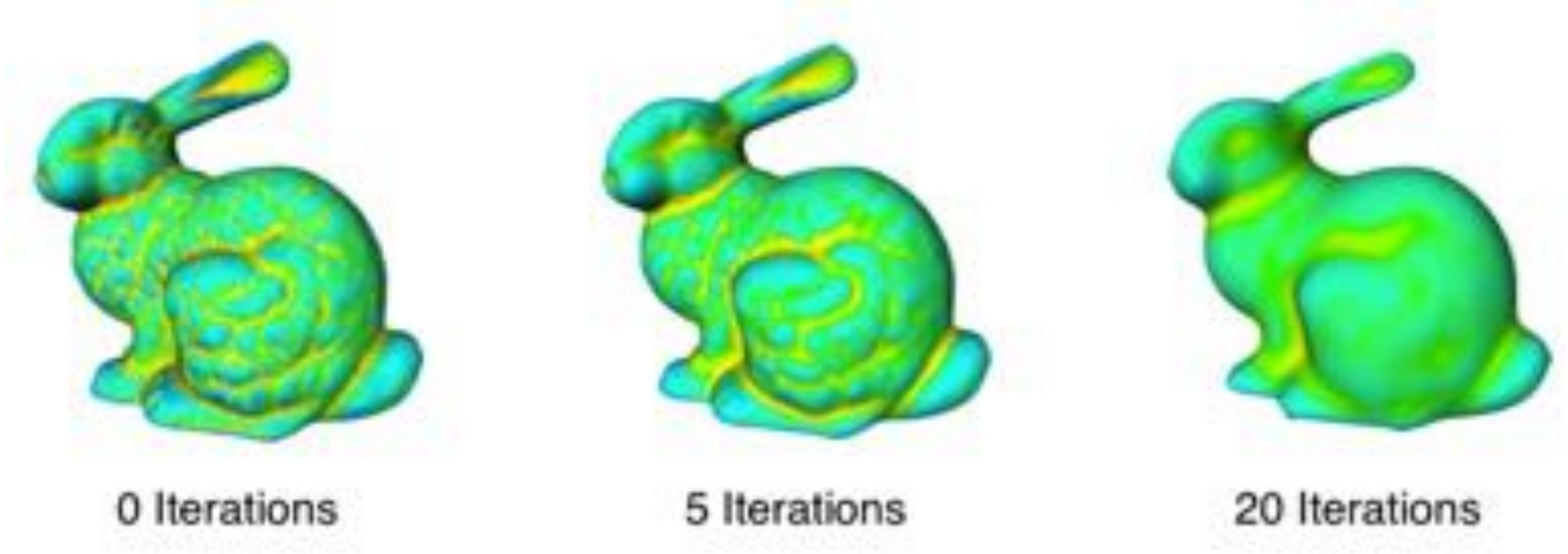
Mesh Smoothing



Outline

- Differential Geometry
- **Discrete Differential Geometry**
- Mesh Quality Measures

Differential Geometry: Surfaces

- Normals
- First/Second Fundamental forms
- Normal curvature
- Principal curvatures
- mean/Gauss Curvatures

Gauss-Bonnet Theorem

- For **any** closed manifold surface with Euler characteristic $\chi = 2 - 2g$

$$\int_{\Omega \in S} K(u,v)\,du\,dv = 2\pi\chi$$

$\int K($  $) = \int K($  $) = \int K($  $) = 4\pi$

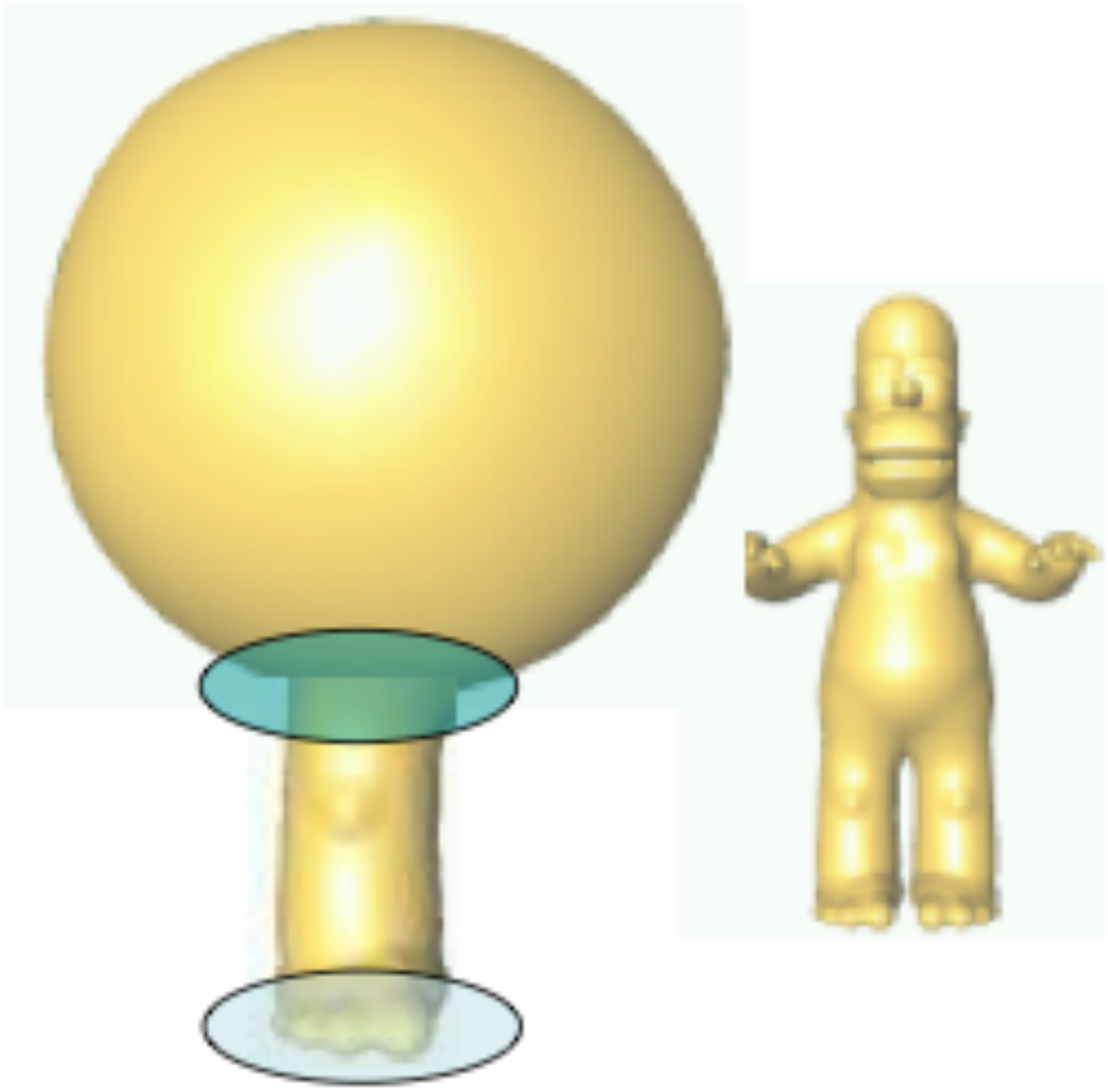
Gauss-Bonnet Theorem

- Sphere

$$\kappa_1 = \kappa_2 = 1/r$$

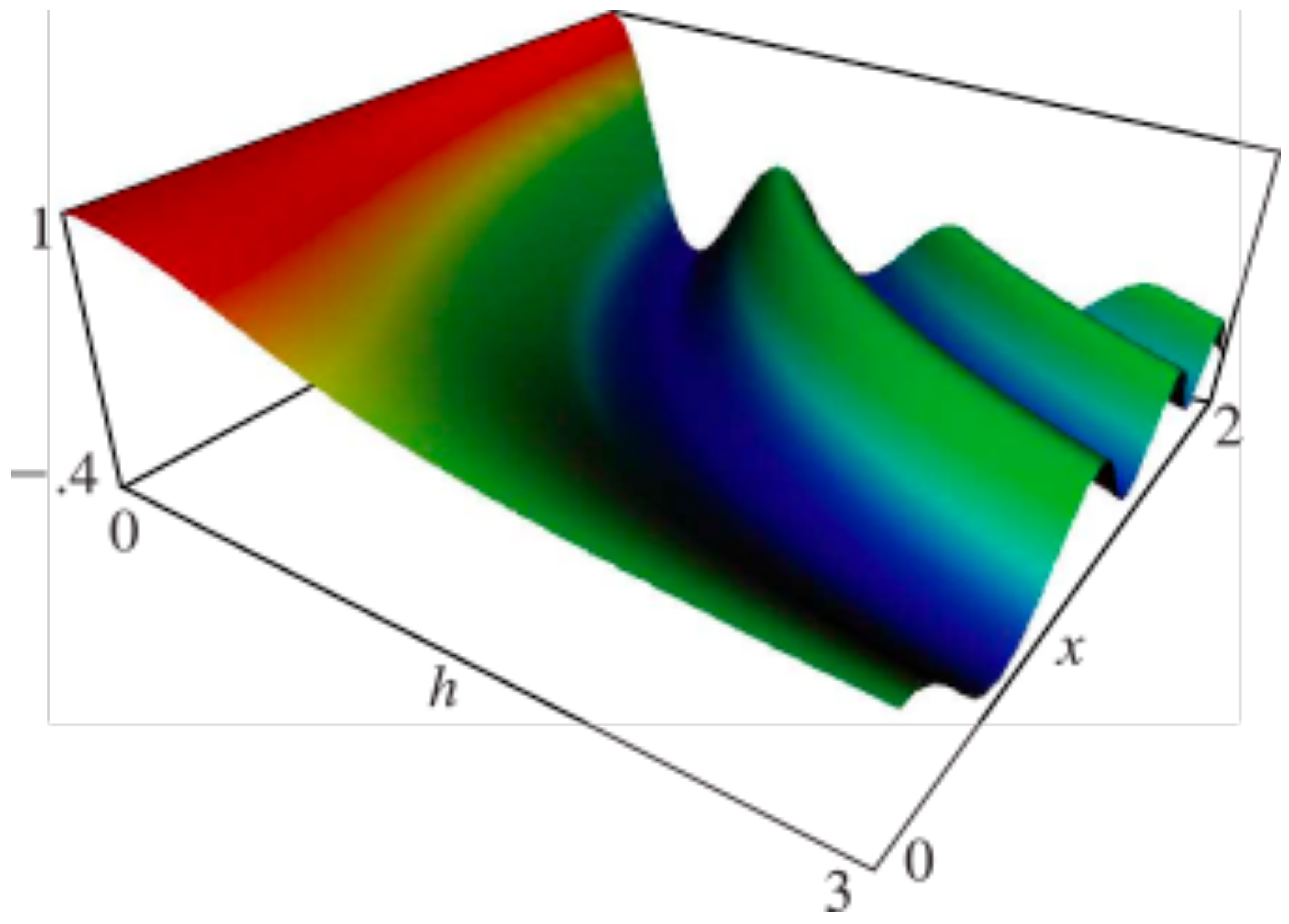
$$K = \kappa_1 \kappa_2 = 1/r^2$$

$$\int K = 4\pi r^2 \cdot \frac{1}{r^2} = 4\pi$$

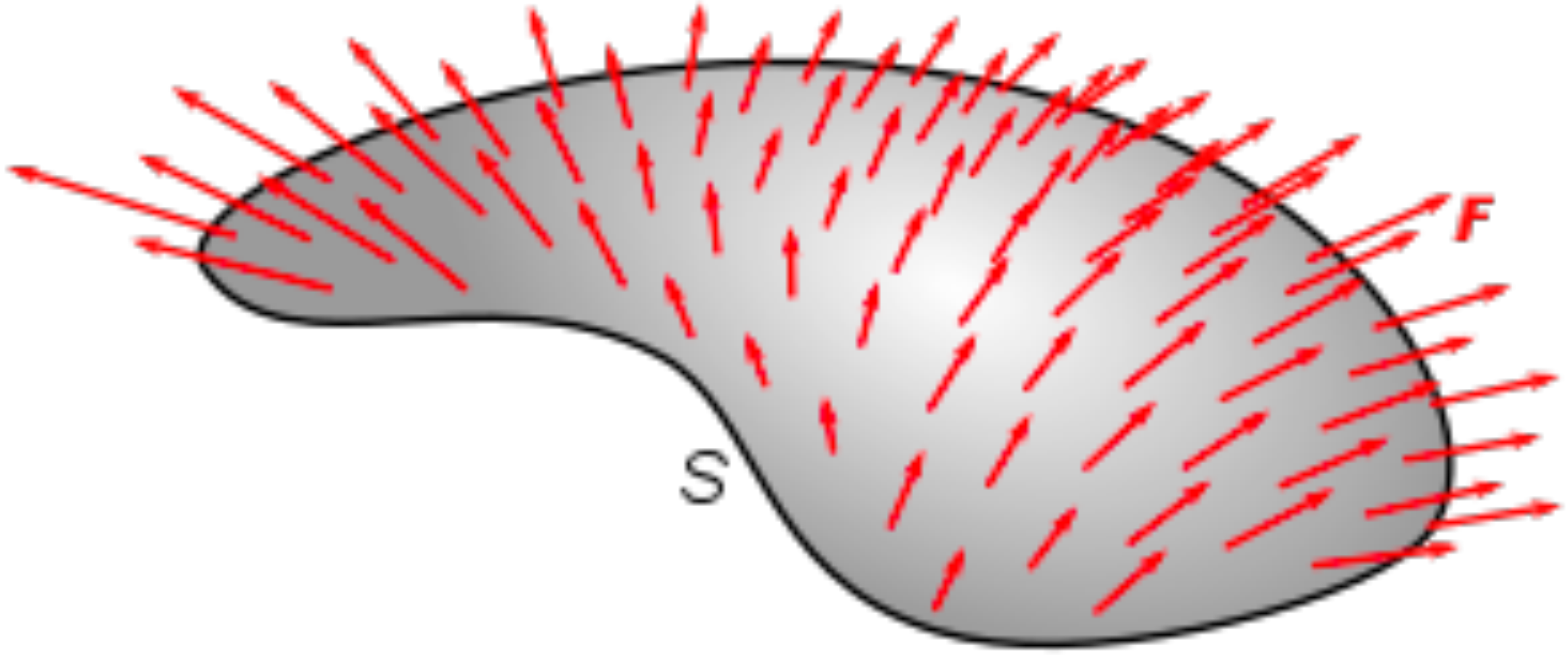


  – when sphere is deformed new positive and negative curvature cancel out!

Functions on Surfaces



scalar values on surfaces



vector values on surfaces

Differential Operators

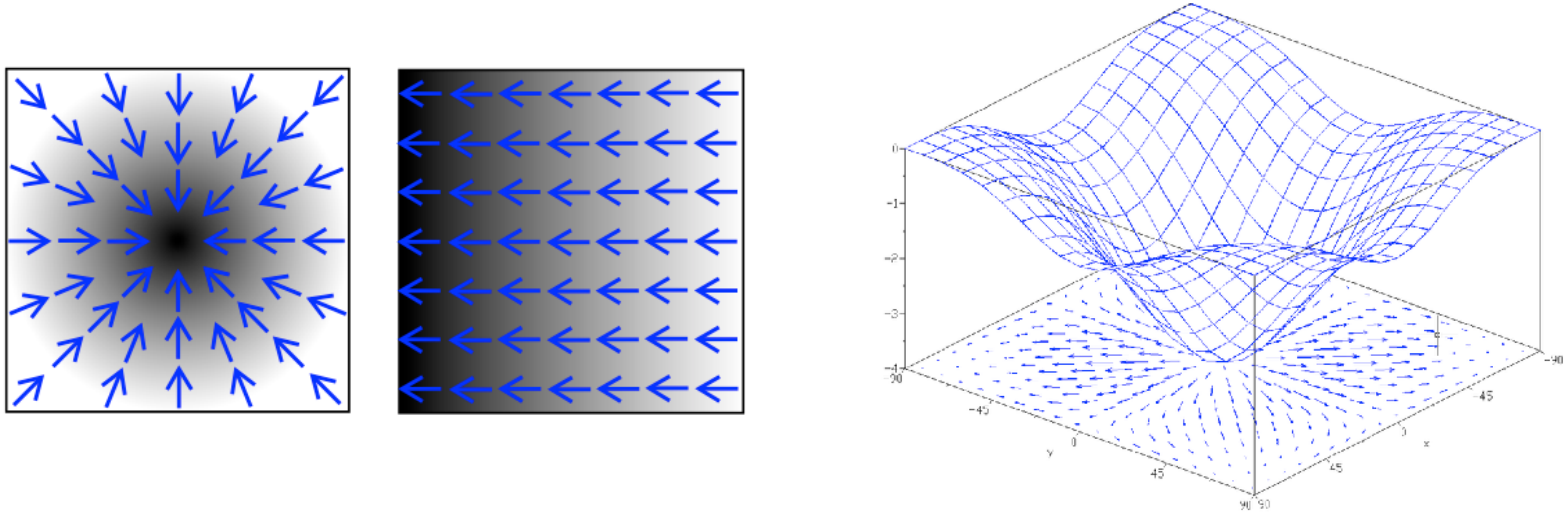
- Gradient (vector)

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

  – points in the direction of steepest ascent

用来measure 向量场的源头||汇的大小,

$$\text{div } g := < \left(\frac{\partial}{\partial x_1}, \ldots \right), g > = \nabla \cdot g$$



Differential Operators

$$\text{div}\; g := <\left(\frac{\partial}{\partial x_1}, \ldots\right), g> = \nabla \cdot g$$

div之定义

i.e. 速度在空间中分布

$$\text{div}\; \mathbf{G} := <\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), (G_x, G_y, G_z)> = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}$$

描述某点附近速度是否闭合

空间变化

Differential Operators

- Divergence

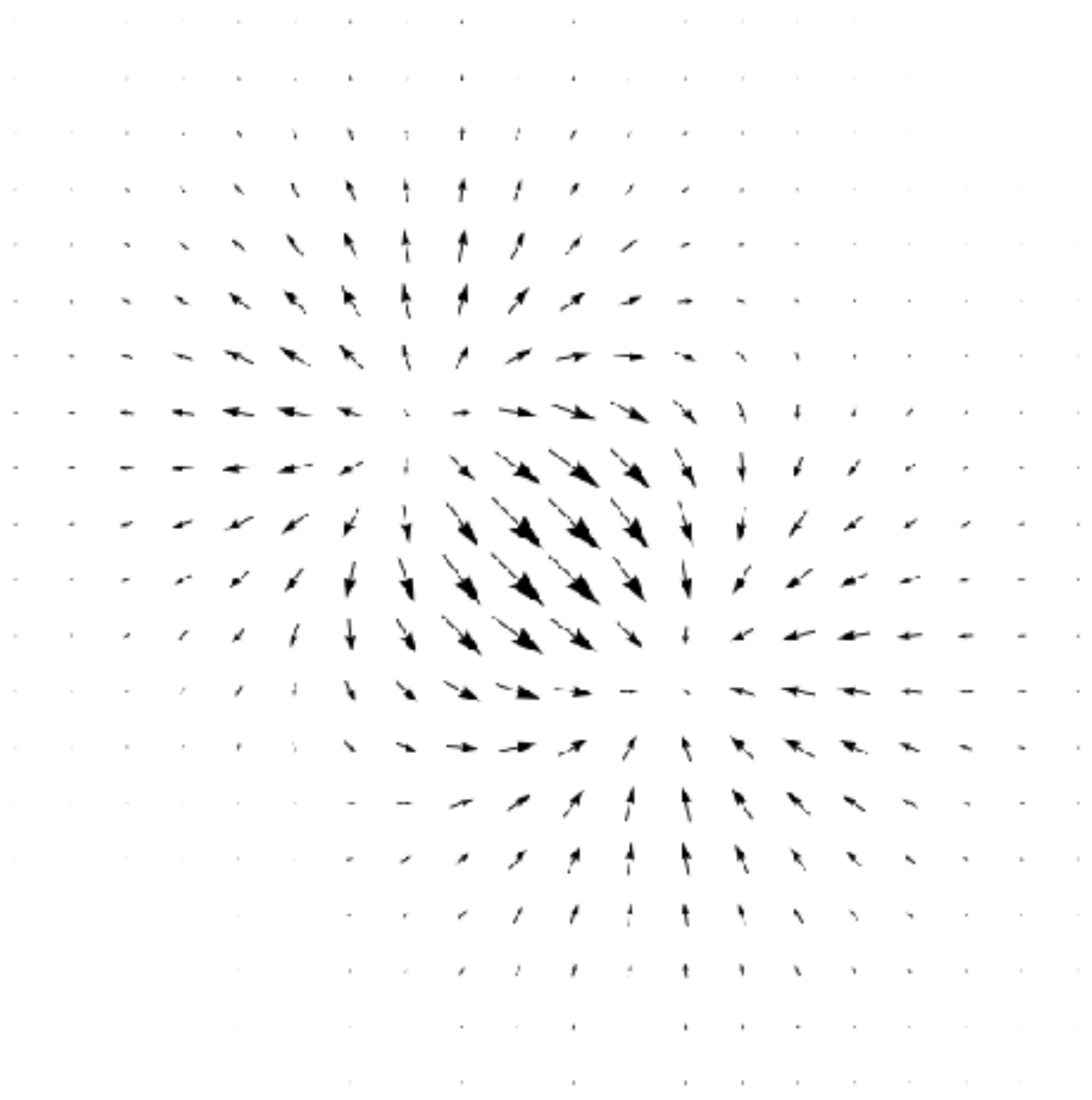
$$\mathrm{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$

  – volume density of outward flux of vector field
  – magnitude of source or sink at given point
  – Example: Incompressible fluid
    - velocity field is divergence-free

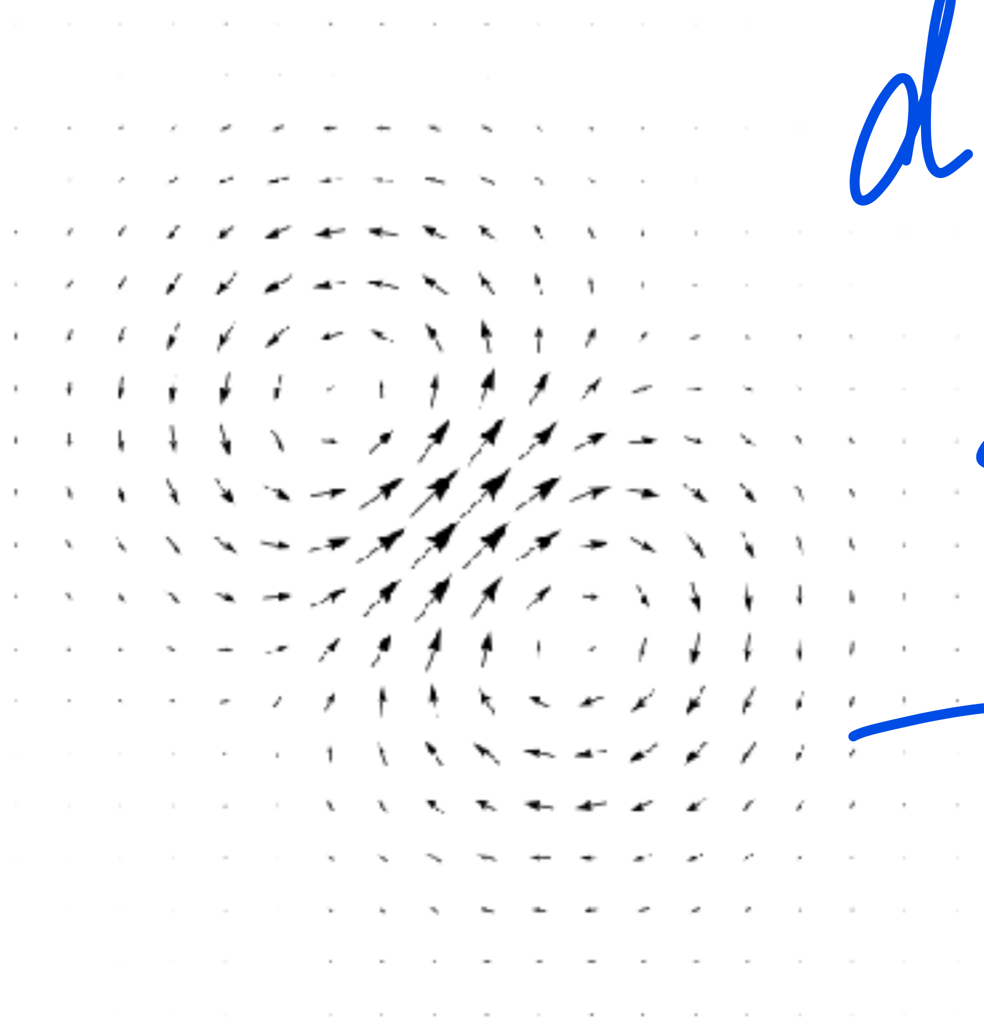
Differential Operators

- Divergence

$$\mathrm{div} F = \nabla \cdot F := \frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}$$



high divergence



low divergence

div $\Leftrightarrow$ $\nabla \cdot F$ $\Leftrightarrow$ sum

sum

∇F 是 vector

$\nabla F \neq \nabla \cdot F$

sum

Laplace Operator

$$\underline{\operatorname{div}\,\nabla f}$$

vector: 所有方向的一阶导

$$\operatorname{div}\,\nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

$$\operatorname{div}\,\nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2} = \underline{\Delta f}$$

scaler

Laplace Operator

$$\Delta f = \operatorname{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

Laplace operator

gradient operator

2nd partial derivatives

function in Euclidean space

eg - 颜色、温度

divergence operator

Cartesian coordinates

Laplace-Beltrami Operator

coord

Extension of Laplace to **functions on manifolds**

$$P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Delta P = -2H\vec{n}$$

scalar vector

normal

mean curvature

Laplace-Beltrami

gradient operator

$n \times n$

$$\Delta_{\mathcal{S}} f = \operatorname{div}_{\mathcal{S}} \nabla_{\mathcal{S}} f$$

function on manifold $\mathcal{S}$

divergence operator

Laplace-Beltrami Operator

Extension of Laplace to **functions on manifolds**

Laplace-Beltrami

gradient operator

mean curvature

[scaler]

n x n

$$\Delta_{\mathcal{S}} \mathbf{x} = \operatorname{div}_{\mathcal{S}} \nabla_{\mathcal{S}} \mathbf{x} = -2H\mathbf{n}$$

n x 3

x, y, z

n x 3

$2\pi - \sum \theta_i$

coordinate function

divergence operator

surface normal

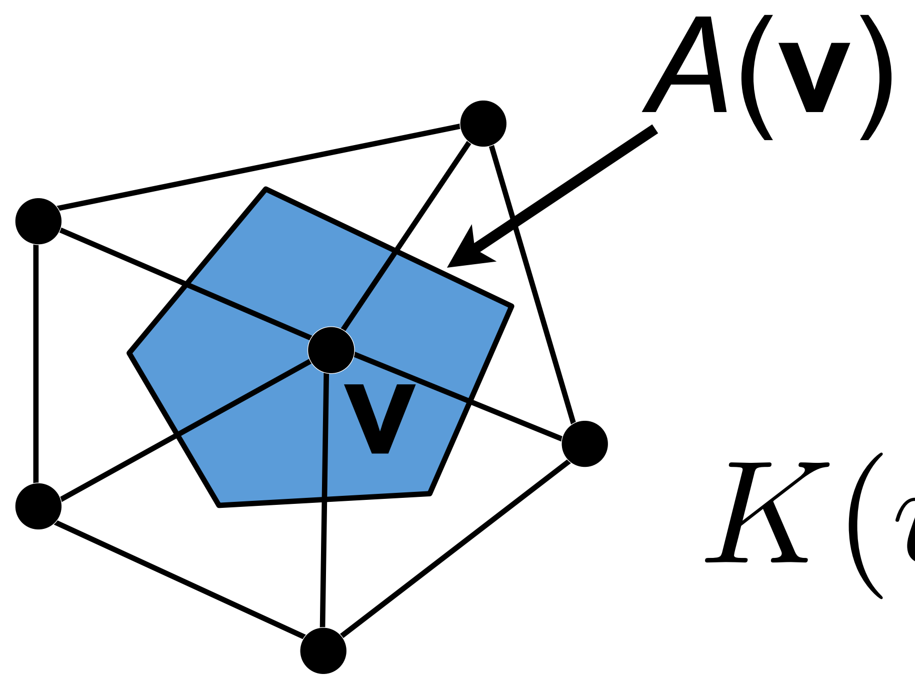
Outline

- Differential Geometry
- **Discrete Differential Geometry**
- Mesh Quality Measures

Discrete Curvatures

- How to discretize curvatures on a mesh?
  - Zero curvature within triangles
  - Infinite curvature at edges / vertices
  - Point-wise definition does not make sense

- Approximate differential properties at point $\mathbf{v}$ as average over local neighborhood $A(\mathbf{v})$
  - $\mathbf{v}$ is a mesh vertex
  - $A(\mathbf{v})$ within one-ring neighborhood



$$K(v) \approx \frac{1}{A(v)} \int_{A(v)} K(\mathbf{x}) \, \mathrm{d}A$$

Area Discretizations

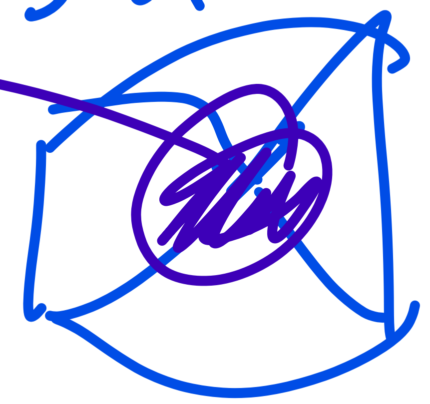
Voronoi Cells

将空间划分为多个region

对于 找这个region

3d

边

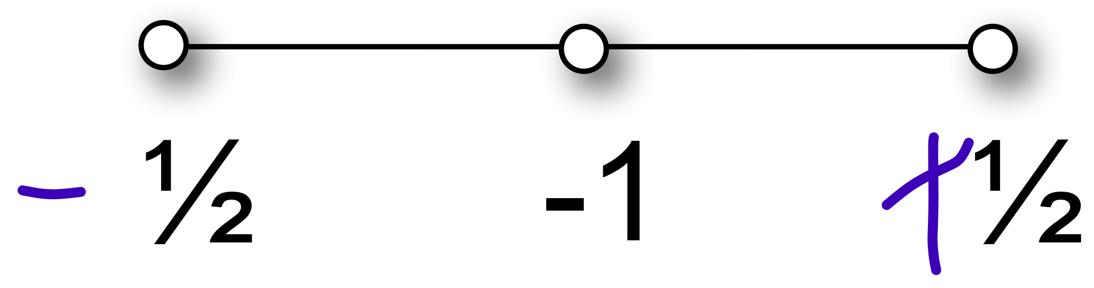


Discrete Curvatures

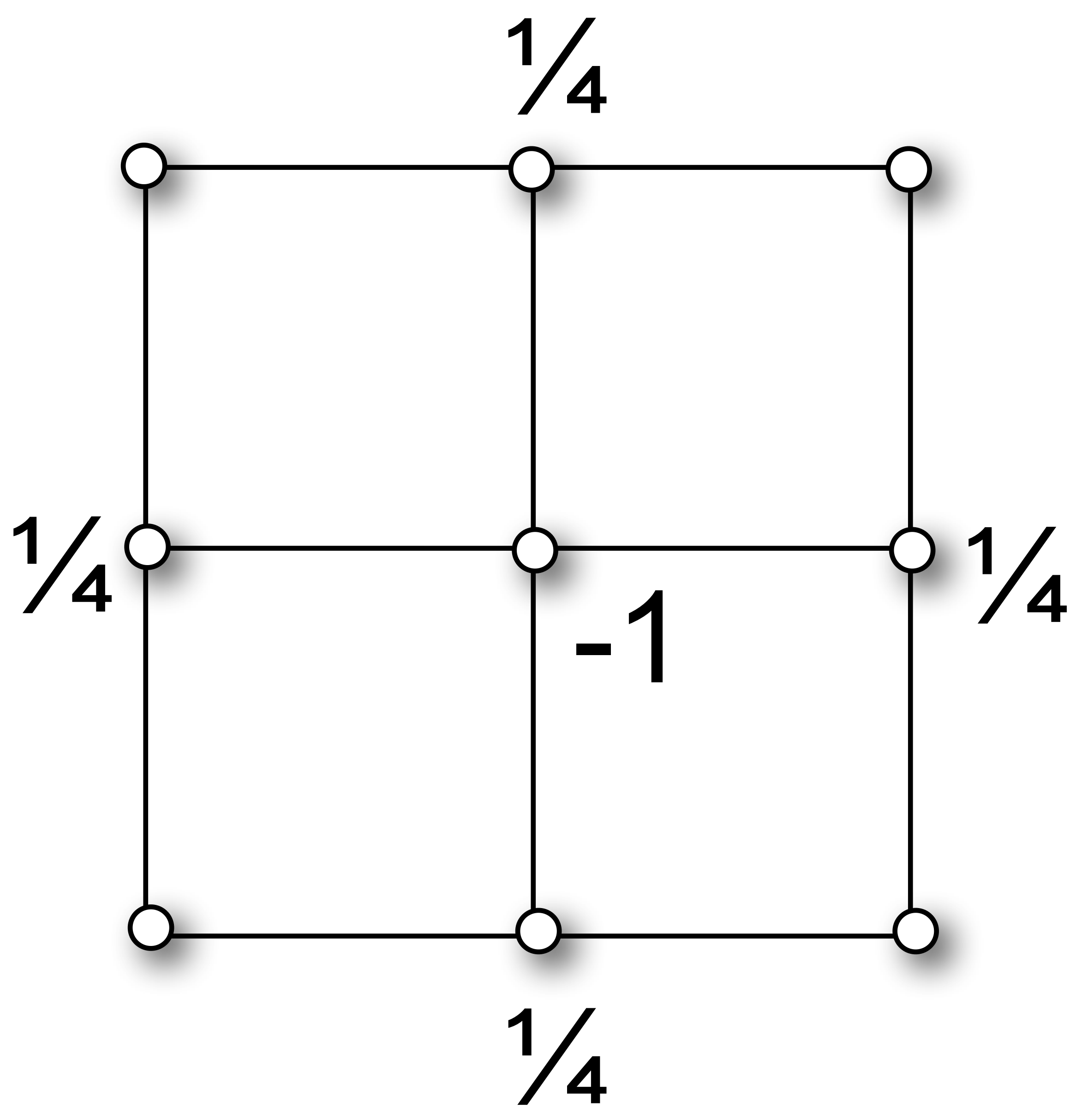
- Which curvatures to discretize?
  - Discretize Laplace-Beltrami operator
  - Laplace-Beltrami gives us mean curvature H
  - Discretize Gaussian curvature K

Laplace Operator on Meshes?

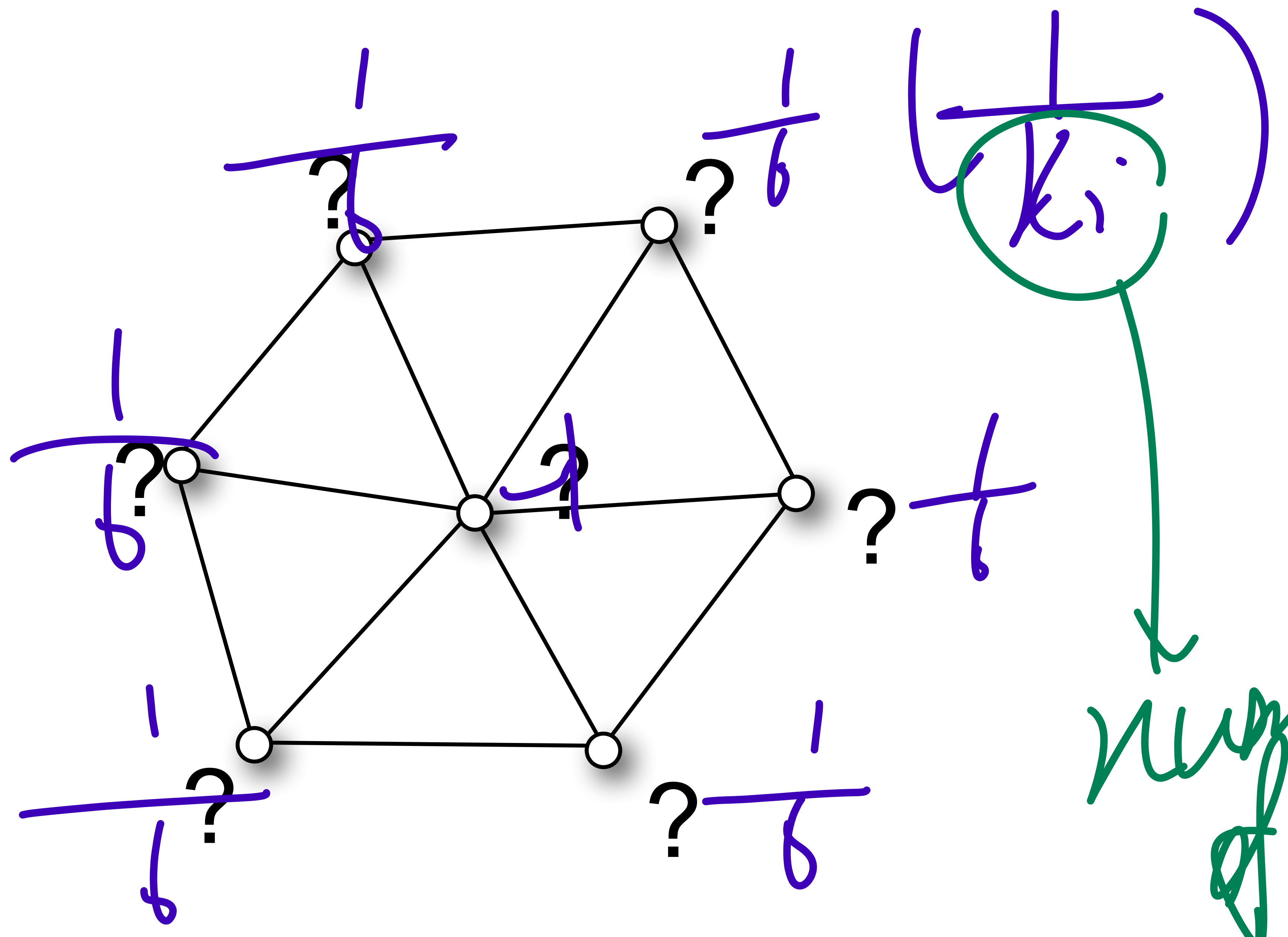
- Extend finite differences to meshes?
  - What weights per vertex / edge?



1D grid



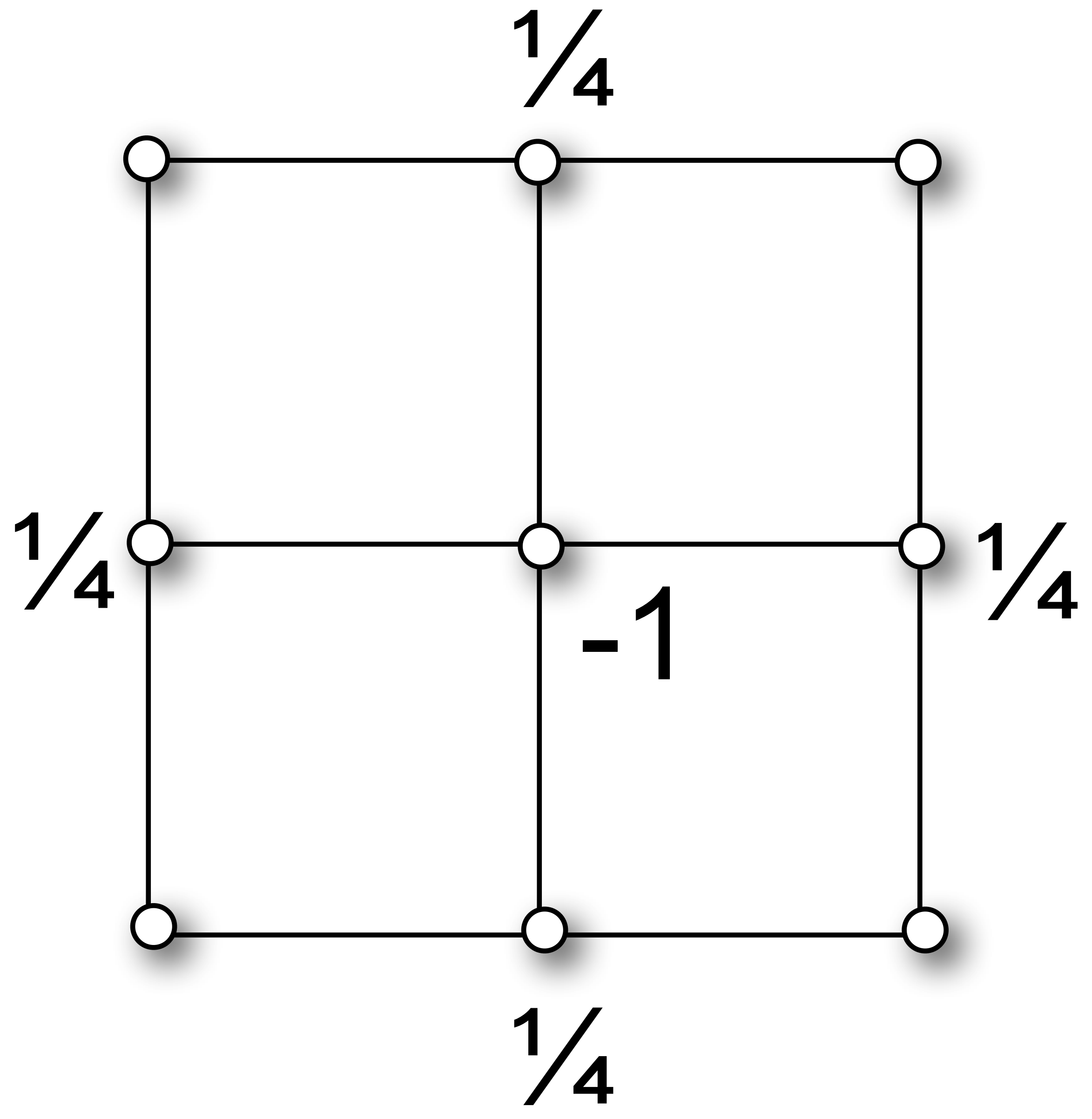
2D grid



2D/3D mesh

number of neighbours

Matrix Version of Neighborhood



2D grid

Uniform Laplacian Matrix

neighbour 均匀 & 固定权重

- Uniform discretization

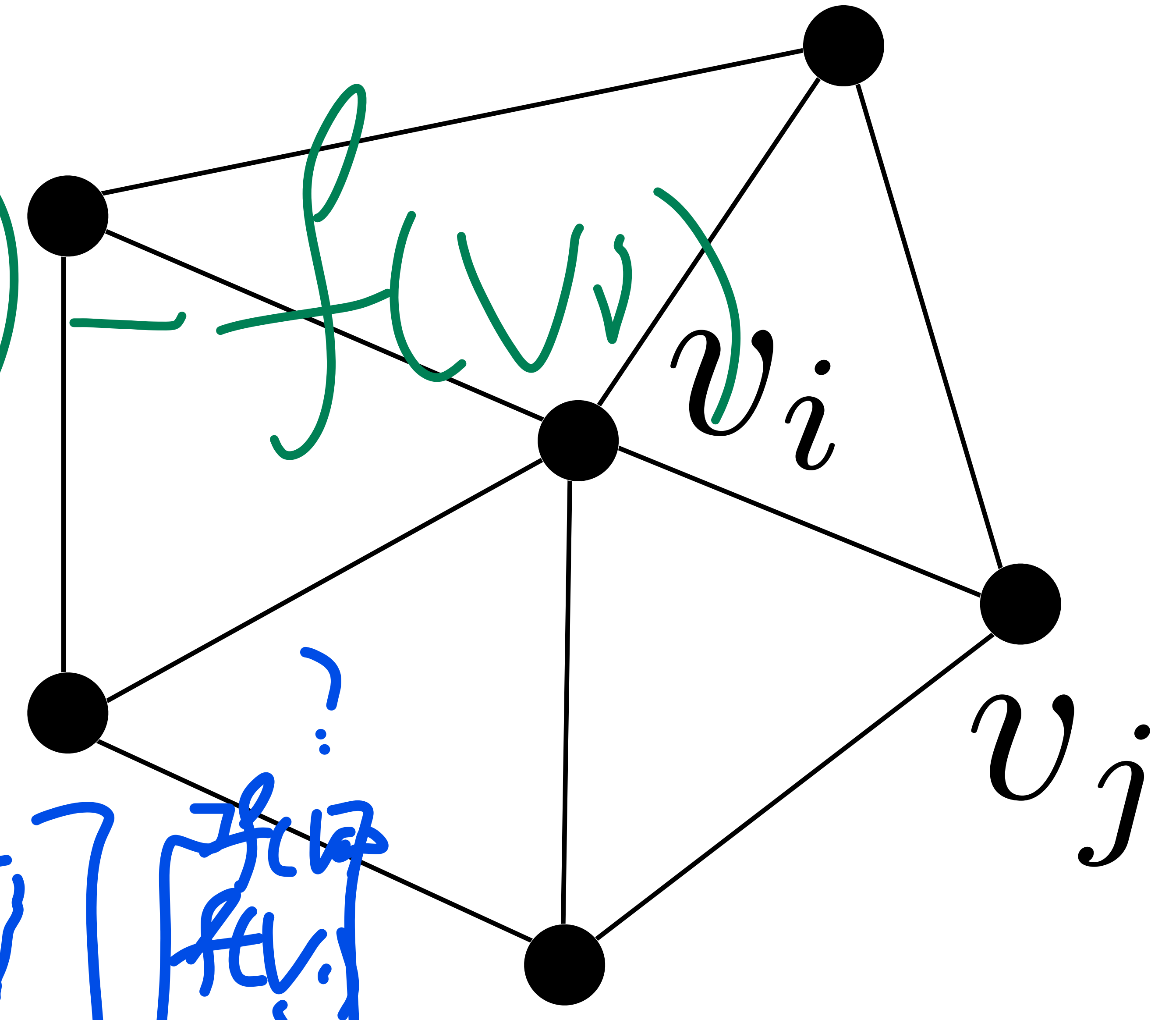
neighbour

$$\Delta_{\text{uni}} f(v_i) := \frac{1}{|\mathcal{N}_1(v_i)|} \sum_{v_j \in \mathcal{N}_1(v_i)} (f(v_j) - f(v_i))$$

随着平滑 迭代↓

$$= \left(\frac{1}{|\ |}\sum_j f(v_j)\right) - f(v_i)$$

- Properties
  - depends only on connectivity
  - simple and efficient
  - bad approximation for irregular triangulations
    - can give non-zero *H* for planar meshes
    - tangential drift for mesh smoothing



$\Delta U \rightarrow = [\ ]\,[f(v_i)]$ $N\times N$ $N\times 1$

Uniform Laplacian Matrix

for CW2, construct matrix for the boundary

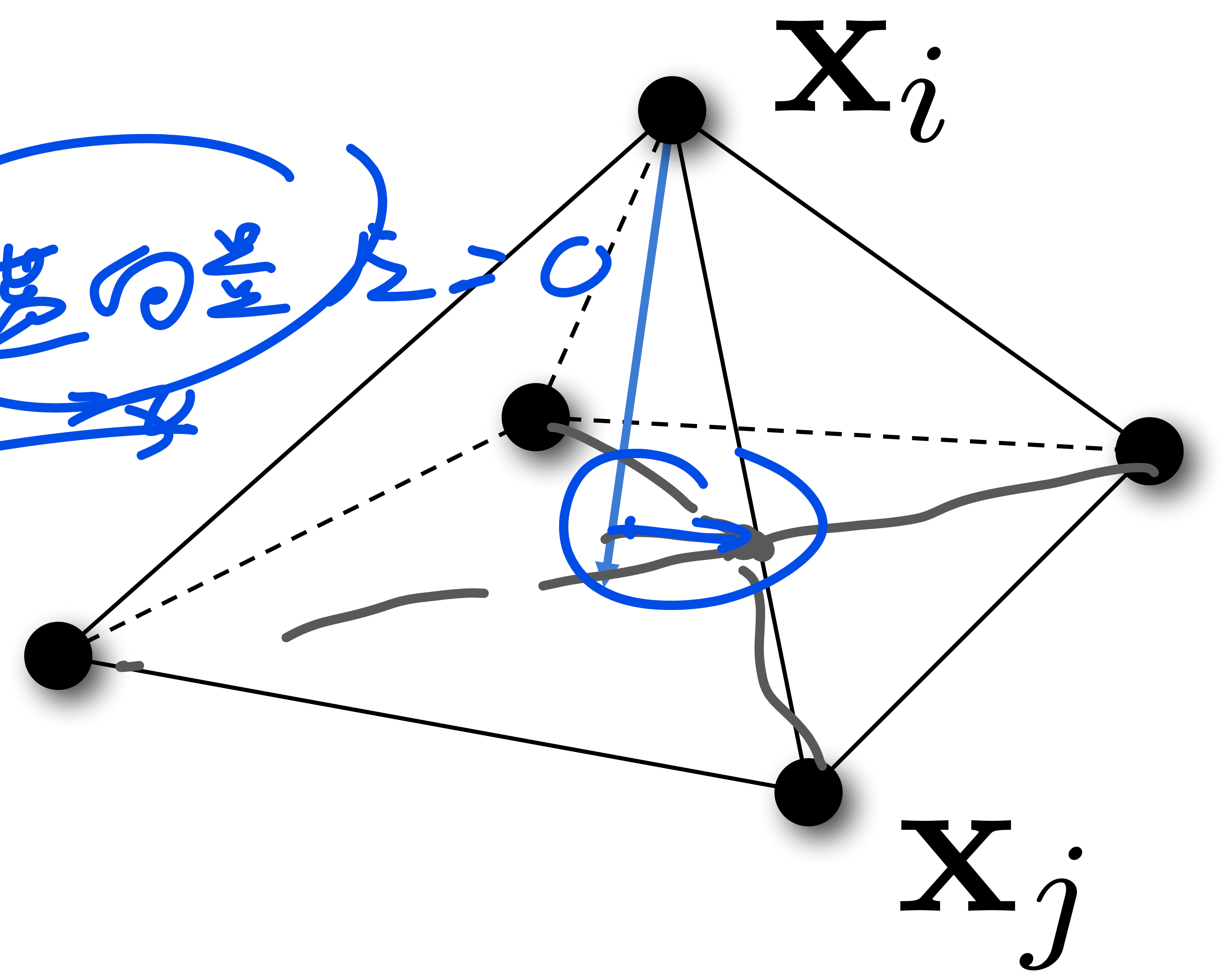
- Uniform discretization

这里是负的，想来的

$$\Delta_{\text{uni}}\mathbf{x}_i := \frac{1}{|\mathcal{N}_1(v_i)|}\sum_{v_j\in\mathcal{N}_1(v_i)}(\mathbf{x}_j - \mathbf{x}_i) \approx -2H\mathbf{n}$$

- Properties
  - depends only on connectivity
  - simple and efficient
  - bad approximation for irregular triangulations
    - can give non-zero H for planar meshes
    - tangential drift for mesh smoothing

这里是flat，应该是=0



Discrete Laplace-Beltrami

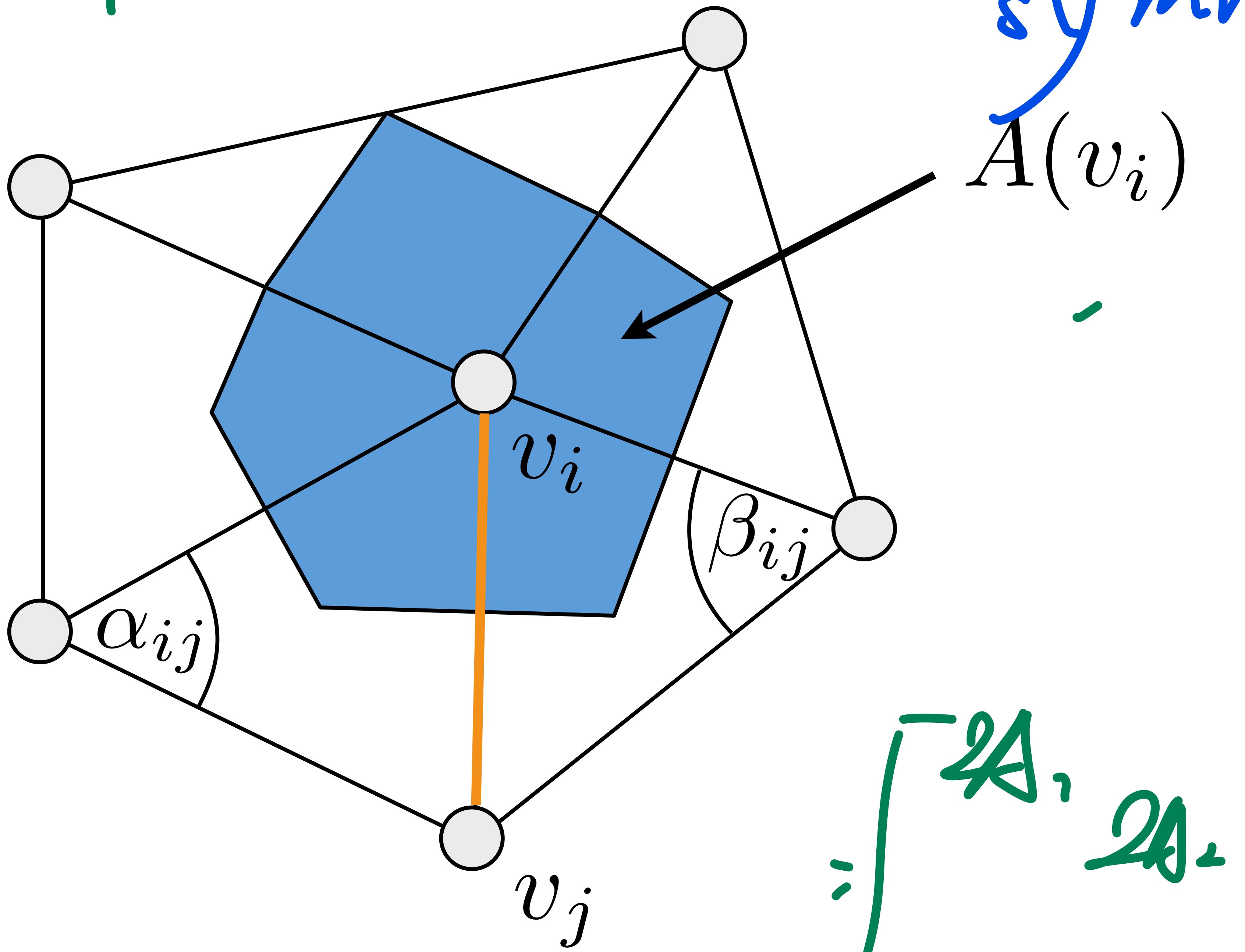
- Cotangent discretization not symmetric

$$\Delta_S f(v_i) := \frac{1}{2A(v_i)} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot \alpha_{ij} + \cot \beta_{ij}) \left(f(v_j) - f(v_i) \right)$$

symmetric

to solve this 不对的.

$M^{-1} C$

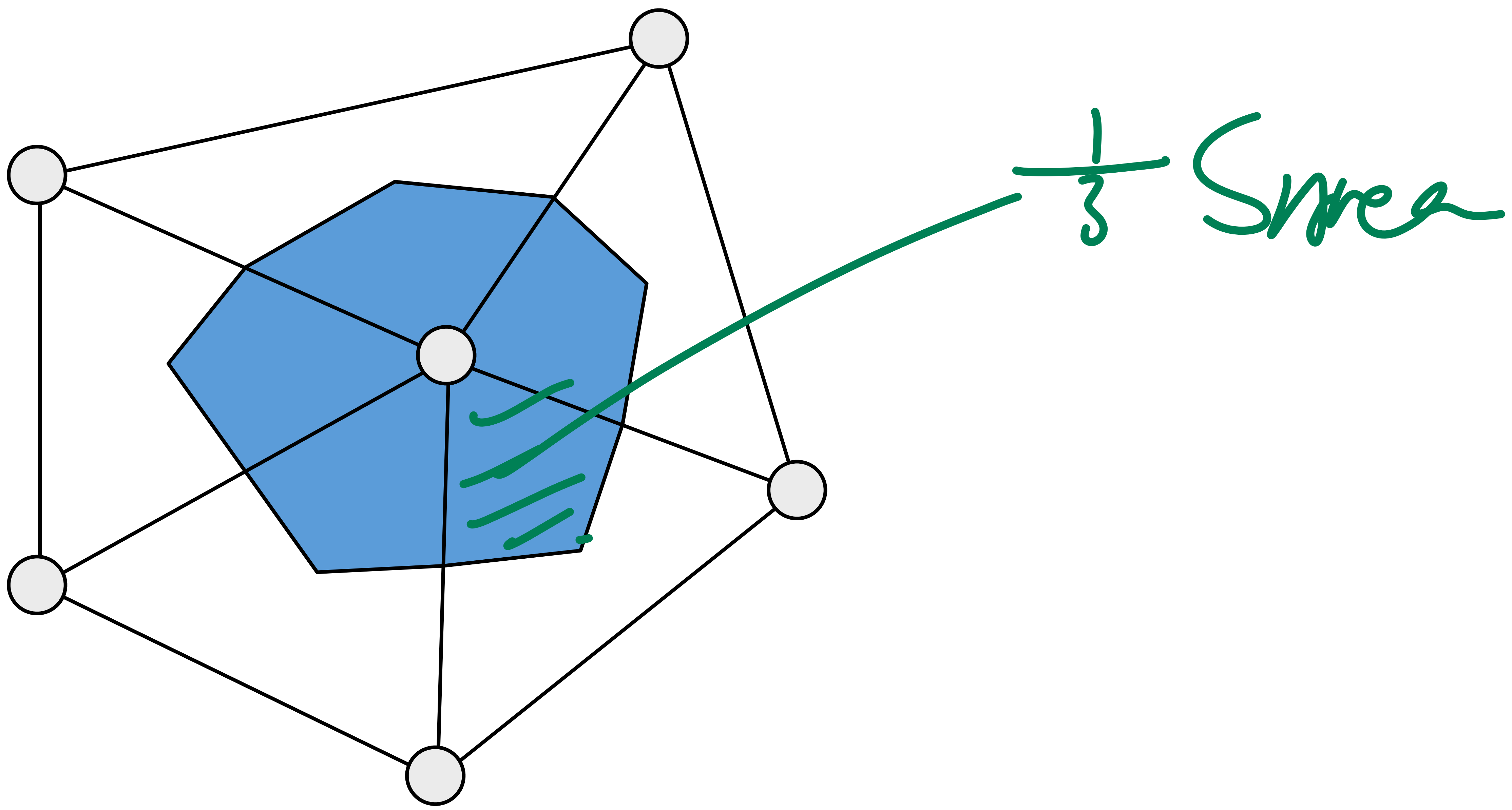


$$\Delta_S \mathbf{x} = -2H\mathbf{n}$$

$= \begin{bmatrix} 2A_1 & & \\ & 2A_2 & \\ & & \ddots \end{bmatrix}$

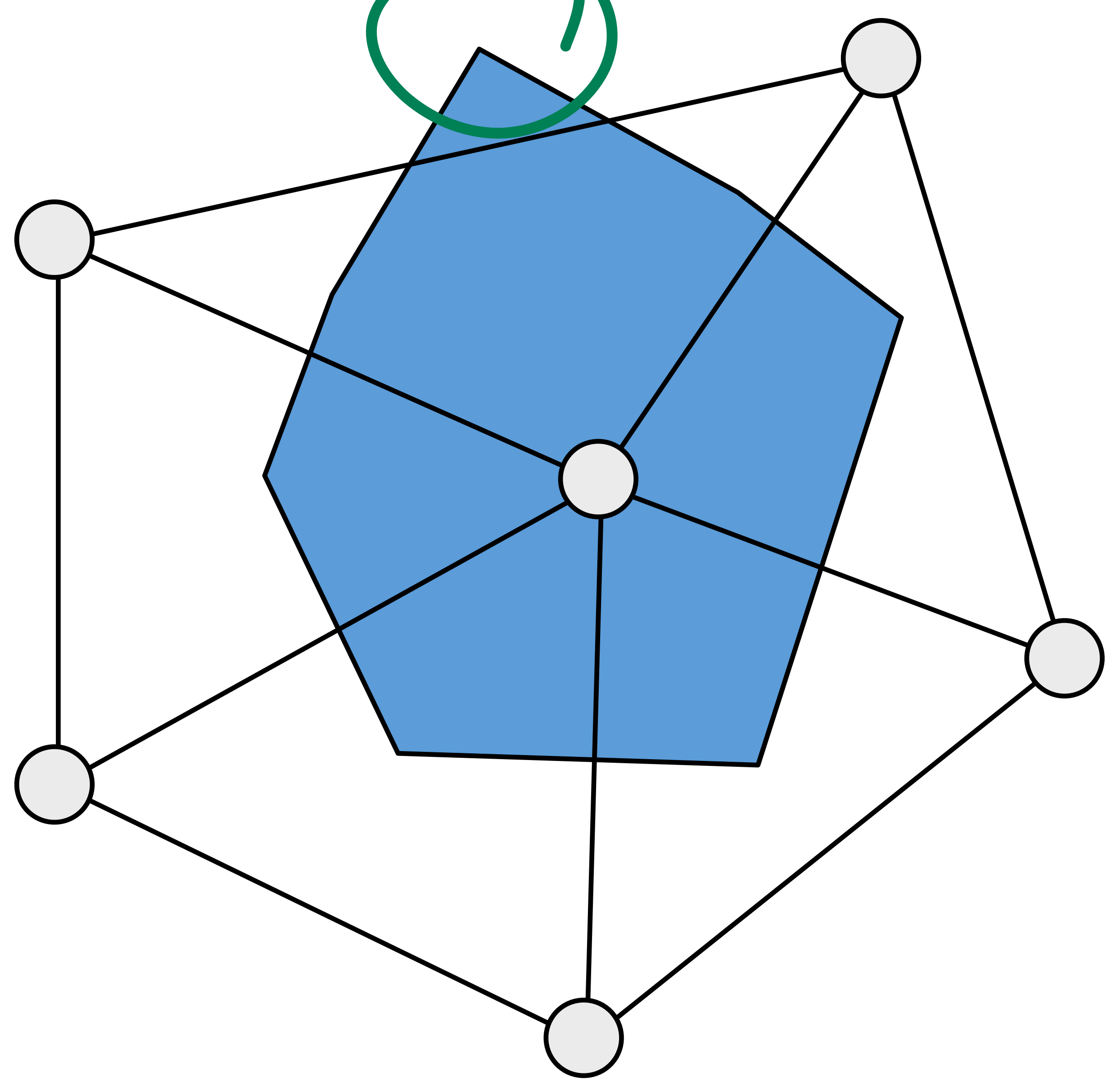
#1: Barycentric Cells

- Connect edge midpoints and triangle barycenters
  - Simple to compute
  - Area is 1/3 of triangle areas



#2: Voronoi Cells 细胞用图1了

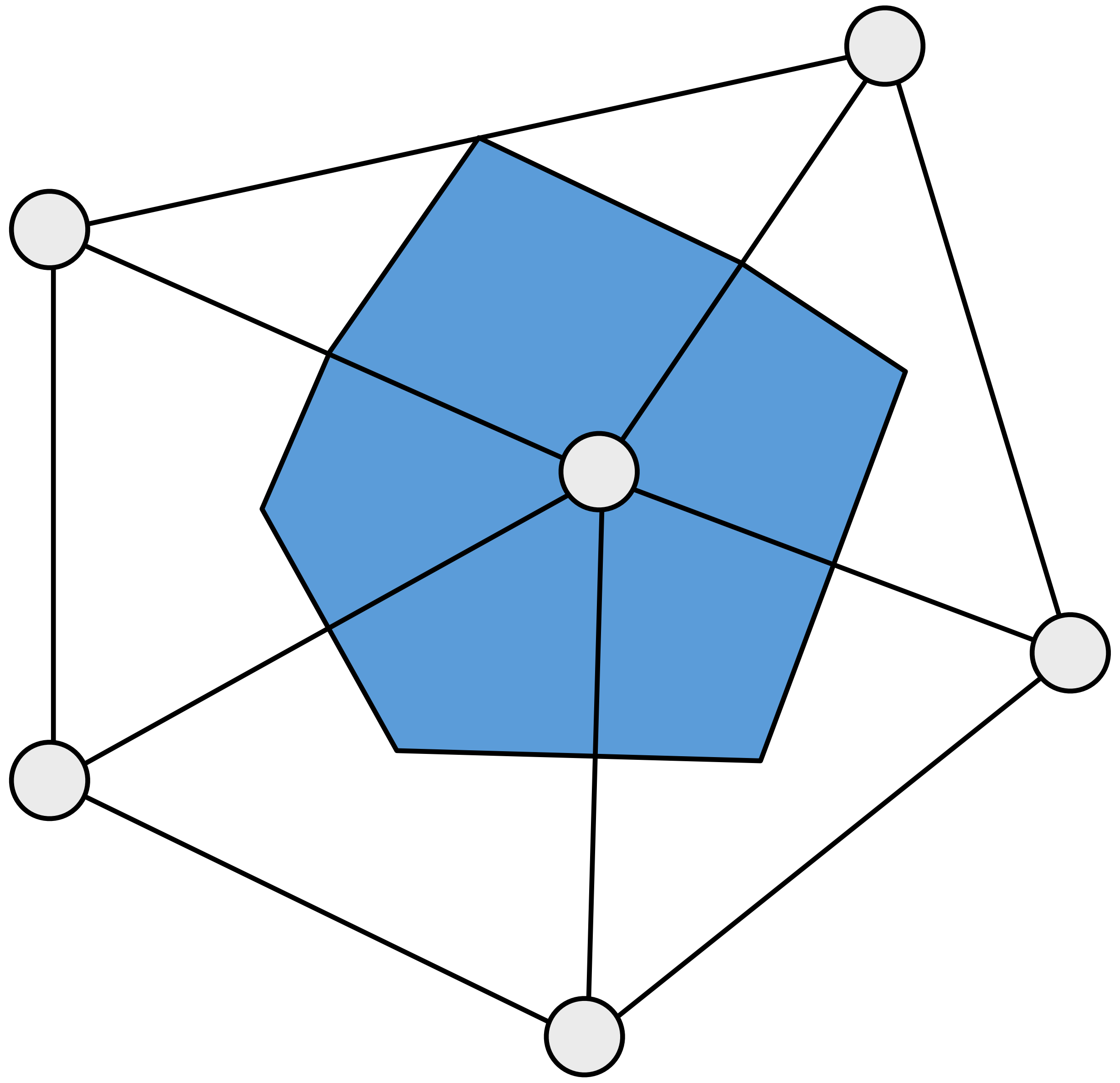
- Connect edge midpoints and
  - Circumcenters for all triangles
  - What about obtuse triangles?



Solve

#3: Mixed Cells

- Connect edge midpoints and
  - Circumcenters for non-obtuse triangles
  - Midpoint of opposite edge for obtuse triangles
  - Better approximation, more complex to compute...



Discrete Laplace-Beltrami

- Cotangent discretization

$$\Delta_{\mathcal{S}} f(v) := \frac{1}{2A(v)} \sum_{v_i \in \mathcal{N}_1(v)} (\cot \alpha_i + \cot \beta_i)\,(f(v_i) - f(v))$$

- Problems
  - weights can become negative (when?)
  - depends on triangulation

- Still the most widely used discretization

Discrete Curvatures

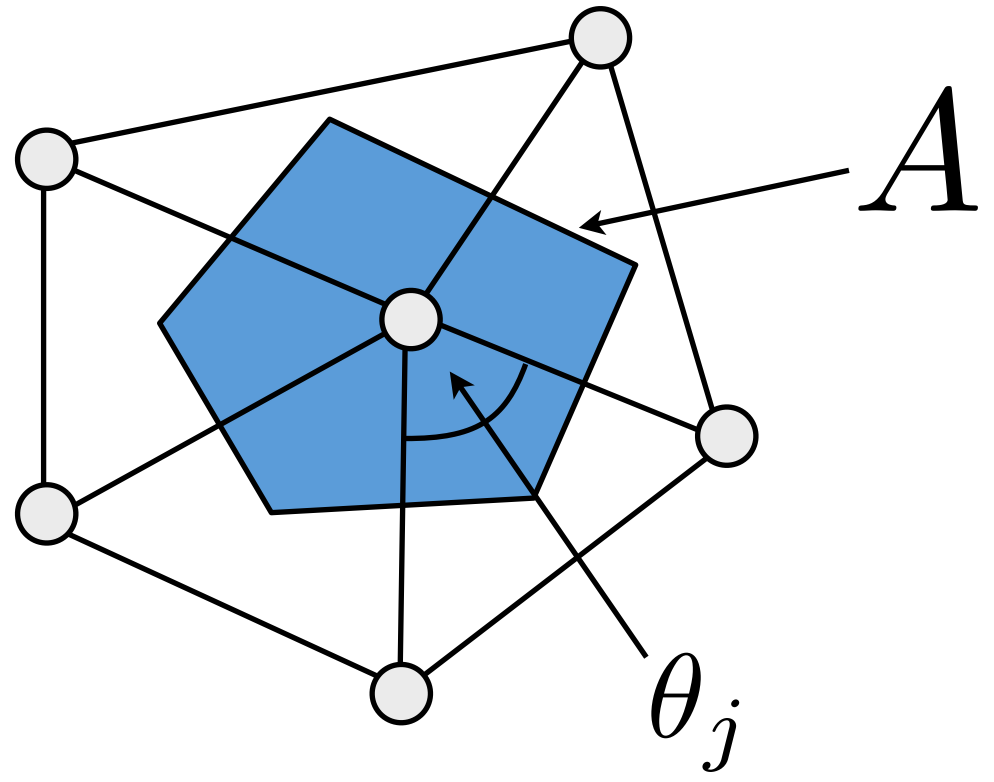
- Mean curvature (absolute value)

$$H = \frac{1}{2} \|\Delta_{\mathcal{S}} \mathbf{x}\|$$

$$\Delta_S \mathbf{x} = -2H\mathbf{n}$$

- Gaussian curvature

$$K = (2\pi - \sum_j \theta_j)/A$$



Literature

- Taubin: *A signal processing approach to fair surface design*, SIGGRAPH 1996.
- Desbrun et al.: *Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow*, SIGGRAPH 1999.
- Meyer et al.: *Discrete Differential-Geometry Operators for Triangulated 2-Manifolds*, VisMath 2002.
- Wardetzky, Mathur, Kaelberer, Grinspun: Discrete Laplace Operators: No free lunch, SGP 2007

Outline

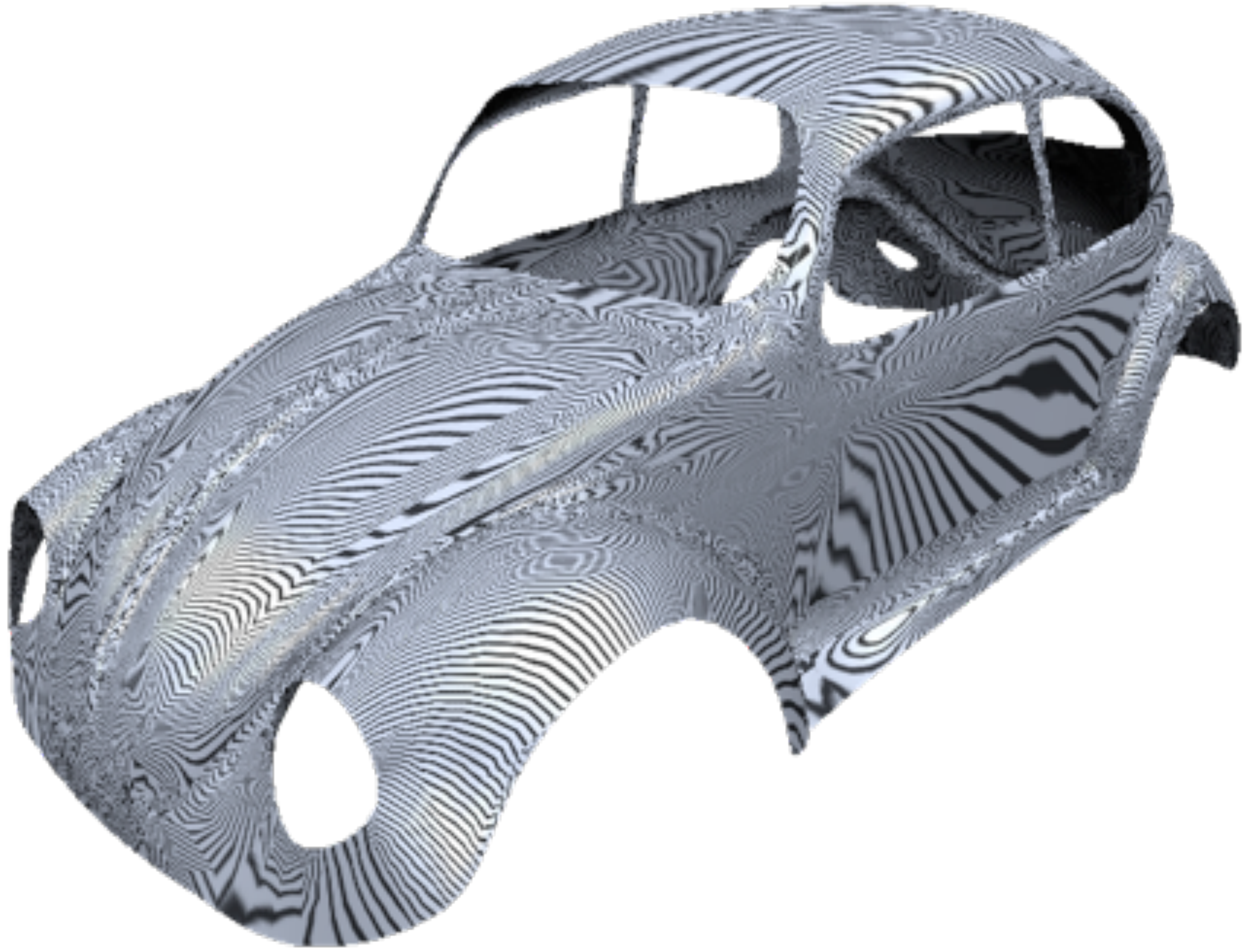
- Differential Geometry
- Discrete Differential Geometry
- **Mesh Quality Measures**

Mesh Quality

- Visual inspection of “sensitive” attributes
  - Specular shading

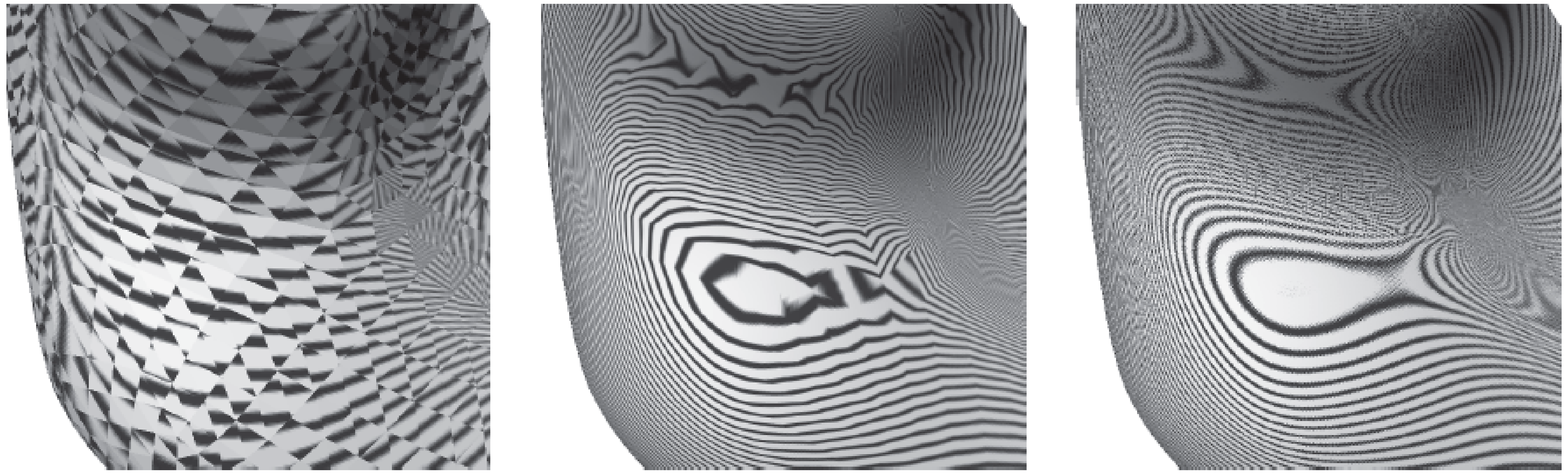
Mesh Quality

- Visual inspection of “sensitive” attributes
  - Specular shading
  - Reflection lines



Mesh Quality

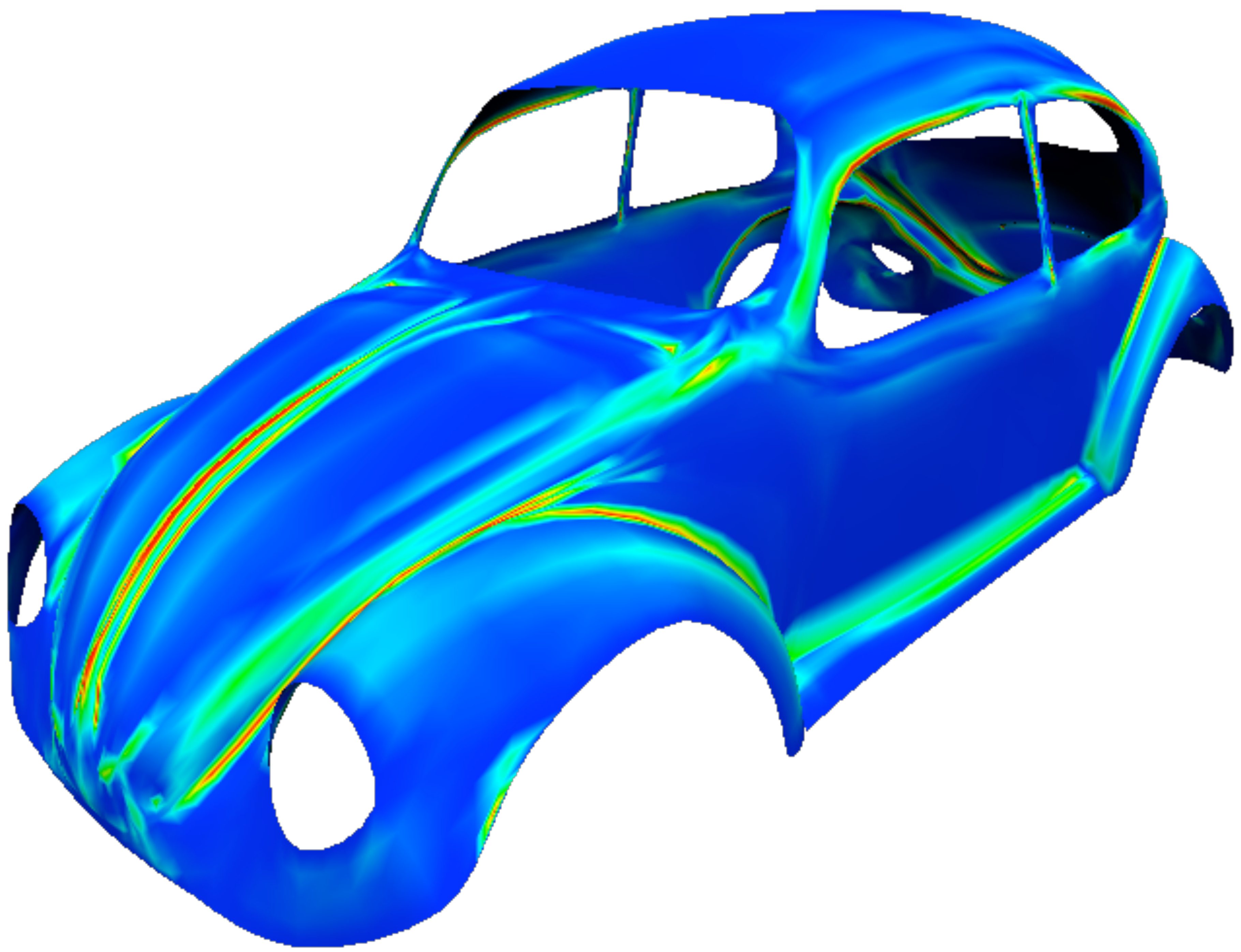
- Visual inspection of “sensitive” attributes
  - Specular shading
  - Reflection lines
    - differentiability one order lower than surface
    - can be efficiently computed using graphics hardware



C^0 C^1 C^2

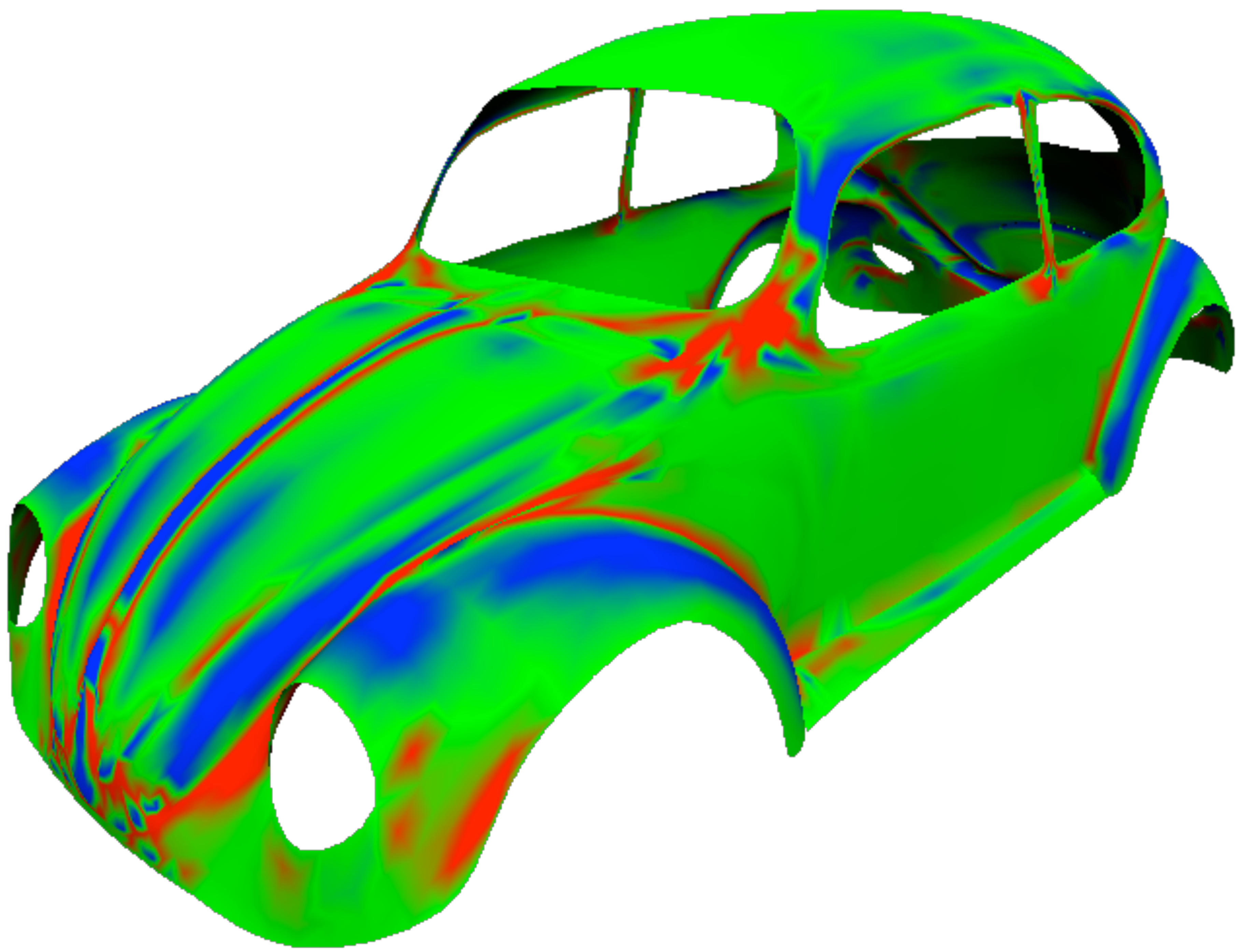
Mesh Quality

- Visual inspection of “sensitive” attributes
  - Specular shading
  - Reflection lines
  - Curvature
    - Mean curvature



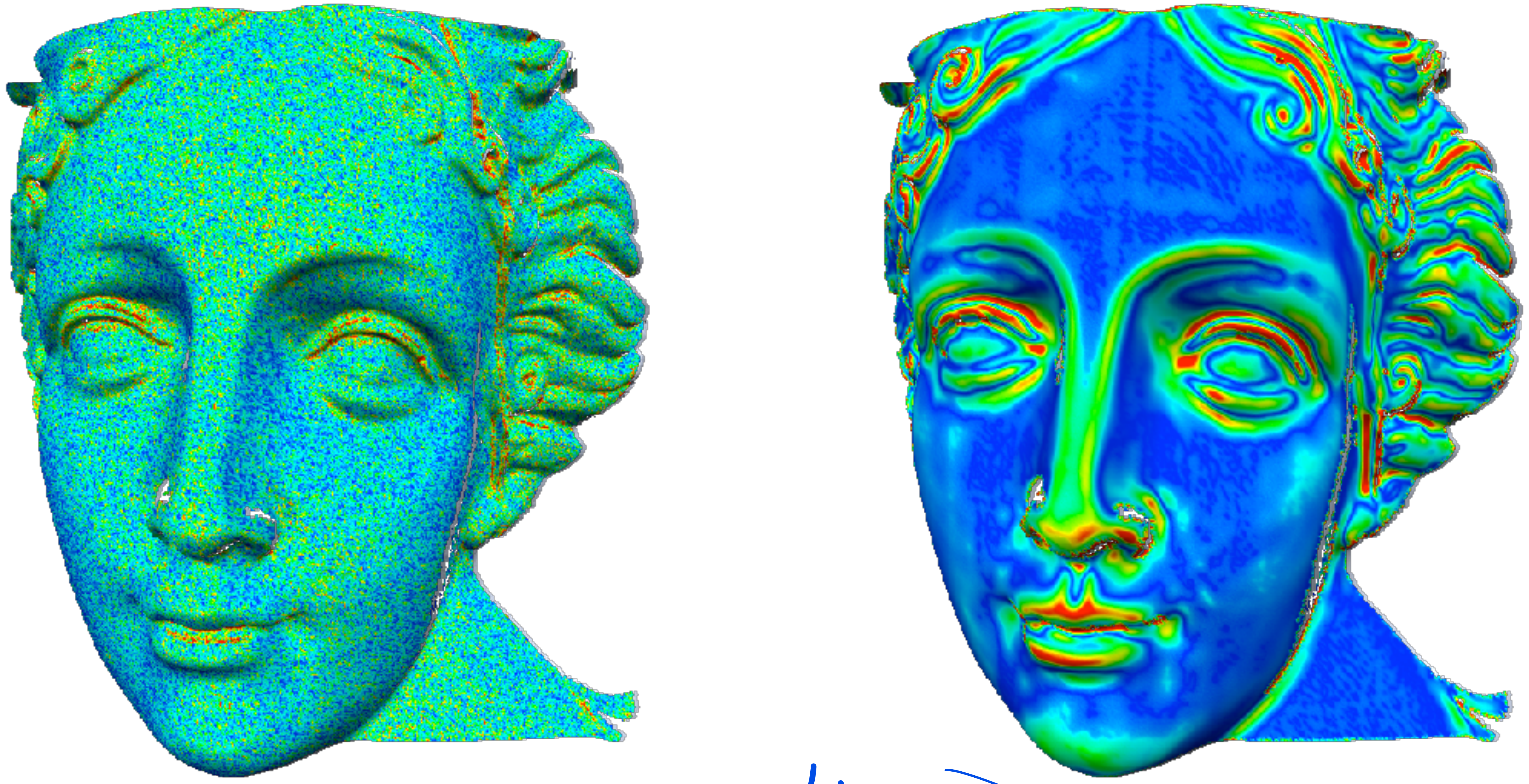
Mesh Quality

- Visual inspection of “sensitive” attributes
  - Specular shading
  - Reflection lines
  - Curvature
    - Mean curvature
    - Gauss curvature



Mesh Quality Criteria

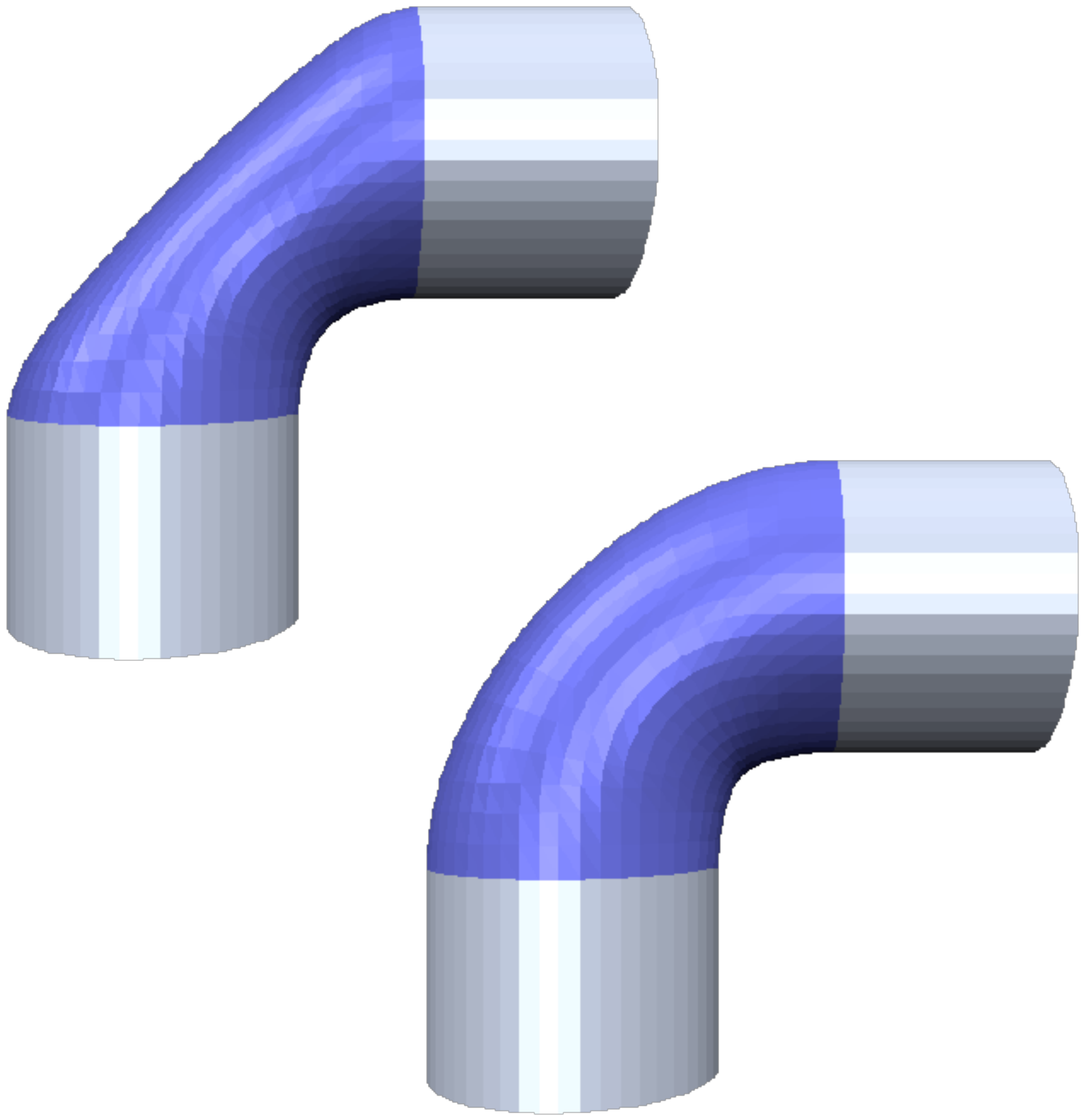
- Smoothness
  - Low geometric noise



smoother
[goal]

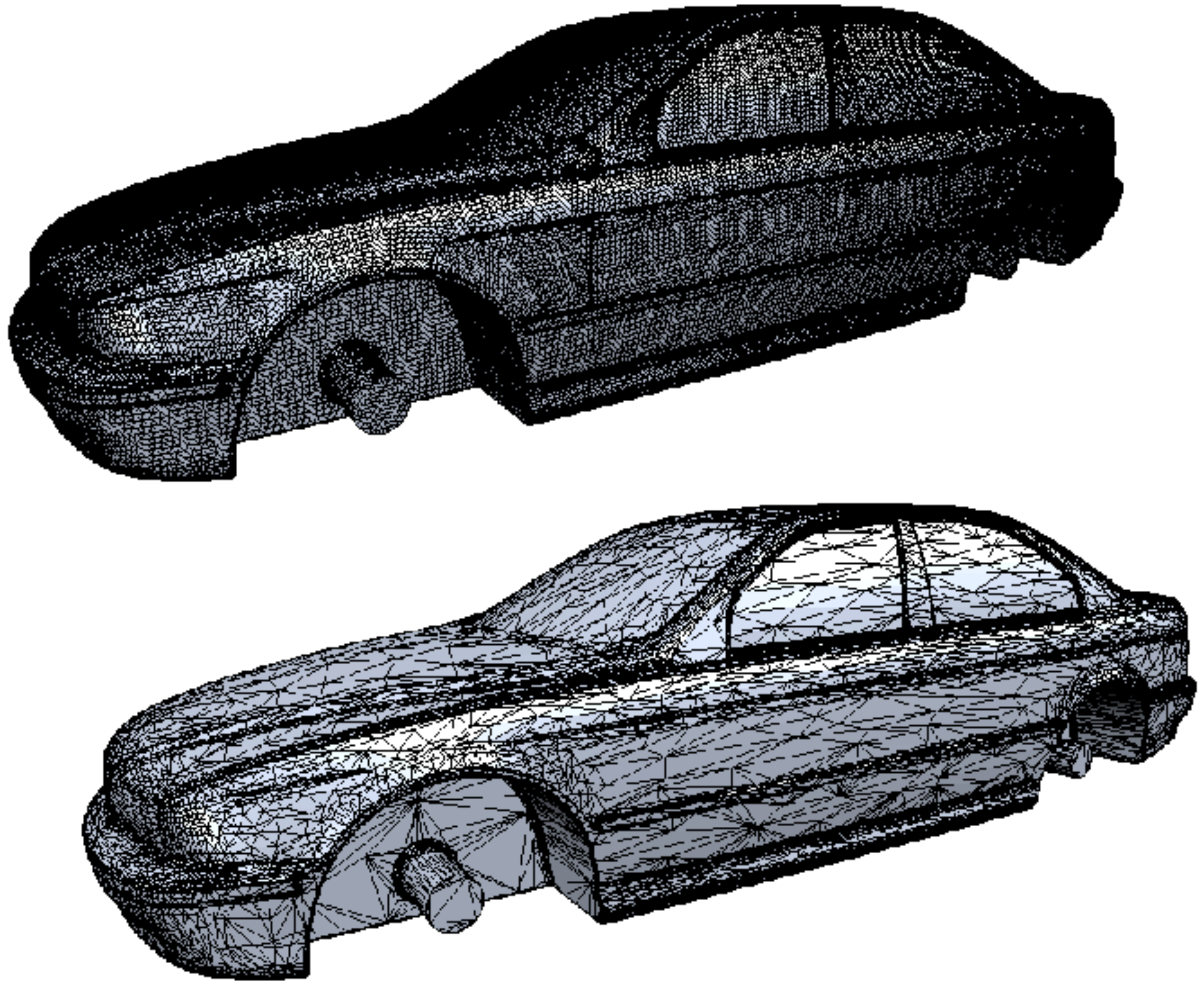
Mesh Quality Criteria

- Smoothness
  - Low geometric noise
- Fairness
  - Simplest shape



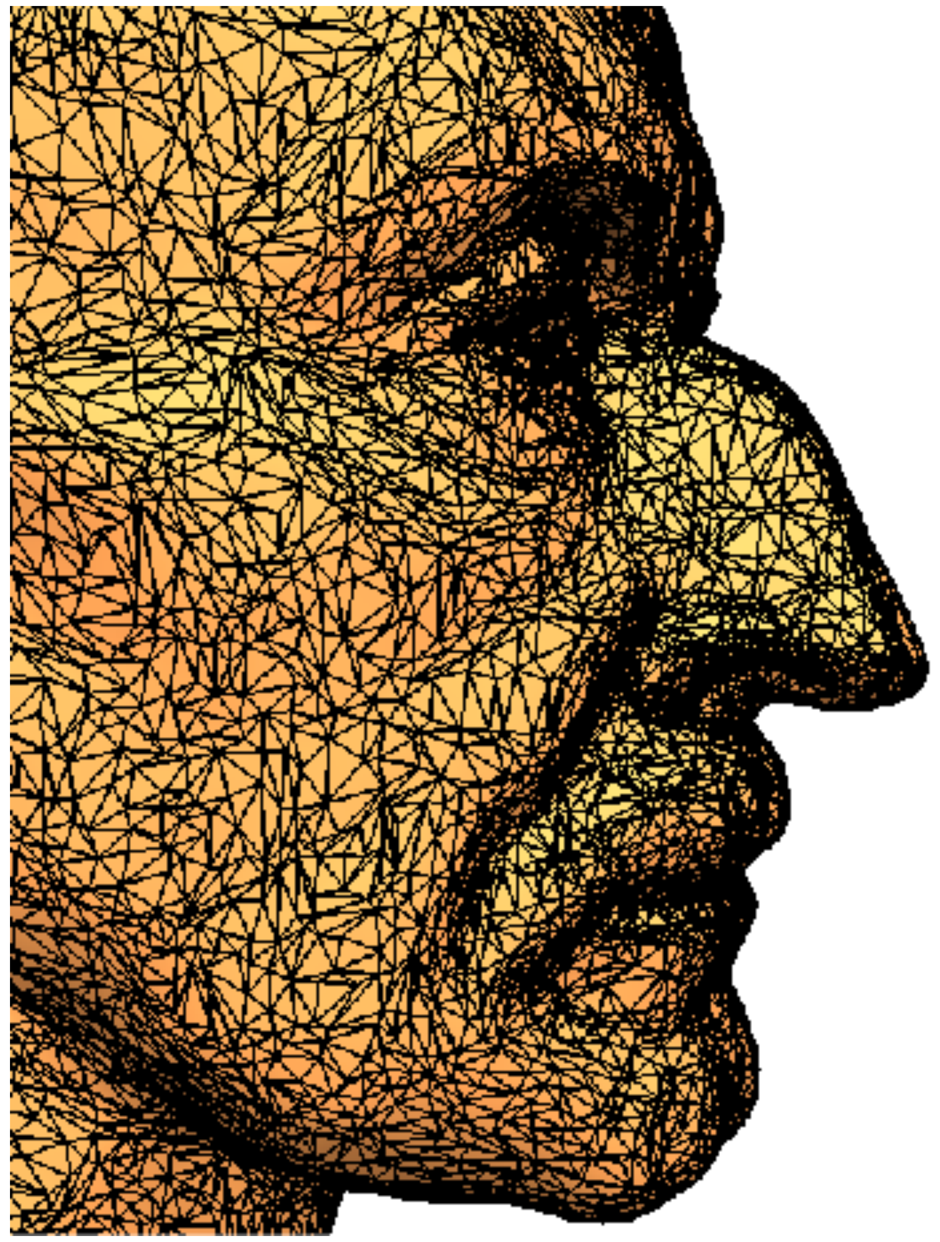
Mesh Quality Criteria

- Smoothness
  - Low geometric noise
- Fairness
  - Simplest shape
- Adaptive tessellation
  - Low complexity



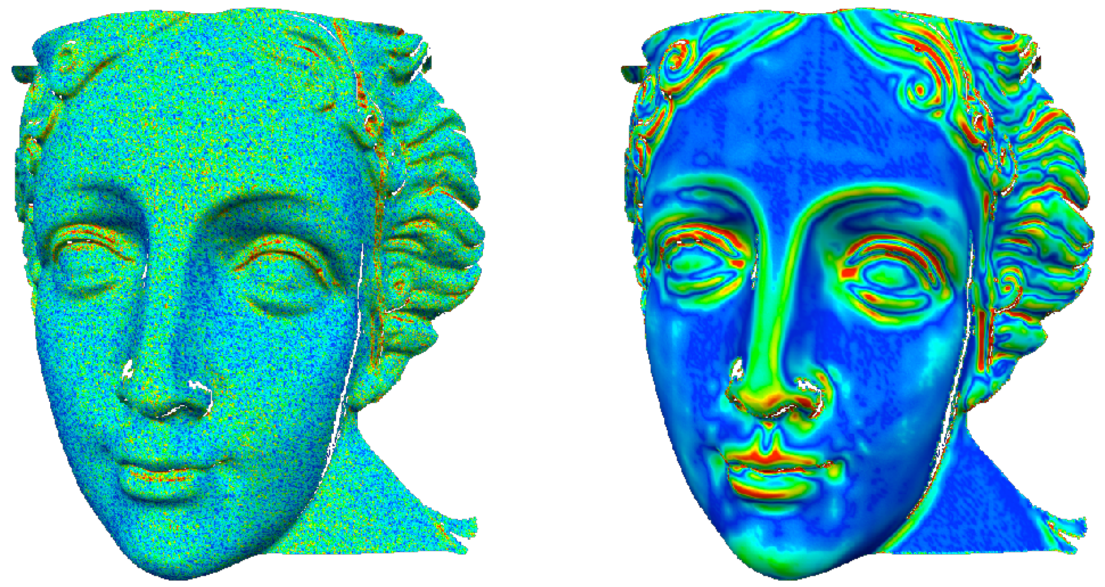
Mesh Quality Criteria

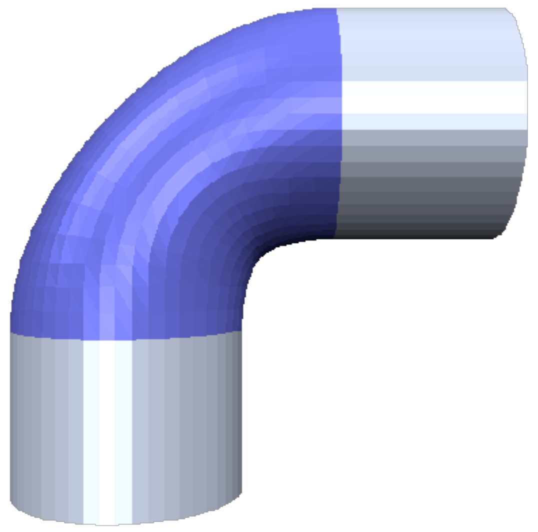
- Smoothness
  - Low geometric noise
- Fairness
  - Simplest shape
- Adaptive tessellation
  - Low complexity
- Triangle shape
  - Numerical robustness

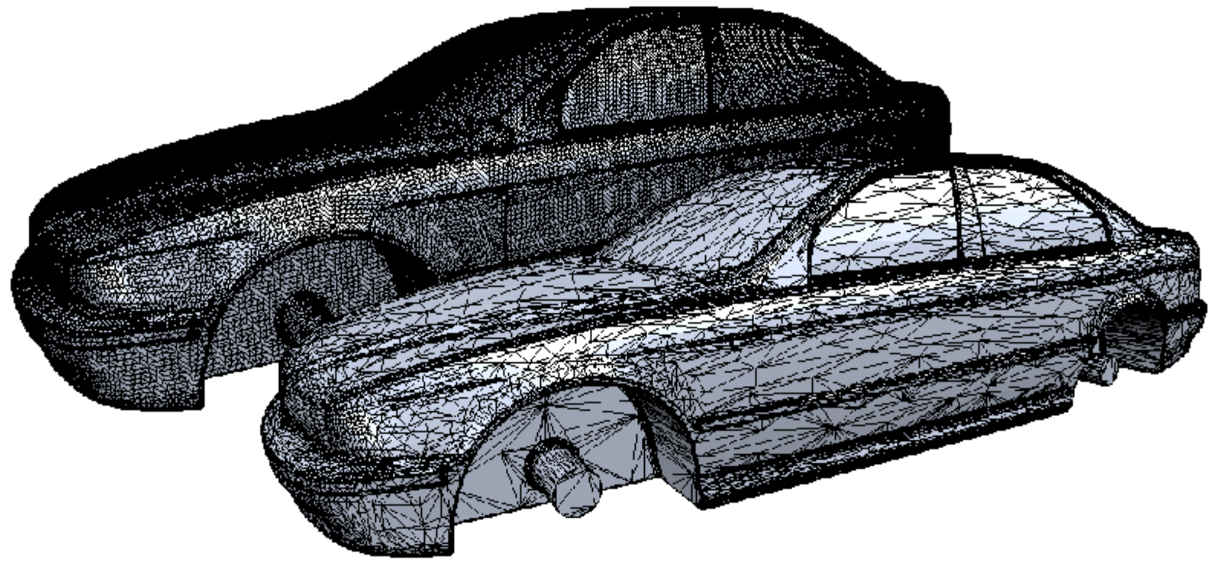


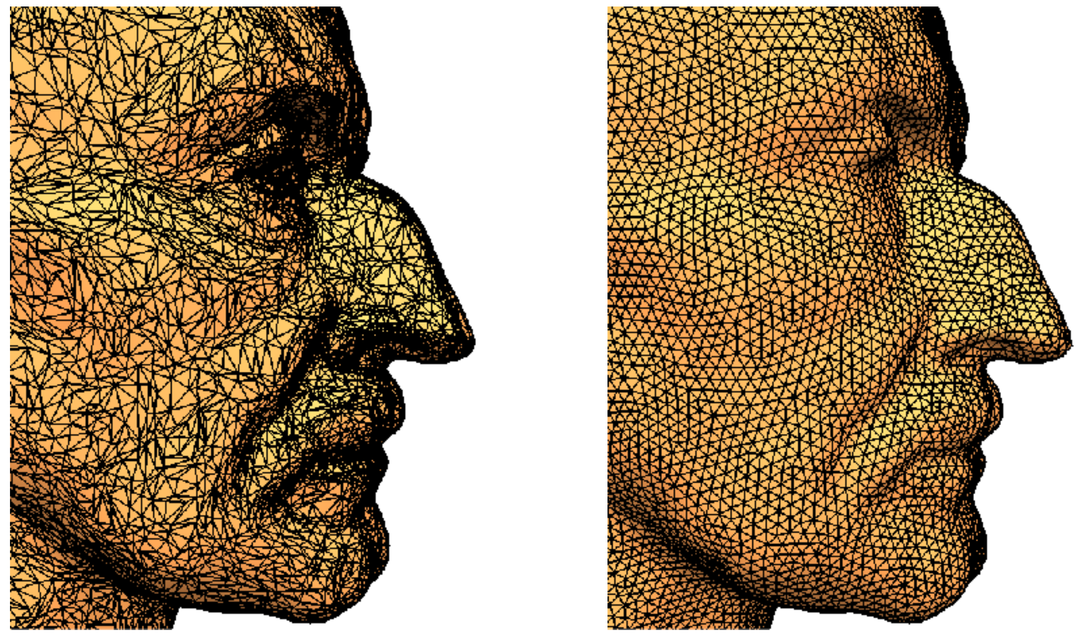
Mesh Optimization

- Smoothness
  ➡ Smoothing
- Fairness
  ➡ Fairing
- Adaptive tessellation
  ➡ Decimation
- Triangle shape
  ➡ Remeshing



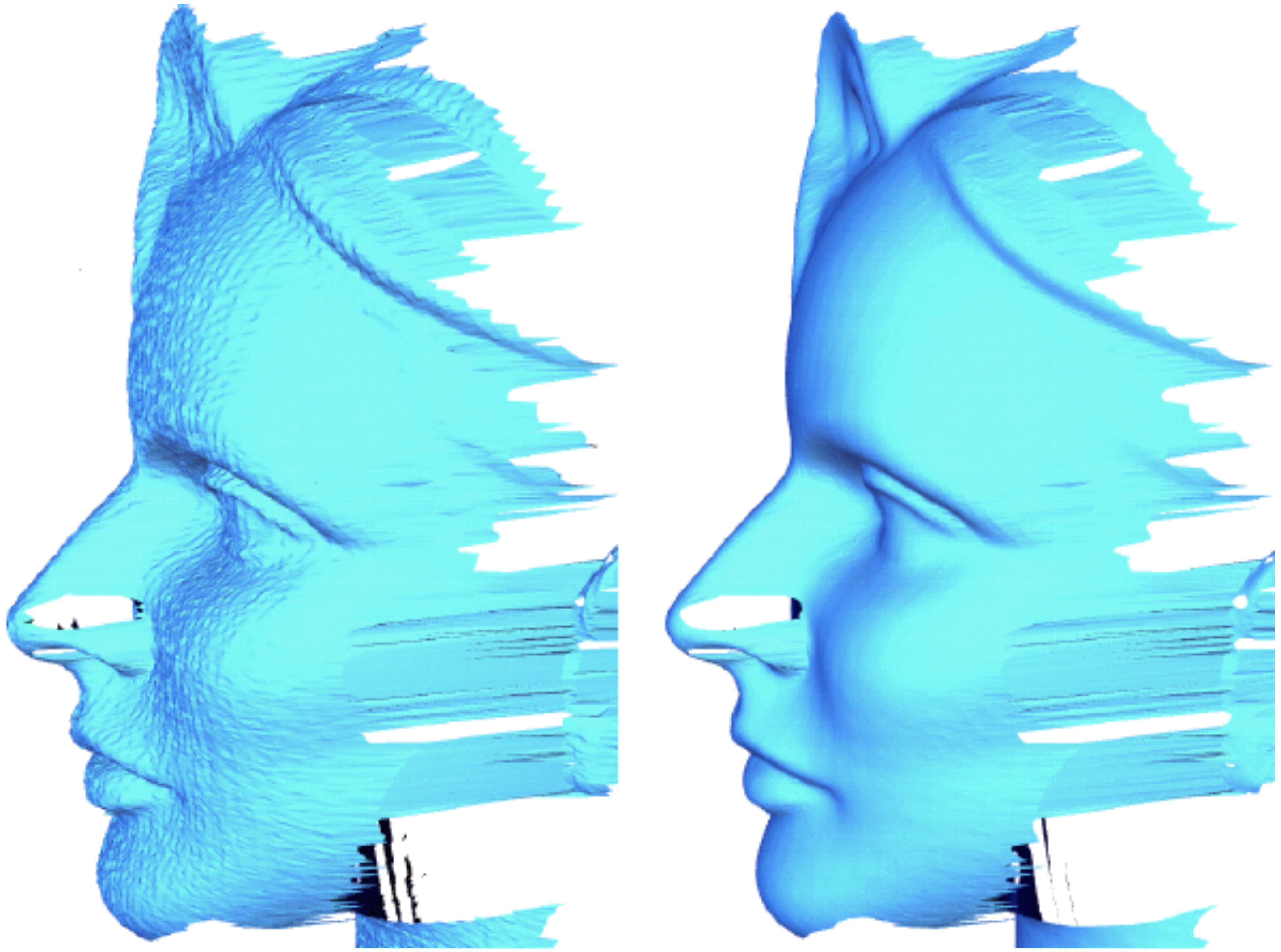






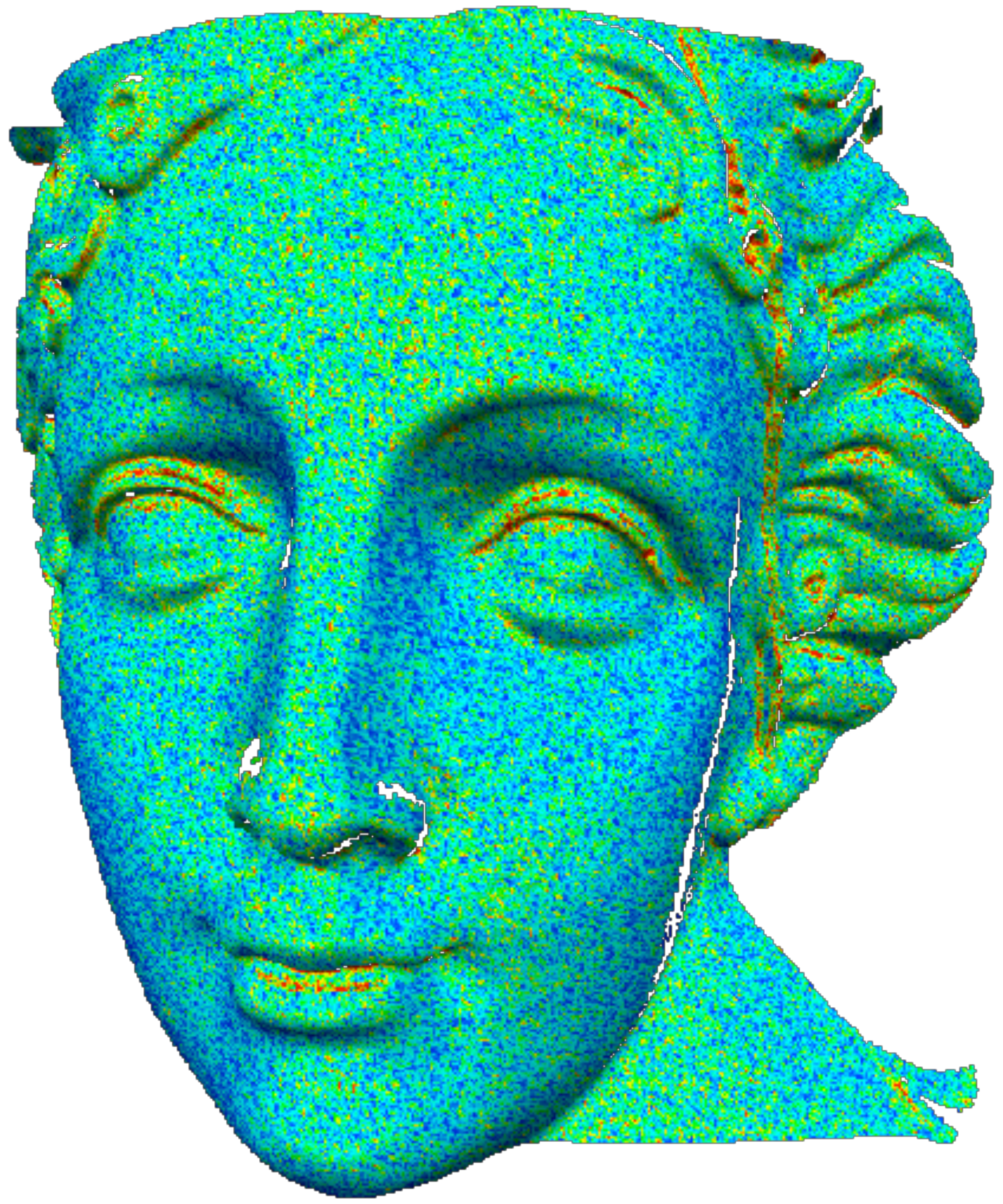
#1: Filter Scanning Noise

- Filter out high frequency noise

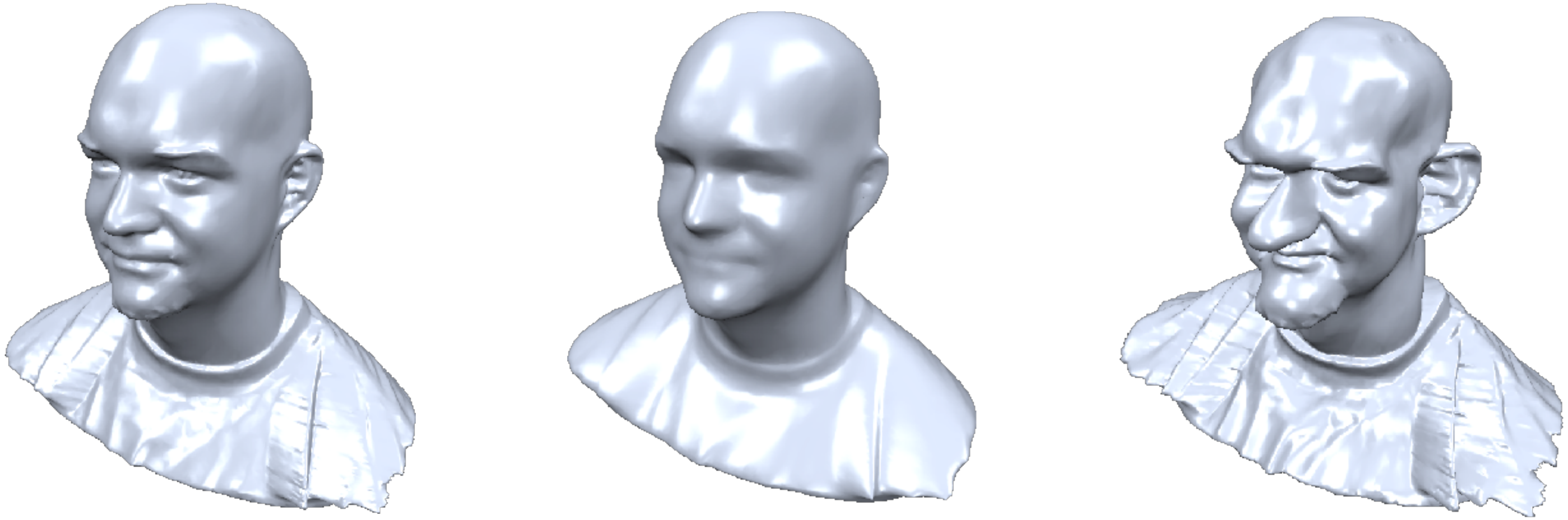


Desbrun, Meyer, Schroeder, Barr
Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow
SIGGRAPH 99

Mesh Smoothing *(reduce noise)*



#2: Feature Modification

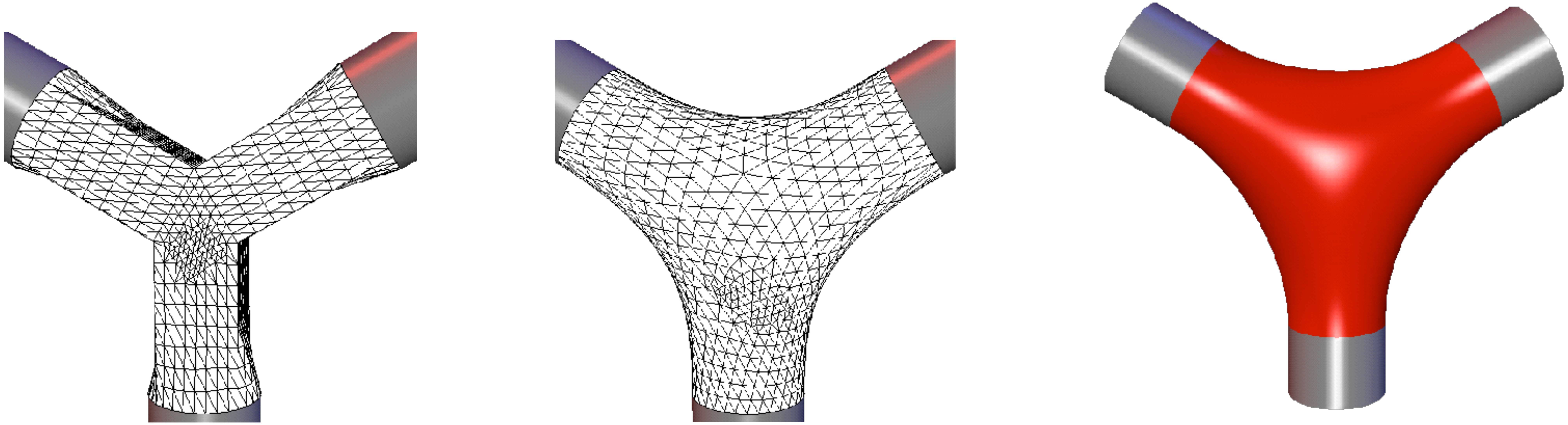


Original Low-Pass Exaggerate

Kim, Rosignac: *Geofilter: Geometric Selection of Mesh Filter Parameters*, Eurographics 05

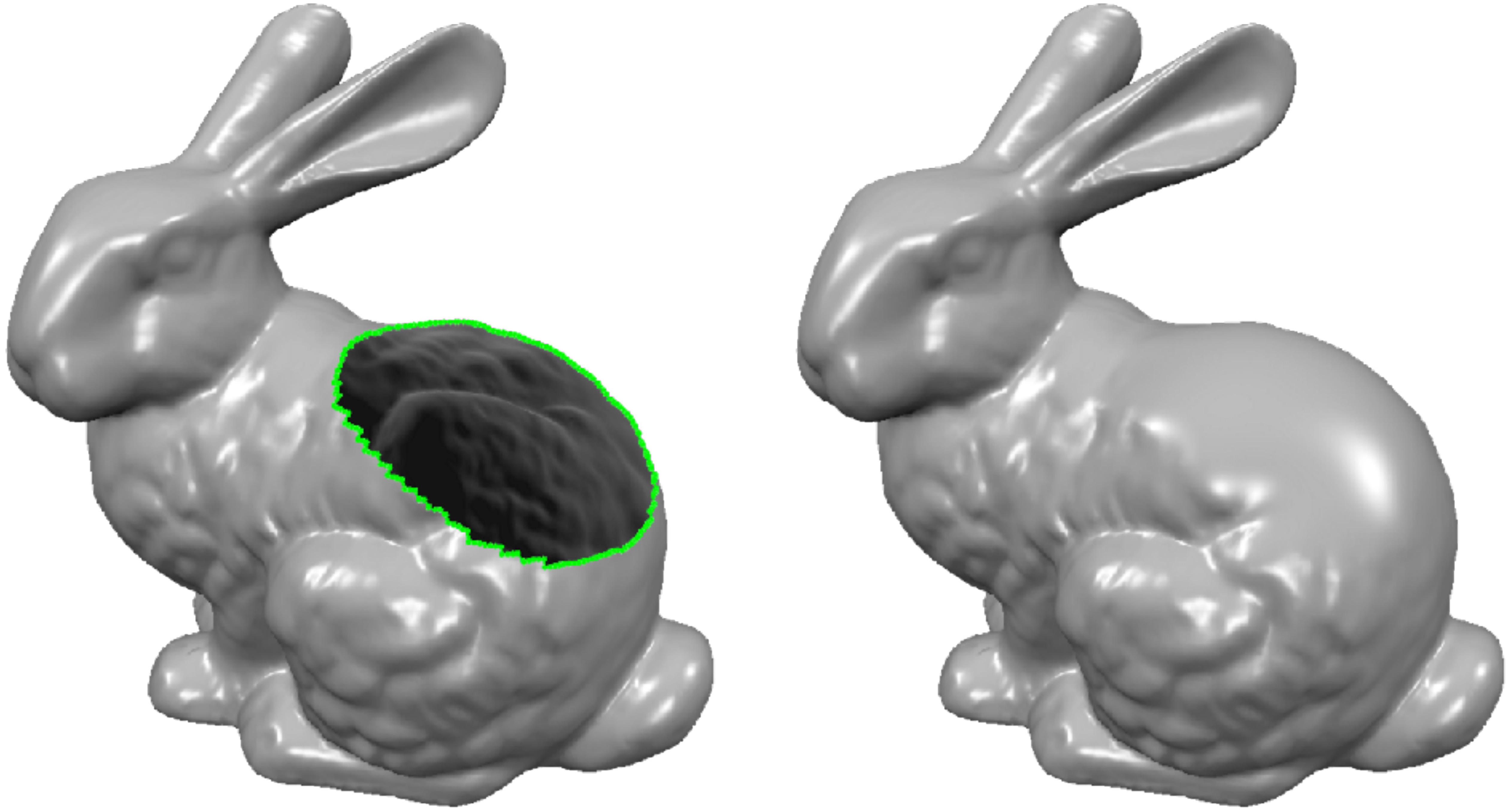
#3: Fair Surface Design

- Fair Surface Design



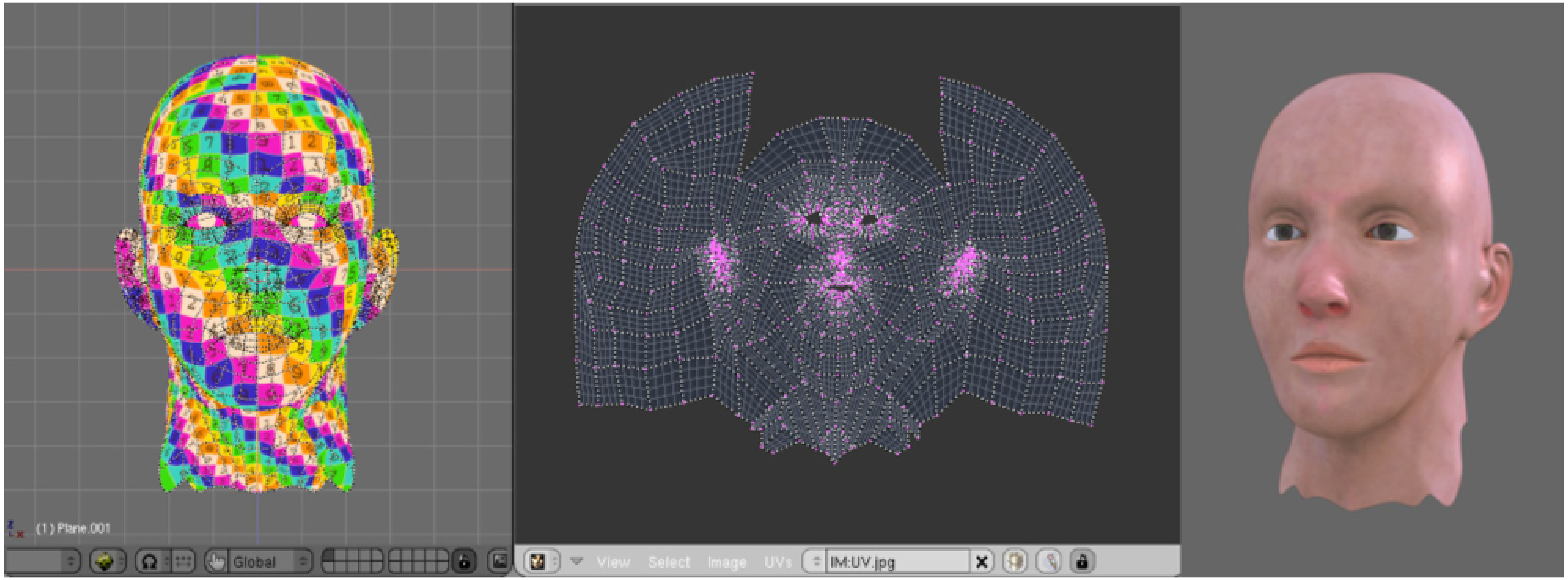
Schneider, Kobbelt: *Geometric fairing of irregular meshes for free-form surface design*, CAGD 18(4), 2001

#4: Hole Filling *(energy minimization)*



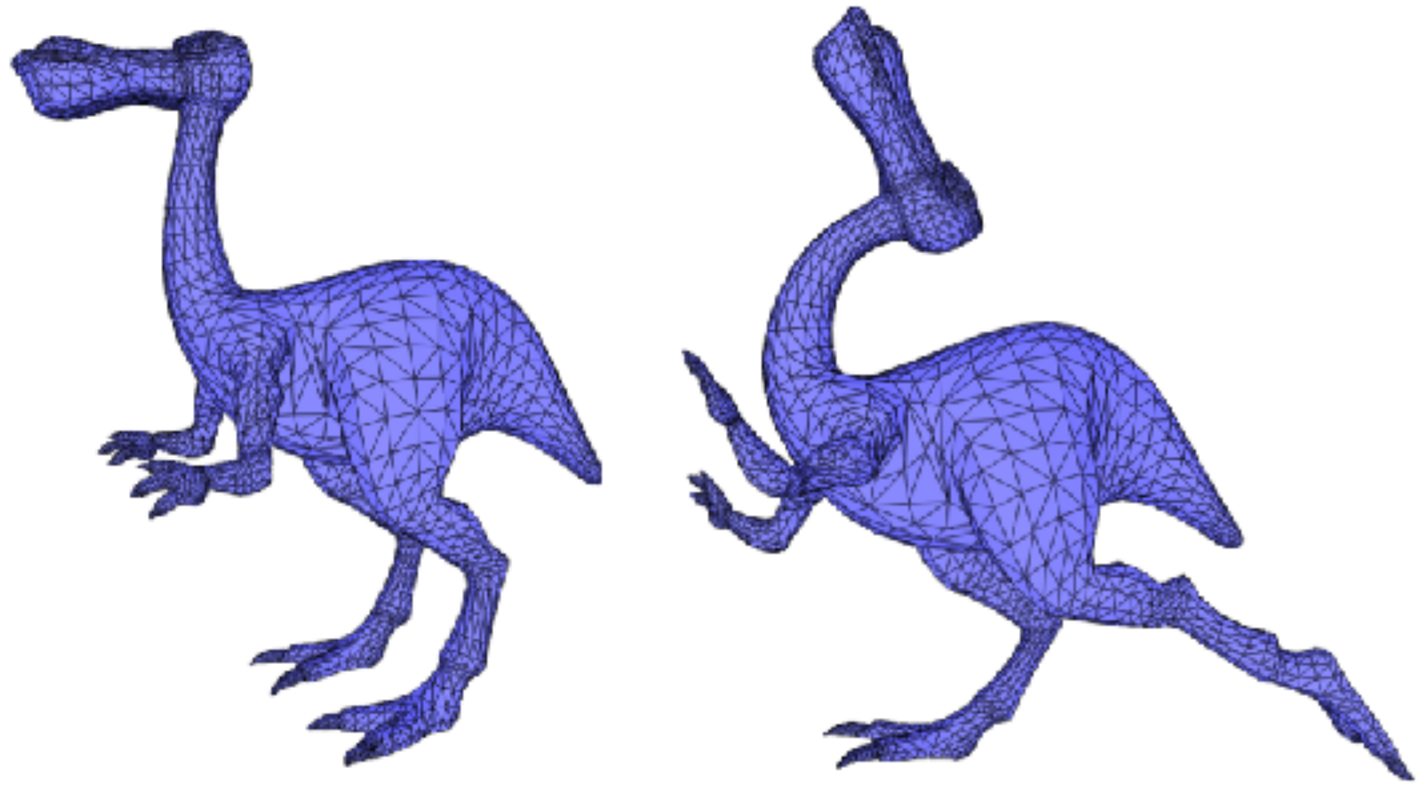
#5: Mesh Parameterization

$$\mathbf{x}(u, v) \qquad\qquad (u, v)$$

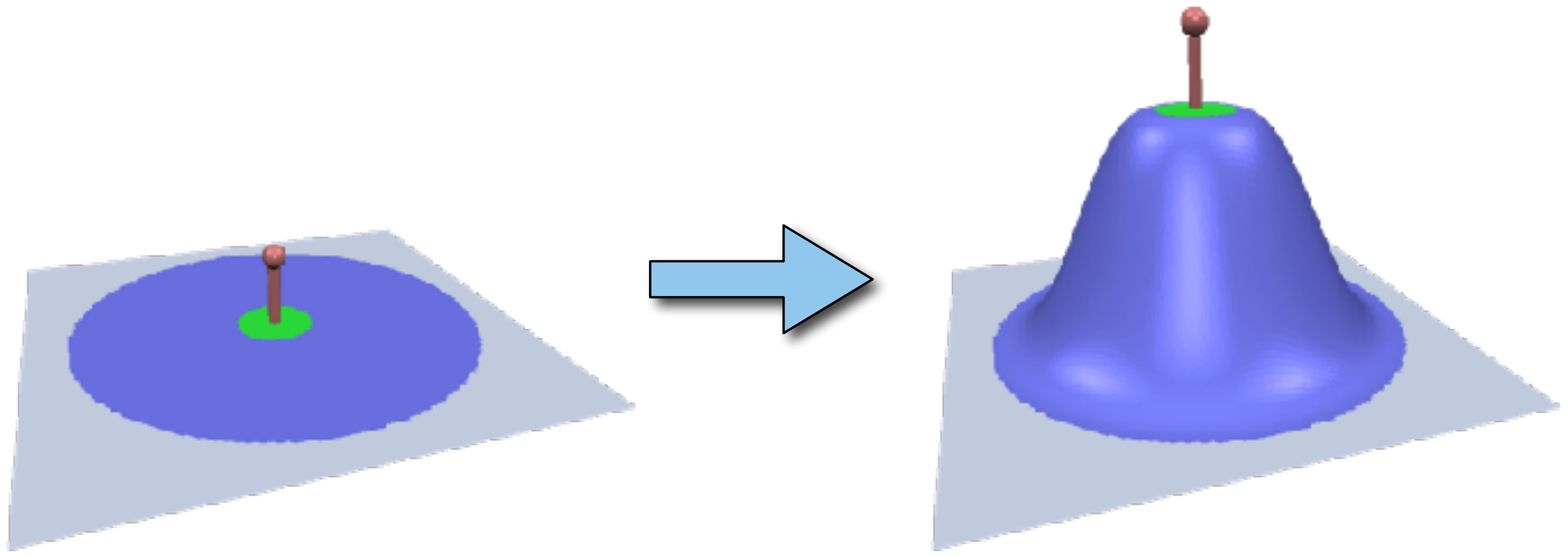


Levy et al.: Least squares conformal maps for automatic texture atlas generation, SIGGRAPH 2002.

#6: Surface Deformation



Olga Sorkine and Marc Alexa. As-rigid-as-possible surface modeling, SGP 2007



Botsch, Kobbelt: *An intuitive framework for real-time freeform modeling*, SIGGRAPH 04

Laplace Operator on Surfaces Δ

- Uniform Laplace operator
  - depends on mesh connectivity only
  - approximates second derivative on surfaces

- Cotan operator (Laplace Beltrami)

not symmetric symmetric

$$\Delta = M^{-1} C$$

$$M = M^T \qquad C = C^T$$

$$\Delta^T \neq \Delta$$

$$c_{ij} = \frac{1}{2}\left(\cot(\alpha_{ij}) + \cot(\beta_{ij})\right)$$

$$c_{ii} = -\sum_{j \in N_1(i)} c_{ij} \qquad \sum_j c_{ij} = 0$$

along each row, sum of it up

Common Problem Types

- Laplace equation (special case of Poisson equation)

$$\Delta f = 0$$

$$M^{-1}Cf = 0 \rightarrow Cf = 0 \quad \text{subject to boundary conditions}$$
$$f(x) = f_0 \quad x \in \partial B$$

Common Problem Types

- Laplace equation (special case of Poisson equation)

$$\Delta f = 0$$

$$M^{-1}Cf = 0 \rightarrow Cf = 0 \quad \text{subject to boundary conditions}$$

$$f(x) = f_0 \quad x \in \partial B$$

- Poisson equation

$$\Delta f = g$$

$$M^{-1}Cf = g \rightarrow Cf = Mg$$

Common Problem Types

- Laplace equation (special case of Poisson equation)

$$\Delta f = 0$$

$$M^{-1}Cf = 0 \rightarrow Cf = 0 \quad \text{subject to boundary conditions}$$

$$f(x) = f_0 \quad x \in \partial B$$

- Poisson equation

$$\Delta f = g$$

$$M^{-1}Cf = g \rightarrow Cf = Mg$$

- Differential equation

$$f_t = \frac{\partial f}{\partial t} = \Delta f$$

Common Problem Types

- Laplace equation (special case of Poisson equation)

$$\Delta f = 0$$

$$M^{-1}Cf = 0 \rightarrow Cf = 0 \quad \text{subject to boundary conditions}$$

$$f(x) = f_0 \quad x \in \partial B$$

not symmetric

- Poisson equation

$$\Delta f = g$$

$$M^{-1}Cf = g \rightarrow Cf = Mg$$

$Ax = \lambda x$

$M^{-1}C x = \lambda x$

$M^{\frac{1}{2}} M^{-\frac{1}{2}} M^{-\frac{1}{2}} C x =$

$M^{-\frac{1}{2}} C x = \lambda M^{\frac{1}{2}} x$

$y = M^{\frac{1}{2}} x$

$x = M^{-\frac{1}{2}} y$

- Differential equation

$$f_t = \frac{\partial f}{\partial t} = \Delta f$$

- Eigen/spectral analysis

$$\Delta \phi_i = \lambda_i \phi_i$$

$$\left(M^{-\frac{1}{2}} C M^{-\frac{1}{2}}\right) M^{\frac{1}{2}} \phi_i = \lambda_i M^{\frac{1}{2}} \phi_i$$

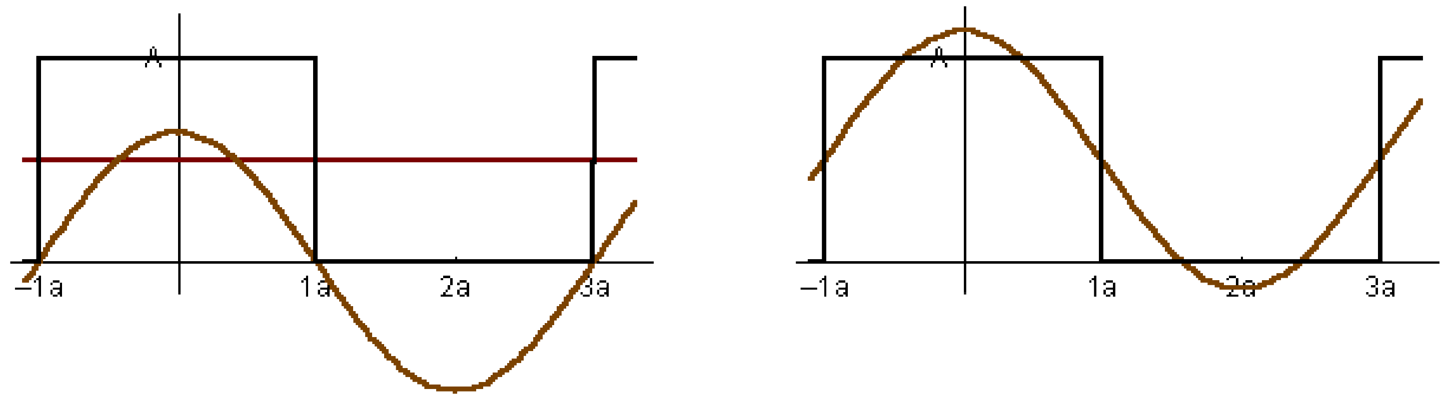
$$\left(M^{-\frac{1}{2}} C M^{-\frac{1}{2}}\right) \xi_i = \lambda_i \xi_i \qquad \rightarrow \phi_i = M^{-\frac{1}{2}} \xi_i$$

Outline

- Spectral Analysis
- Diffusion Flow
- Energy Minimization

Fourier Transform

- Represent a function as a weighted sum of sines and cosines



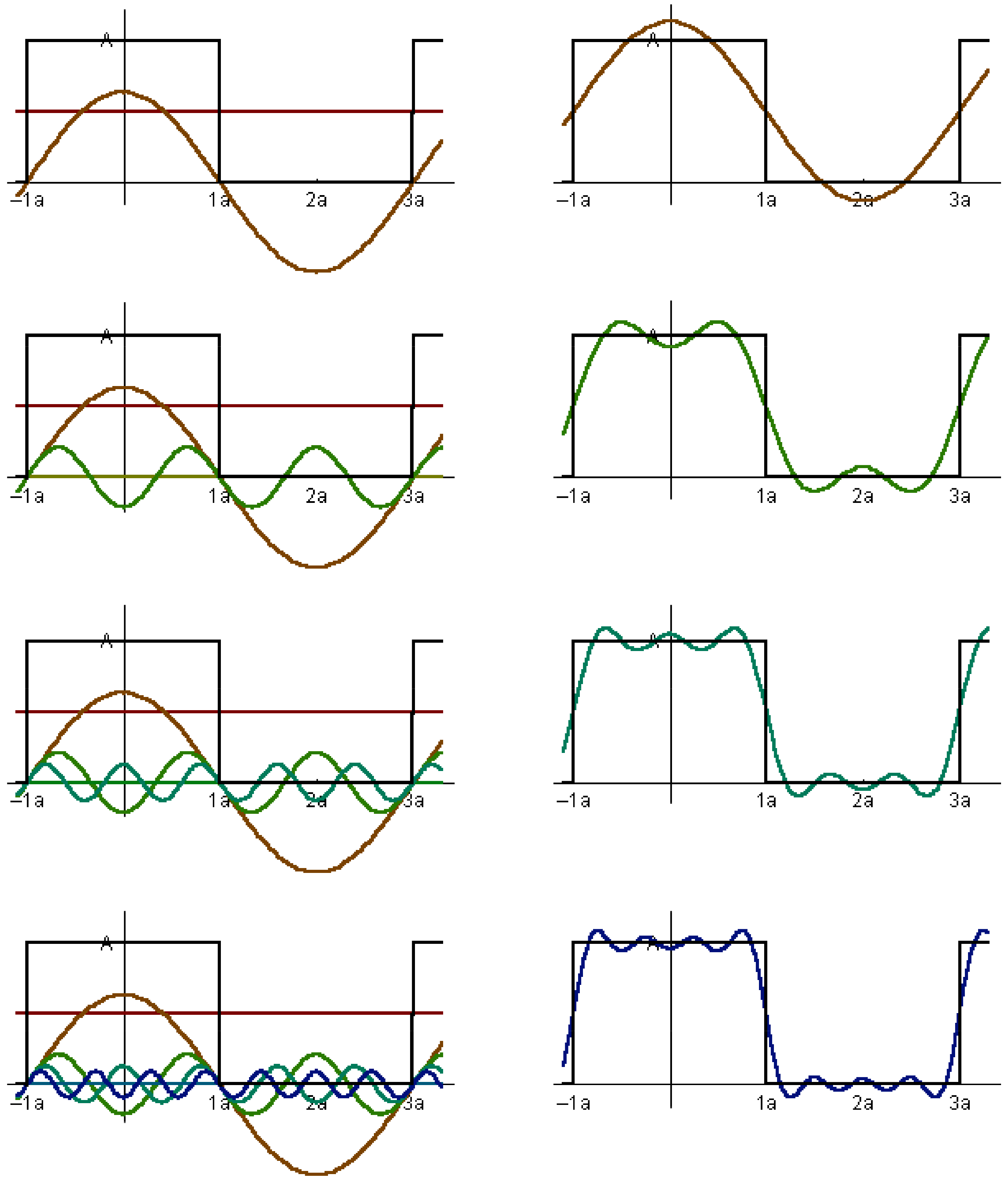
Joseph Fourier 1768 - 1830

$$f(x) = a_0 + a_1 \cos(x)$$

Fourier Transform

- Represent a function as a weighted sum of sines and cosines

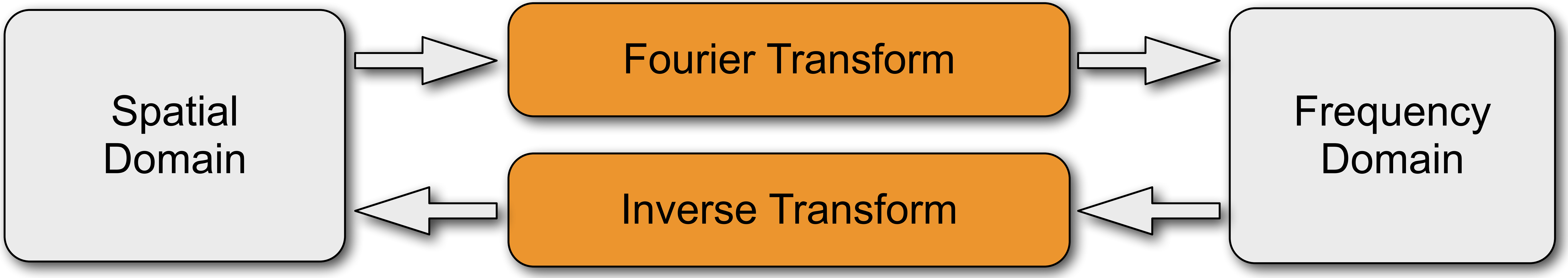
Joseph Fourier 1768 - 1830



$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x) + a_3 \cos(5x) + a_4 \cos(7x) + \ldots$$

Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} \omega x} \mathrm{d}x$$



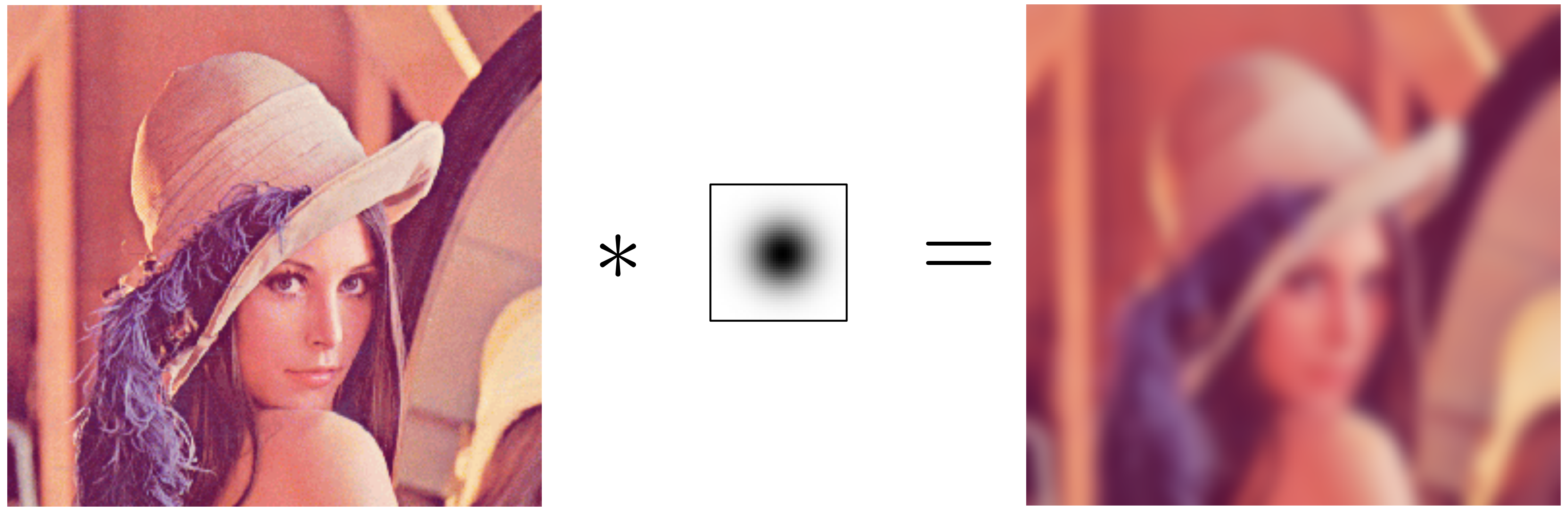
$$f(x) = \int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{2\pi \mathrm{i} \omega x}\, \mathrm{d}\omega$$

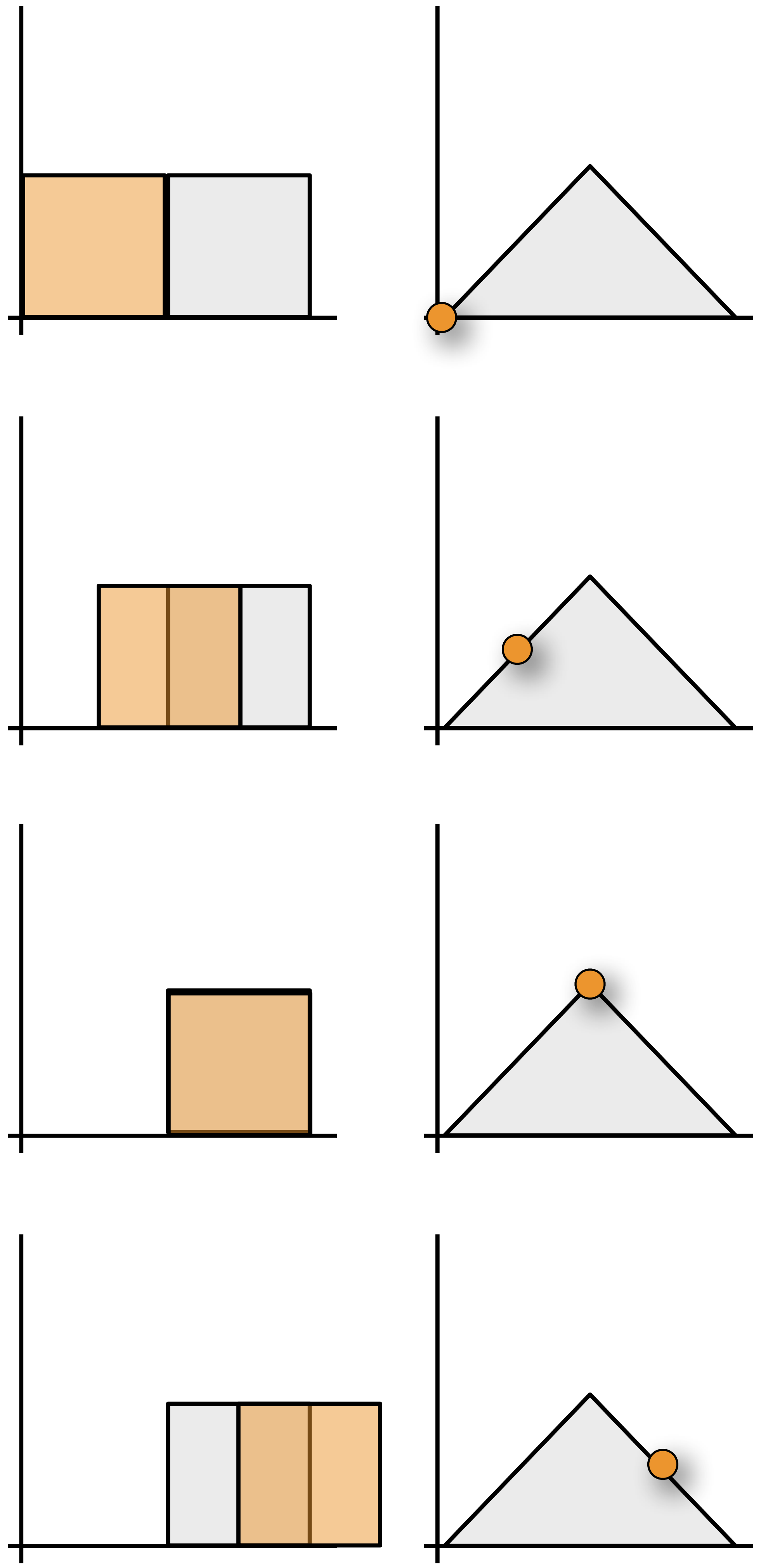
Convolution

- Smooth signal by convolution with a kernel function

$$h(x) = f * g := \int f(y) \cdot g(x-y)\,\mathrm{d}y$$

- Example: Gaussian blurring





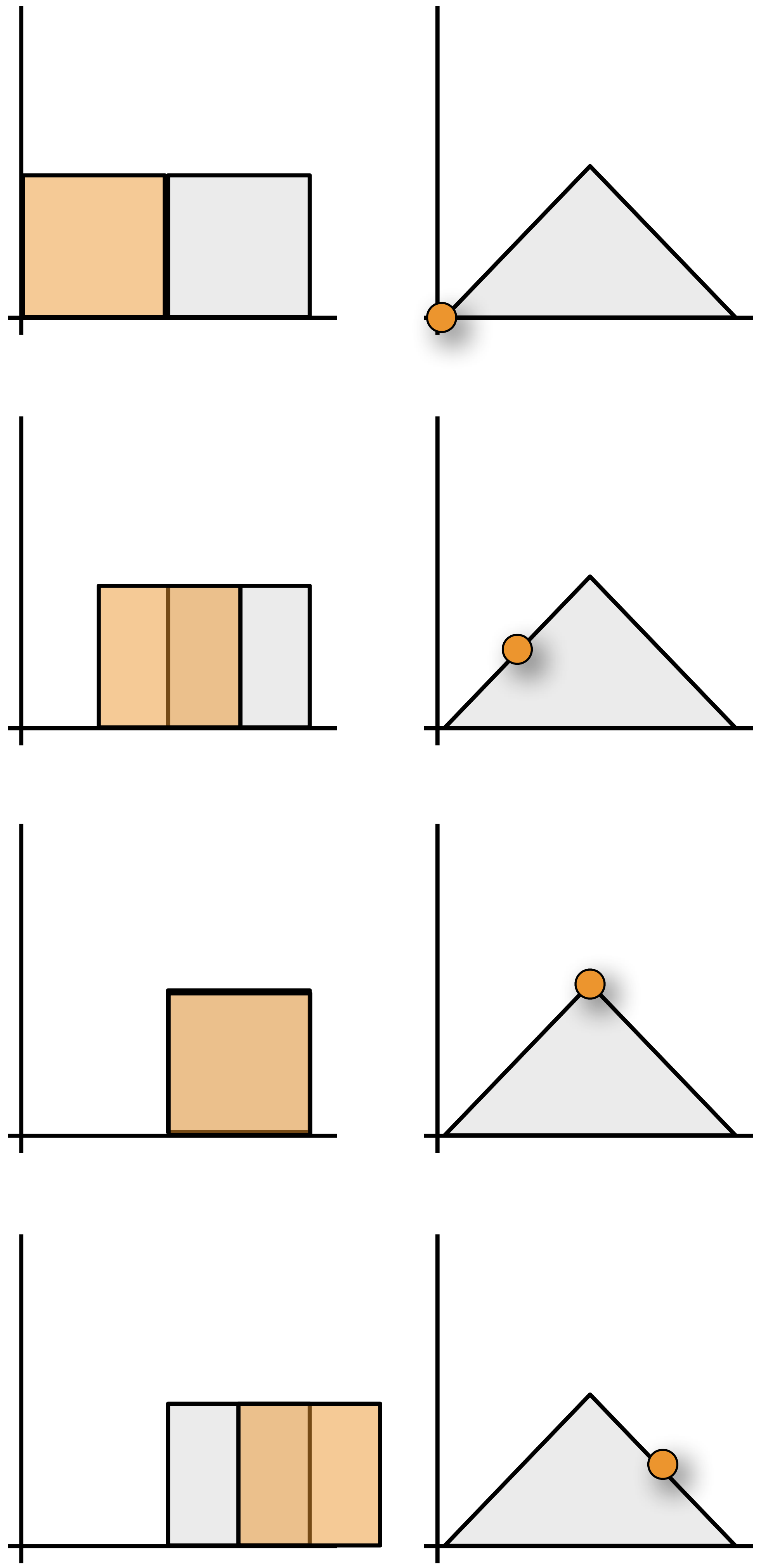
Convolution

- Smooth signal by convolution with a kernel function

$$h(x) = f * g := \int f(y) \cdot g(x-y) \, \mathrm{d}y$$

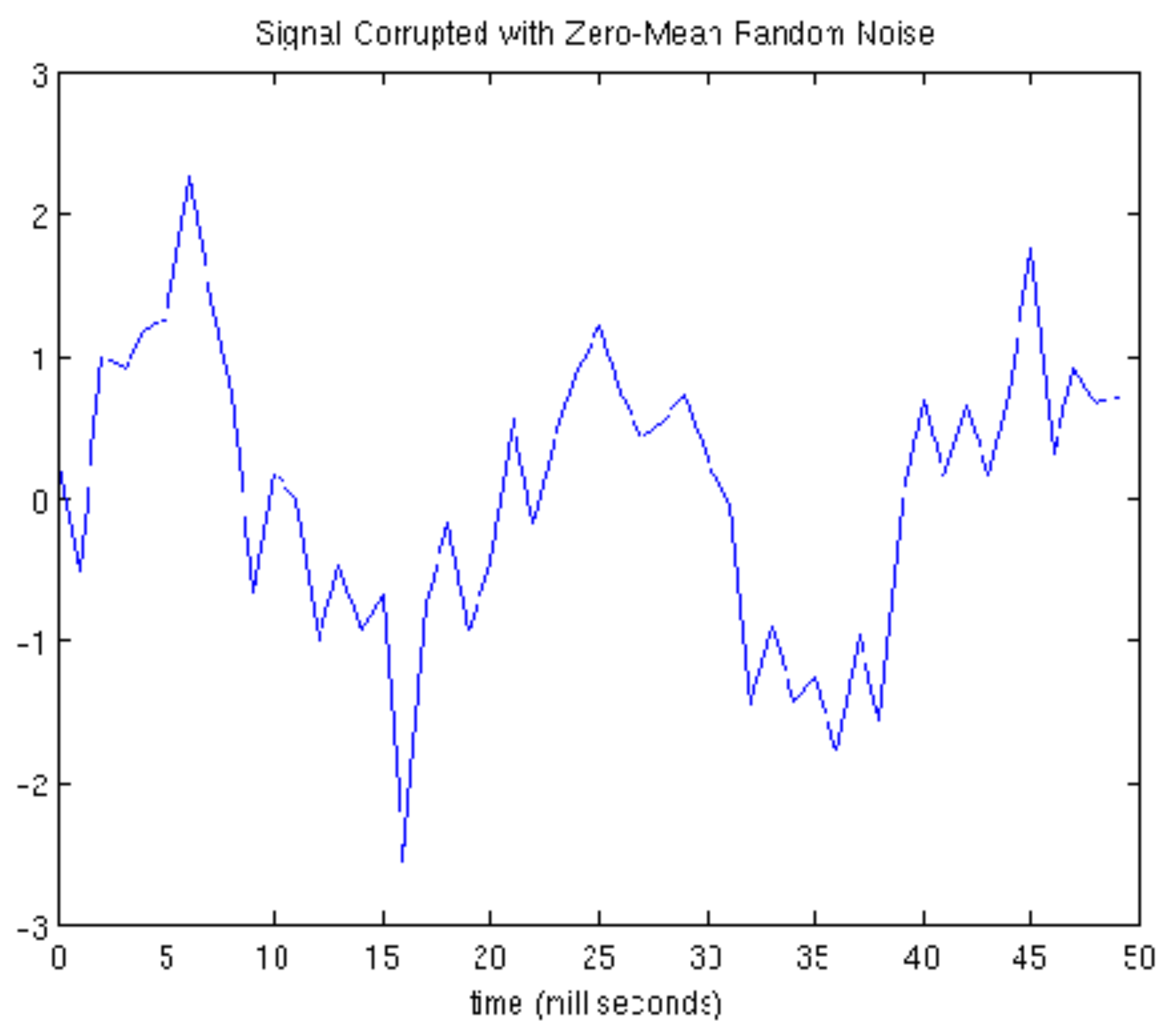
- Convolution in spatial domain ⇔ Multiplication in frequency domain

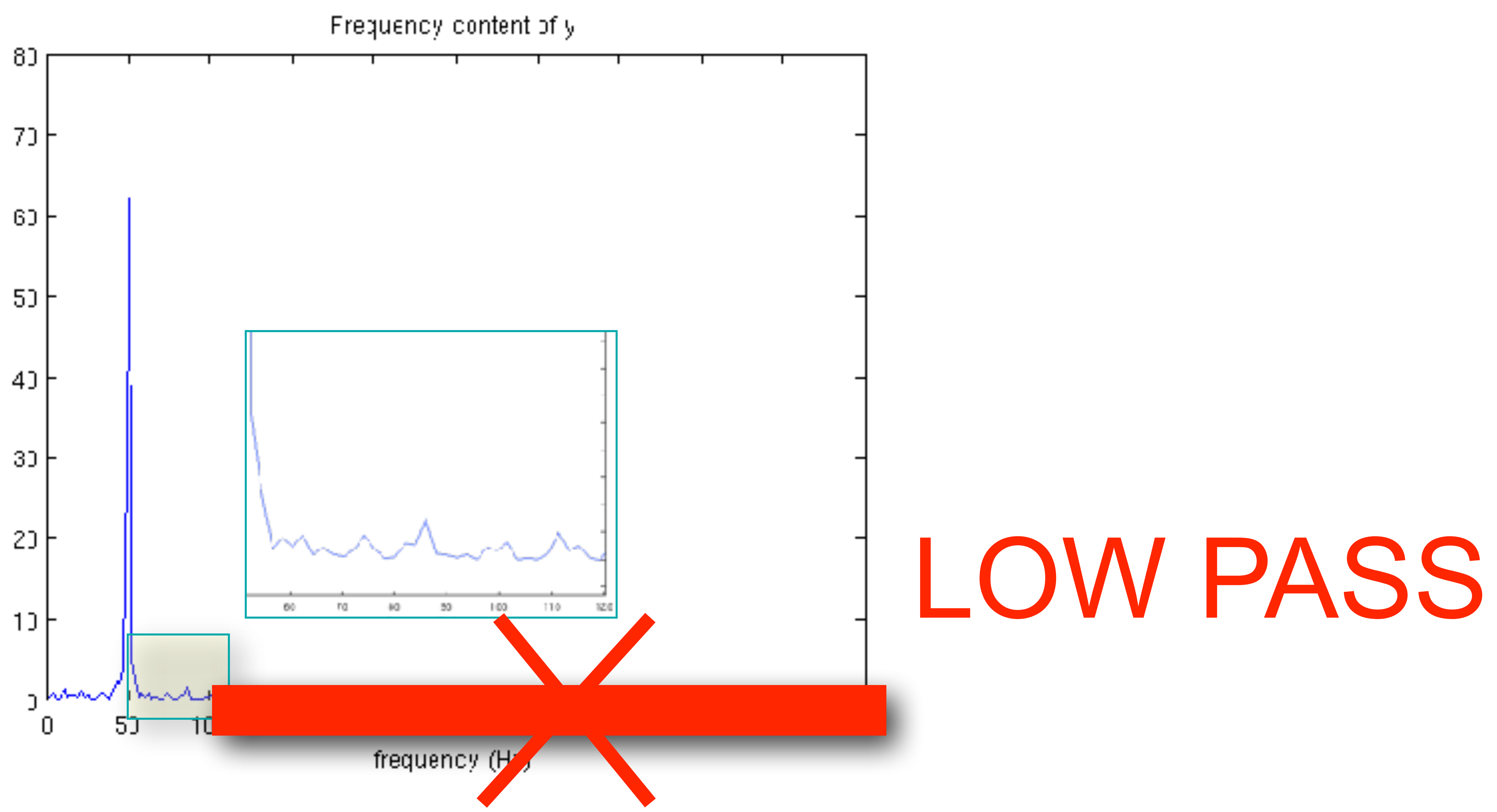
$$H(\omega) = F(\omega) \cdot G(\omega)$$



Fourier Analysis

- Low-pass filter discards high frequencies





spatial domain frequency domain

Fourier Transform

- Spatial domain *f(x)* → Frequency domain *F(ω)*

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} \omega x} \mathrm{d}x$$

- Multiply by low-pass filter *G(ω)*

$$F(\omega) \;\leftarrow\; F(\omega) \cdot G(\omega)$$

- Frequency domain *F(ω)* → Spatial domain *f(x)*

$$f(x) = \int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{2\pi \mathrm{i} \omega x}\, \mathrm{d}\omega$$

Fourier Transform

- Consider L^2 function space with inner product

$$\langle f, g \rangle := \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, \mathrm{d}x$$

- Complex "waves" build an orthonormal basis

$$e_\omega(x) \; := \; \mathrm{e}^{-2\pi \mathrm{i} \omega x} \; = \; \cos(2\pi\omega x) - \mathrm{i}\sin(2\pi\omega x)$$

- Fourier transform is a change of basis

$$f(x) = \int_{-\infty}^{\infty} \langle f, e_\omega \rangle \; e_\omega \, \mathrm{d}\omega \qquad\qquad f(x) = \sum_{\omega=-\infty}^{\infty} \langle f, e_\omega \rangle \; e_\omega \, \mathrm{d}\omega$$

Fourier Analysis on Meshes?

- Only applicable to parametric patches $f(u, v)$
- Generalize frequency concept to meshes!
- Complex waves are Eigenfunctions of Laplace

$$\Delta\left(e^{2\pi \mathrm{i}\omega x}\right) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} e^{2\pi \mathrm{i}\omega x} = -(2\pi\omega)^2 e^{2\pi \mathrm{i}\omega x}$$

➡ ***Use Eigenfunctions of discrete Laplace-Beltrami***

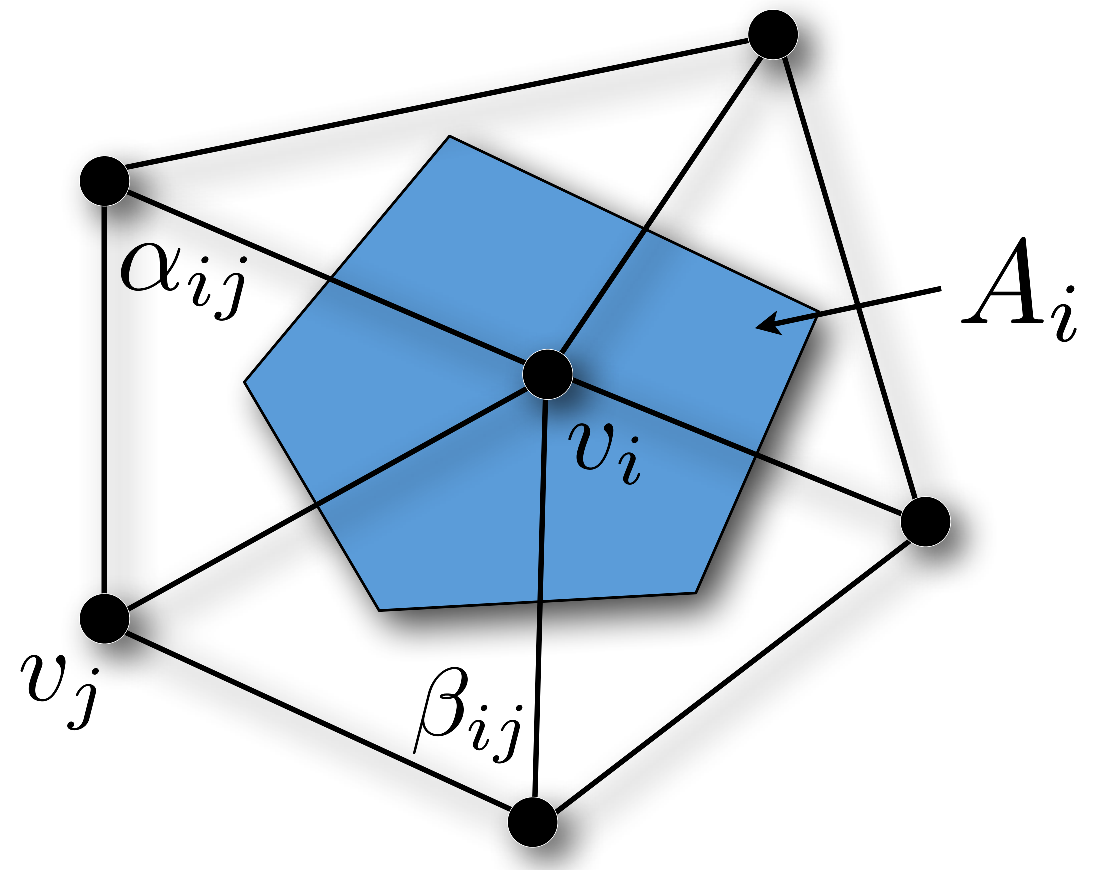
Discrete Laplace-Beltrami

- Function values sampled at mesh vertices

$$\mathbf{f} = [f_1, f_2, \ldots, f_n] \in \mathbb{R}^n$$

- Discrete Laplace-Beltrami (per vertex)

$$\Delta_{\mathcal{S}} f(v_i) := \frac{1}{2A_i} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij}) \left(f(v_j) - f(v_i)\right)$$

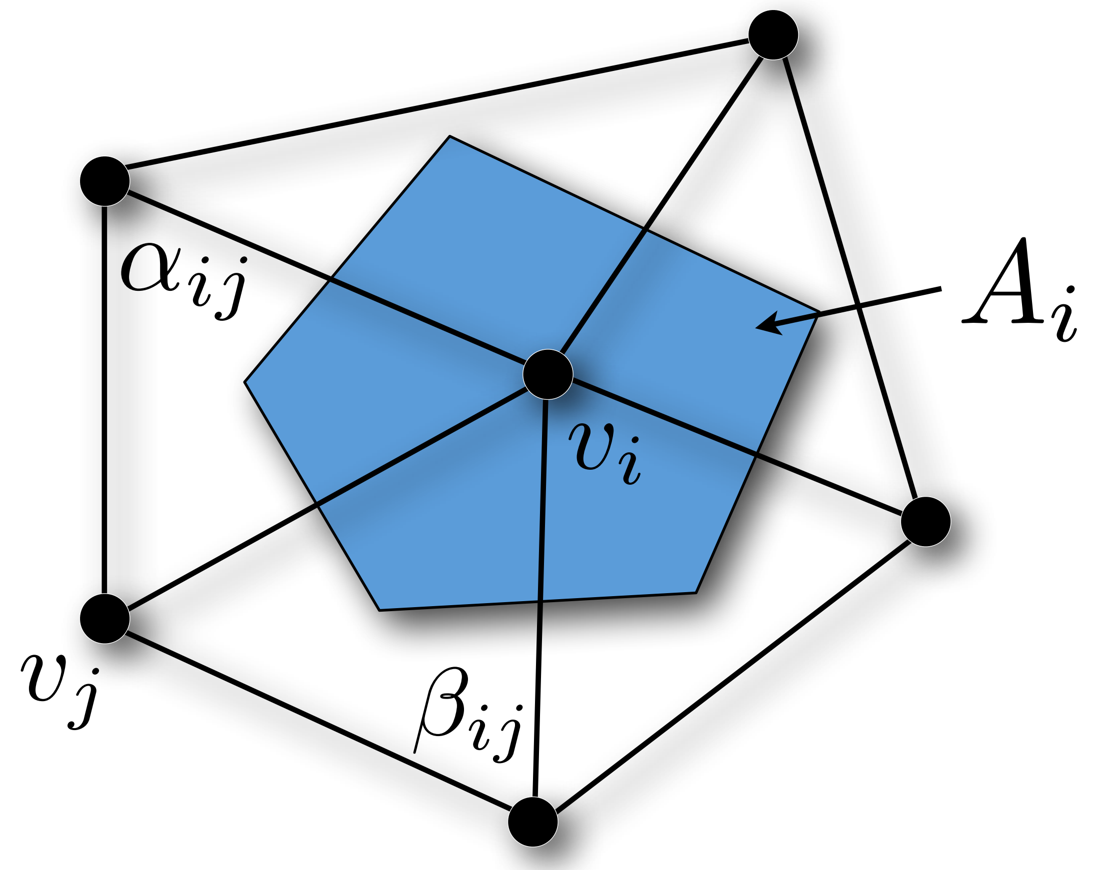


Discrete Laplace-Beltrami

- Discrete Laplace *operator* (per mesh)
  - Sparse matrix

$$\mathbf{L} = \mathbf{M}^{-1}\mathbf{C} \in \mathbb{R}^{n \times n}$$

$$\begin{pmatrix} \vdots \\ \Delta_{\mathcal{S}} f(v_i) \\ \vdots \end{pmatrix} = \mathbf{L} \cdot \begin{pmatrix} \vdots \\ f(v_i) \\ \vdots \end{pmatrix}$$



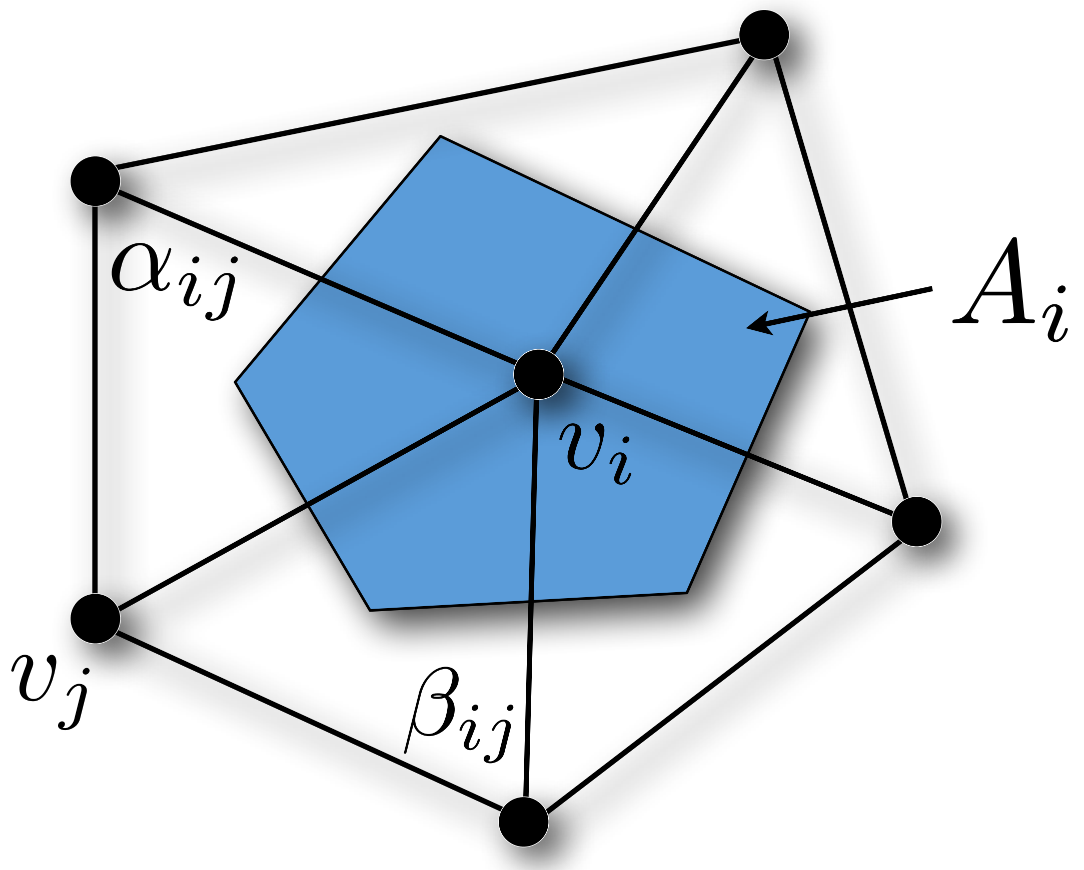
Discrete Laplace-Beltrami

$$\mathbf{L} = \mathbf{M}^{-1}\mathbf{C} \in \mathbb{R}^{n\times n}$$

- Discrete Laplace *operator* (per mesh)
  - Sparse matrix

$$\mathbf{C}_{ij} = \begin{cases} \cot\alpha_{ij} + \cot\beta_{ij}, & i \neq j\,,\ j \in \mathcal{N}_1(v_i) \\ -\sum_{v_j\in\mathcal{N}_1(v_i)}(\cot\alpha_{ij} + \cot\beta_{ij}) & i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{M}^{-1} = \operatorname{diag}\left(\ldots, \frac{1}{2A_i}, \ldots\right)$$



  - Note fraction 1/2 can either be in C or in M matrix

Discrete Laplace-Beltrami

- Discrete function sampled at mesh vertices

$$\mathbf{f} = [f_1, f_2, \ldots, f_n] \in \mathbb{R}^n$$

- Discrete Laplace-Beltrami (per vertex)

$$\Delta_{\mathcal{S}} f(v_i) := \frac{1}{2A_i} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij})\,(f(v_j) - f(v_i))$$

- Discrete Laplace-Beltrami matrix **L**
  - Eigenvectors are “natural vibrations”
  - Eigenvalues are “natural frequencies”

$$\Delta \mathbf{e}_i = \lambda_i \mathbf{e}_i$$

Discrete Laplace-Beltrami

$\mathbf{e}_2$ $\mathbf{e}_3$

- Discrete Laplace-Beltrami matrix **L**
  - Eigenvectors are “natural vibrations”
  - Eigenvalues are “natural frequencies”

Spectral Analysis

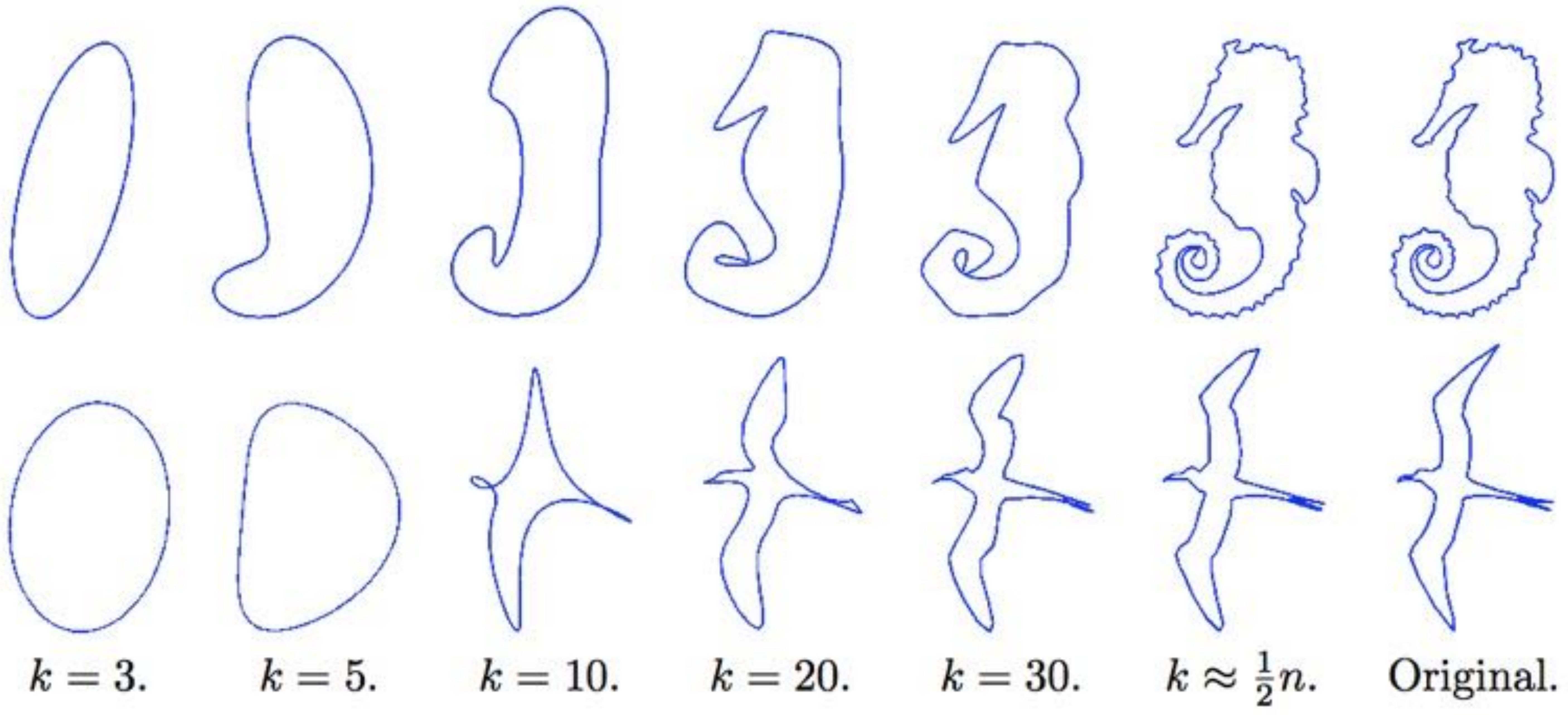
1. Setup Laplace-Beltrami matrix $\mathbf{L} = \Delta$
2. Compute *k* smallest eigenvectors {$\mathbf{e}_1$,..., $\mathbf{e}_k$}
3. Reconstruct mesh from those (component-wise)

$$\mathbf{x} := [x_1, \ldots, x_n] \qquad \mathbf{y} := [y_1, \ldots, y_n] \qquad \mathbf{z} := [z_1, \ldots, z_n]$$

$$\mathbf{x} \leftarrow \sum_{i=1}^{k} \left(\mathbf{x}^T \mathbf{e}_i\right) \mathbf{e}_i \qquad \mathbf{y} \leftarrow \sum_{i=1}^{k} \left(\mathbf{y}^T \mathbf{e}_i\right) \mathbf{e}_i \qquad \mathbf{z} \leftarrow \sum_{i=1}^{k} \left(\mathbf{z}^T \mathbf{e}_i\right) \mathbf{e}_i$$

Spectral Analysis

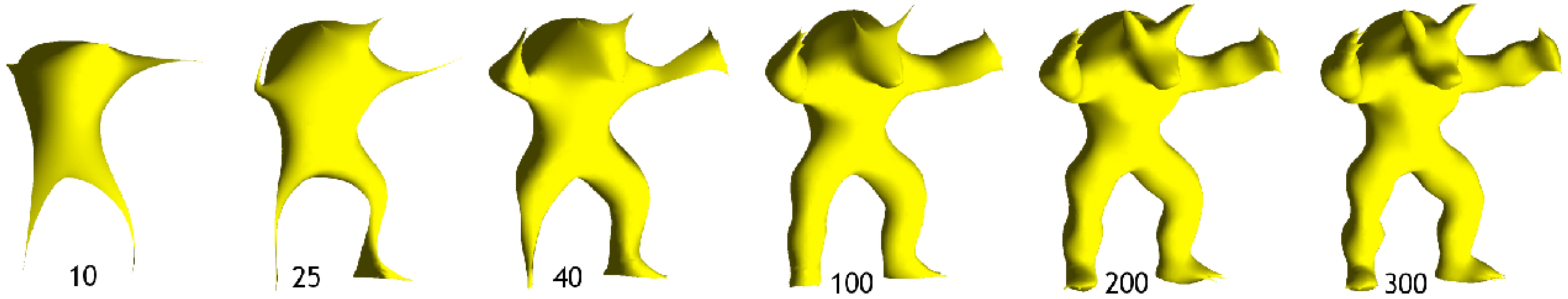
1. Setup Laplace-Beltrami matrix **L**
2. Compute k smallest eigenvectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$
3. Reconstruct mesh from those



Spectral Analysis

1. Setup Laplace-Beltrami matrix **L**
2. Compute k smallest eigenvectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$
3. Reconstruct mesh from those

Too complex for large meshes!



Bruno Levy: *Laplace-Beltrami Eigenfunctions: Towards an algorithm that understands geometry*, Shape Modeling and Applications, 2006

Review: Curves

Differential Geometry of Curves

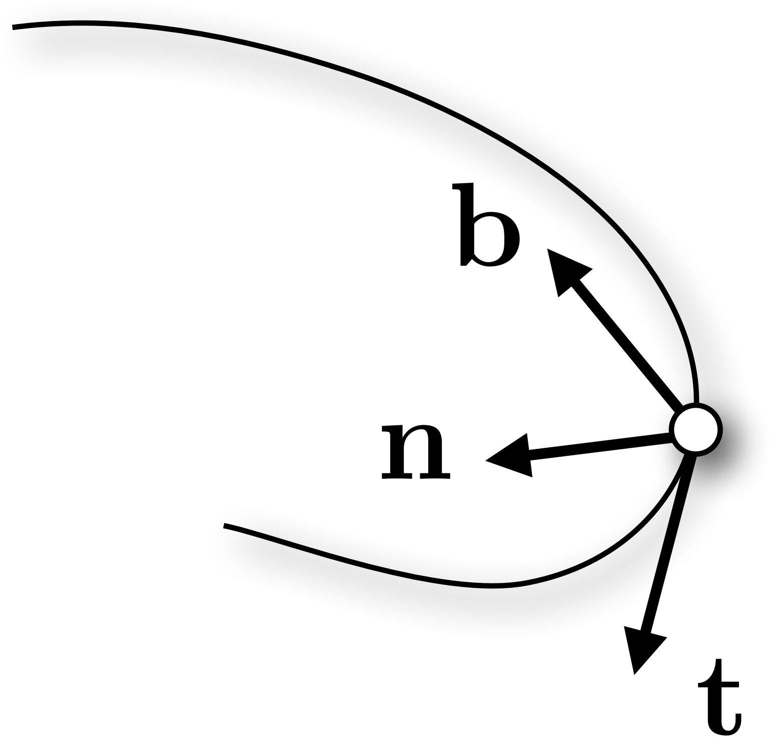
- chord length and arc length
- arc-length parameterization
- Frenet frame

$$s = s(t) = \int_a^t \|\mathbf{x}_t\| \, \mathrm{d}t$$

$$\mathbf{t} = \frac{\mathbf{x}_t}{\|\mathbf{x}_t\|}$$

$$\mathbf{n} = \mathbf{b} \times \mathbf{t}$$

$$\mathbf{b} = \frac{\mathbf{x}_t \times \mathbf{x}_{tt}}{\|\mathbf{x}_t \times \mathbf{x}_{tt}\|}$$



Frenet frame

Review: Curves

Differential Geometry of Curves

– ***Frenet-Serret***: Derivatives w.r.t. arc length *s*

$$\begin{aligned} \mathbf{t}_s &= & +\kappa\mathbf{n} & \\ \mathbf{n}_s &= -\kappa\mathbf{t} & & +\tau\mathbf{b} \\ \mathbf{b}_s &= & -\tau\mathbf{n} & \end{aligned}$$

– **Curvature** (*deviation from straight line*)

$$\kappa = \|\mathbf{x}_{ss}\|$$

– **Torsion** (*deviation from planarity*)

$$\tau = \frac{1}{\kappa^2}\det([\mathbf{x}_s, \mathbf{x}_{ss}, \mathbf{x}_{sss}])$$

Review: Surface

Differential Geometry of Surfaces

– **First fundamental form**

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix} \qquad \left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

- **Angle**

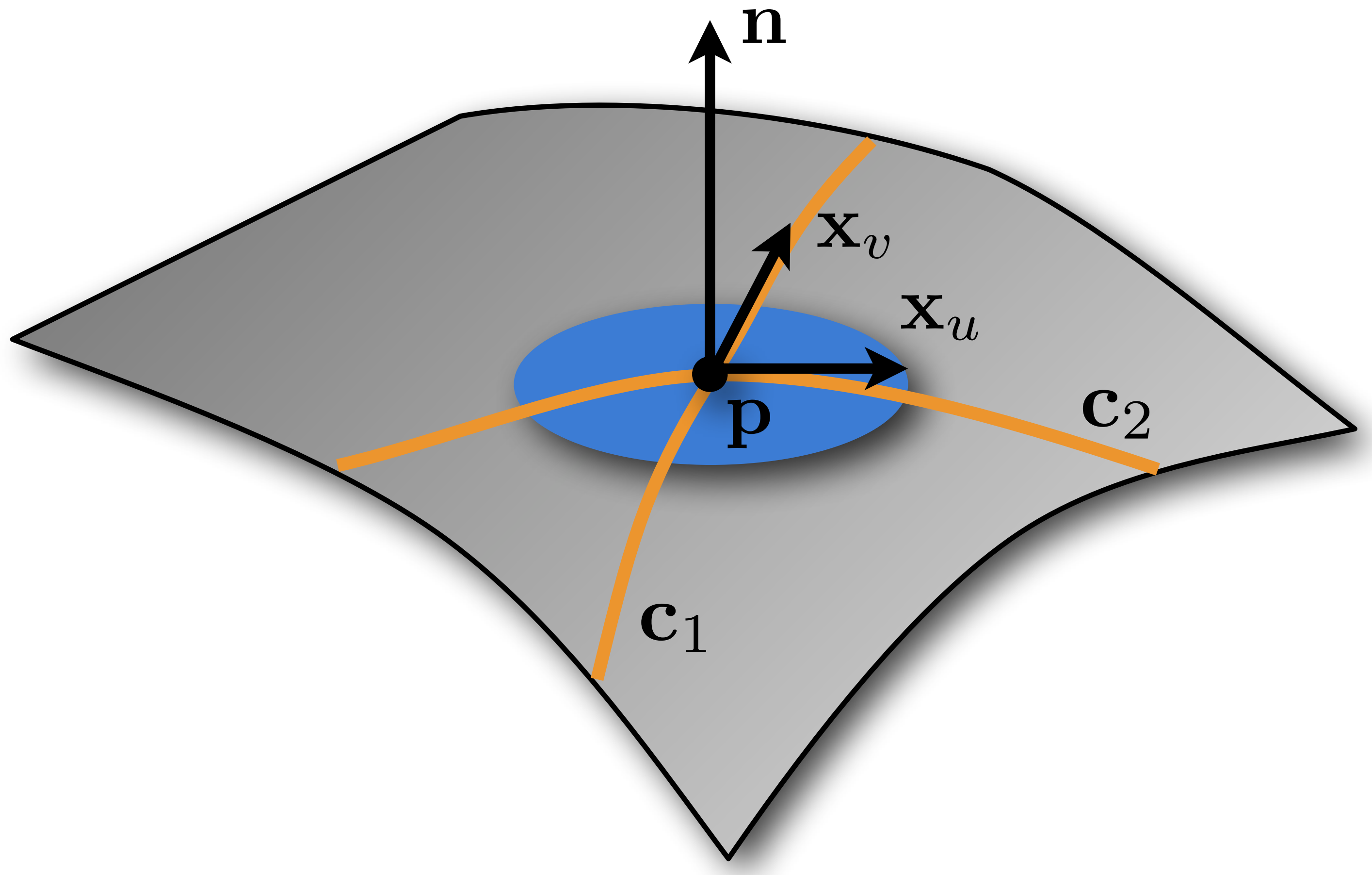
$$\mathbf{t}_1^T \mathbf{t}_2 = \langle (\alpha_1, \beta_1) \, , \, (\alpha_1, \beta_1) \rangle$$

- **Length**

$$\mathrm{d}s^2 = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$$

- **Area**

$$\mathrm{d}A = \sqrt{EG - F^2}\mathrm{d}u\,\mathrm{d}v$$

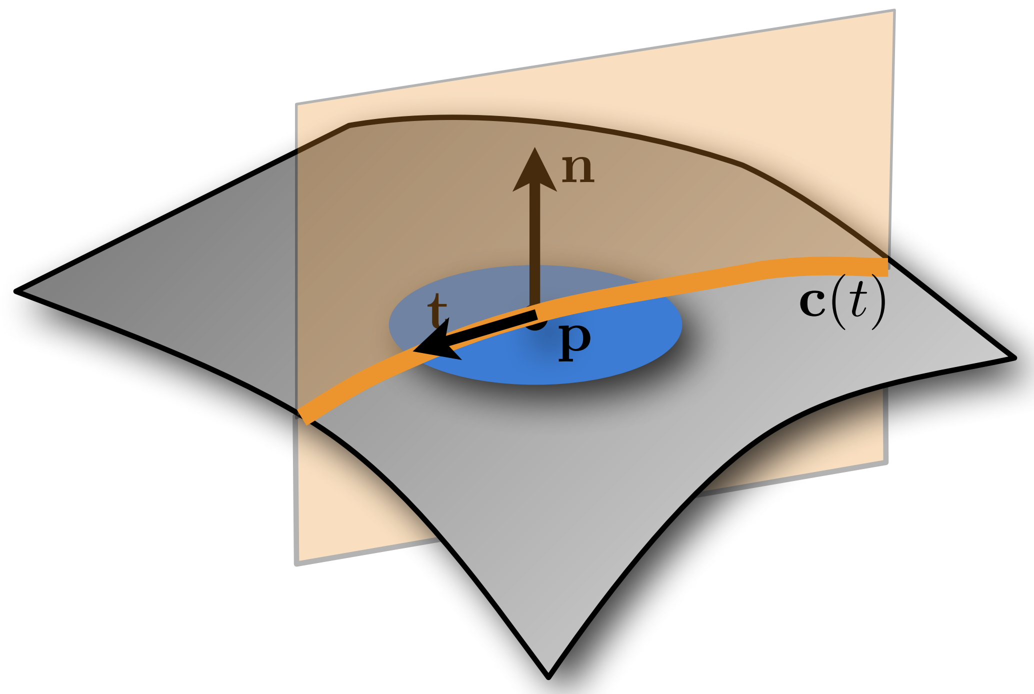


Review: Surface

Differential Geometry of Surfaces

– **Second fundamental form**

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{pmatrix}$$



- **normal curvature**

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \, \mathbf{II} \, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \, \mathbf{I} \, \bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}$$

- **principal curvatures**

$$\kappa_1 = \max_{\phi} \kappa_n(\phi) \qquad \kappa_2 = \min_{\phi} \kappa_n(\phi)$$

- **mean and Gaussian curvature**

$$H = \frac{\kappa_1 + \kappa_2}{2} \qquad K = \kappa_1 \cdot \kappa_2$$

Outline

- Spectral Analysis
- **Diffusion Flow**
- Energy Minimization

Diffusion Flow on Height Fields

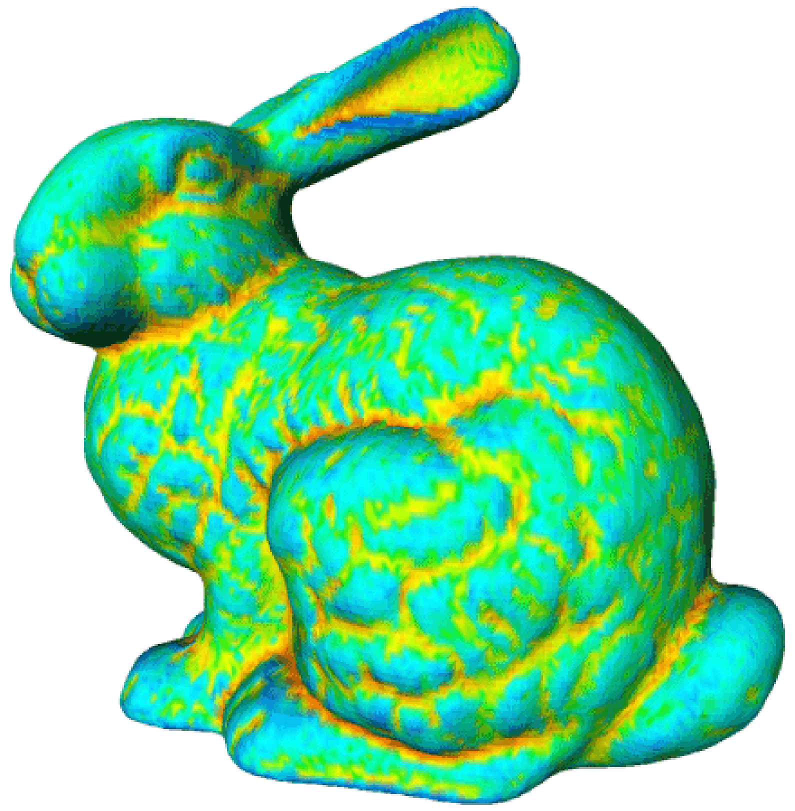
diffusion constant

$$\frac{\partial f}{\partial t} = \lambda \Delta f$$

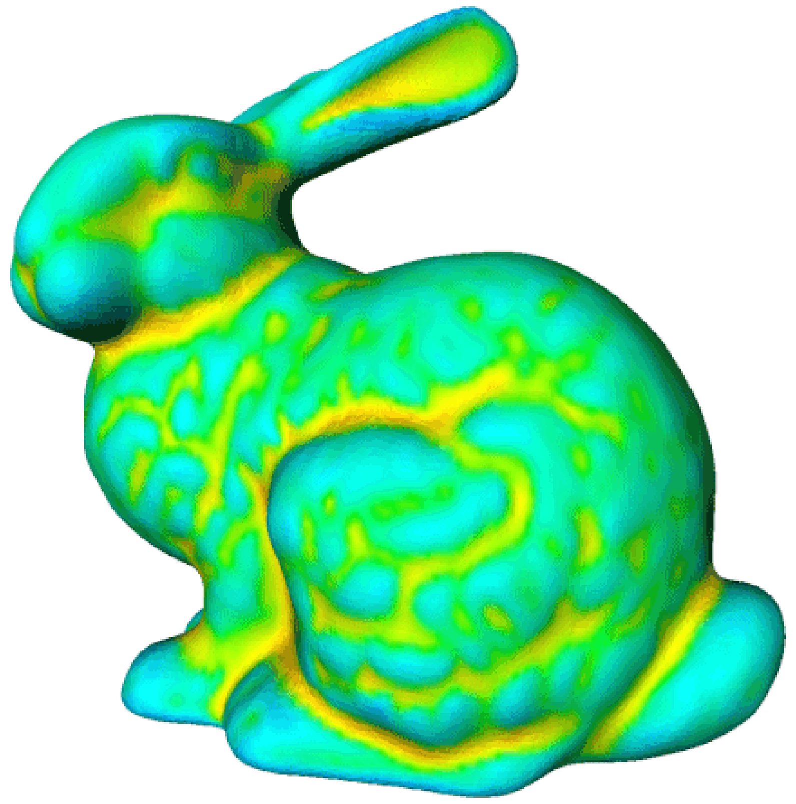
Laplace operator

Diffusion Flow on Meshes

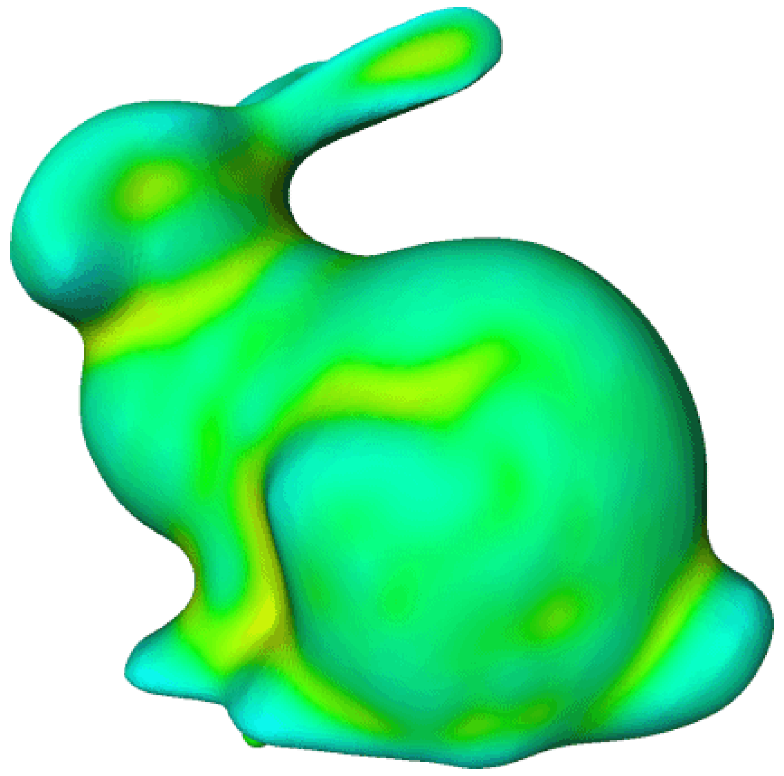
- Iterate $\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda \Delta \mathbf{p}_i$



0 Iterations



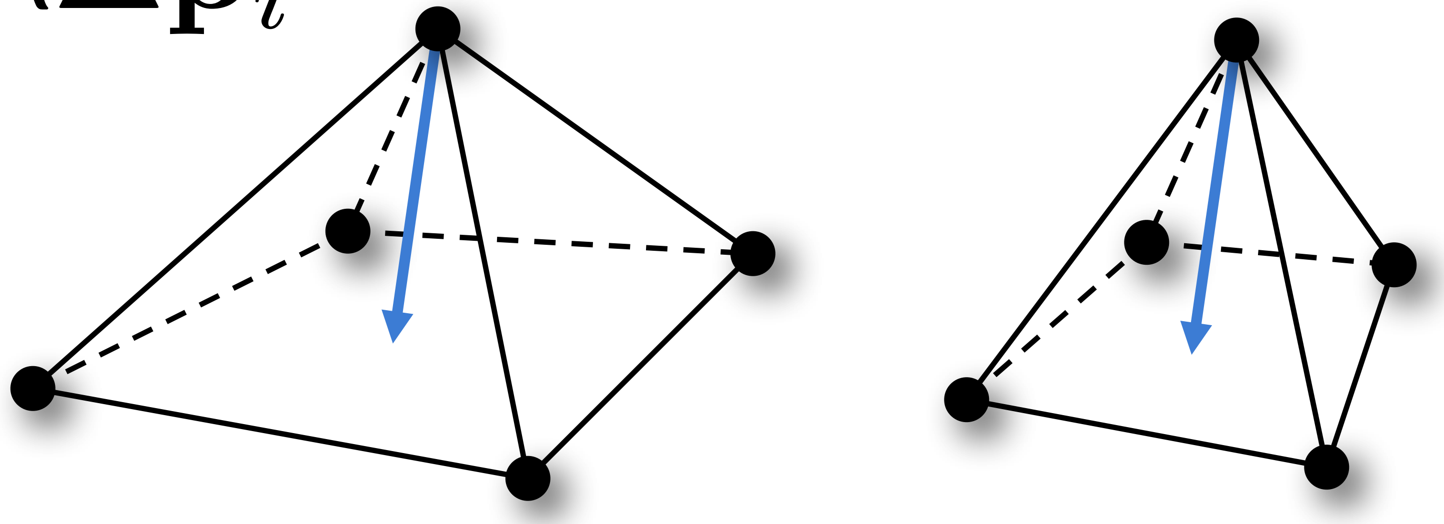
5 Iterations



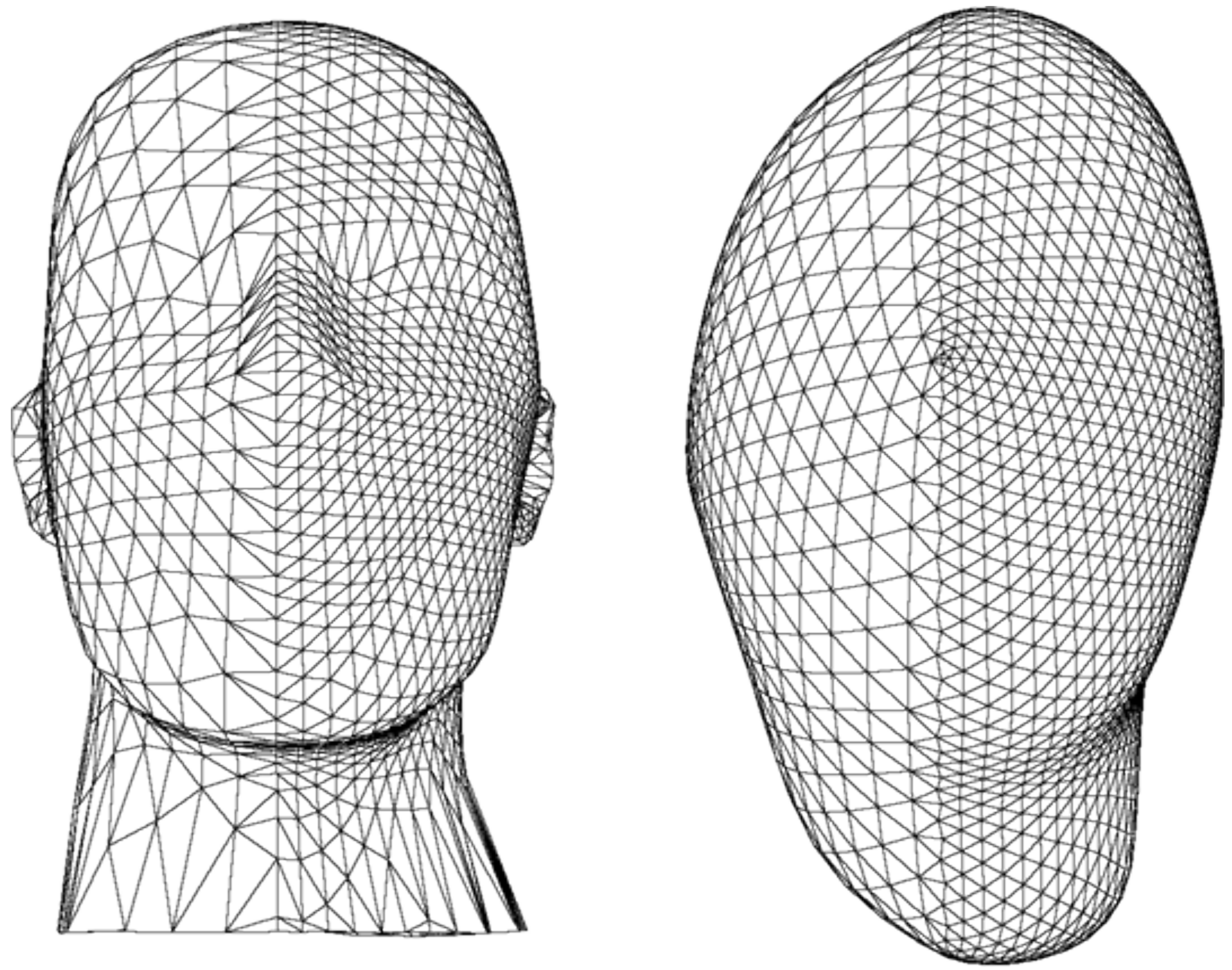
20 Iterations

Uniform Laplace Discretization

$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda \Delta \mathbf{p}_i$$



- Smoothes geometry and triangulation
- Can be non-zero even for planar triangulations
- Vertex drift can lead to distortions
- Might be desired for mesh regularization



Desbrun et al., Siggraph 1999

Curvature Flow

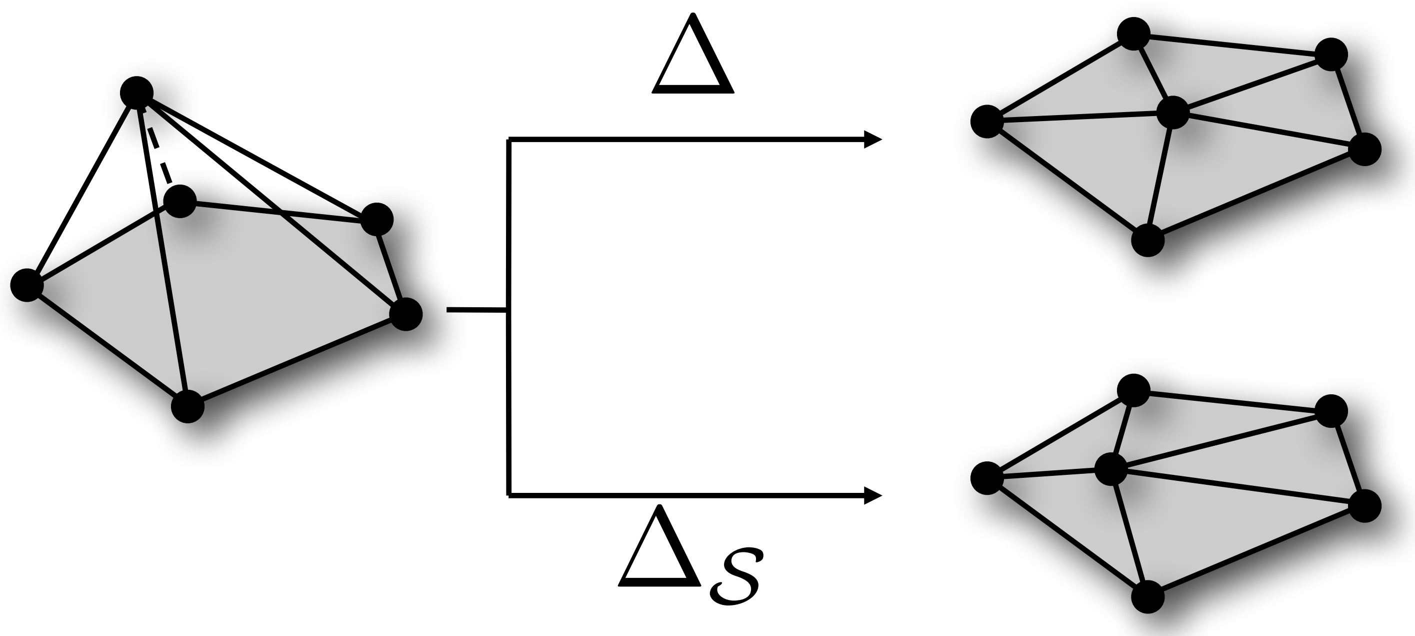
- Use diffusion flow with Laplace-Beltrami

$$\frac{\partial \mathbf{p}}{\partial t} = \lambda \Delta_{\mathcal{S}} \mathbf{p}$$

- Laplace-Beltrami is parallel to surface normal

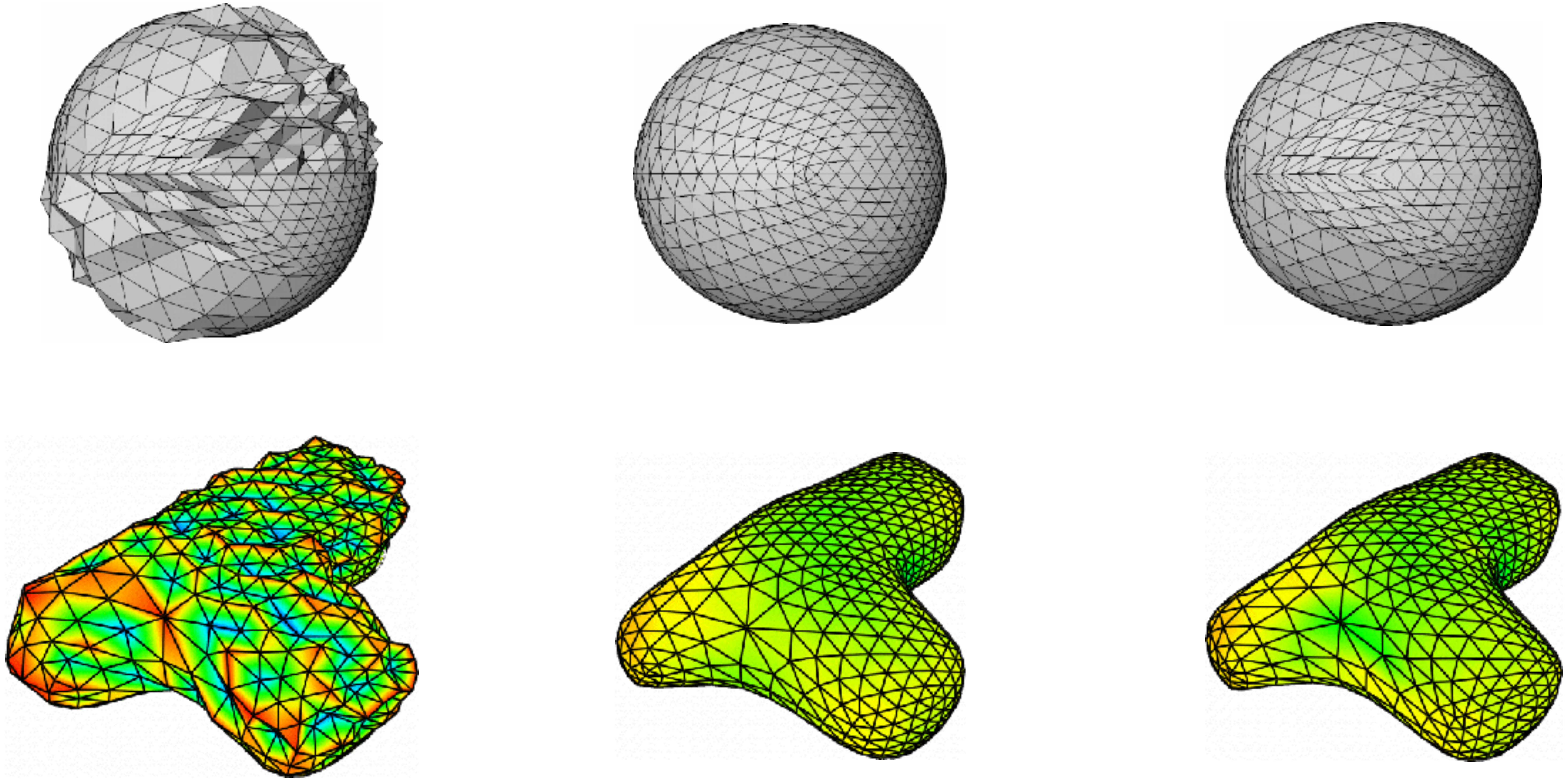
$$\frac{\partial \mathbf{p}}{\partial t} = -2\lambda H \mathbf{n}$$

➡ Avoids vertex drift on surface



Comparison

Original | Uniform Laplace | Laplace-Beltrami



Numerical Integration

- Write update $\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \lambda \Delta \mathbf{p}_i^{(t)}$ in matrix notation

$$\mathbf{P}^{(t)} = \left(\mathbf{p}_1^{(t)}, \ldots, \mathbf{p}_n^{(t)}\right)^T \in \mathbb{R}^{n \times 3}$$

$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda \Delta \mathbf{p}_i$$

- Corresponds to **explicit** integration

$$\mathbf{P}^{(t+1)} = (\mathbf{I} + \lambda \mathbf{L})\, \mathbf{P}^{(t)}$$

Requires small λ for stability!

- **Implicit** integration is *unconditionally stable*

$$(\mathbf{I} - \lambda \mathbf{L})\, \mathbf{P}^{(t+1)} = \mathbf{P}^{(t)}$$

Implementation

- Solve linear system each iteration

$$(\mathbf{I} - \lambda \mathbf{L})\mathbf{P}^{(t+1)} = \mathbf{P}^{(t)}$$

- Matrix $L = M^{-1}L_w$ is not symmetric because of M
  - ➡ Symmetrize by multiplying M from left

$$(\mathbf{M} - \lambda \mathbf{L}_w)\mathbf{P}^{(t+1)} = \mathbf{M}\mathbf{P}^{(t)}$$

- Solve sparse symmetric positive definite system
  - ➡ Iterative conjugate gradients, sparse Cholesky

Outline

- Spectral Analysis
- Diffusion Flow
- **Energy Minimization**

Fairness

- Idea: Penalize "unaesthetic behavior"
- **Measure** fairness
  - Principle of the simplest shape
  - Physical interpretation
- **Minimize** some fairness functional
  - Surface area, curvature
  - Membrane energy, thin plate energy

Non-Linear Energies

- Membrane energy (surface area)

$$\int_{\mathcal{S}} \mathrm{d}A \;\rightarrow\; \min \quad \text{with} \quad \delta\mathcal{S} = \mathbf{c}$$

- Thin-plate surface (curvature)

$$\int_{\mathcal{S}} \kappa_1^2 + \kappa_2^2 \,\mathrm{d}A \;\rightarrow\; \min \quad \text{with} \quad \delta\mathcal{S} = \mathbf{c}, \quad \mathbf{n}(\delta\mathcal{S}) = \mathbf{d}$$

- Too complex ... simplify energies

Membrane Surfaces

- Surface parameterization

$$\mathbf{x} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

- Membrane energy (surface area)

$$\min_{\mathbf{x}} \int_{\Omega} \|\mathbf{x}_u\|^2 + \|\mathbf{x}_v\|^2 \mathrm{d}u\mathrm{d}v$$

Variational Calculus in 1D

- 1D membrane energy

$$L(f) = \int_a^b f'^2(x) \, \mathrm{d}x \;\rightarrow\; \min$$

- Add test function *u* with *u*(*a*) = *u*(*b*) = 0

$$L(f + \lambda u) = \int_a^b (f' + \lambda u')^2 = \int_a^b f'^2 + 2\lambda f' u' + \lambda^2 u'^2$$

- If *f* minimizes *L*, the following has to vanish

$$\left. \frac{\partial L(f + \lambda u)}{\partial \lambda} \right|_{\lambda = 0} = \int_a^b 2 f' u' \stackrel{!}{=} 0$$

Variational Calculus in 1D

- Has to vanish for any *u* with *u*(*a*) = *u*(*b*) = 0

$$\int_a^b f'u' = \underbrace{[f'u]_a^b}_{=0} - \int_a^b f''u \stackrel{!}{=} 0 \quad \forall u$$

$$\int_0^1 x'y = [xy]_0^1 - \int_0^1 xy'$$

- Only possible if

$$f'' = \Delta f = 0$$

➡ ***Euler-Lagrange equation***

Bivariate Variational Calculus

- Find minimum of functional

$$\underset{f}{\operatorname{argmin}} \int_{\Omega} L\left(f_{uu}, f_{vv}, f_u, f_v, f, u, v\right)$$

- Euler-Lagrange PDE defines the minimizer *f*

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial u}\frac{\partial L}{\partial f_u} - \frac{\partial}{\partial v}\frac{\partial L}{\partial f_v} + \frac{\partial^2}{\partial u^2}\frac{\partial L}{\partial f_{uu}} + \frac{\partial^2}{\partial u \partial v}\frac{\partial L}{\partial f_{uv}} + \frac{\partial^2}{\partial v^2}\frac{\partial L}{\partial f_{vv}} = 0$$

– Again, subject to suitable boundary constraints

Membrane Surfaces

- Surface parameterization

$$\mathbf{x} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

- Membrane energy (surface area)

$$\min_{\mathbf{x}} \int_{\Omega} \|\mathbf{x}_u\|^2 + \|\mathbf{x}_v\|^2 \mathrm{d}u\mathrm{d}v$$

- Variational calculus

$$\Delta \mathbf{x} = 0$$

Laplace-Beltrami Operator

- Extension of Laplace to functions on manifolds

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{S}} \mathbf{x} = \mathrm{div}_{\mathcal{S}} \nabla_{\mathcal{S}} \mathbf{x} = -2H\mathbf{n}$$

$$\nabla \mathbf{x} = [\partial \mathbf{x}/\partial u \quad \partial \mathbf{x}/\partial v]$$

$$\mathrm{div}\mathbf{y} := \nabla \cdot \mathbf{y} = \partial \mathbf{y}_u/\partial u + \partial \mathbf{y}_v/\partial v$$

$$\mathrm{div}(\nabla \mathbf{x}) = \Delta \mathbf{x}$$

Thin-Plate Surfaces

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$$

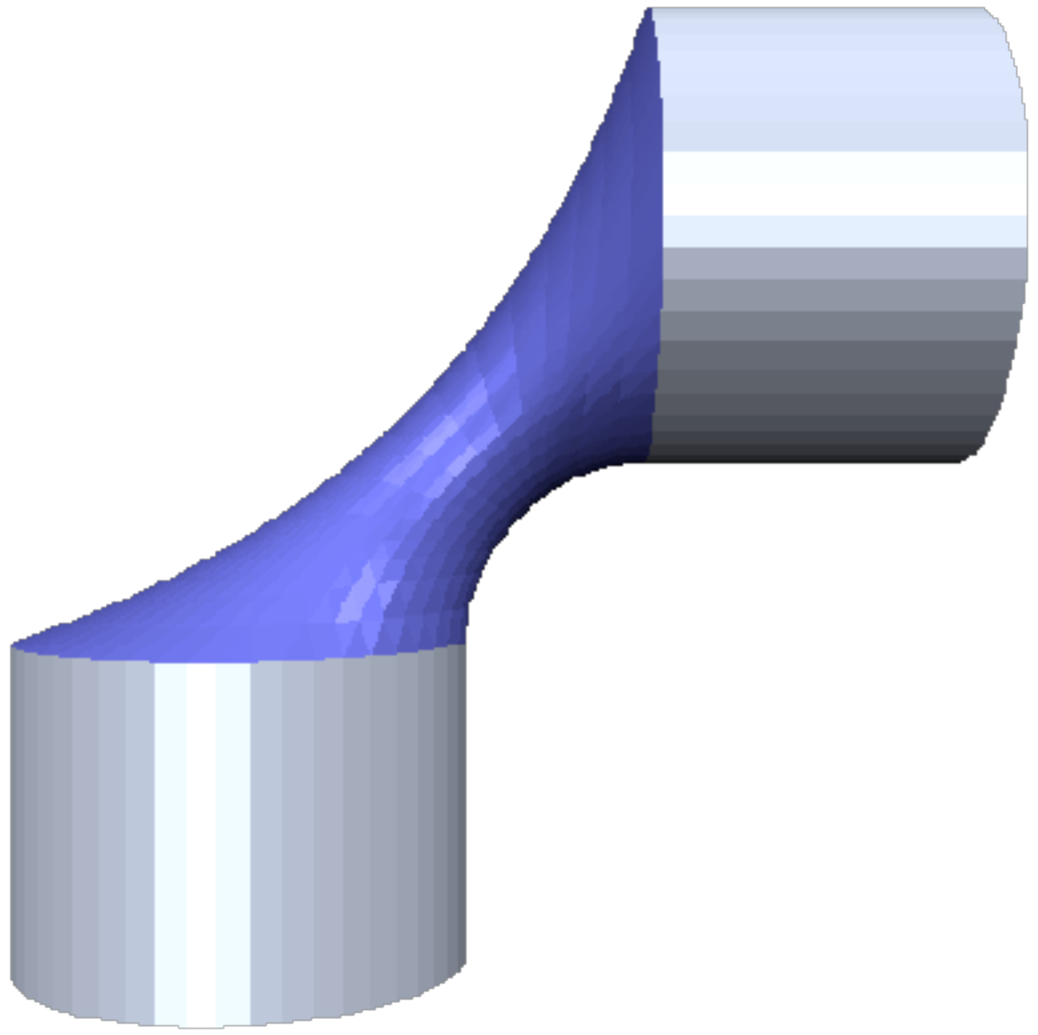
- Thin-plate energy (curvature)

$$\int_\Omega \|\mathbf{p}_{uu}\|^2 + 2\,\|\mathbf{p}_{uv}\|^2 + \|\mathbf{p}_{vv}\|^2 \,\mathrm{d}u\mathrm{d}v \;\to\; \min$$

- Variational calculus

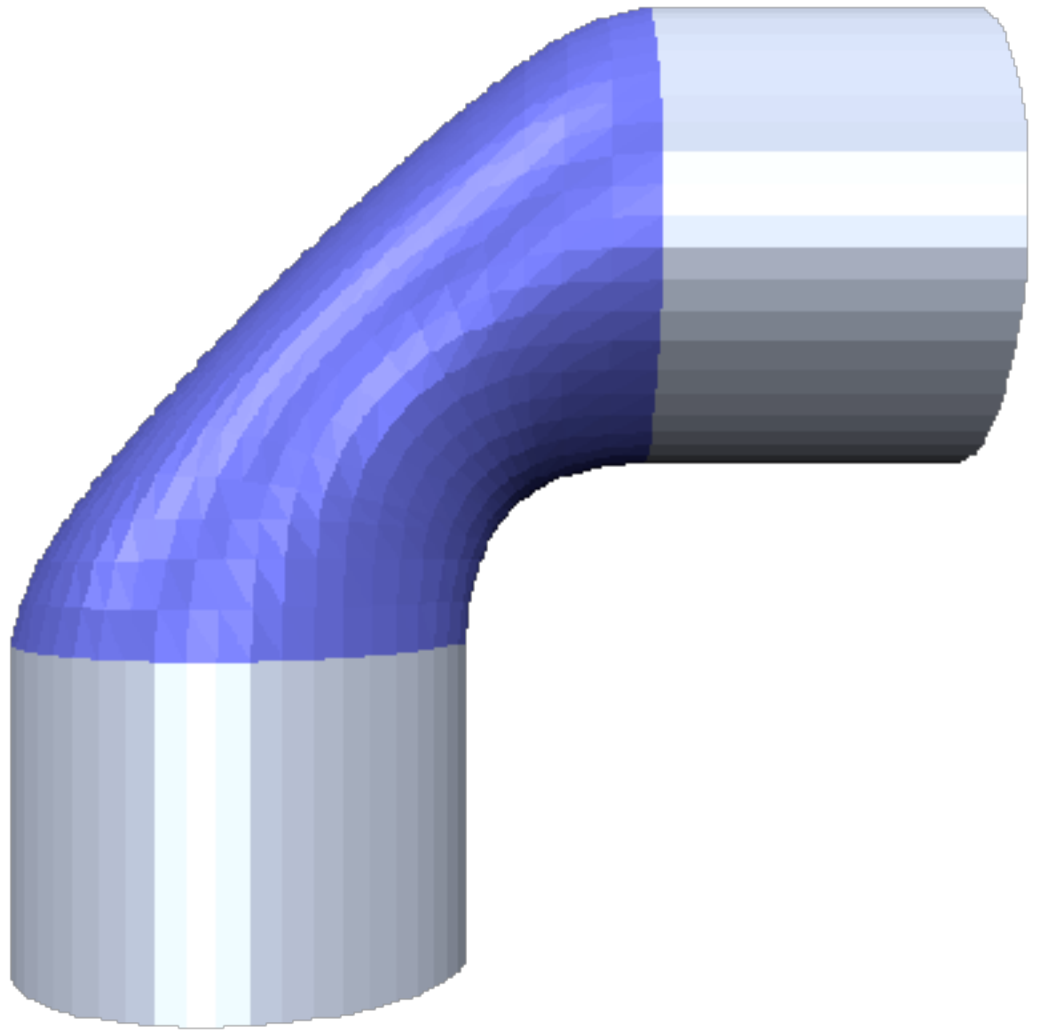
$$\Delta^2 \mathbf{p} = 0$$

Energy Functionals



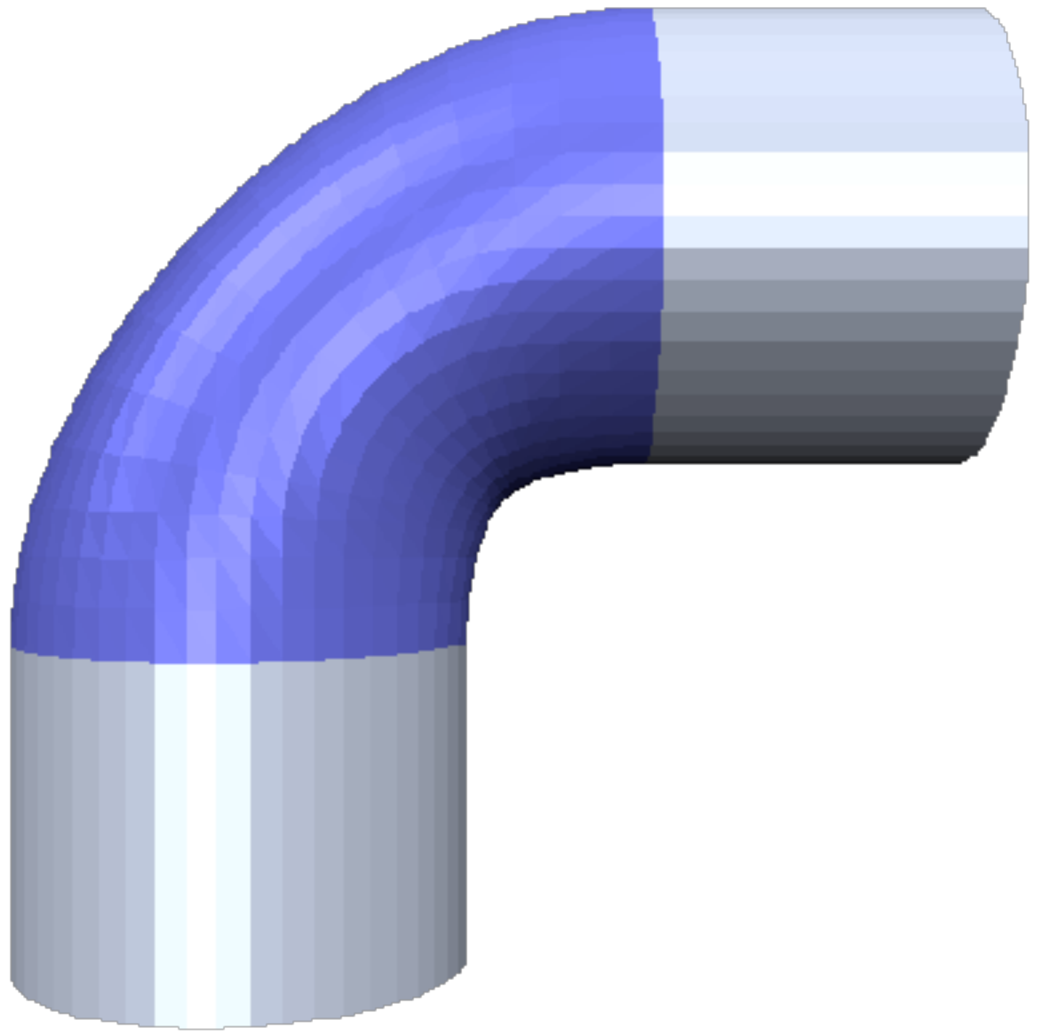
Membrane

$$\Delta_{\mathcal{S}} \mathbf{p} = 0$$



Thin Plate

$$\Delta_{\mathcal{S}}^2 \mathbf{p} = 0$$



$$\Delta_{\mathcal{S}}^3 \mathbf{p} = 0$$

Analysis

- Minimizer surfaces satisfy Euler-Lagrange PDE

$$\Delta_{\mathcal{S}}^{k} \mathbf{p} = 0$$

- They are ***stationary surfaces of Laplacian flows***

$$\frac{\partial \mathbf{p}}{\partial t} = \Delta_{\mathcal{S}}^{k} \mathbf{p}$$

- Explicit flow integration corresponds to iterative solution of linear system

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- Book: Chapter 4 (Botsch et al.),
  Chapter 5 (CohenOr et al.)
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