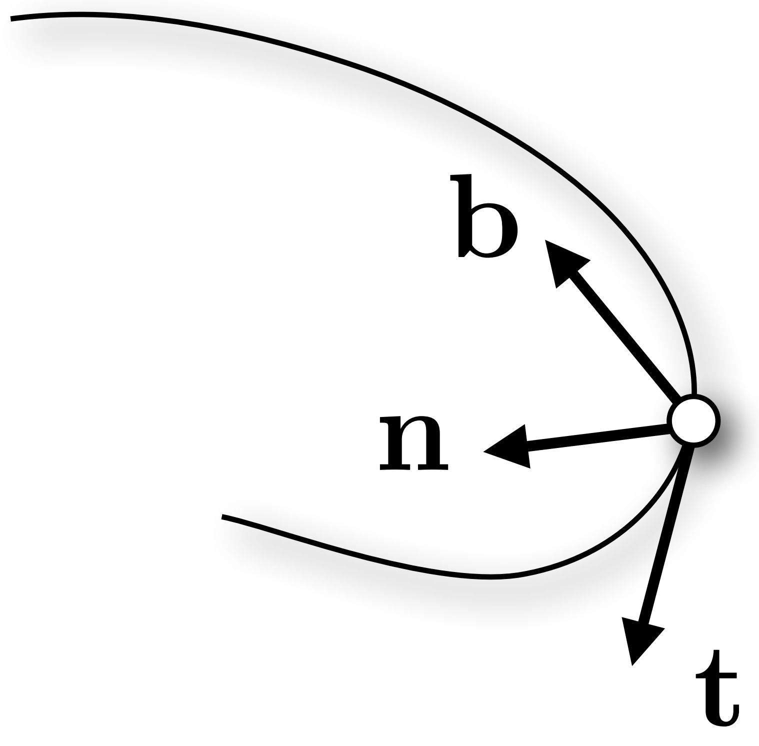
tangent | principal normal | binormal

The Frenet Frame

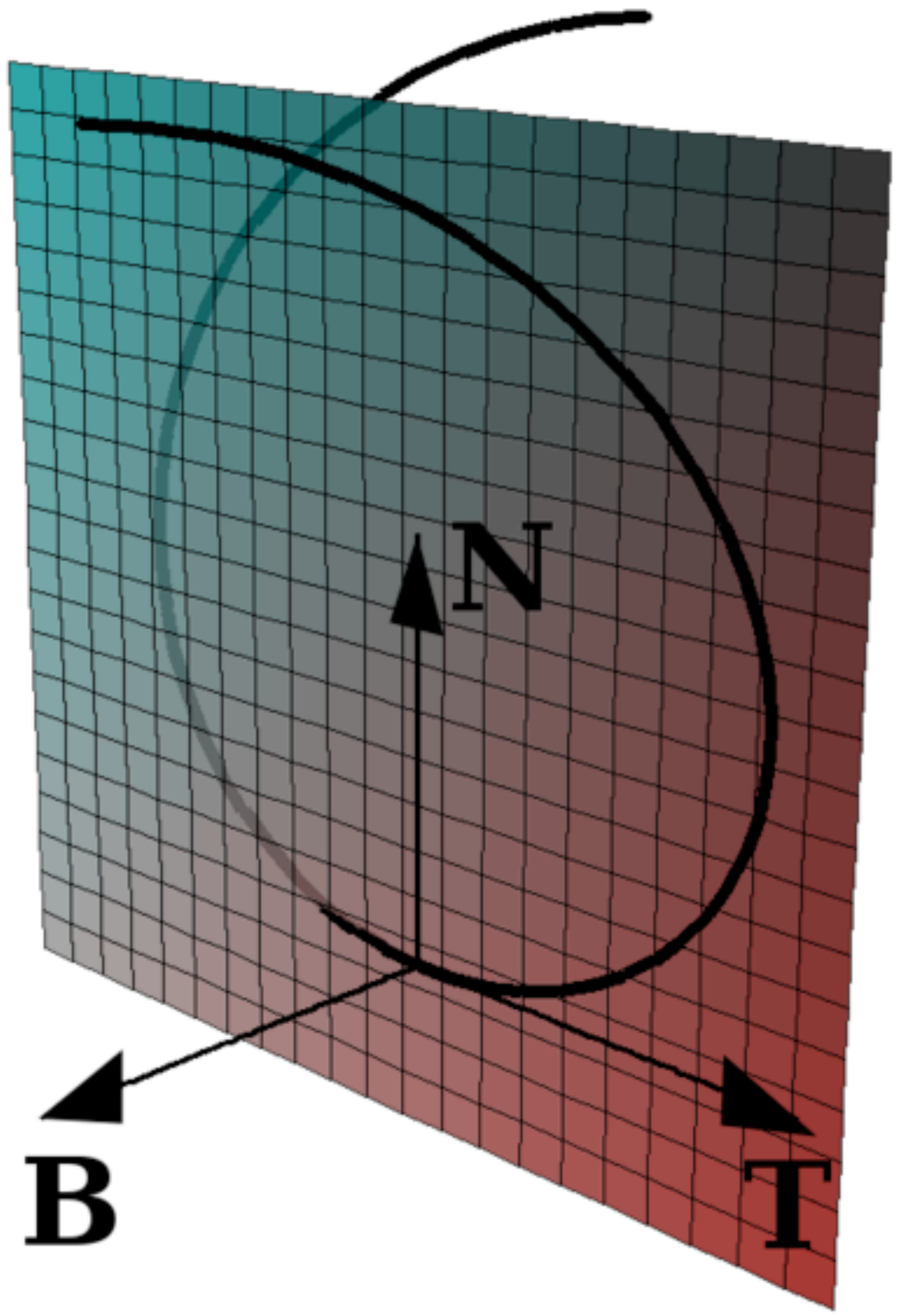
- Orthonormalization of local frame



local affine frame



Frenet frame



osculating plane

The Frenet Frame

- *Frenet-Serret*: Derivatives w.r.t. arc length *s*

$$\begin{aligned} \mathbf{t}_s &= & &+\kappa\mathbf{n} & \\ \mathbf{n}_s &= -\kappa\mathbf{t} & & &+\tau\mathbf{b} \\ \mathbf{b}_s &= & &-\tau\mathbf{n} & \end{aligned}$$

- Curvature (*deviation from straight line*)

$$\kappa = \|\mathbf{x}_{ss}\|$$

- Torsion (*deviation from planarity*)

$$\tau = \frac{1}{\kappa^2}\det([\mathbf{x}_s, \mathbf{x}_{ss}, \mathbf{x}_{sss}])$$

Curvature and Torsion

- Planes defined by **x** and two vectors:
  - *osculating plane*: vectors **t** and **n**
  - *normal plane*: vectors **n** and **b**
  - *rectifying plane*: vectors **t** and **b**
- Osculating circle
  - second order contact with curve
  - center $\mathbf{c} = \mathbf{x} + (1/\kappa)\mathbf{n}$
  - radius $1/\kappa$

Curvature and Torsion

- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
- Independent of parameterization
  - intrinsic properties of the curve
- Euclidean invariants
  - invariant under rigid motion
- Define curve uniquely up to rigid motion (Fundamental theorem of curves)

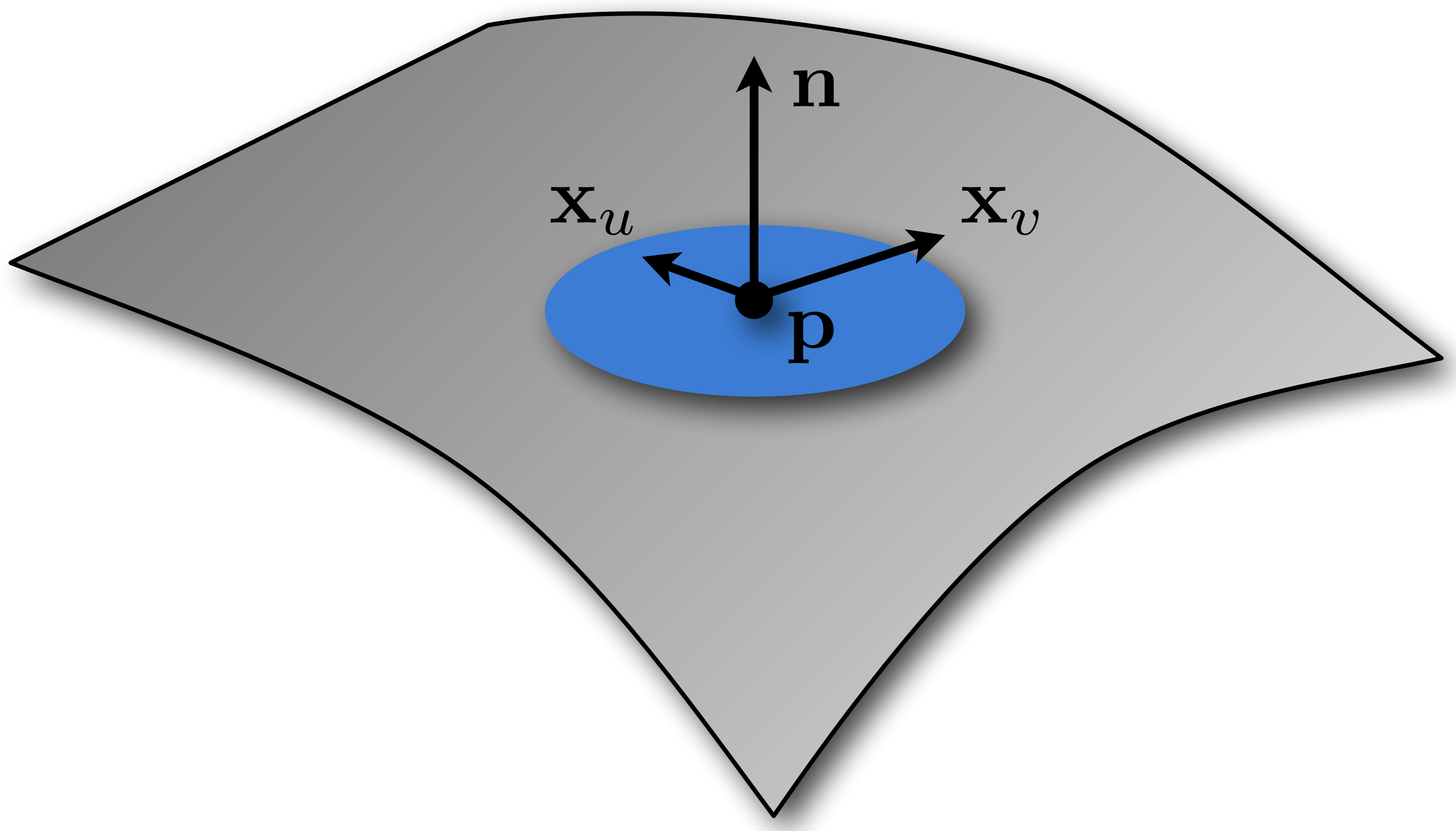
Differential Geometry of Surfaces

- Recap: multivariate chain rule
- Parameterization/map $\mathbf{x}(u, v)$
- Directional derivatives $\mathbf{x}_u, \mathbf{x}_v$
- Normal $\mathbf{n}(u, v) = \mathbf{x}_u(u, v) \wedge \mathbf{x}_v(u, v)$ and tangent plane $T_p(S)$
- First fundamental form $I_p(\mathbf{v}) = < \mathbf{v}, \mathbf{v} >_p$ for $\mathbf{v} \in T_p(S)$ given by E, F, G
- Second fundamental form $II_p(\mathbf{v}) = - < dN(\mathbf{v}), \mathbf{v} >_p$ for $\mathbf{v} \in T_p(S)$ given by e, f, g

Parametric Surfaces

- Continuous surface

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$



- Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

- Assume *regular* parameterization

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$

Angles on Surface

- Curve [*u*(*t*), *v*(*t*)] in *uv*-plane defines curve on the surface **x***(u,v)*

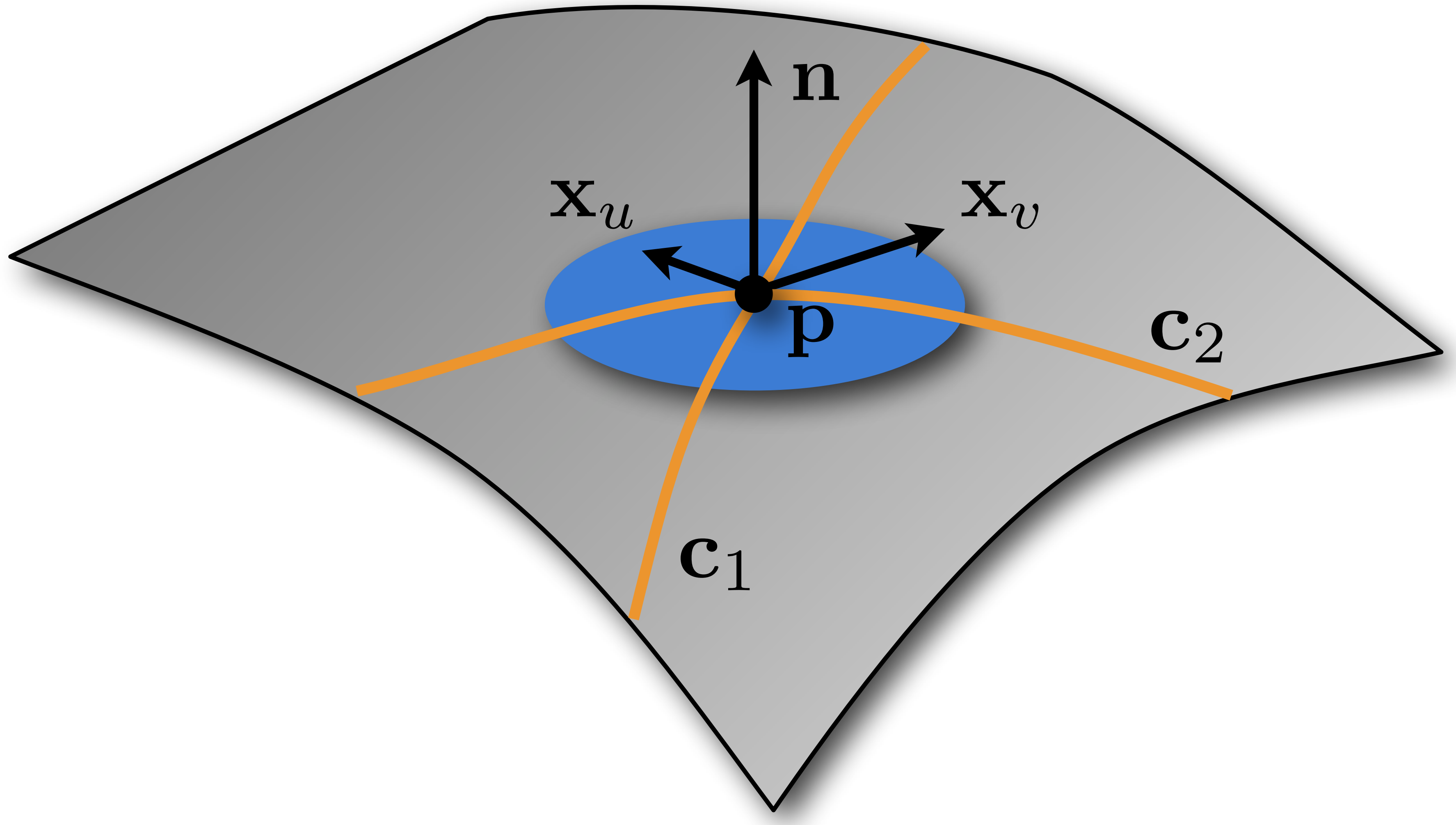
$$\mathbf{c}(t) \;=\; \mathbf{x}(u(t)\,, v(t))$$

- Two curves $\mathbf{c}_1$ and $\mathbf{c}_2$ intersecting at **p**
  – Angle of intersection?
  – Two tangents $\mathbf{t}_1$ and $\mathbf{t}_2$

$$\mathbf{t}_i \;=\; \alpha_i \mathbf{x}_u + \beta_i \mathbf{x}_v$$

  – Compute inner product

$$\mathbf{t}_1^T \mathbf{t}_2 \;=\; \cos\theta \,\|\mathbf{t}_1\|\,\|\mathbf{t}_2\|$$



Angles on Surface

- Curve [*u*(*t*), *v*(*t*)] in *uv*-plane defines curve on the surface **x***(u,v)*

$$\mathbf{c}(t) = \mathbf{x}(u(t), v(t)) \qquad c' = u' x_u + v' x_v$$

- Two curves $\mathbf{c}_1$ and $\mathbf{c}_2$ intersecting at **p**

$$\begin{aligned}
\mathbf{t}_1^T \mathbf{t}_2 &= (\alpha_1 \mathbf{x}_u + \beta_1 \mathbf{x}_v)^T (\alpha_2 \mathbf{x}_u + \beta_2 \mathbf{x}_v) \\
&= \alpha_1 \alpha_2 \mathbf{x}_u^T \mathbf{x}_u + (\alpha_1 \beta_2 + \alpha_2 \beta_1) \mathbf{x}_u^T \mathbf{x}_v + \beta_1 \beta_2 \mathbf{x}_v^T \mathbf{x}_v \\
&= (\alpha_1, \beta_1) \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}
\end{aligned}$$

First Fundamental Form

- First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

- Defines *inner product on tangent space*

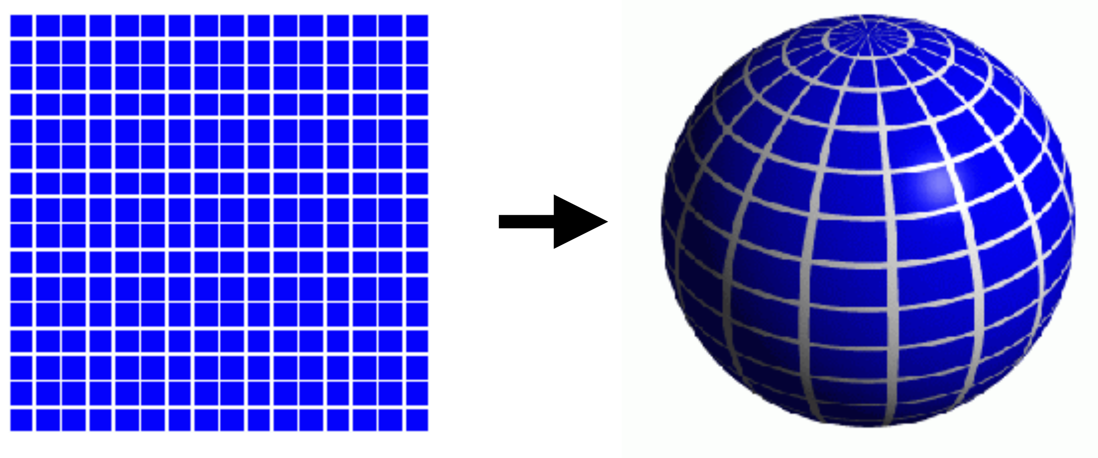
$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

First Fundamental Form

- First fundamental form **I** allows to measure (*with respect to surface metric*)
  - Angle $$\mathbf{t}_1^T \mathbf{t}_2 = \langle (\alpha_1, \beta_1)\ ,\ (\alpha_1, \beta_1) \rangle$$
  - Length $$\begin{aligned} \mathrm{d}s^2 &= \langle (\mathrm{d}u, \mathrm{d}v), (\mathrm{d}u, \mathrm{d}v) \rangle \\ &= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 \end{aligned}$$
  - Area $$\begin{aligned} \mathrm{d}A &= \|\mathbf{x}_u \times \mathbf{x}_v\| \,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{\mathbf{x}_u^T \mathbf{x}_u \cdot \mathbf{x}_v^T \mathbf{x}_v - (\mathbf{x}_u^T \mathbf{x}_v)^2}\,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{EG - F^2}\mathrm{d}u\,\mathrm{d}v \end{aligned}$$

Example: Sphere

- Spherical parameterization



$$\mathbf{x}(u,v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}, \quad (u,v) \in [0, 2\pi) \times [0, \pi)$$

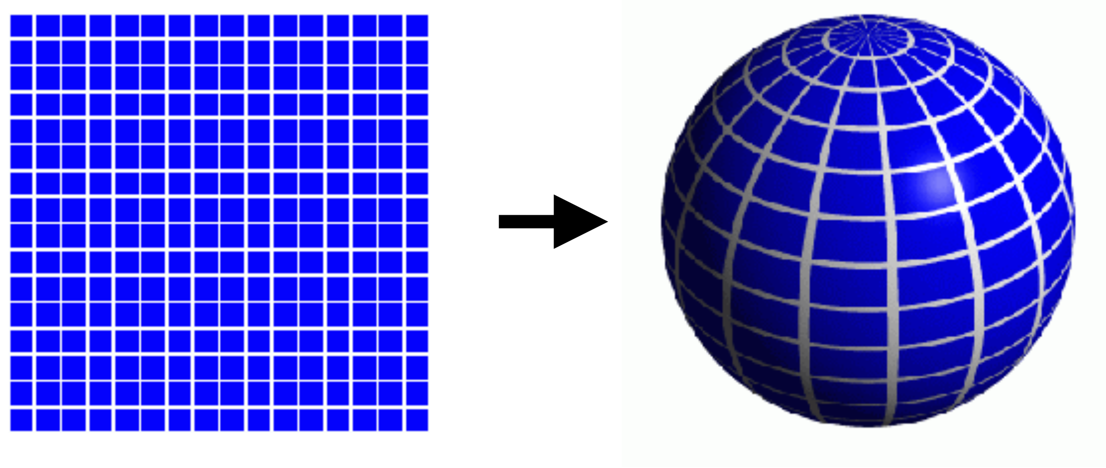
- Tangent vectors

$$\mathbf{x}_u(u,v) = \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix} \quad \mathbf{x}_v(u,v) = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ -\sin v \end{pmatrix}$$

- First fundamental form

$$\mathbf{I} = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

Example: Sphere



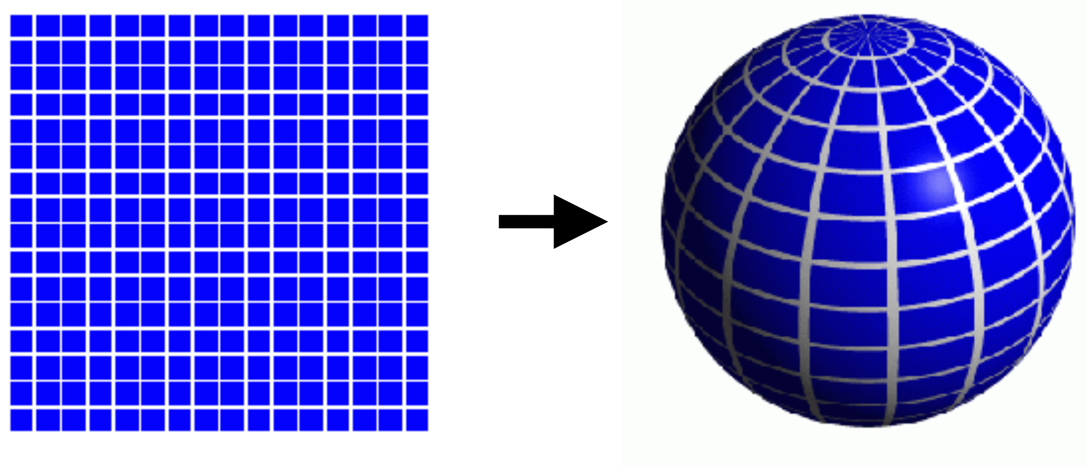
- Length of equator i.e., v = $\pi/2$

$$\int_0^{2\pi} 1\,\mathrm{d}s = \int_0^{2\pi} \sqrt{E\left(u_t\right)^2 + 2F u_t v_t + G\left(v_t\right)^2}\,\mathrm{d}t$$

$$= \int_0^{2\pi} \sin v\,\mathrm{d}t$$

$$= 2\pi \sin v = 2\pi$$

Example: Sphere



- Area of a sphere

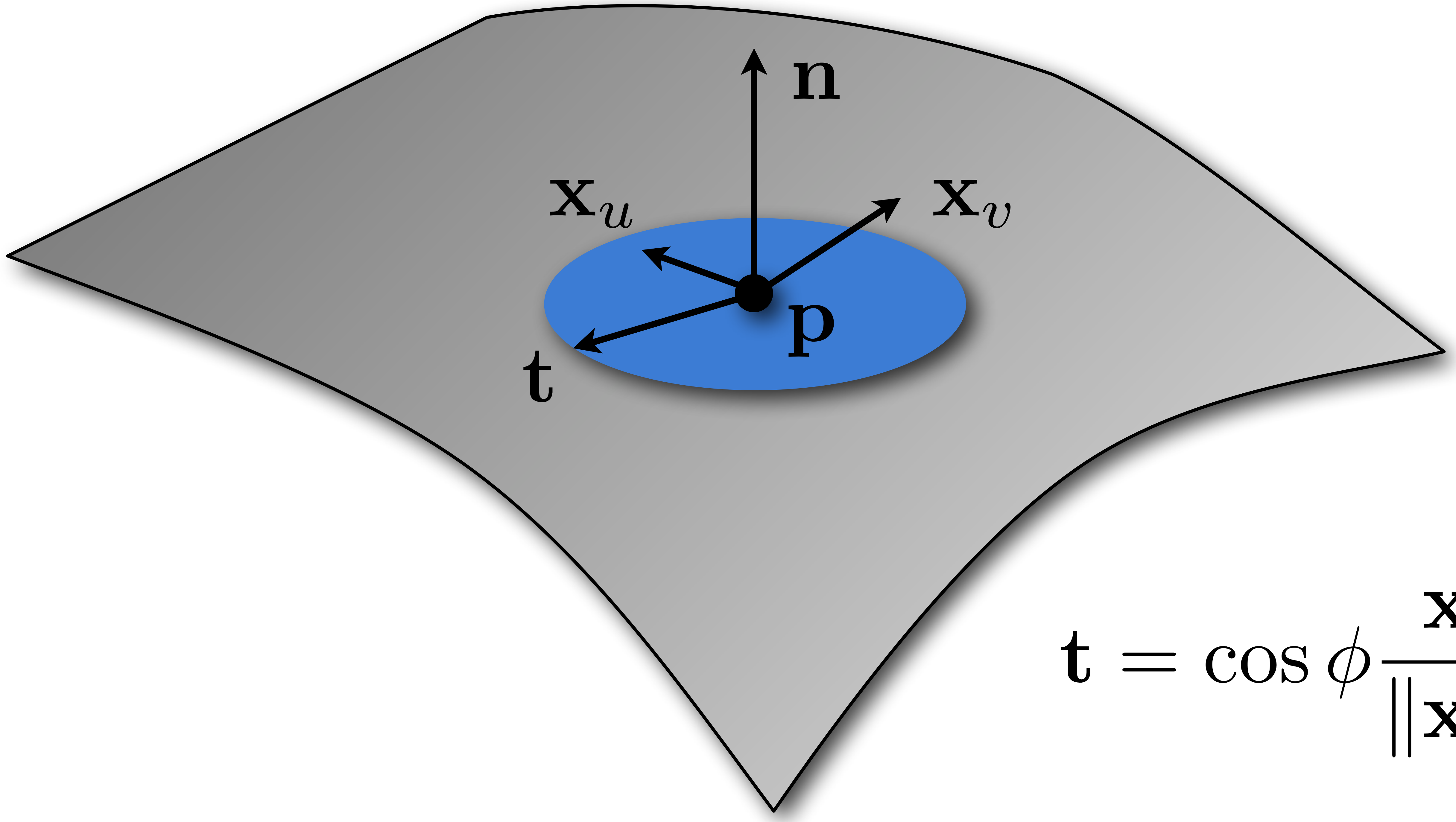
$$\int_0^{\pi}\int_0^{2\pi} 1\,\mathrm{d}A = \int_0^{\pi}\int_0^{2\pi}\sqrt{EG-F^2}\,\mathrm{d}u\,\mathrm{d}v$$

$$= \int_0^{\pi}\int_0^{2\pi}\sin v\,\mathrm{d}u\,\mathrm{d}v$$

$$= 4\pi$$

Normal Curvature

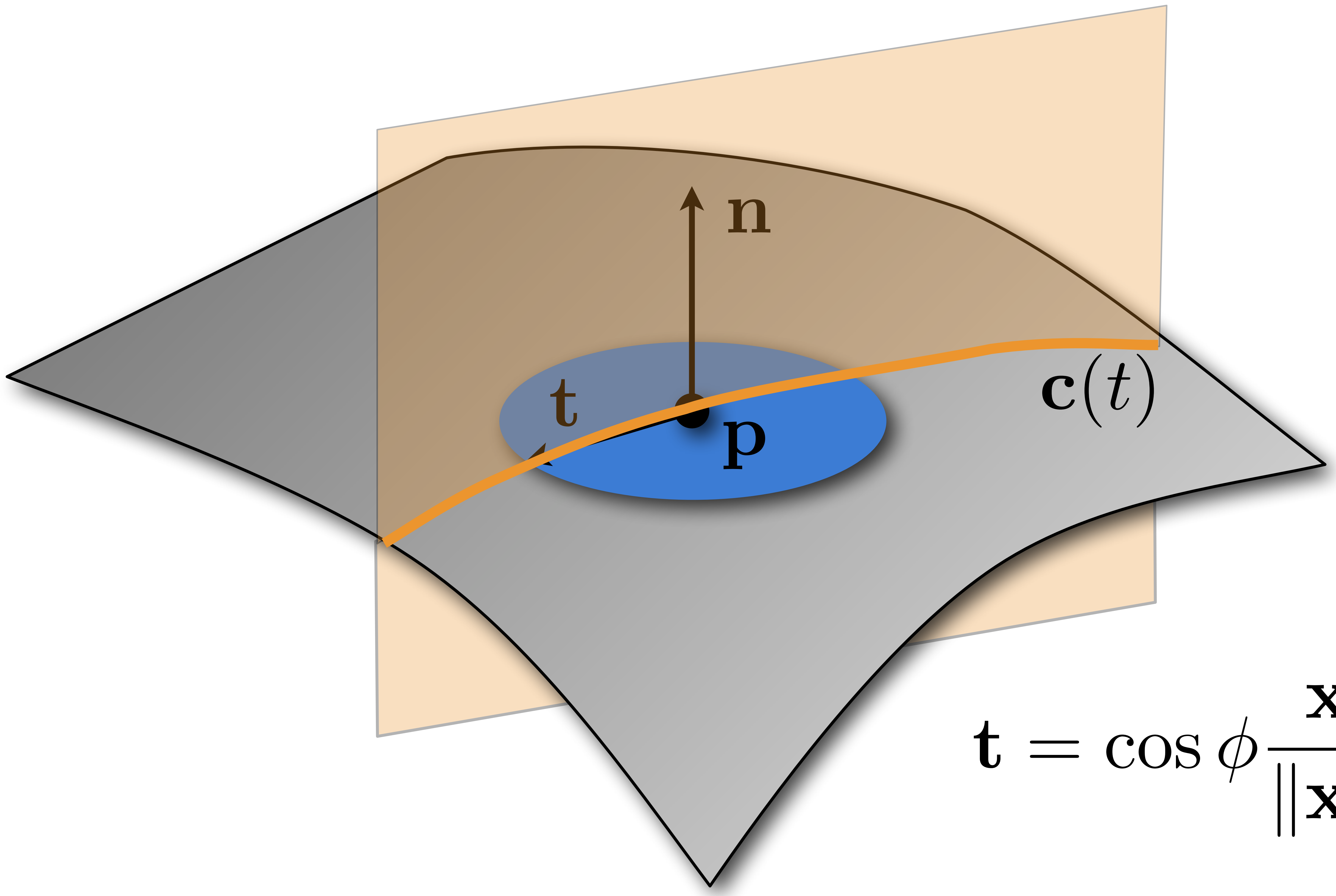
- Tangent vector $\mathbf{t} \in T_p(S)$



$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Normal Curvature

- .. defines intersection plane, yielding curve **c**(*t*)



$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Normal Curvature

- Normal curvature $\kappa_n(\mathbf{t})$ is defined as curvature of the normal curve $\mathbf{c}(t)$ at point $\mathbf{p} = \mathbf{x}(u, v)$.

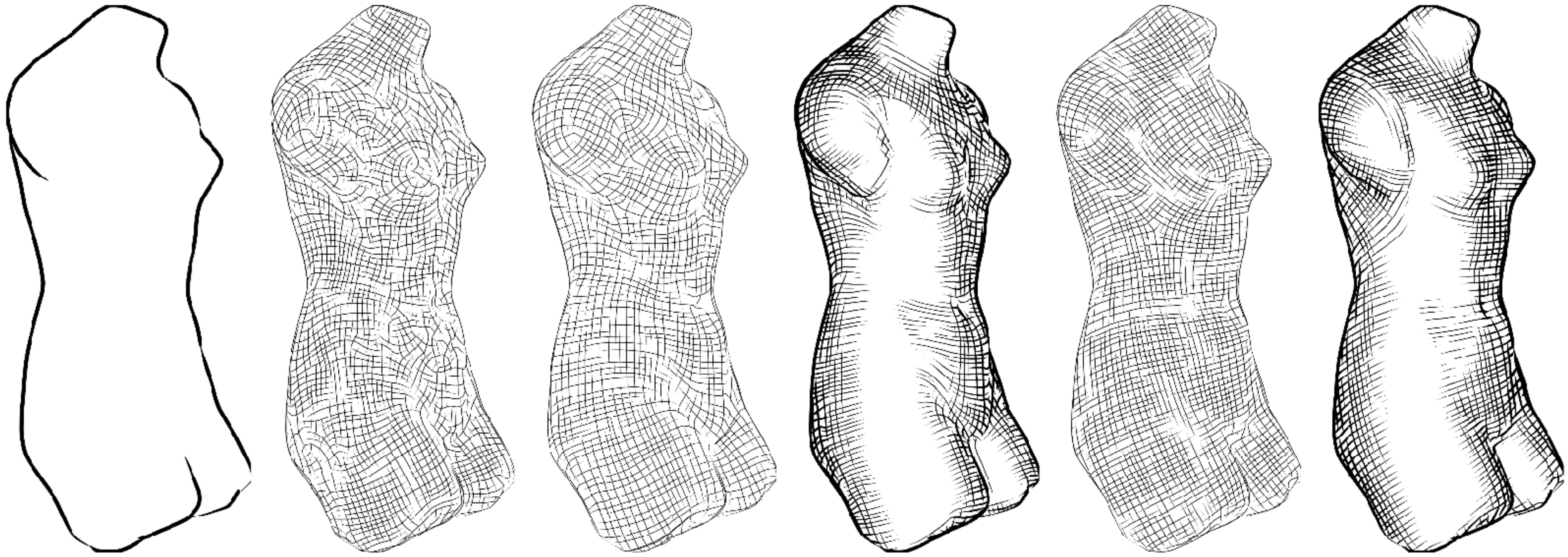
- With **second fundamental** form

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{pmatrix}$$

normal curvature can be computed as

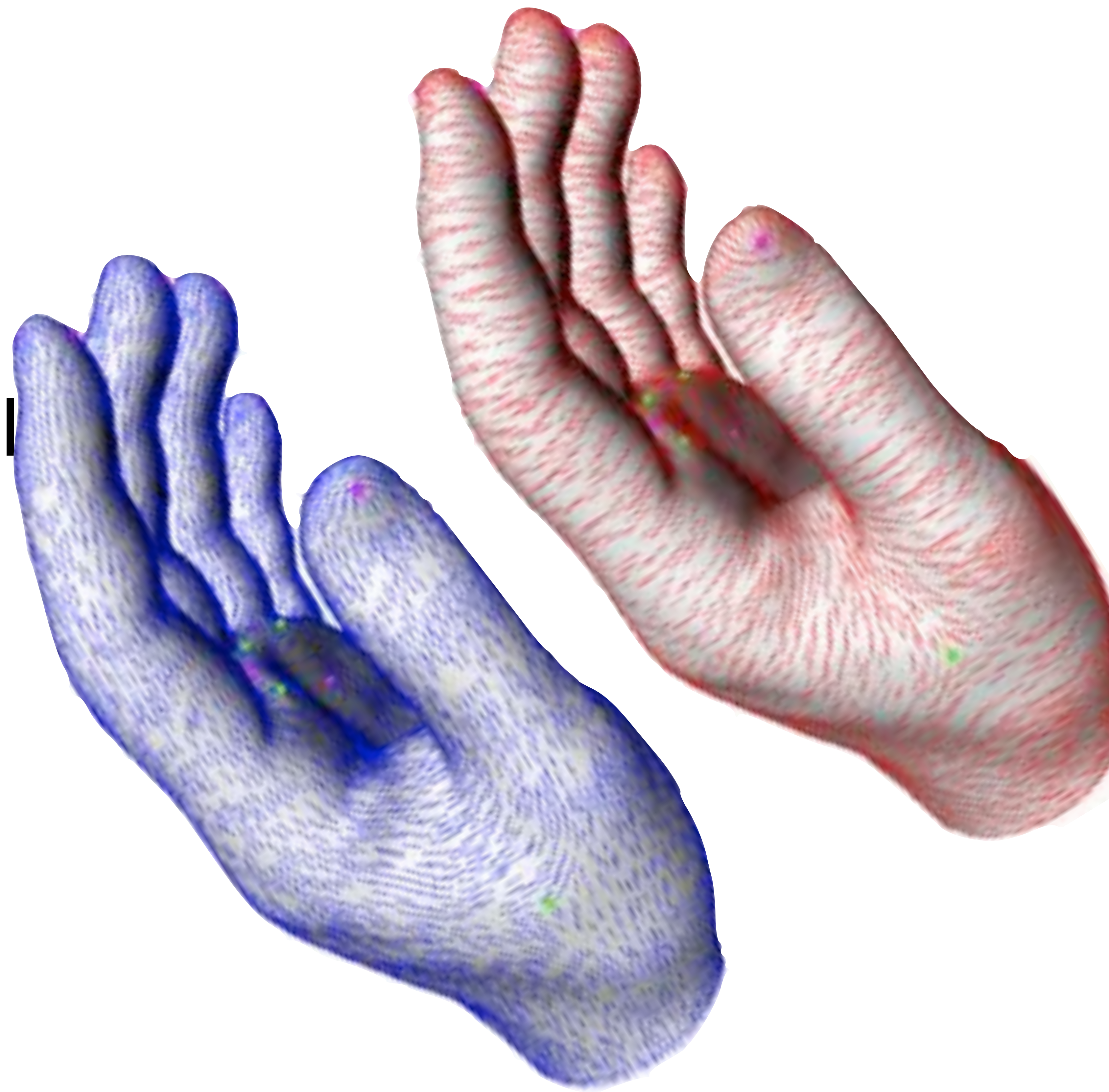
$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \, \mathbf{II} \, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \, \mathbf{I} \, \bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2} \qquad \begin{aligned} \mathbf{t} &= a\mathbf{x}_u + b\mathbf{x}_v \\ \bar{\mathbf{t}} &= (a, b) \end{aligned}$$

Principal Curvature Directions



Surface Curvature(s)

- *Principal curvatures*
  - Maximum curvature $$\kappa_1 = \max_{\phi} \kappa_n(\phi)$$
  - Minimum curvature $$\kappa_2 = \min_{\phi} \kappa_n(\phi)$$
  - Euler theorem: $$\kappa_n(\phi) = \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi$$
  - Corresponding *principal directions* $\mathbf{e}_1$, $\mathbf{e}_2$ are orthogonal



Surface Curvature(s)

- *Principal curvatures*
  - Maximum curvature $$\kappa_1 = \max_{\phi} \kappa_n(\phi)$$
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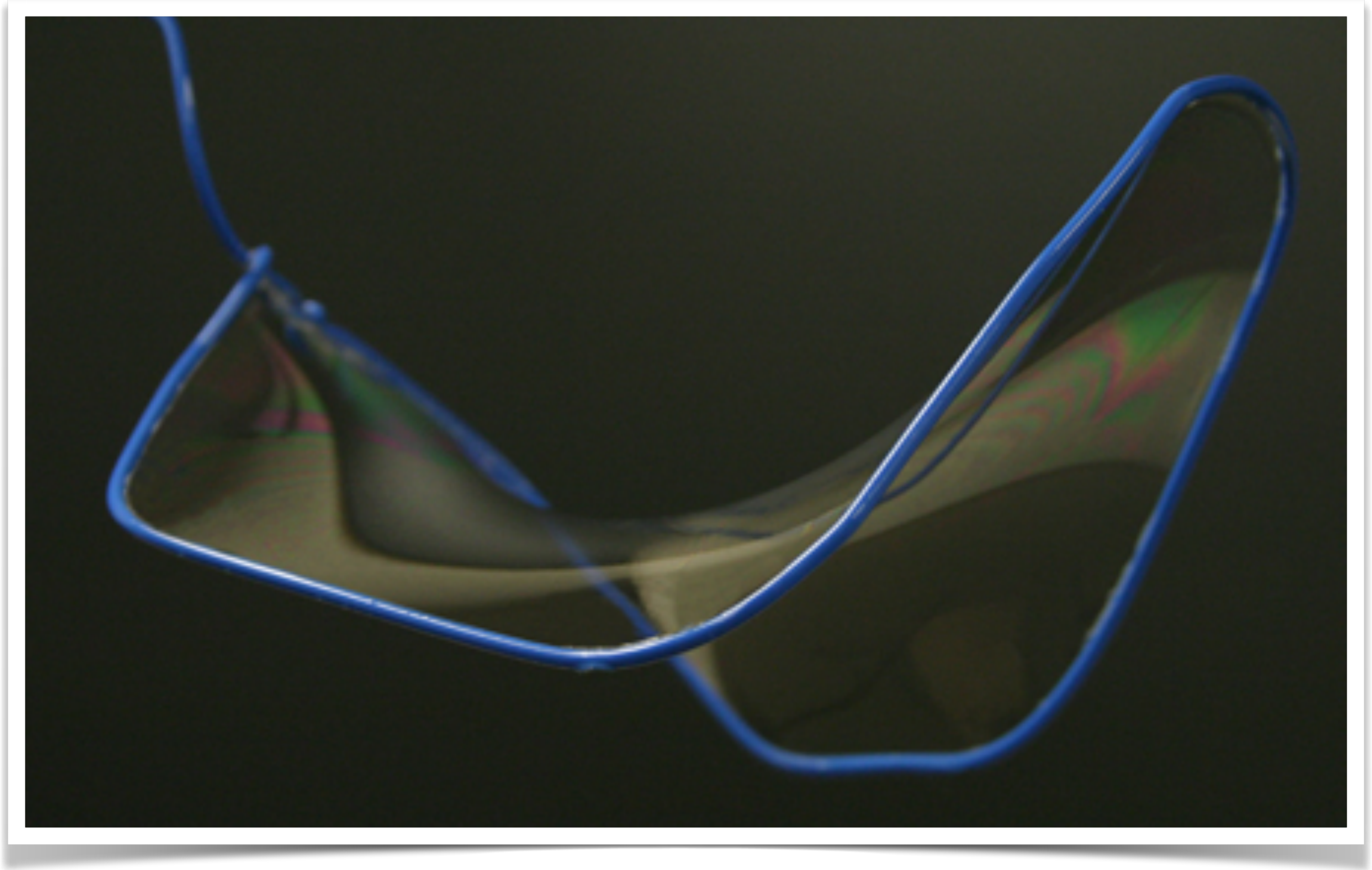
  - Euler theorem: $$\kappa_n(\phi) = \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi$$
  - Corresponding *principal directions* $\mathbf{e}_1$, $\mathbf{e}_2$ are orthogonal
- Special curvatures
  - Mean curvature $$H = \frac{\kappa_1 + \kappa_2}{2}$$
  - Gaussian curvature $$K = \kappa_1 \cdot \kappa_2$$

Curvature of Surfaces

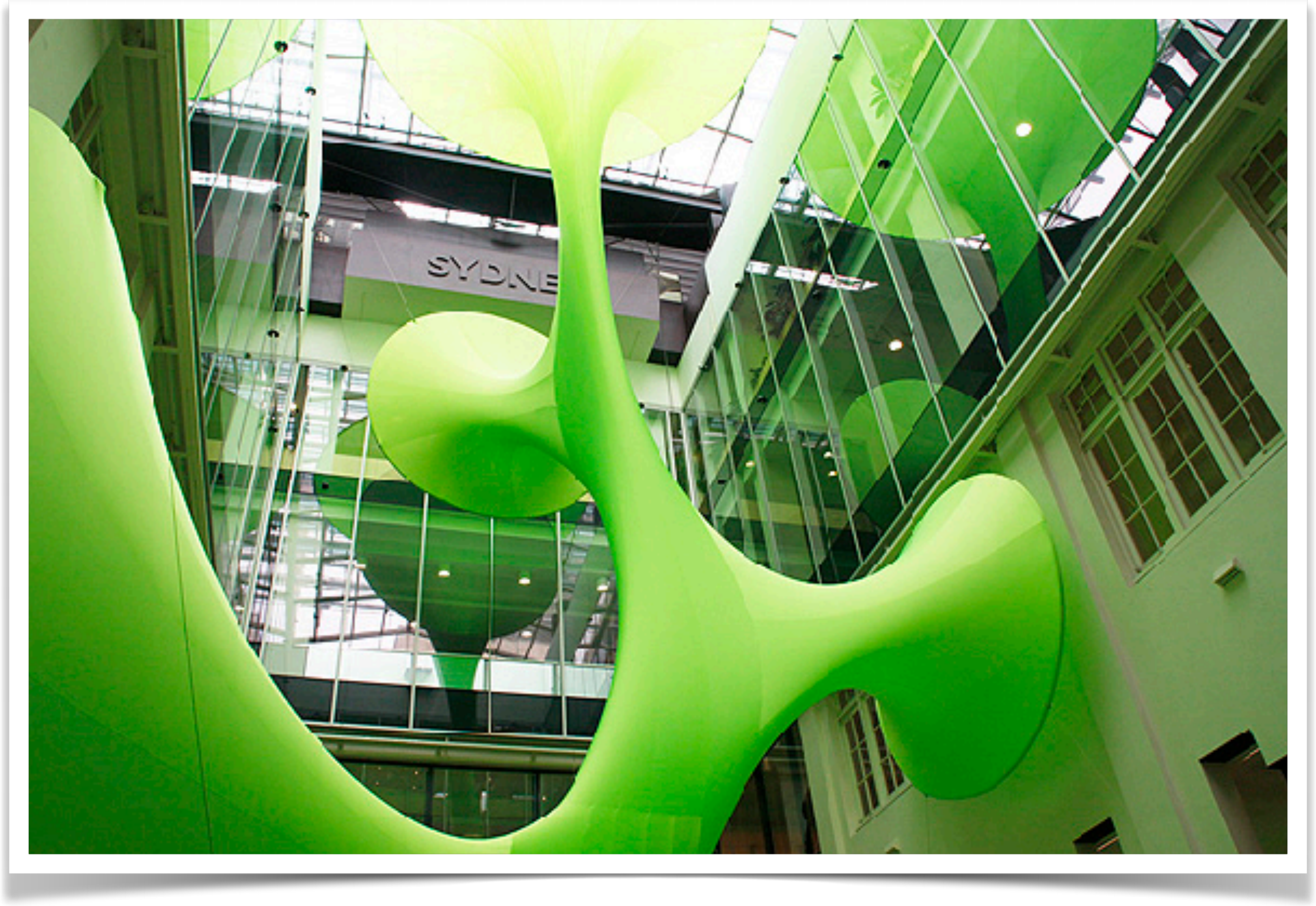
- Mean curvature $$H = \frac{\kappa_1 + \kappa_2}{2}$$
  - *H = 0* everywhere → minimal surface

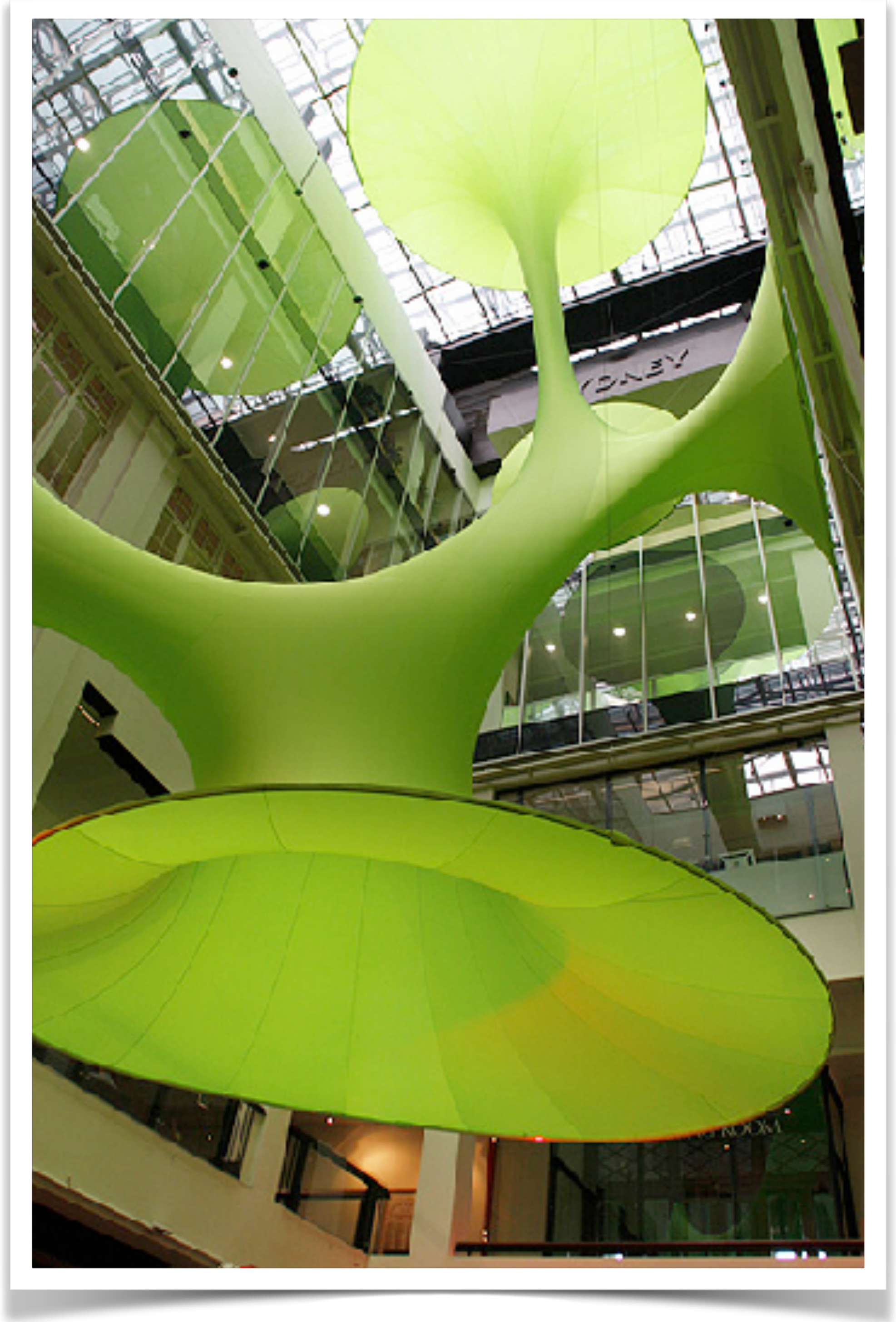


soap films

Curvature of Surfaces

- Mean curvature $$H = \frac{\kappa_1 + \kappa_2}{2}$$
  - *H = 0* everywhere → minimal surface





Green Void, Sydney
Architects: LAVA

Curvature of Surfaces

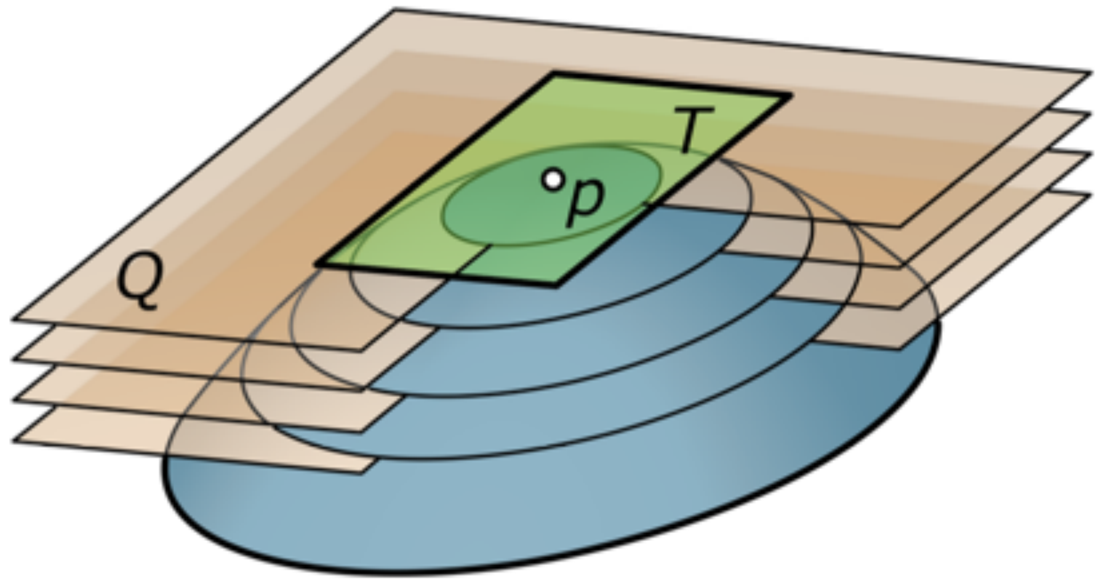
- Gaussian curvature $K = \kappa_1 \cdot \kappa_2$
  - K = 0 everywhere → developable surface

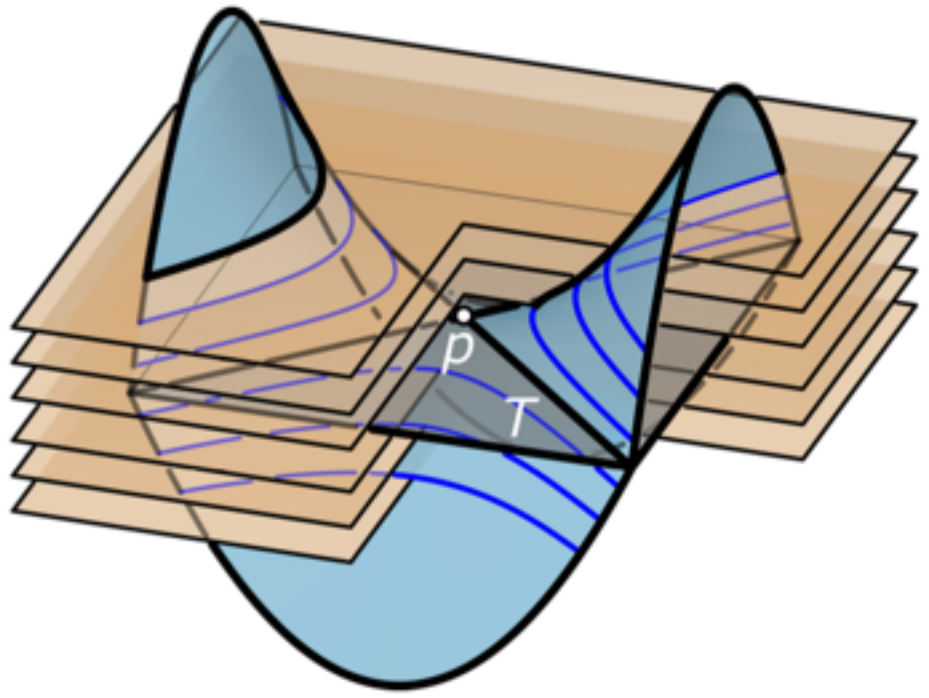
Disney Concert Hall, L.A.
Architects: Gehry Partners

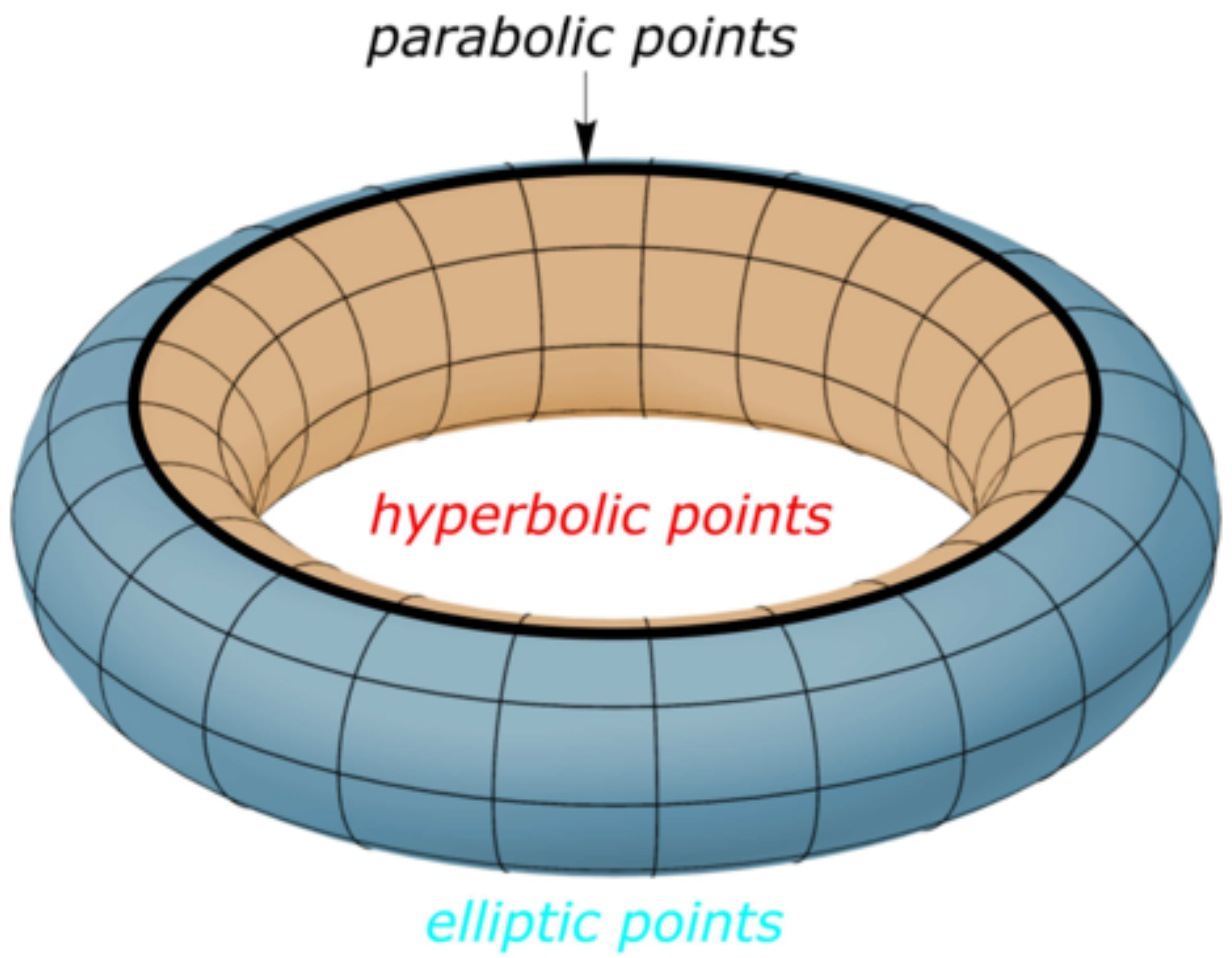
Timber Fabric
IBOIS, EPFL

Classification

- A point **x** on the surface is called
  - *elliptic*, if *K > 0*
  - *hyperbolic*, if *K < 0*
  - *parabolic*, if *K = 0*
  - *umbilic*, if $\kappa_1 = \kappa_2$

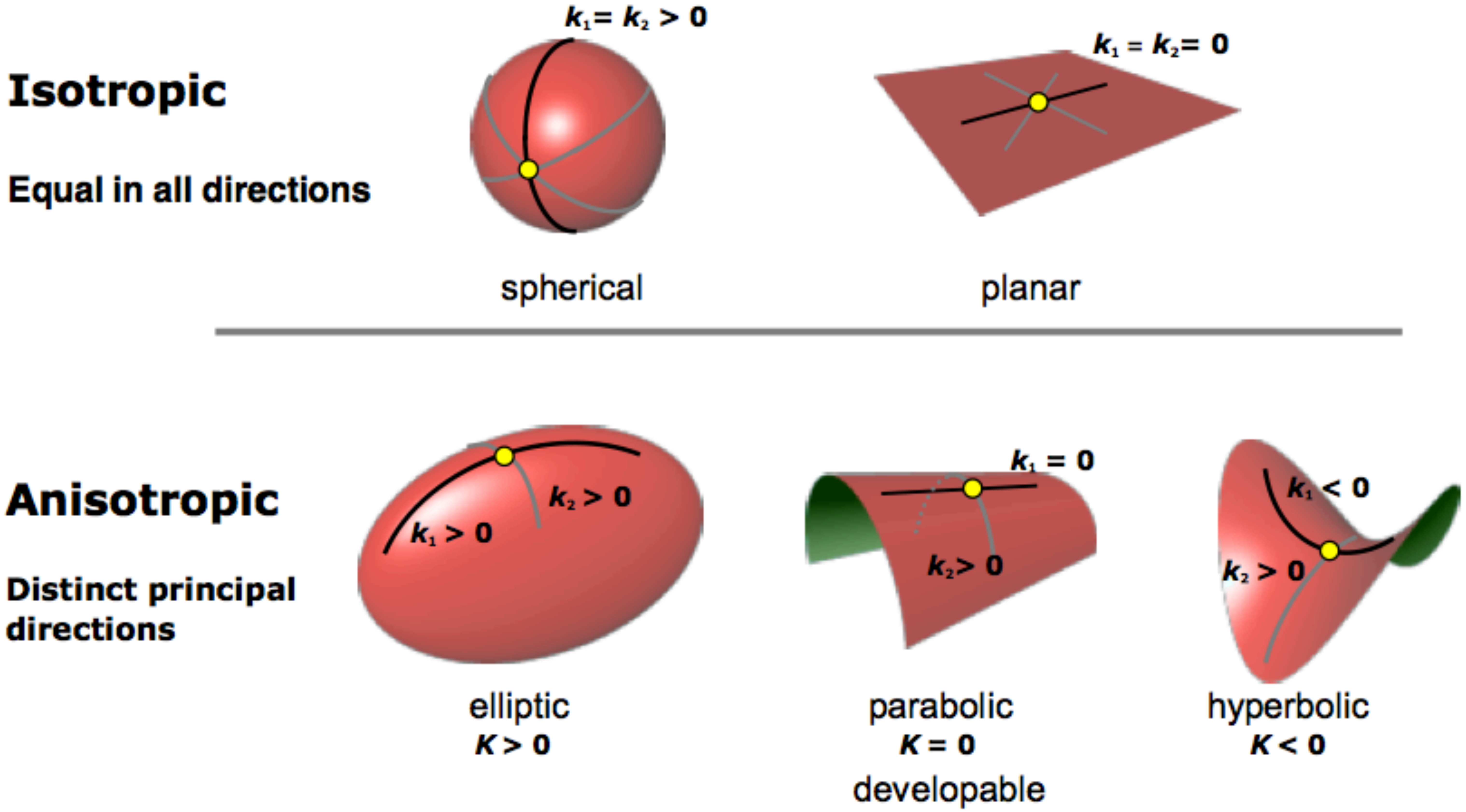






Classification

- A point **x** on the surface is called



Intrinsic Geometry

- Properties of the surface that only depend on the first fundamental form
  - length
  - angles
  - Gaussian curvature (Theorema Egregium)

Gauss-Bonnet Theorem

- For **any** closed manifold surface with Euler characteristic $\chi = 2 - 2g$

$$\int_{\Omega \in S} K(u,v)\,du\,dv = 2\pi\chi$$

$\int K($  $) = \int K($  $) = \int K($  $) = 4\pi$

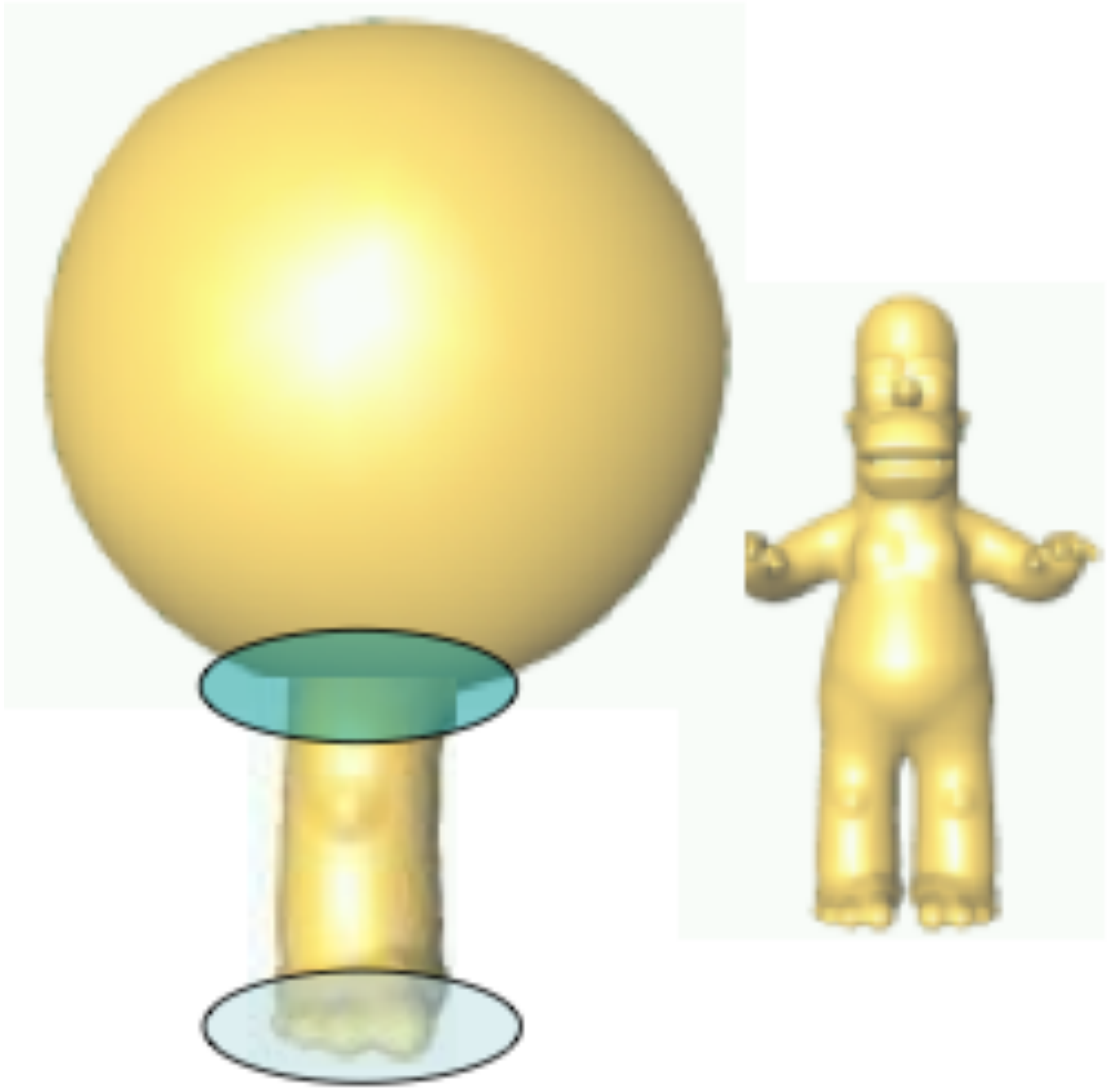
Gauss-Bonnet Theorem

- Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

$$K = \kappa_1 \kappa_2 = 1/r^2$$

$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$



  – when sphere is deformed new positive and negative curvature cancel out!

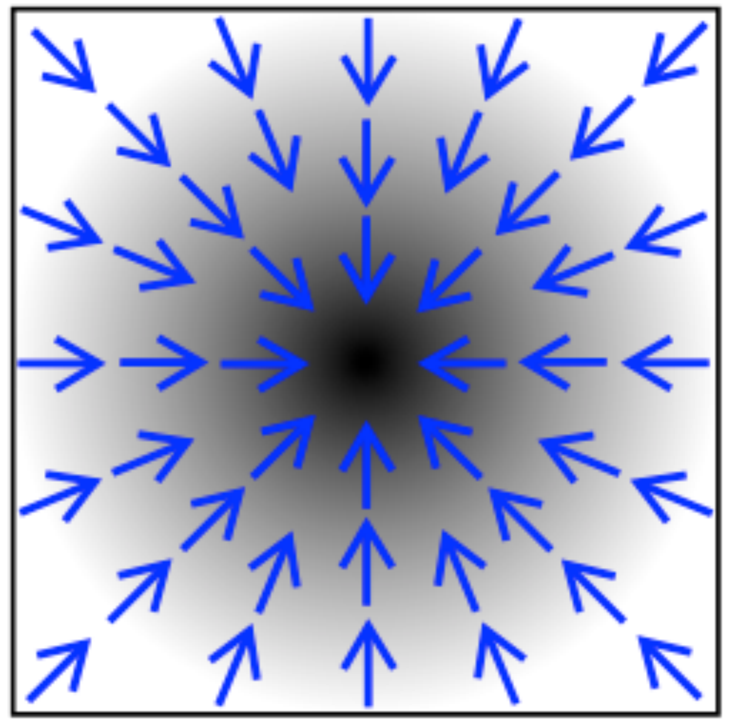
Differential Operators

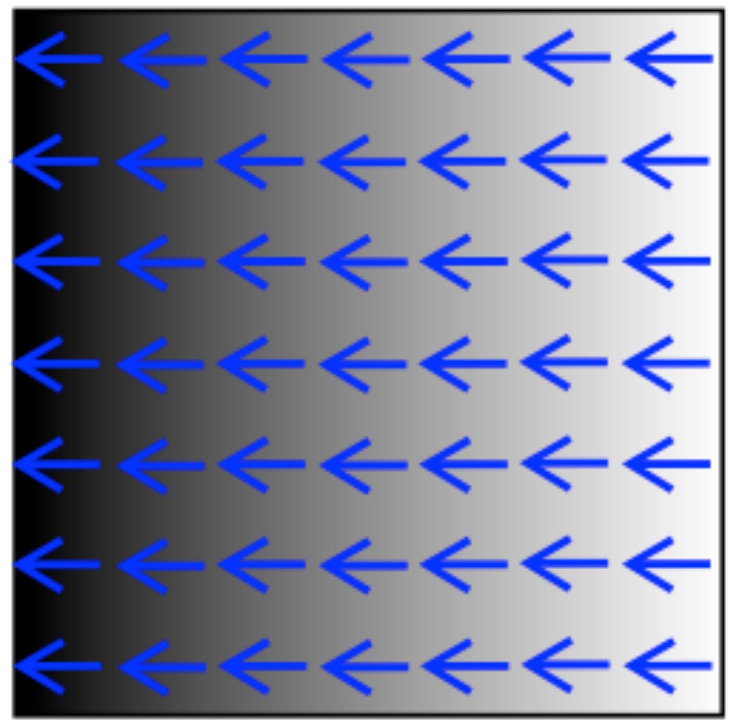
- Gradient

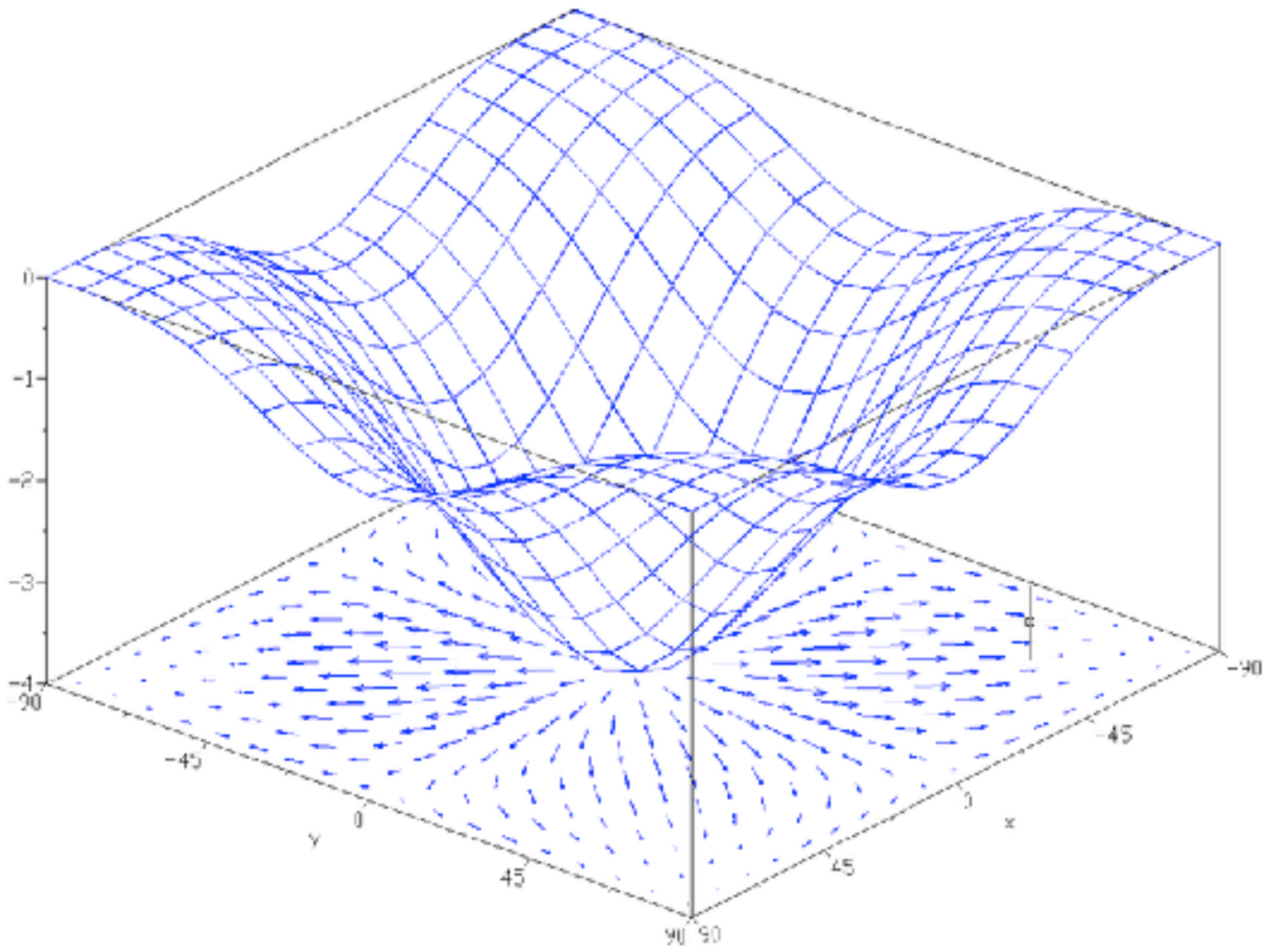
$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

$$\text{div} f := < \left(\frac{\partial}{\partial x_1}, \ldots\right) \cdot f >$$

  - points in the direction of steepest ascent







Differential Operators

- Divergence

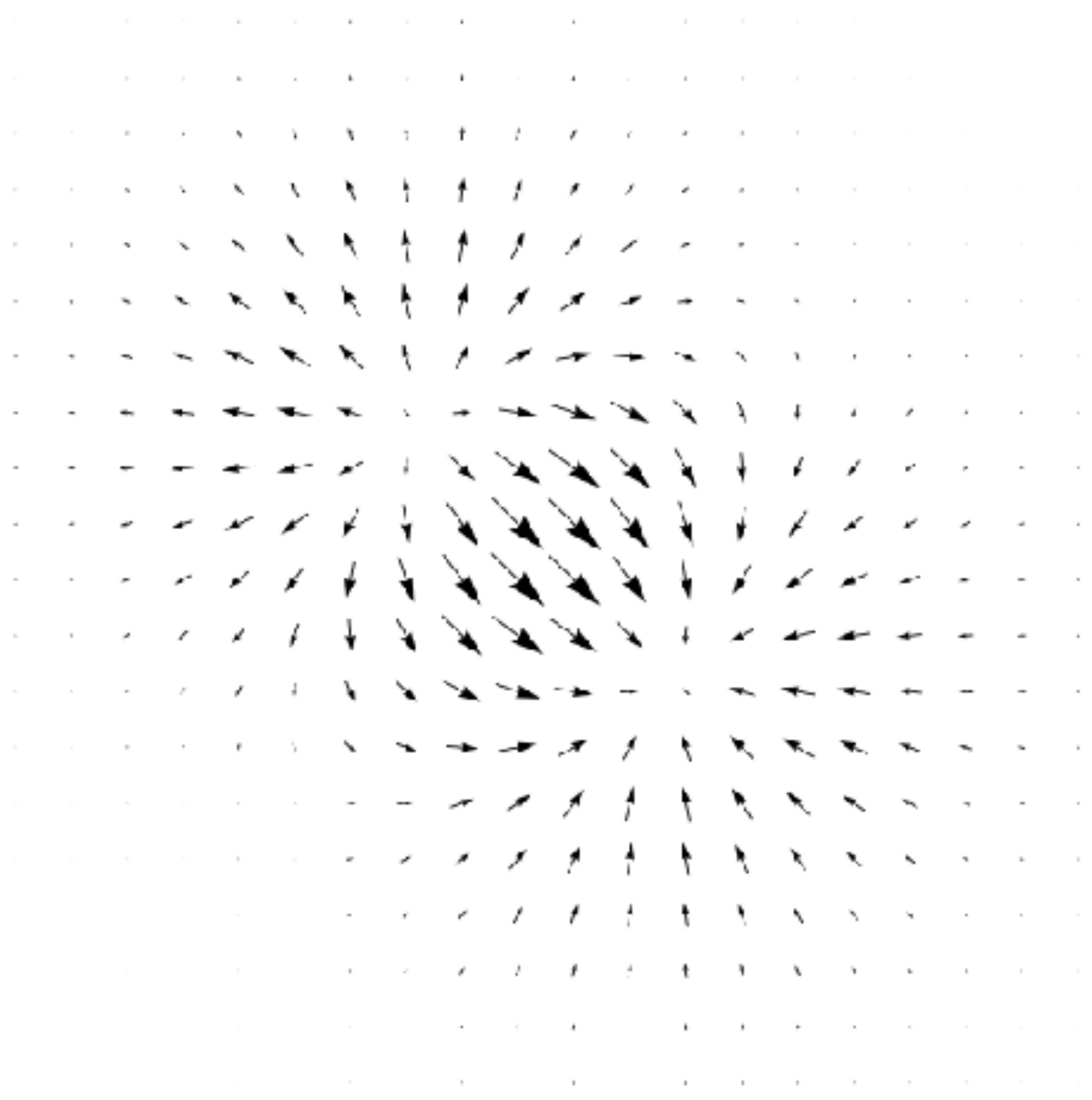
$$\mathrm{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$

  – volume density of outward flux of vector field
  – magnitude of source or sink at given point
  – Example: Incompressible fluid
    - velocity field is divergence-free

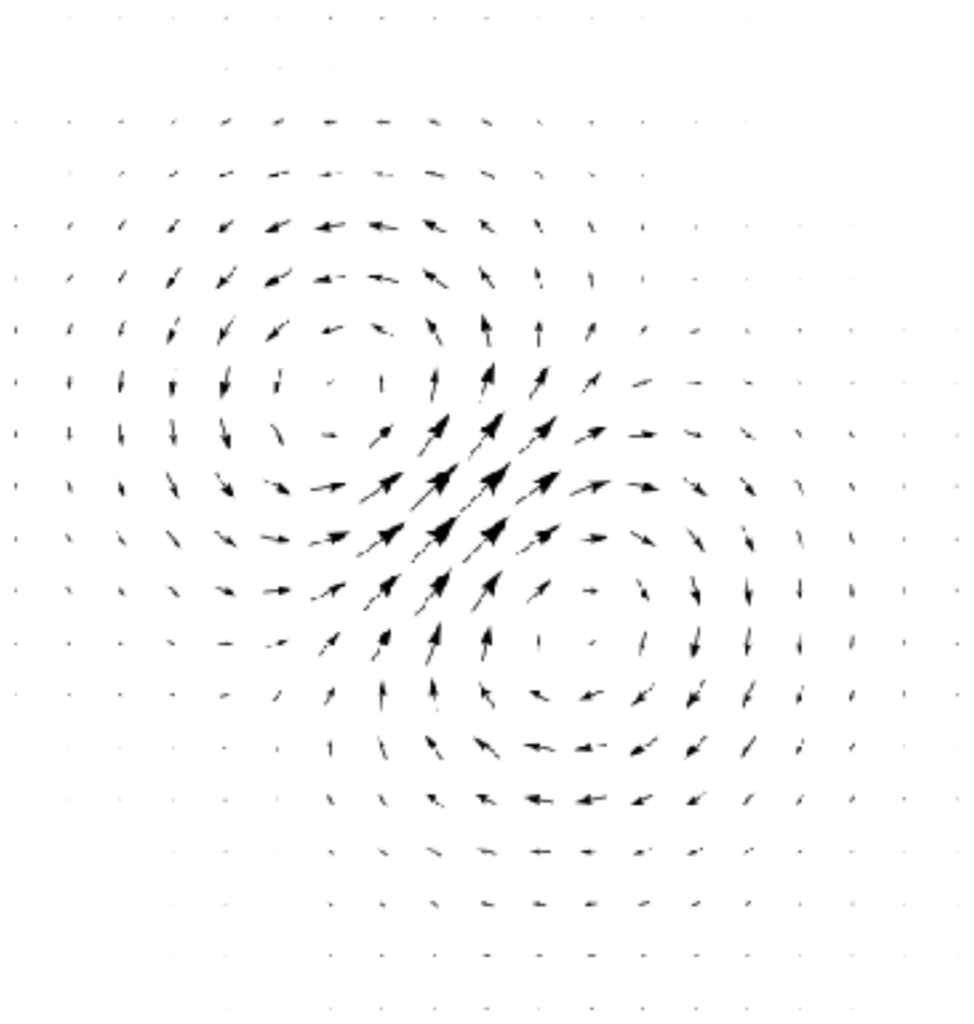
Differential Operators

- Divergence

$$\mathrm{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$



high divergence



low divergence

Literature

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- M. Do Carmo: ***Differential Geometry of Curves and Surfaces***, Prentice Hall, 1976.
- A. Pressley: ***Elementary Differential Geometry***, Springer, 2010