EULER CHARACTERISTICS TOPOLOGICAL

① $V - E + F = 2(1-g)$ $(=0)$

DVU

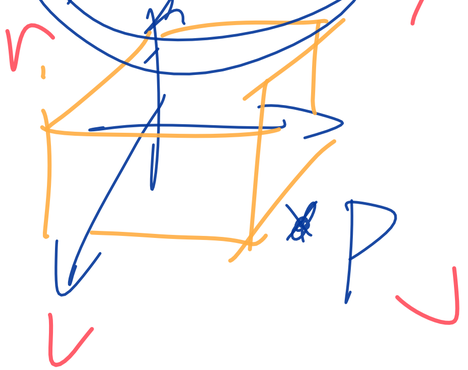
GENUS (亏)

GENUS 定义在下面 CHAPTER 有

② $3F = 2E$ (∵ 边被 2个△共享)

$(F = 2V)$

NERF ← 很多 CAM



$\sigma(p) \Rightarrow$ density (多少光可以通过)

p

$c(p) \Rightarrow$ color

空间中之点 p

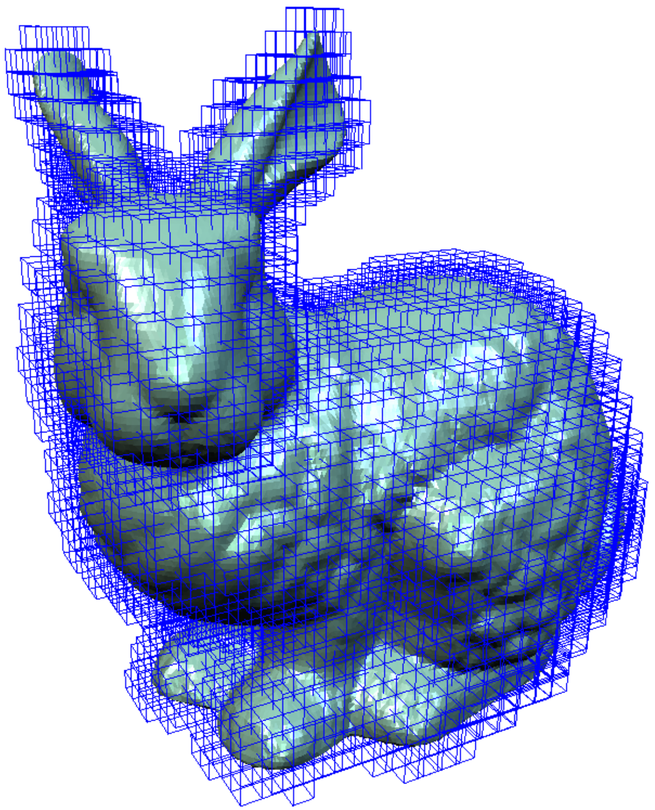
what we r optimizing

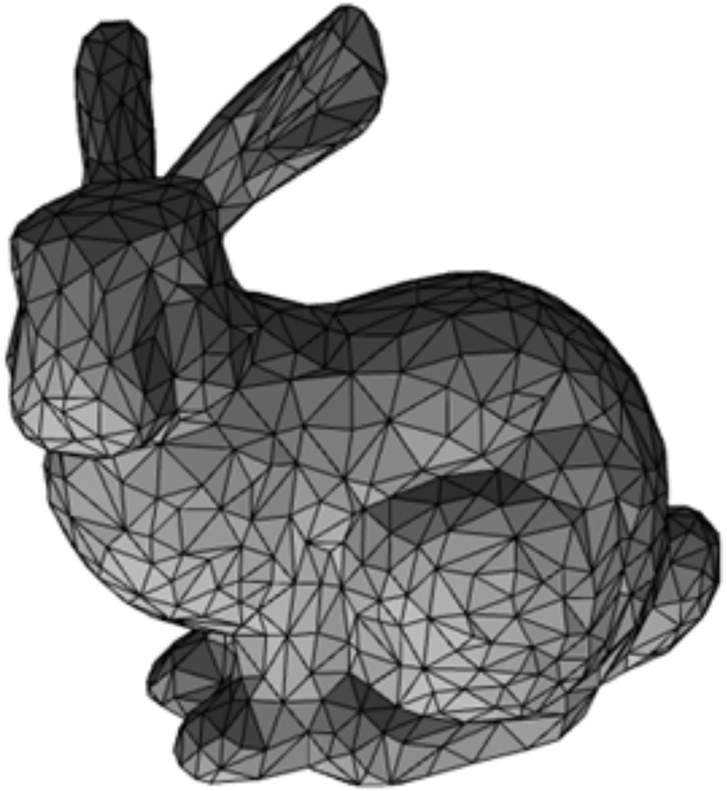
$F(\sigma, \sigma, \text{CAMERA}) = \text{Image}_I$

image formation model

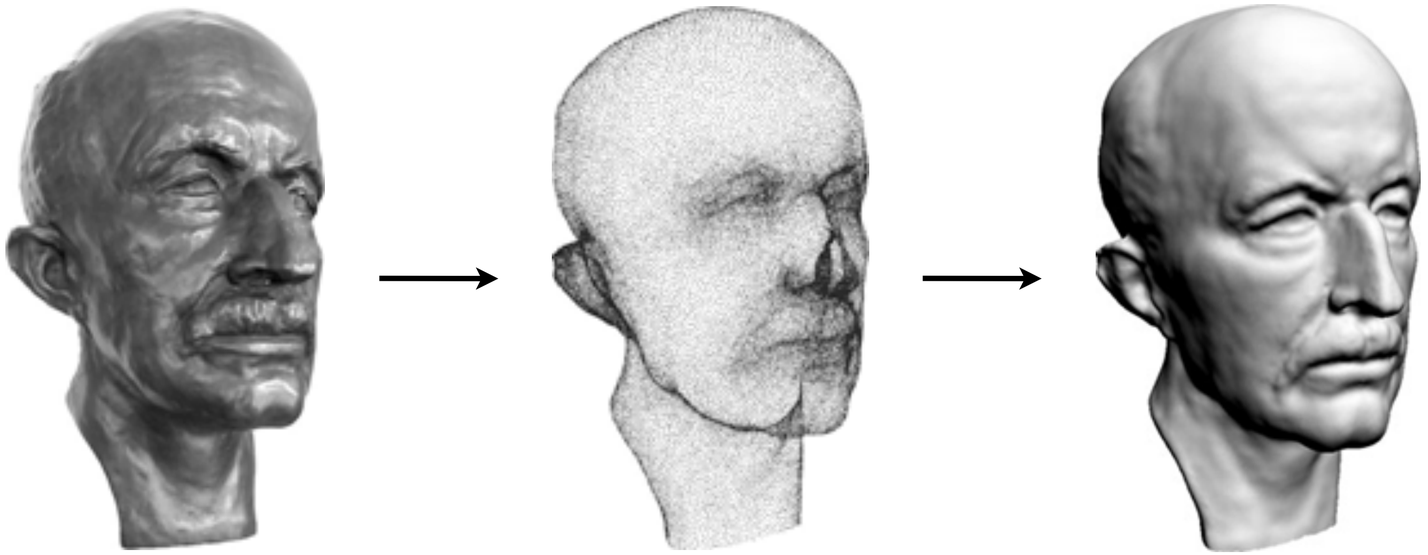
$[\text{Image}^{GT}]$

Implicit and Explicit Representations



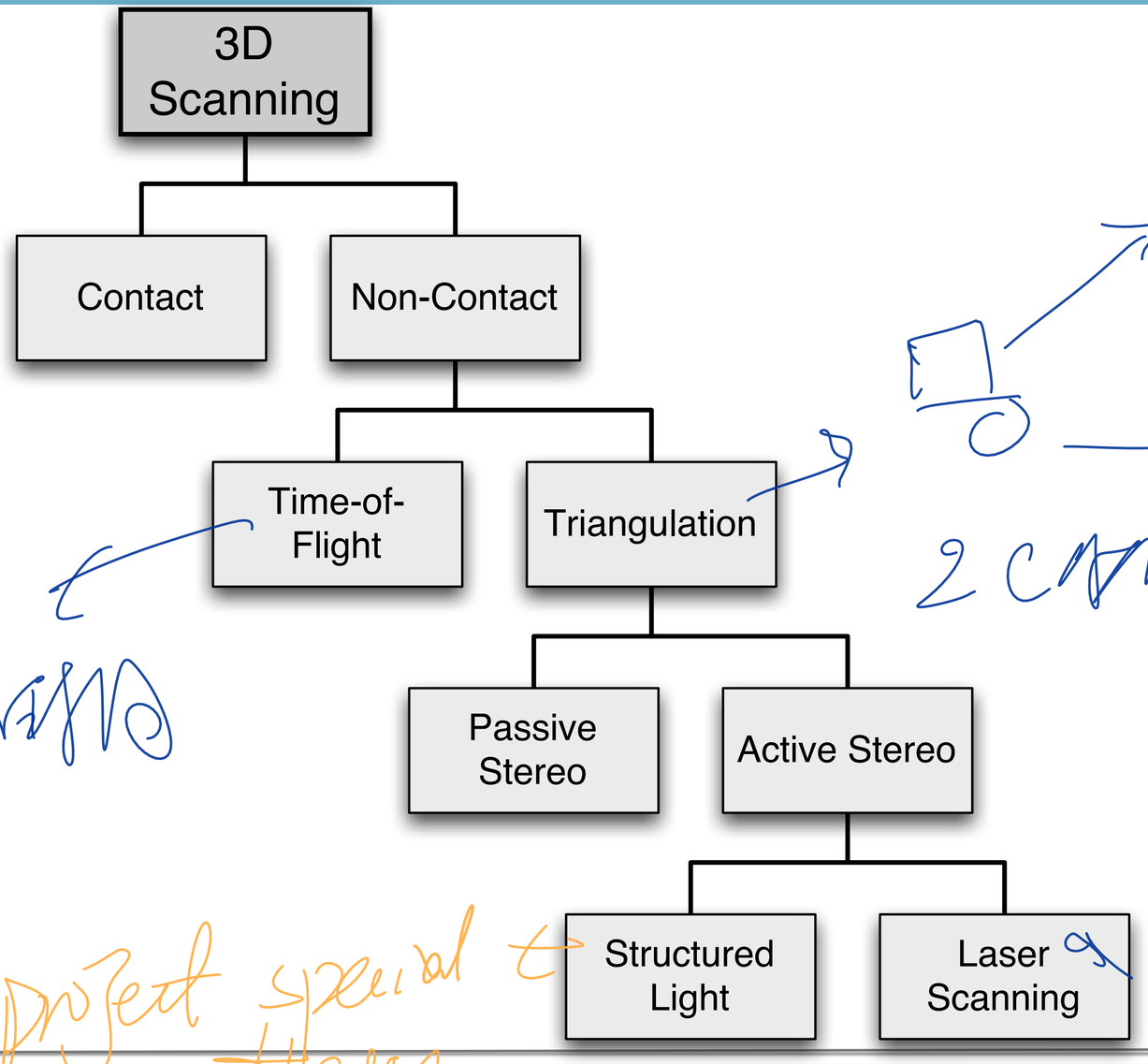


Last Week



physical model → acquired point cloud → reconstructed model

3D Scanning Techniques



scan

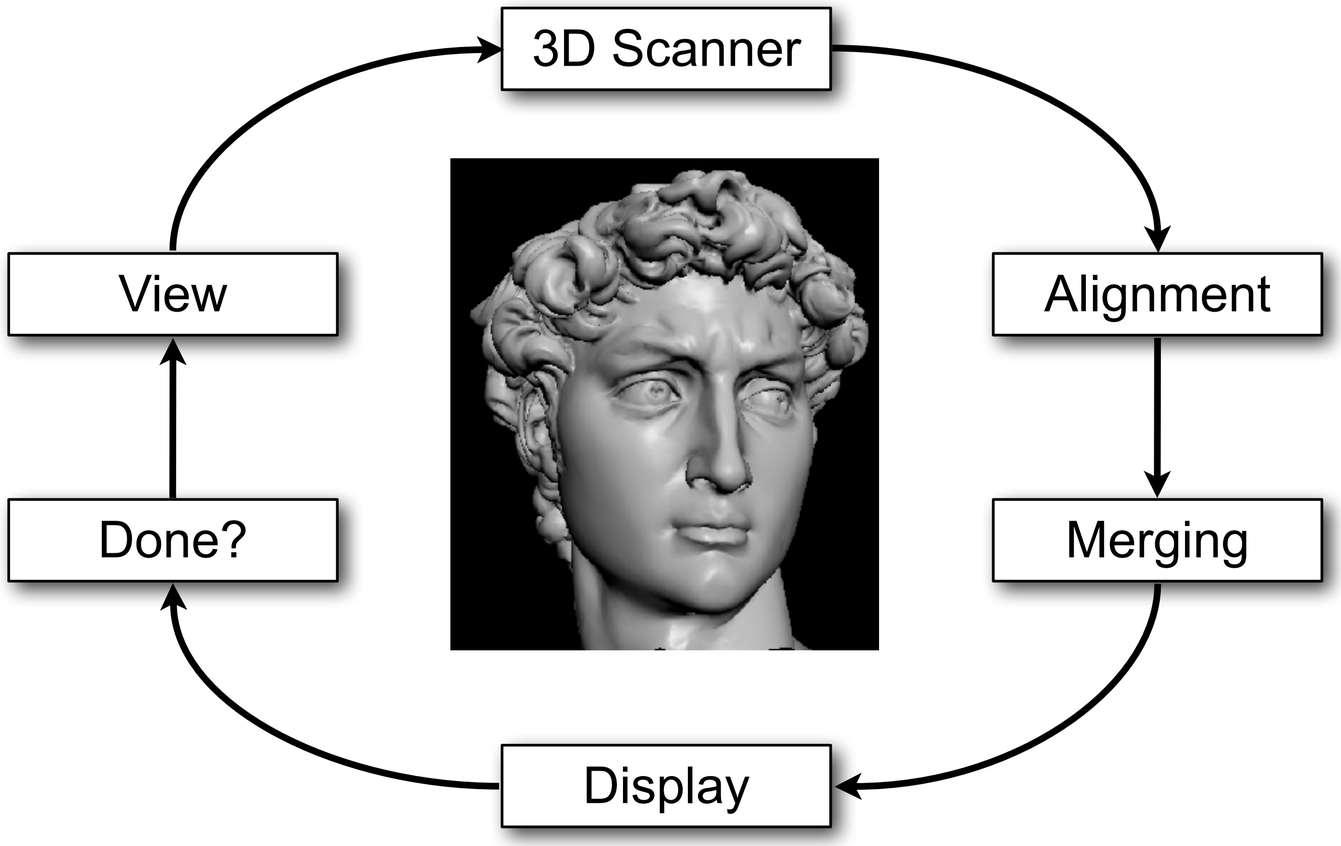
scan

2 cam 1 point

project special pattern

Last Time

- 3D Model Acquisition Pipeline



Recap

Basic ICP Algorithm (rigidly align Q to P)

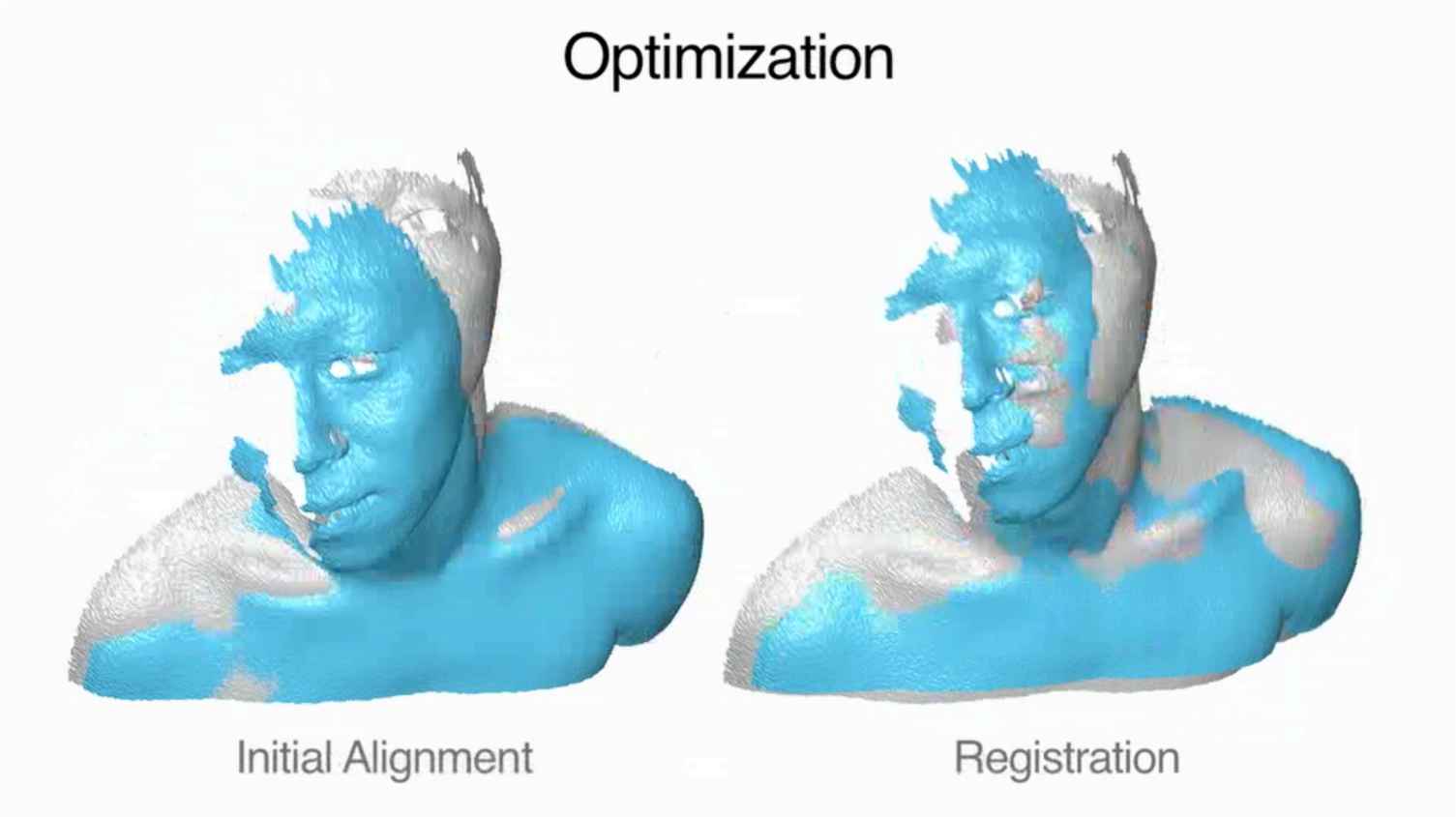
Iterate until convergence:

1. **Select** a subset of points $\mathbf{p}_i$
2. **Match** each $\mathbf{p}_i$ to closest point $\mathbf{q}_i$ on other scan
3. **Reject** "bad" pairs ($\mathbf{p}_i$, $\mathbf{q}_i$)
4. **Compute** rotation **R** and translation **t** to minimize

$$\min_{\mathbf{R},\mathbf{t}} \sum_i \|\mathbf{p}_i - \mathbf{R}\mathbf{q}_i - \mathbf{t}\|^2$$

5. **Iterate** after scan alignment: $\mathbf{q}_i \leftarrow \mathbf{R}\mathbf{q}_i + \mathbf{t}$

Non-rigid Registration



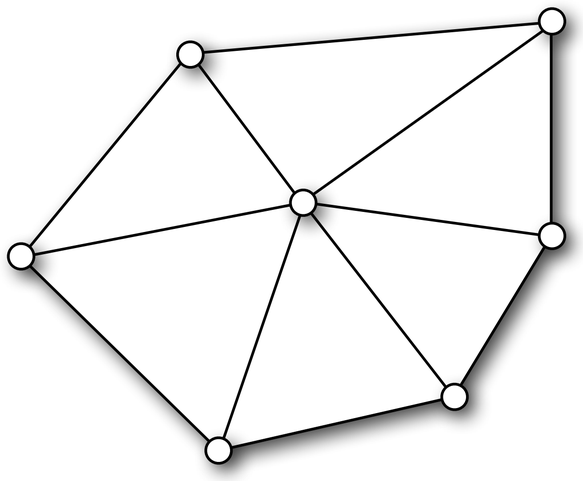
Triangle Meshes

- **Topology**: vertices, edges, triangles

$$\mathcal{V} = \{v_1, \ldots, v_n\}$$
$$\mathcal{E} = \{e_1, \ldots, e_k\} \ , \quad e_i \in \mathcal{V} \times \mathcal{V}$$
$$\mathcal{F} = \{f_1, \ldots, f_m\} \ , \quad f_i \in \mathcal{V} \times \mathcal{V} \times \mathcal{V}$$

- **Geometry**: vertex positions

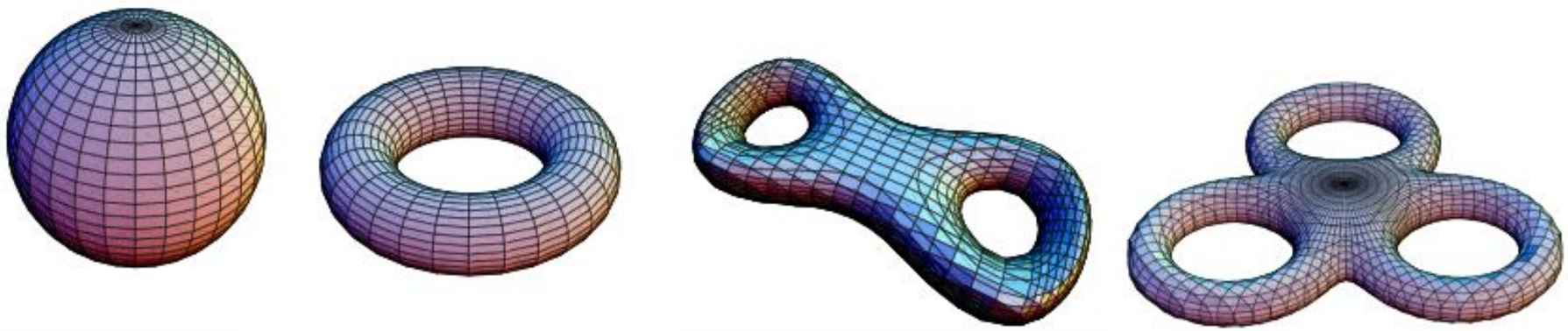
$$\mathcal{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \ , \quad \mathbf{p}_i \in \mathbb{R}^3$$



Global Connectivity: Genus

Genus:
Maximal number of closed simple cutting curves that do not disconnect the graph into multiple components.

Informally, the number of holes or handles.



Genus 0 Genus 1 Genus 2 Genus 3

Euler-Poincare Formula

- For a closed polygonal mesh of genus *g*, the relation of the number *V* of vertices, *E* of edges, and *F* of faces is given by *Euler's formula* χ

$$V - E + F = 2(1 - g)$$

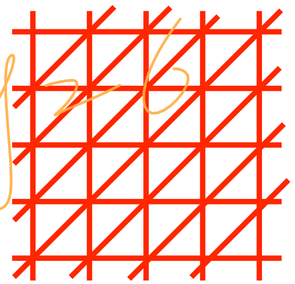
- The term $2(1 - g)$ is called the ***Euler characteristic***

Consequences of Euler Characteristics

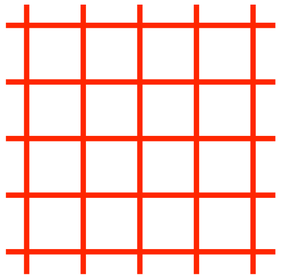
- Triangle meshes
  - $F \approx 2V$
  - $E \approx 3V$
  - Average valence = 6

$n \Rightarrow 3V = 2F \Rightarrow n = 6$

从一个顶点出发几条线



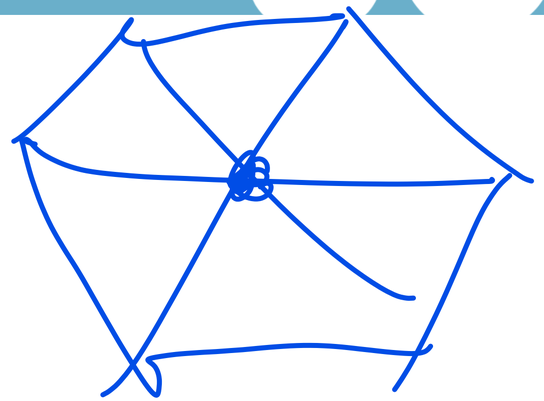
- Quad meshes
  - $F \approx V$
  - $E \approx 2V$
  - Average valence = 4



2-Manifold Surfaces

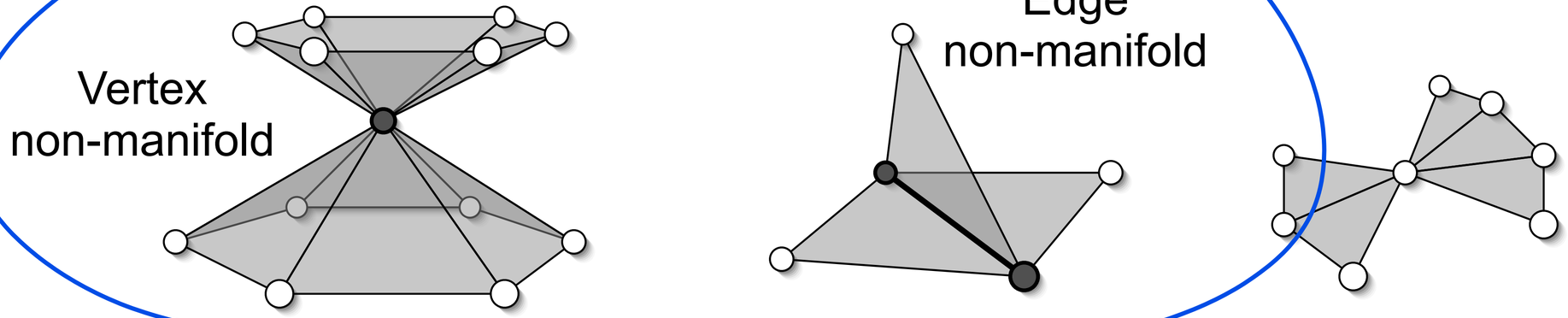
- Local neighborhoods are disk-shaped

$$\mathbf{f}(D_\varepsilon[u,v]) = D_\delta[\mathbf{f}(u,v)]$$



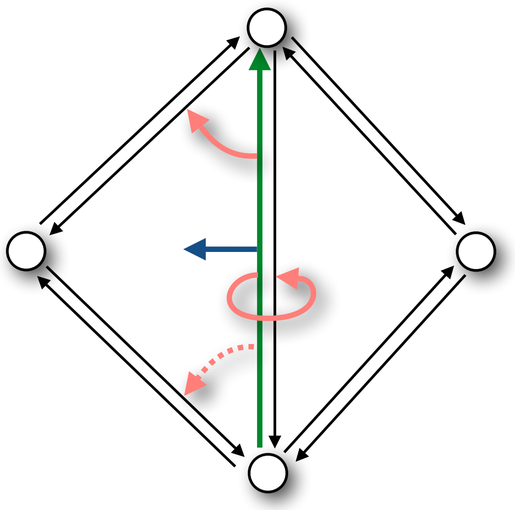
- Guarantees meaningful neighbor enumeration
  - required by most algorithms

- **Non-manifold** examples:



Halfedge-Based Connectivity

- Vertex
  - position
  - 1 halfedge
- Halfedge
  - 1 vertex
  - 1 face
  - 1, 2, or 3 halfedges
- Face
  - 1 halfedge



96 to 144 B/v

no case distinctions during traversal

Halfedge-Based Libraries

- CGAL
  - www.cgal.org
  - Computational geometry
  - Free for non-commercial use

- OpenMesh
  - www.openmesh.org
  - Mesh processing
  - Free, LGPL licence

Overview

$p + \lambda \vec{d}$

- **Surface representations**
  - Explicit vs. Implicit

$y = Ax + b$

- Explicit representations
  - Triangle meshes
- Implicit representations
  - Signed distance fields
- Conversions
  - Explicit ↔ Implicit

量子一走刚

Distance Field

IMPLICIT	EXPLICIT
$F(x,y) =$	$c(t)$ curve
$F(x,y,z) \begin{cases} >0 \\ =0 \\ <0 \end{cases}$	$x(u,v)$ surface

MARCHING CUBE

$F(p) = n^T p + d$

fast $p + \lambda u + \beta v$

Explicit vs. Implicit

Explicit:

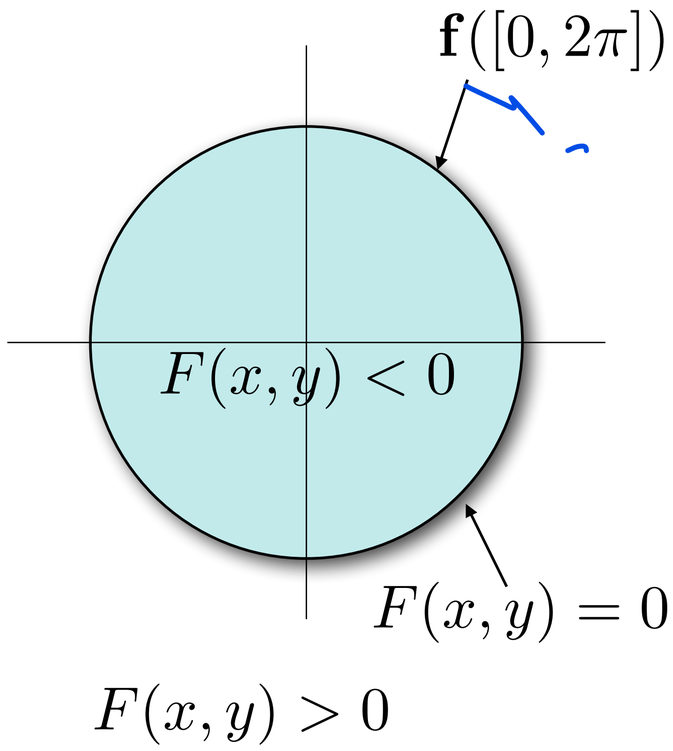
– Range of parametrization function

$$\mathbf{f}(x) = (r\cos(x), r\sin(x))^T$$

Implicit:

– Kernel of implicit function

$$F(x,y) = \sqrt{x^2+y^2} - r$$



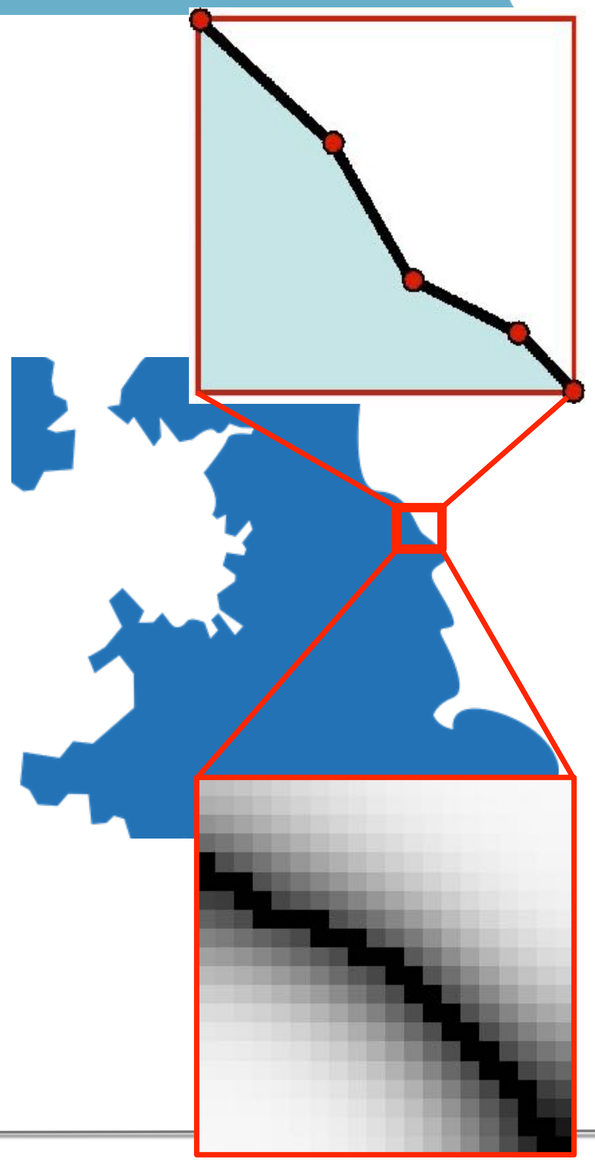
Explicit vs. Implicit

- Explicit:
  - Range of parametrization function
  - Piecewise approximation!

$$\mathbf{f}(x) =$$

- Implicit:
  - Kernel of implicit function
  - Piecewise approximation!

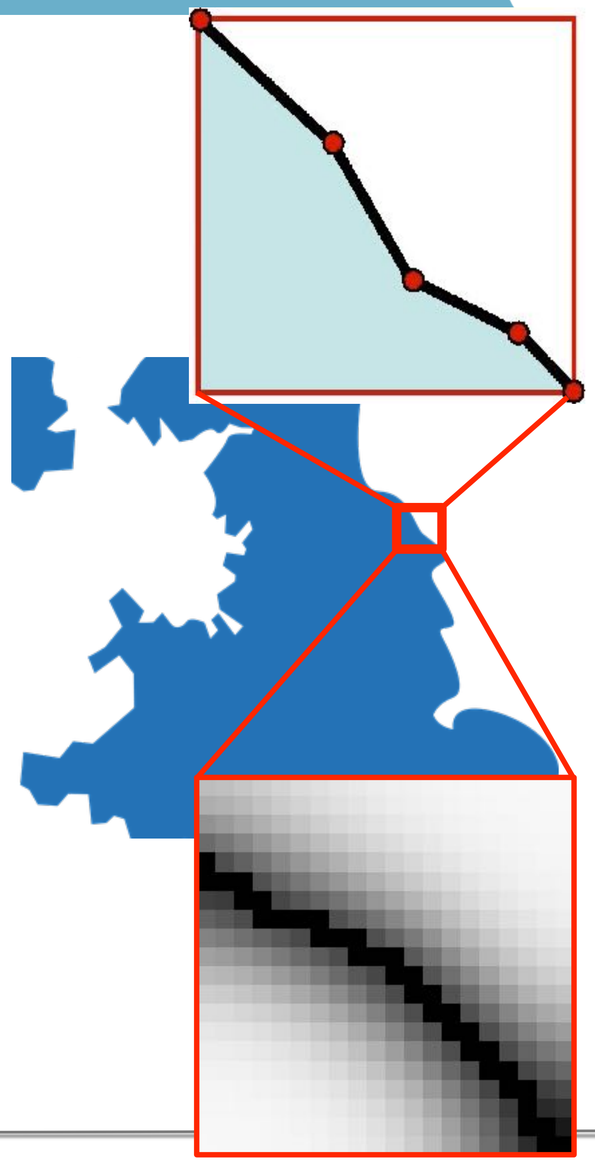
$$F(x, y) =$$



Explicit vs. Implicit

- **Explicit:**
  - Range of parametrization function
  - Piecewise approximation!
  - E.g., splines, triangle mesh
  - Easy enumeration
  - Easy geometry modification

- **Implicit:**
  - Kernel of implicit function
  - Piecewise approximation!
  - E.g., scalar-valued 3D grid
  - Easy in/out test
  - Easy topology modification (CSG)



Overview

- Surface representations
  - Explicit vs. Implicit
- **Explicit representations**
  - Triangle meshes
- Implicit representations
  - Signed distance fields
- Conversions
  - Explicit ↔ Implicit

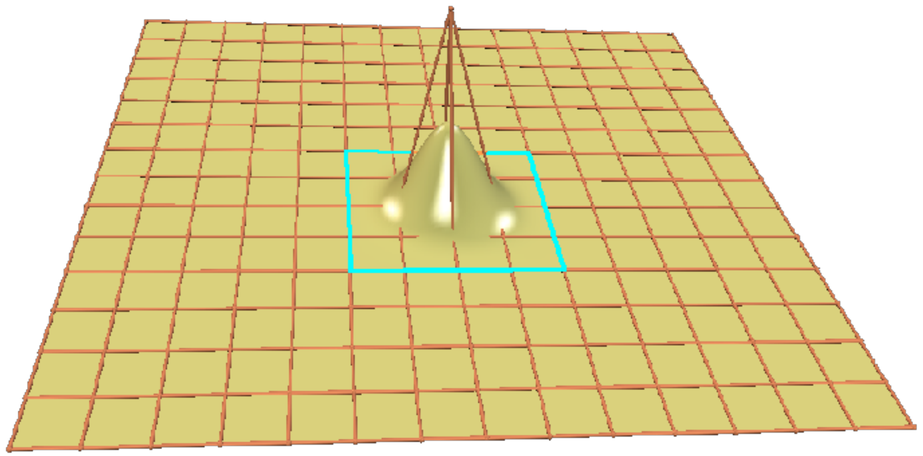
Polynomial Approximation

- Polynomials of degree *p*, elements of size *h*
  - Approximation error is $O(h^{p+1})$
- Improve approximation quality by
  - increasing *p* ... higher order polynomials
  - decreasing *h* ... smaller / more segments
- Issues
  - smoothness of the target data
  - smoothness conditions between segments $\max_t f^{p+1}(t)$

Spline Surfaces

Piecewise polynomial approximation

$$\mathbf{f}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{c}_{ij} N_i^n(u) N_j^m(v)$$

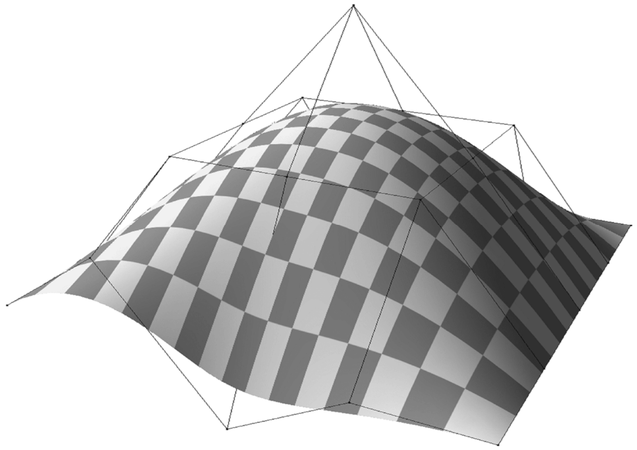


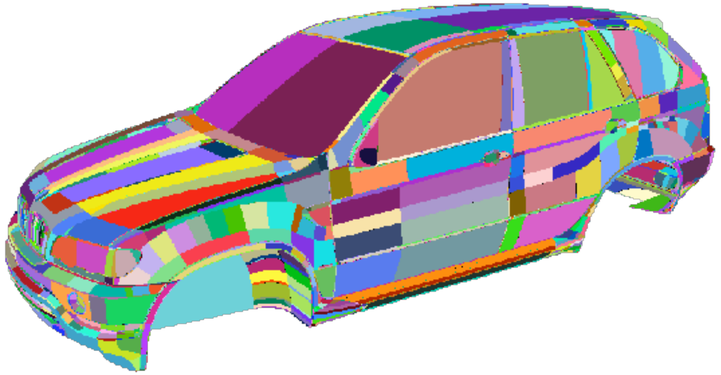
Spline Surfaces

- Piecewise polynomial approximation

- Topological constraints
  - Rectangular patches
  - Regular control mesh

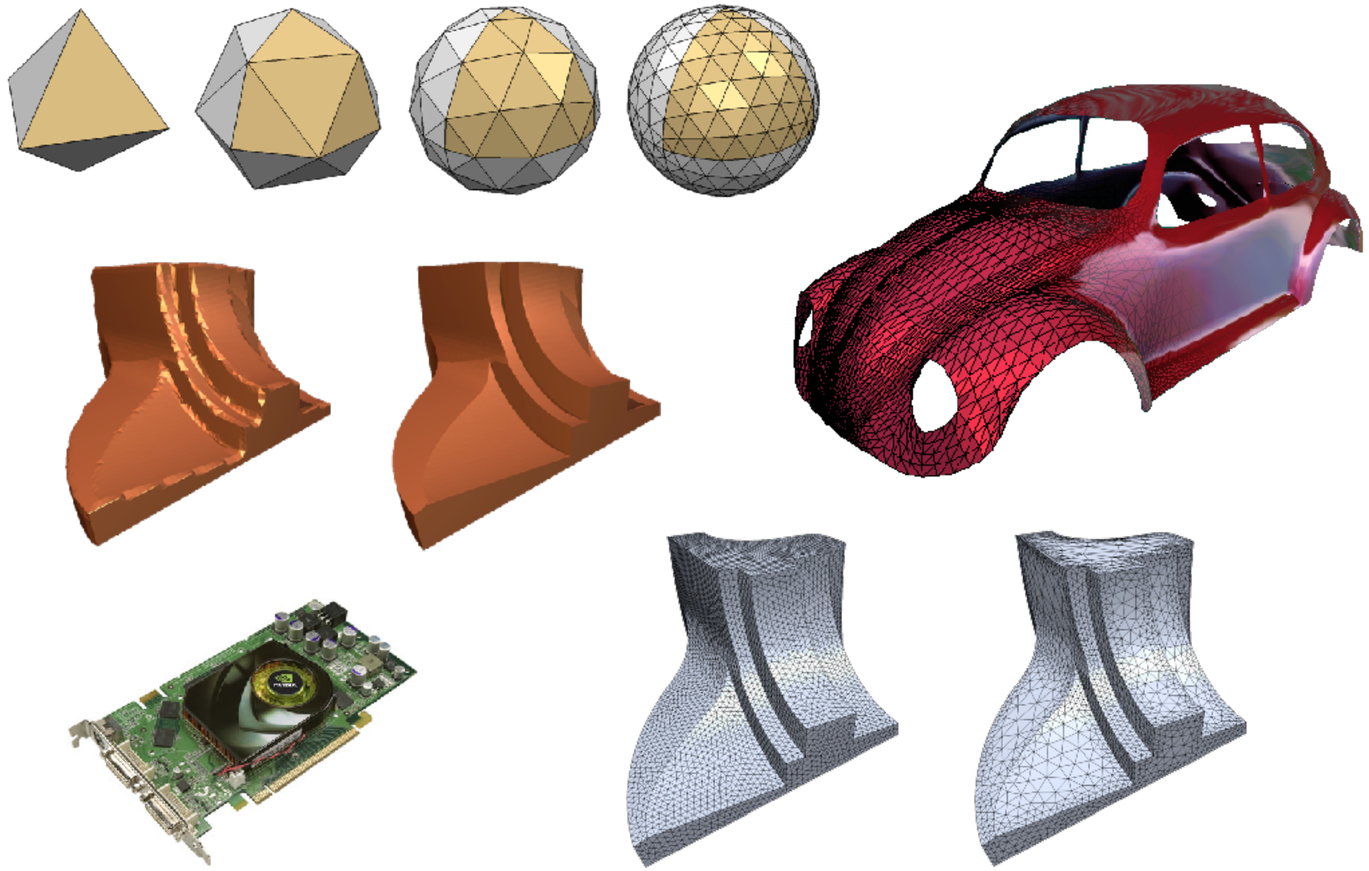
- Geometric constraints
  - Large number of patches
  - Continuity between patches
  - Trimming





Polygon Meshes

- Polygonal meshes for surface approximation
  - approximation $O(h^2)$
  - arbitrary topology
  - piecewise smooth surfaces
  - adaptive refinement
  - efficient rendering



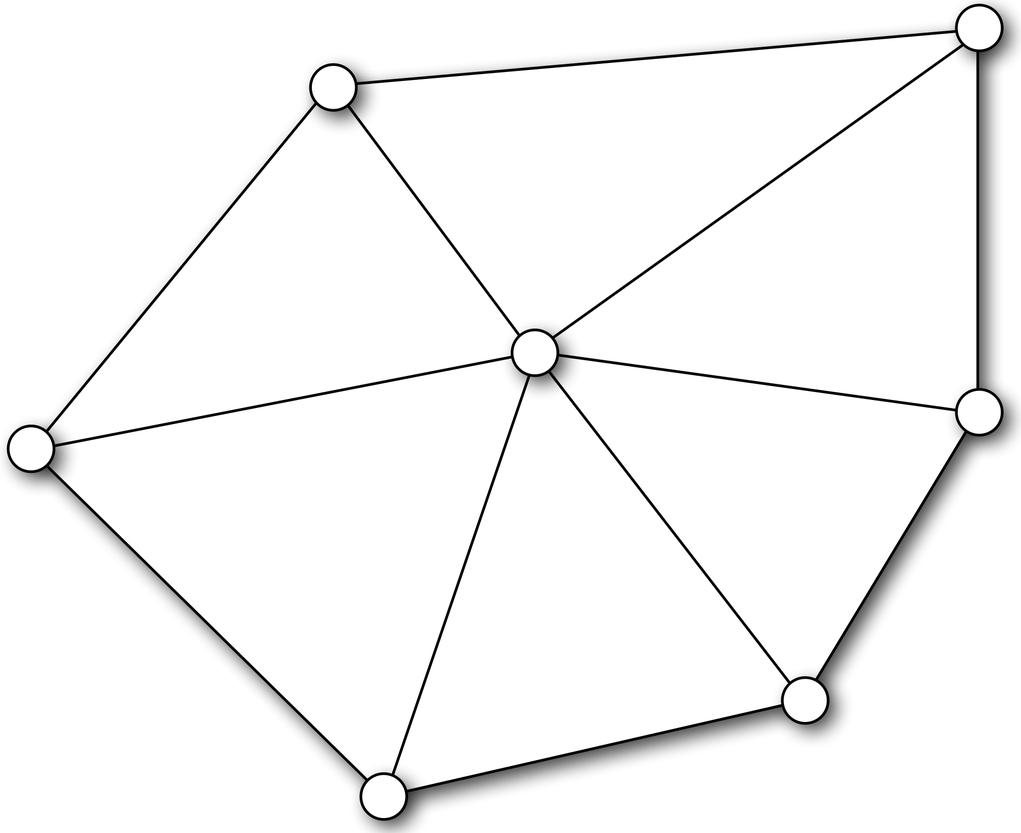
Triangle Meshes

- Topology: vertices, edges, triangles

$$\mathcal{V} = \{v_1, \ldots, v_n\}$$
$$\mathcal{E} = \{e_1, \ldots, e_k\} \ , \quad e_i \in \mathcal{V} \times \mathcal{V}$$
$$\mathcal{F} = \{f_1, \ldots, f_m\} \ , \quad f_i \in \mathcal{V} \times \mathcal{V} \times \mathcal{V}$$

- Geometry: vertex positions

$$\mathcal{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \ , \quad \mathbf{p}_i \in \mathbb{R}^3$$



Overview

- Surface representations
  - Explicit vs. Implicit
- Explicit representations
  - Triangle meshes
- **Implicit representations**
  - Signed distance fields
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  - Explicit ↔ Implicit

Implicit vs Explicit Functions

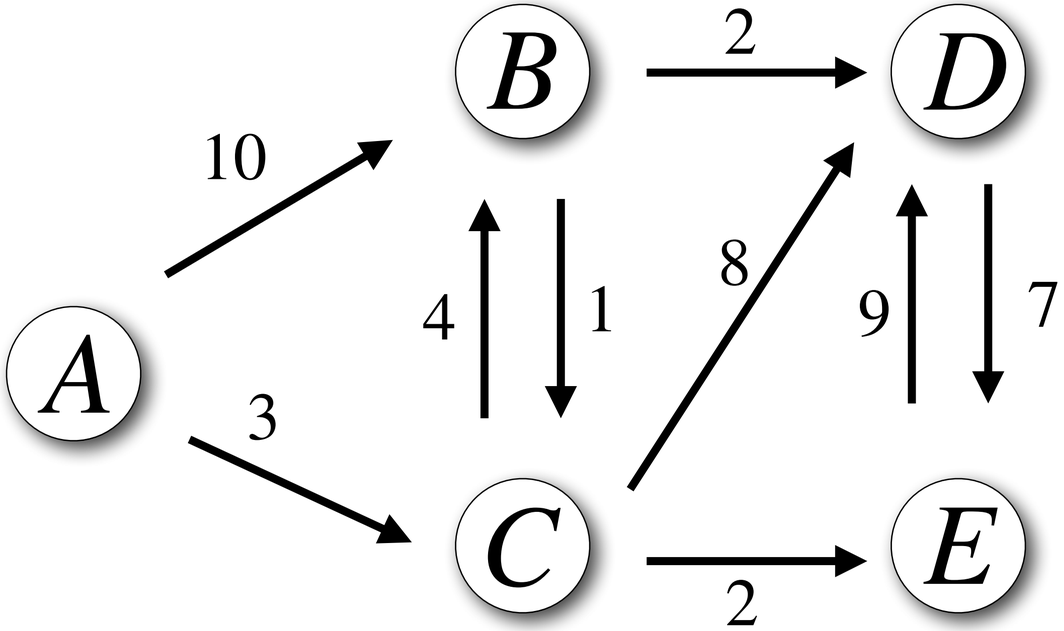
Implicit

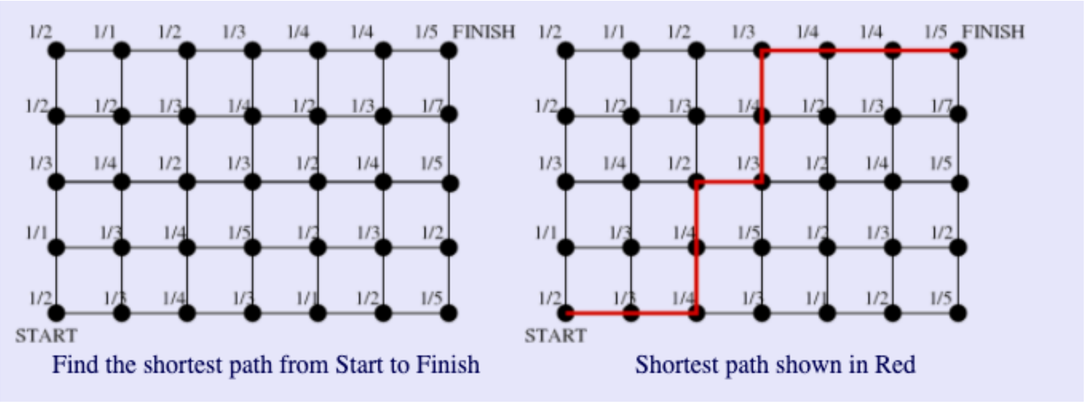
Explicit

Implicit →Explicit

Explicit →Implicit

Dijkstra's Algorithm (Recap)

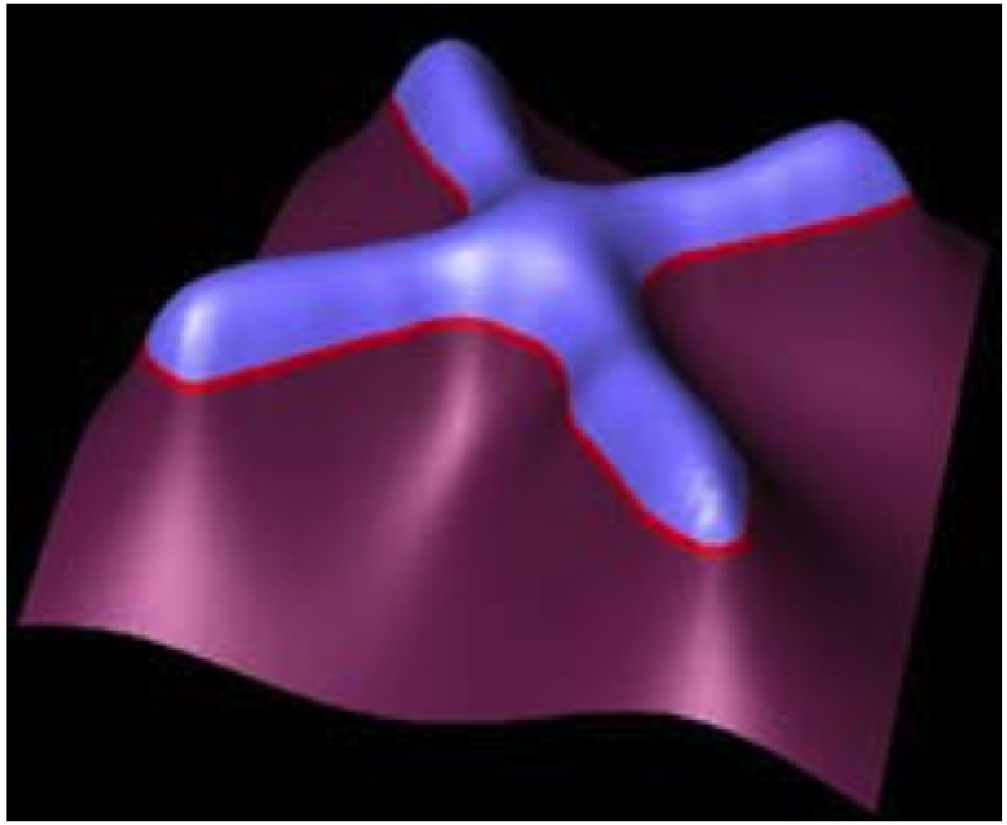




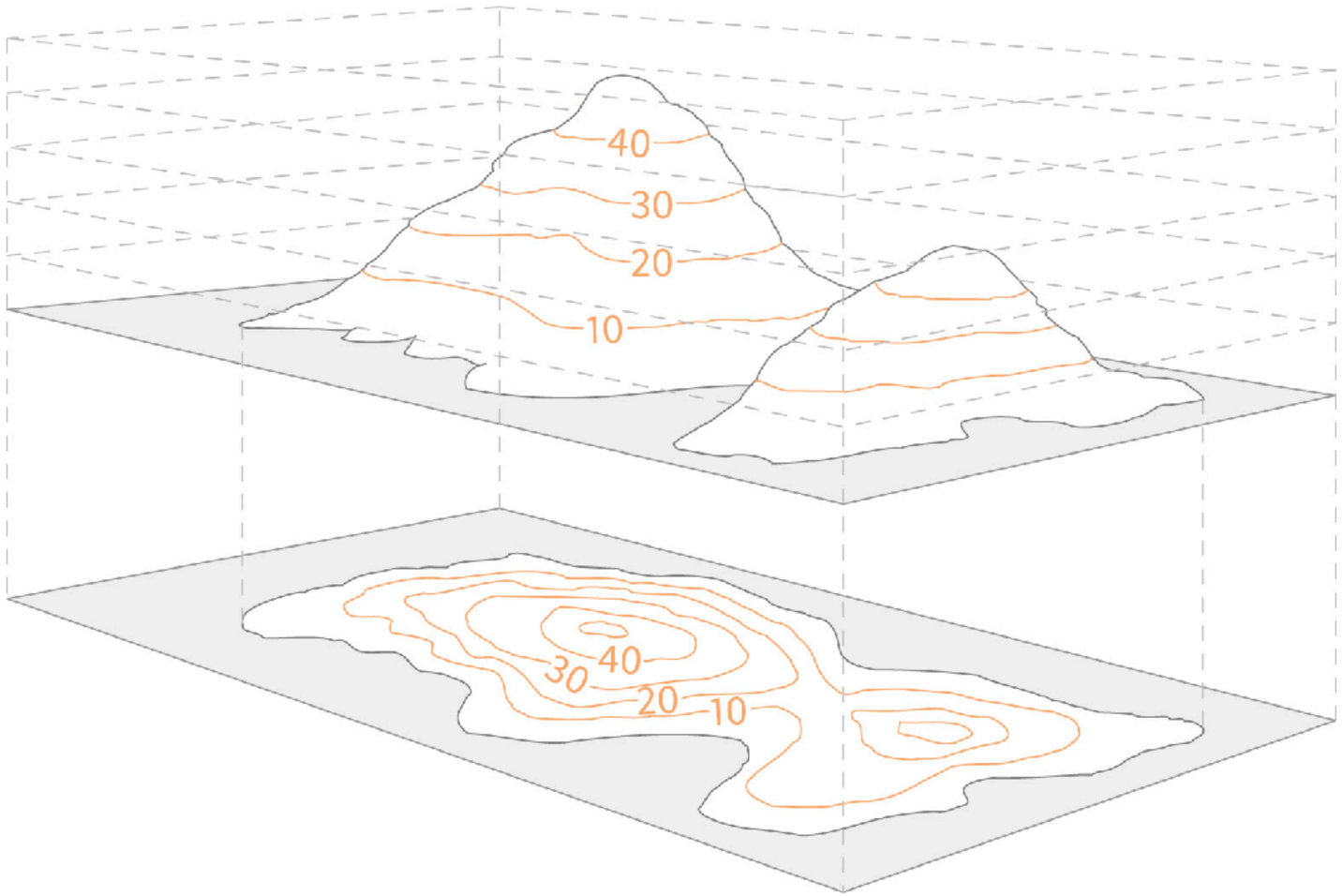
Explicit →Implicit

Implicit Representations

Level set of 2D function defines 1D curve



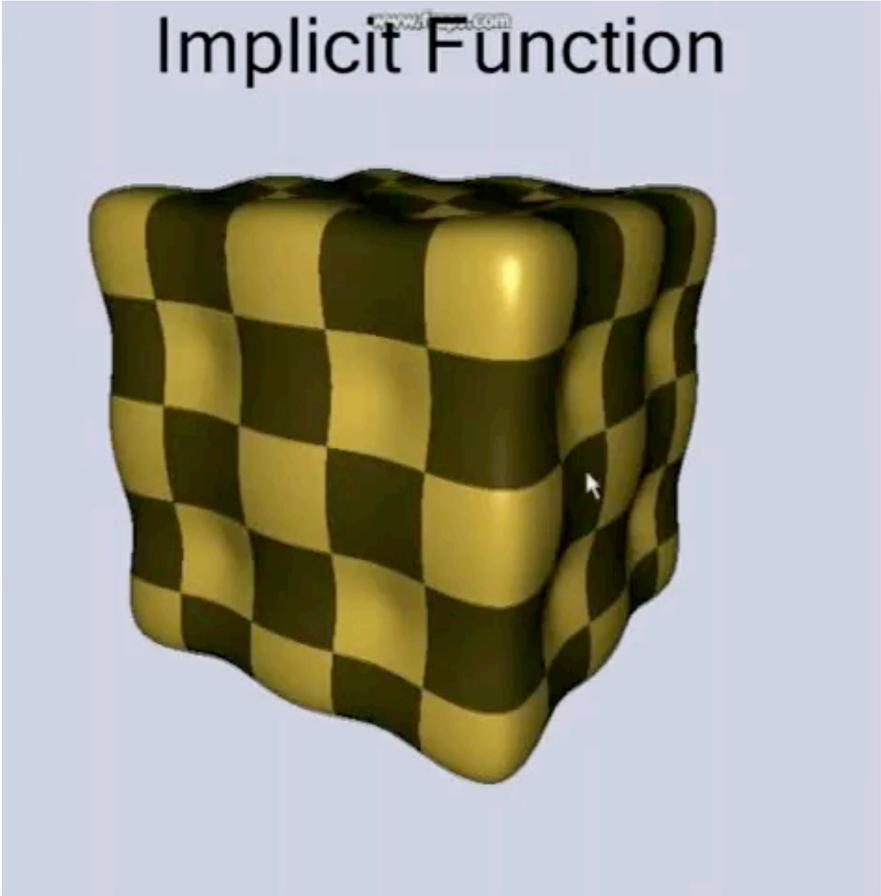
Contour Map



https://getoutside.ordnancesurvey.co.uk/guides/understanding-map-contour-lines-for-beginners/

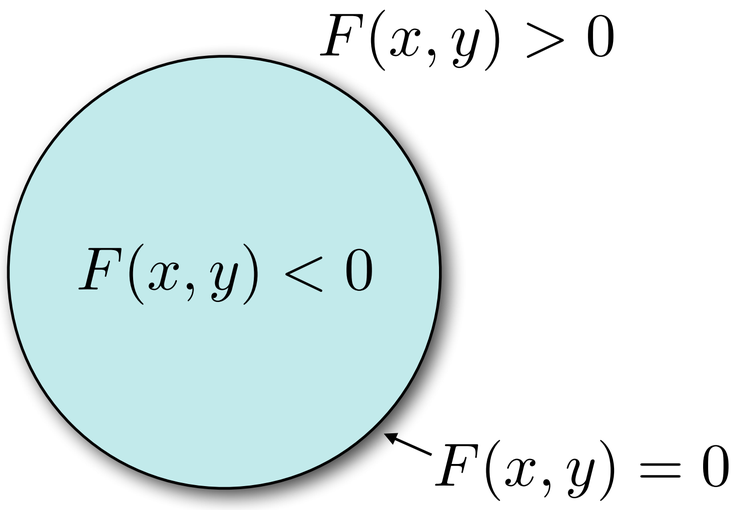
Implicit Representations

Level set of 3D function defines 2D surface



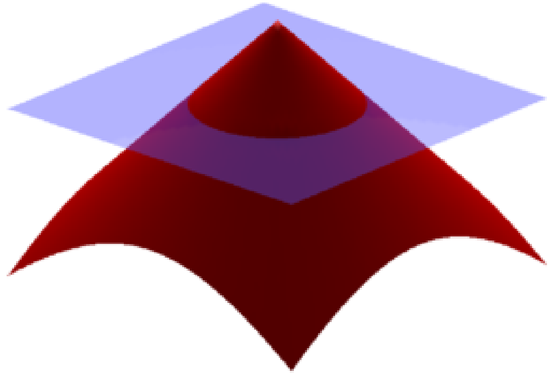
Implicit Representations

- General implicit function:
  - Interior: $F(x,y,z) < 0$
  - Exterior: $F(x,y,z) > 0$
  - Surface: $F(x,y,z) = 0$
- Gradient **∇F is orthogonal to level set**
- Special case
  - Signed distance function (SDF)
  - Gradient ∇F is unit surface normal



Signed Distance Function

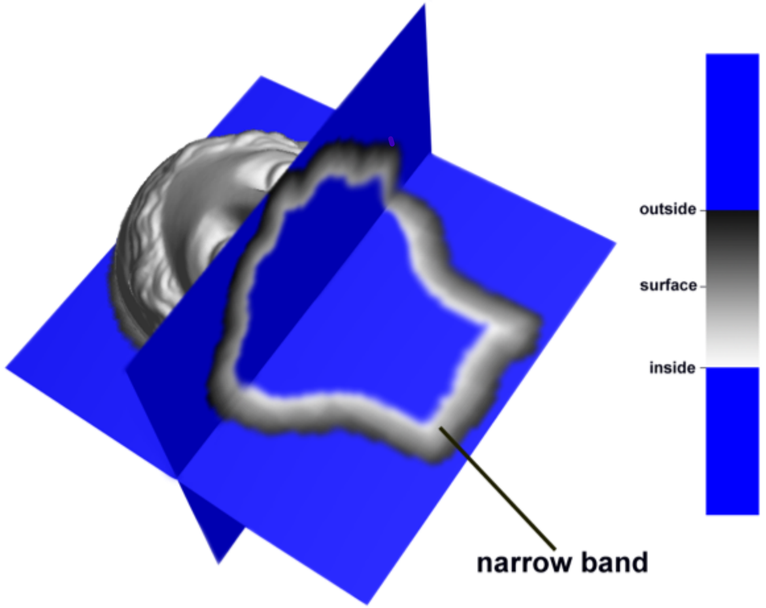
- SDF of 2D circle?
- General shapes



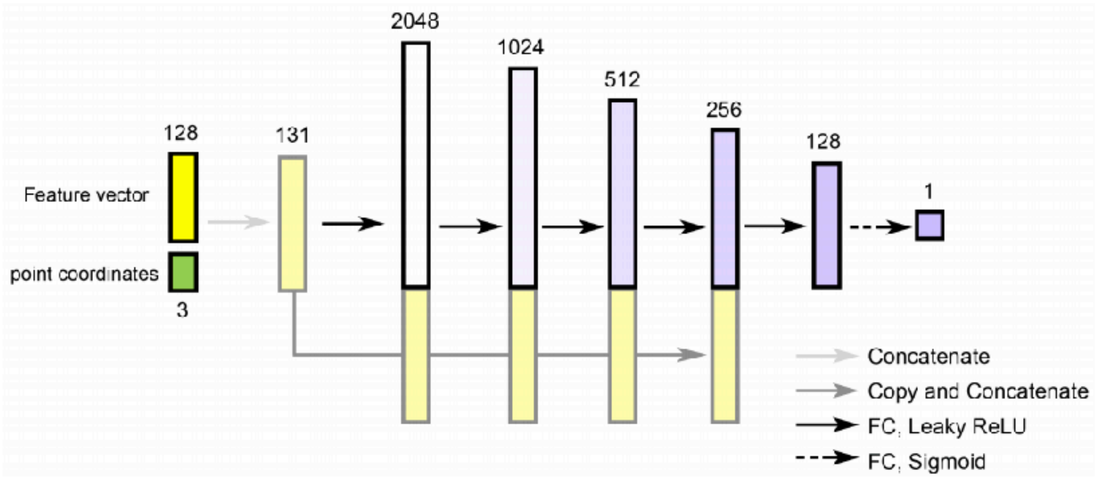
(TSDF)

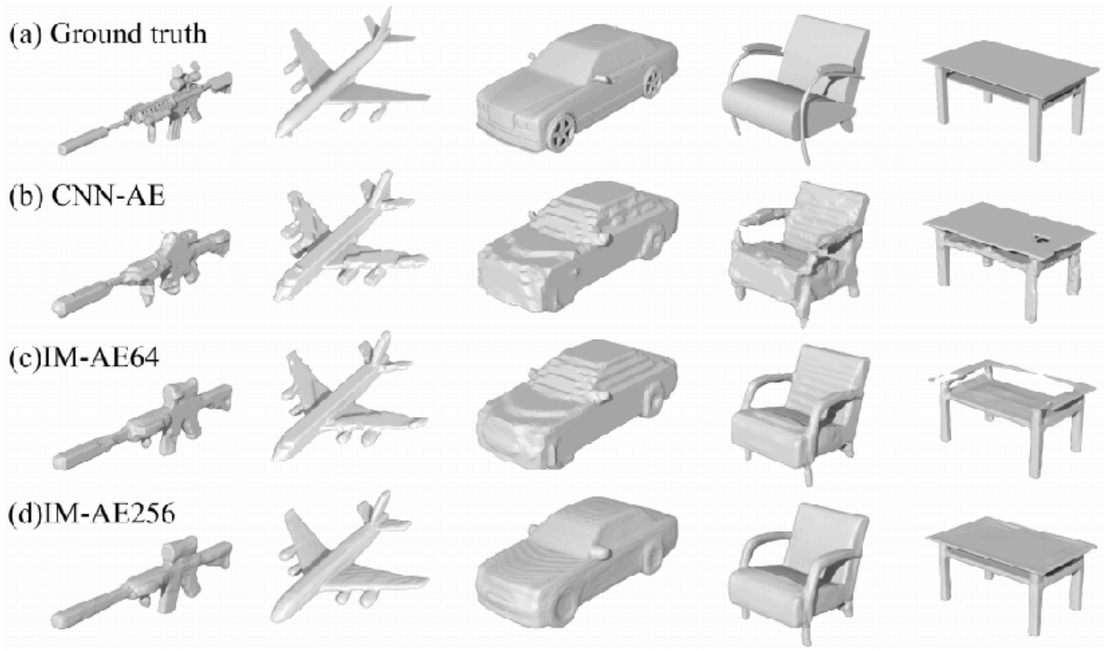
↓ truncated

(不管太远了)



Learning an SDF





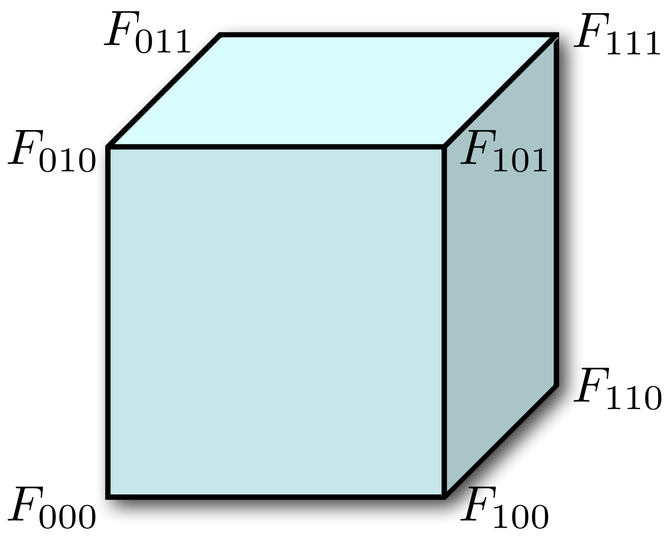
$$f_\theta(\mathbf{p}, \mathbf{z})$$

$$\mathbb{R}^{3+f} \rightarrow \mathbb{R}$$

[Chen and Zhang, 2018]

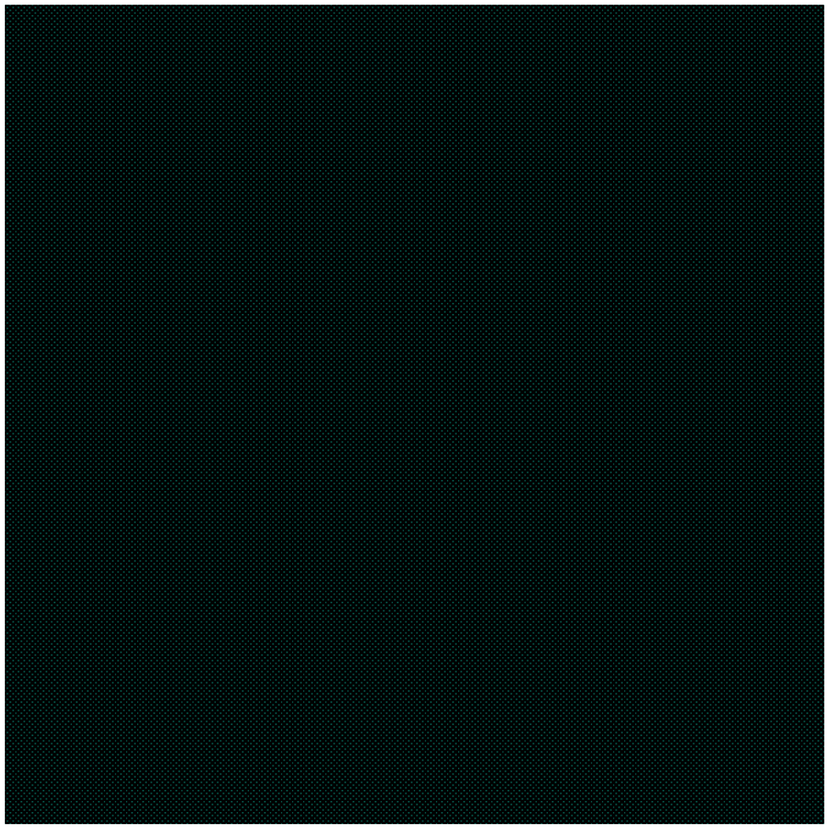
SDF Discretization

- Regular Cartesian 3D grid
  - Compute signed distance at nodes
  - Tri-linear interpolation within cells

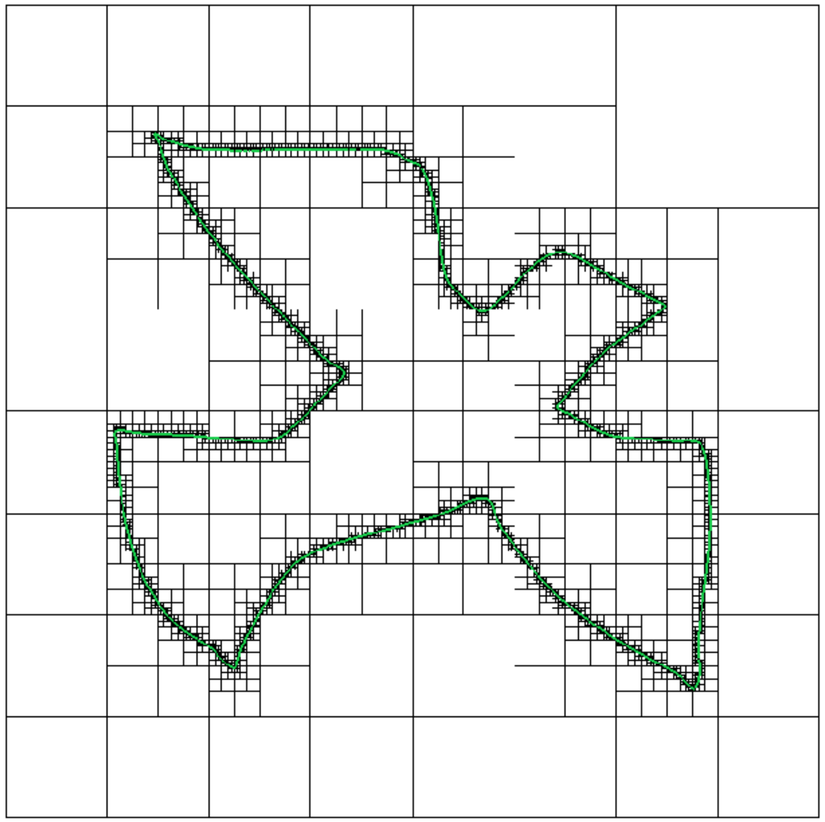


$$
\begin{array}{lcccc}
F_{000} & (1-u) & (1-v) & (1-w) & + \\
F_{100} & u & (1-v) & (1-w) & + \\
F_{010} & (1-u) & v & (1-w) & + \\
F_{001} & (1-u) & (1-v) & w & + \\
\vdots & & & & \\
F_{111} & u & v & w &
\end{array}
$$

3-Color Octree

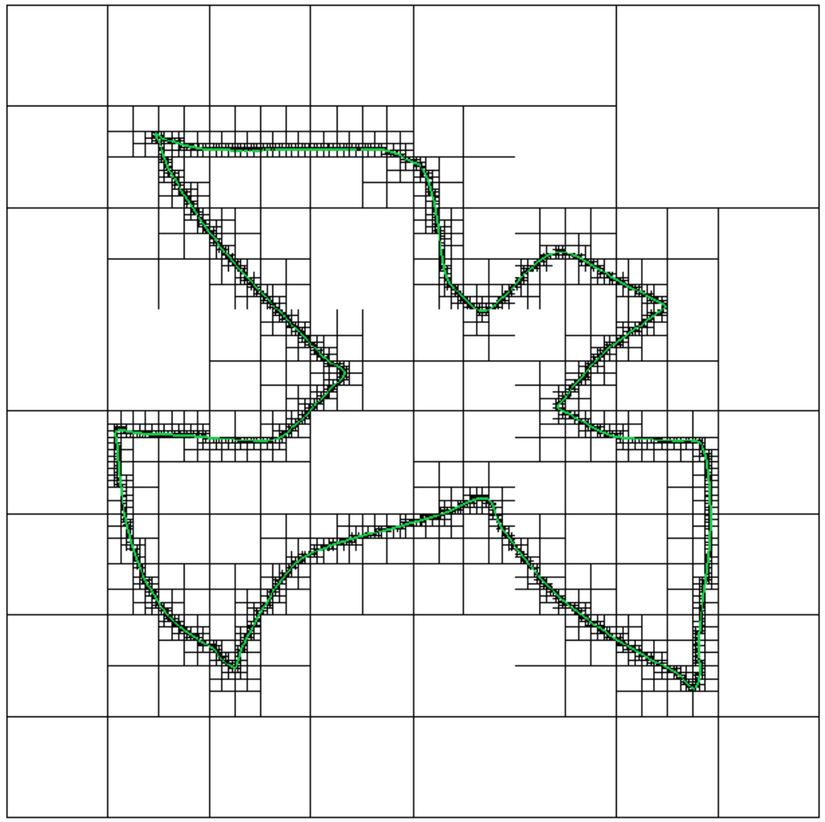


1048576 cells

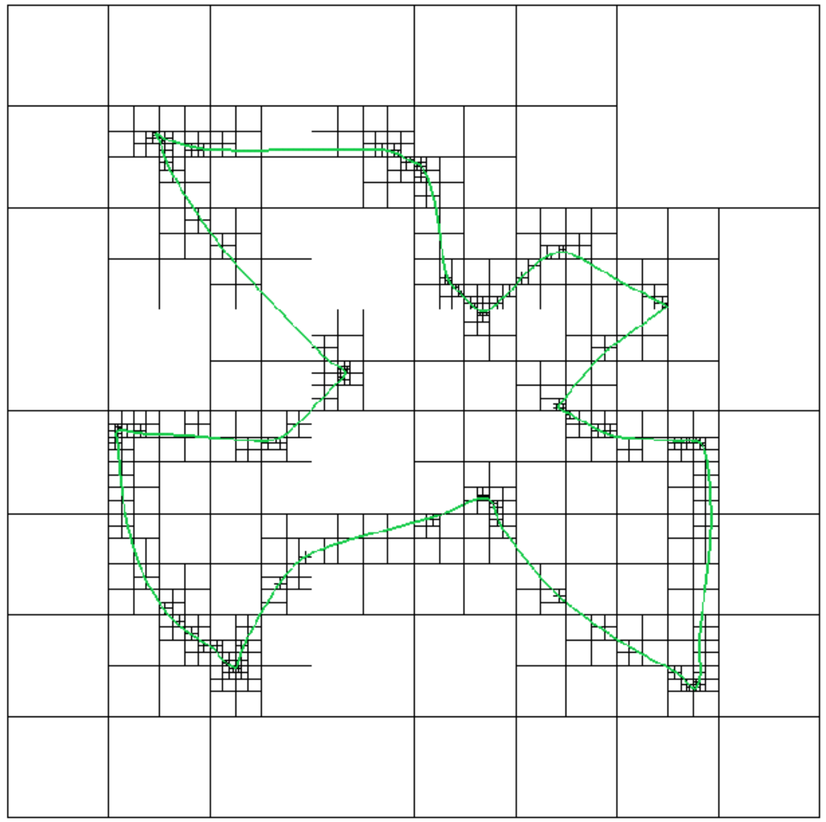


12040 cells

Adaptively Sampled Distance Fields

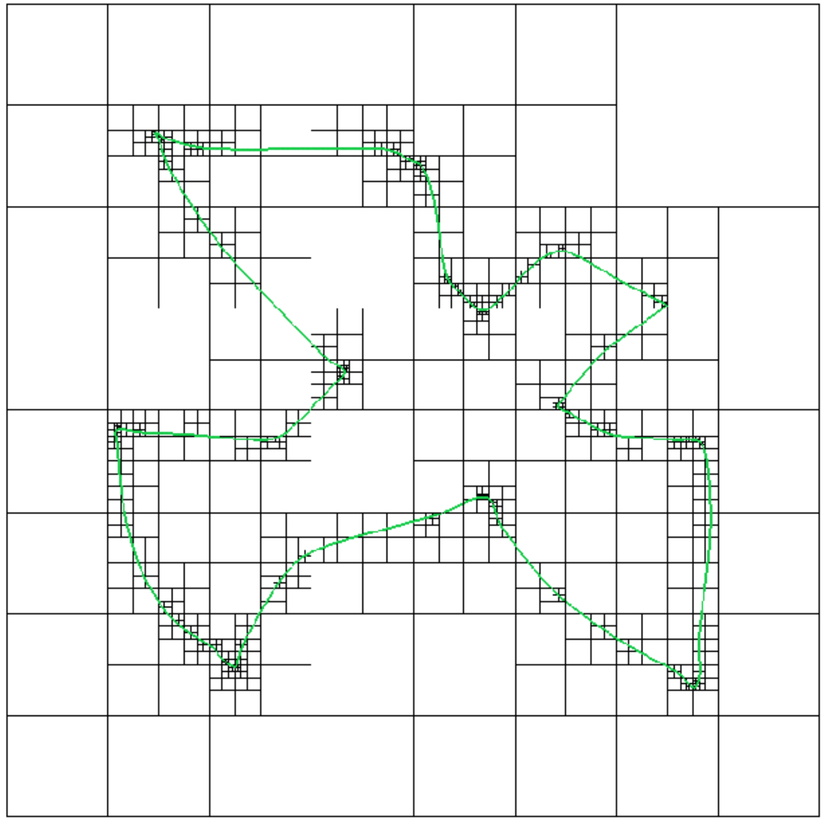


12040 cells

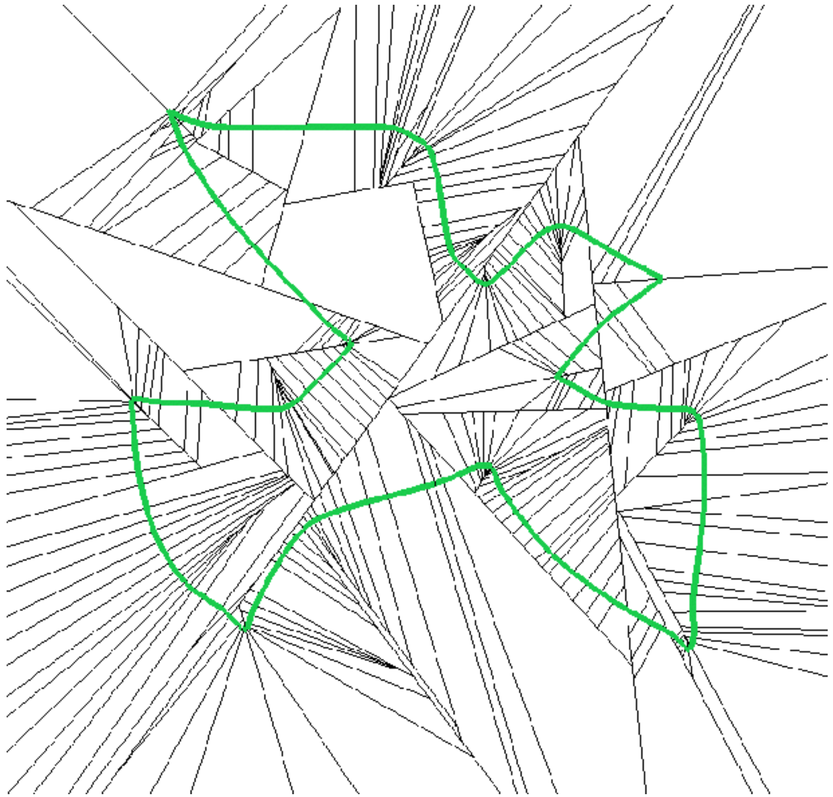


895 cells

Binary Space Partitions



895 cells



254 cells

Regularity vs. Complexity

- Implicit surface discretizations
  - Uniform, regular voxel grids
  - Adaptive, 3-color octrees
    - surface-adaptive refinement
    - feature-adaptive refinement
  - Irregular hierarchies
    - binary space partition (BSP)

Regularity vs. Complexity

- Implicit surface discretizations
  - Uniform, regular voxel grids $O(h^{-3})$
  - Adaptive, 3-color octrees
    - surface-adaptive refinement $O(h^{-2})$
    - feature-adaptive refinement $O(h^{-1})$
  - Irregular hierarchies
    - binary space partition (BSP) $O(h^{-1})$

Literature

- Frisken et al., *“Adaptively Sampled Distance Fields: A general representation of shape for computer graphics”*, SIGGRAPH 2000
- Wu & Kobbelt, *“Piecewise Linear Approximation of Signed Distance Fields”*, VMV 2003

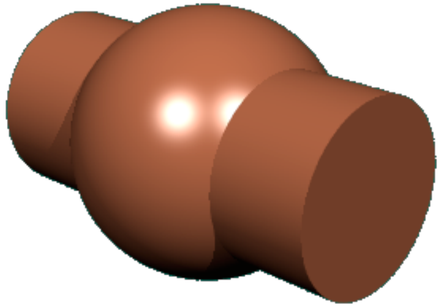
Implicit Representations

- Natural representation for volumetric data,
  e.g., CT scans, density fields, etc.
- Advantageous when modeling shapes with complex and/or changing topology (e.g., fluids)
- Very suitable representation for *Constructive Solid Geometry* (CSG)

Constructive Solid Geometry

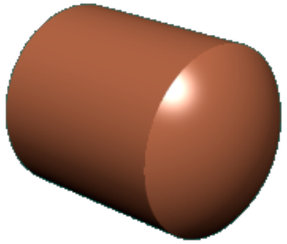
- Union

$$F_{C \cup S}(\cdot) = \min\{F_C(\cdot), F_S(\cdot)\}$$



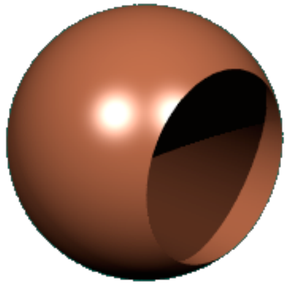
- Intersection

$$F_{C \cap S}(\cdot) = \max\{F_C(\cdot), F_S(\cdot)\}$$



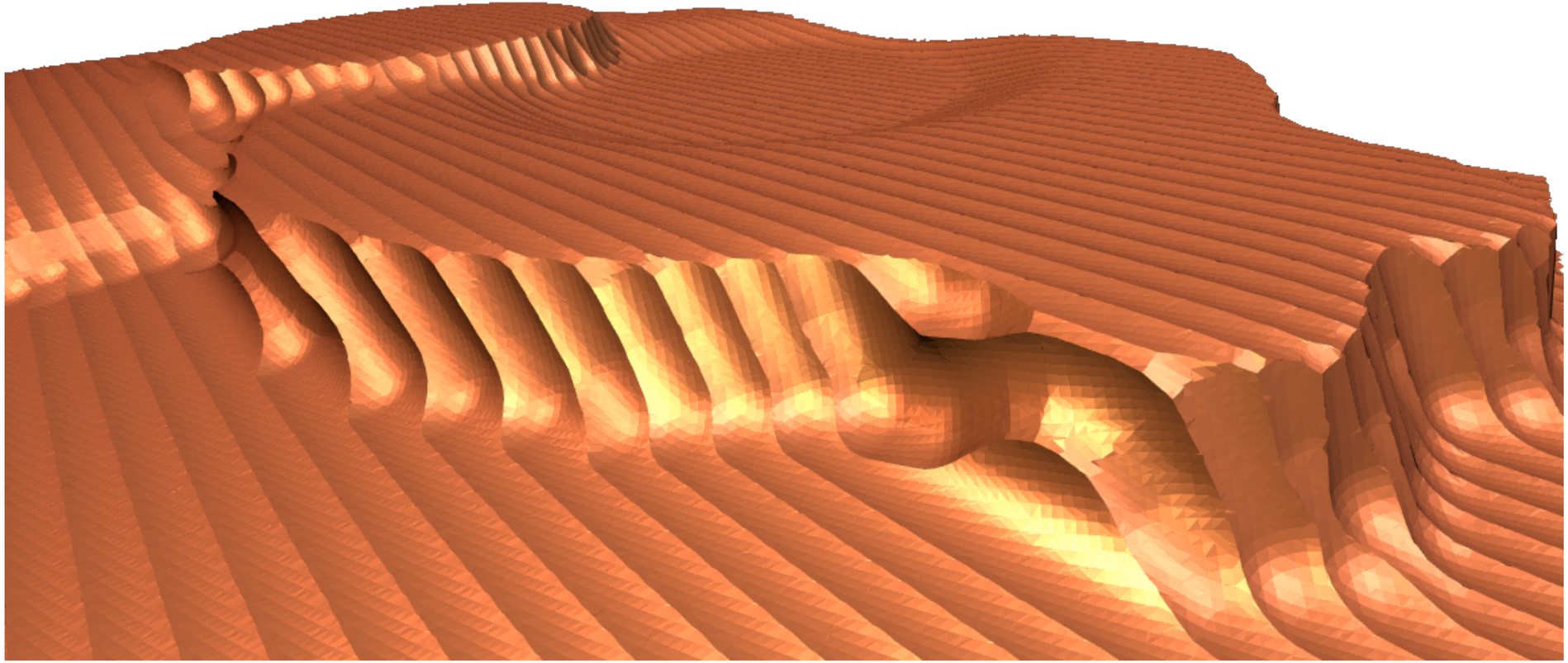
- Difference

$$F_{S \setminus C}(\cdot) = \max\{-F_C(\cdot), F_S(\cdot)\}$$



CSG Example: Milling

- Union
- Intersection
- Difference



Overview

- Surface representations
  - Explicit vs. Implicit
- Explicit representations
  - Triangle meshes
- Implicit representations
  - Signed distance fields
- **Conversions**
  - Explicit ↔ Implicit

Conversions

- **Explicit to Implicit**
  - Compute signed distance at grid points
  - Compute distance point-mesh
  - Fast marching

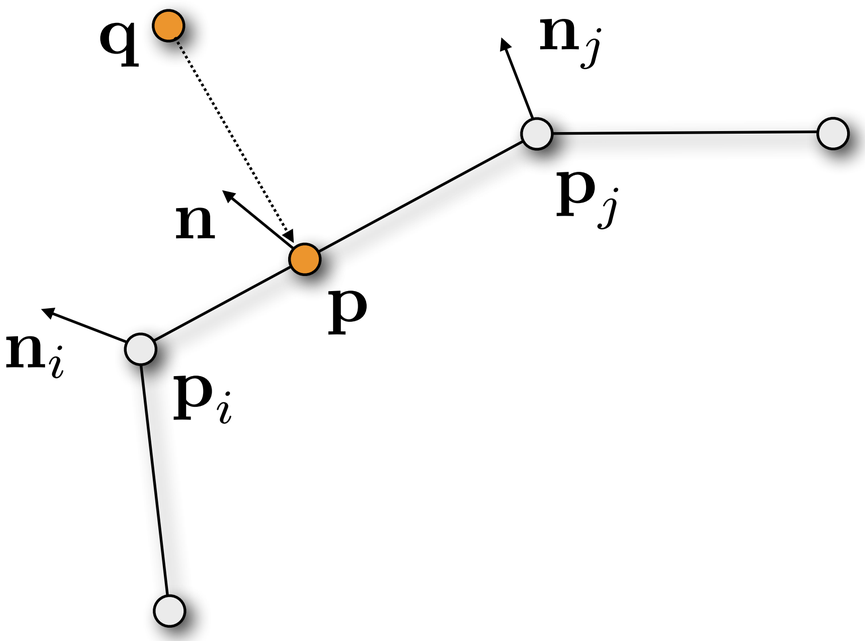
- Implicit to Explicit
  - Extract zero-level iso-surface $F(x,y,z)=0$
  - Other iso-surfaces $F(x,y,z)=C$
  - Medical imaging, simulations, measurements, ...

Signed Distance Computation

- Find closest mesh triangle
  - Use spatial hierarchies (octree, BSP tree)
- Distance Point-Triangle
  - Distance to plane, edge, or vertex
  - See http://www.geometrictools.com
- Inside or outside?
  - Based on interpolated surface normals

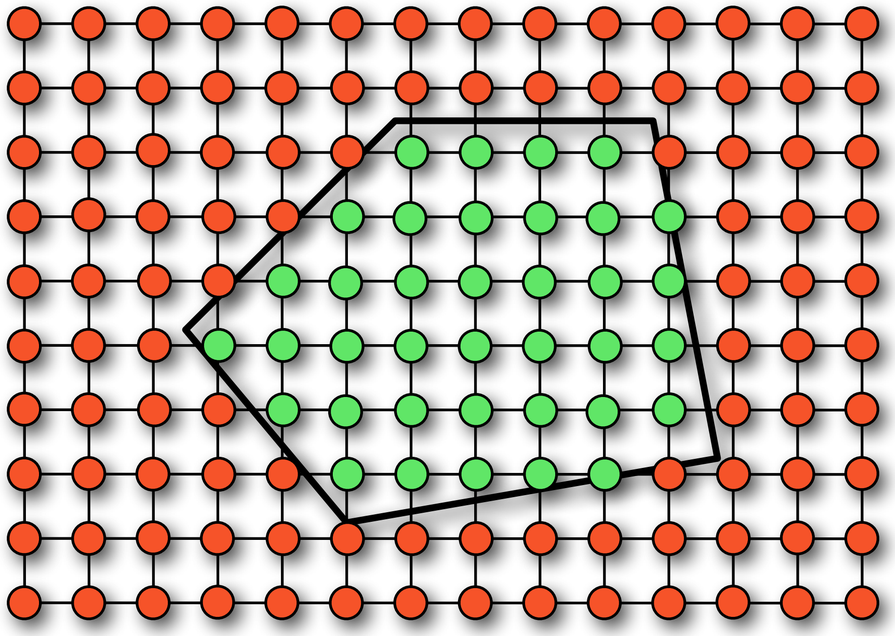
Signed Distance Heuristic

1. Closest point $\mathbf{p} = \alpha \mathbf{p}_i + (1-\alpha)\mathbf{p}_j$
2. Interpolated normal $\mathbf{n} = \alpha \mathbf{n}_i + (1-\alpha)\mathbf{n}_j$
3. Inside if $(\mathbf{q}-\mathbf{p})^T \mathbf{n} < 0$



Fast Marching Techniques

1. Initialize with exact distance in mesh's vicinity
2. Fast-march outwards
3. Fast-march inwards



Literature

- Schneider, Eberly, *“Geometric Tools for Computer Graphics”*, Morgan Kaufmann, 2002
- Sethian, *“Level Set and Fast Marching Methods”*, Cambridge University Press, 1999

Conversions

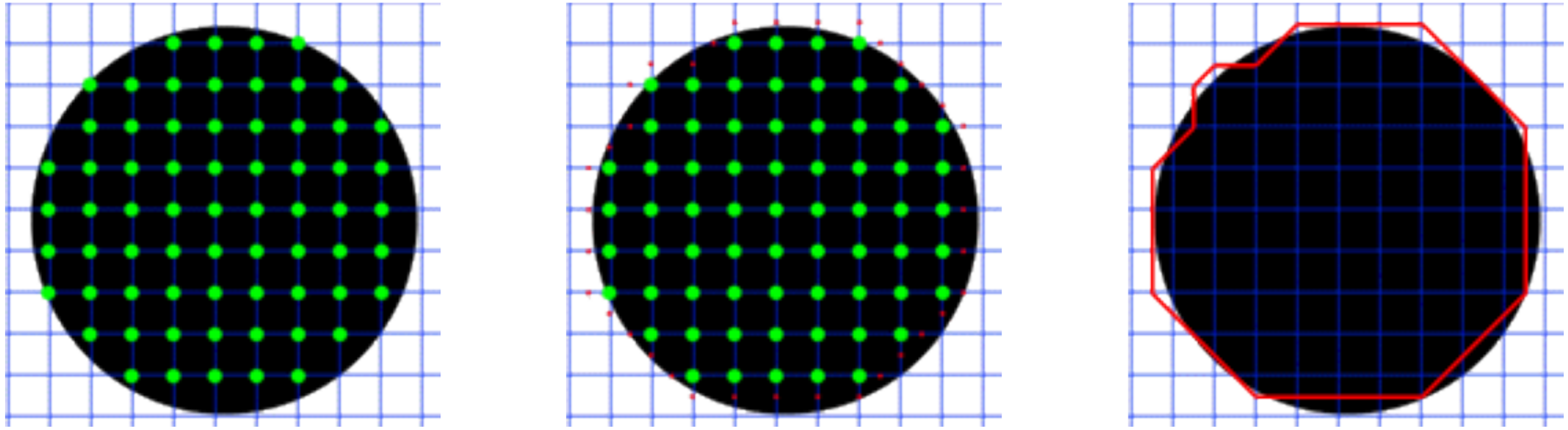
- Explicit to Implicit
  - Compute signed distance at grid points
  - Compute distance point-mesh
  - Fast marching

- **Implicit to Explicit**
  - Extract zero-level iso-surface $F(x,y,z)=0$
  - Other iso-surfaces $F(x,y,z)=C$
  - Medical imaging, simulations, measurements, ...

2D: Marching Squares

1. Classify grid nodes as inside/outside
   - Is $F(\mathbf{x}_{ij}) > 0 \quad \text{or} \quad < 0?$
2. Classify cell: 2^4 configurations
   - In/out for each corner
3. Compute intersection points
   - Linear interpolation along edges
4. Connect them by edges
   - Look-up table for edge configuration

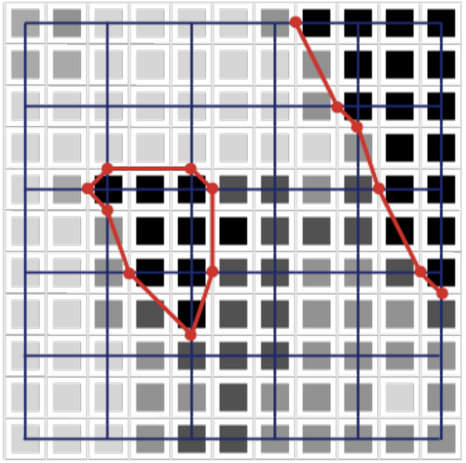
Marching Cube in 2D

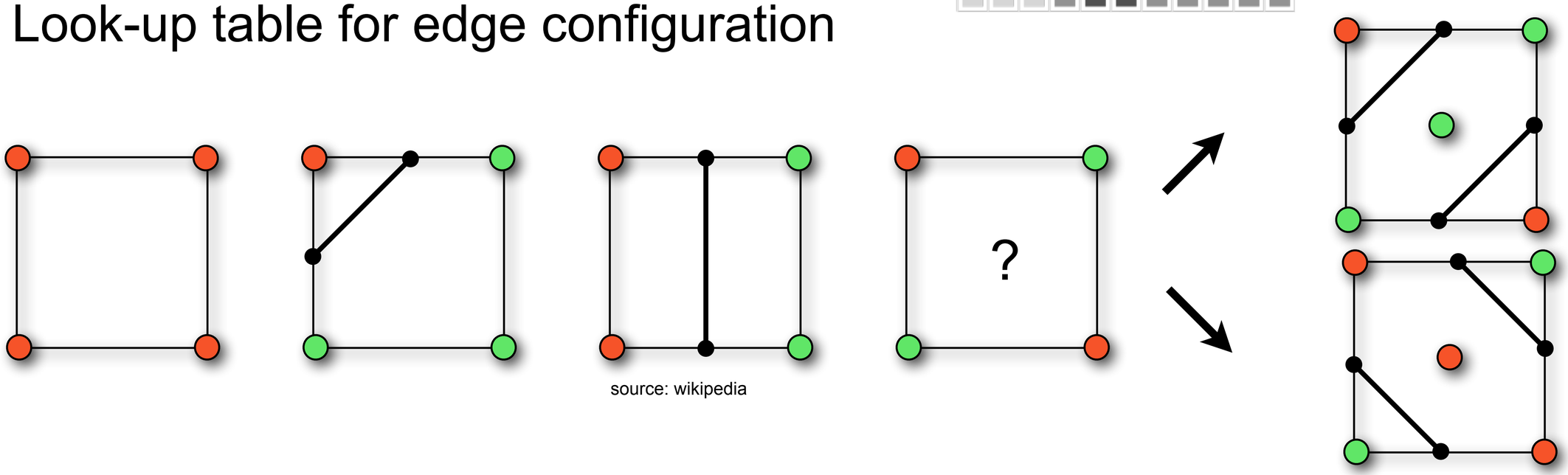


[http://www.exaflop.org/docs/marchcubes/]

2D: Marching Squares

- Classify grid nodes as inside/outside
- Classify cell: $2^4 = 16$ configurations
- Linear interpolation along edges
- Look-up table for edge configuration





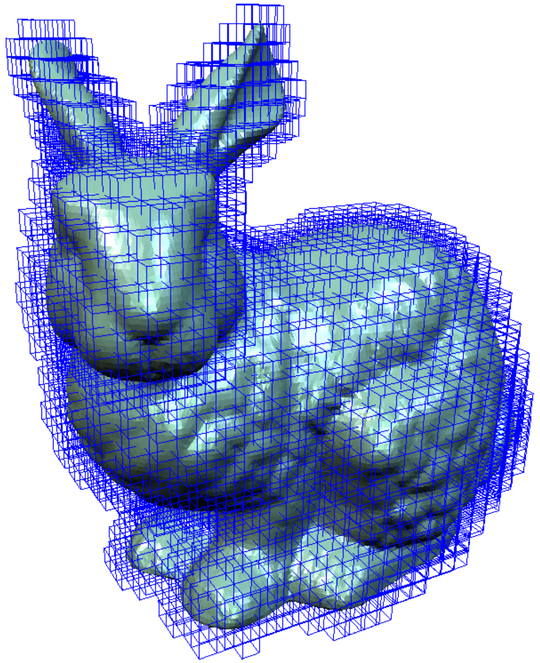
source: wikipedia

3D: Marching Cubes

1. Classify grid nodes as inside/outside
   - Is $F(\mathbf{x}_{ijk}) > 0 \quad \text{or} \quad < 0?$
2. Classify cell: 2^3 configurations
   - In/out for each corner
3. Compute intersection points
   - Linear interpolation along edges
4. Connect them by triangles
   - Look-up table for path configuration
   - Disambiguation by modified table [Montani 94]

Marching Cubes

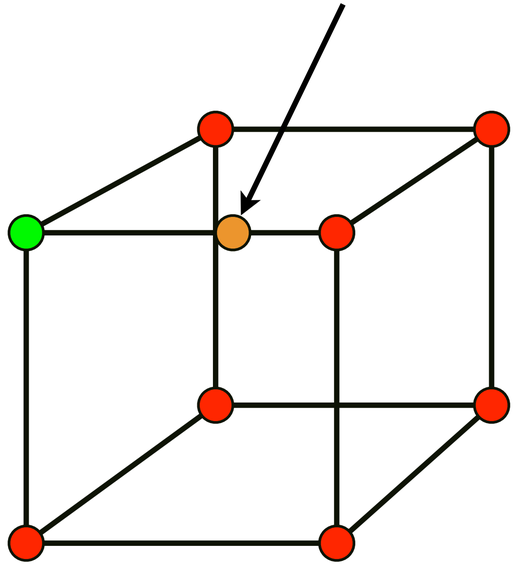
- Classify grid nodes $\mathbf{x}_{i,j,k}$ based on $F_{i,j,k} = F(\mathbf{x}_{i,j,k})$
  - Inside or outside
- Classify all cubes based on $F_{i,j,k}$
  - Inside, outside, or intersecting
- Refined only intersected cells
  - 3-color adaptive octree
  - $O(h^{-2})$ complexity



Intersection Points

- Linear interpolation along edges

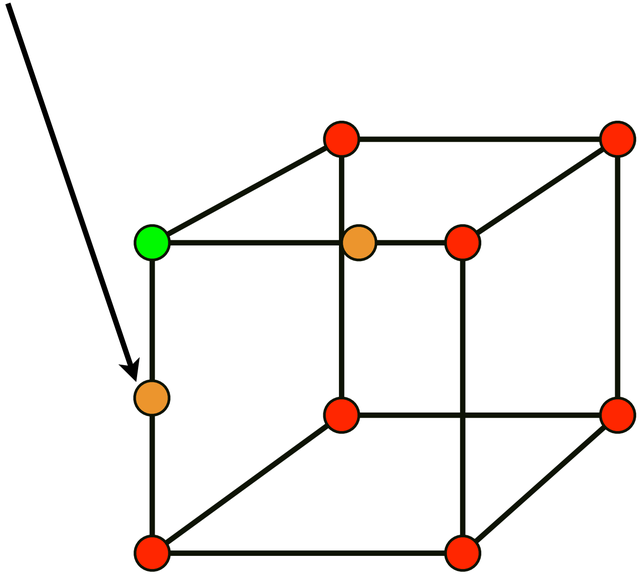
$$\frac{\mathbf{x}_{i,j,k} \cdot |F_{i+1,j,k}| \; + \; \mathbf{x}_{i+1,j,k} \cdot |F_{i,j,k}|}{|F_{i,j,k}| + |F_{i+1,j,k}|}$$



Intersection Points

- Linear interpolation along edges

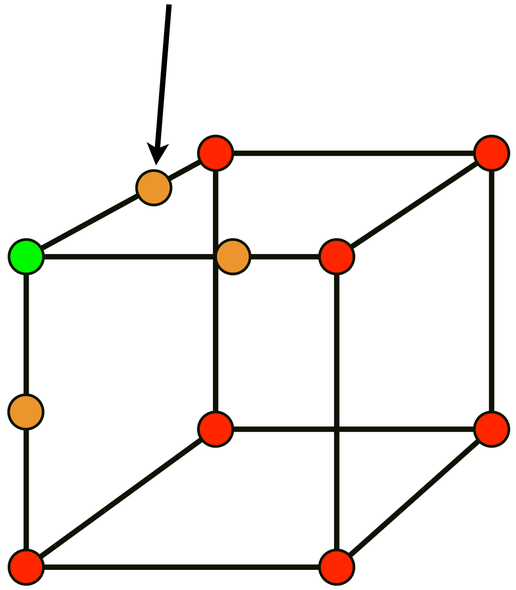
$$\frac{\mathbf{x}_{i,j,k} \cdot |F_{i,j+1,k}| \; + \; \mathbf{x}_{i,j+1,k} \cdot |F_{i,j,k}|}{|F_{i,j,k}| + |F_{i,j+1,k}|}$$



Intersection Points

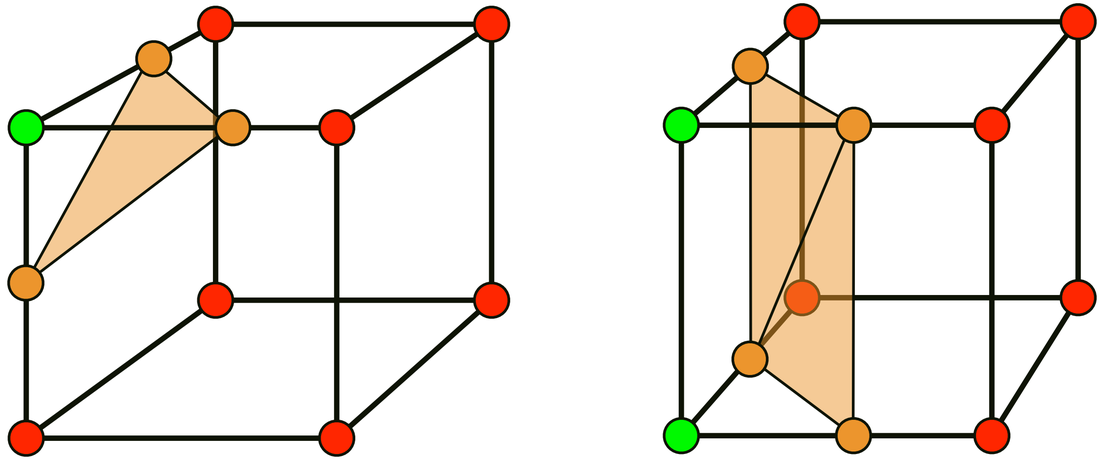
- Linear interpolation along edges

$$\frac{\mathbf{x}_{i,j,k} \cdot |F_{i,j,k+1}| \; + \; \mathbf{x}_{i,j,k+1} \cdot |F_{i,j,k}|}{|F_{i,j,k}| + |F_{i,j,k+1}|}$$



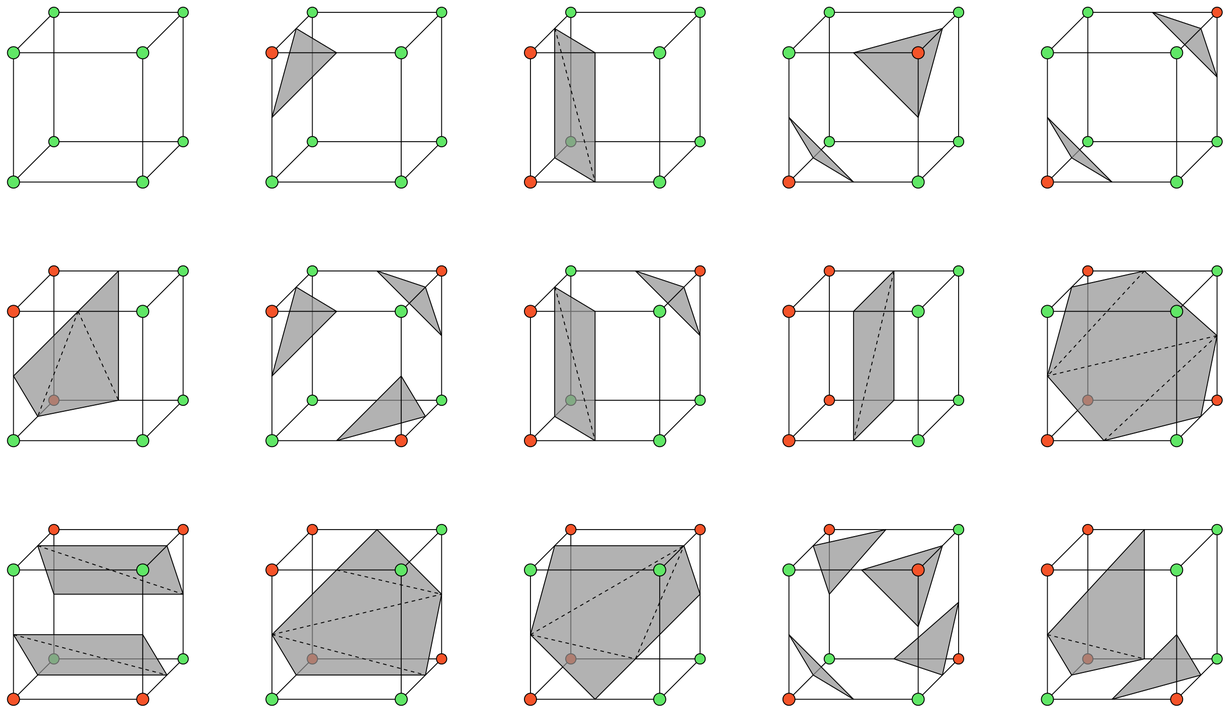
Intersection Points

- Linear interpolation along edges
- Lookup table for patch configuration



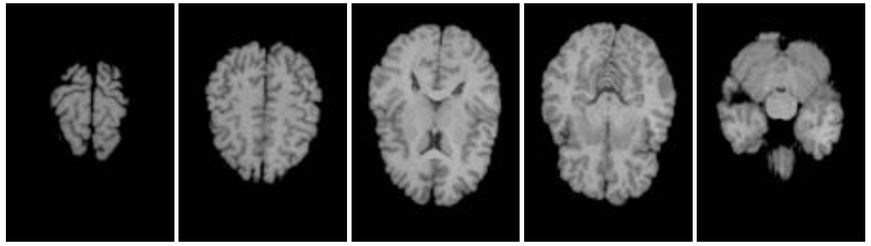
Marching Cubes

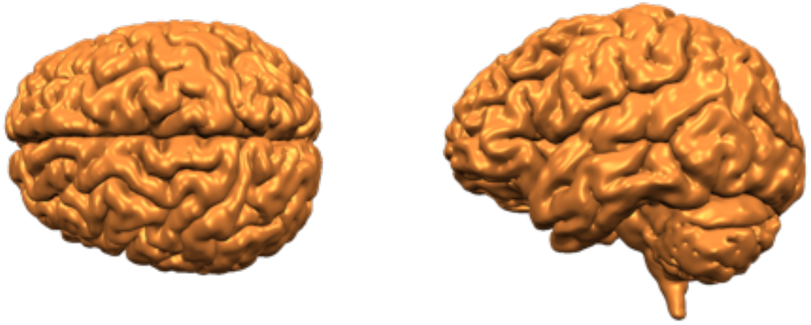
- Look-up table with 2^8 entries
  - 15 representative cases shown
  - Others follow by symmetry



Marching Cubes

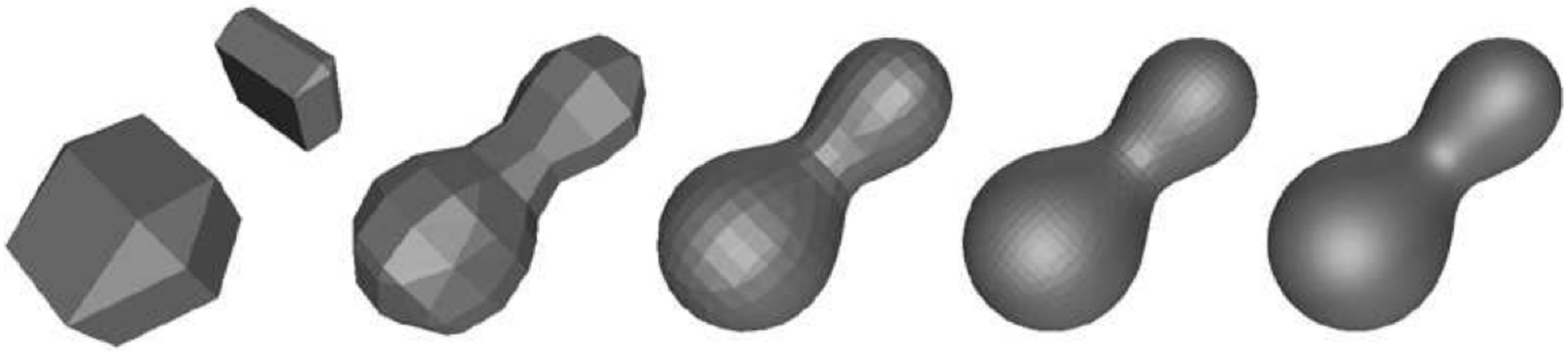
THE algorithm for isosurface extraction from medical scans (CT, MRI)





Marching Cubes

Effect of grid size



Grid size=10
70 Facets

Grid size=5
220 Facets

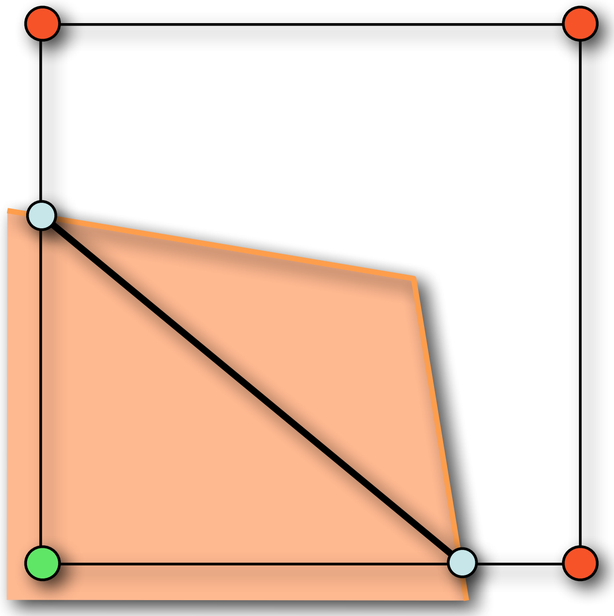
Grid size=2
1700 Facets

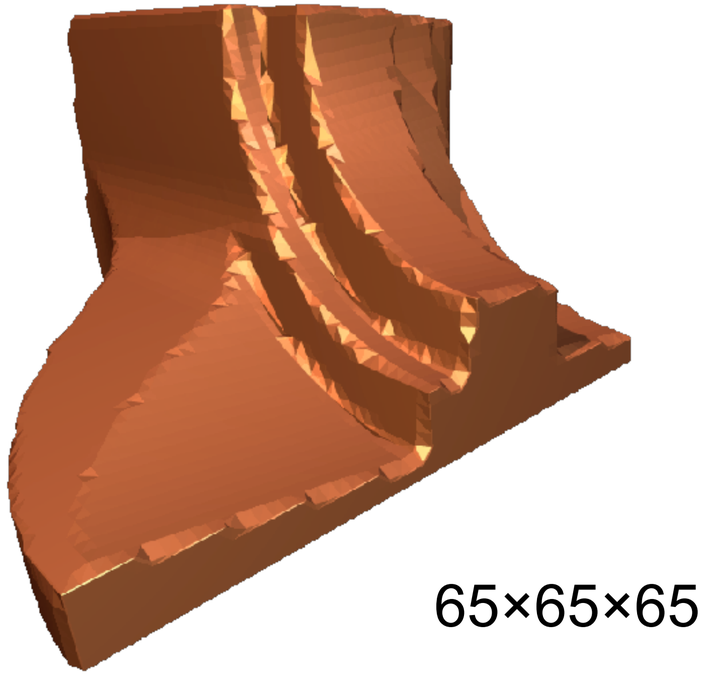
Grid size=1
6800 Facets

Grid size=0.5
27000 Facets

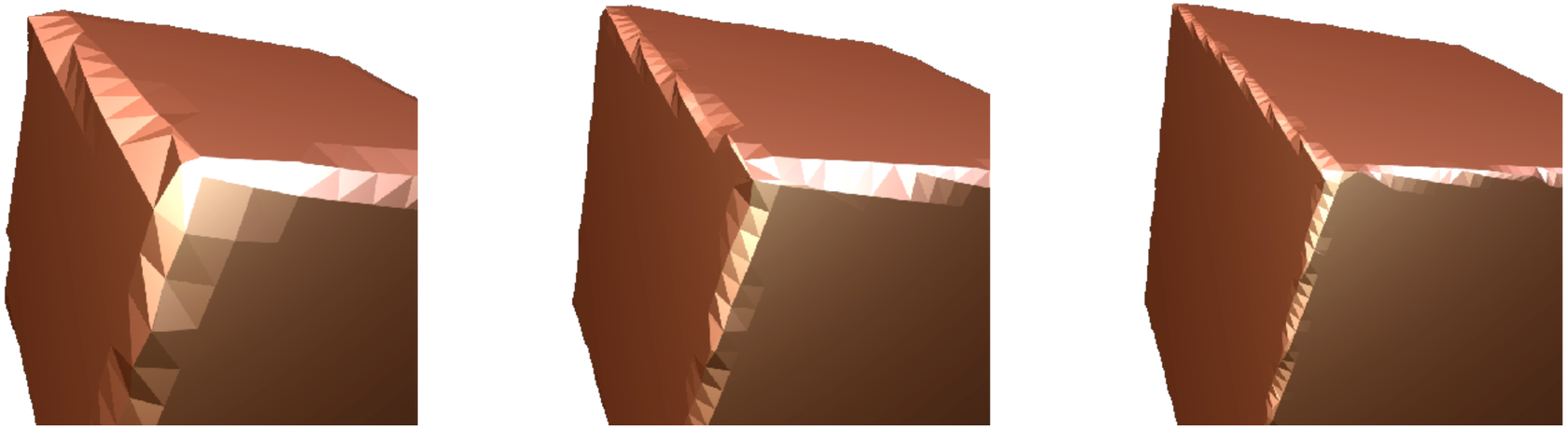
Marching Cubes

- Sample points restricted to edges of regular grid
- Alias artifacts at sharp features





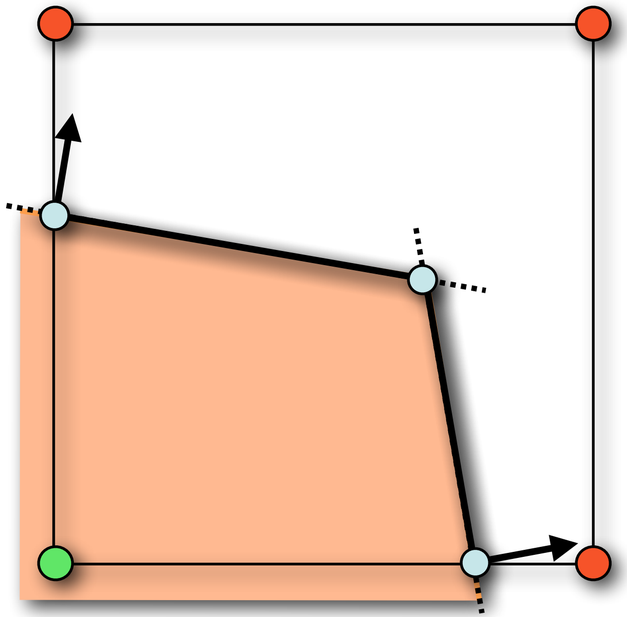
Increasing Resolution?

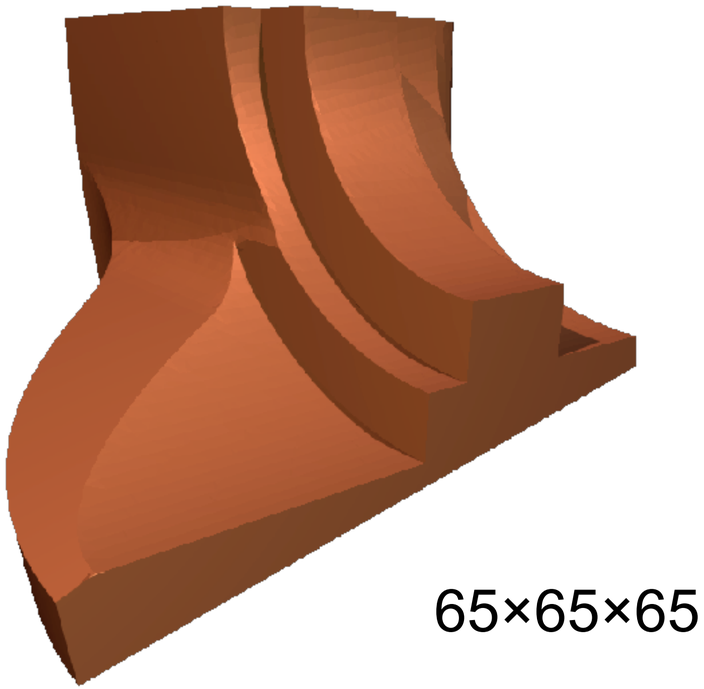


Does not remove alias problems!

Extended Marching Cubes

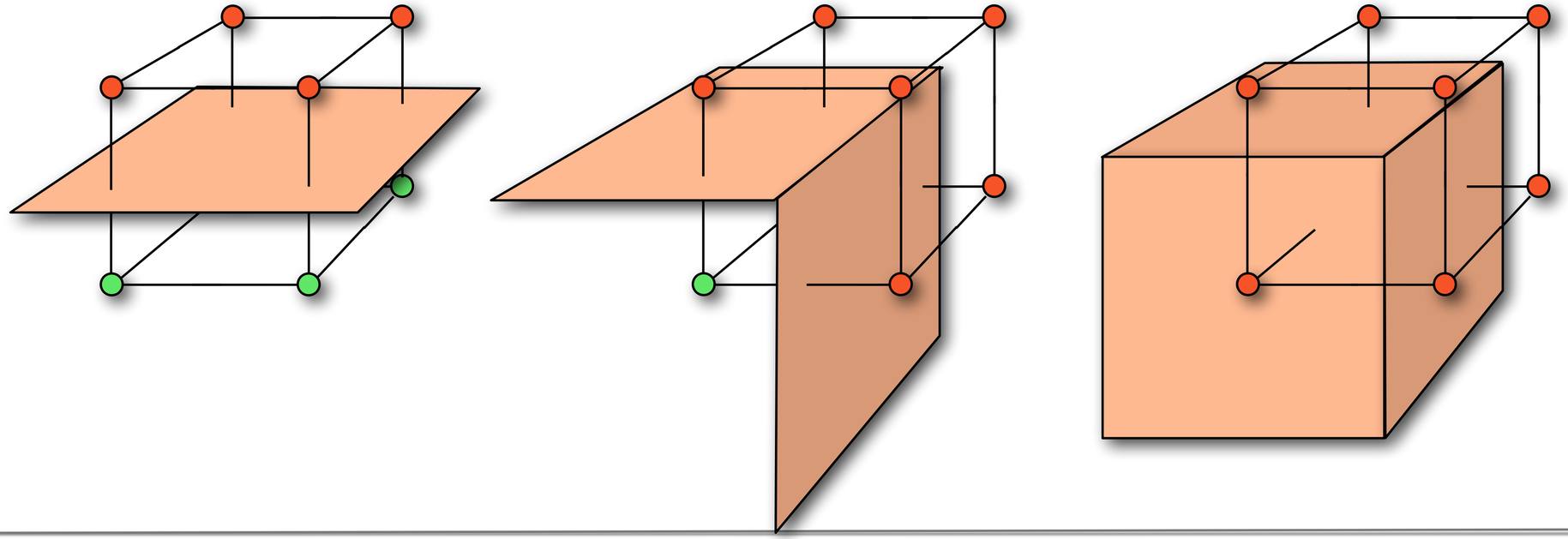
- Locally extrapolate distance gradient
- Place samples on estimated feature





Extended Marching Cubes

- Feature detection
  - Based on angle between normals $\mathbf{n}_i$
  - Classify into edges / corners



Extended Marching Cubes

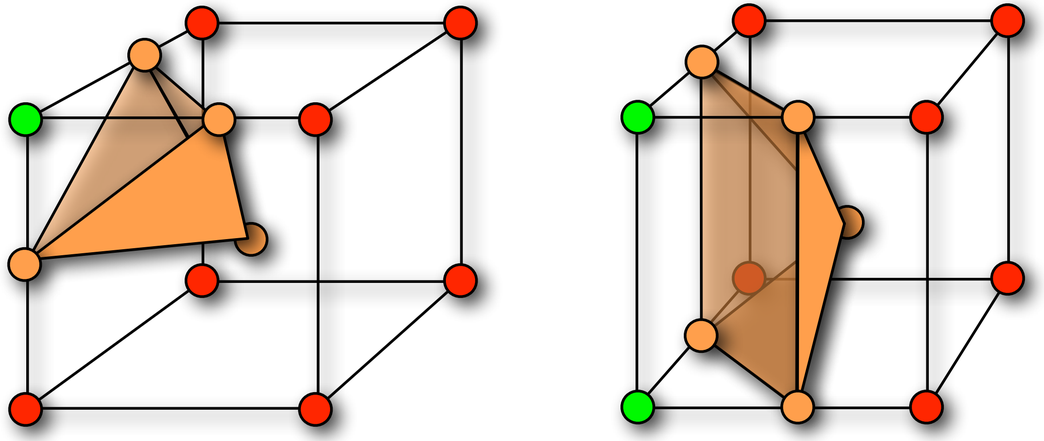
- Feature sampling
  - Intersect tangent planes ($\mathbf{s}_i$, $\mathbf{n}_i$)

$$\begin{pmatrix} \vdots \\ \mathbf{n}_i \\ \vdots \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathbf{n}_i^T \mathbf{s}_i \\ \vdots \end{pmatrix}$$

  - Over- or under-determined system
  - Solve by SVD pseudo-inverse

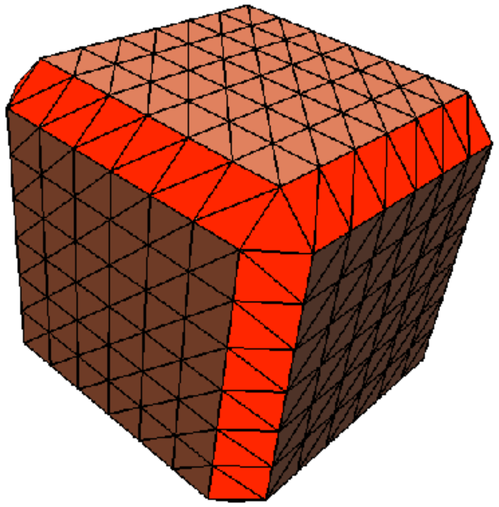
Extended Marching Cubes

- Feature sampling
  - Intersect tangent planes ($\mathbf{s}_i$, $\mathbf{n}_i$)
  - Triangle fans centered at feature point

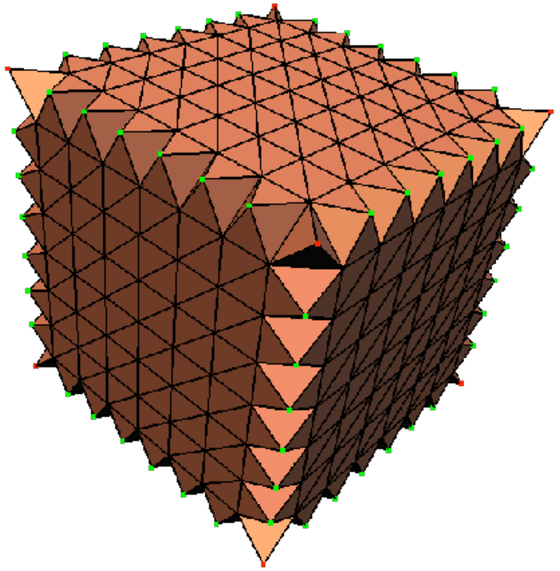


Extended Marching Cubes

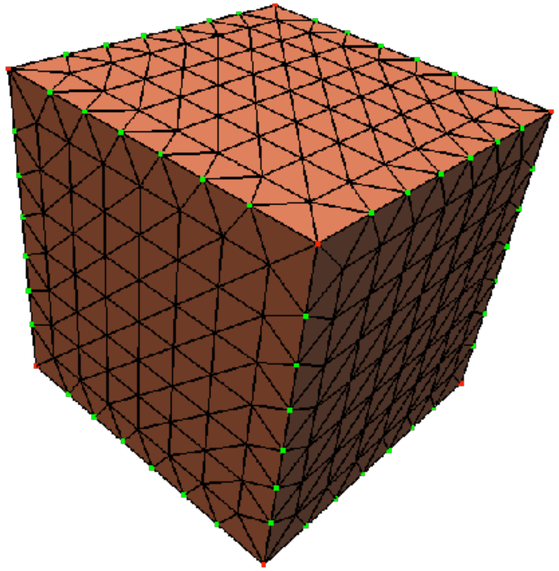
- Feature reconstruction



Feature Detection

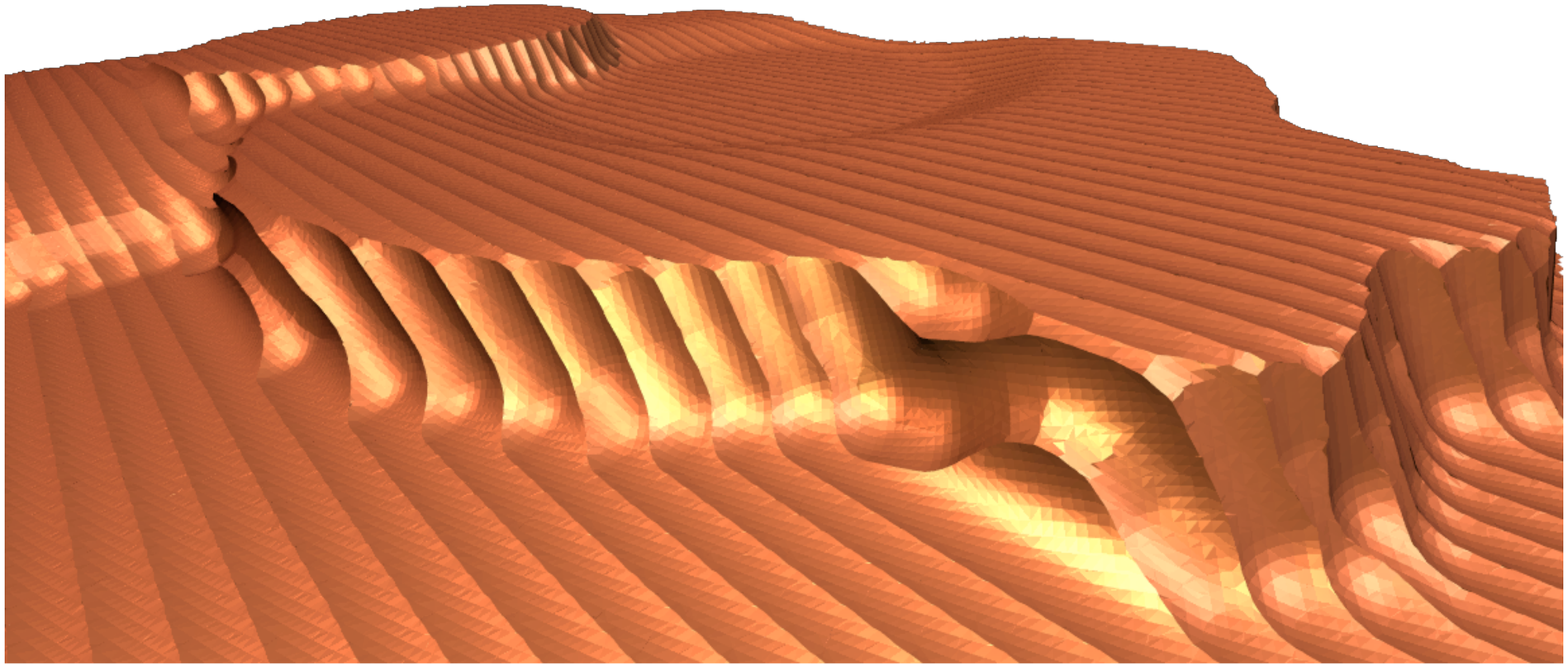


Feature Sampling



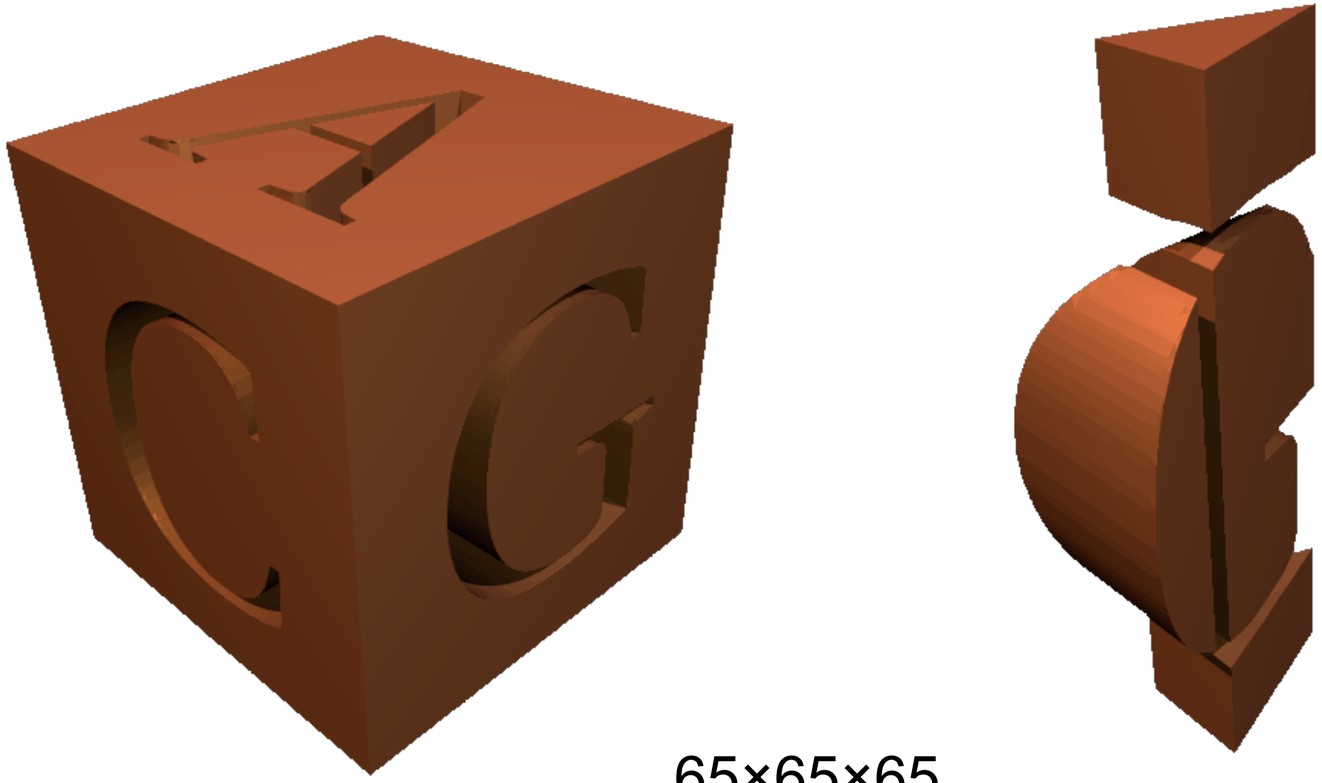
Edge Flipping

Milling Simulation



257×257×257

CSG Modeling



65×65×65

Marching Cubes

+ Result is watertight, closed 2-manifold surface!
+ Easy to parallelize
- Uniform (over-)sampling (→ mesh decimation)
- Degenerate triangles (→ remeshing)
- MC does not preserve features
+ EMC preserves features, but...
  - about 10% more triangles
  - 20-40% computational overhead

Literature

- Lorensen & Cline, *"Marching Cubes: A High Resolution 3D Surface Construction Algorithm"*, SIGGRAPH 1987
- Montani et al., *"A modified look-up table for implicit disambiguation of Marching Cubes"*, Visual Computer 1994
- Kobbelt et al., *"Feature Sensitive Surface Extraction from Volume Data"*, SIGGRAPH 2001

local property of Curve =

$\alpha(\theta) \to (\cos\theta, \sin\theta)$

$\alpha'(\theta) \to (-\sin\theta, \cos\theta)$ (切线)

$\|\alpha''(\theta)\| = 1$

对应的 $\alpha(\theta) \to (\cos 2\theta, \sin 2\theta)$

$\|\alpha''(\theta)\| = 2$

在这里的中心位置

$t \to \alpha(t)$

在曲线中那动的的点

表示是单位速率曲线

$\text{ARC length} = \int_a^b \|\alpha'(t)\| dt$

$\|\alpha'(s)\| = 1$

$\alpha(s)$ [tangent]

$\alpha'(s)$

代表 tangent 变动的速率

$\alpha''(s) = \kappa(s) n(s)$

$\kappa(s)$: curvature

$n(s)$: normal

measure

beat

$\nabla s = \frac{1}{\kappa(s)}$

BINORMAL

$b(s) = t(s) \wedge n(s)$

cross

parallel

$b' = t' \wedge n + t \wedge n'$

Frenet frame

$t' = k \wedge n$

$b' = \tau n$

$n = b \wedge t$

$= b' \wedge t + b \wedge t'$

$= \tau (n \wedge t) + k (b \wedge n)$

$= -\tau \vec{b} - k \vec{t}$

$= \boxed{\tau(s)} n(s)$

↓

用参 measure twist

$x = \cos s$

$y = \sin s$ (check $|\alpha(s)| = 1$ ✓)

$\alpha(s) = x(s)\ y(s)\ z(s)$

① $t = \alpha' = x'\ y'\ z'$

$|\alpha'| = 1$

② $t' = k n$

$n = t' / k$ (要normalize)

得到 k & $\vec{n}$

③ $b = t \wedge n$

④ $\tau = \dfrac{b'}{n}$