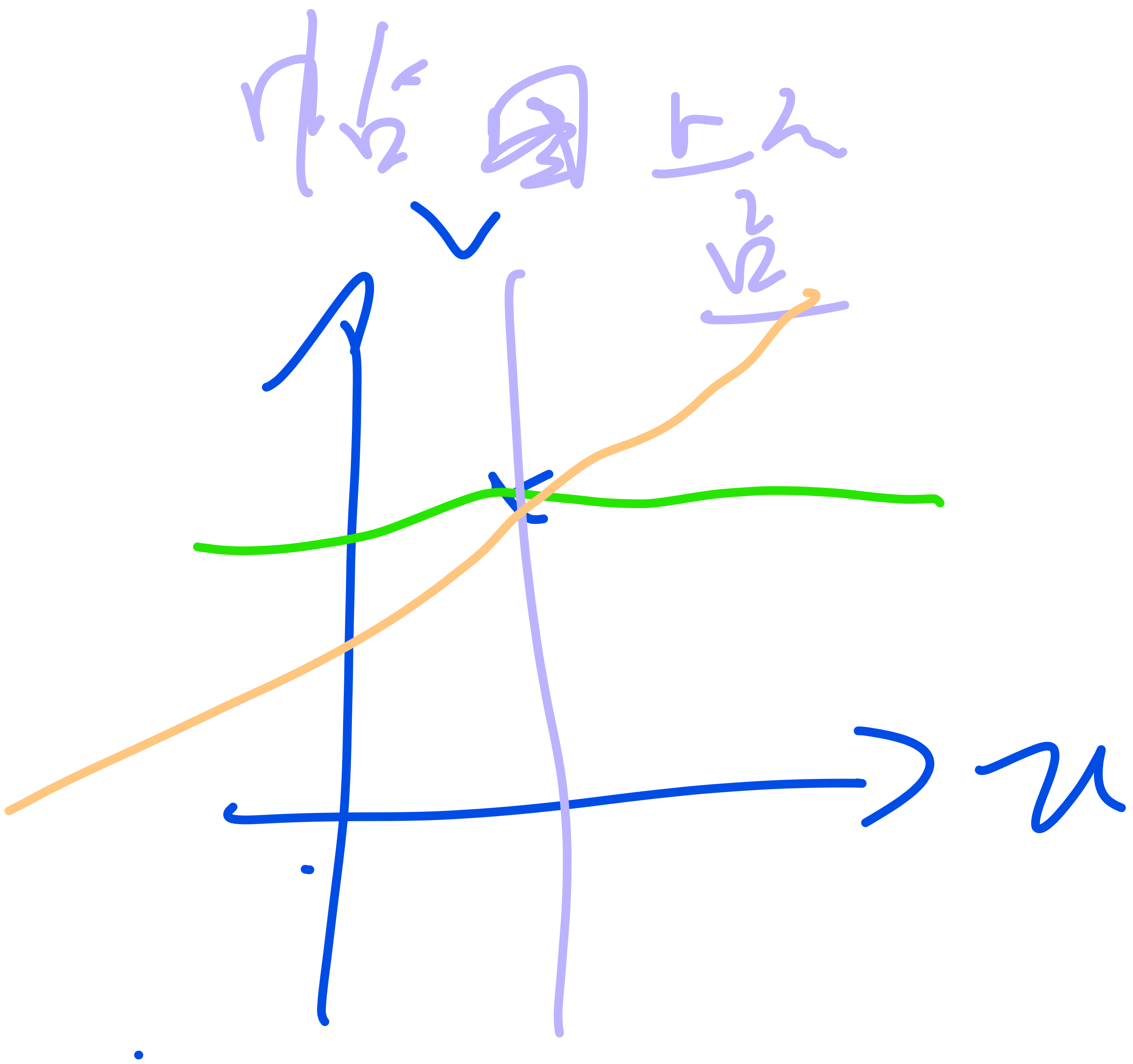
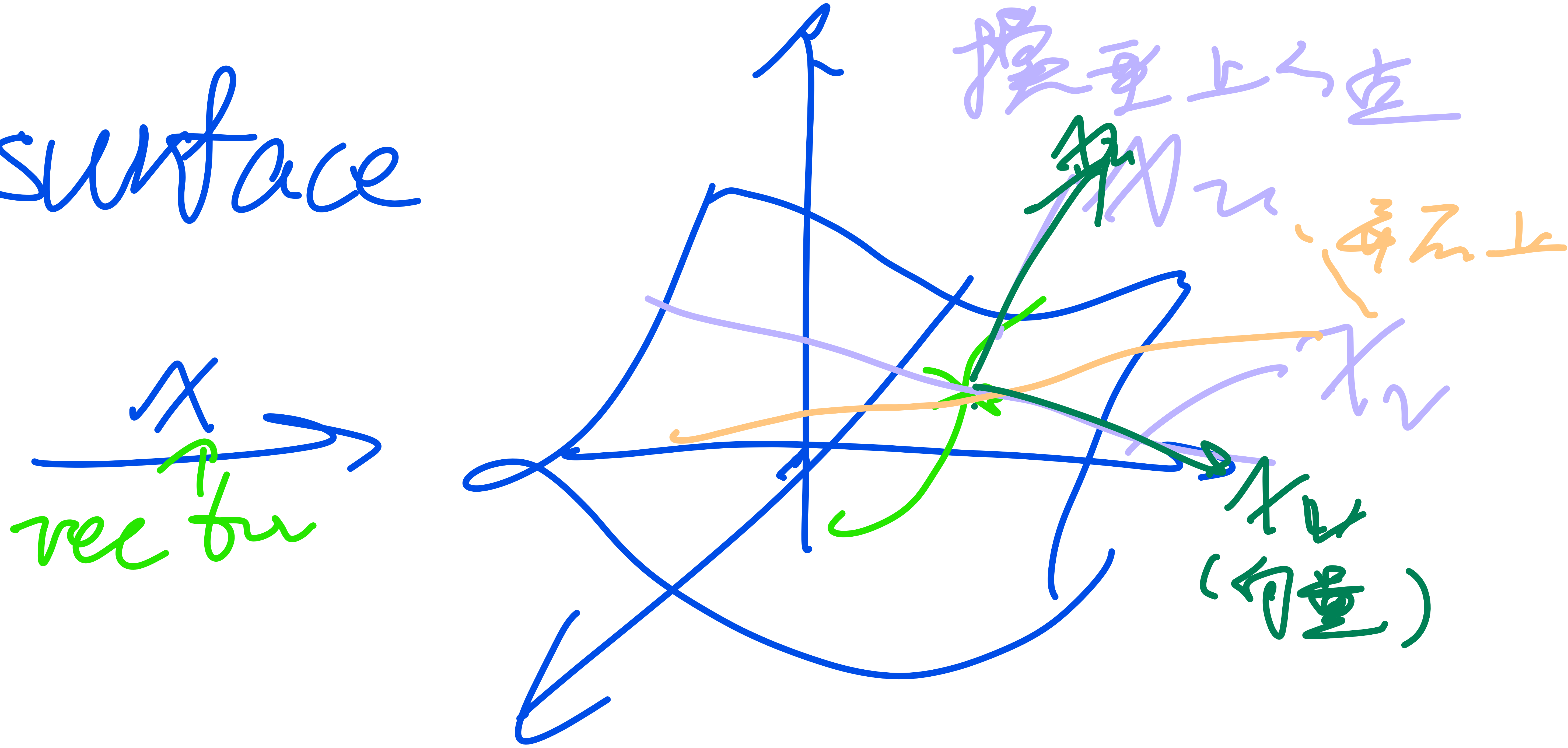


surface



$$u, v \longrightarrow x(u,v),\ y(u,v),\ z(u,v)$$

Q、橙色的线 (经过X点)，那么它在3d的 tangent 方向一定在 X_u, X_v define 的平面上 (那么 $N(u,v) = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}$)

$|a| = 1$ $\quad |a(t)| = 1$

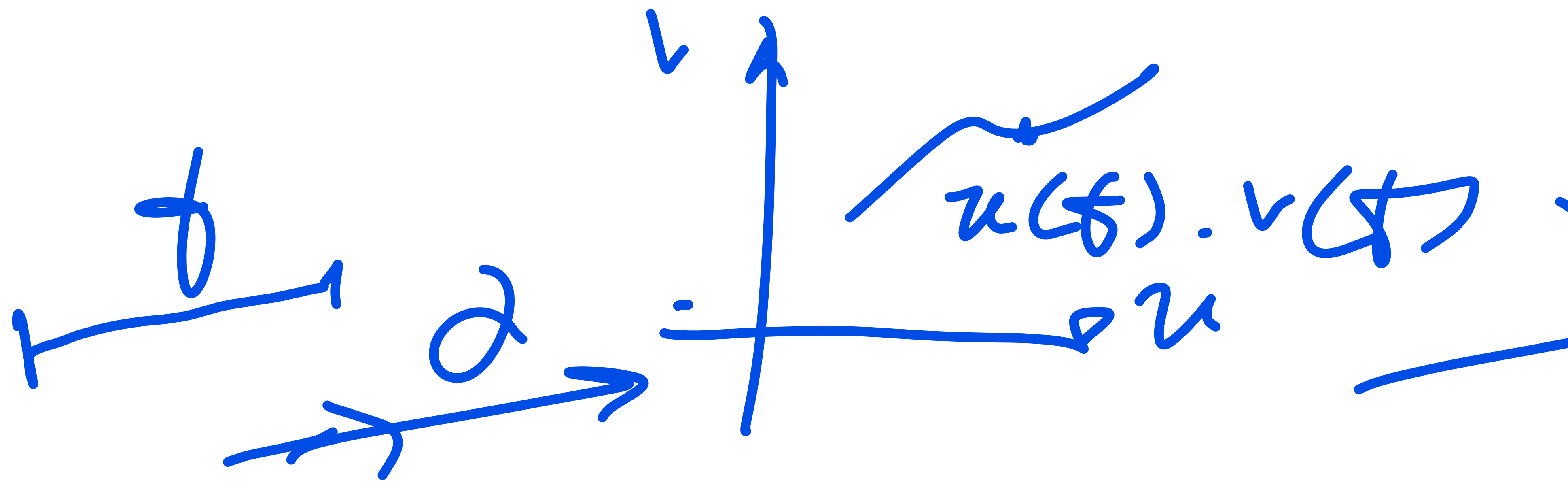
$\Rightarrow a'(t) \cdot a(t) = 0$?

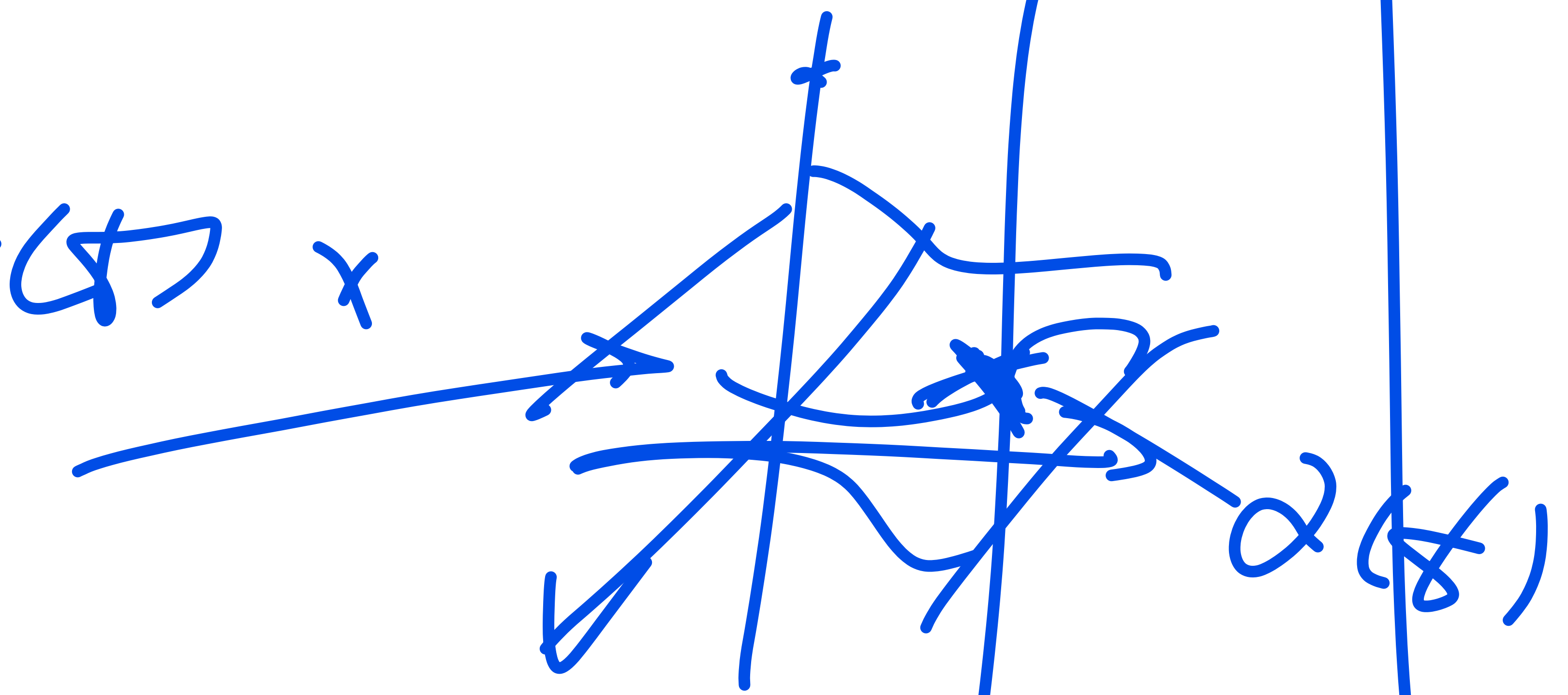
dot product $\langle a, b \rangle$

$\langle a, b\rangle^2 + \|a \wedge b\|^2$

$= (|a||b|\cos\theta)^2 + (|a||b|\sin\theta)^2 = (|a||b|)^2$

Bt





$\alpha(t) = X(u(t), v(t))$

$\alpha'(t) = X_u \cdot u' + v' X_v$ (chain rule)

$|\alpha'(t)|^2 =$

$\langle X_u u' + X_v v', X_u u' + X_v v' \rangle$

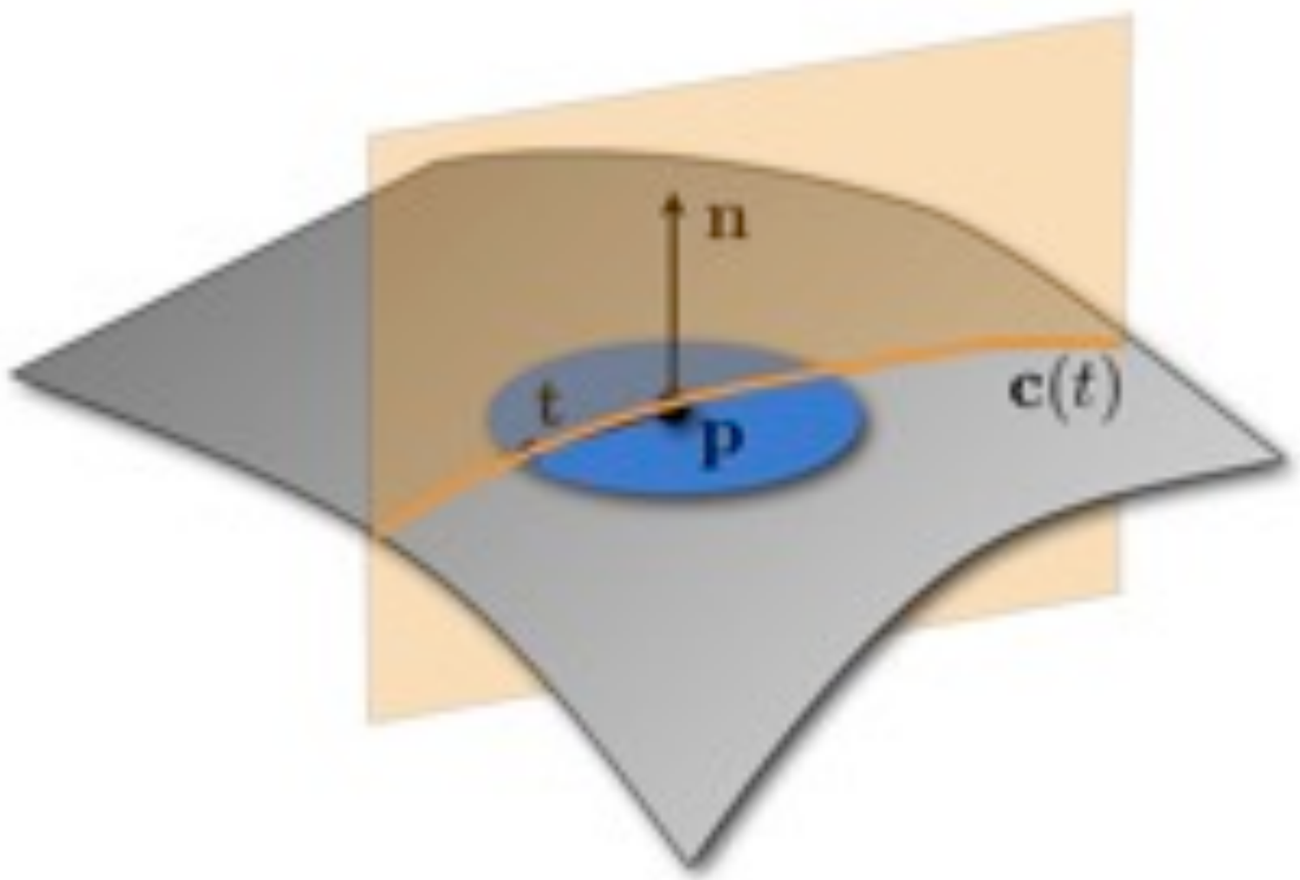
$= \underbrace{\langle X_u, X_u\rangle (u')^2}_{E} + \underbrace{2\langle X_u, X_v\rangle u'v'}_{F} + \underbrace{\langle X_v, X_v\rangle (v')^2}_{G}$

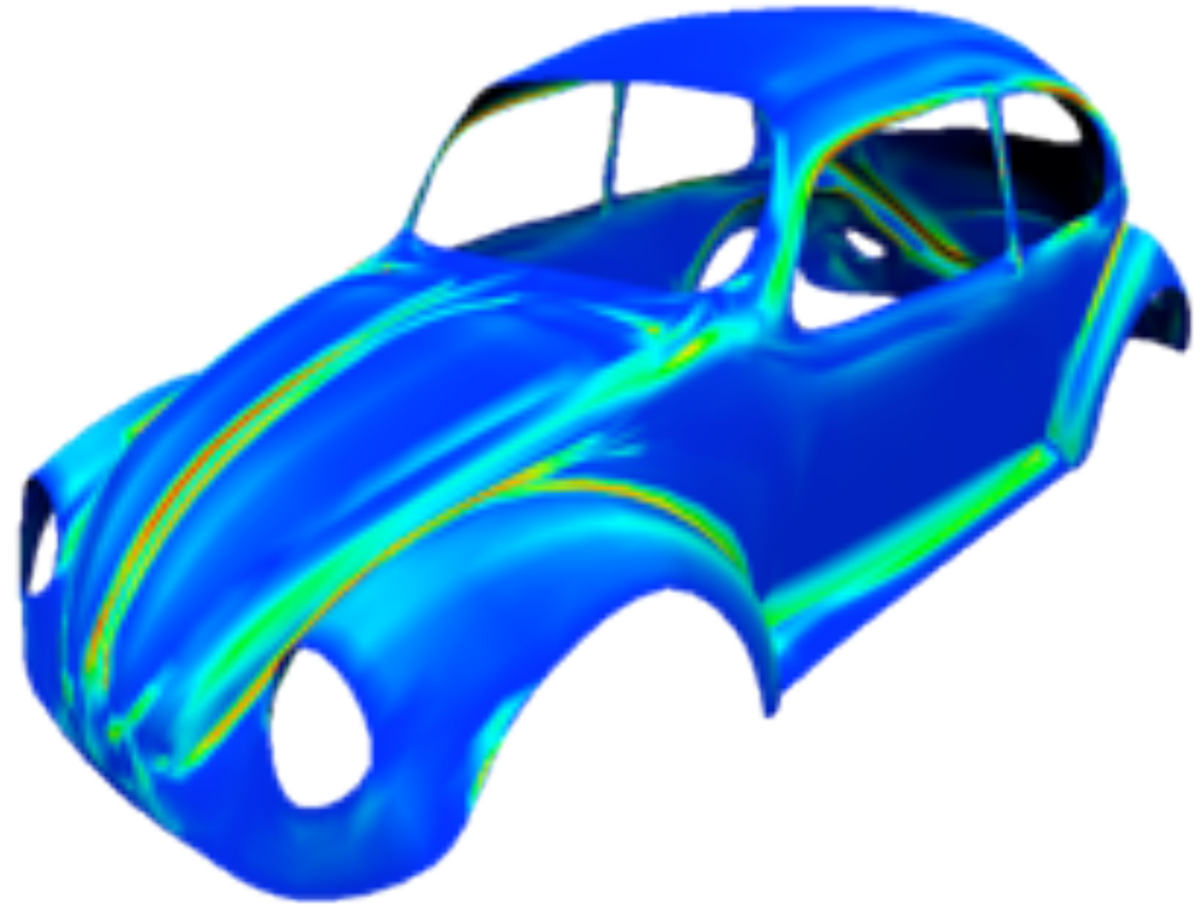
first fundamental form

$\begin{bmatrix} E & F \\ F & G \end{bmatrix}$

(如果 X_u, X_v 是单位正交 $\Rightarrow E = G = 1$, $F = 0$)

Differential Geometry





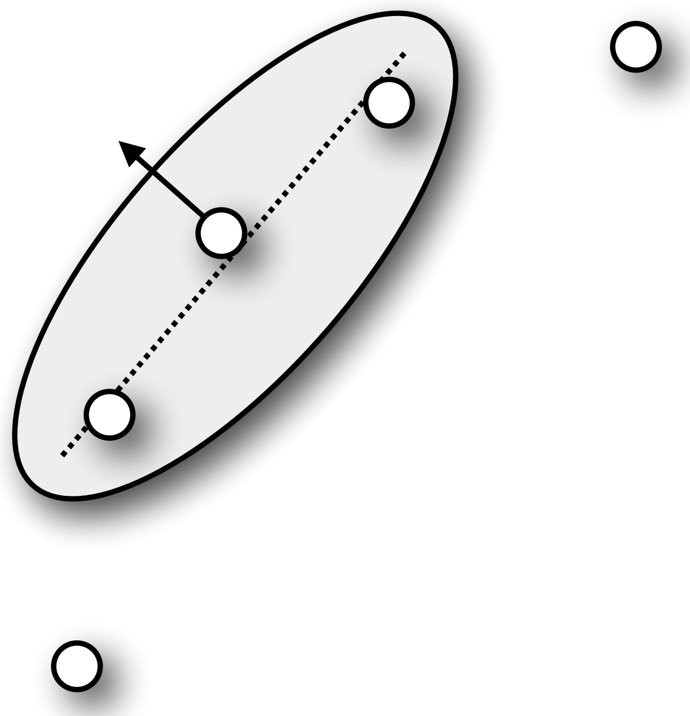
Last Time

- Explicit Reconstruction
  - Fast Marching Method
- Implicit Reconstruction
  - SDF from point clouds
  - (SDF from range scans)
  - Fast Marching Cubes

Normal Estimation

– Find normal $\mathbf{n}_i$ for each sample point $\mathbf{p}_i$

1. Examine local neighborhood for each point
   – Set of k nearest neighbors
2. Compute best approximating tangent plane
   – Covariance analysis
3. Determine normal orientation
   – MST propagation

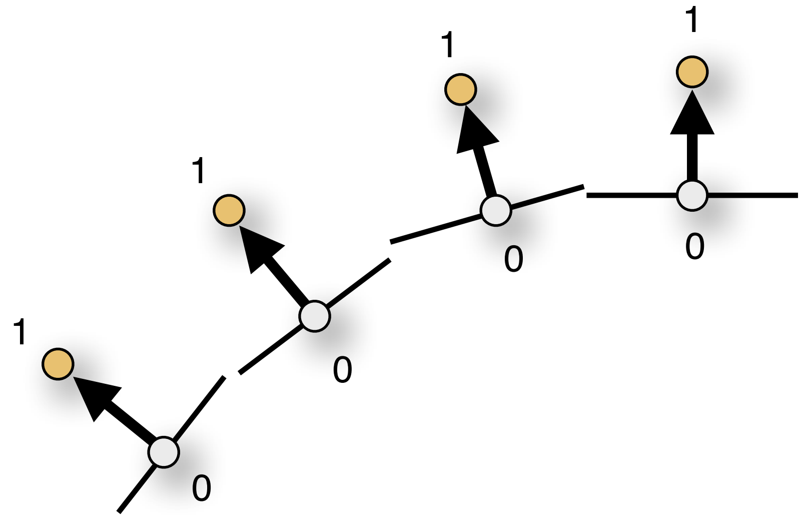


Implicit Reconstruction

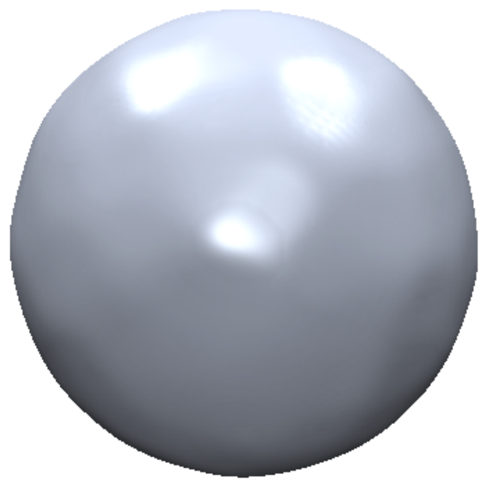
- Scattered data interpolation problem
  - On-surface constraints $\mathrm{dist}(\mathbf{p}_i) = 0$
  - Off-surface constraints $\mathrm{dist}(\mathbf{p}_i + \mathbf{n}_i) = 1$
- Radial basis functions (RBFs)

$$\mathrm{dist}(\mathbf{x}) = \sum_i w_i \cdot \varphi(\|\mathbf{x} - \mathbf{c}_i\|)$$

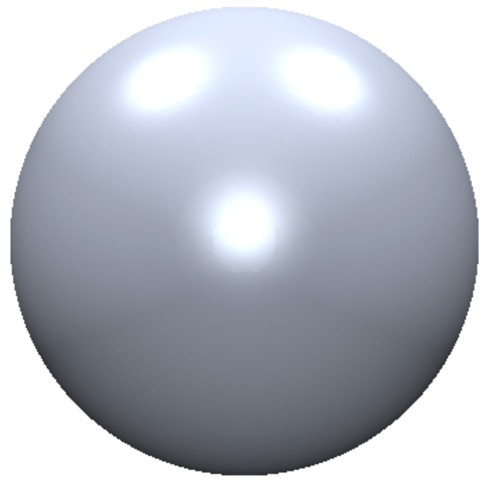
  - Solve symmetric linear system for weights w_i



Hoppe ‘92



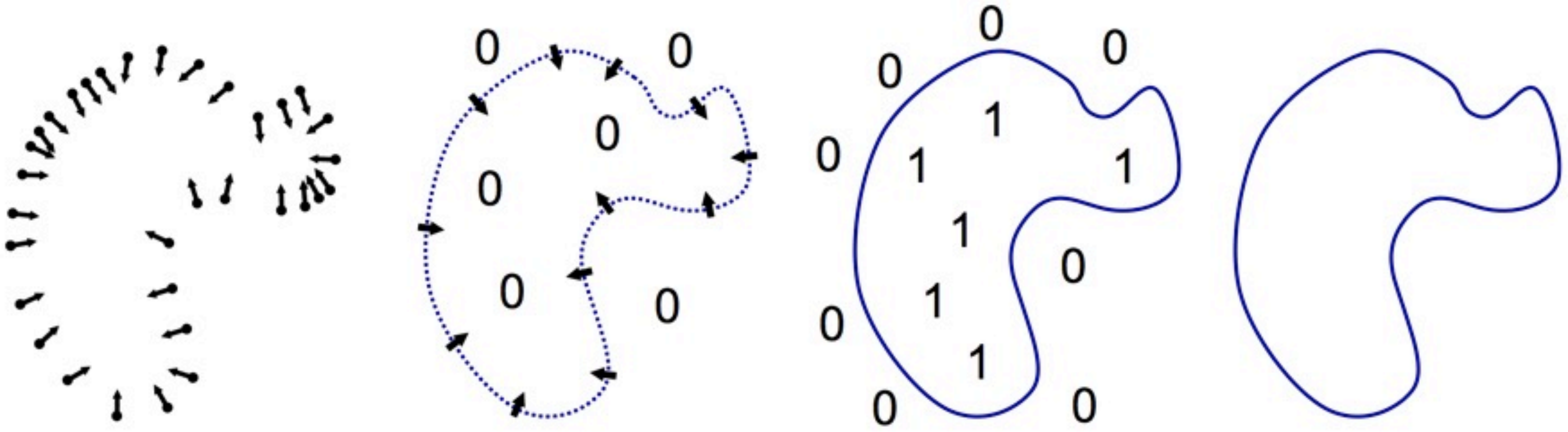
Compact RBF Wendland C^2



Global RBF Triharmonic

Poisson Surface Reconstruction

– relationship between normal field and gradient of indicator function



Oriented points | Indicator gradient | Indicator function | Surface

$\overrightarrow{V}$ $\nabla \chi_M$ χ_M ∂M

Outline

- Differential Geometry
- Discrete Differential Geometry
- Mesh Quality Measures

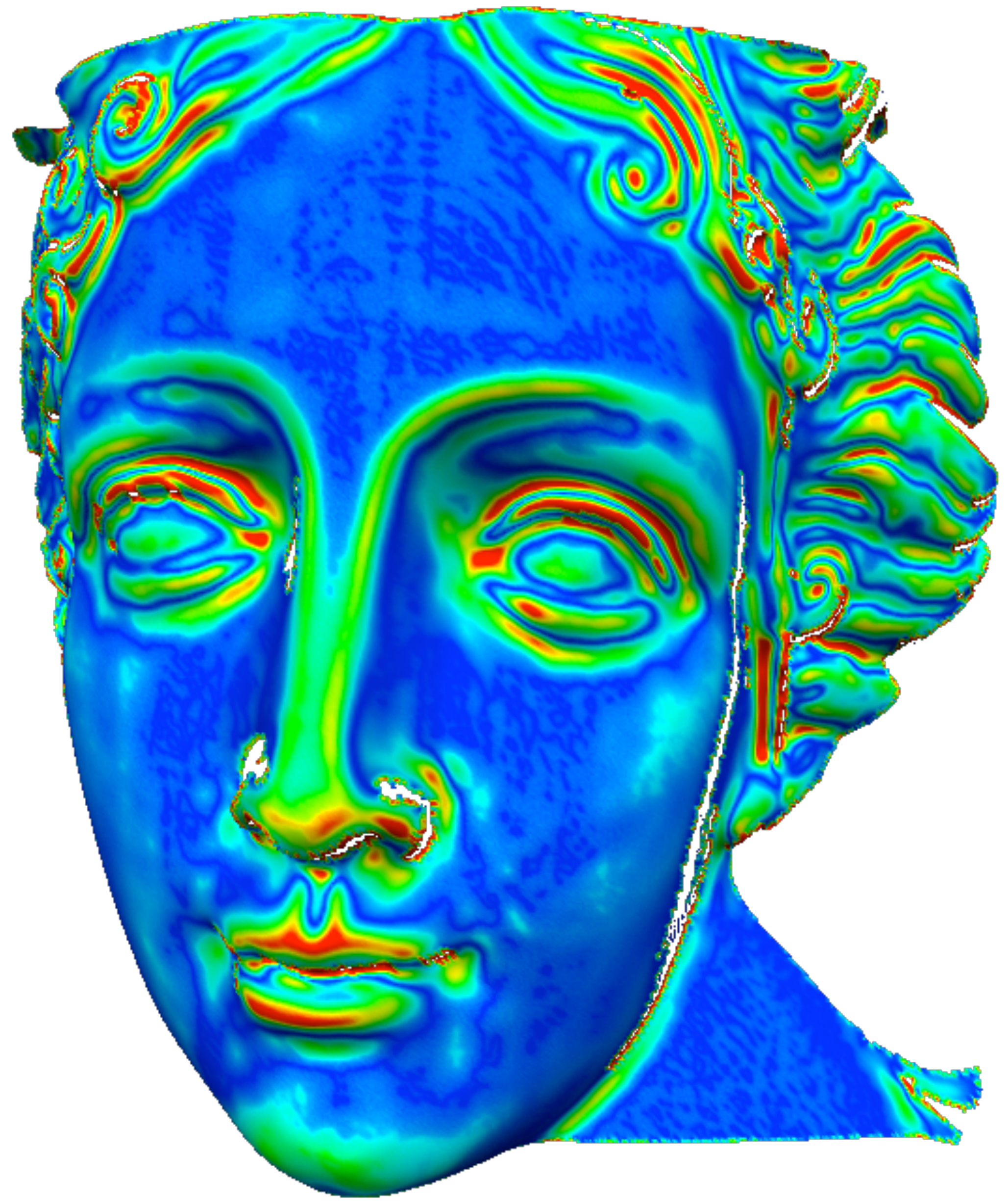
Motivation

Motivation

- We need differential geometry to compute

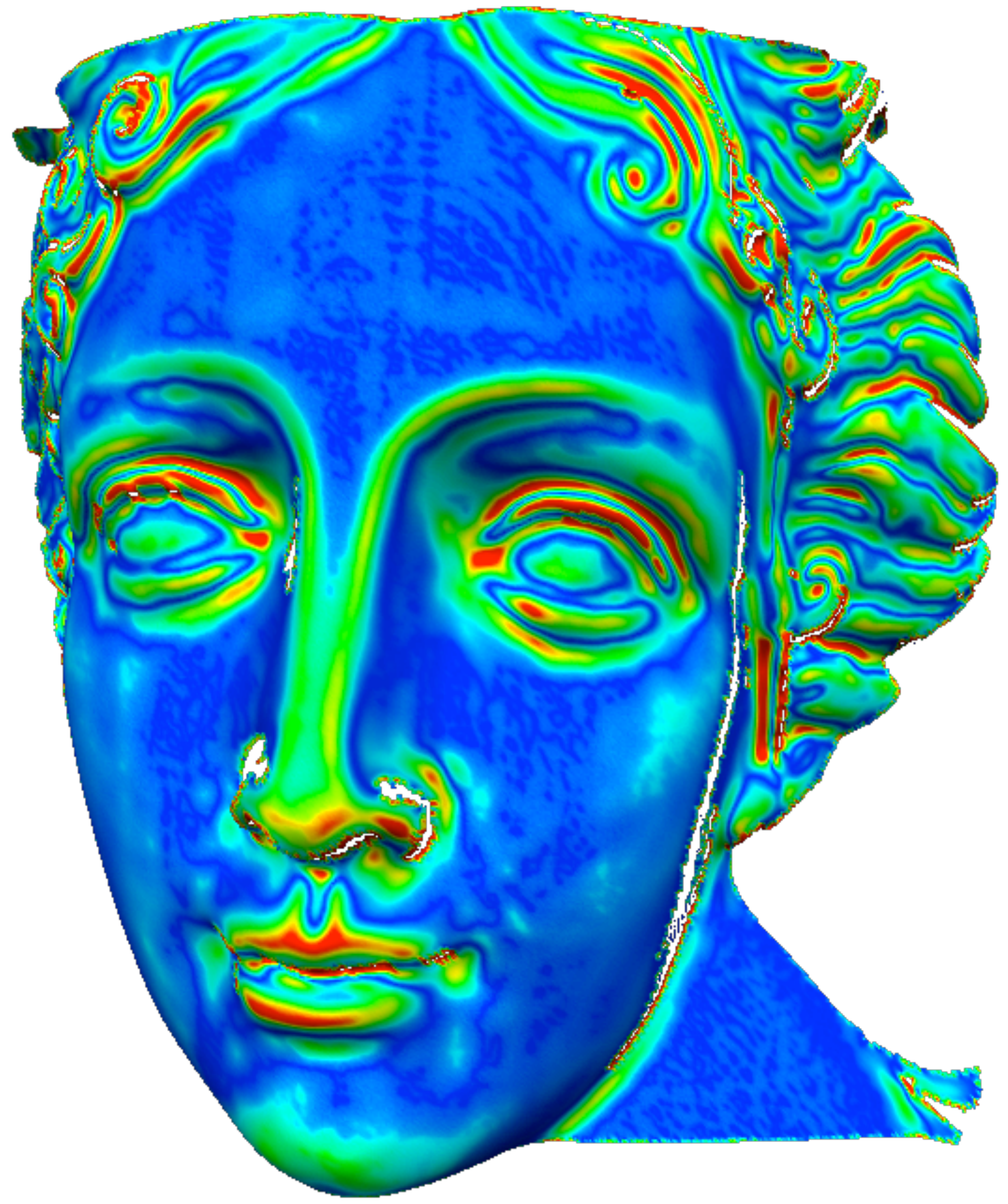
Motivation

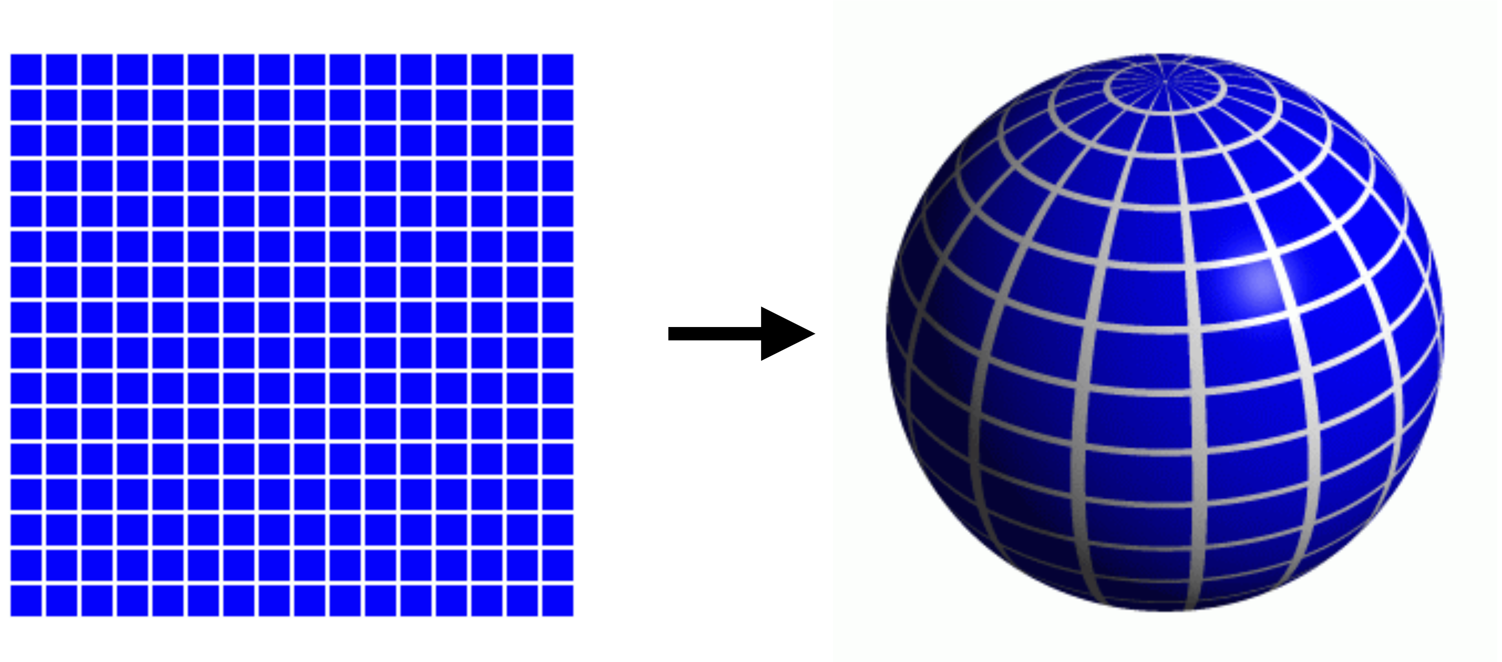
- We need differential geometry to compute
  - surface curvature



Motivation

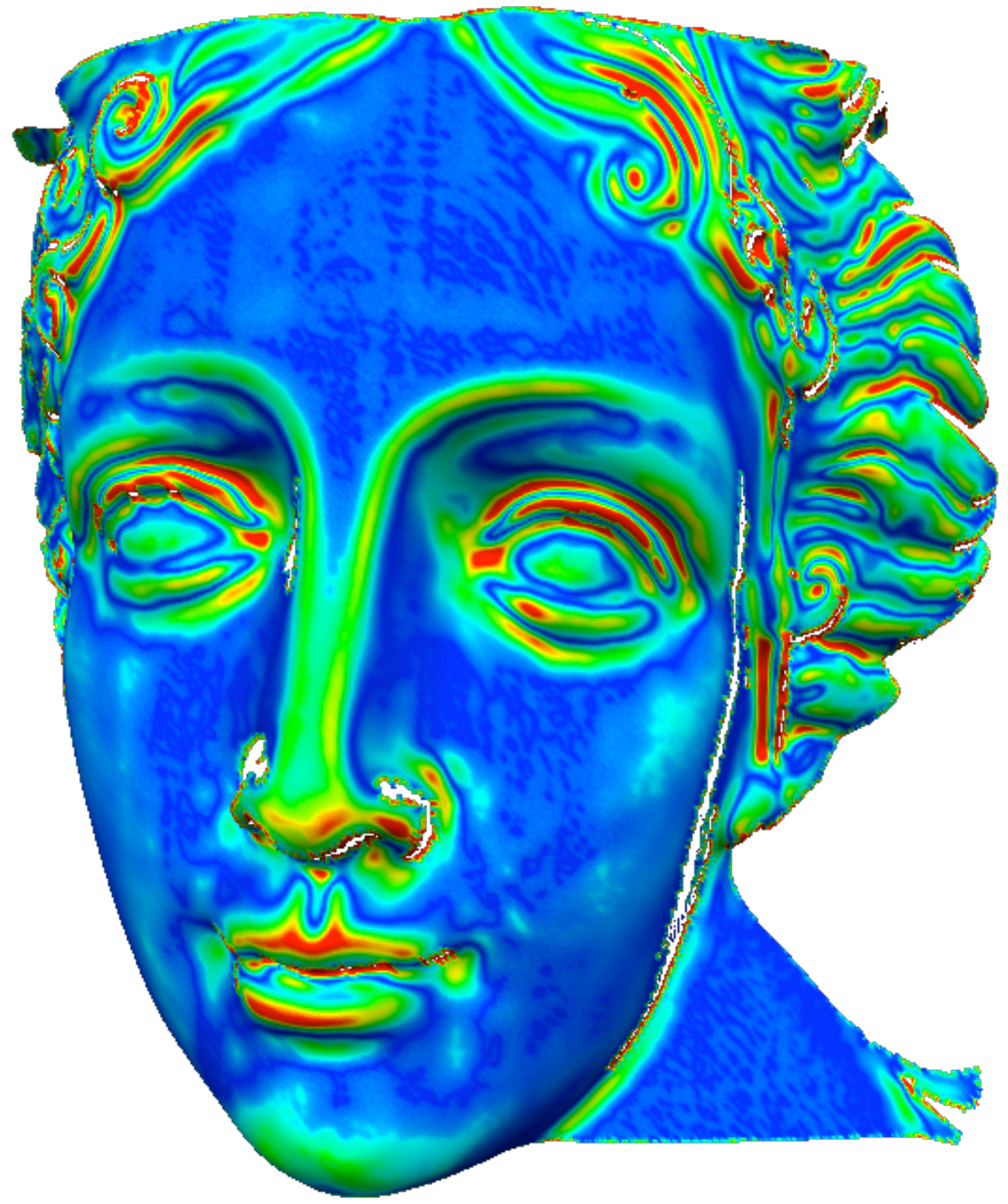
- We need differential geometry to compute
  - surface curvature
  - parameterization distortion

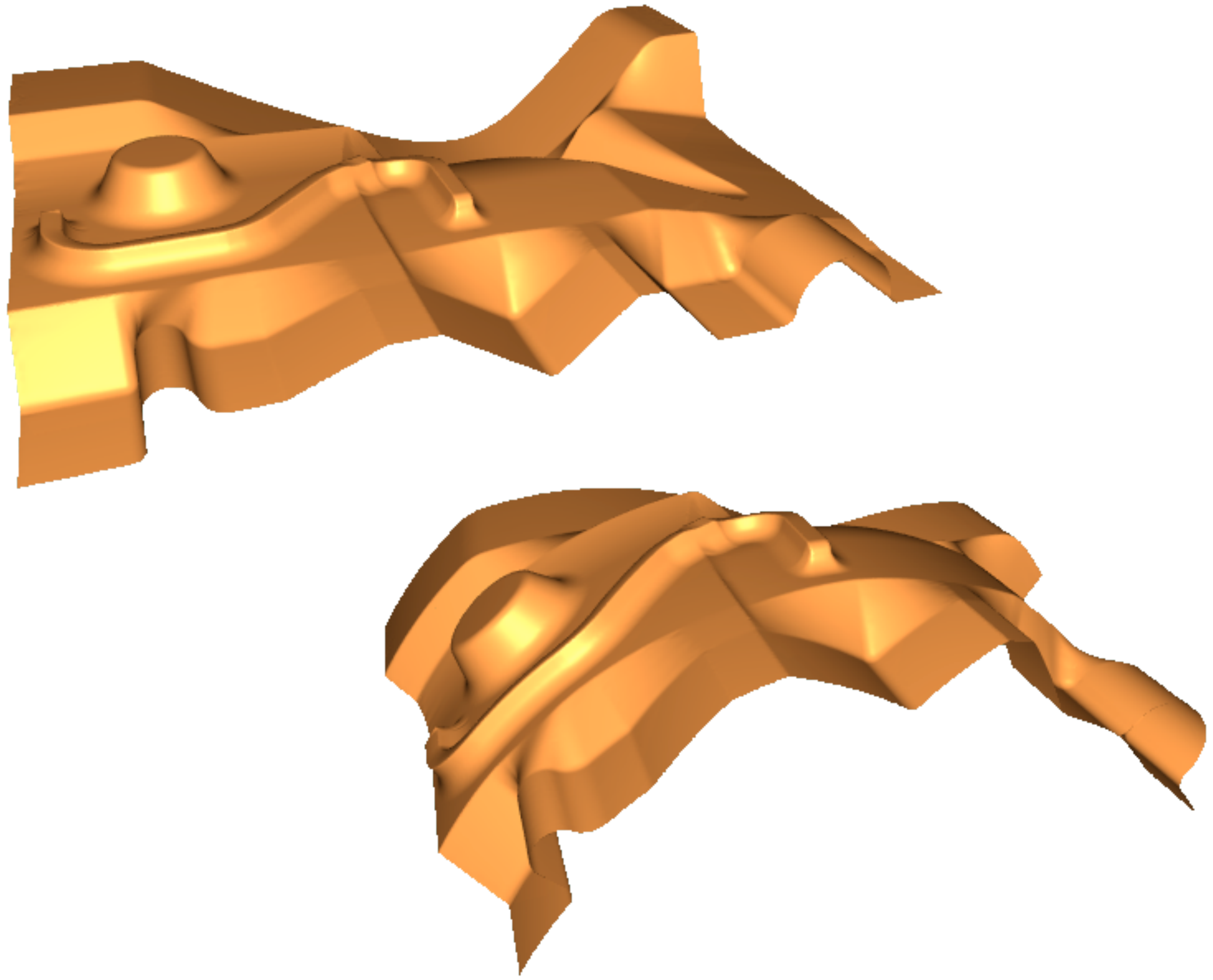


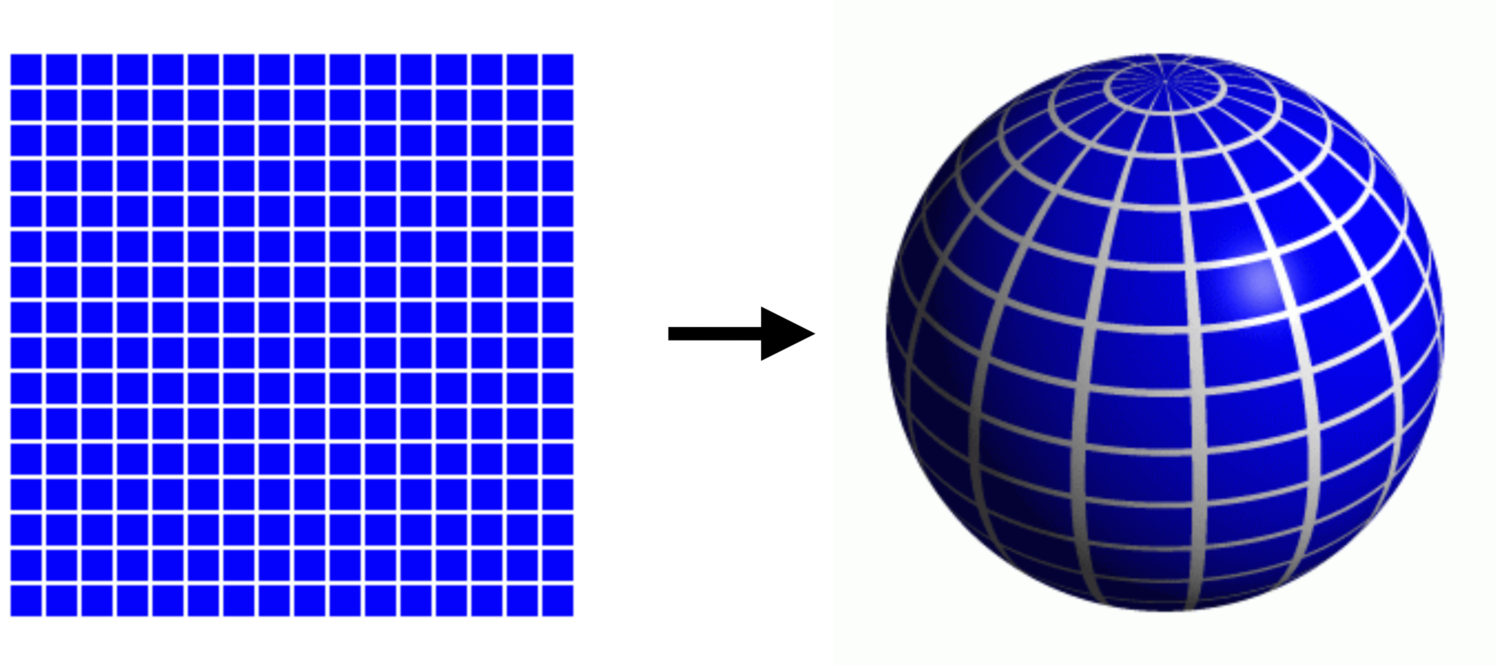


Motivation

- We need differential geometry to compute
  - surface curvature
  - parameterization distortion
  - deformation energies

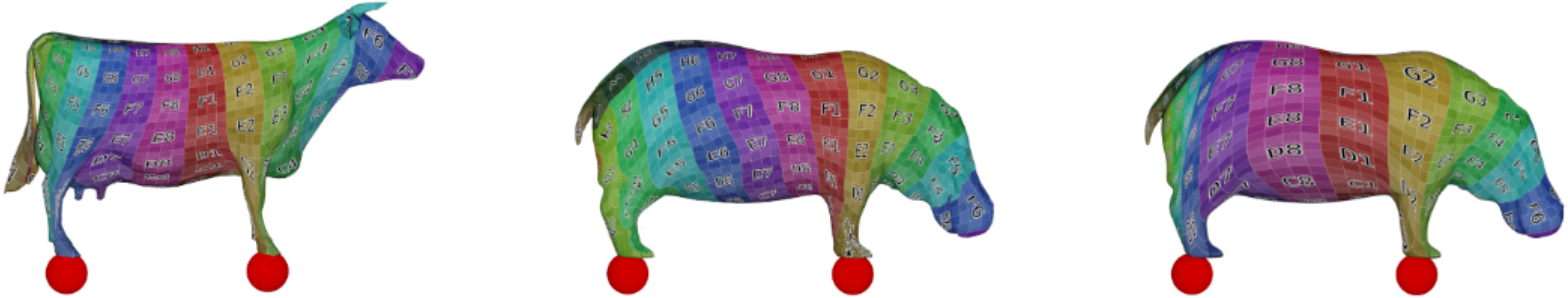




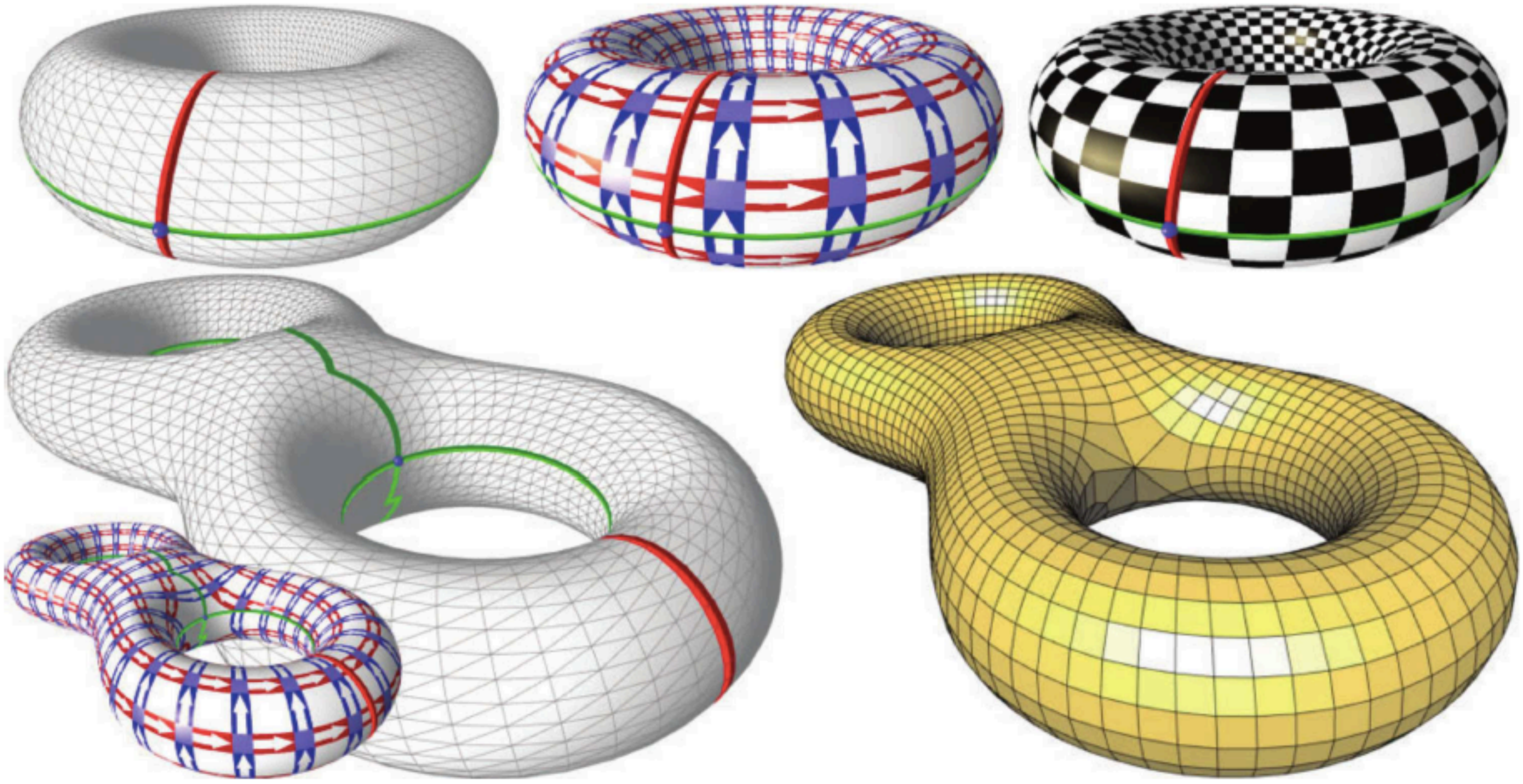


Texture Mapping

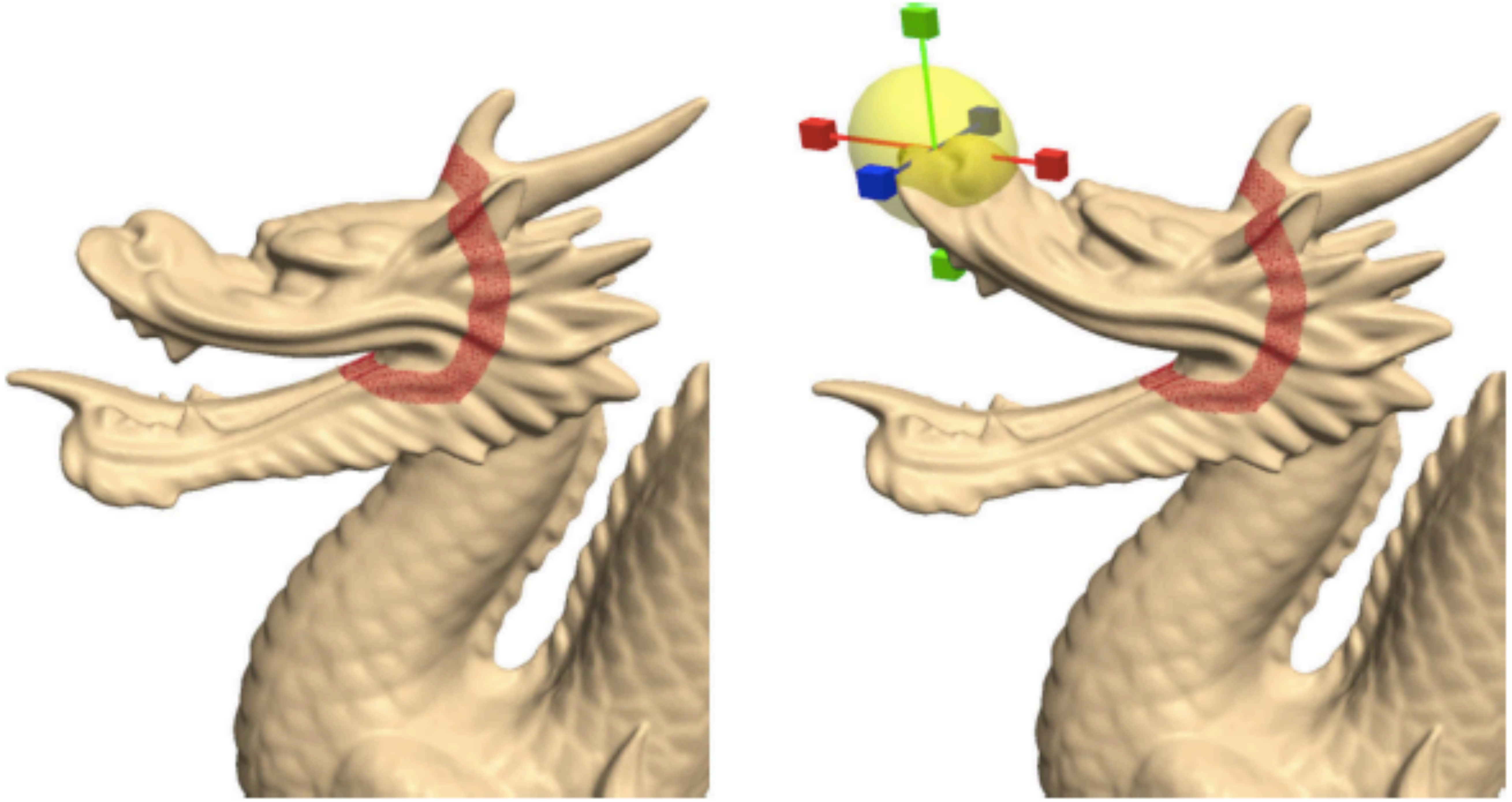
Texture Mapping

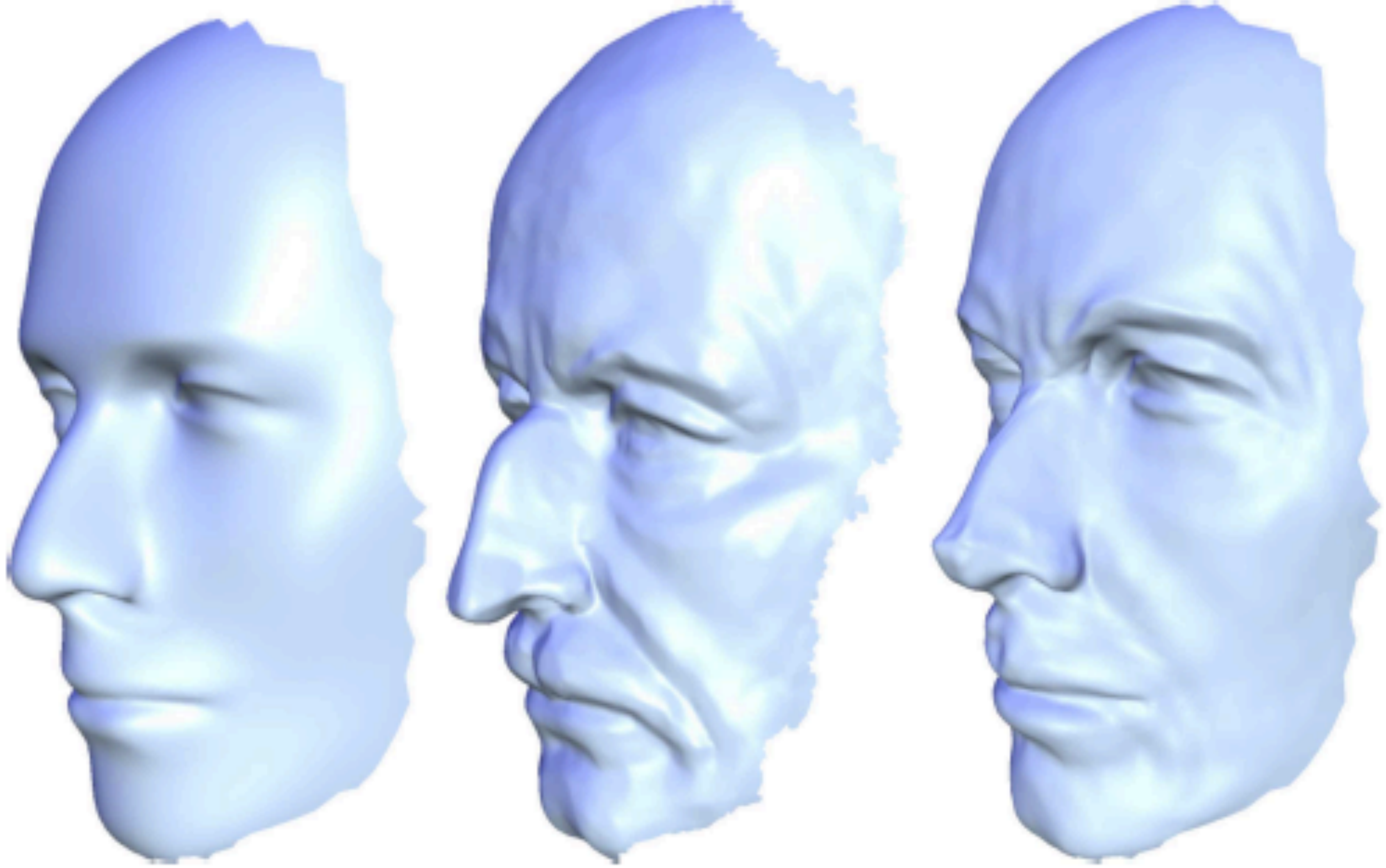


Parameterization + Meshing



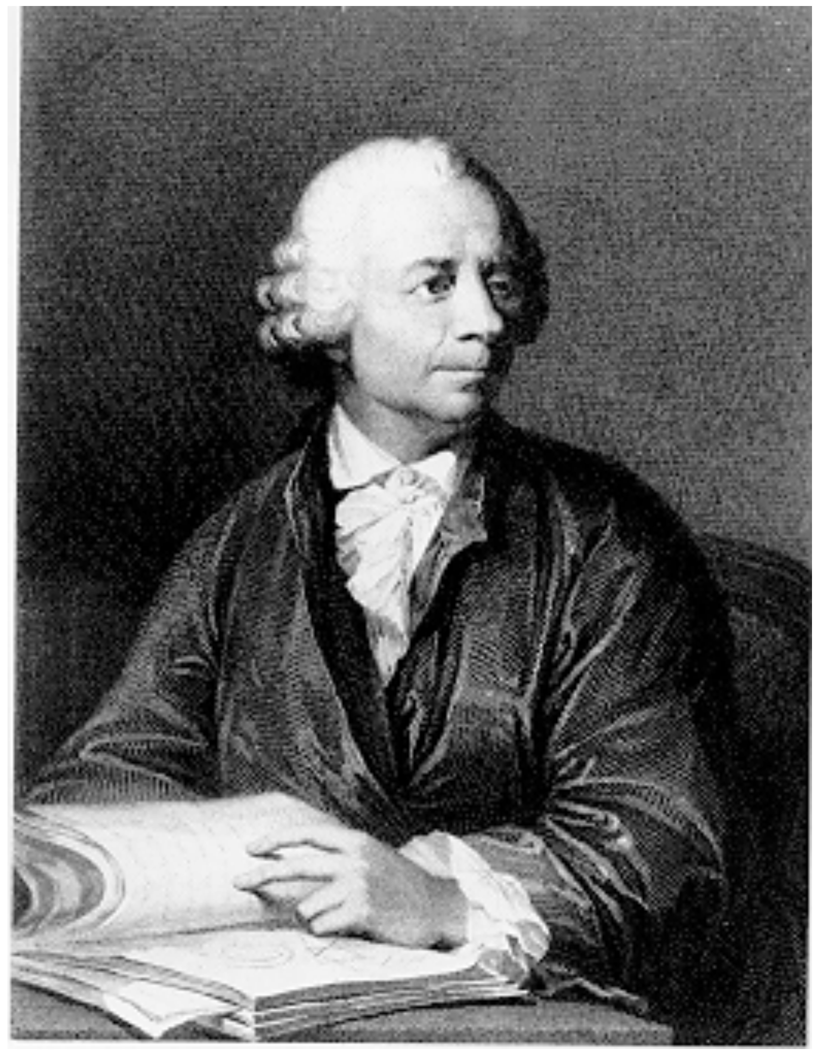
Surface Deformation





Differential Geometry

- Manfredo P. do Carmo: ***Differential Geometry of Curves and Surfaces***, Prentice Hall, 1976
- Andrew Pressley: ***Elementary Differential Geometry***, Springer, 2010



Leonard Euler (1707 - 1783)

Carl Friedrich Gauss (1777 - 1855)

Overview — Curves

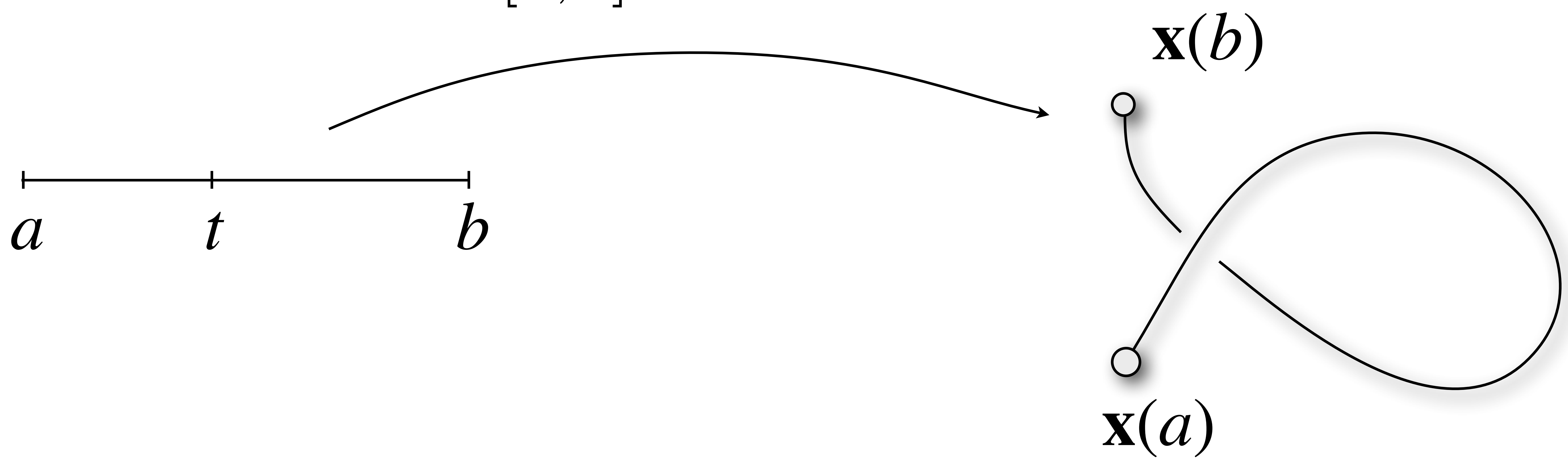
- **Parameterized** curves + **trace** of a curve
- **Ambiguity** due to parameterization
- Instantaneous **velocity** and **acceleration**
- constant **speed** versus constant **velocity**
- measuring curve length
- constant speed: velocity orthogonal to acceleration
- no self-intersection; regular parameterization

Overview (cont.)

- radius of curvature
- Frenet frame; osculating plane
- tangent, normal, binormal
- ——- end curves

Parametric Curves

$$\mathbf{x} : [a, b] \subset \mathbb{R} \to \mathbb{R}^3$$



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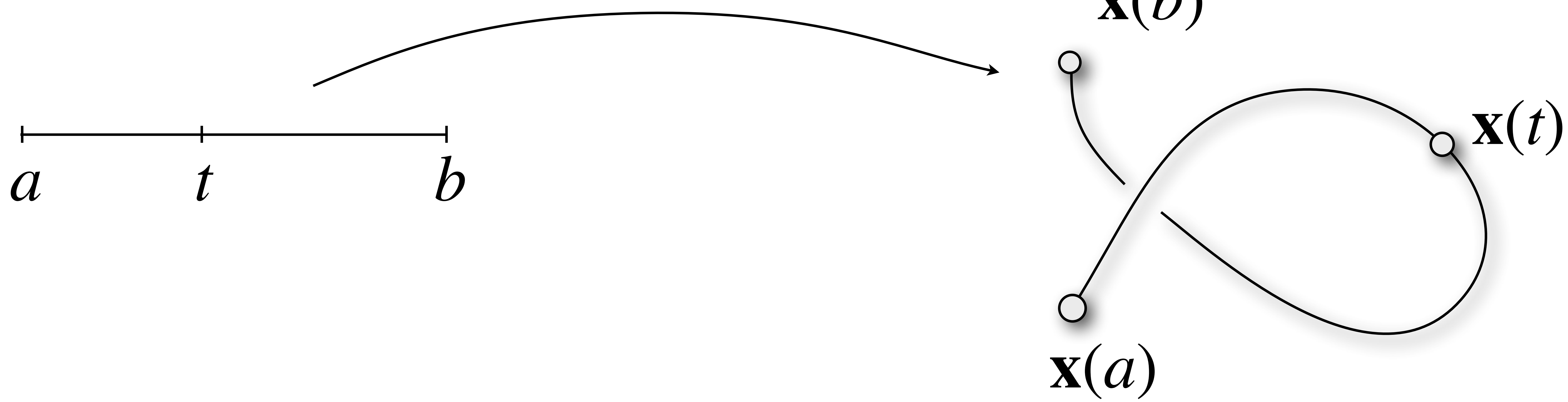

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

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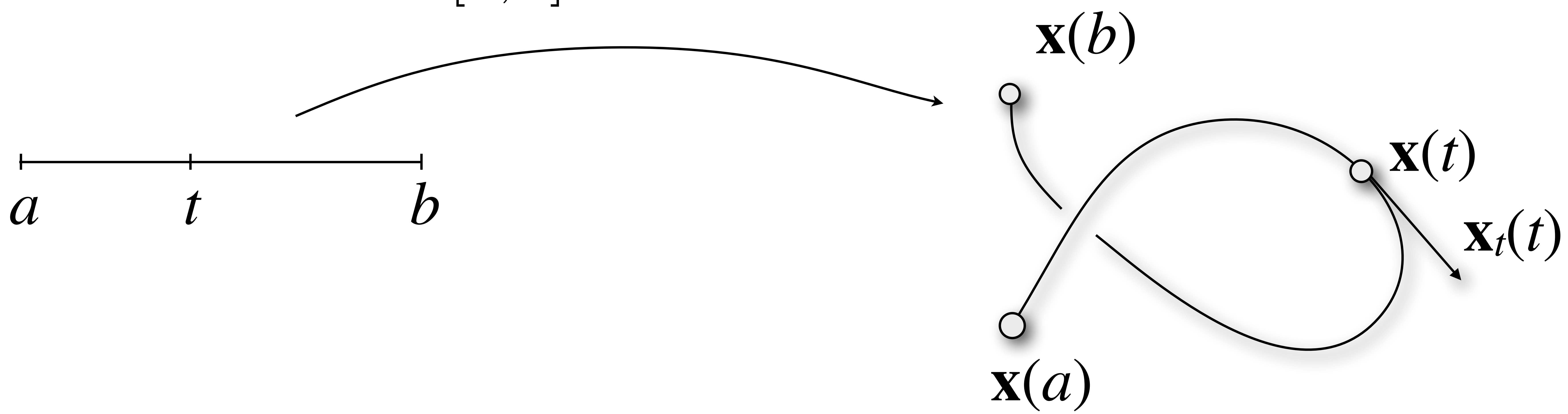
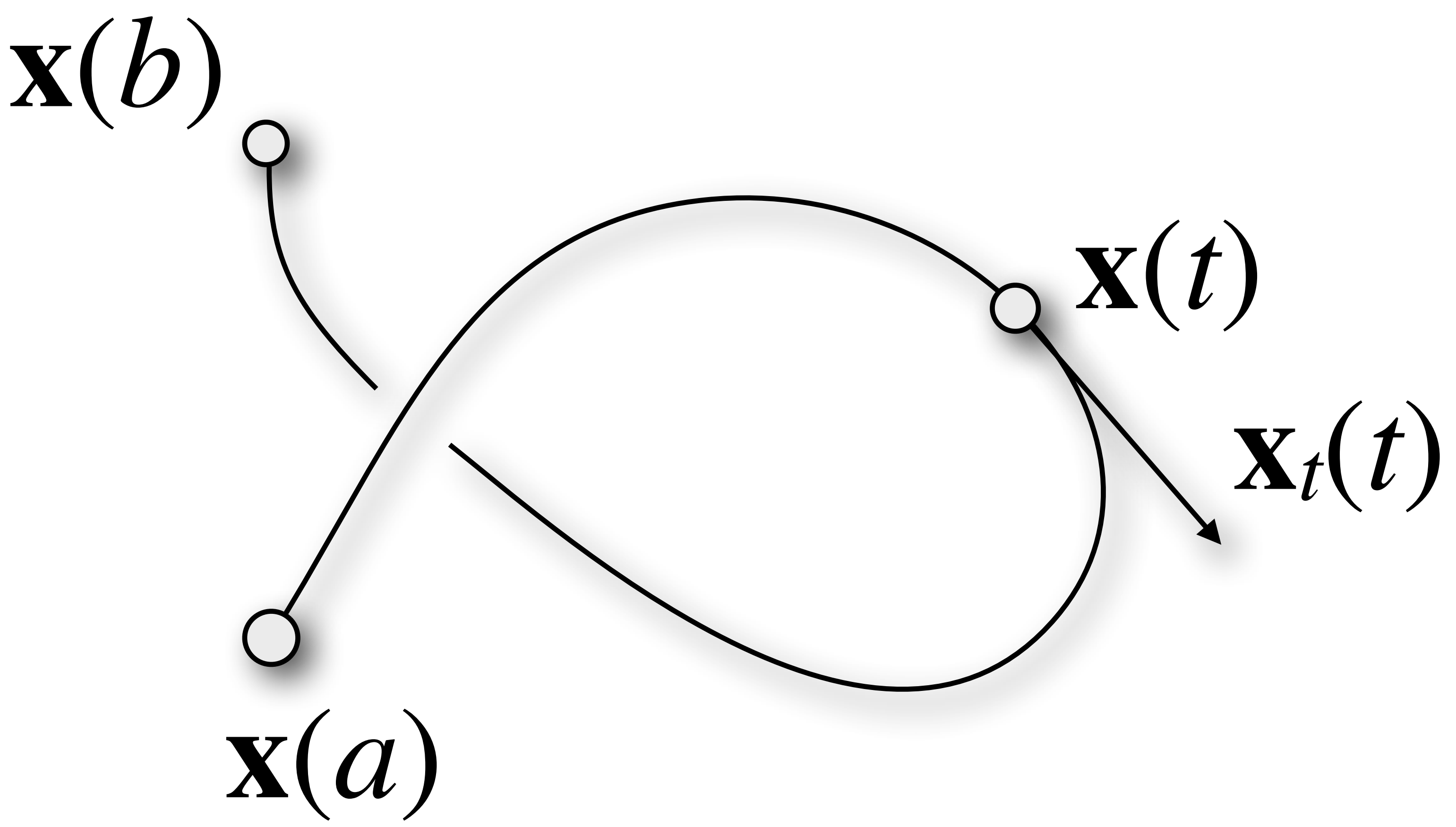


$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad \mathbf{x}_t(t) := \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \begin{pmatrix} \mathrm{d}x(t)/\mathrm{d}t \\ \mathrm{d}y(t)/\mathrm{d}t \\ \mathrm{d}z(t)/\mathrm{d}t \end{pmatrix}$$

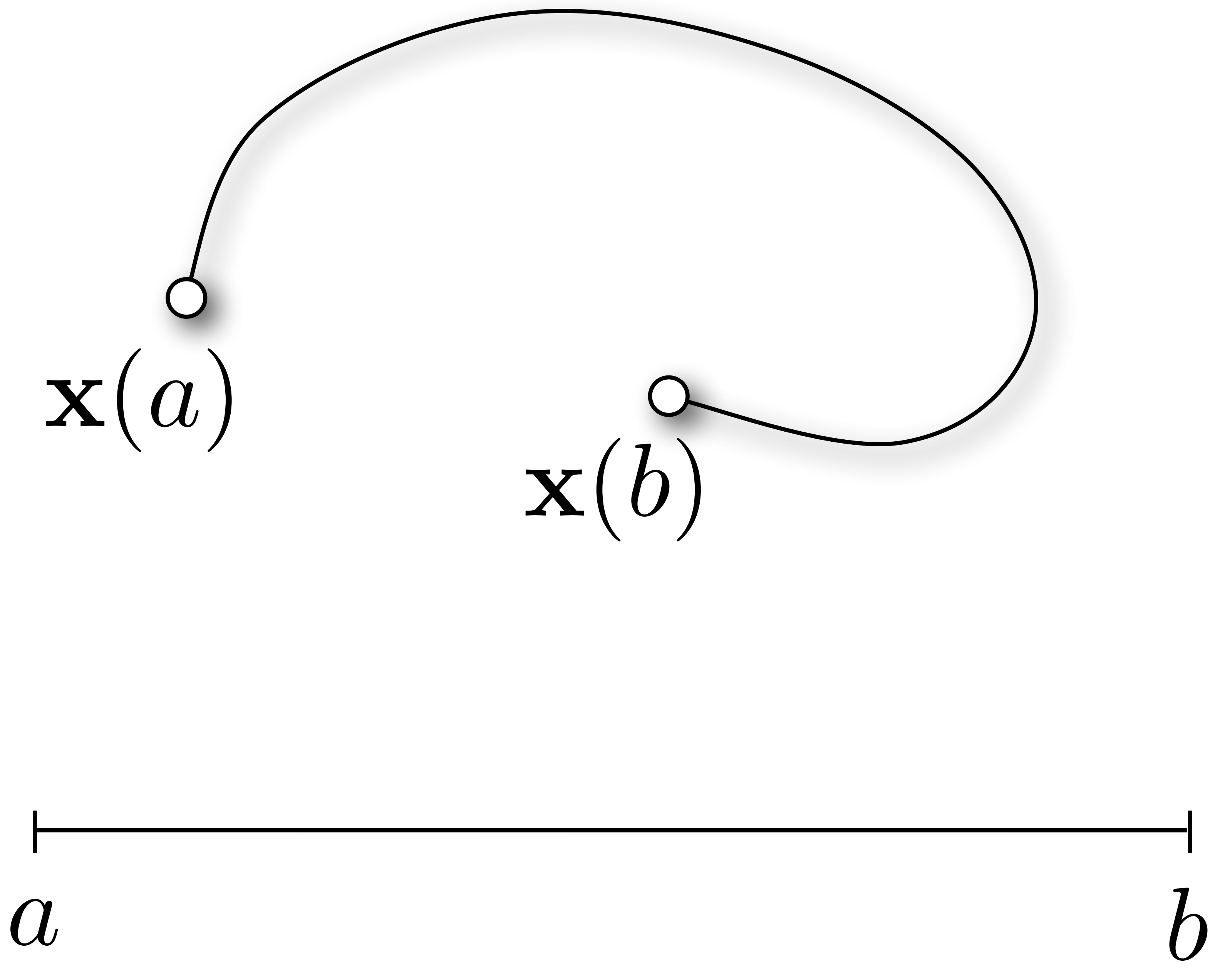
Parametric Curves

- A parametric curve $\mathbf{x}(t)$ is
  - *simple* — $\mathbf{x}(t)$ is injective (no self-intersections)
  - *differentiable* — $\mathbf{x}_t(t)$ is defined for all $t \in [a, b]$
  - *regular* — $\mathbf{x}_t(t) \neq 0$ for all $t \in [a, b]$



Length of a Curve

- Let $t_i = a + i\Delta t$ and $\mathbf{x}_i = \mathbf{x}(t_i)$



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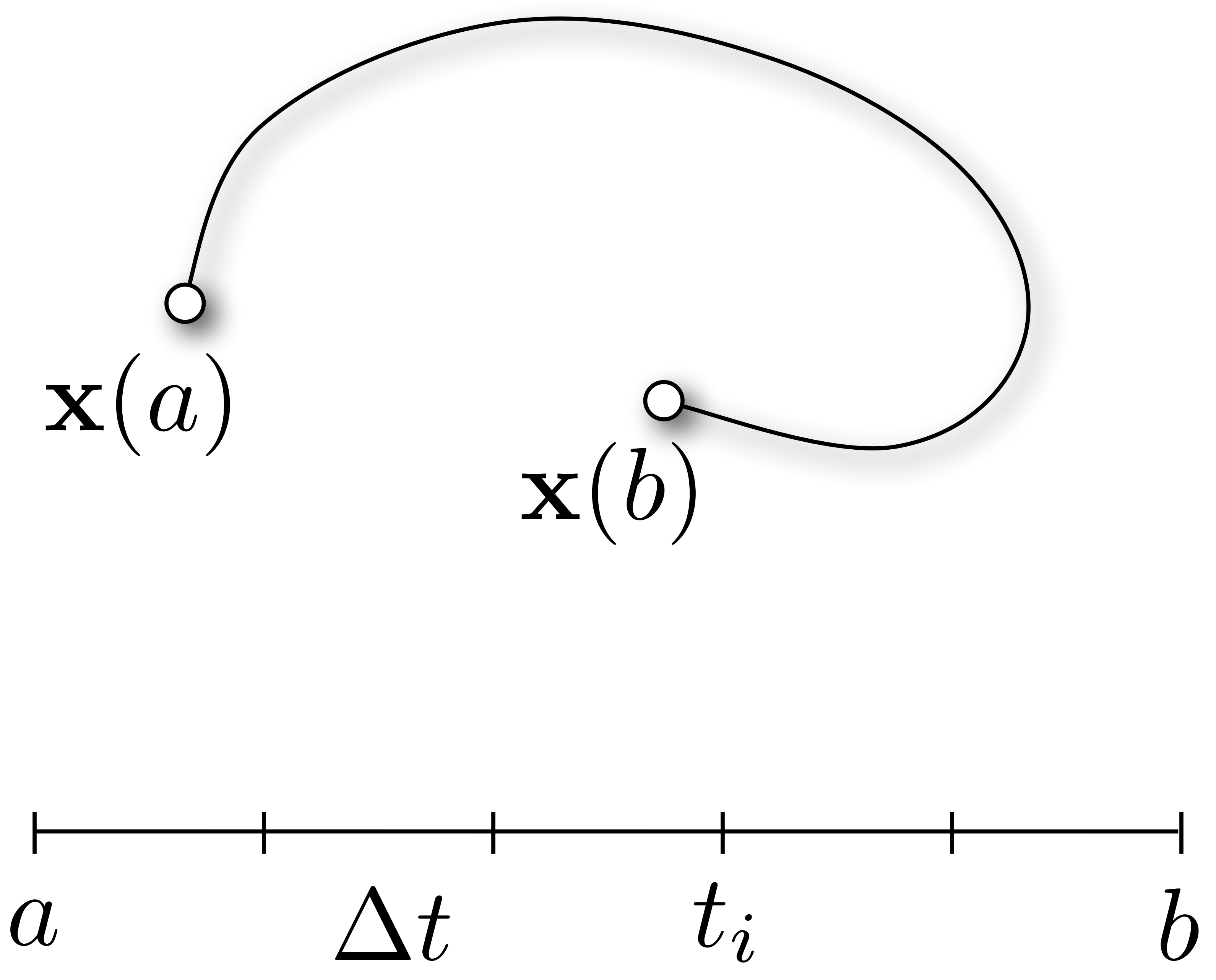

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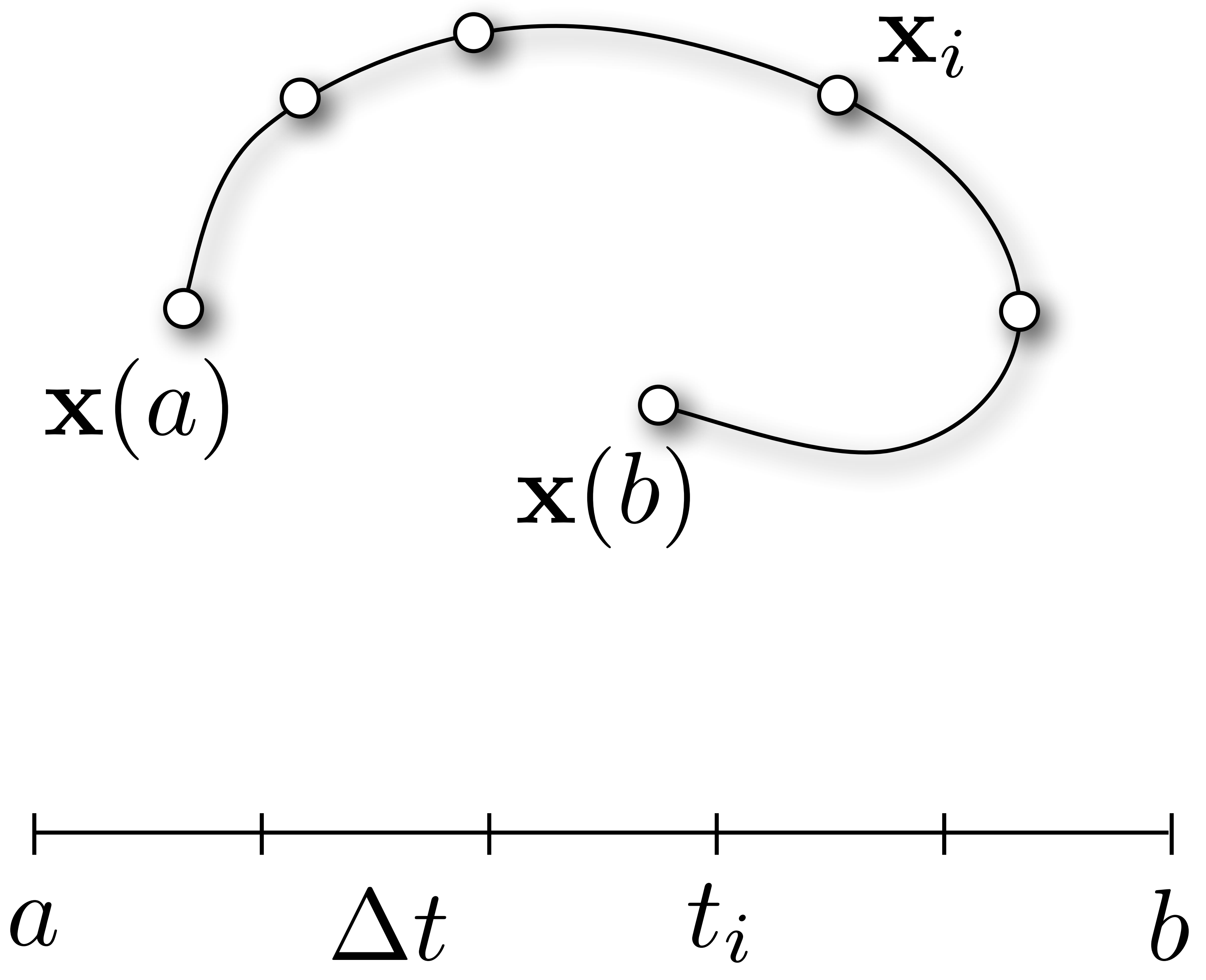

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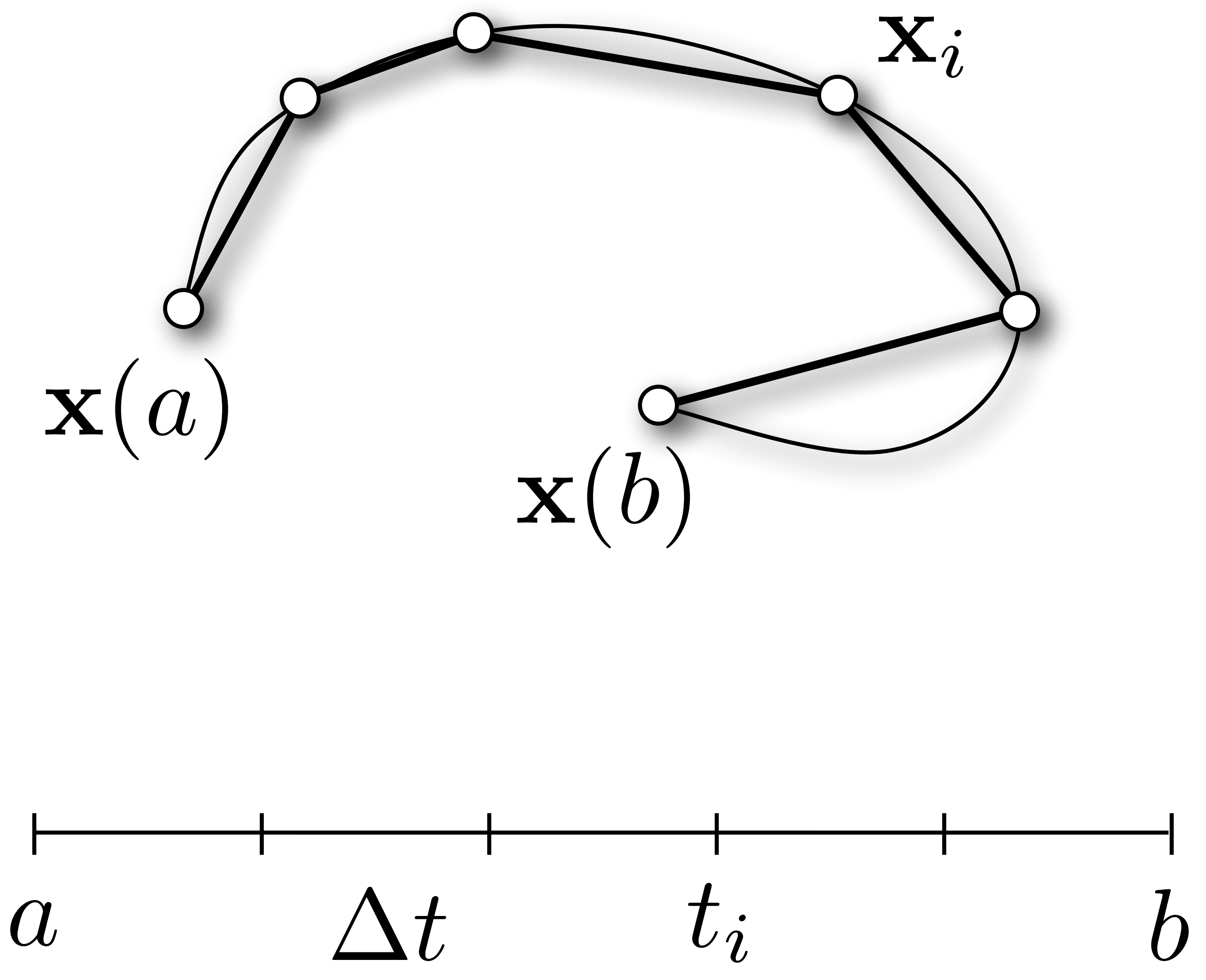
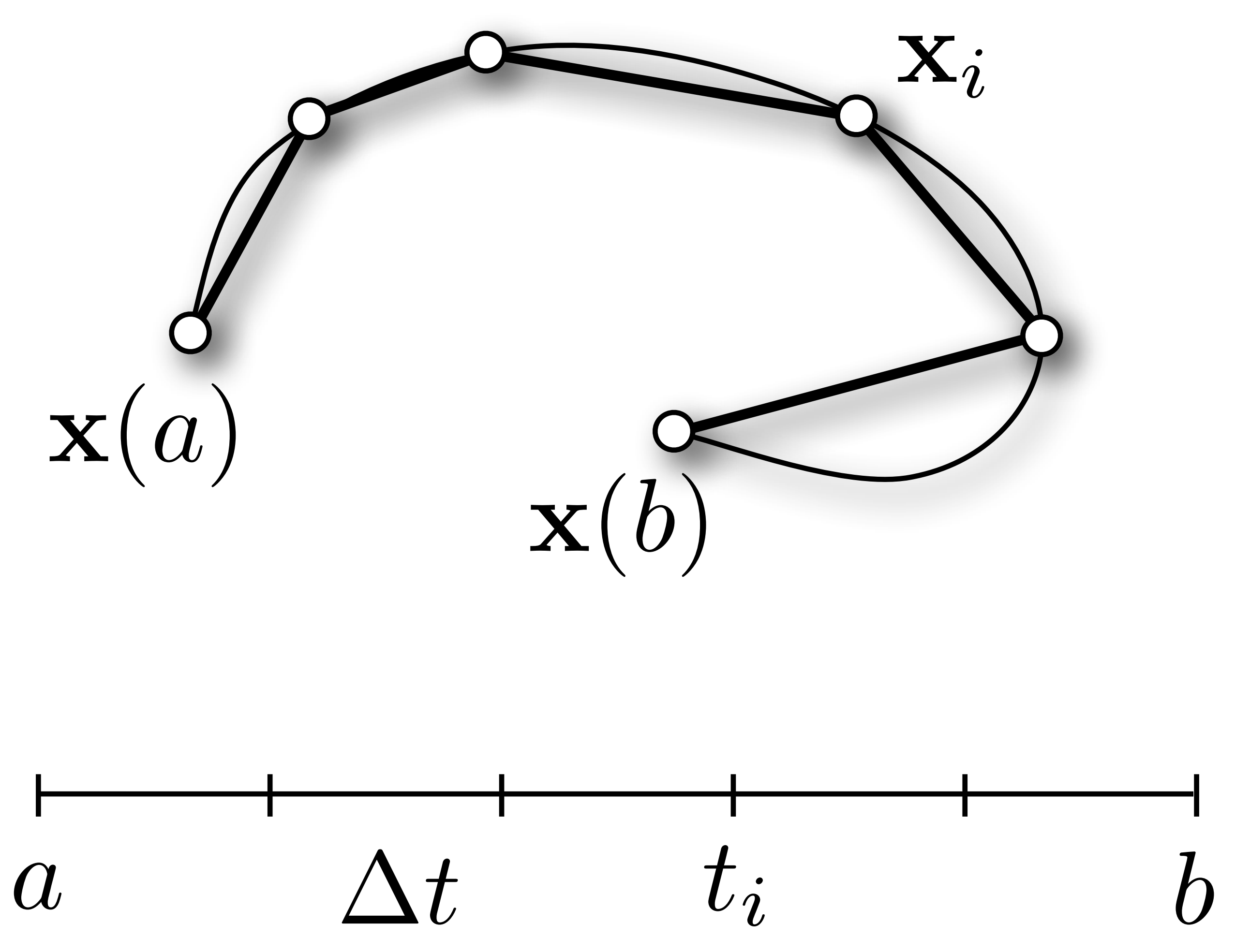


Length of a Curve

- Polyline *chord length*

$$S = \sum_i \|\Delta \mathbf{x}_i\| = \sum_i \left\| \frac{\Delta \mathbf{x}_i}{\Delta t} \right\| \Delta t \,, \quad \Delta \mathbf{x}_i := \|\mathbf{x}_{i+1} - \mathbf{x}_i\|$$



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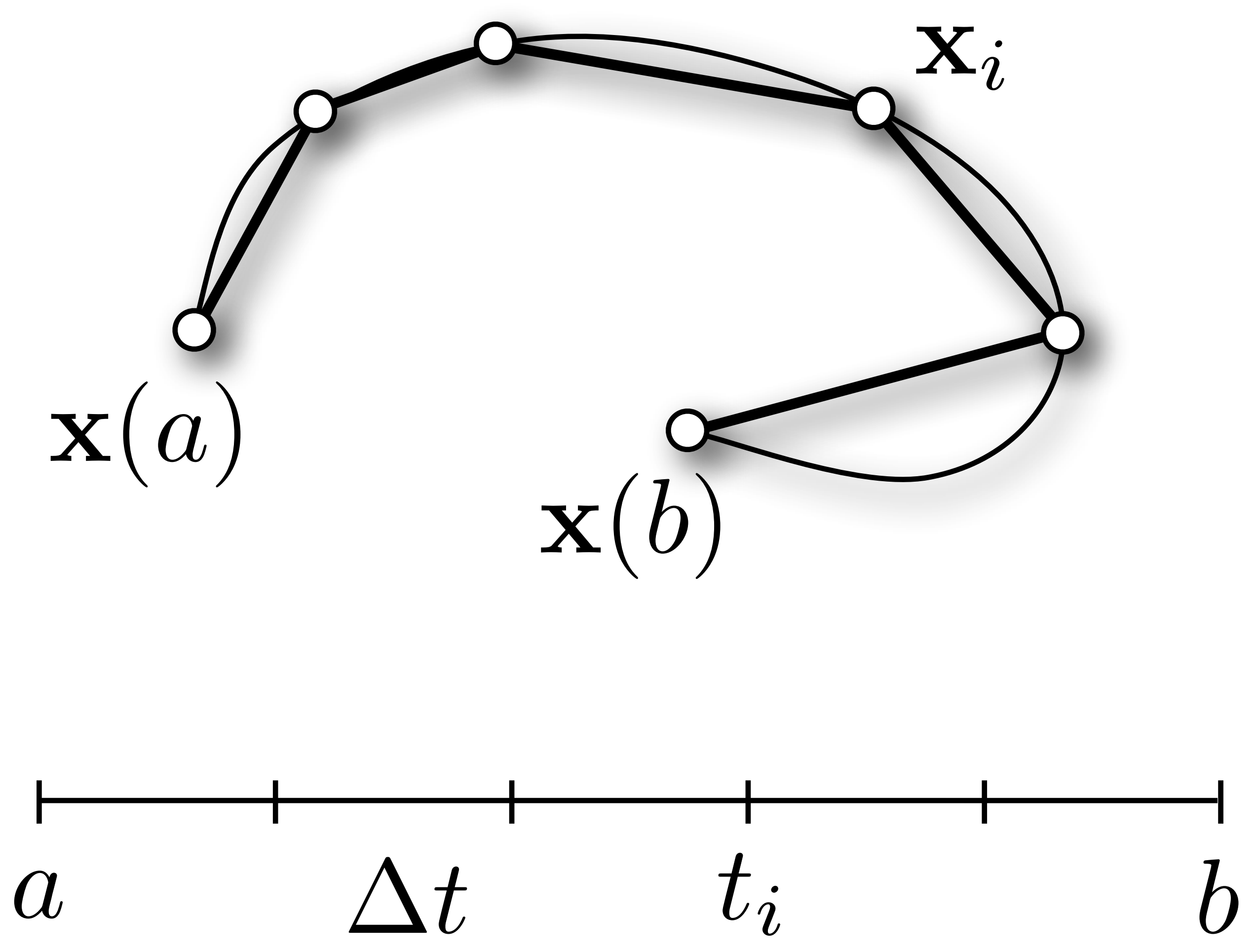
- Curve *arc length* $(\Delta t \to 0)$

$$s = s(t) = \int_a^t \|\mathbf{x}_t\|\,\mathrm{d}t$$



Re-Parameterization

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- Mapping of parameter domain

$$u : [a, b] \to [c, d]$$

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$$u : [a, b] \rightarrow [c, d]$$

- Re-parameterization w.r.t. *u(t)*

$$[c, d] \rightarrow \mathbb{R}^3 , \quad t \mapsto \mathbf{x}(u(t))$$

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- Re-parameterization w.r.t. *u(t)*

$$[c, d] \to \mathbb{R}^3 , \quad t \mapsto \mathbf{x}(u(t))$$

- Derivative (chain rule)

$$\frac{\mathrm{d}\mathbf{x}(u(t))}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t} = \mathbf{x}_u(u(t))\ u_t(t)$$

Re-Parameterization

- Example

$$\mathbf{f} : \left[0, \frac{1}{2}\right] \rightarrow \mathbb{R}^2 \quad , \quad t \mapsto (4t, 2t)$$

$$\phi : \left[0, \frac{1}{2}\right] \rightarrow [0, 1] \quad , \quad t \mapsto 2t$$

$$\mathbf{g} : [0, 1] \rightarrow \mathbb{R}^2 \quad , \quad t \mapsto (2t, t)$$

Re-Parameterization

- Example

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$$\mathbf{g} : [0, 1] \rightarrow \mathbb{R}^2 \quad , \quad t \mapsto (2t, t)$$

$$\Rightarrow \quad \mathbf{g}(\phi(t)) = \mathbf{f}(t)$$

Arc Length Parameterization

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- Mapping of parameter domain:

$$t \;\mapsto\; s(t) = \int_a^t \|\mathbf{x}_t\| \, \mathrm{d}t$$

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$$t \;\mapsto\; s(t) = \int_a^t \|\mathbf{x}_t\| \, \mathrm{d}t$$

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$$\mathbf{x}(s) \;=\; \mathbf{x}(s(t)) \qquad\qquad \mathrm{d}s \;=\; \|\mathbf{x}_t\| \, \mathrm{d}t$$

- Special properties of resulting curve

$$\|\mathbf{x}_s(s)\| = 1 \,, \quad \mathbf{x}_s(s) \cdot \mathbf{x}_{ss}(s) = 0$$

The Frenet Frame

- Taylor expansion

$$\mathbf{x}(t+h) \;=\; \mathbf{x}(t) \;+\; \mathbf{x}_t(t)\,h \;+\; \frac{1}{2}\mathbf{x}_{tt}(t)\,h^2 \;+\; \frac{1}{6}\mathbf{x}_{ttt}(t)\,h^3 \;+\; \ldots$$

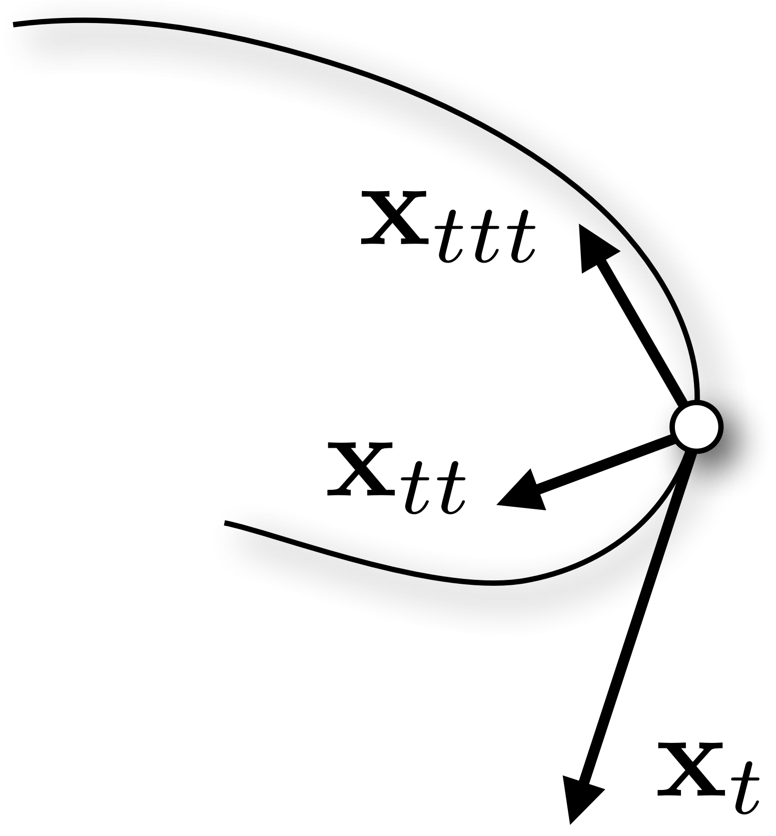
- Define local frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ (*Frenet frame*)

$$\mathbf{t} \;=\; \frac{\mathbf{x}_t}{\|\mathbf{x}_t\|} \qquad \mathbf{n} \;=\; \mathbf{b} \times \mathbf{t} \qquad \mathbf{b} \;=\; \frac{\mathbf{x}_t \times \mathbf{x}_{tt}}{\|\mathbf{x}_t \times \mathbf{x}_{tt}\|}$$

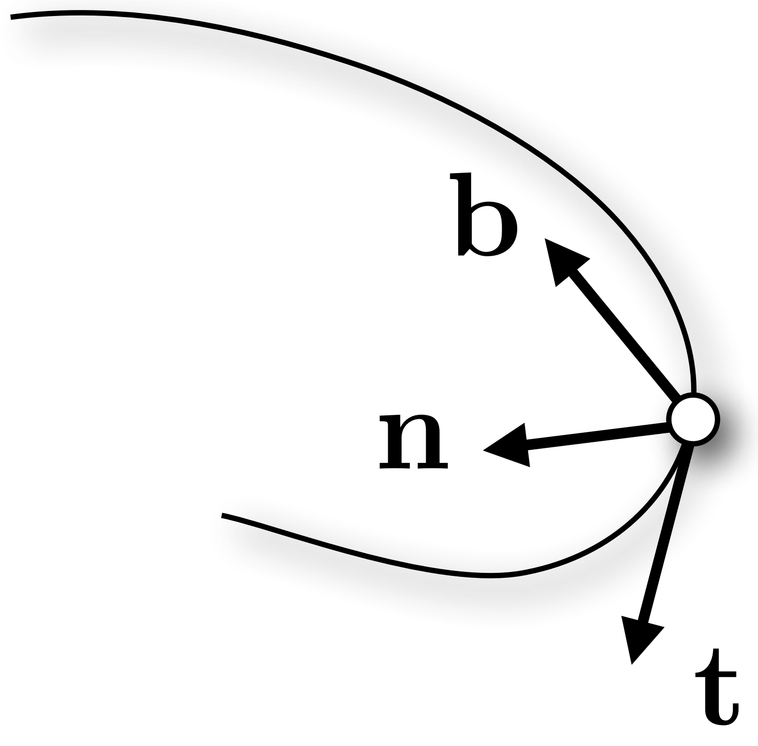
tangent | principal normal | binormal

The Frenet Frame

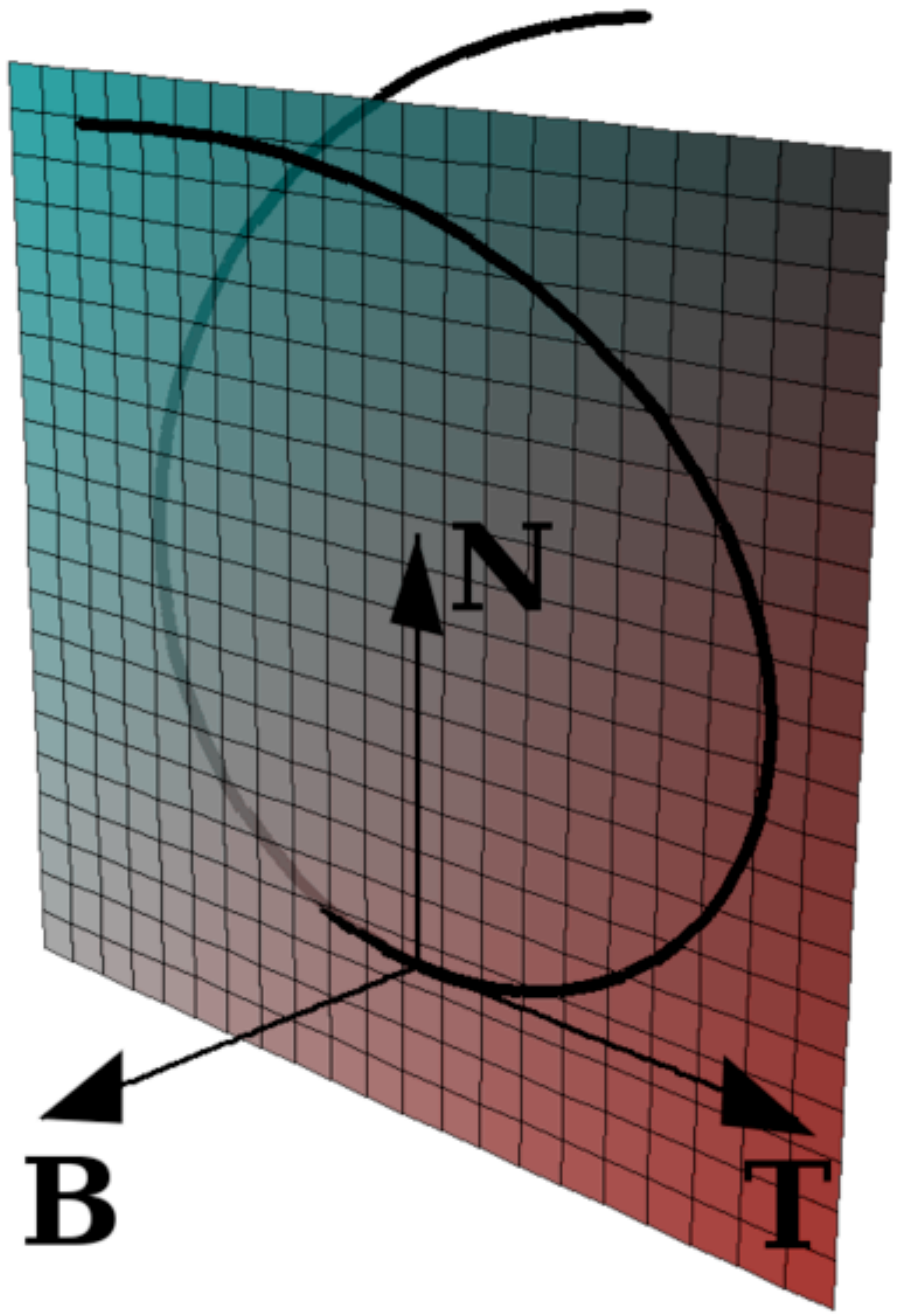
- Orthonormalization of local frame



local affine frame



Frenet frame



The Frenet Frame

The Frenet Frame

- *Frenet-Serret*: Derivatives w.r.t. arc length *s*

$$\begin{aligned} \mathbf{t}_s &= && +\kappa\mathbf{n} && \\ \mathbf{n}_s &= -\kappa\mathbf{t} && && +\tau\mathbf{b} \\ \mathbf{b}_s &= && -\tau\mathbf{n} && \end{aligned}$$

The Frenet Frame

- *Frenet-Serret*: Derivatives w.r.t. arc length *s*

$$\begin{aligned}
\mathbf{t}_s &= \phantom{-\kappa\mathbf{t}} \quad +\kappa\mathbf{n} \\
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\mathbf{b}_s &= \phantom{-\kappa\mathbf{t}} \quad -\tau\mathbf{n}
\end{aligned}$$

- Curvature (*deviation from straight line*)

$$\kappa = \|\mathbf{x}_{ss}\|$$

The Frenet Frame

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- Curvature (*deviation from straight line*)

$$\kappa = \|\mathbf{x}_{ss}\|$$

- Torsion (*deviation from planarity*)

$$\tau = \frac{1}{\kappa^2}\det([\mathbf{x}_s, \mathbf{x}_{ss}, \mathbf{x}_{sss}])$$

Curvature and Torsion

- Planes defined by $\mathbf{x}$ and two vectors:
  - *osculating plane*: vectors $\mathbf{t}$ and $\mathbf{n}$
  - *normal plane*: vectors $\mathbf{n}$ and $\mathbf{b}$
  - *rectifying plane*: vectors $\mathbf{t}$ and $\mathbf{b}$
- Osculating circle
  - second order contact with curve
  - center $\mathbf{c} = \mathbf{x} + (1/\kappa)\mathbf{n}$
  - radius $1/\kappa$

Curvature and Torsion

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- Curvature: Deviation from straight line

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- Independent of parameterization

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Curvature and Torsion

- Curvature: Deviation from straight line
- Torsion: Deviation from planarity
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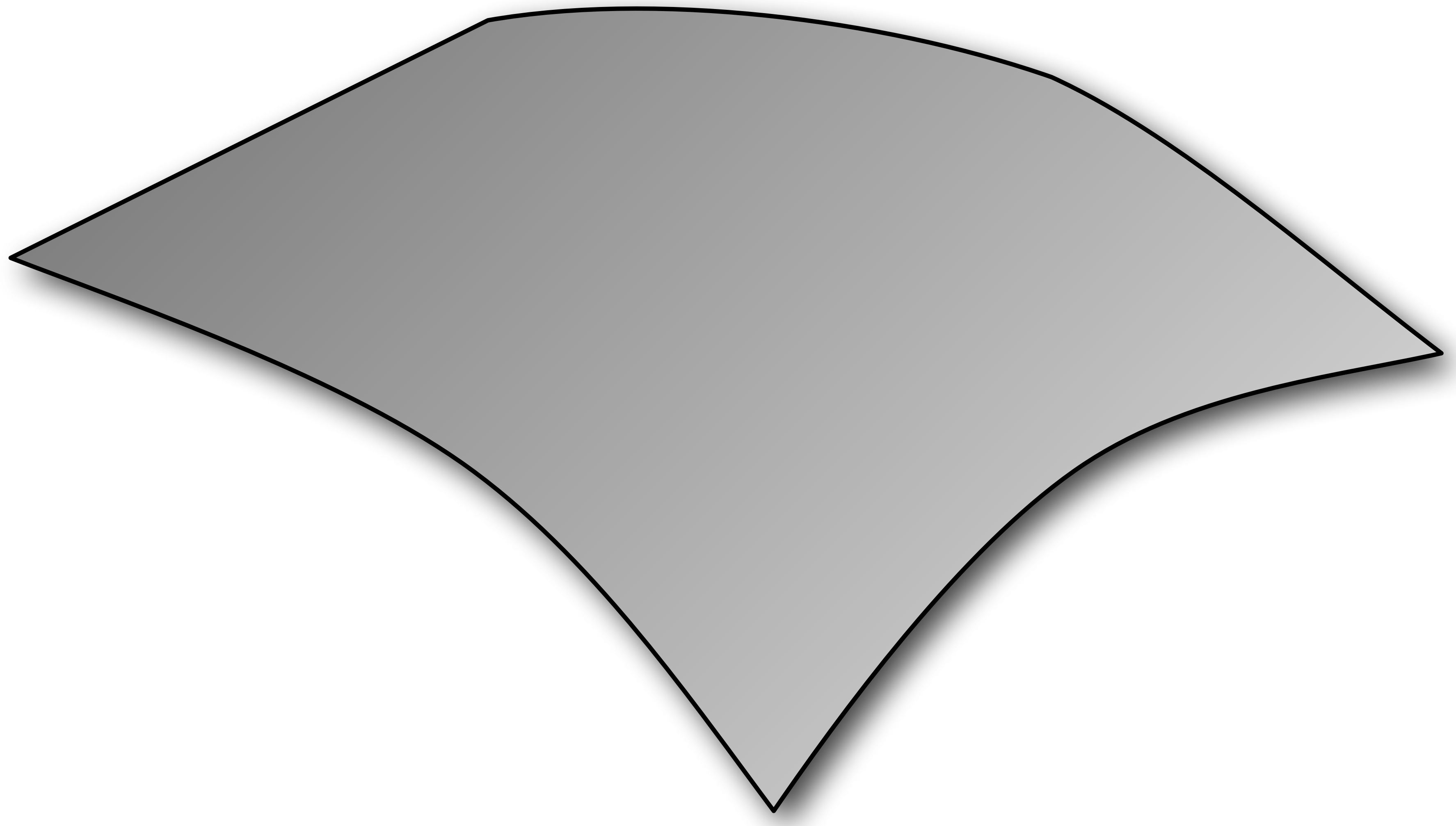
  - intrinsic properties of the curve
- Euclidean invariants
  - invariant under rigid motion
- Define curve uniquely up to rigid motion

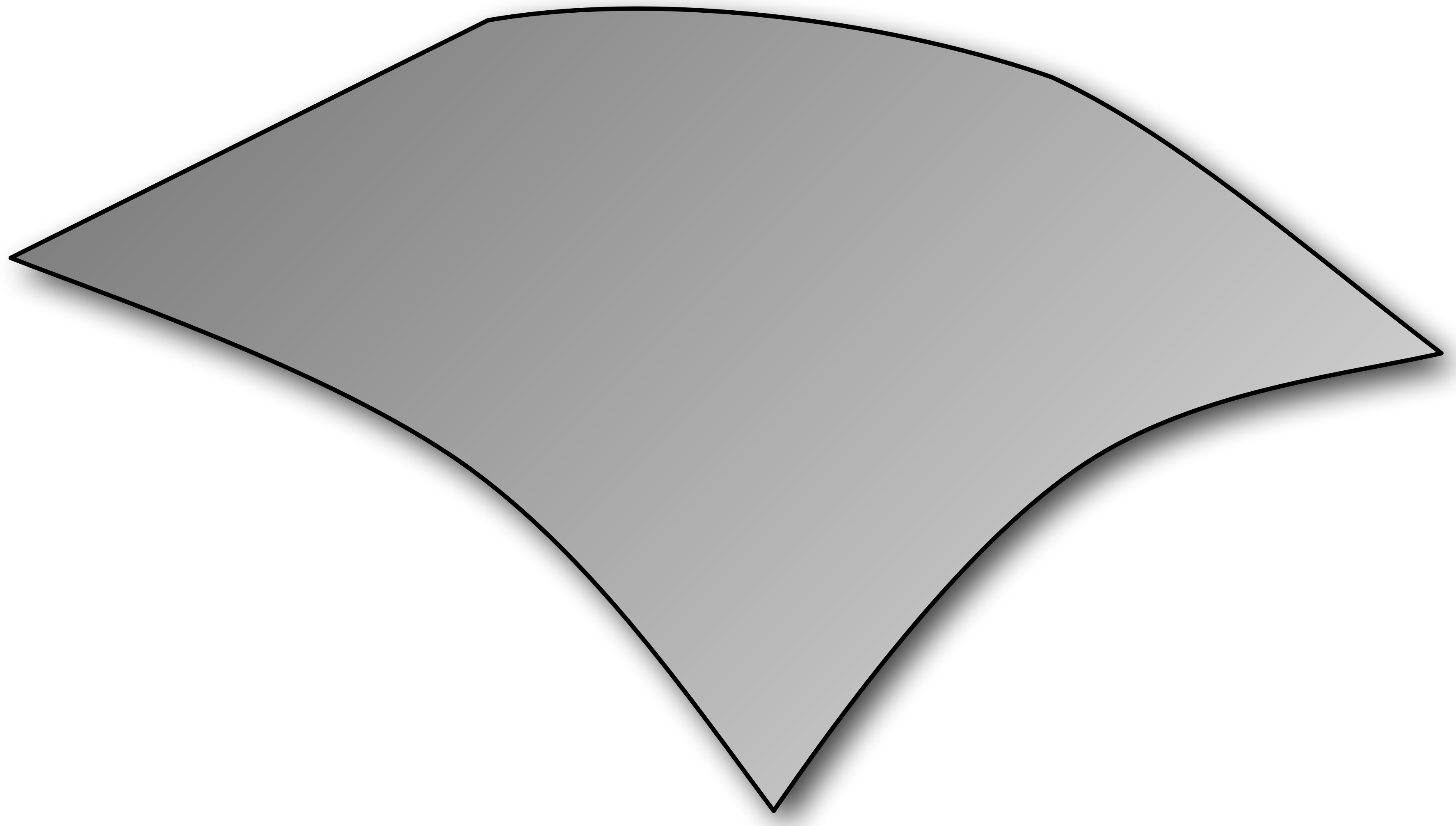
Parametric Surfaces



Parametric Surfaces

- Continuous surface

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$



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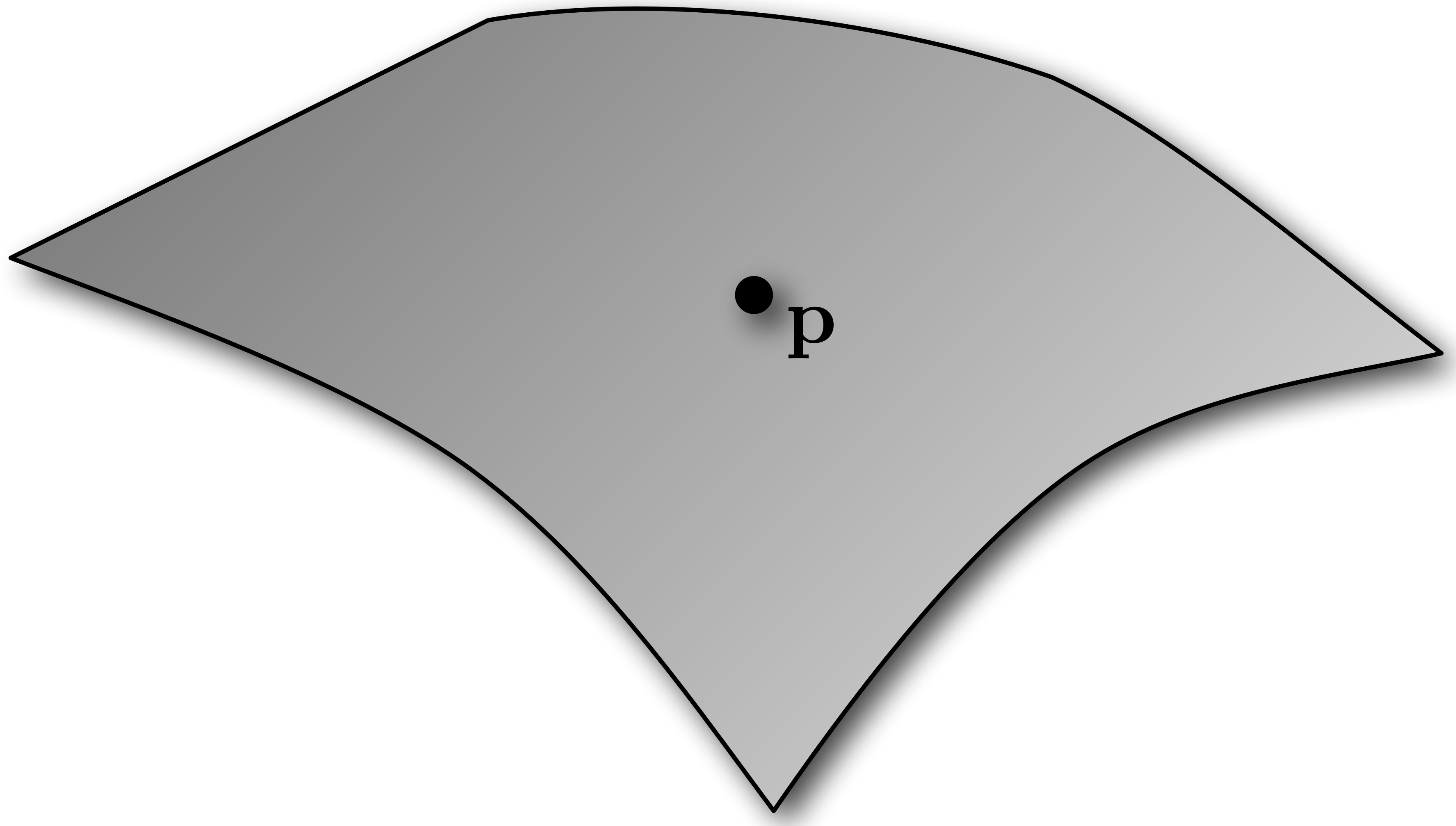


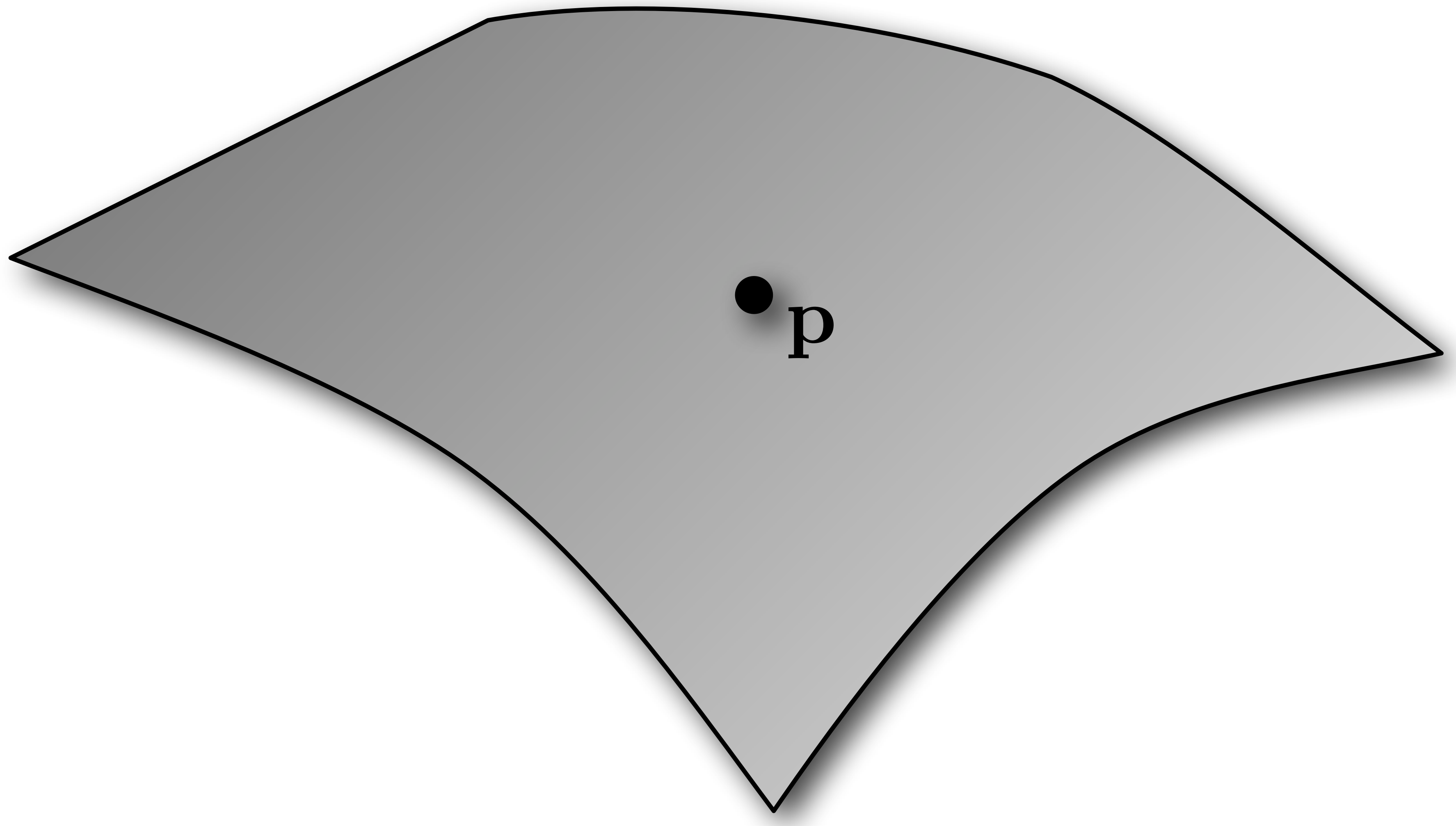
Parametric Surfaces

- Continuous surface

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

- Normal vector

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$



Parametric Surfaces

- Continuous surface

$$\mathbf{x}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

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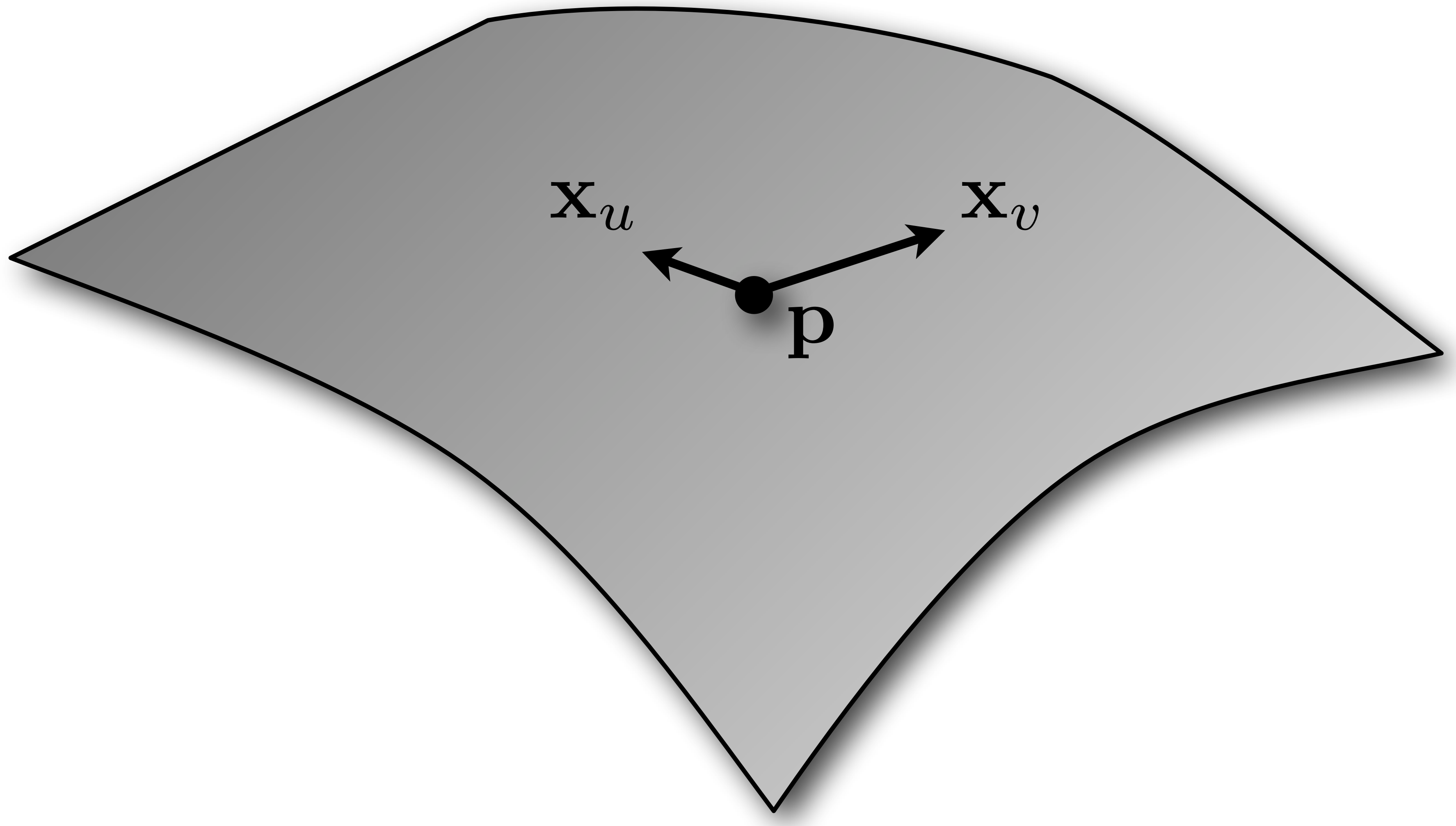

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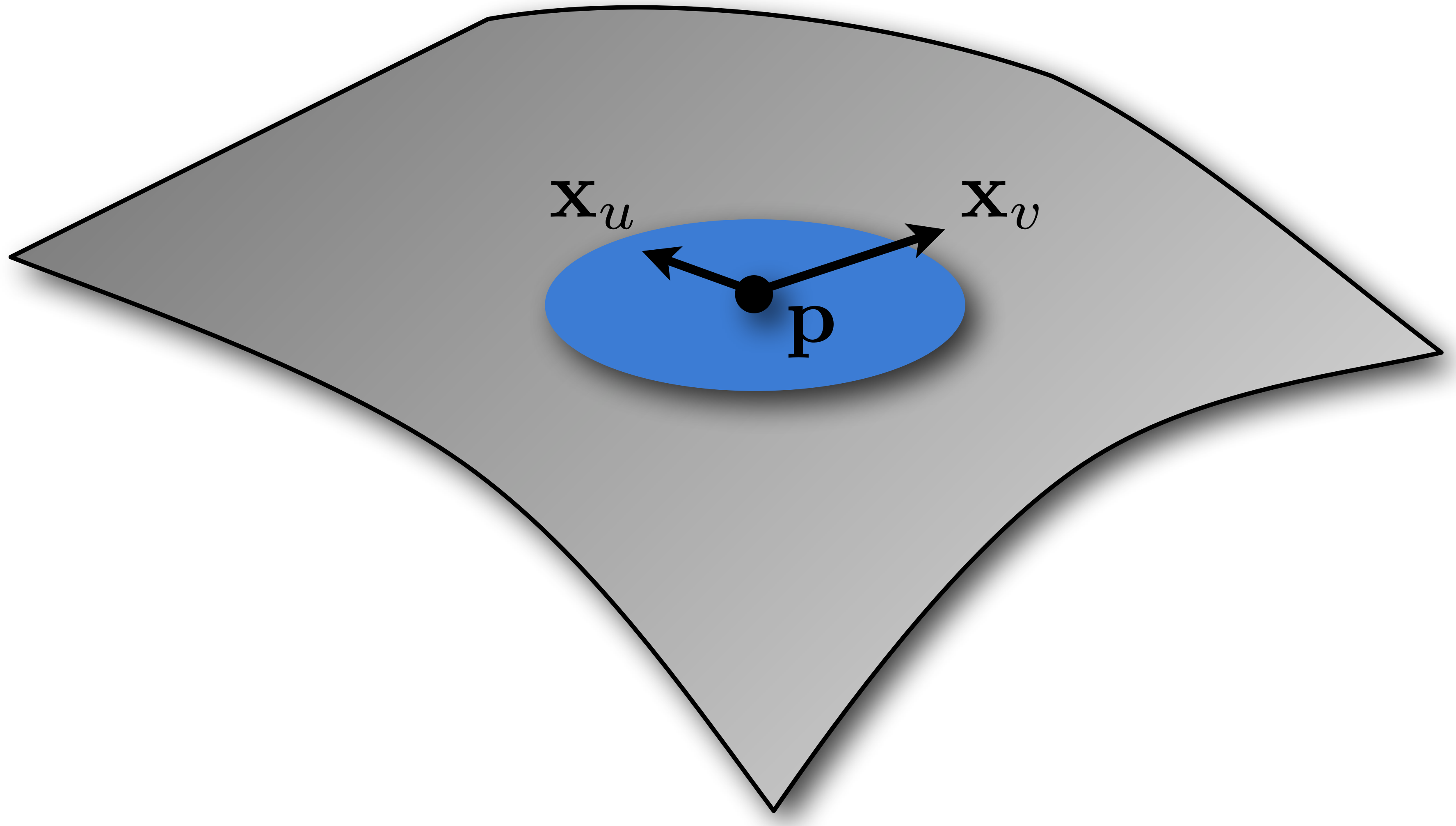

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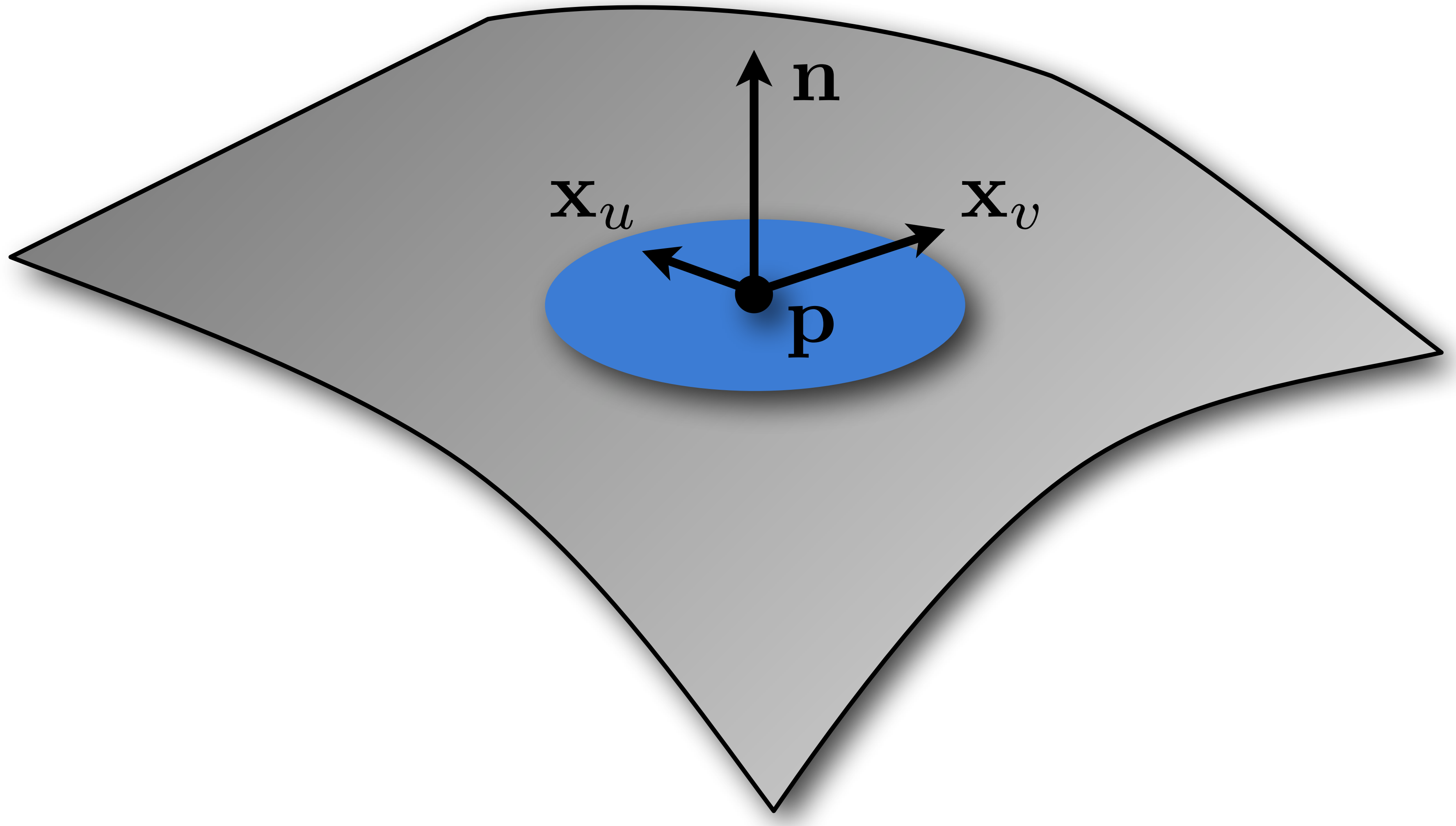

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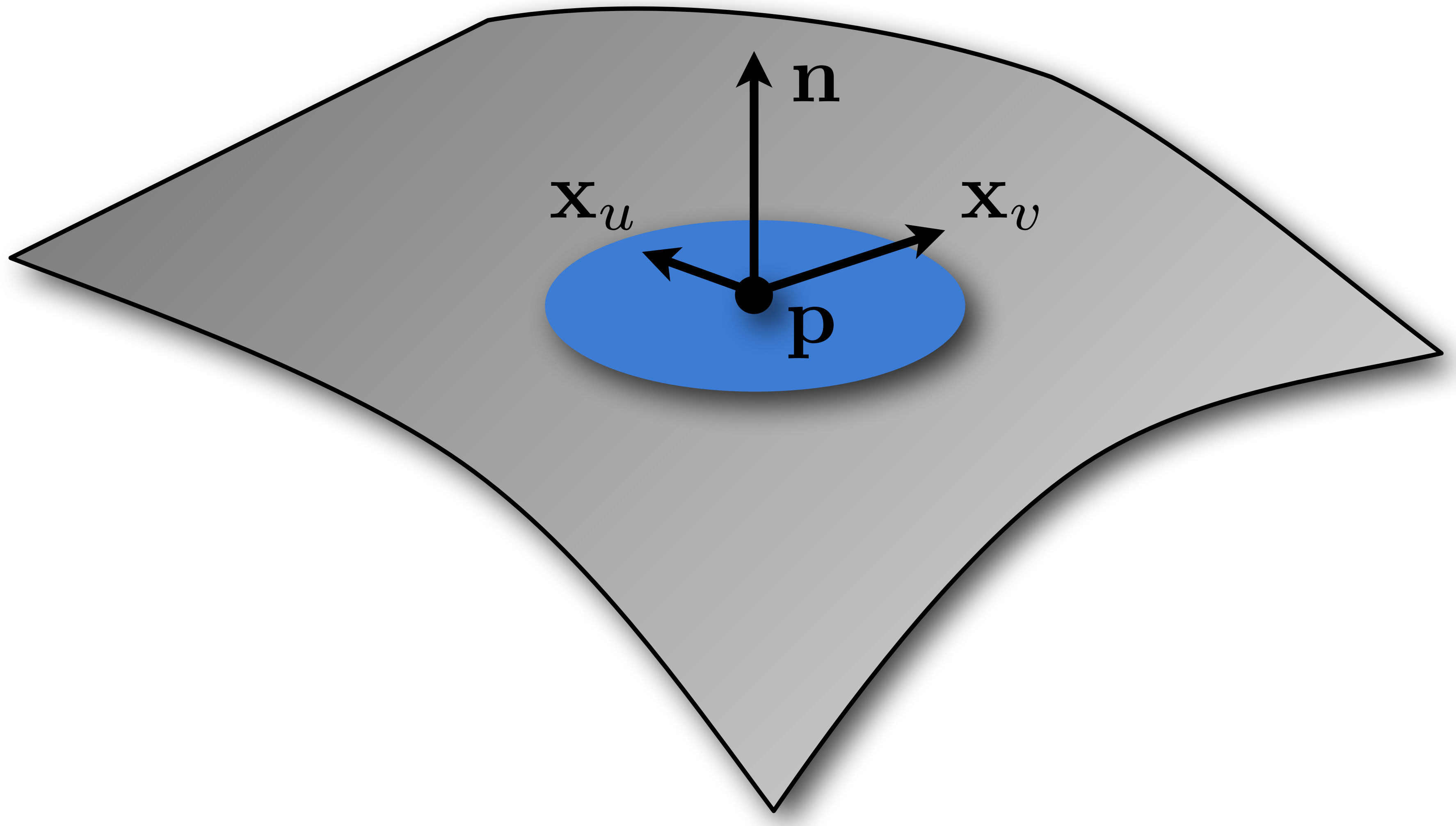
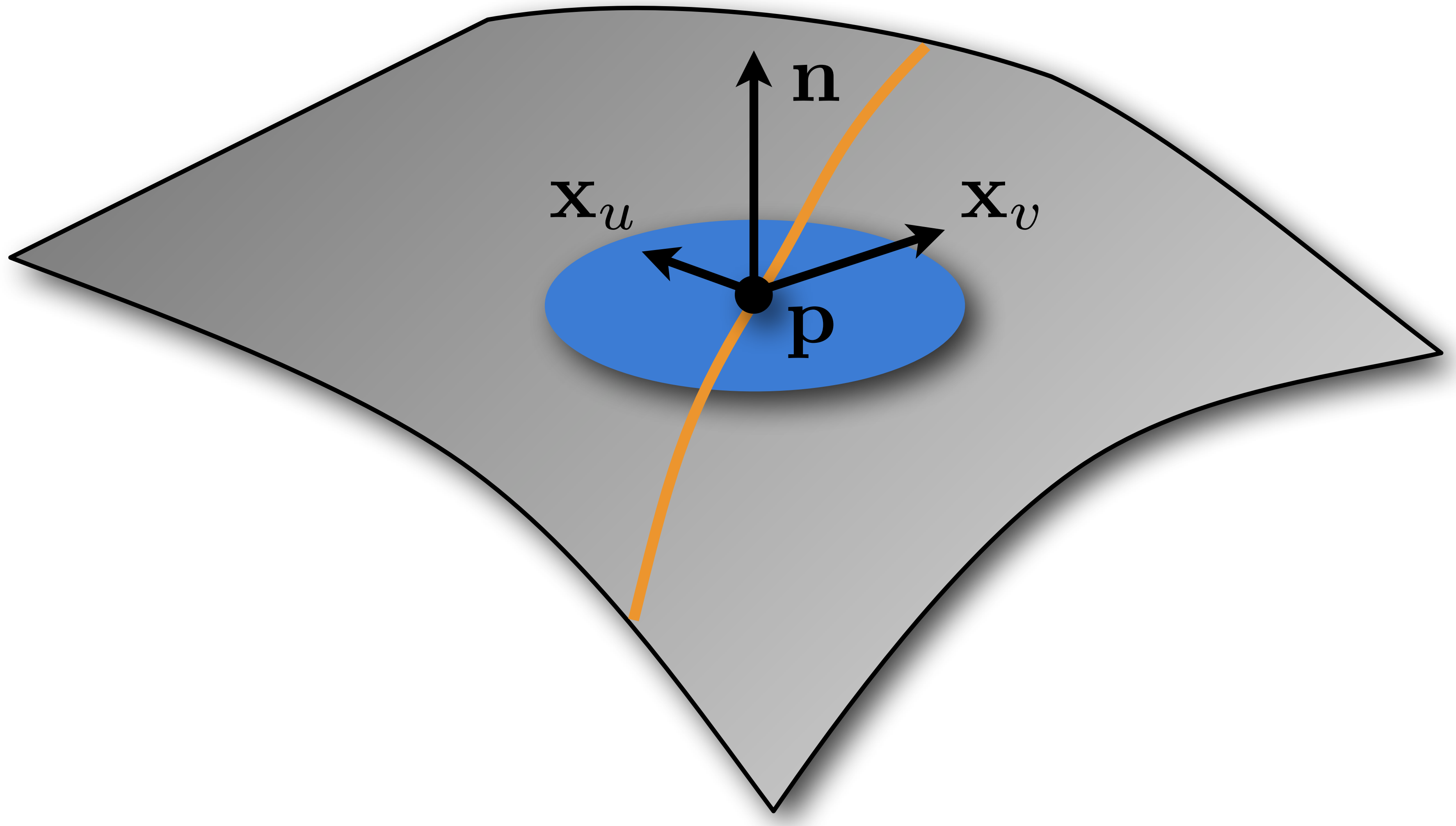
- Assume *regular* parameterization

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$



Angles on Surface



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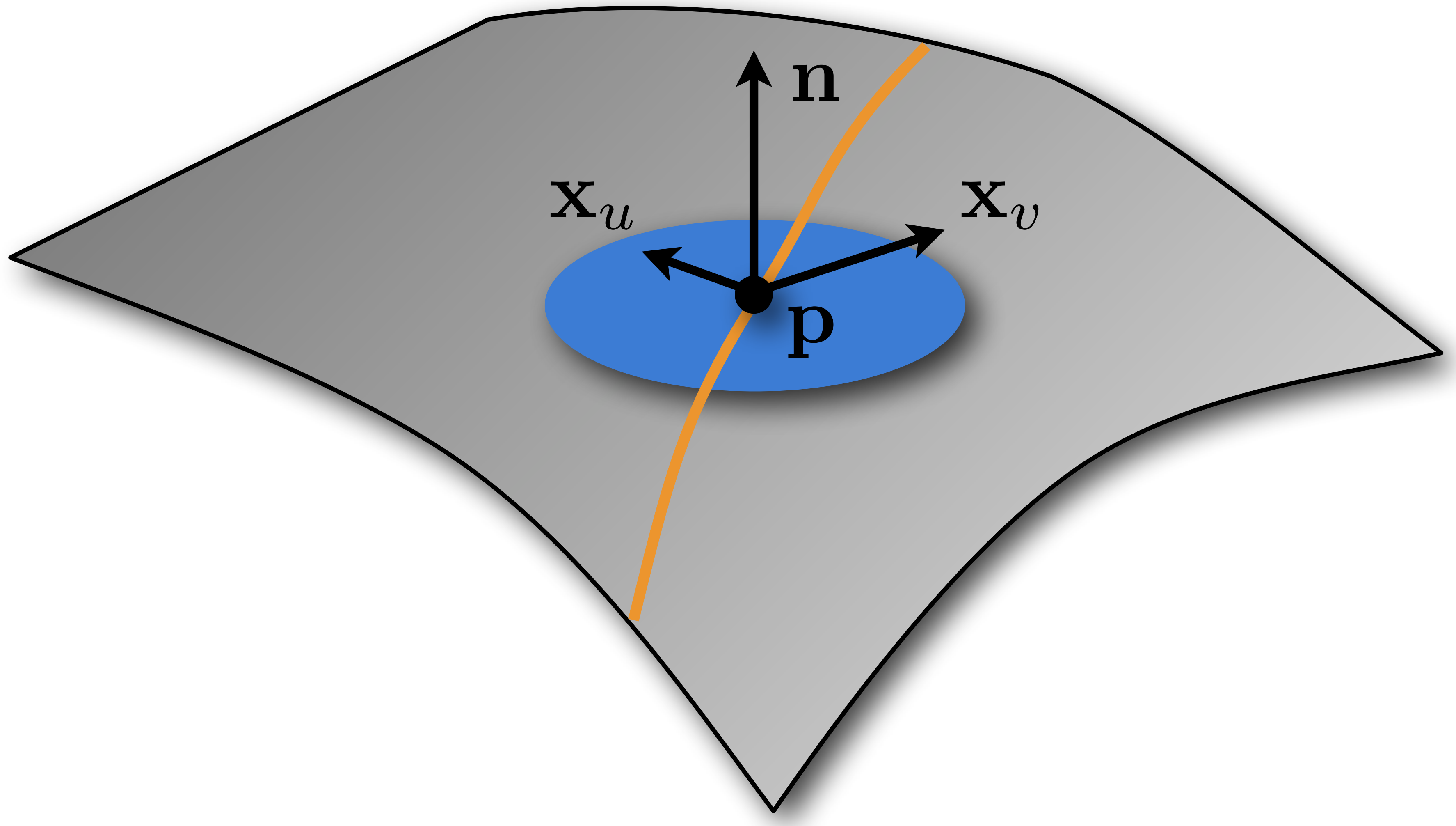

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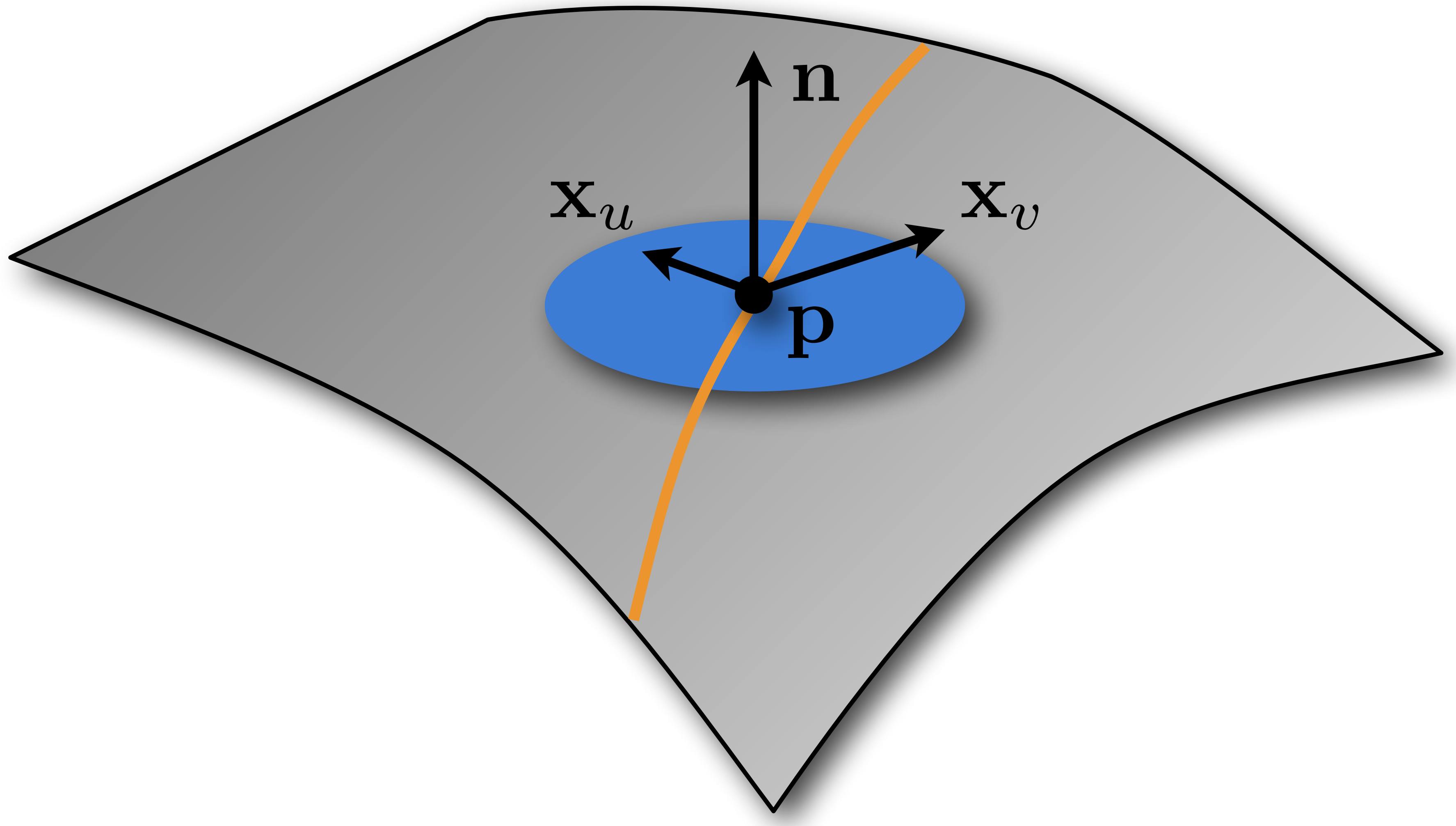

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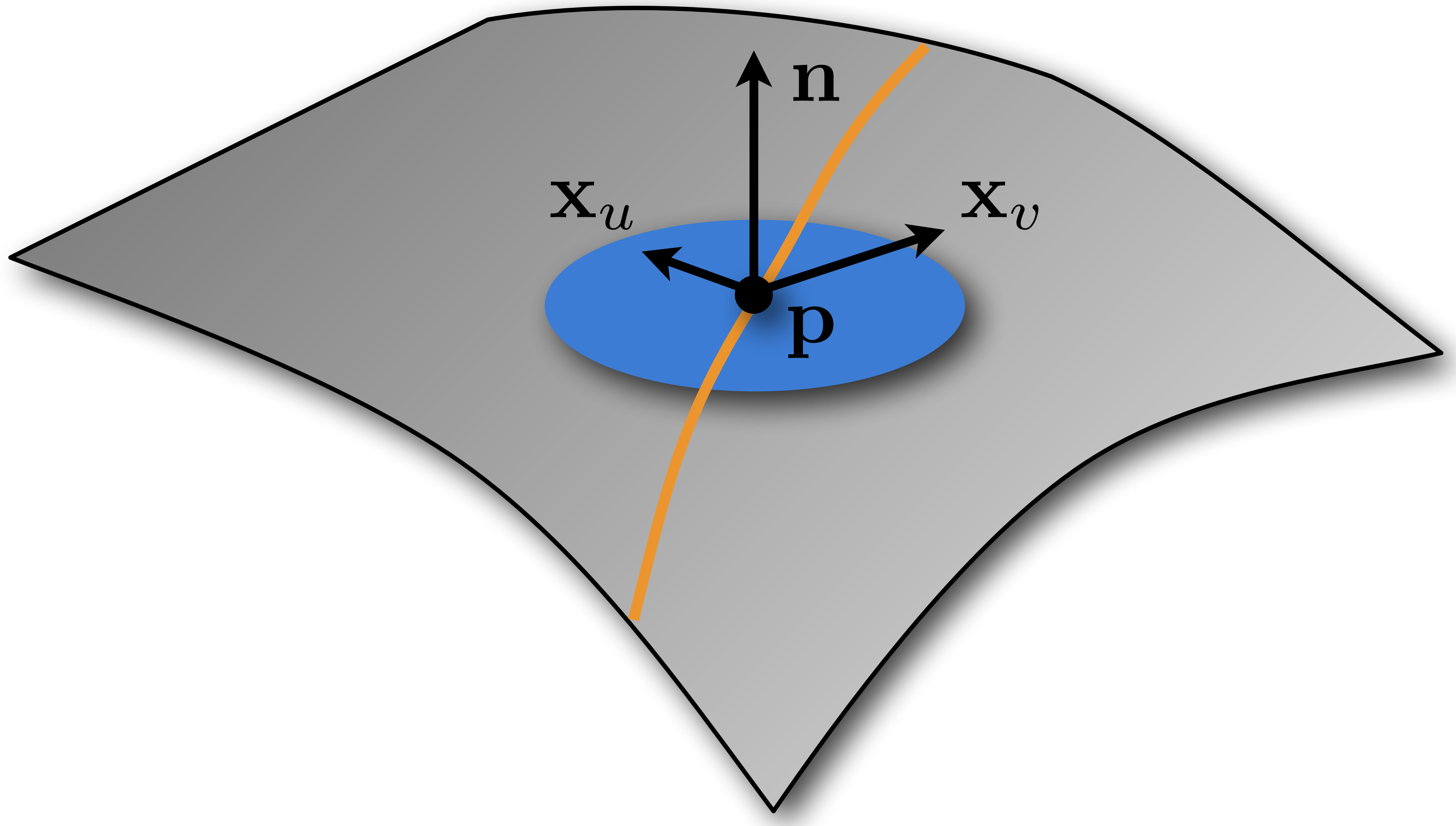

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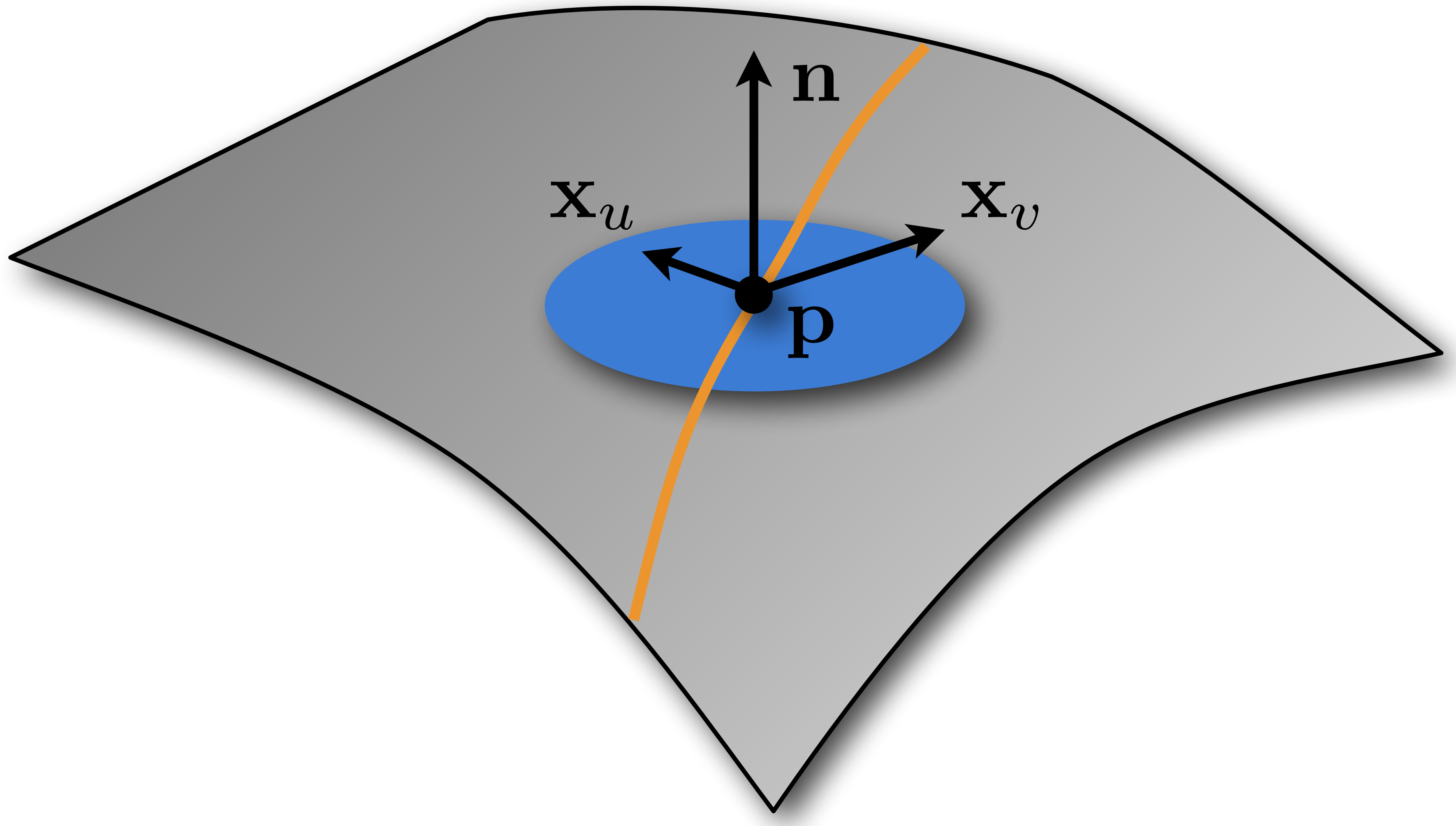

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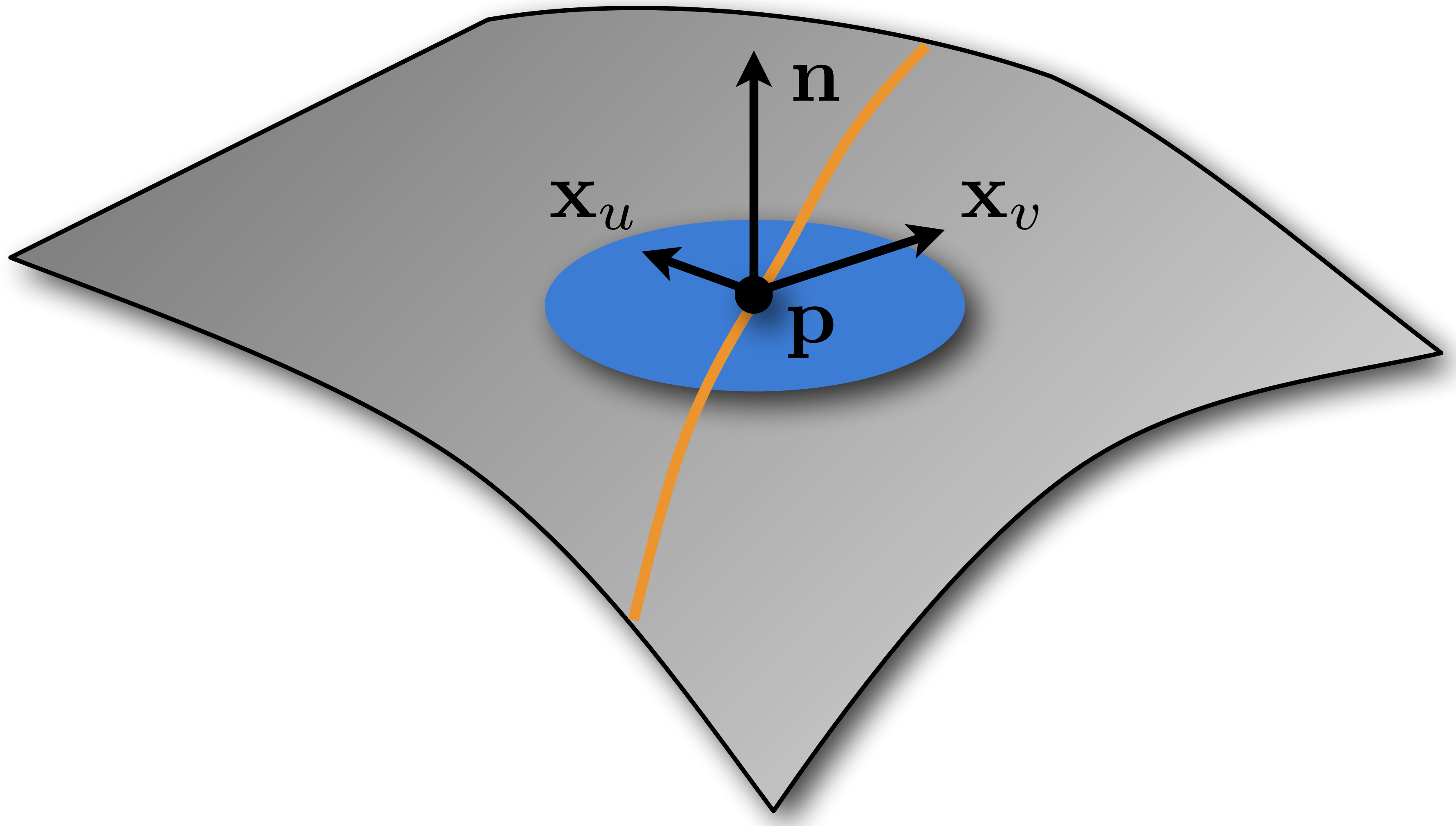

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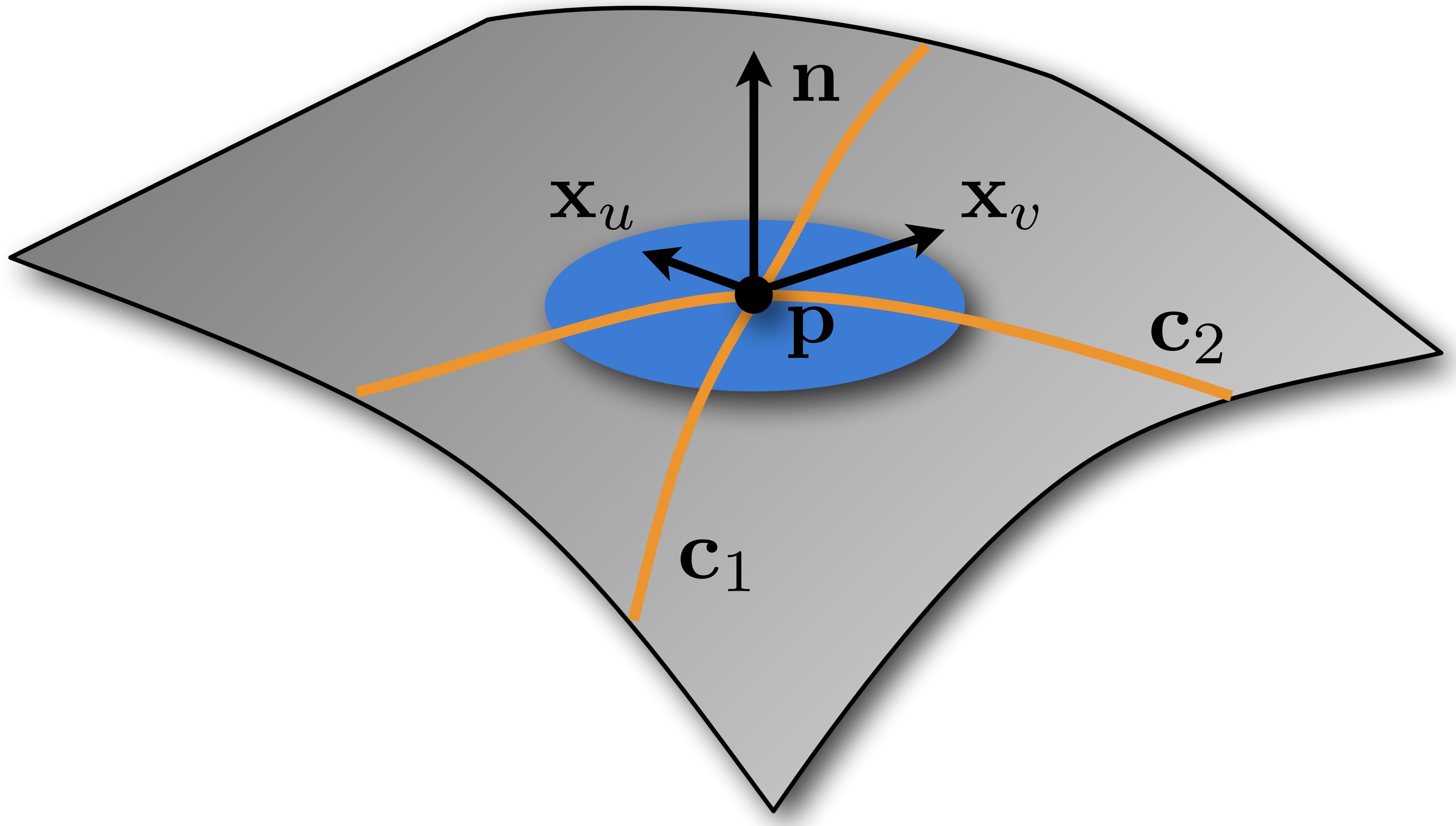
$$\mathbf{t}_i = \alpha_i \mathbf{x}_u + \beta_i \mathbf{x}_v$$

  - Compute inner product

$$\mathbf{t}_1^T \mathbf{t}_2 = \cos\theta \, \|\mathbf{t}_1\| \, \|\mathbf{t}_2\|$$



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$$\mathbf{t}_1^T \mathbf{t}_2 \; = \; (\alpha_1 \mathbf{x}_u + \beta_1 \mathbf{x}_v)^T \, (\alpha_2 \mathbf{x}_u + \beta_2 \mathbf{x}_v)$$

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$$\begin{aligned}
\mathbf{t}_1^T\mathbf{t}_2 &= (\alpha_1\mathbf{x}_u + \beta_1\mathbf{x}_v)^T\,(\alpha_2\mathbf{x}_u + \beta_2\mathbf{x}_v) \\
&= \alpha_1\alpha_2\mathbf{x}_u^T\mathbf{x}_u + (\alpha_1\beta_2 + \alpha_2\beta_1)\,\mathbf{x}_u^T\mathbf{x}_v + \beta_1\beta_2\mathbf{x}_v^T\mathbf{x}_v \\
&= (\alpha_1, \beta_1)\begin{pmatrix}\mathbf{x}_u^T\mathbf{x}_u & \mathbf{x}_u^T\mathbf{x}_v \\ \mathbf{x}_u^T\mathbf{x}_v & \mathbf{x}_v^T\mathbf{x}_v\end{pmatrix}\begin{pmatrix}\alpha_2 \\ \beta_2\end{pmatrix}
\end{aligned}$$

First Fundamental Form

- First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

- Defines inner product on tangent space

$$\left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

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$$\begin{aligned} \mathrm{d}s^2 &= \langle(\mathrm{d}u, \mathrm{d}v), (\mathrm{d}u, \mathrm{d}v)\rangle \\ &= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 \end{aligned}$$

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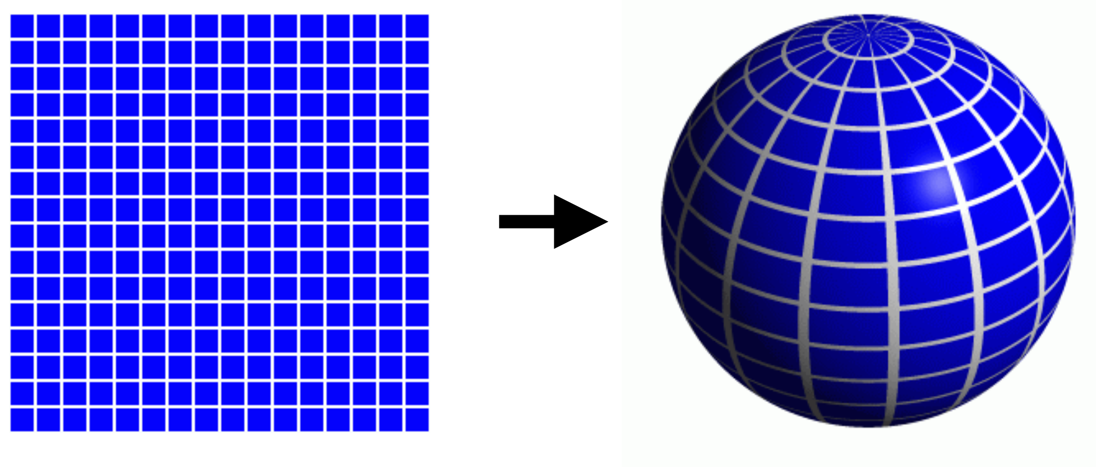
  - Area
    $$\begin{aligned} \mathrm{d}A &= \|\mathbf{x}_u \times \mathbf{x}_v\| \,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{\mathbf{x}_u^T \mathbf{x}_u \cdot \mathbf{x}_v^T \mathbf{x}_v - (\mathbf{x}_u^T \mathbf{x}_v)^2}\,\mathrm{d}u\,\mathrm{d}v \\ &= \sqrt{EG - F^2}\mathrm{d}u\,\mathrm{d}v \end{aligned}$$

Example: Sphere

- Spherical parameterization



$$\mathbf{x}(u,v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}, \quad (u,v) \in [0,2\pi) \times [0,\pi)$$

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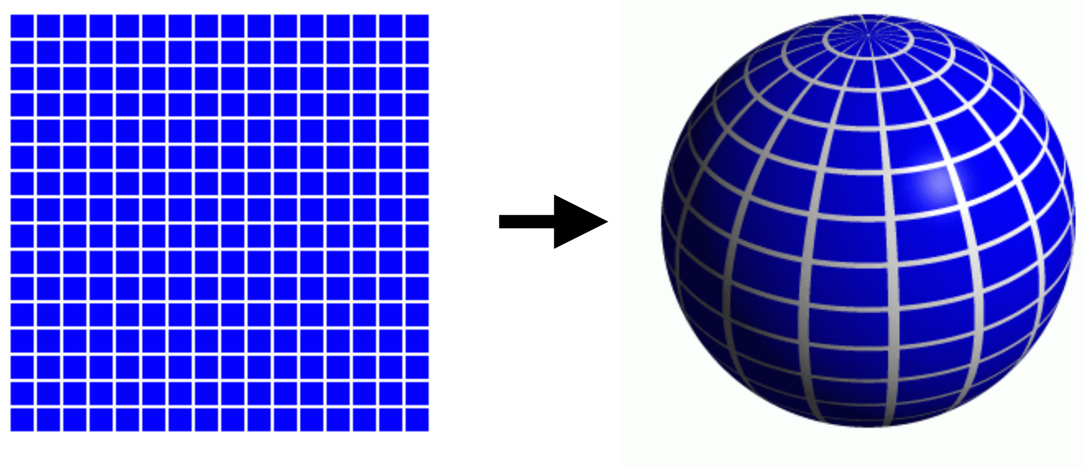

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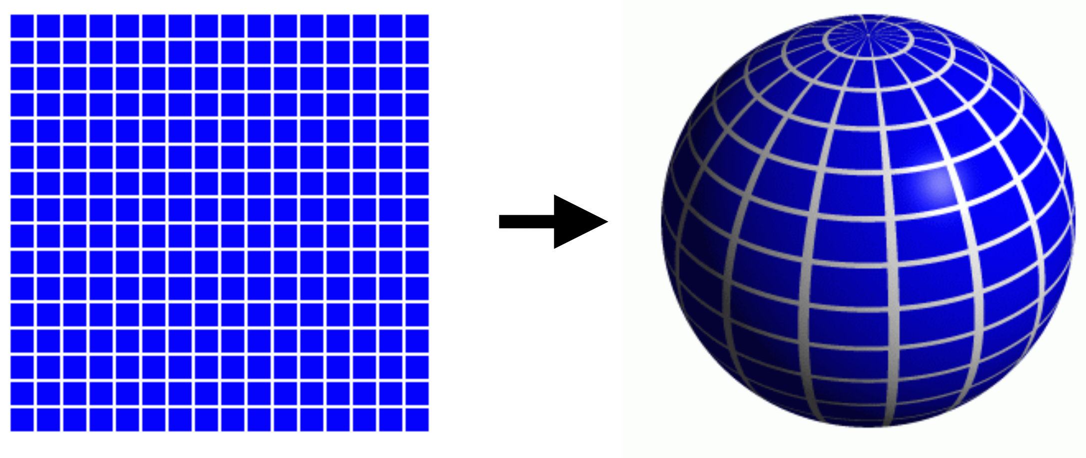
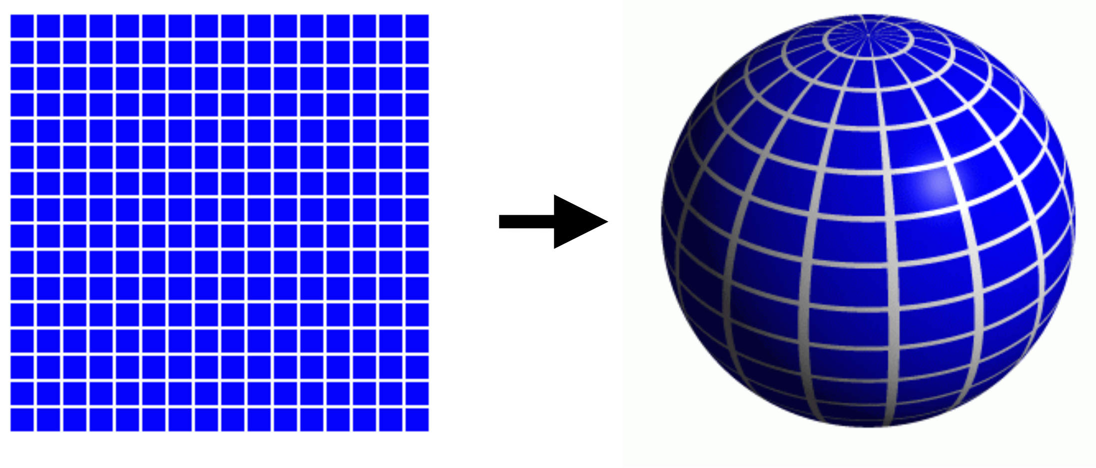


- Tangent vectors

$$\mathbf{x}_u(u,v) = \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix}$$

Example: Sphere



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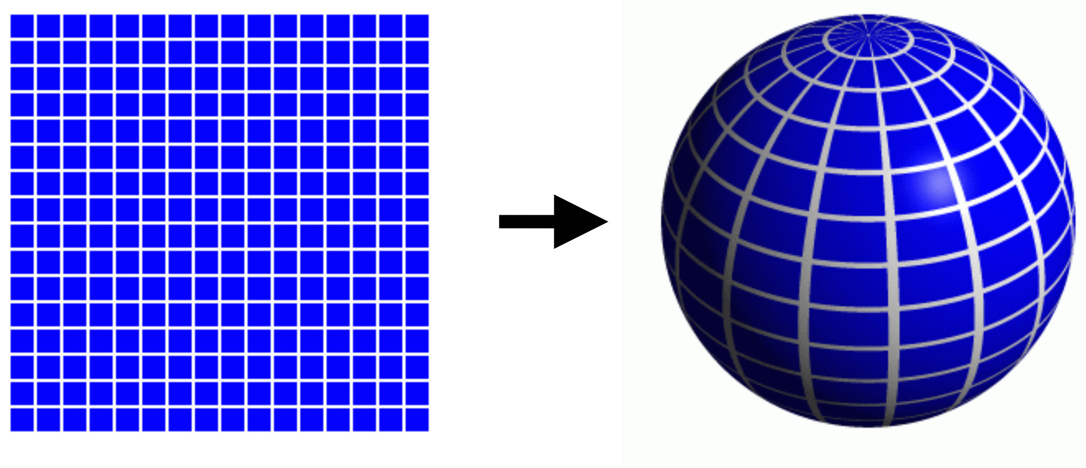

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- First fundamental form

$$\mathbf{I} = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}$$

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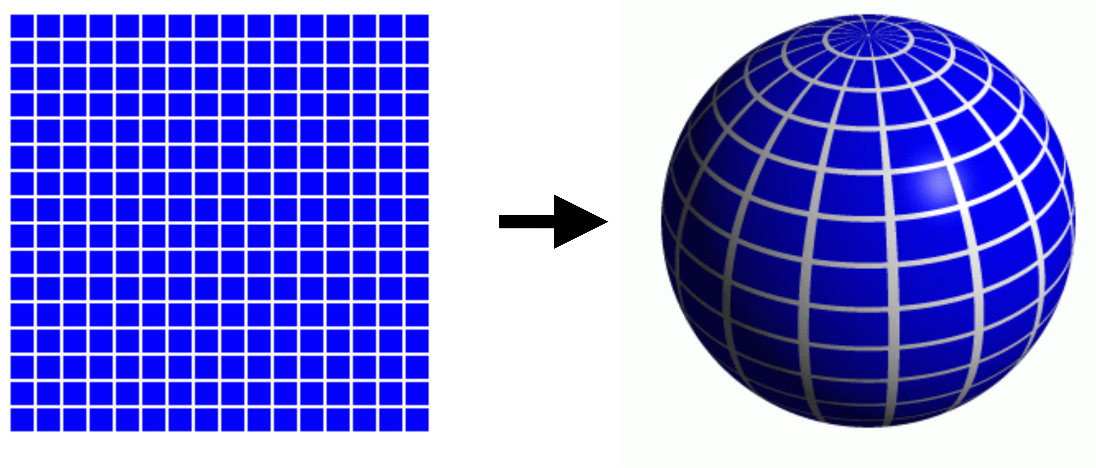

- Length of equator $\mathbf{x}(t, \pi / 2)$

$$\int_0^{2\pi} 1 \, \mathrm{d}s \;=\; \int_0^{2\pi} \sqrt{E\,(u_t)^2 + 2F u_t v_t + G\,(v_t)^2}\,\mathrm{d}t$$

Example: Sphere

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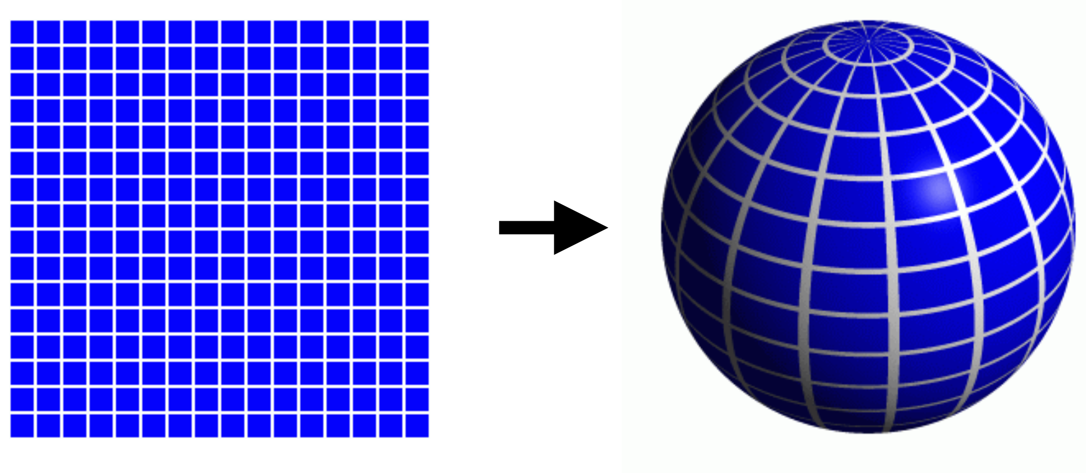

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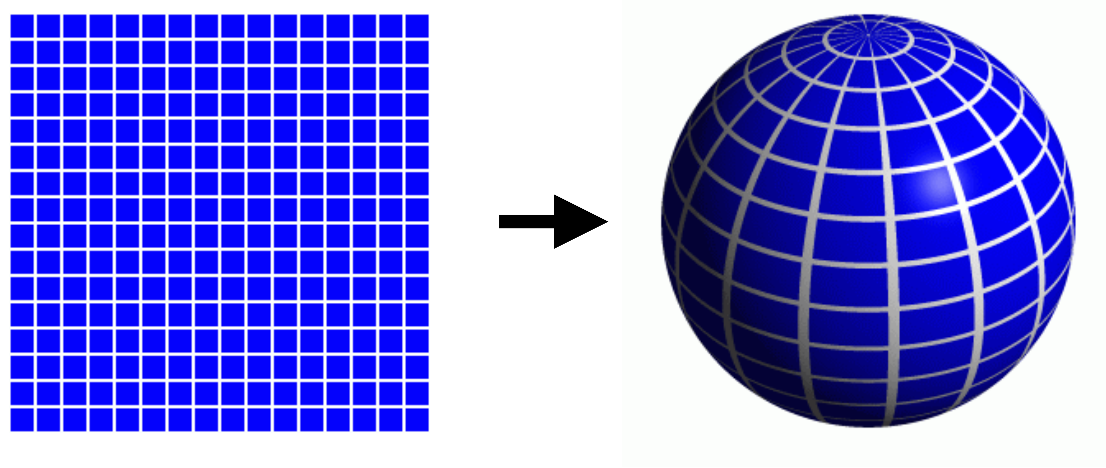

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$$= 2\pi \sin v = 2\pi$$

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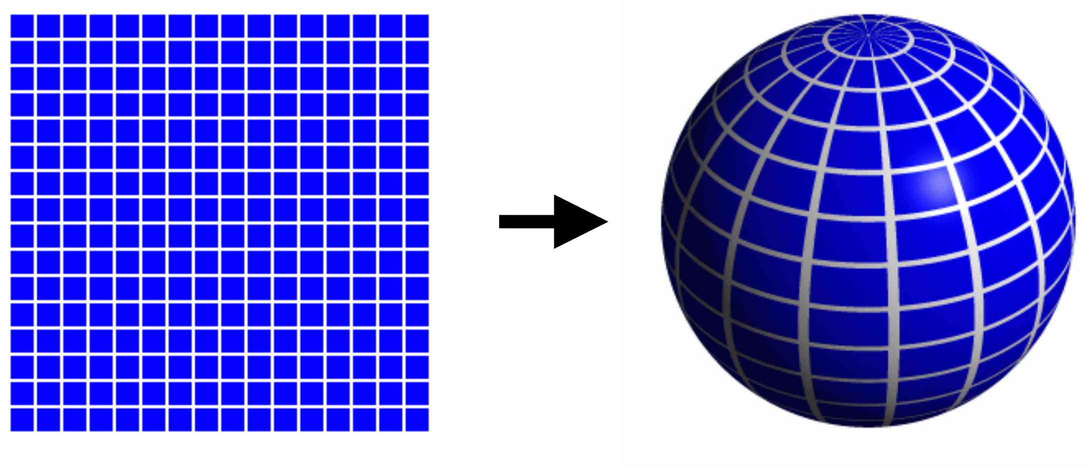

- Area of a sphere

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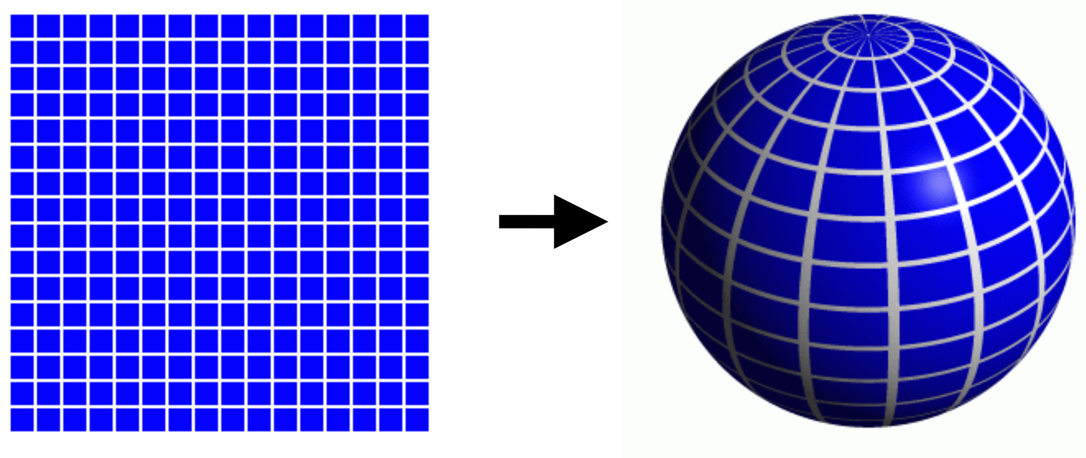

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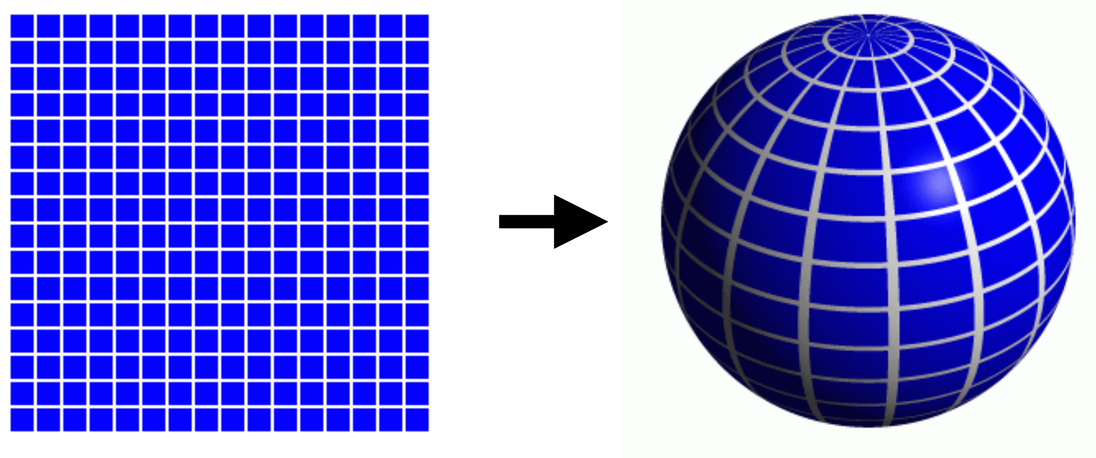

$$
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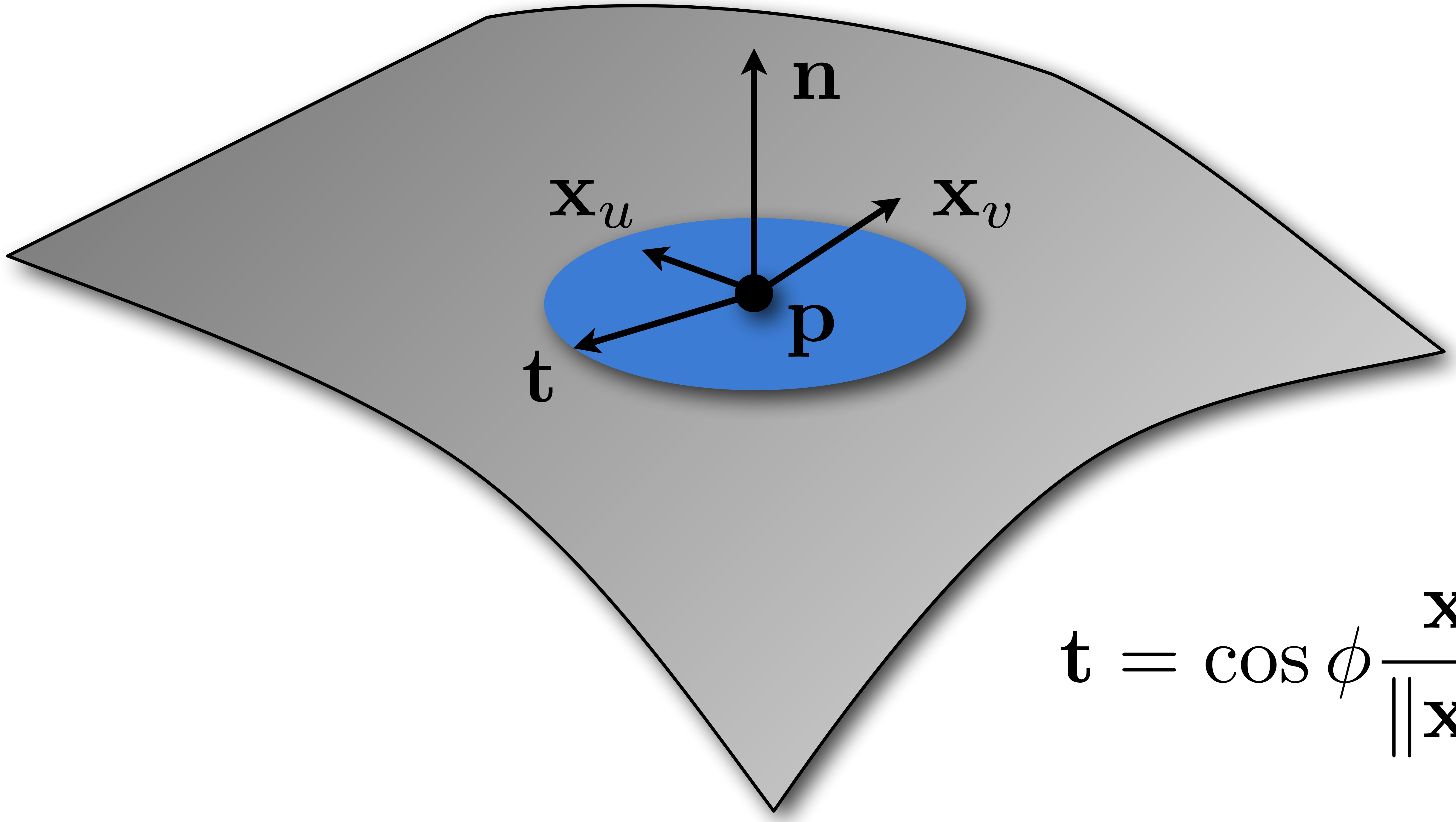
\int_0^{\pi}\int_0^{2\pi} 1\,\mathrm{d}A &= \int_0^{\pi}\int_0^{2\pi} \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_0^{\pi}\int_0^{2\pi} \sin v\,\mathrm{d}u\,\mathrm{d}v \\
&= 4\pi
\end{aligned}
$$

Normal Curvature

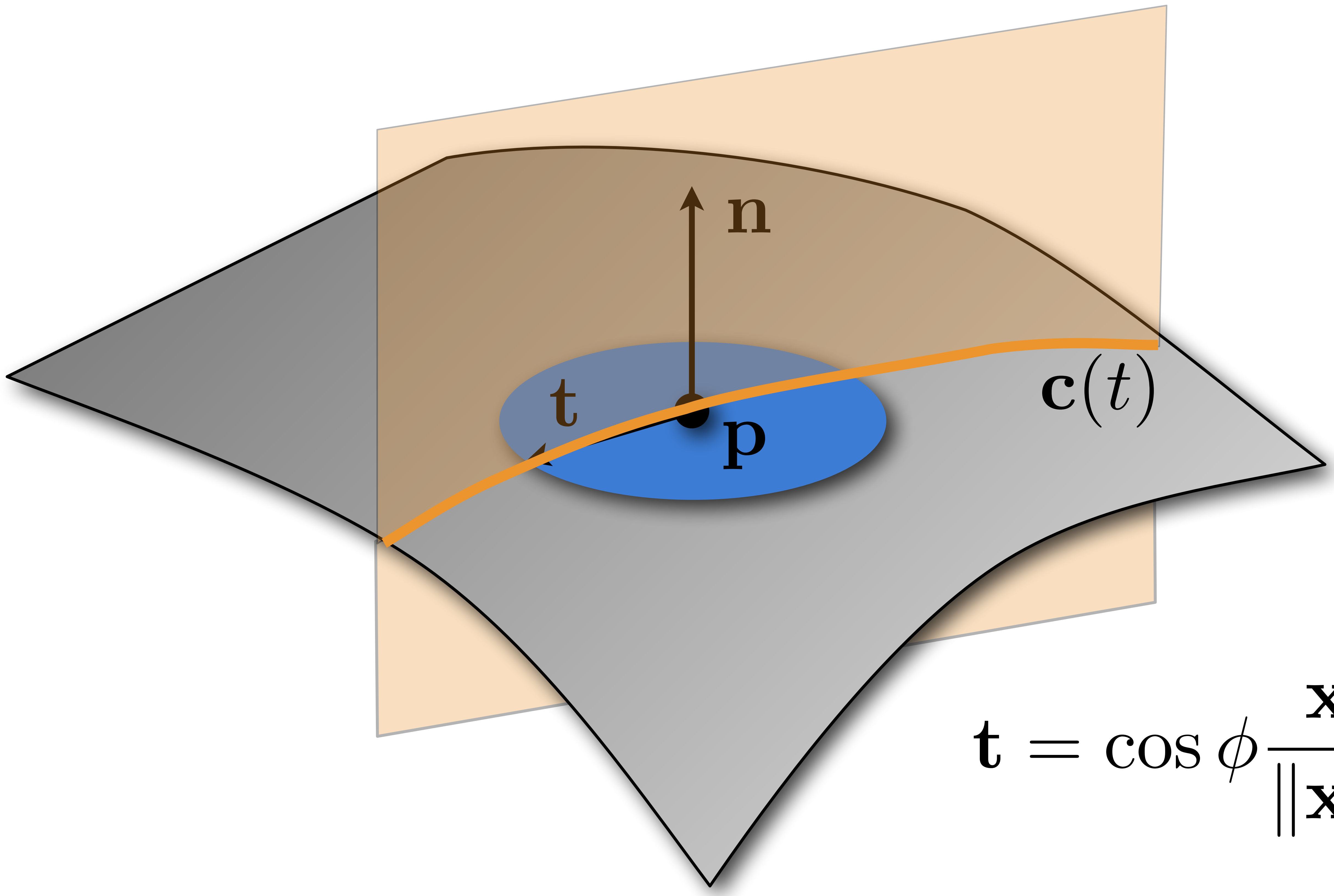
- Tangent vector $\mathbf{t}$...



$$\mathbf{t} = \cos\phi \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} + \sin\phi \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

Normal Curvature

- .. defines intersection plane, yielding curve $\mathbf{c}(t)$



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- With **second fundamental** form

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normal curvature can be computed as

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \, \mathbf{II} \, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \, \mathbf{I} \, \bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}$$

$$\begin{aligned} \mathbf{t} &= a\mathbf{x}_u + b\mathbf{x}_v \\ \bar{\mathbf{t}} &= (a, b) \end{aligned}$$

Surface Curvature(s)

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- Special curvatures
  - Mean curvature $$H = \frac{\kappa_1 + \kappa_2}{2}$$
  - Gaussian curvature $$K = \kappa_1 \cdot \kappa_2$$

Curvature of Surfaces

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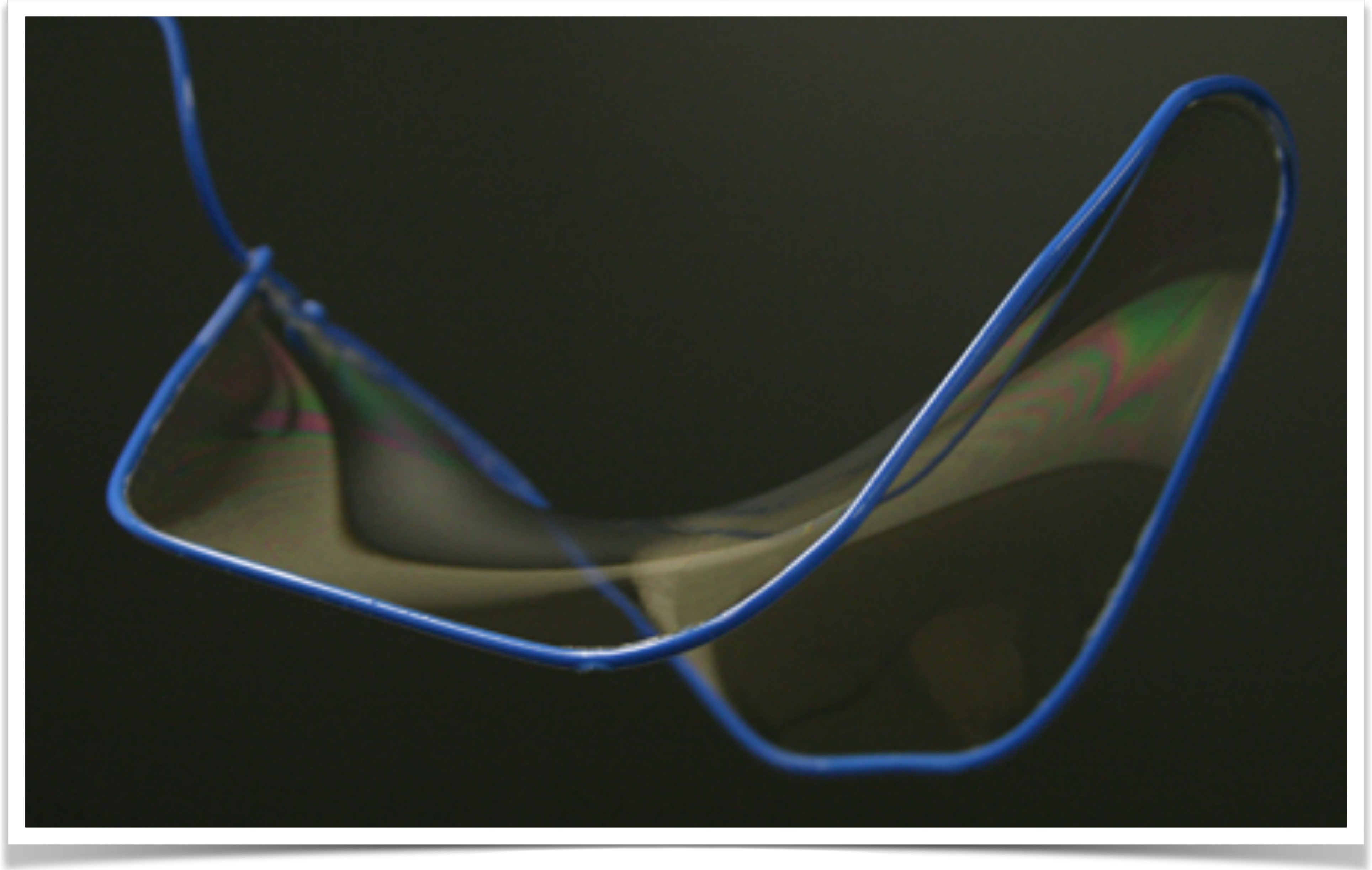

soap films

Curvature of Surfaces

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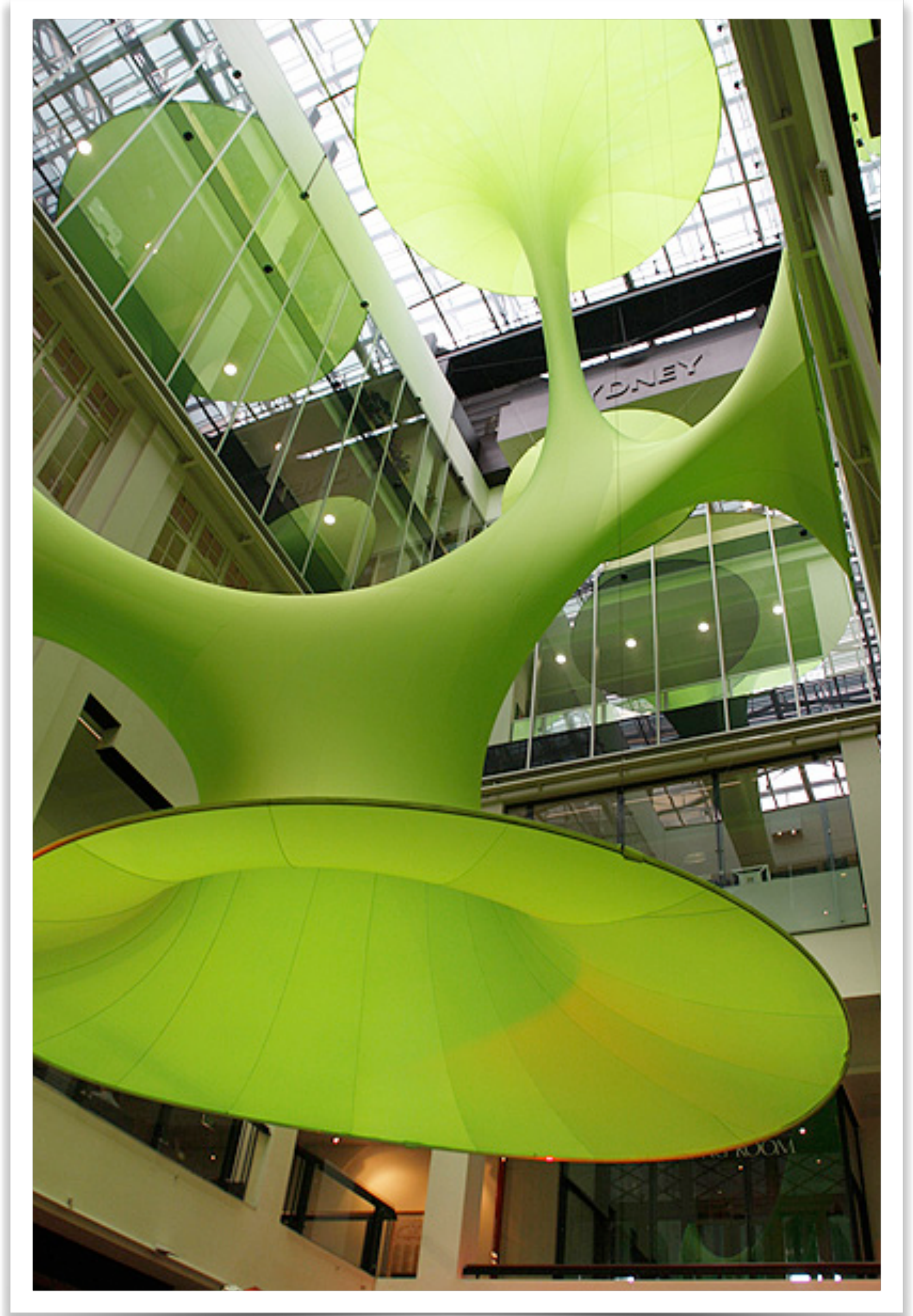
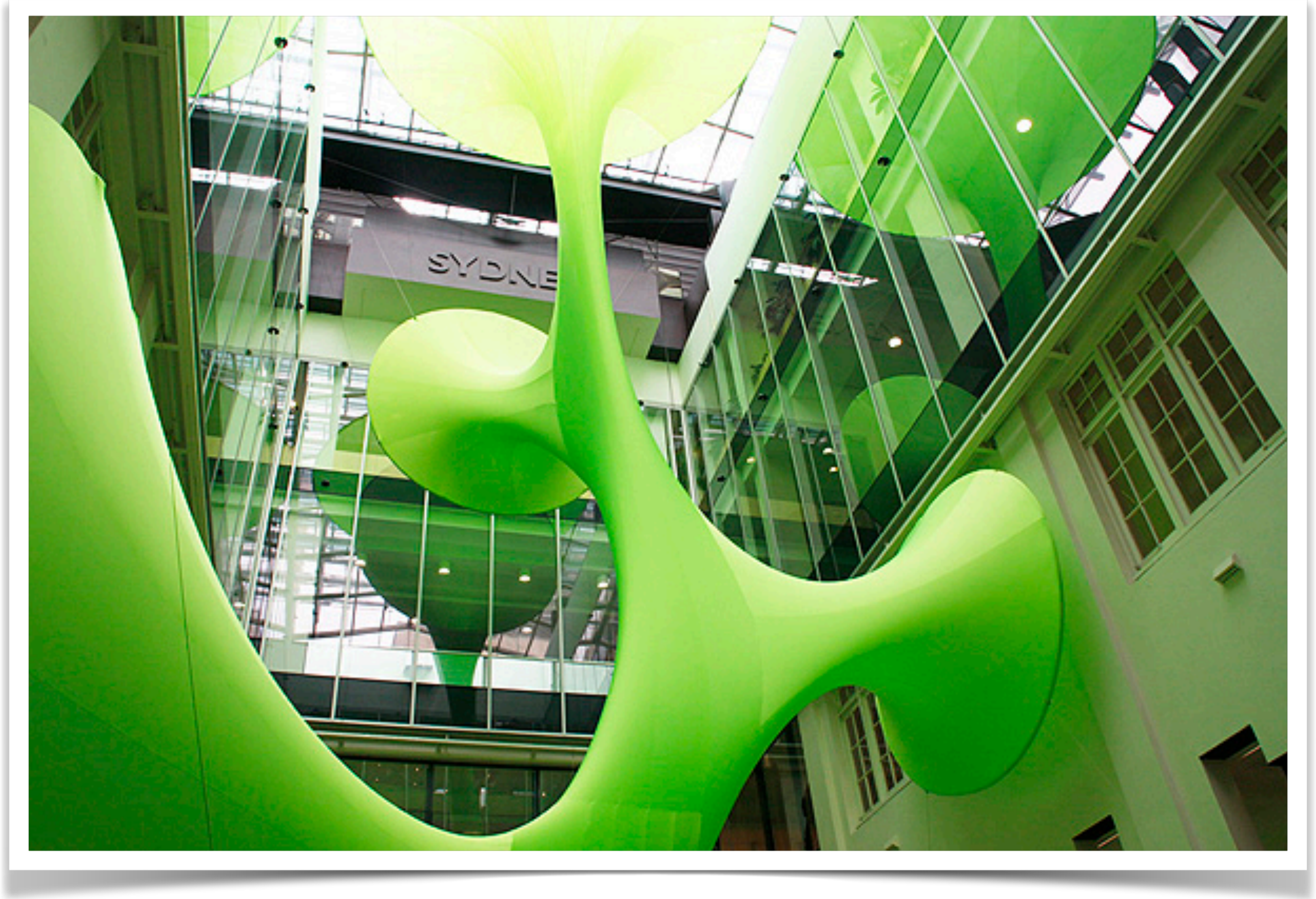
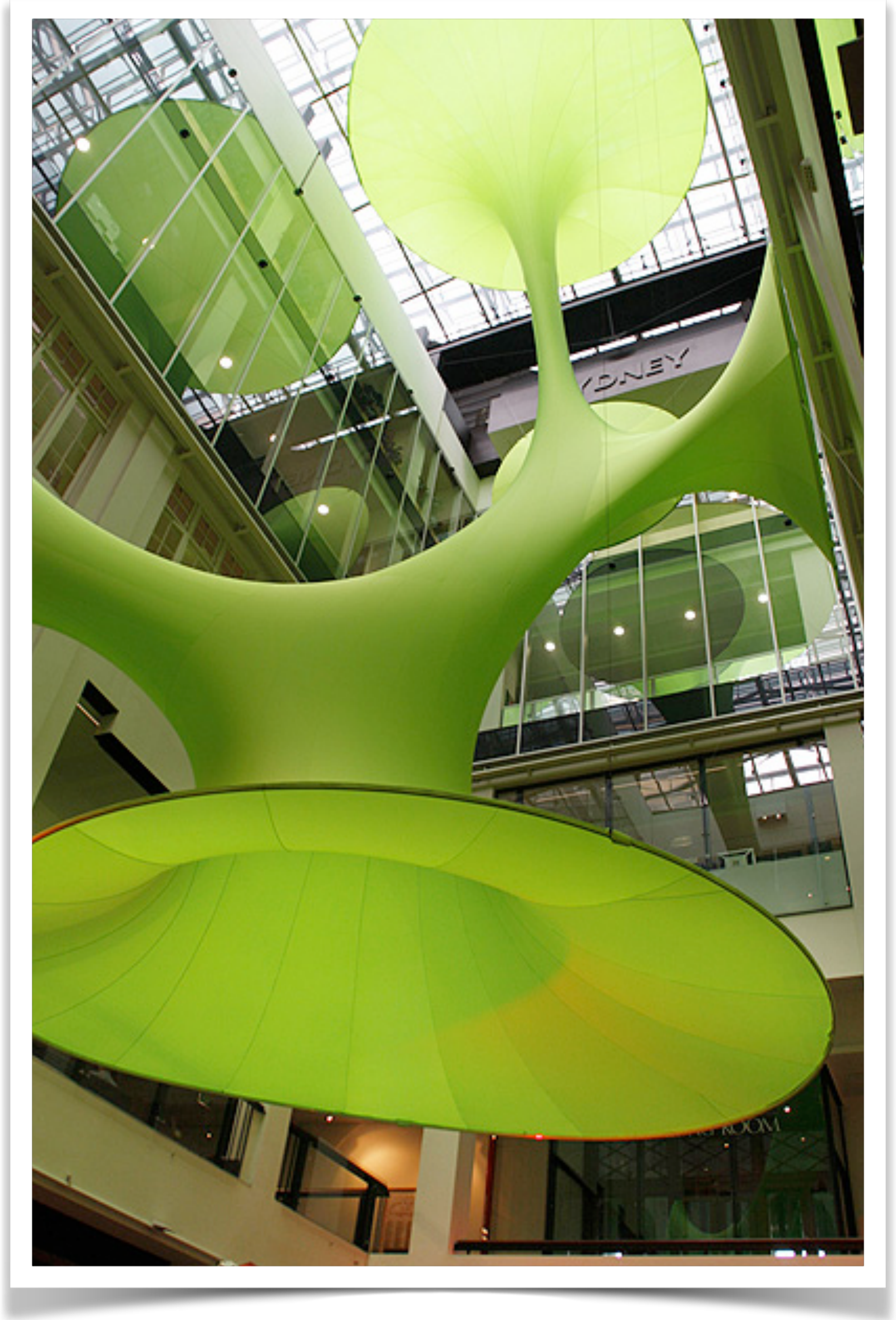


Green Void, Sydney
Architects: LAVA

Curvature of Surfaces

- Mean curvature $$H = \frac{\kappa_1 + \kappa_2}{2}$$
  - $H = 0$ everywhere → minimal surface





Green Void, Sydney
Architects: LAVA

Curvature of Surfaces

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Disney Concert Hall, L.A.
Architects: Gehry Partners

Timber Fabric
IBOIS, EPFL

Intrinsic Geometry

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- Properties of the surface that only depend on the first fundamental form

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  - Gaussian curvature (Theorema Egregium)

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  - angles
  - Gaussian curvature (Theorema Egregium)

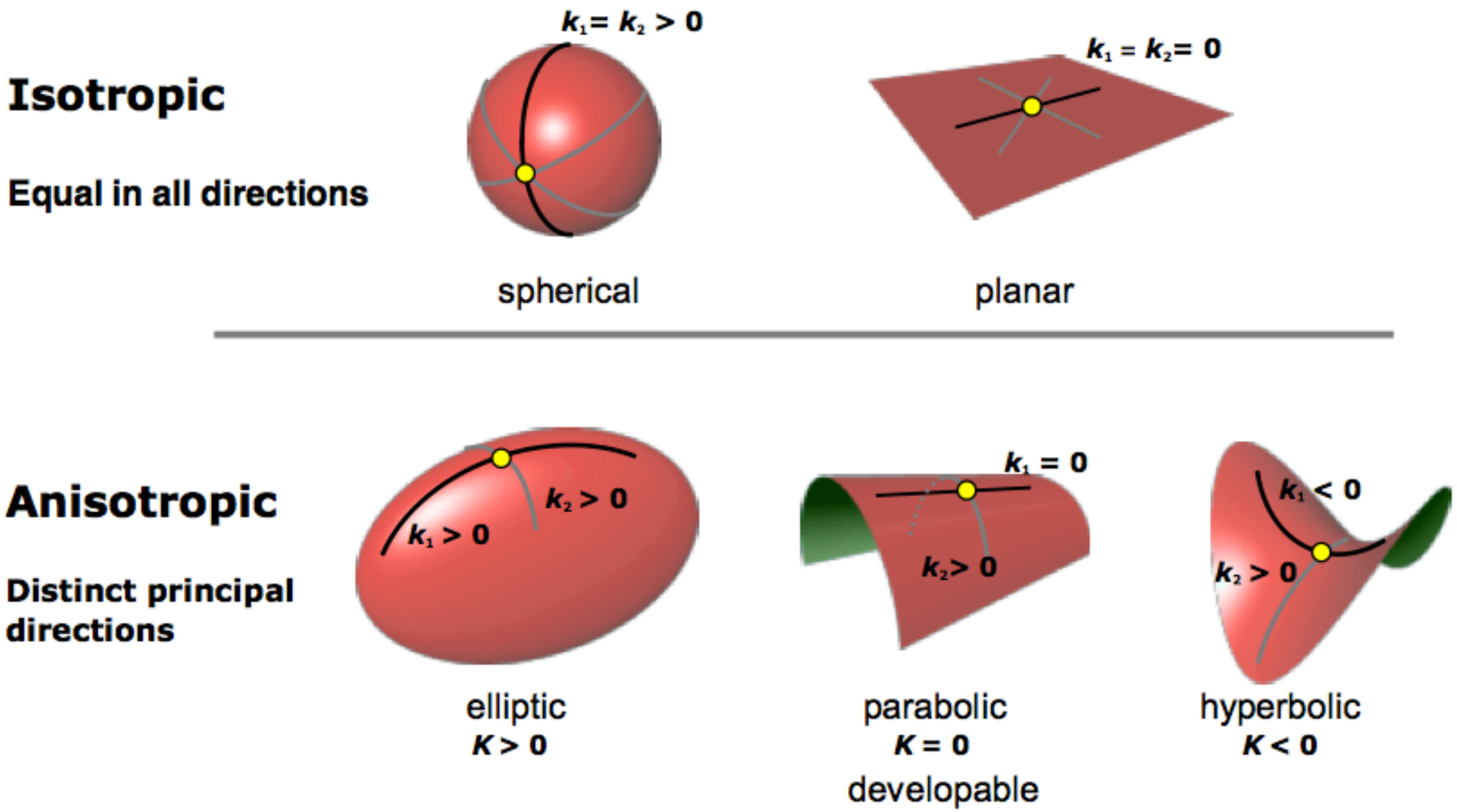
$$K = \lim_{r \to 0} \frac{6\pi r - 3C(r)}{\pi r^3}$$

Classification

- A point $\mathbf{x}$ on the surface is called
  - *elliptic*, if $K > 0$
  - *hyperbolic*, if $K < 0$
  - *parabolic*, if $K = 0$
  - *umbilic*, if $\kappa_1 = \kappa_2$

Classification

- A point $\mathbf{x}$ on the surface is called



Gauss-Bonnet Theorem

- For **any** closed manifold surface with Euler characteristic $\chi = 2\text{-}2g$

$$\int K = 2\pi\chi$$

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$$\int K = 2\pi\chi$$

$\int K($  $) = \int K($  $) = \int K($  $) = 4\pi$

Gauss-Bonnet Theorem

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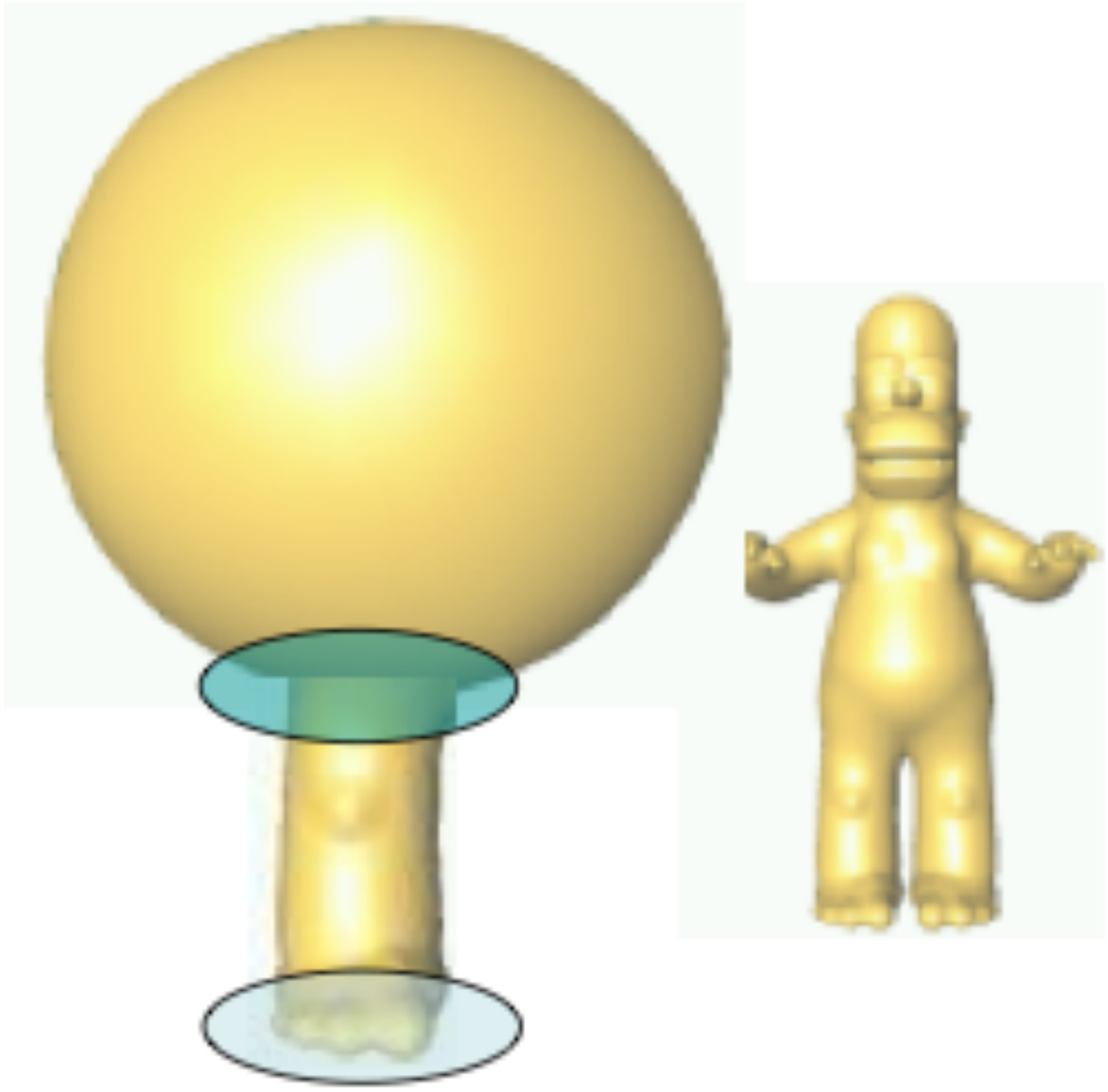

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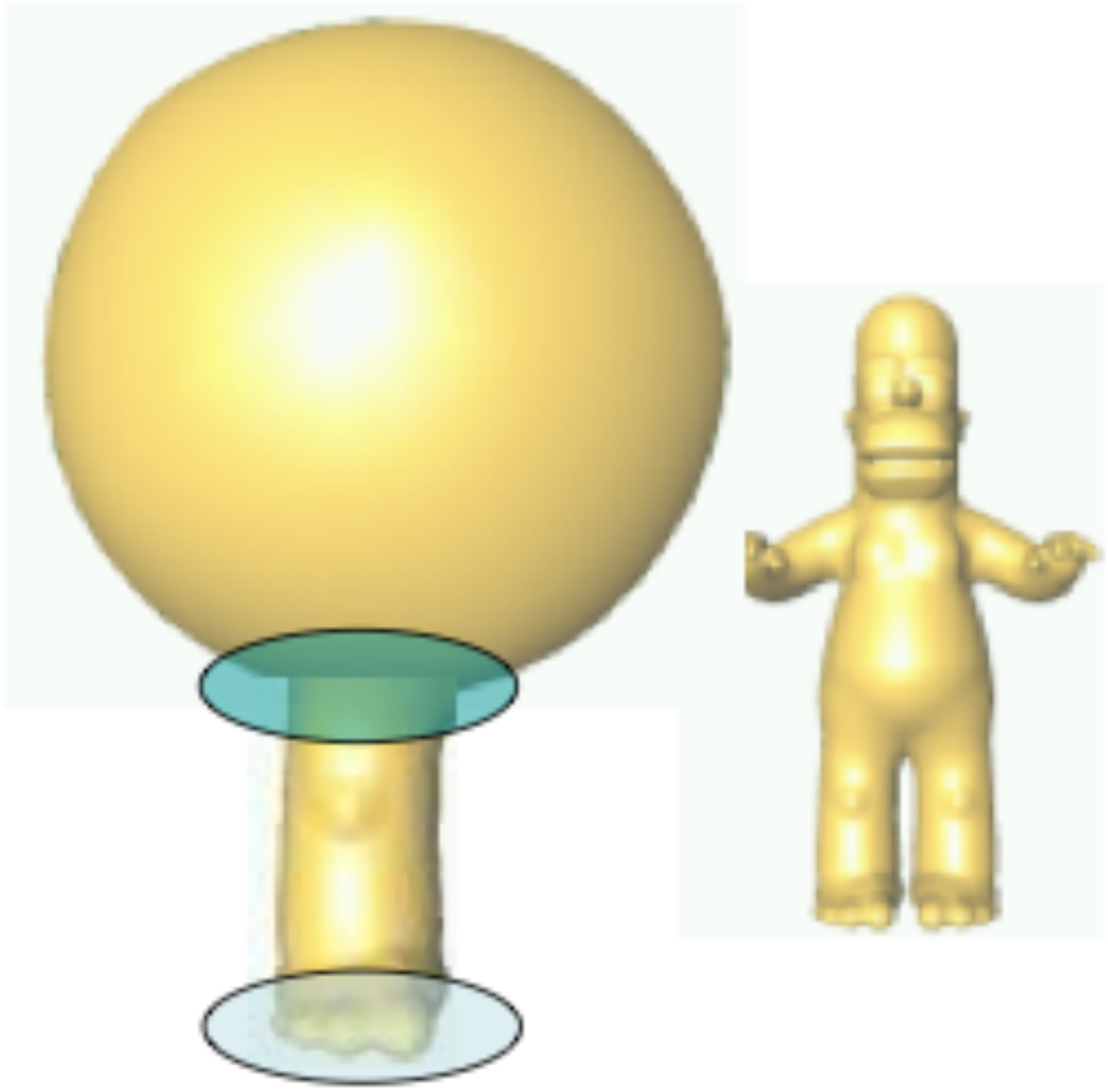


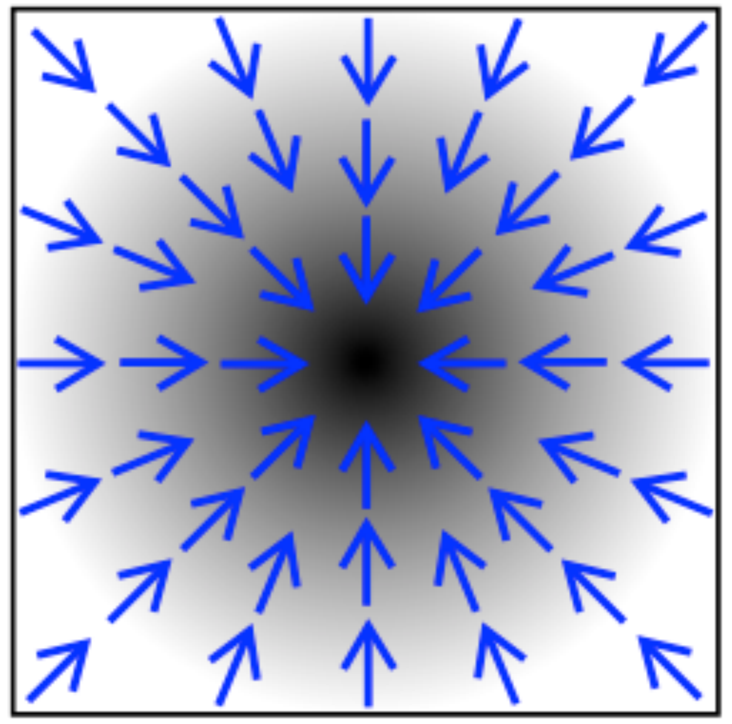
  – when sphere is deformed new positive and negative curvature cancel out!

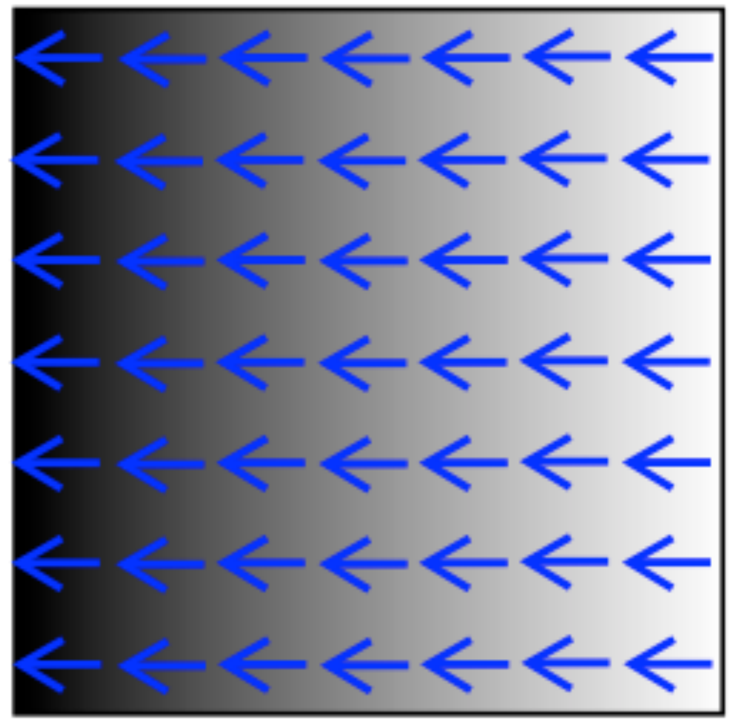
Differential Operators

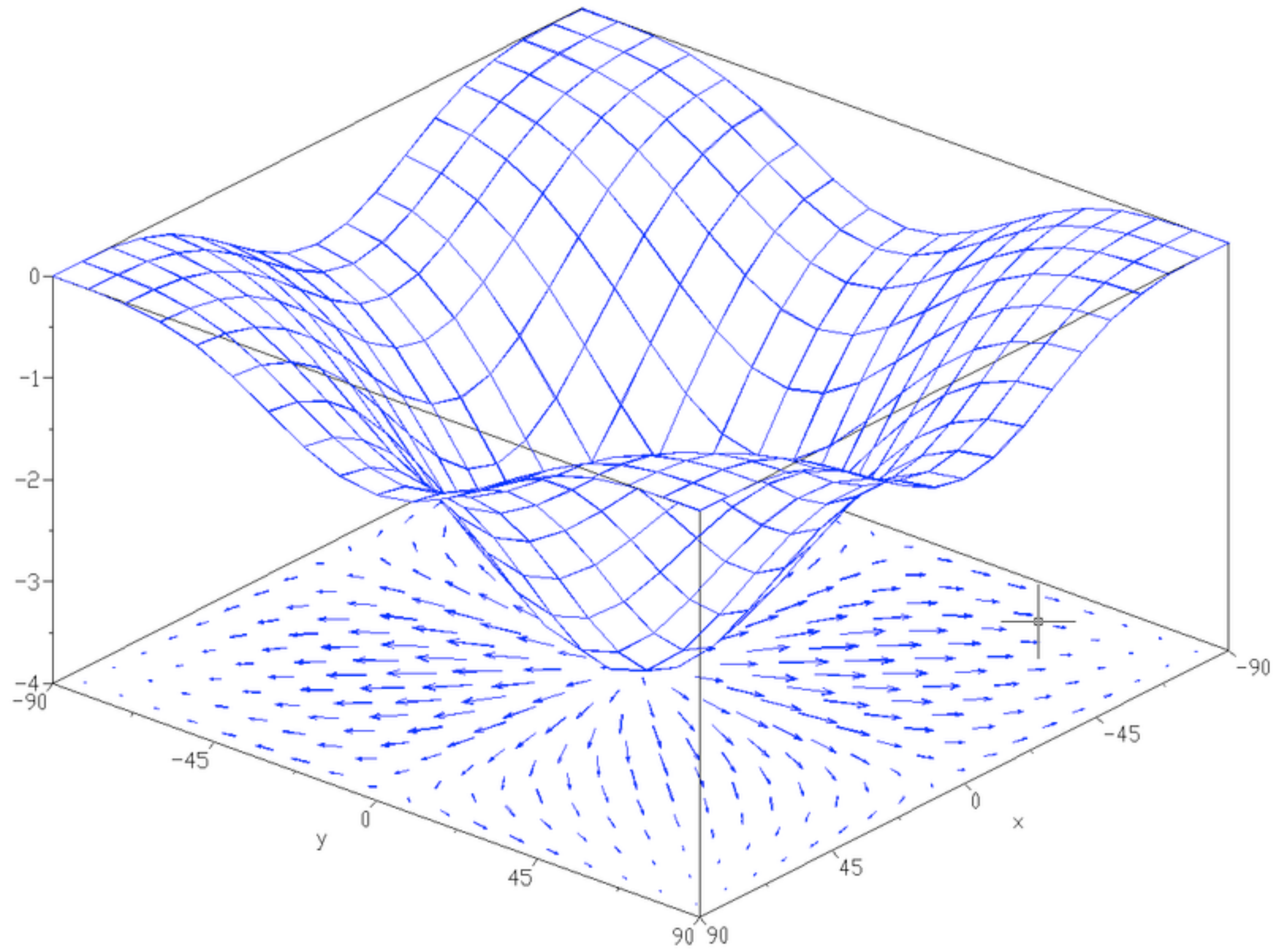
- Gradient

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

  - points in the direction of steepest ascent







Differential Operators

- Divergence

$$\mathrm{div}F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$

  - volume density of outward flux of vector field
  - magnitude of source or sink at given point
  - Example: Incompressible fluid
    - velocity field is divergence-free

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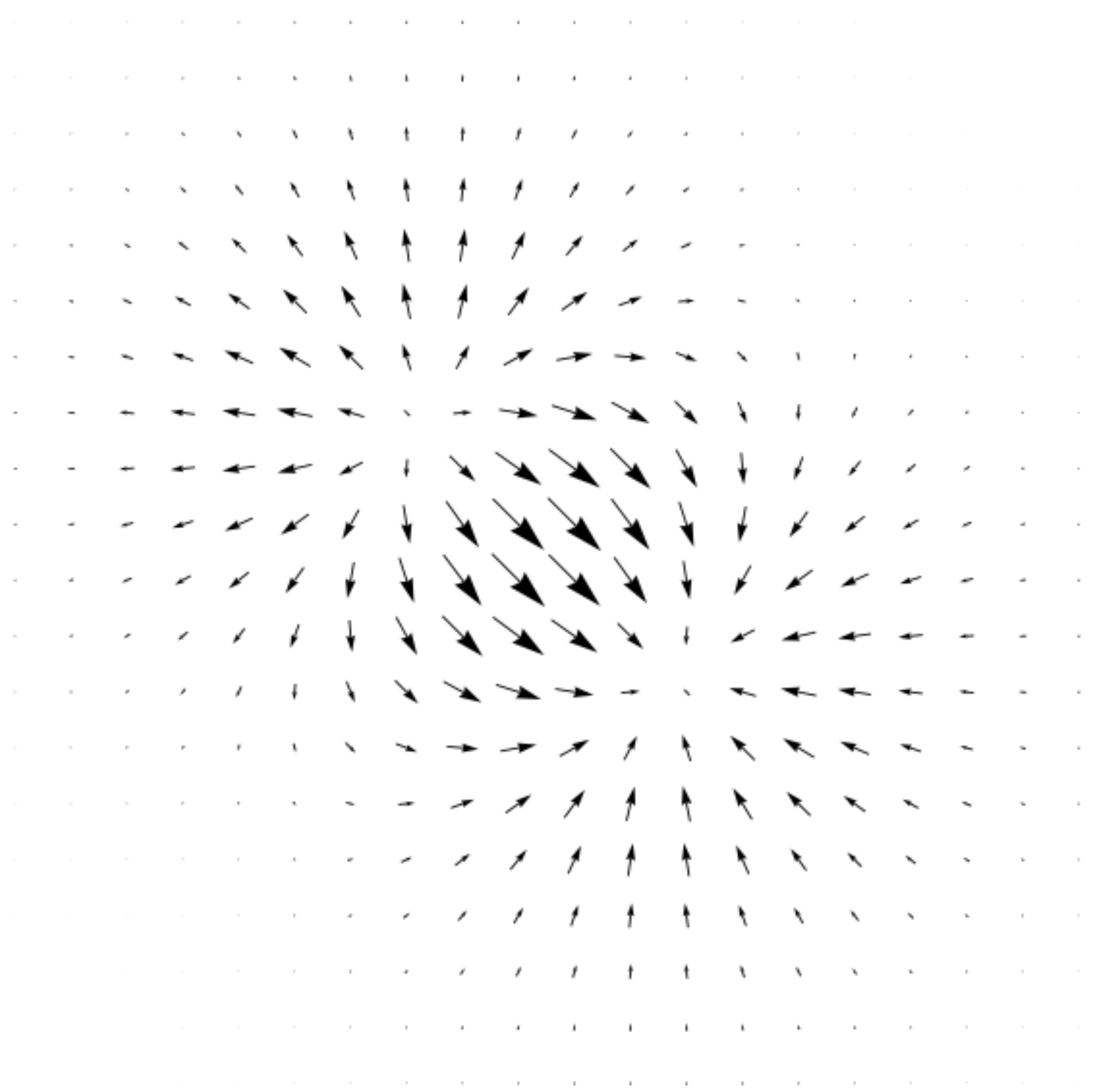

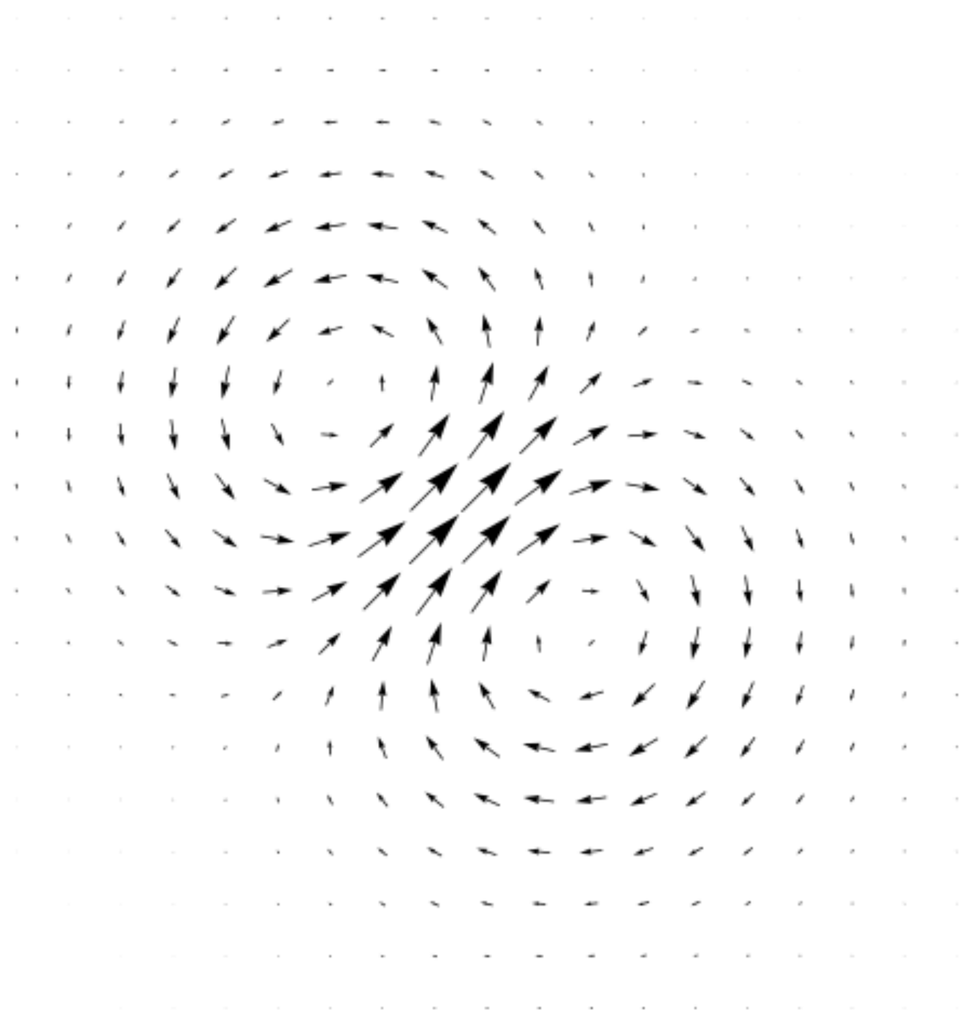

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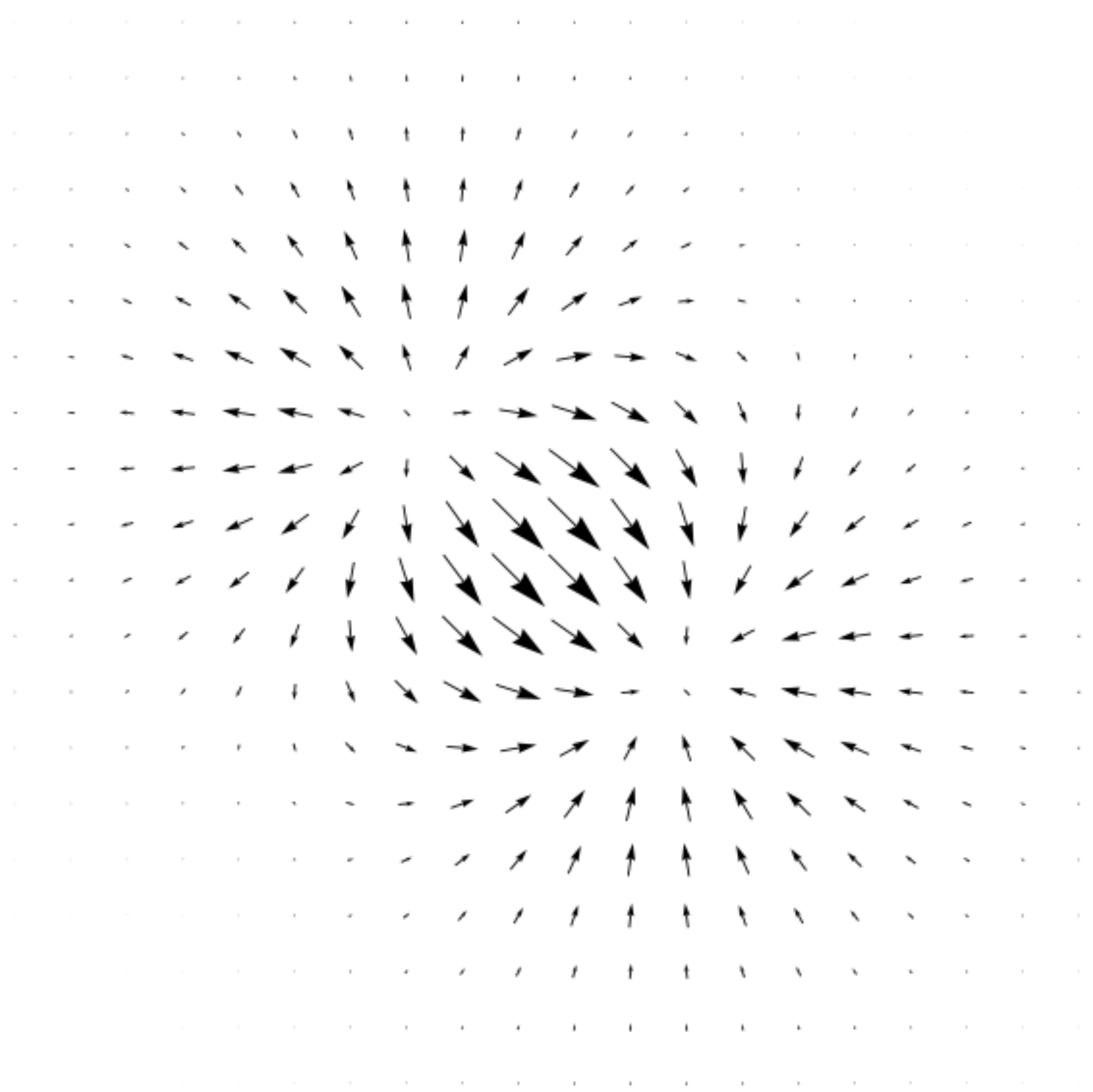
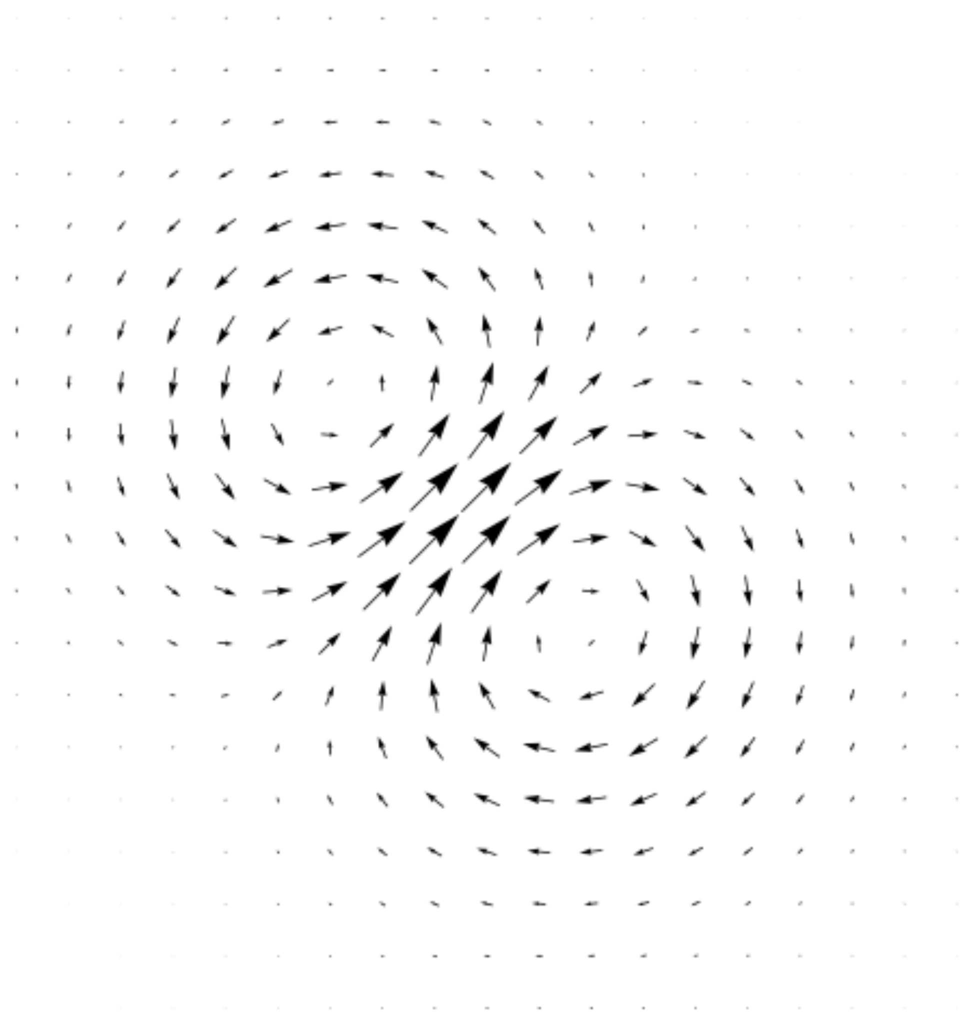


high divergence



low divergence

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