

COMP0026: Image Processing

Image Filtering



D. 从ISI的开始才是b的內容 -

COMP0026: Image Processing

Image Filtering



```
I = imread('moon.tif');
h = fspecial('unsharp');
I2 = imfilter(I,h);
imshow(I), title('Original Image')
figure, imshow(I2), title('Filtered Image')
```

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Lectures will be Recorded

Image Filtering

input image → filter function → output image

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Functions

linear/non-linear

Image Filtering

input image → filter function → output image

Functions

linear/non-linear

Neighborhoods

local/non-local

Goals

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- Low-level data (e.g., pixel) processing operations

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- Smoothing and noise reduction

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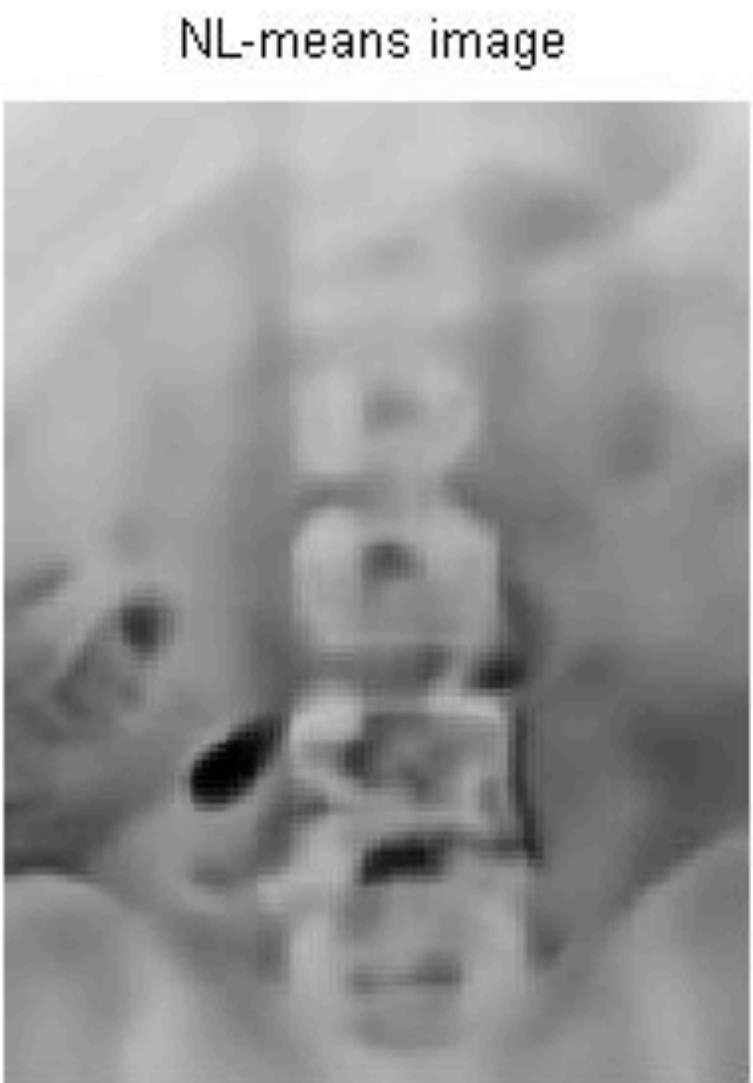
- Low-level data (e.g., pixel) processing operations
- Smoothing and noise reduction
- Feature extraction and enhancement

Non-local Filtering

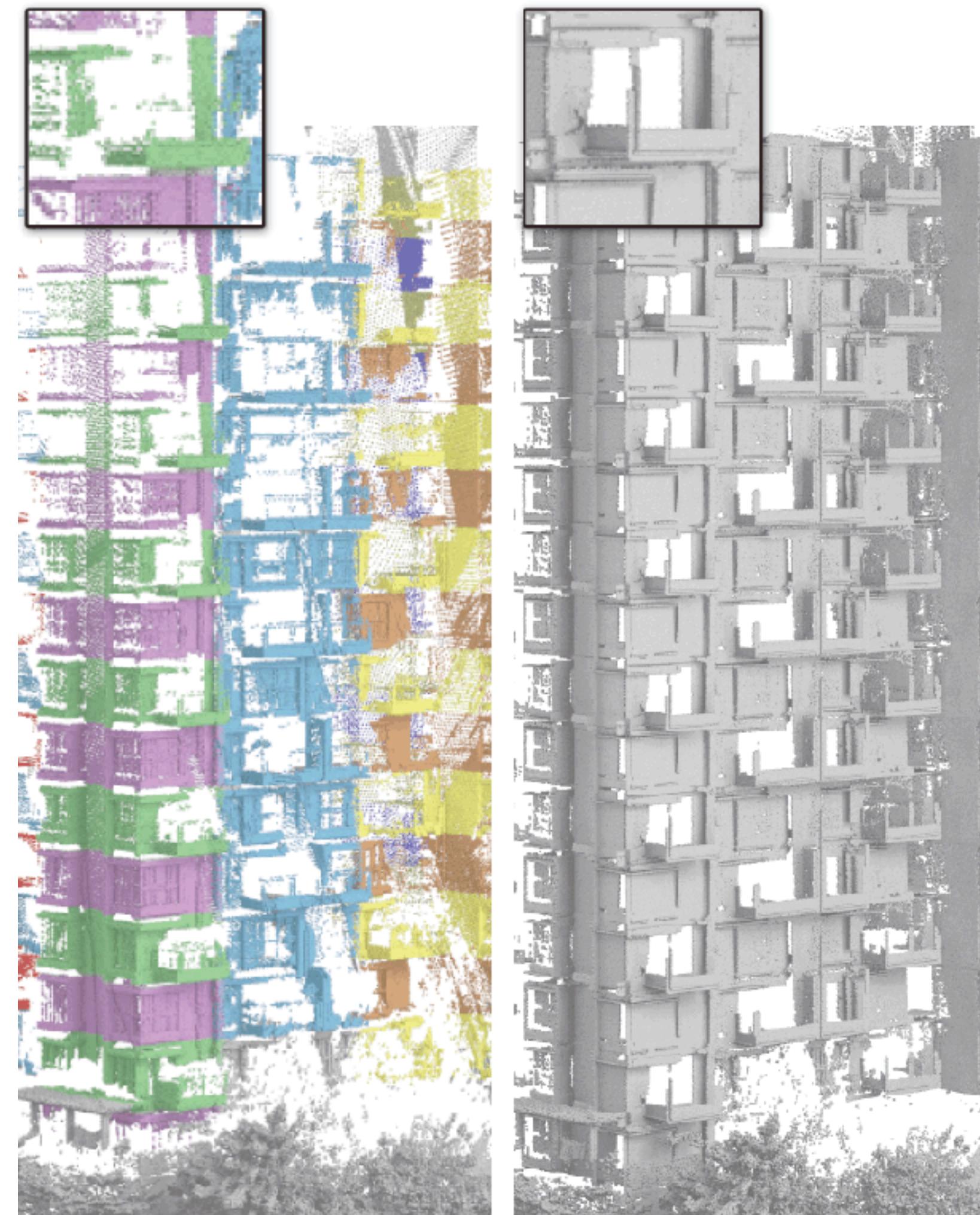
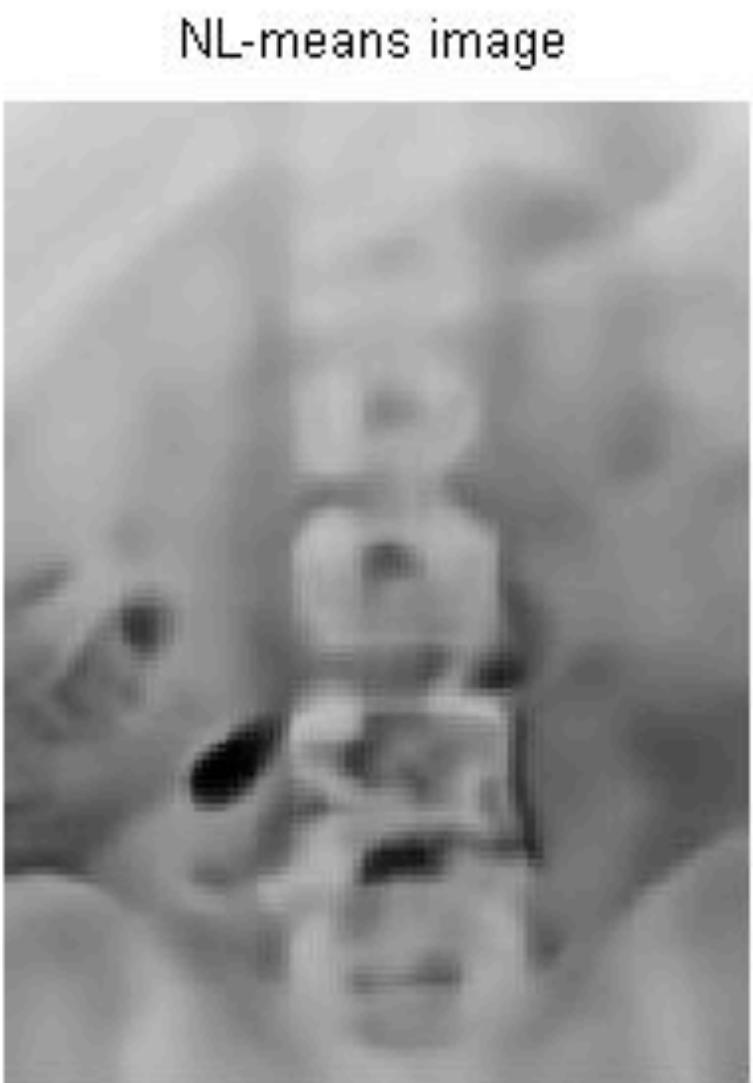


Image Filtering

Non-local Filtering

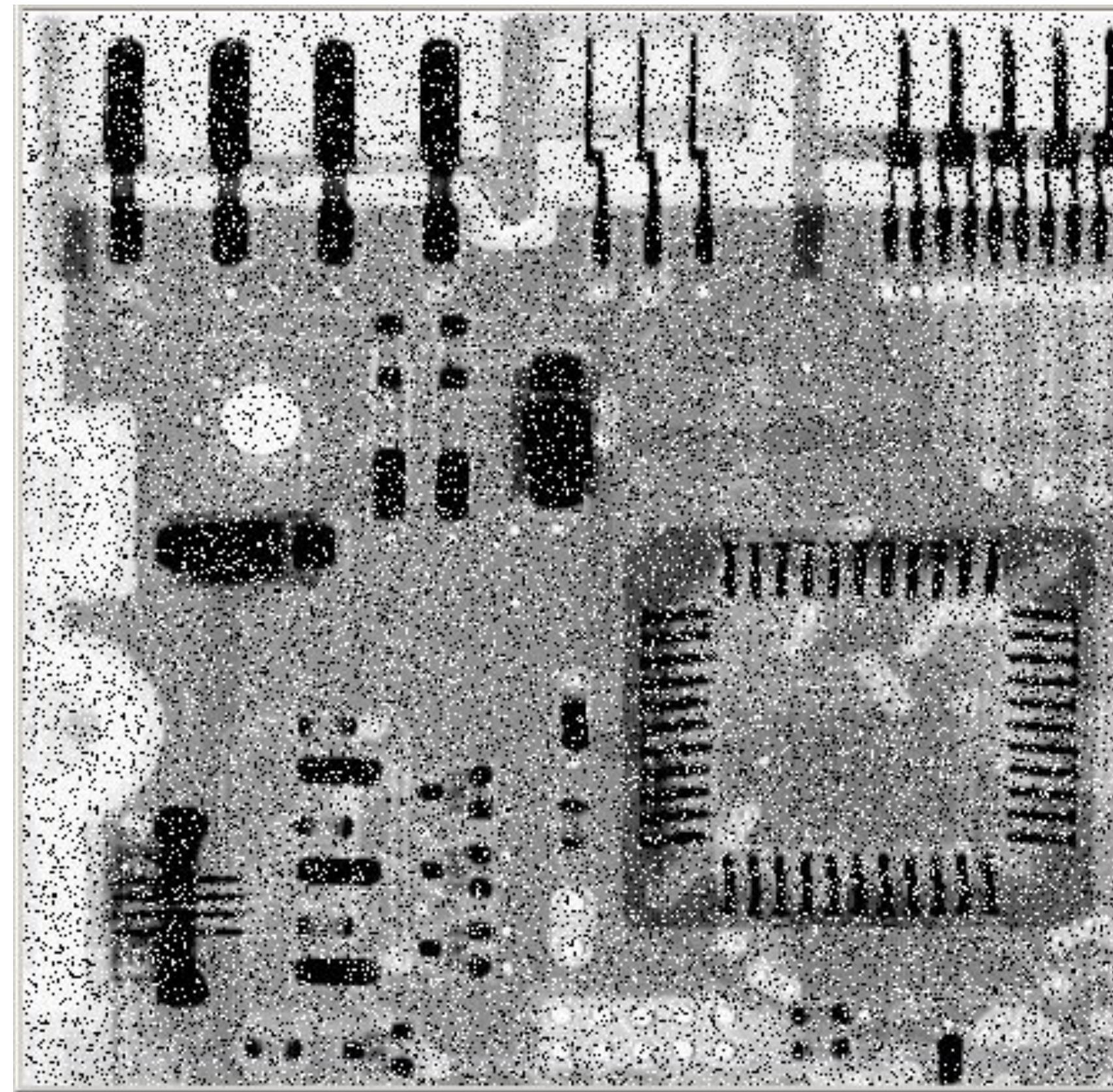


Non-local Filtering



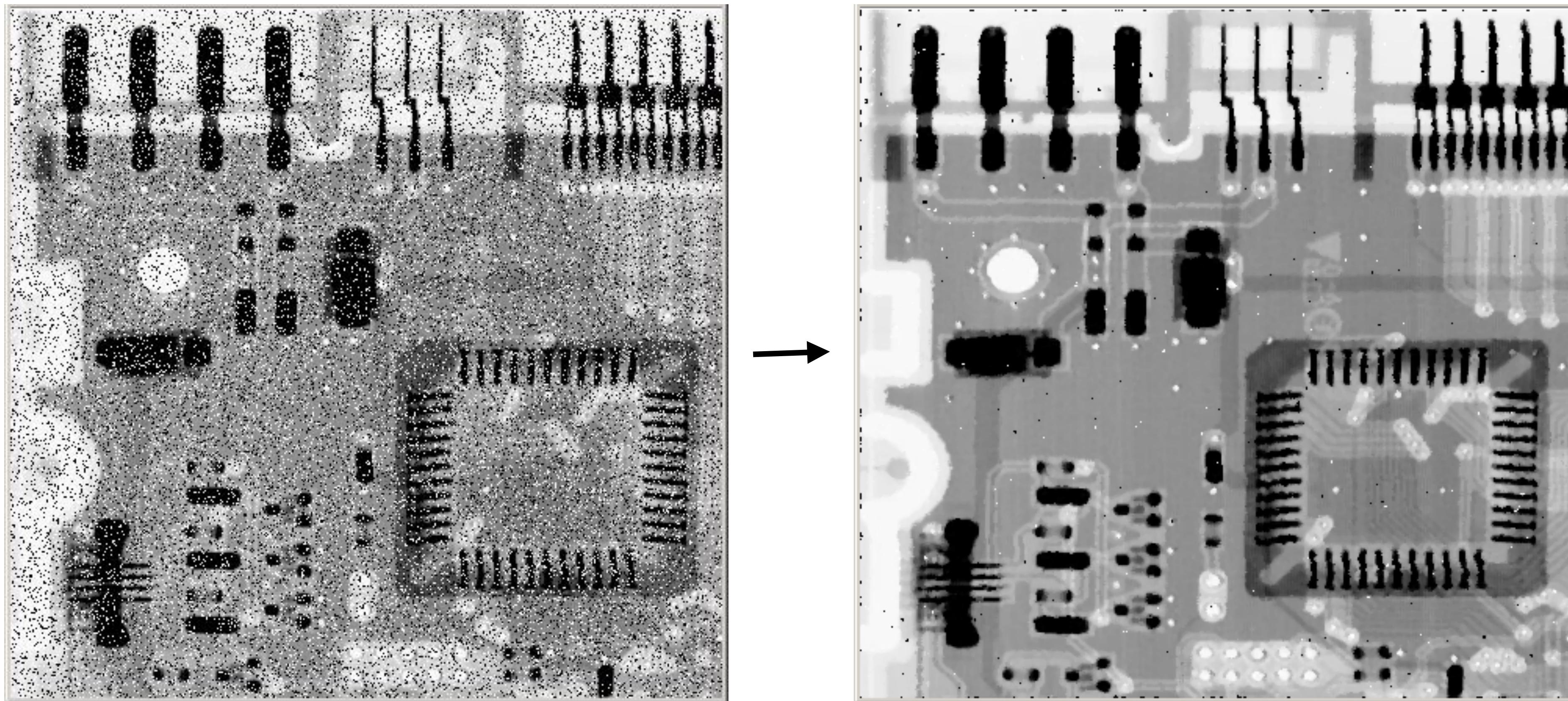
Denoising: Remove “Noise”

(if you know how to model that noise specifically!)



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(if you know how to model that noise specifically!)



median filter

Outline

- **Linear filtering**
- **Convolution operations**
 - Smoothing
 - Low-pass and high-pass filters
 - Sharpen
- **Filtering in the Fourier domain**
- **Rank filters**

Linear Mappings

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- Equivalently, L is *linear* if

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$$L(I_1 + I_2) = L(I_1) + L(I_2)$$

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$$L(\alpha I) = \alpha L(I) \quad \forall \alpha$$

(i.e., L preserves linear combinations)

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(i.e., L preserves linear combinations)

$$L(\alpha I_1 + \beta I_2) = \alpha L(I_1) + \beta L(I_2)$$

Convolutional Filters

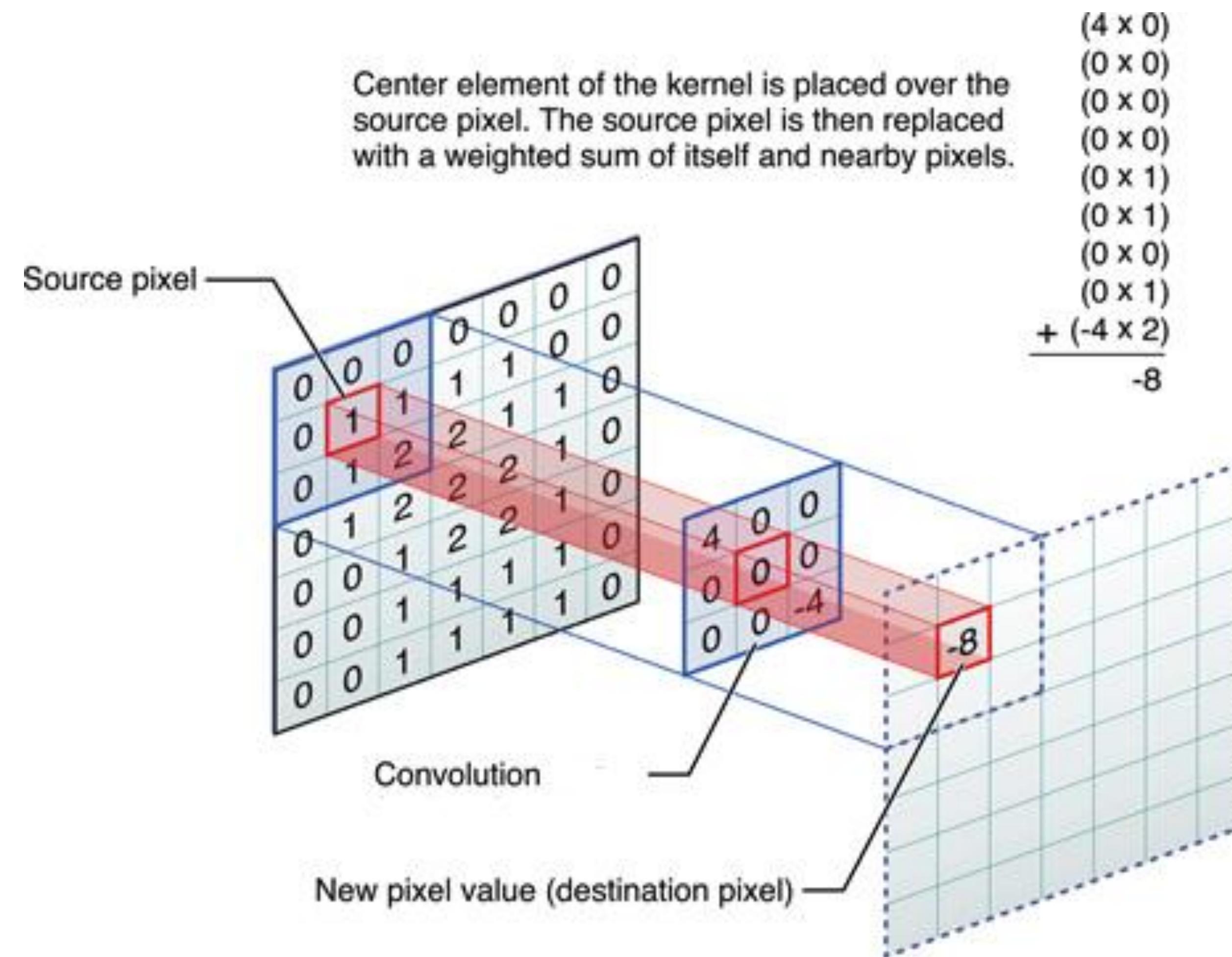
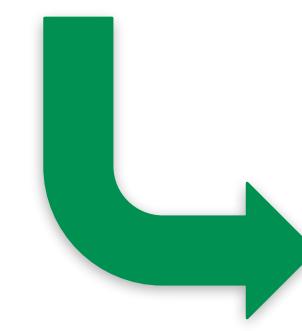
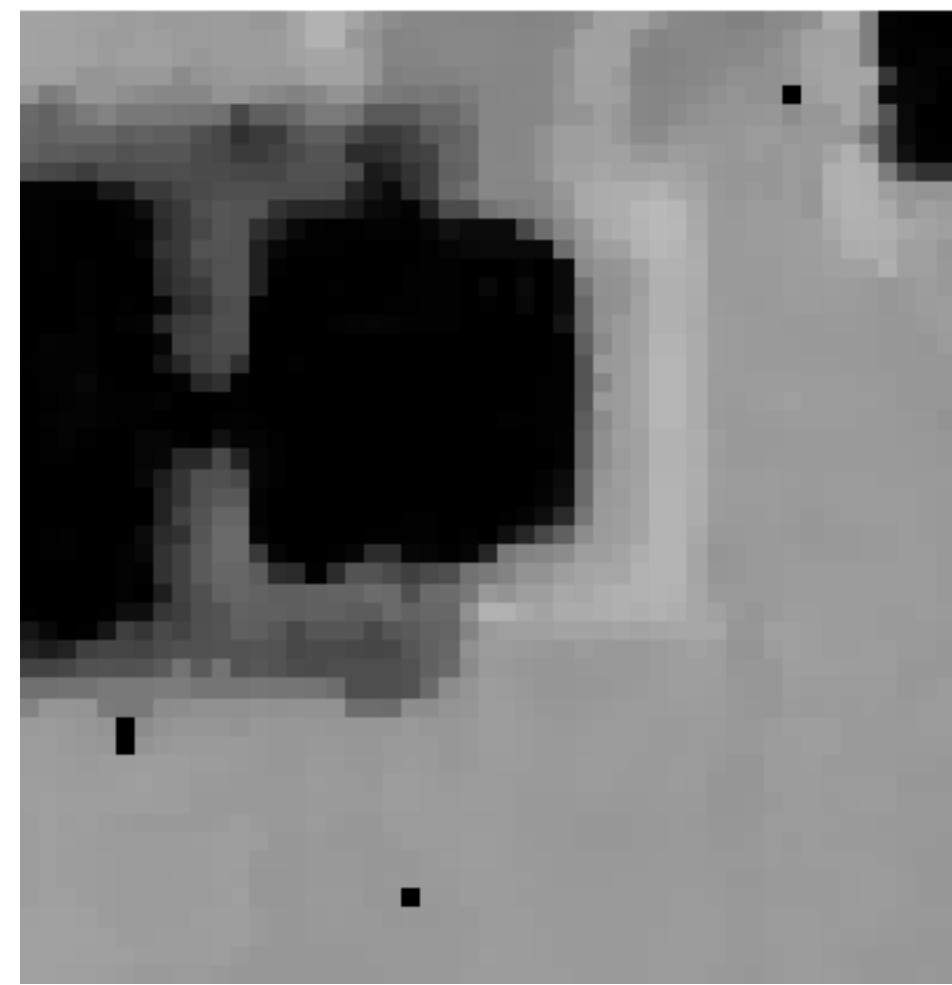
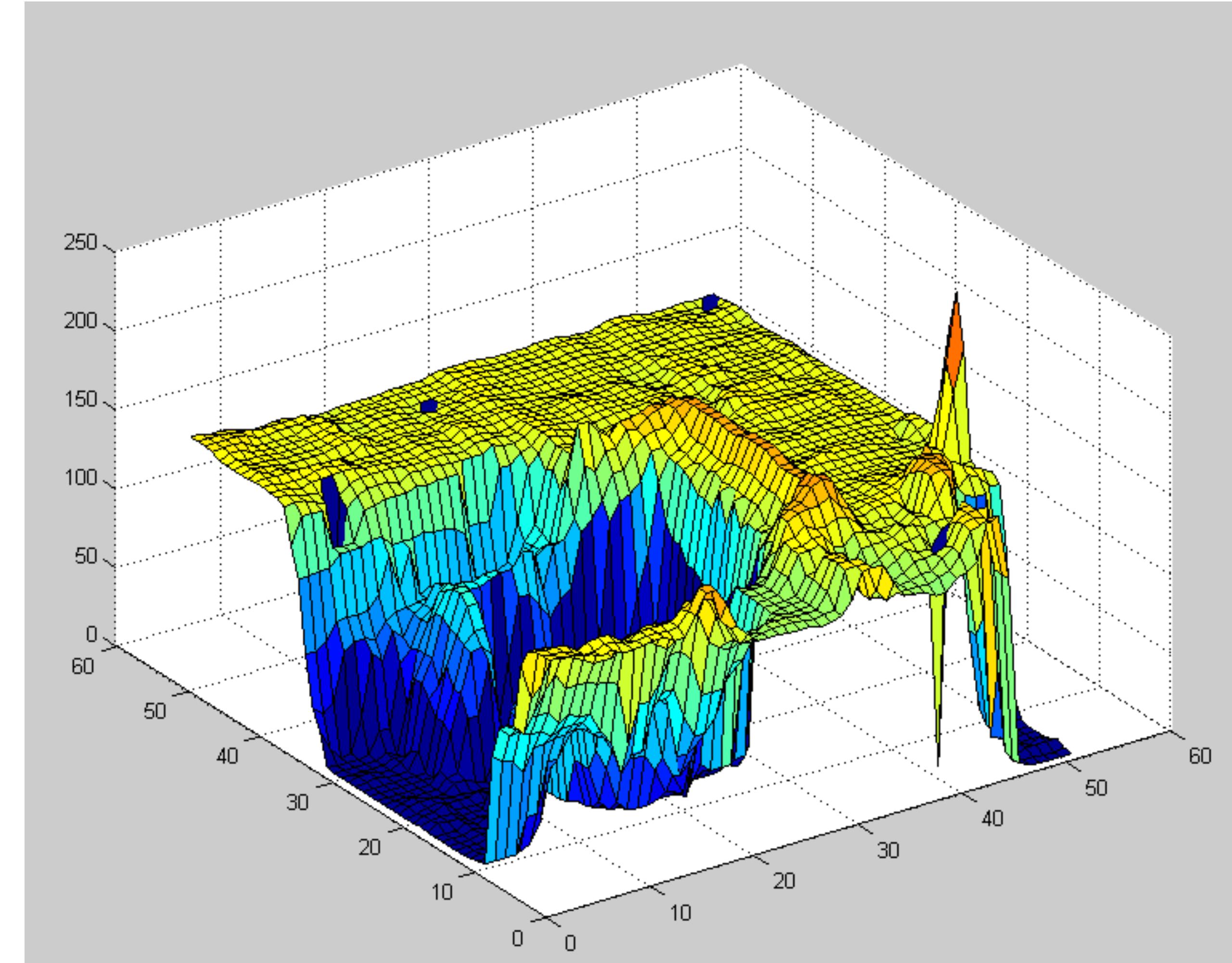


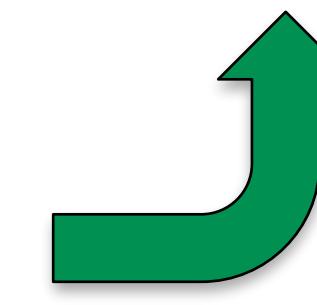
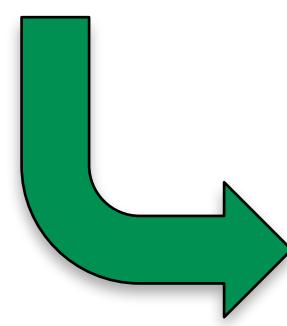
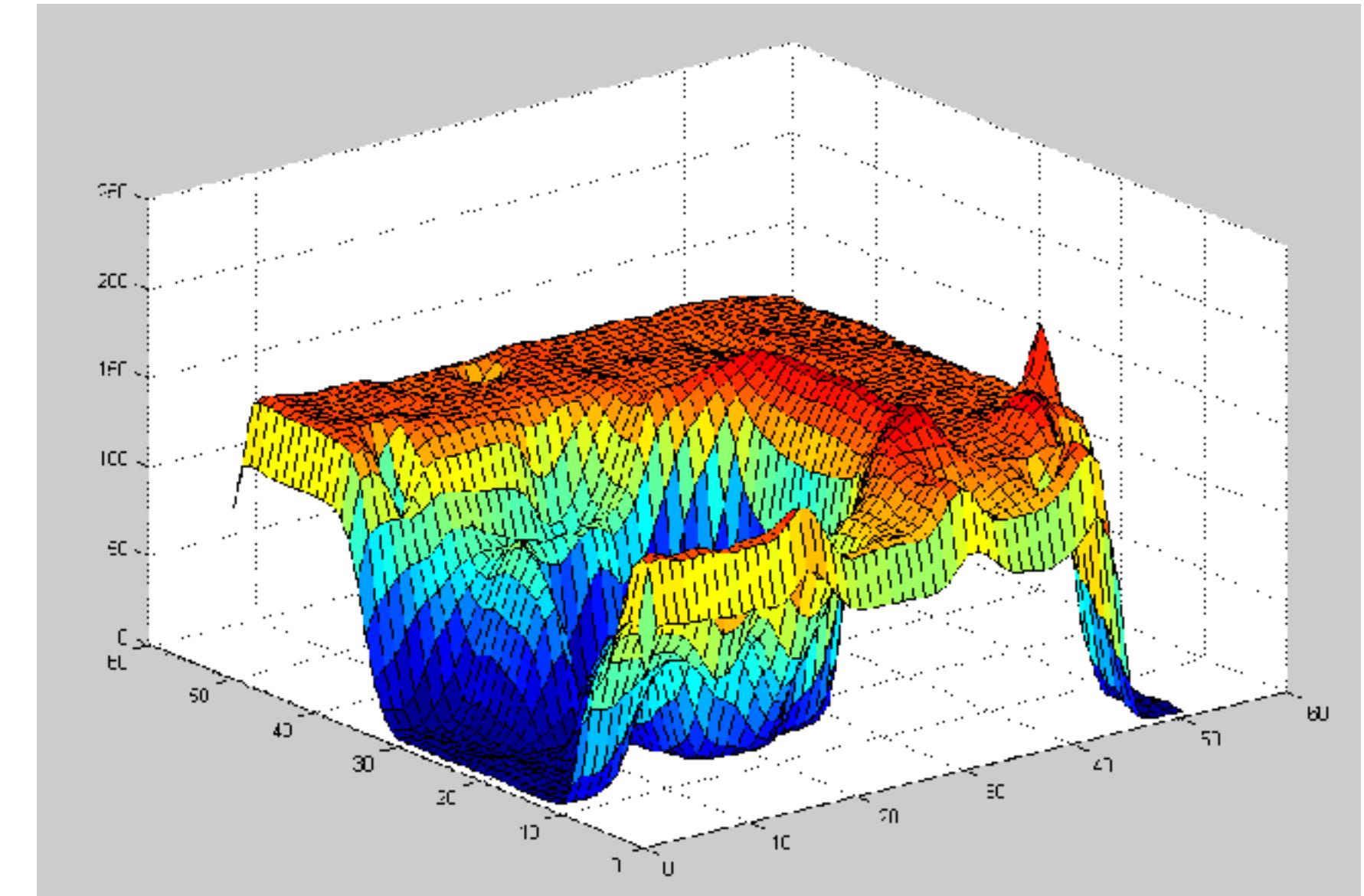
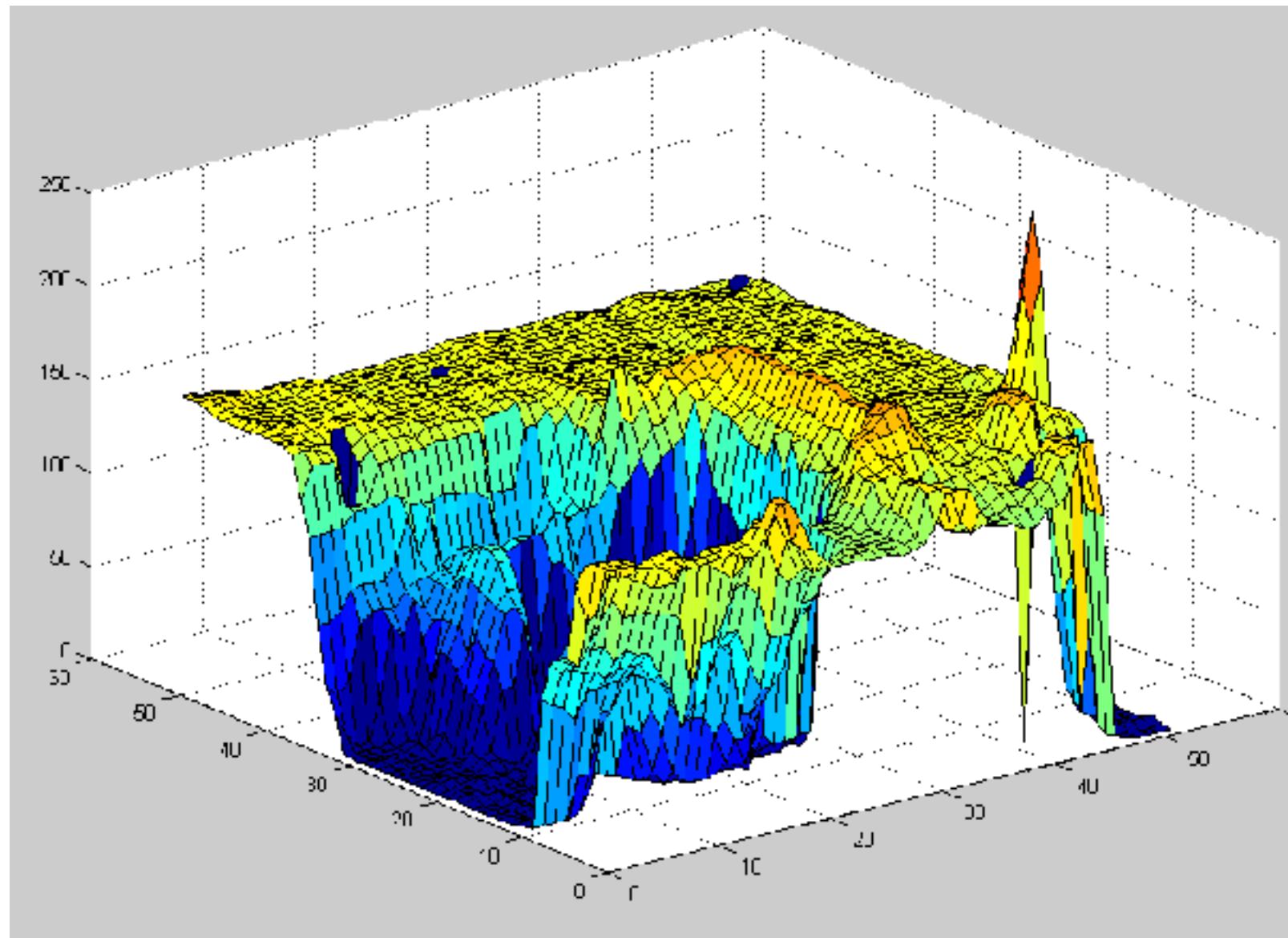
Image ~ Heightfield



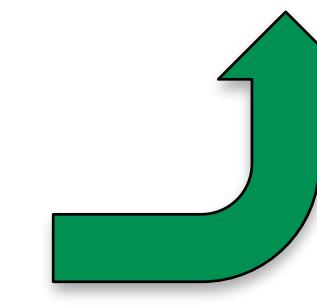
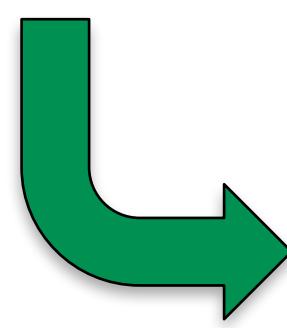
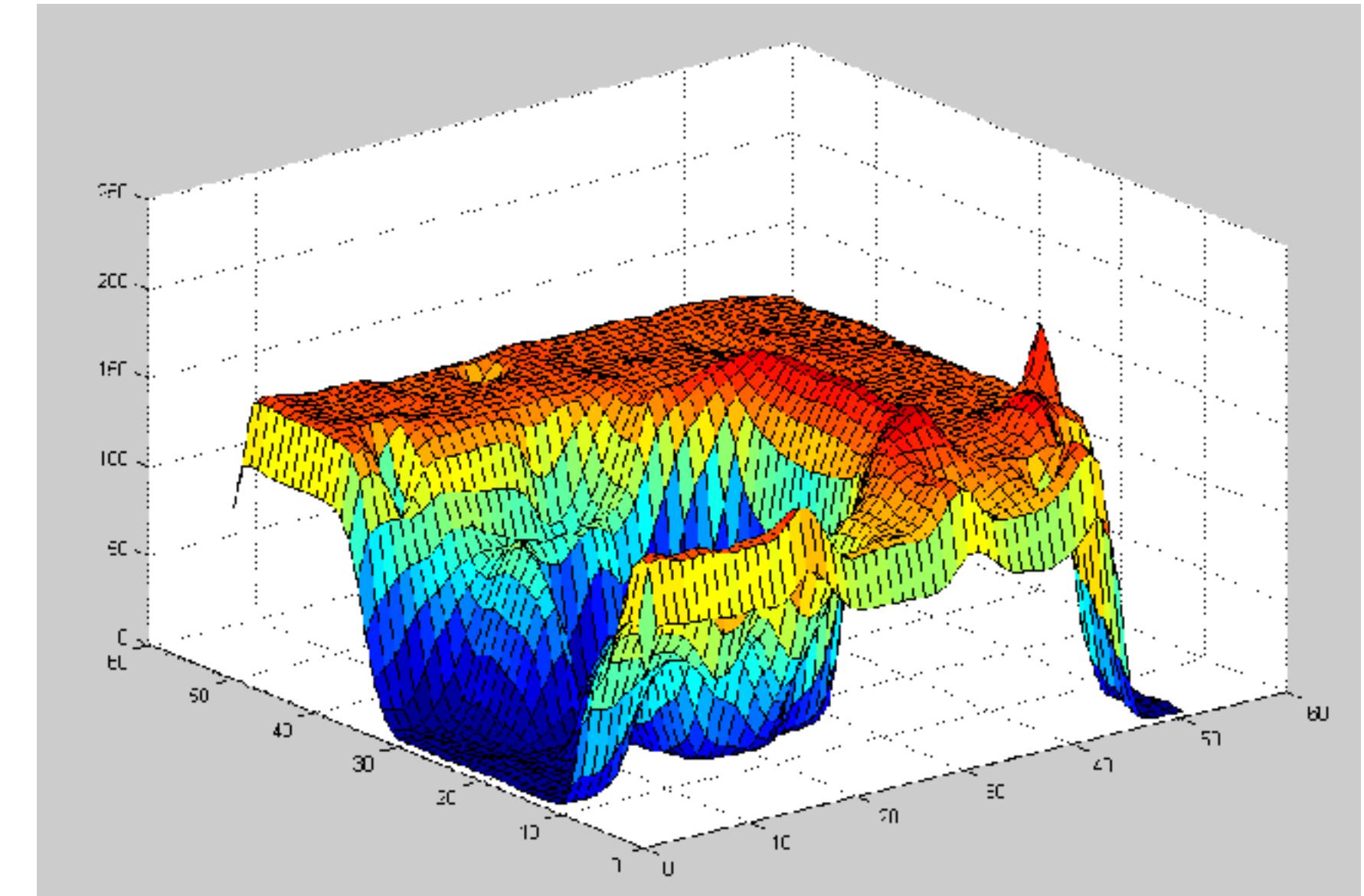
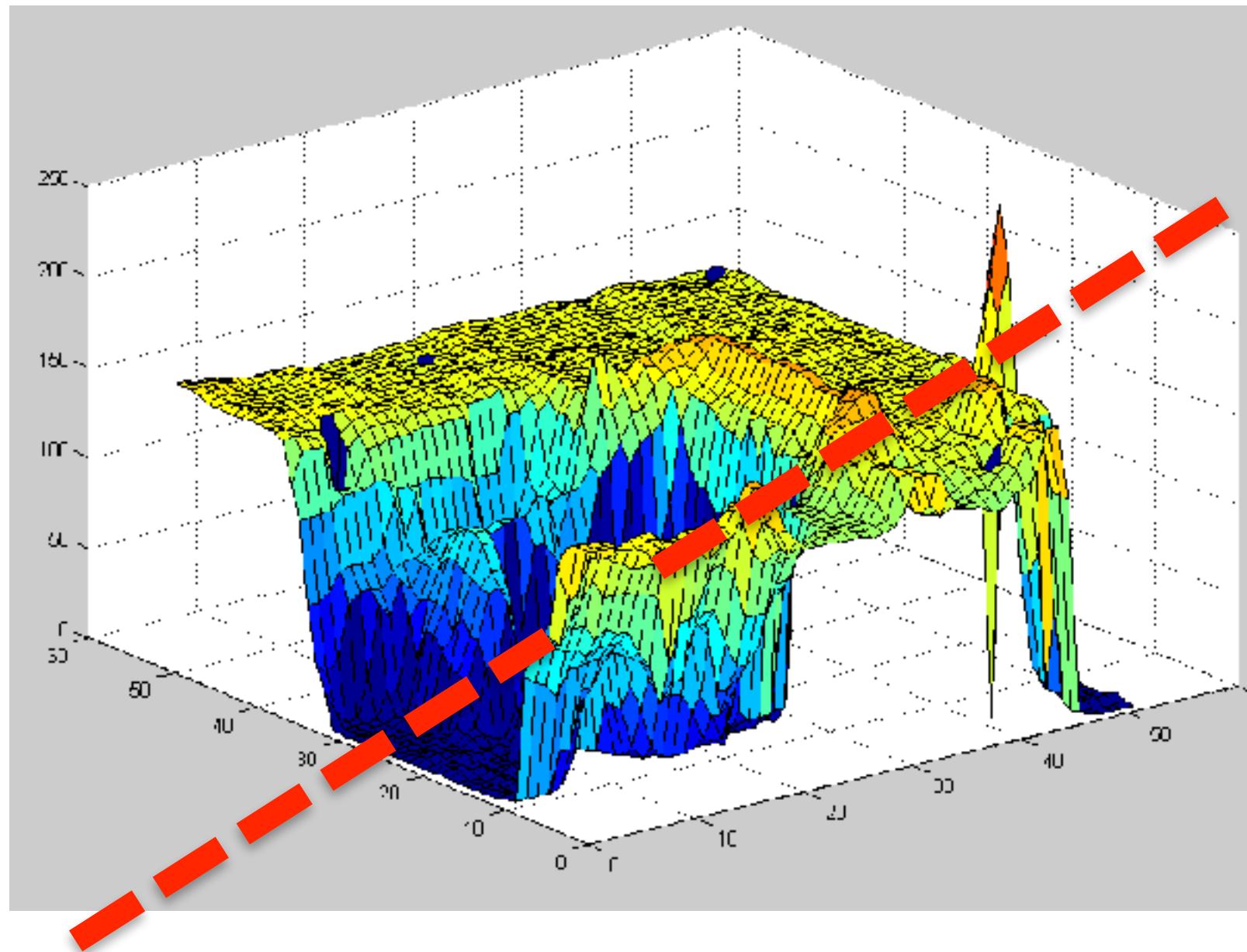
$z = I$'s intensity



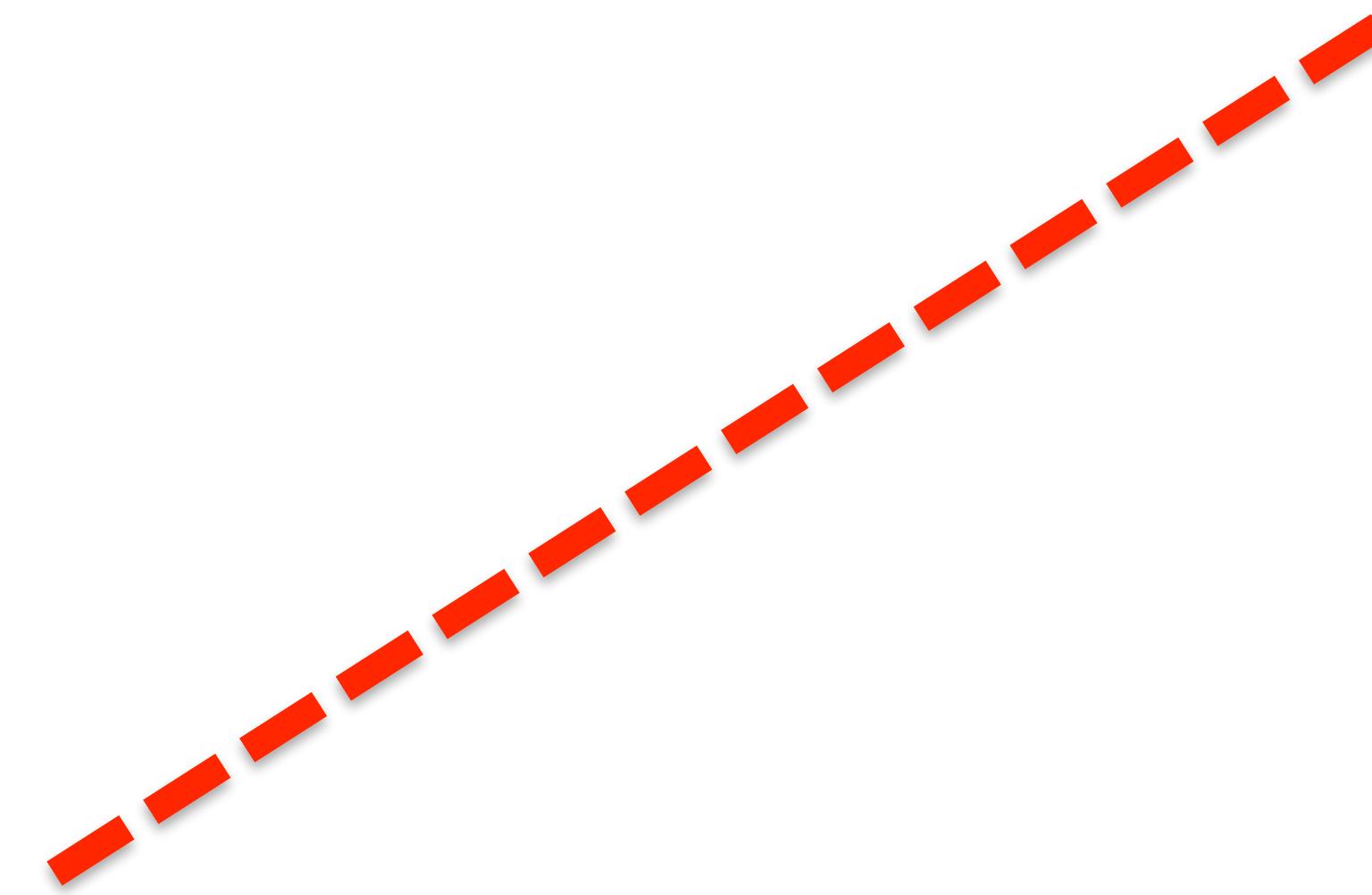
Filter: Let Intensity Through Selectively



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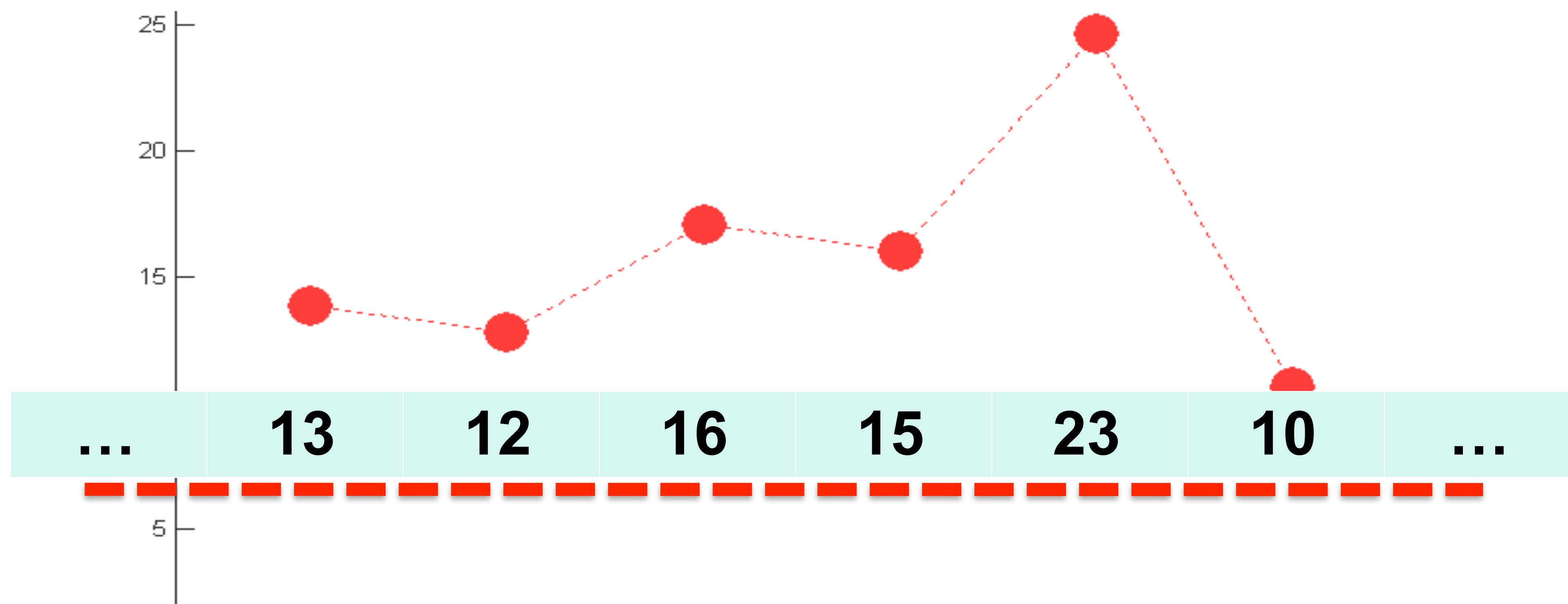
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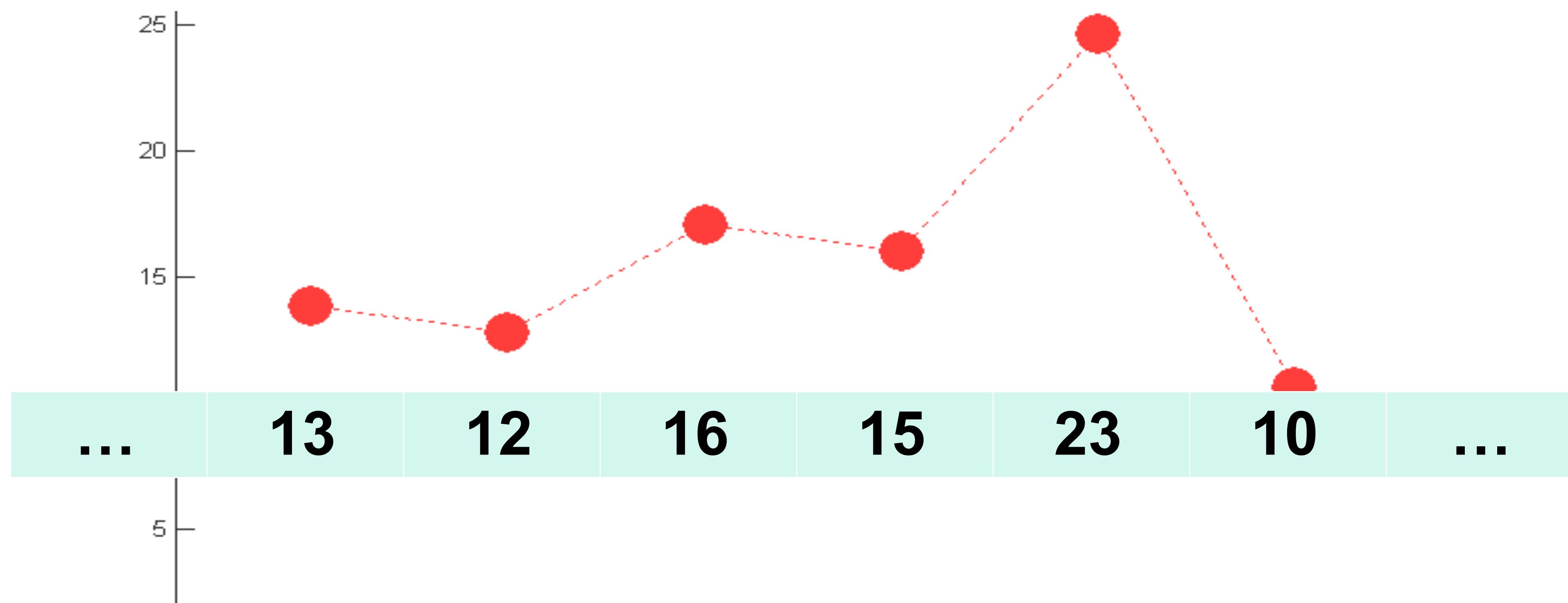
Filter: Let Intensity Through Selectively



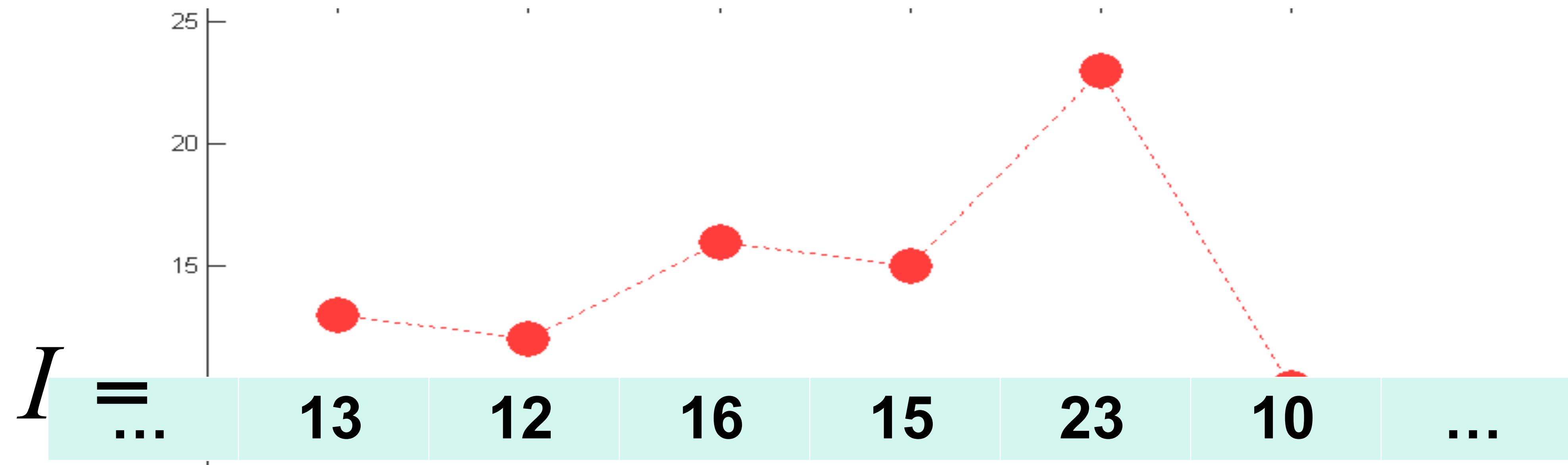
A Row of Pixel Intensities



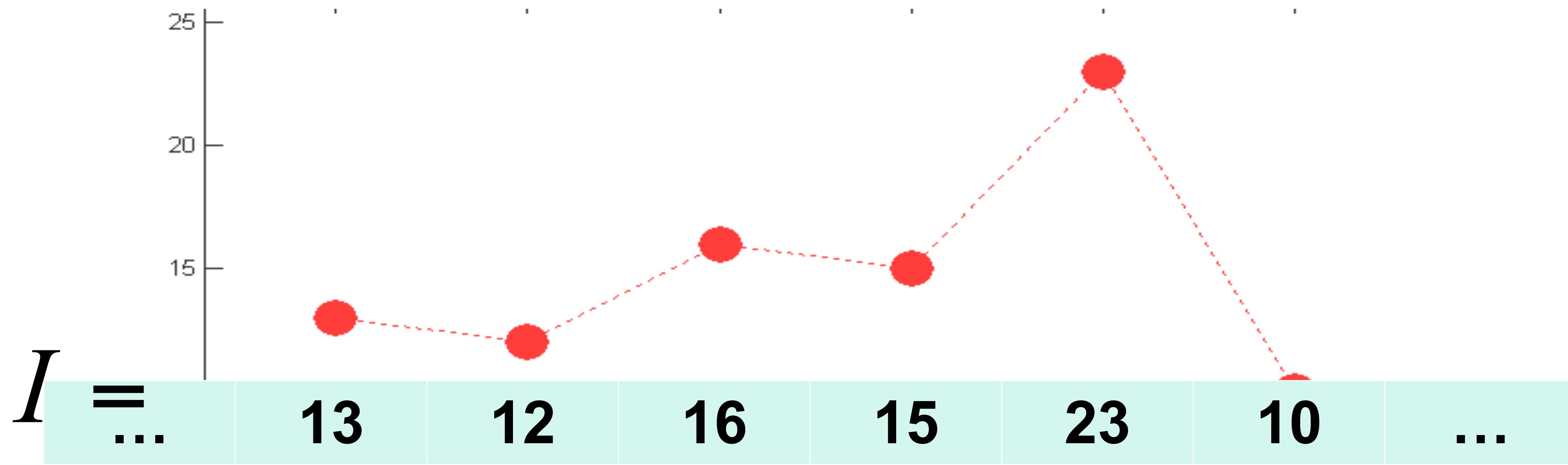
A Row of Pixel Intensities



Linear Operations: Weighted Sum

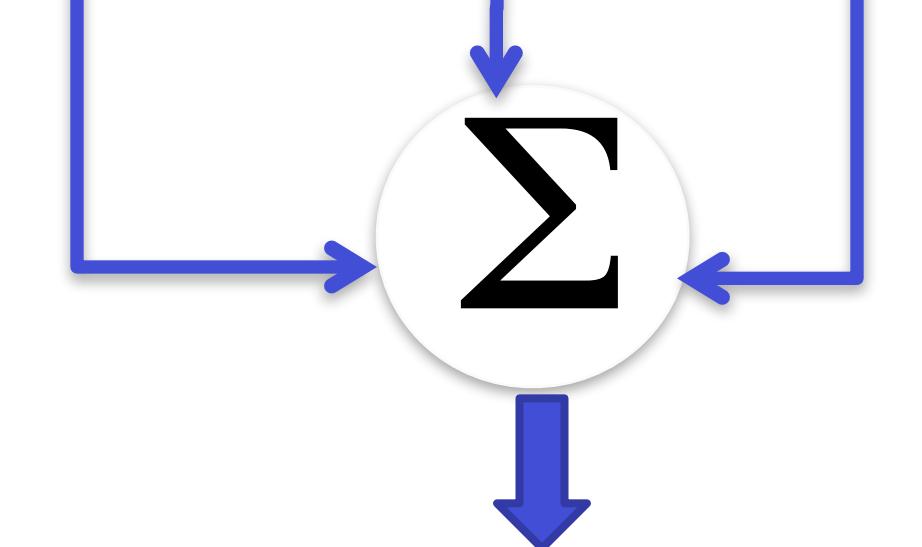


$$I' = \dots \quad ?? \quad ?? \quad ?? \quad ?? \quad \dots$$



$K =$

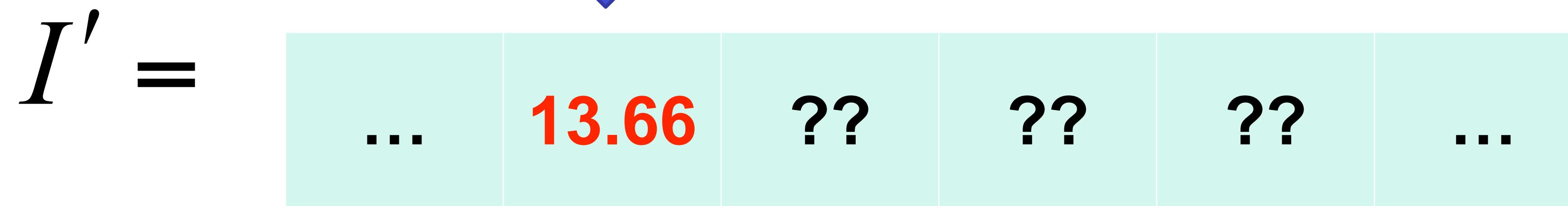
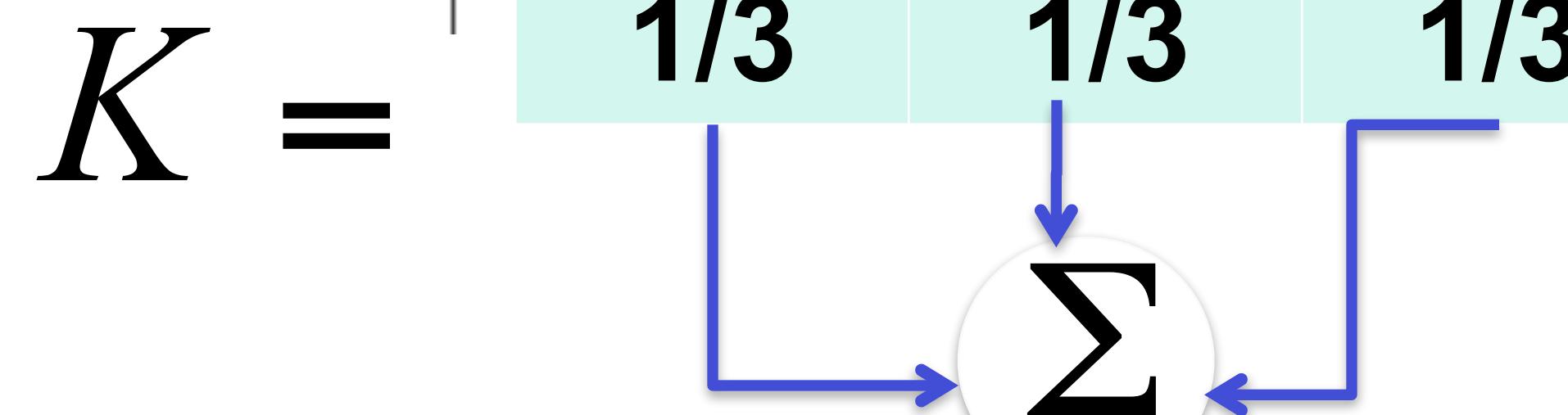
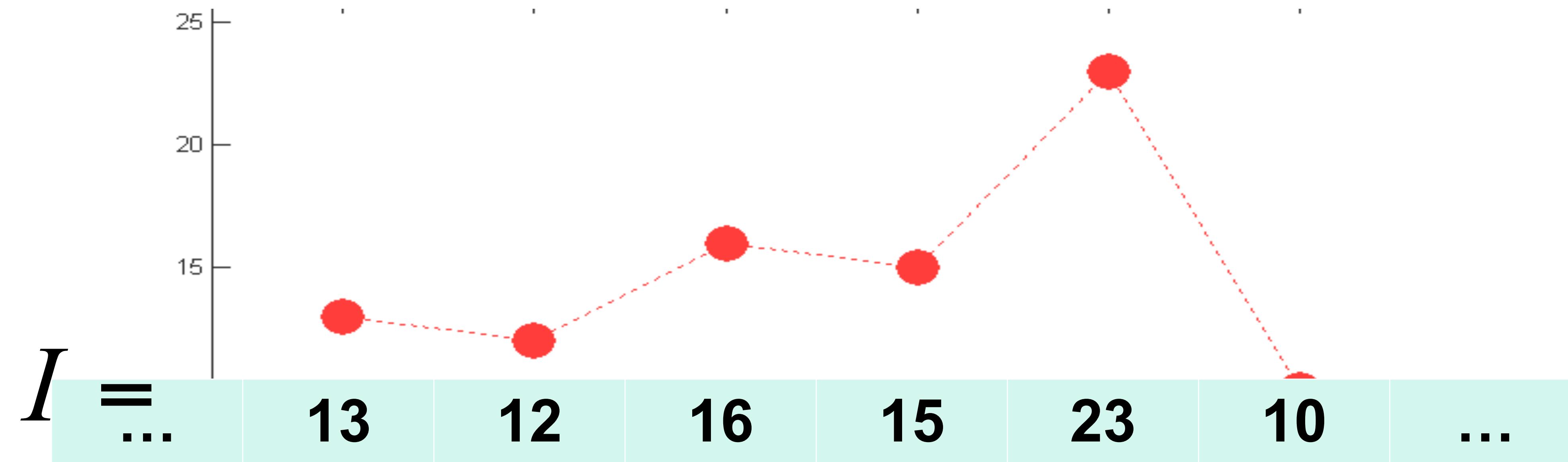
| | | |
|-------|-------|-------|
| $1/3$ | $1/3$ | $1/3$ |
|-------|-------|-------|

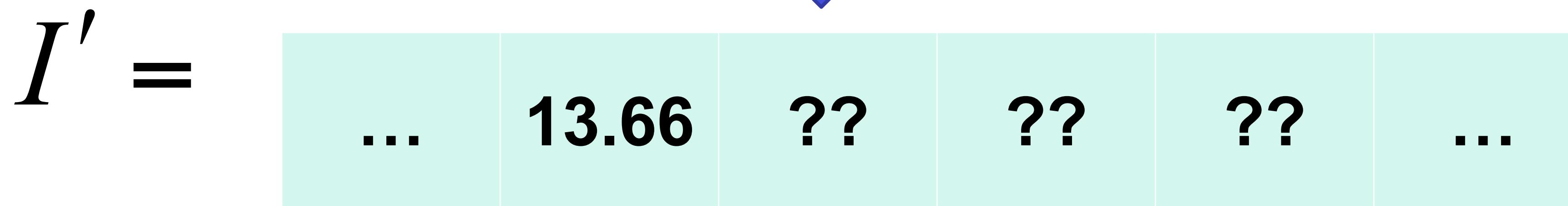
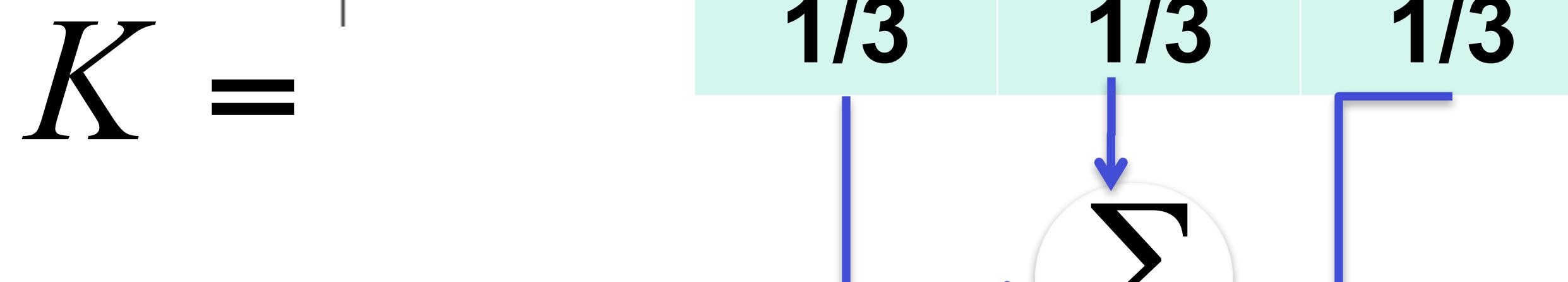
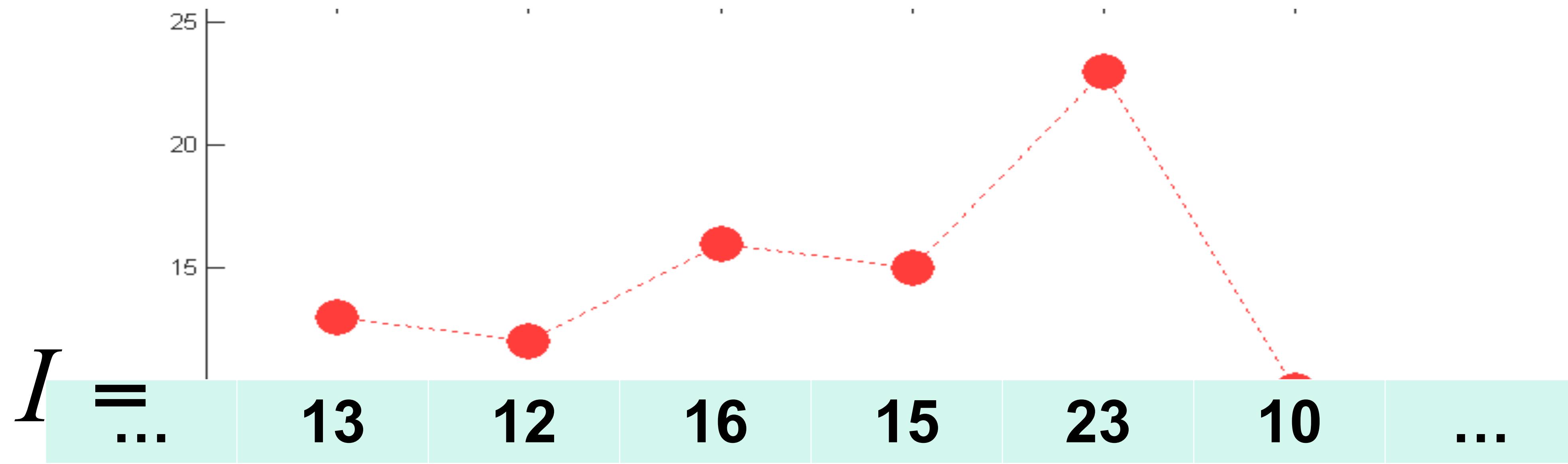


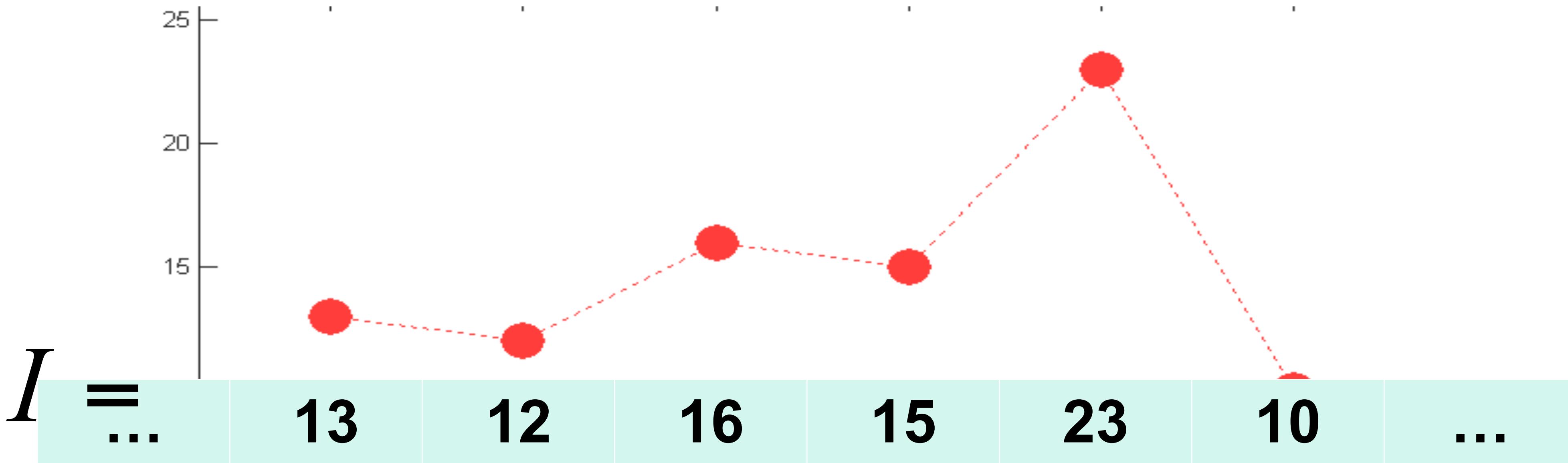
Neighborhood: $N(x) = x \pm 1$
 Operation: Average

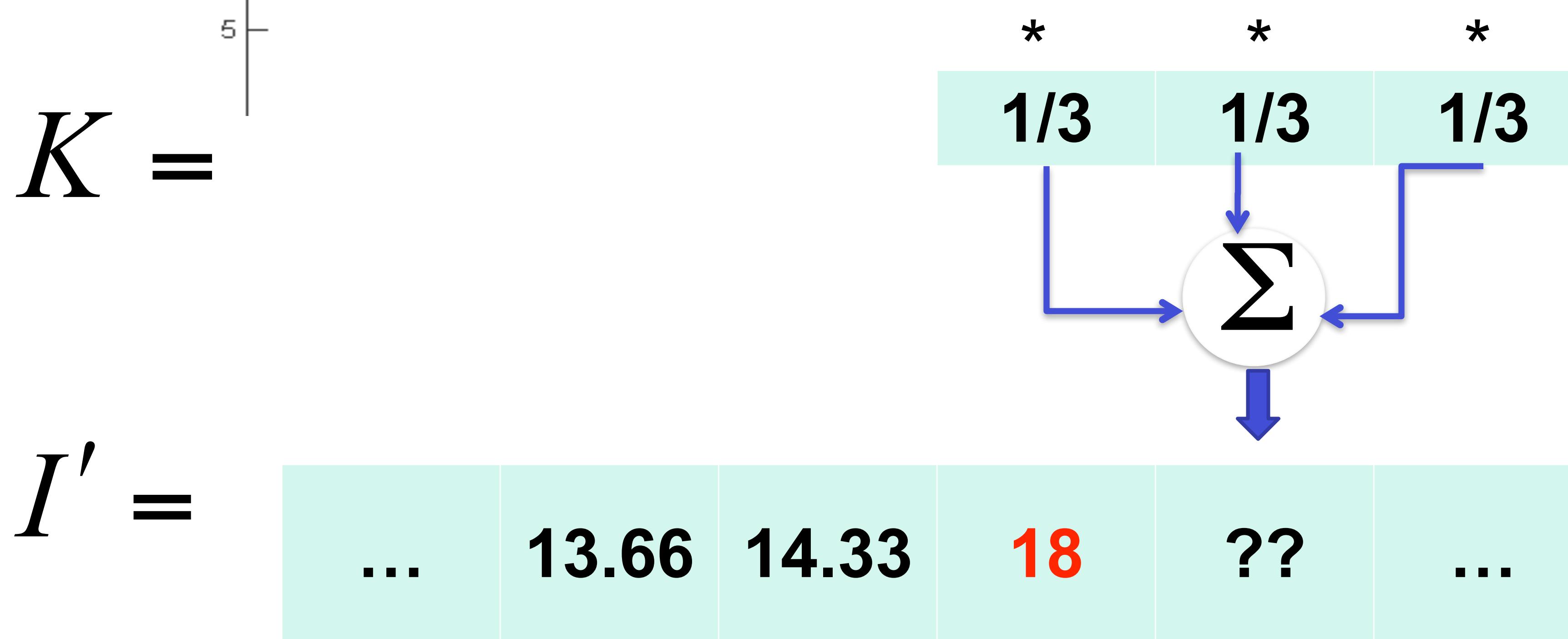
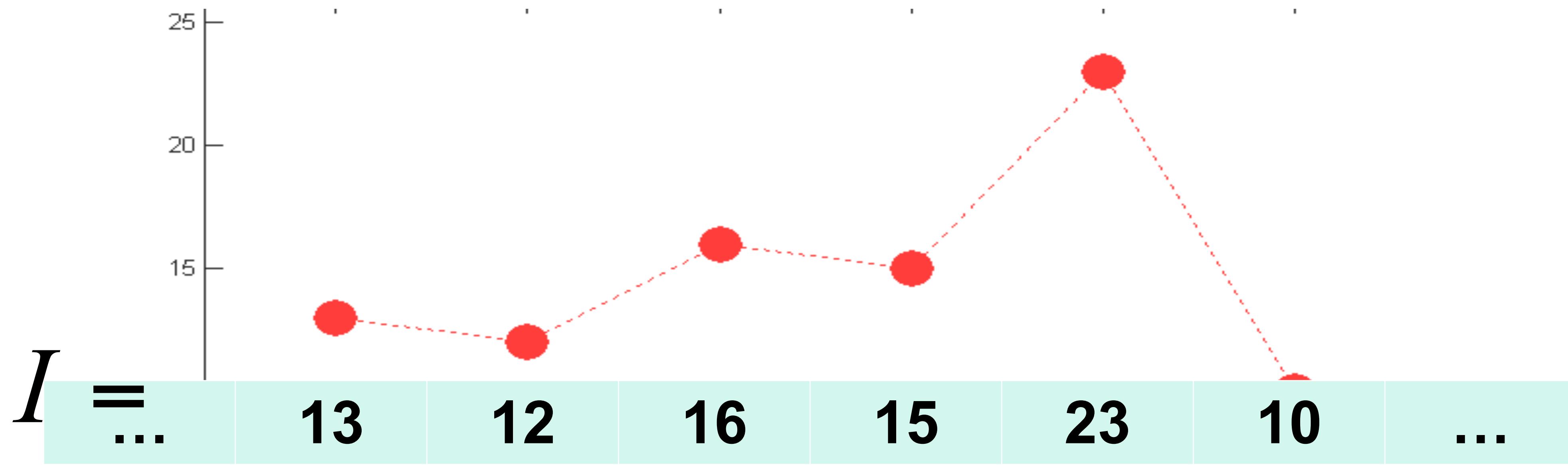
$I' = \dots$

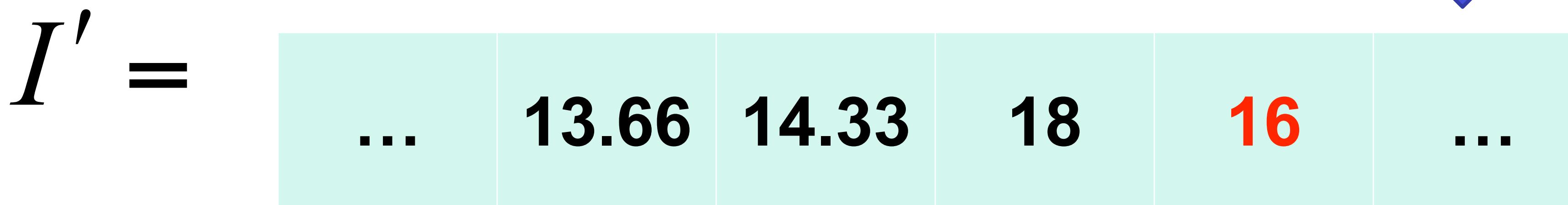
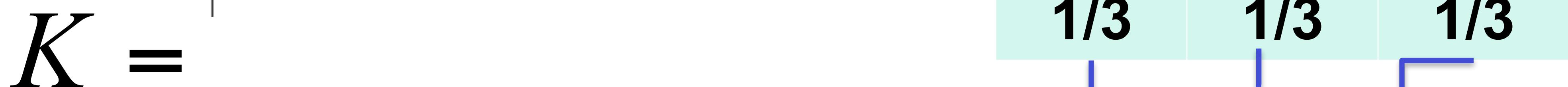
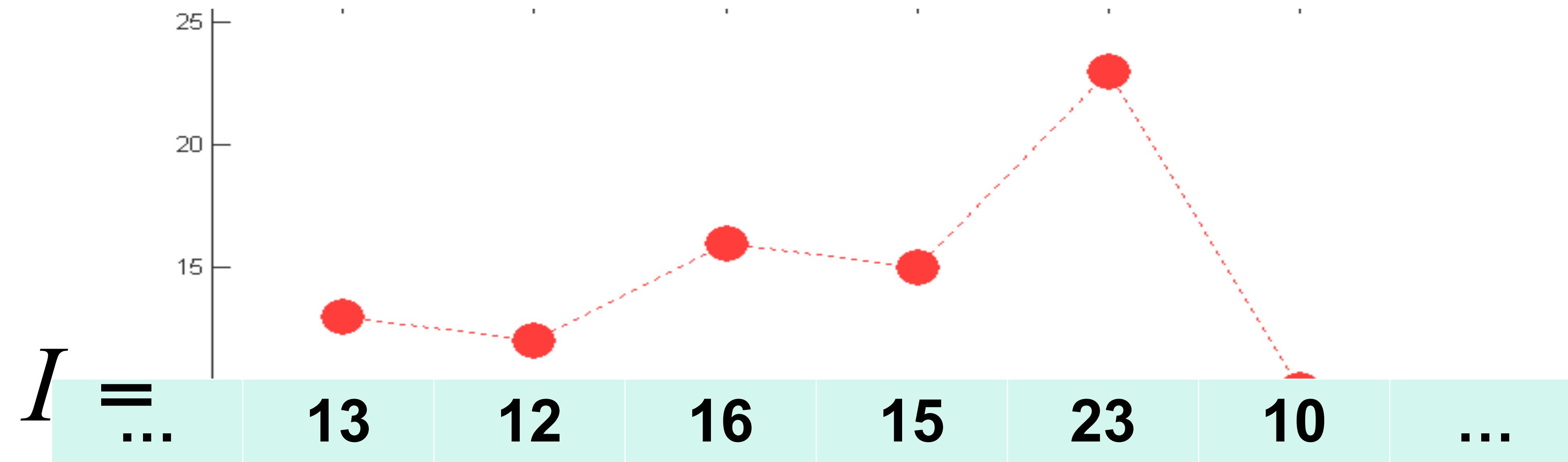
| | | | | | |
|----|----|----|----|----|-----|
| ?? | ?? | ?? | ?? | ?? | ... |
|----|----|----|----|----|-----|











Correlation

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- Linear operation in 1D:

$$I'(x) = \sum_{i=-a}^{a} K(i)I(x+i)$$

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- $N(x)$ is a **neighbourhood** of size $(2a+1)$

Correlation

- Linear operation in 1D:

$$I'(x) = \sum_{i=-a}^a K(i)I(x+i)$$


- Linear operation in 2D:

$$I'(x, y) = \sum_{i=-a}^a \sum_{j=-b}^b K(i, j)I(x+i, y+j)$$

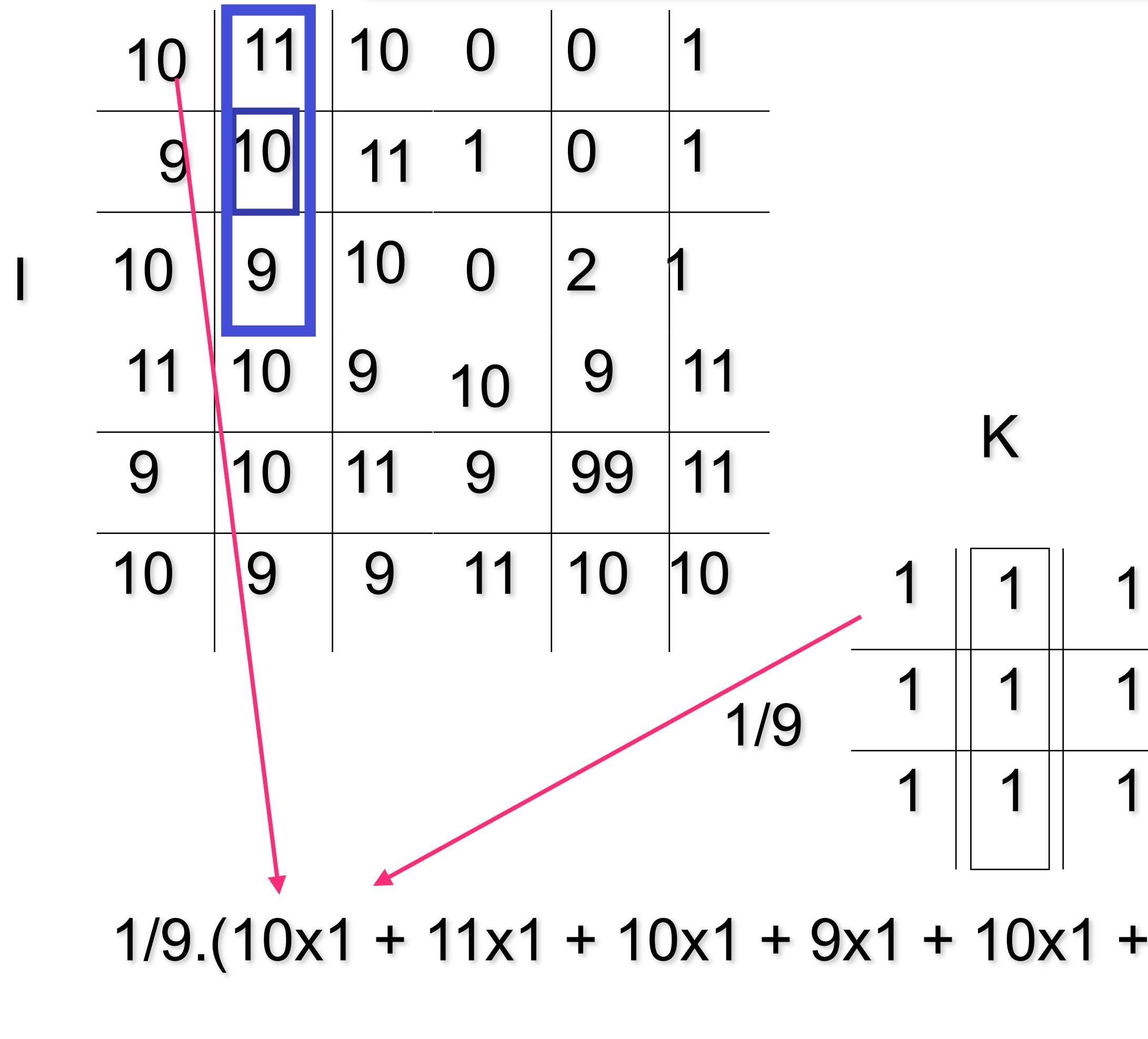

- I is the input image; I' is the output of the operation
- K is the *kernel* of the operation; many choices!
- $N(x,y)$ is a neighbourhood of size $(2a+1) \times (2b+1)$

| | | |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 1 | 1 |
| 1 | 1 | 1 |

$$\frac{1}{9} \cdot (10x_1 + 11x_1 + 10x_1 + 9x_1 + 10x_1 + 11x_1 + 10x_1 + 9x_1 + 10x_1) =$$
$$\frac{1}{9} \cdot (90) = 10$$

| | | | | | |
|---|----|---|---|---|---|
| X | X | X | X | X | X |
| X | 10 | | | | X |
| X | | | | | X |
| X | | | | | X |
| X | | | | | X |
| X | X | X | X | X | X |

$$I'(x, y) = \sum_{i=-a}^a \sum_{j=-b}^b K(i, j) I(x + i, y + j)$$



Correlation with a Unit-Impulse Image

$$I = \begin{array}{cccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}$$

$$K = \begin{array}{ccc} 5 & 6 & 7 \end{array}$$

$$I' = \begin{array}{cccccccc} - & - & - & - & - & - & - & - \end{array}$$

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Correlation reveals copy of K , rotated by 180°

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Correlation reveals copy of K , rotated by 180°

Would be nice to get result that is **NOT** rotated!

Correlation/Convolution

$$I'(x, y) = \sum_{i=-a}^{a} \sum_{j=-b}^{b} K(i, j) I(x + i, y + j)$$

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Convolution

Correlation/Convolution

$$I'(x, y) = \sum_{i=-a}^a \sum_{j=-b}^b K(i, j) I(x + i, y + j)$$

Convolution

$$\begin{aligned} I'(x, y) &= \sum_{i=-a}^a \sum_{j=-b}^b K(i, j) I(x - i, y - j) \\ &= \sum_{i=-a}^a \sum_{j=-b}^b K(-i, -j) I(x + i, y + j) \end{aligned}$$

Correlation/Convolution

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Convolution

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$$K(i, j) = K(-i, -j) \Leftrightarrow \text{convolution} \equiv \text{correlation}$$

Correlation/Convolution

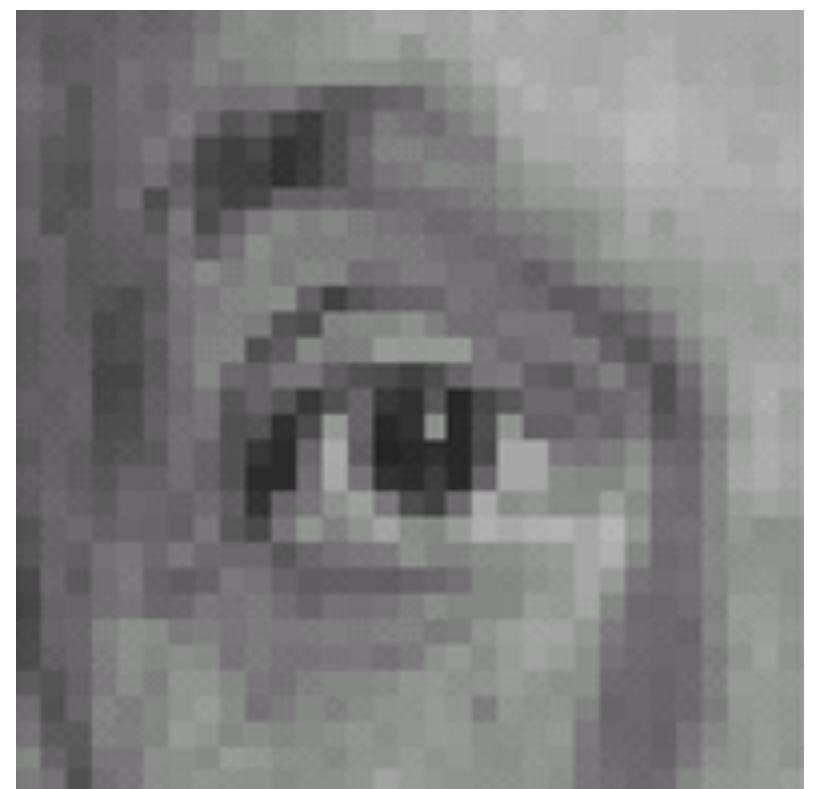
$$I'(x, y) = \sum_{i=-a}^a \sum_{j=-b}^b K(i, j) I(x + i, y + j)$$

Convolution

$$I'(x, y) = \sum_{i=-a}^a \sum_{j=-b}^b K(i, j) I(x - i, y - j)$$

$$I' = K * I$$

Linear Filtering (warm-up)



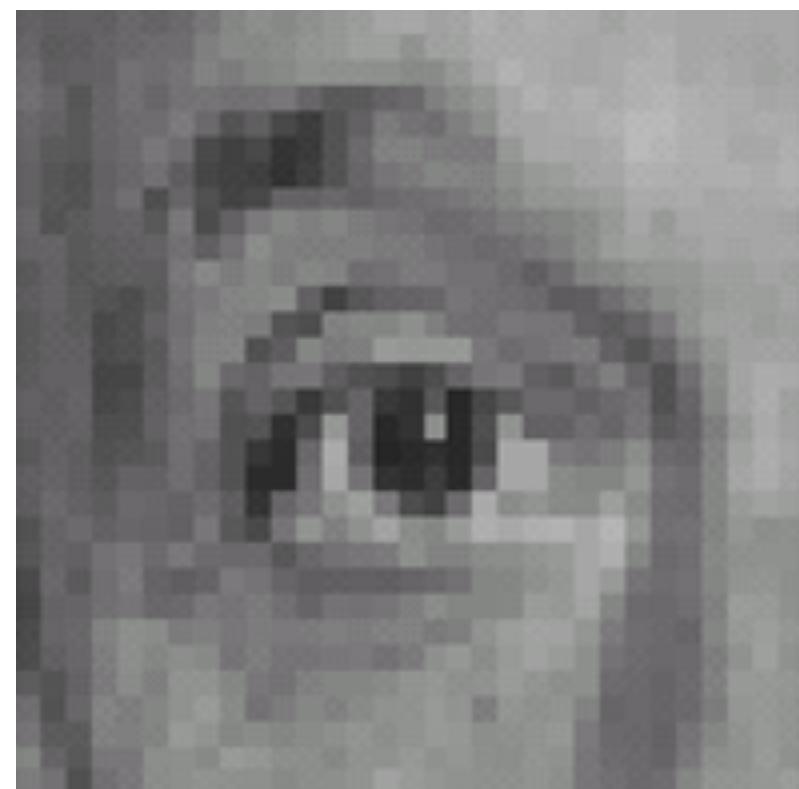
Original

| | | |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |

?

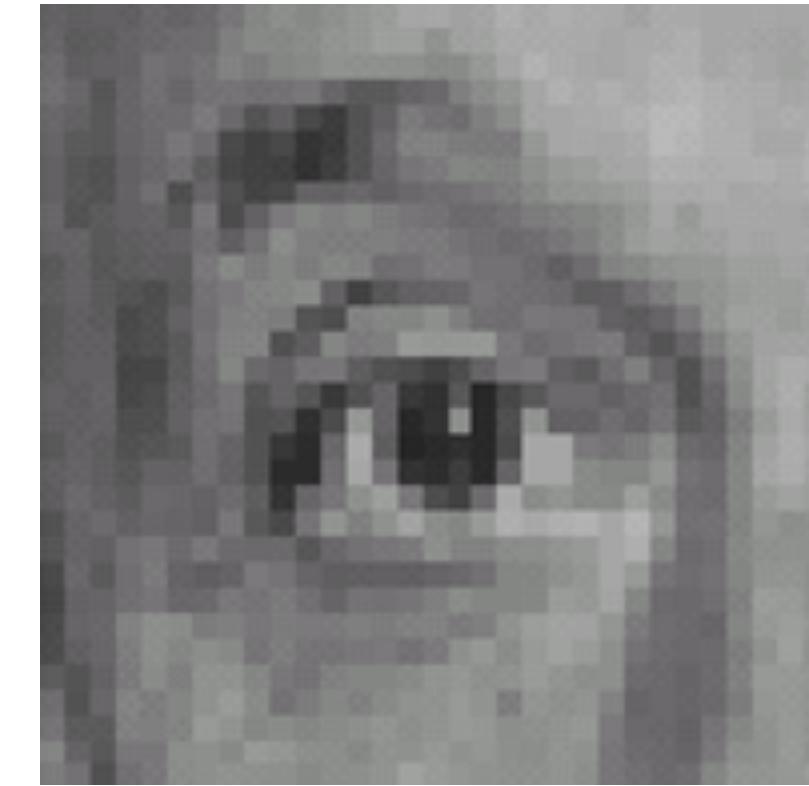
Slide credit: D.A. Forsyth

Linear Filtering (warm-up)



Original

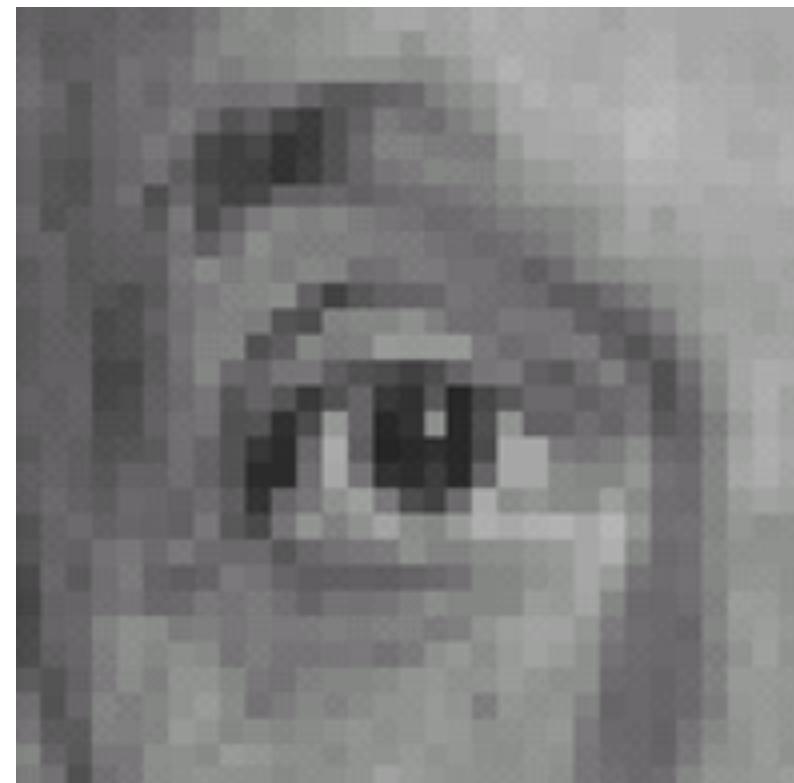
| | | |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |



Filtered
(no change)

Linear Filtering (Convolution)

$$I'(x, y) = \sum_{i=-a}^a \sum_{j=-b}^b K(i, j) I(x - i, y - j)$$



Original

| | | |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 0 |
| 0 | 0 | 0 |

?

Linear Filtering (Convolution)

$$I'(x, y) = \sum_{i=-a}^a \sum_{j=-b}^b K(i, j) I(x - i, y - j)$$



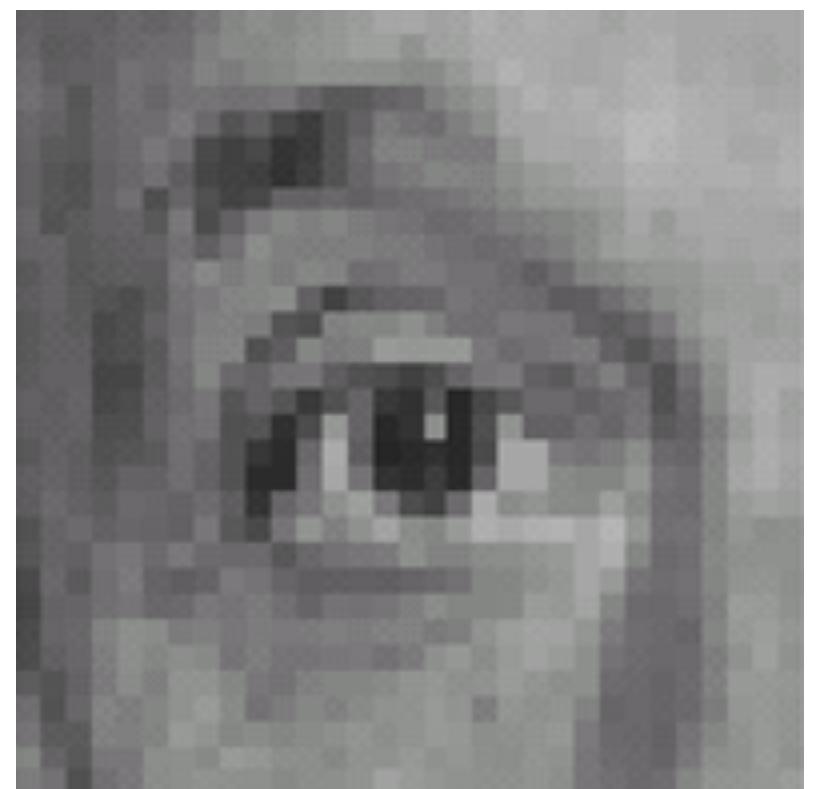
Original

| | | |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 0 |
| 0 | 0 | 0 |



Shifted left
By 1 pixel

Linear Filtering



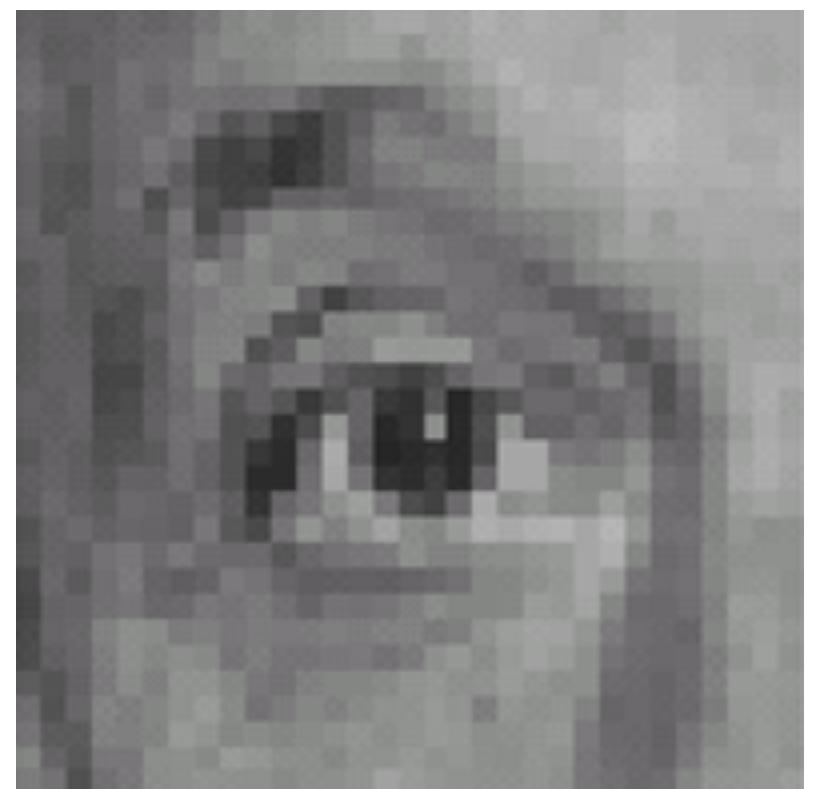
Original

$$\frac{1}{9} \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

?

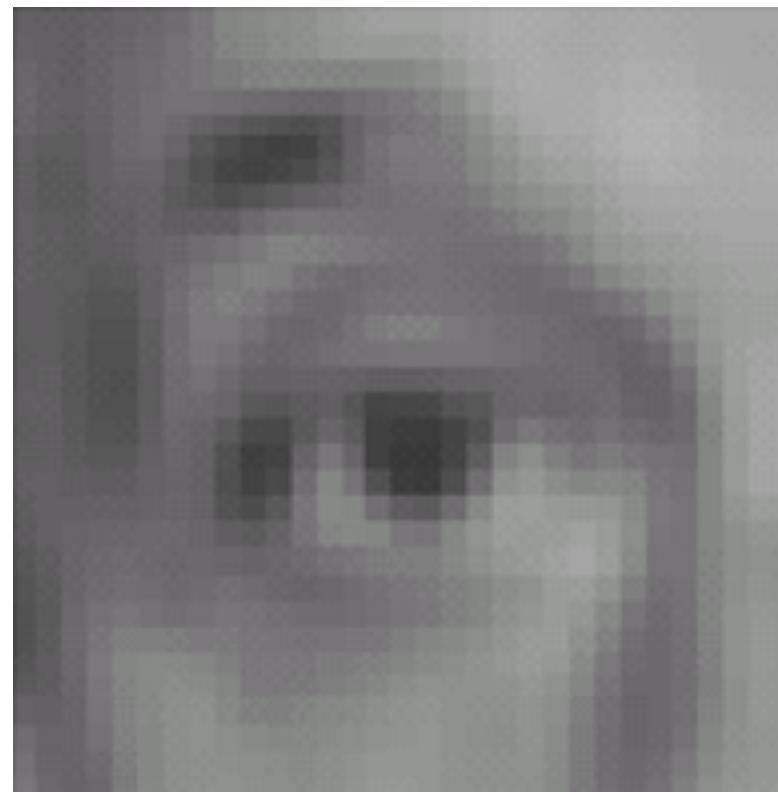
Image Filtering

Linear Filtering



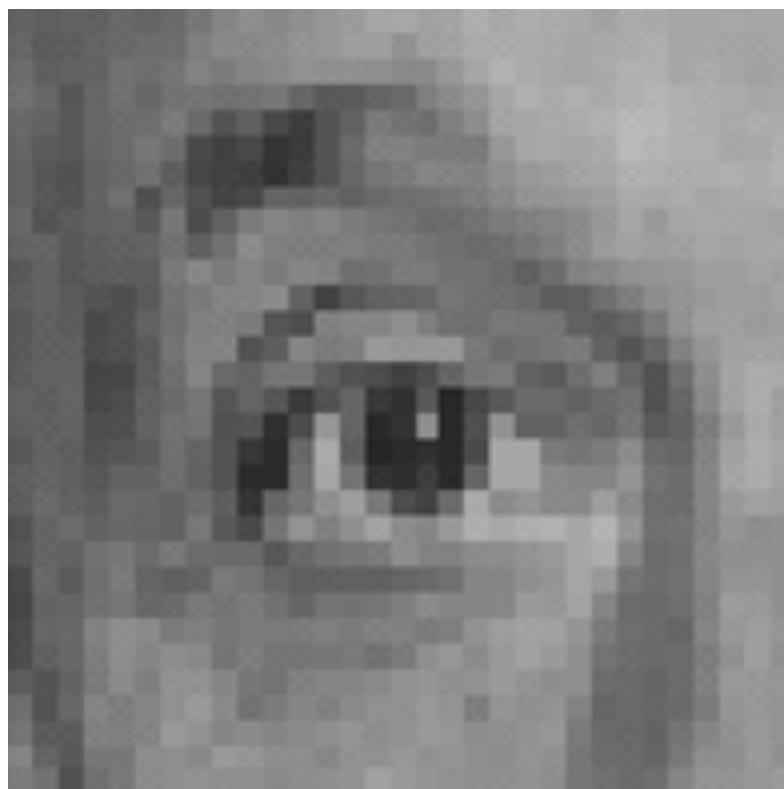
Original

$$\frac{1}{9} \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$



Blur (with a
box filter)

Linear Filtering



Original

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{matrix}$$

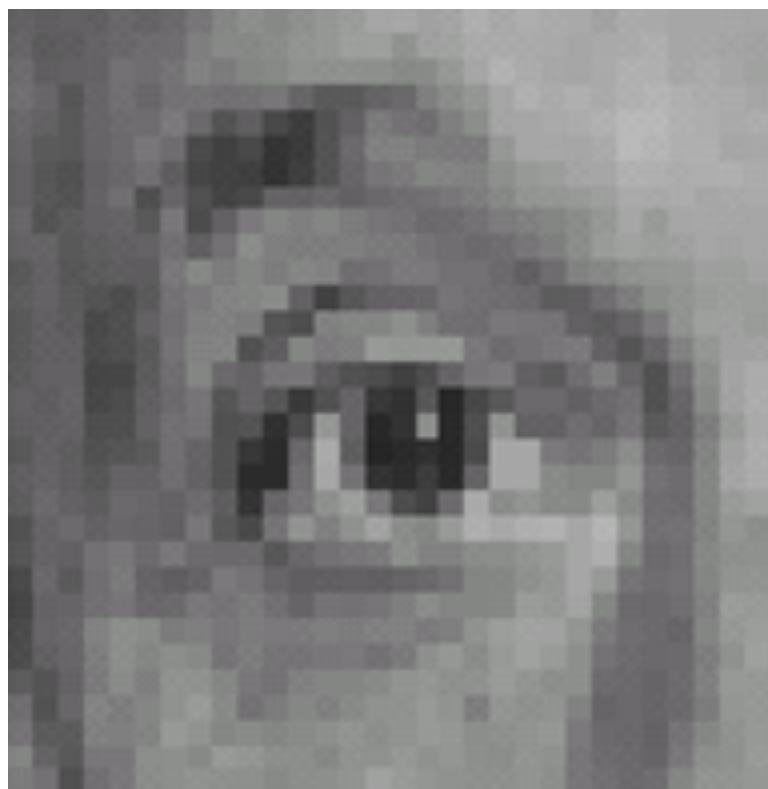
-

$$\frac{1}{9} \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

?

(Note that filter sums to 1)

Linear Filtering



Original

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{matrix}$$

-

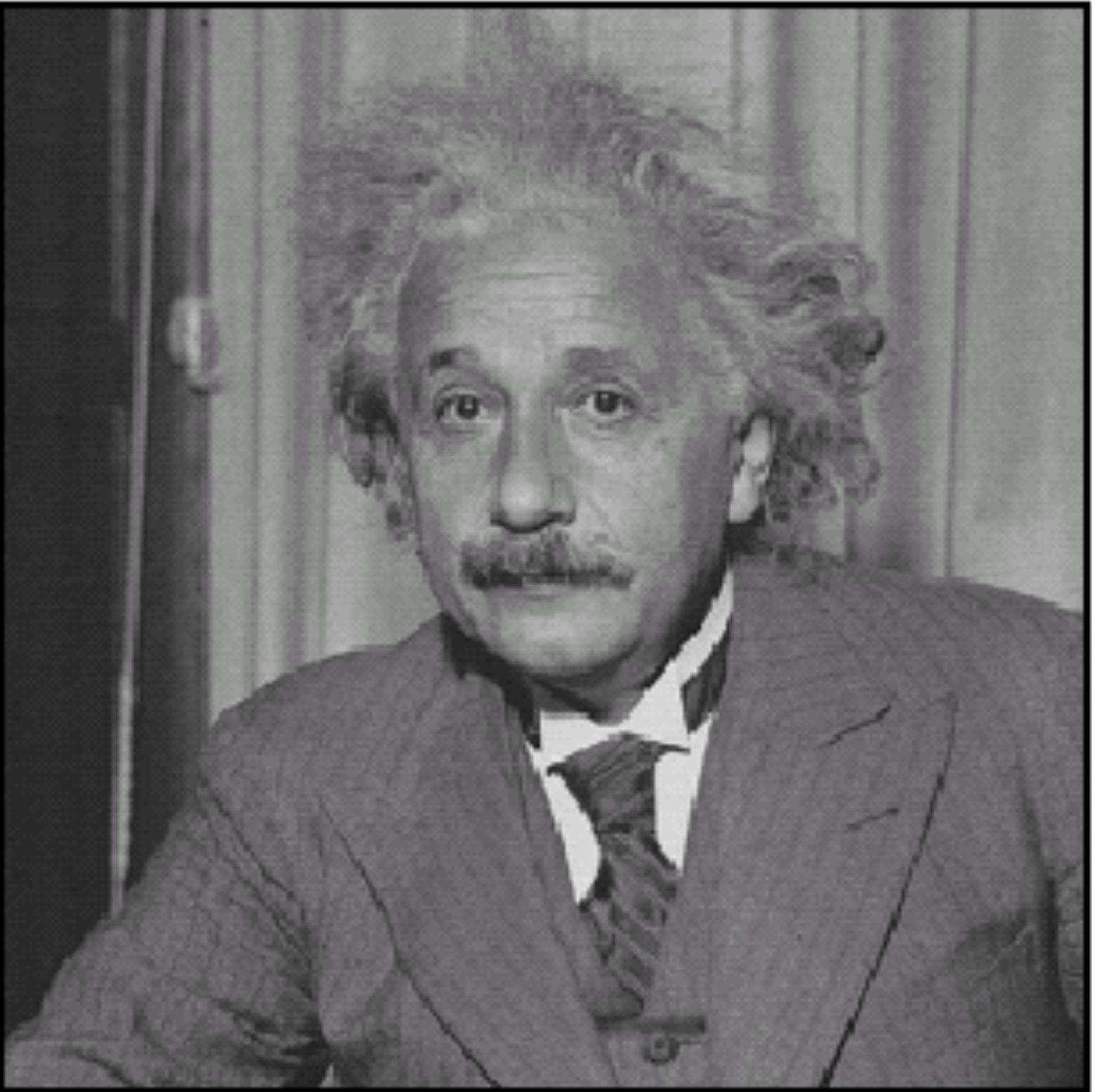
$$\frac{1}{9} \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$



Sharpening filter

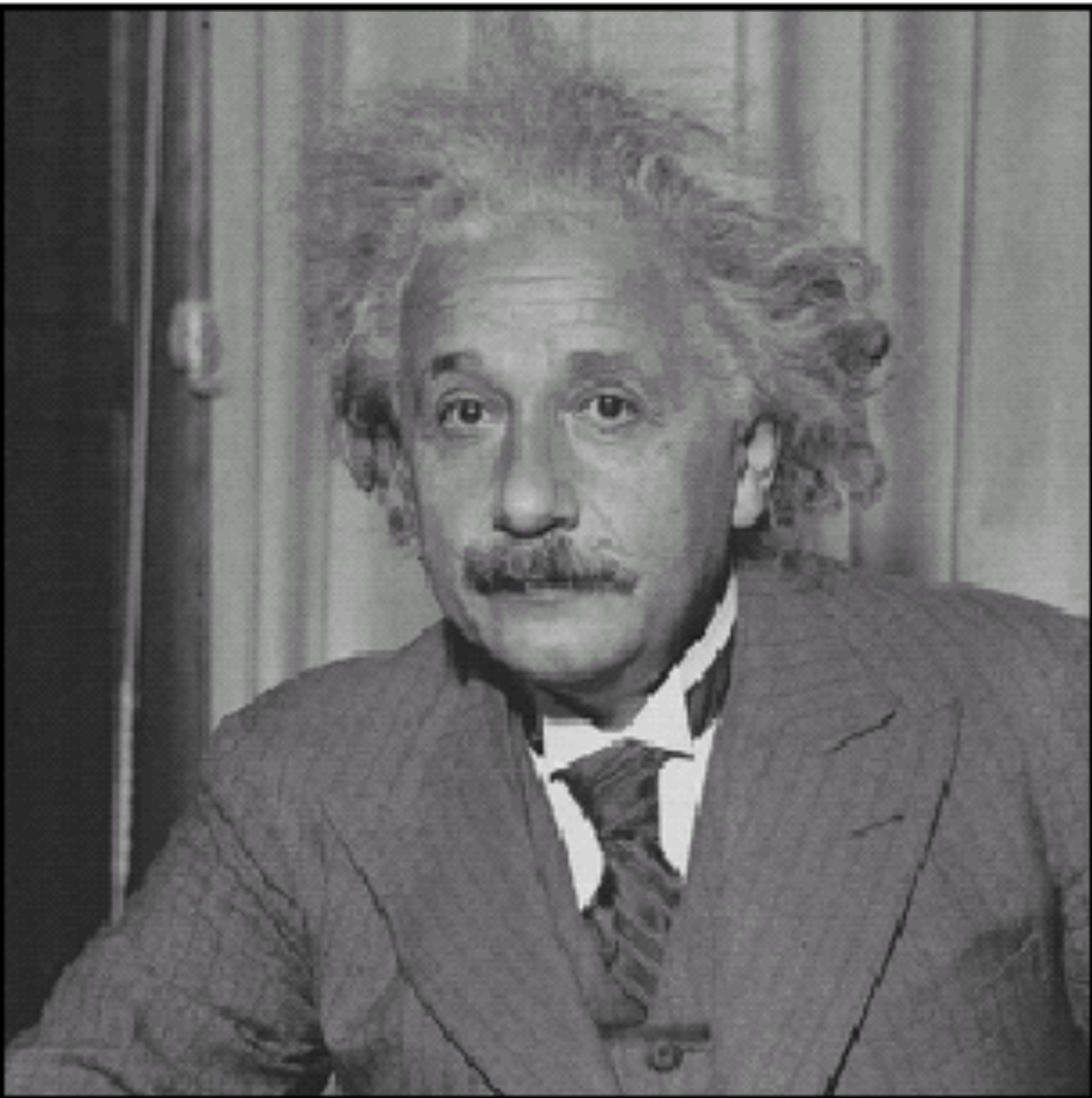
- Amplifies differences with local average
- Also known as Laplacian

Sharpening

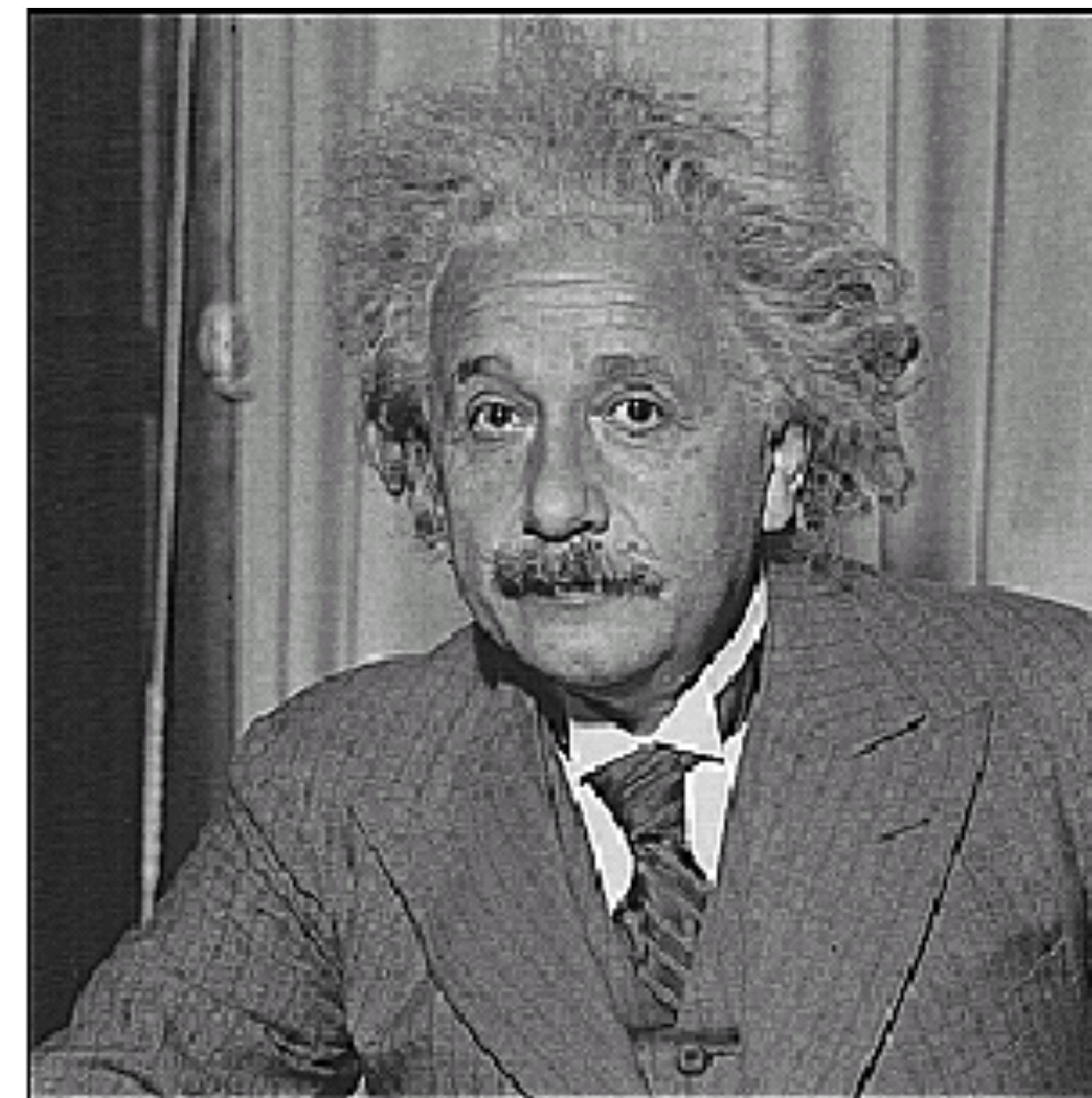


before

Sharpening



before



after

Example

```
K=ones(9,9);
```

```
I2=conv2(I,K);
```



Example

```
K=ones(9,9);
```

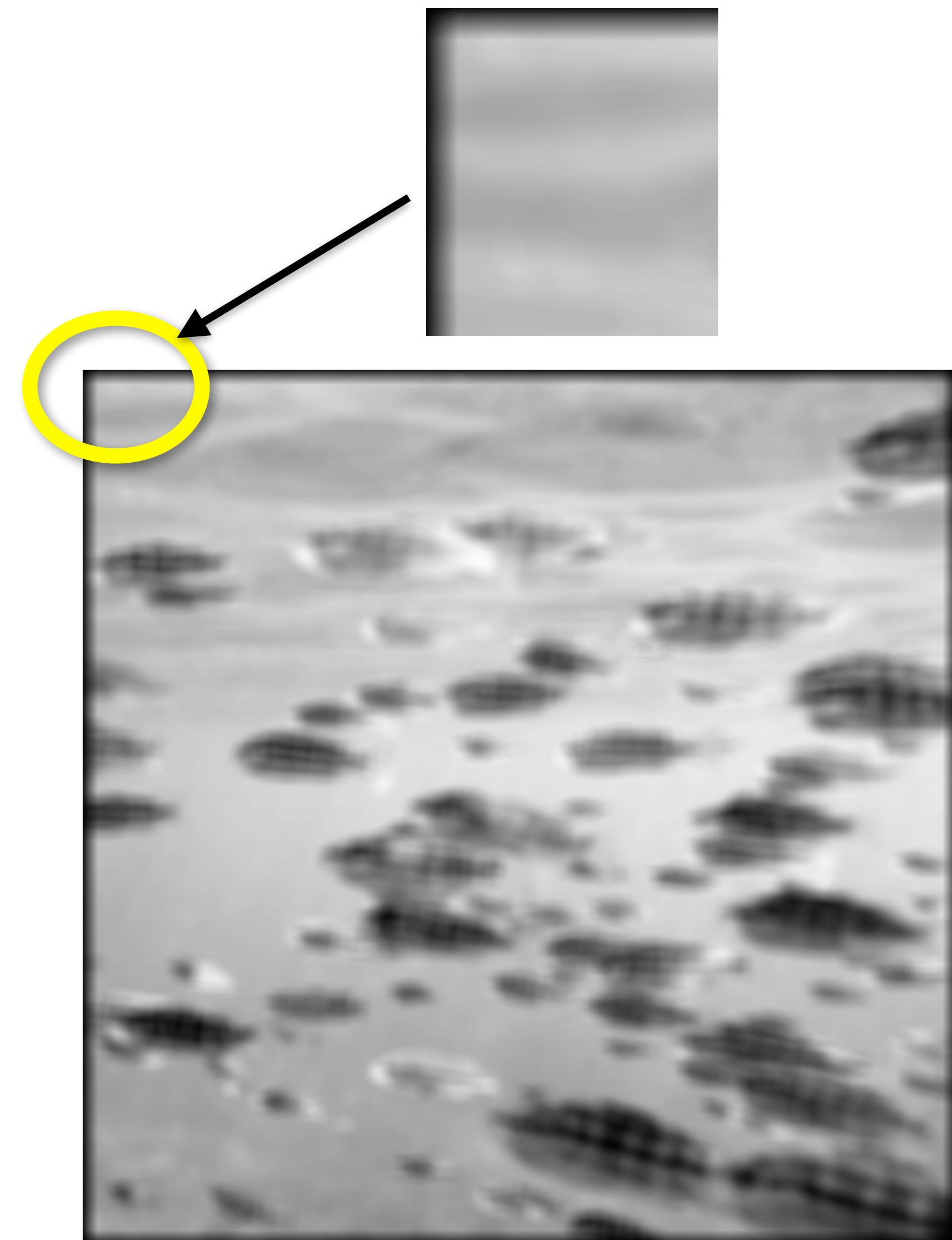
```
I2=conv2(I,K);
```



Example

```
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```

```
I2=conv2(I,K);
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‘Boring’ Details

- When filter window falls off the edge of the image?
 - need to extrapolate
 - methods:
 - clip filter (black)
 - wrap around
 - copy edge
 - reflect across edge
 - vary filter near edge



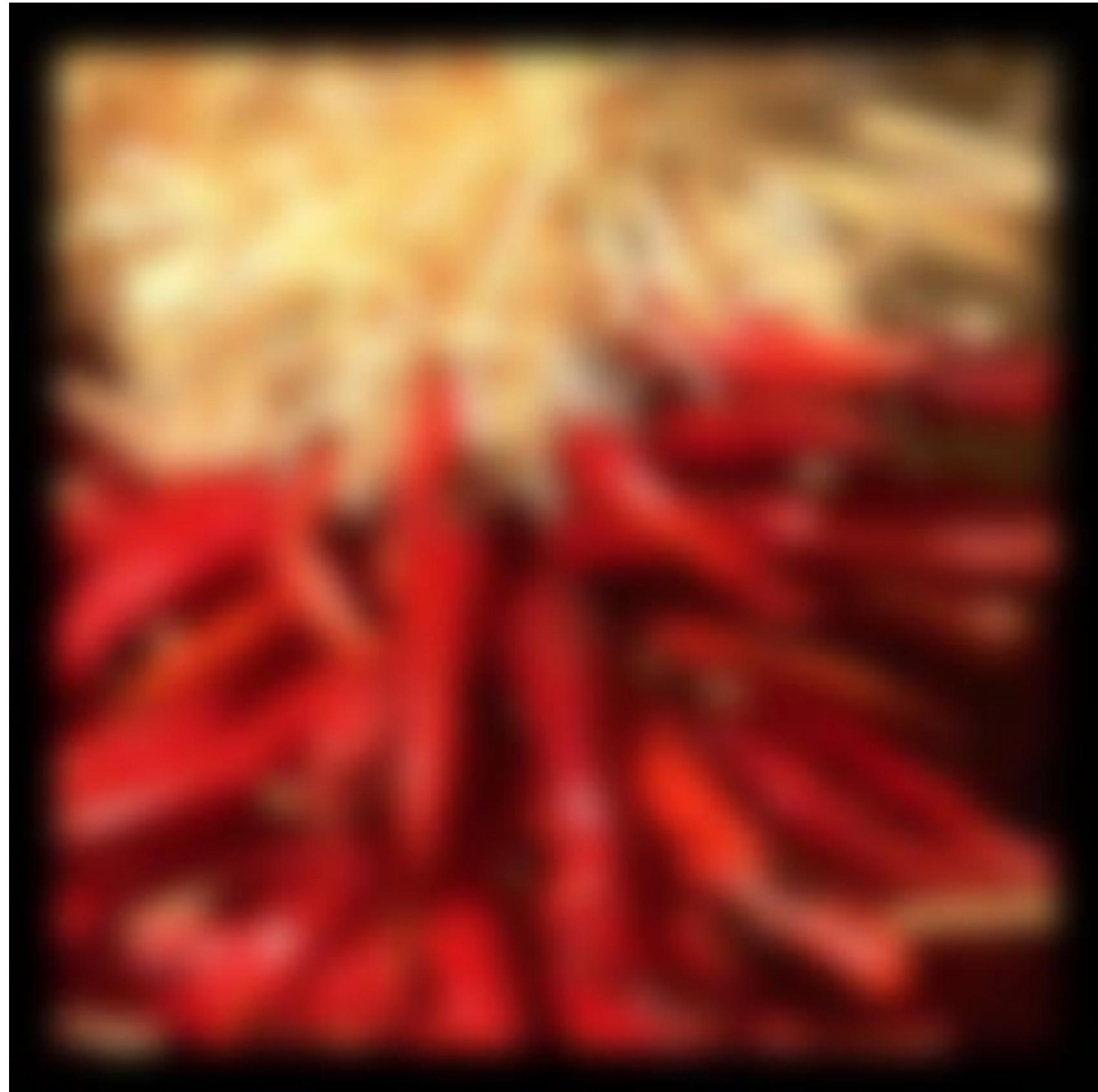
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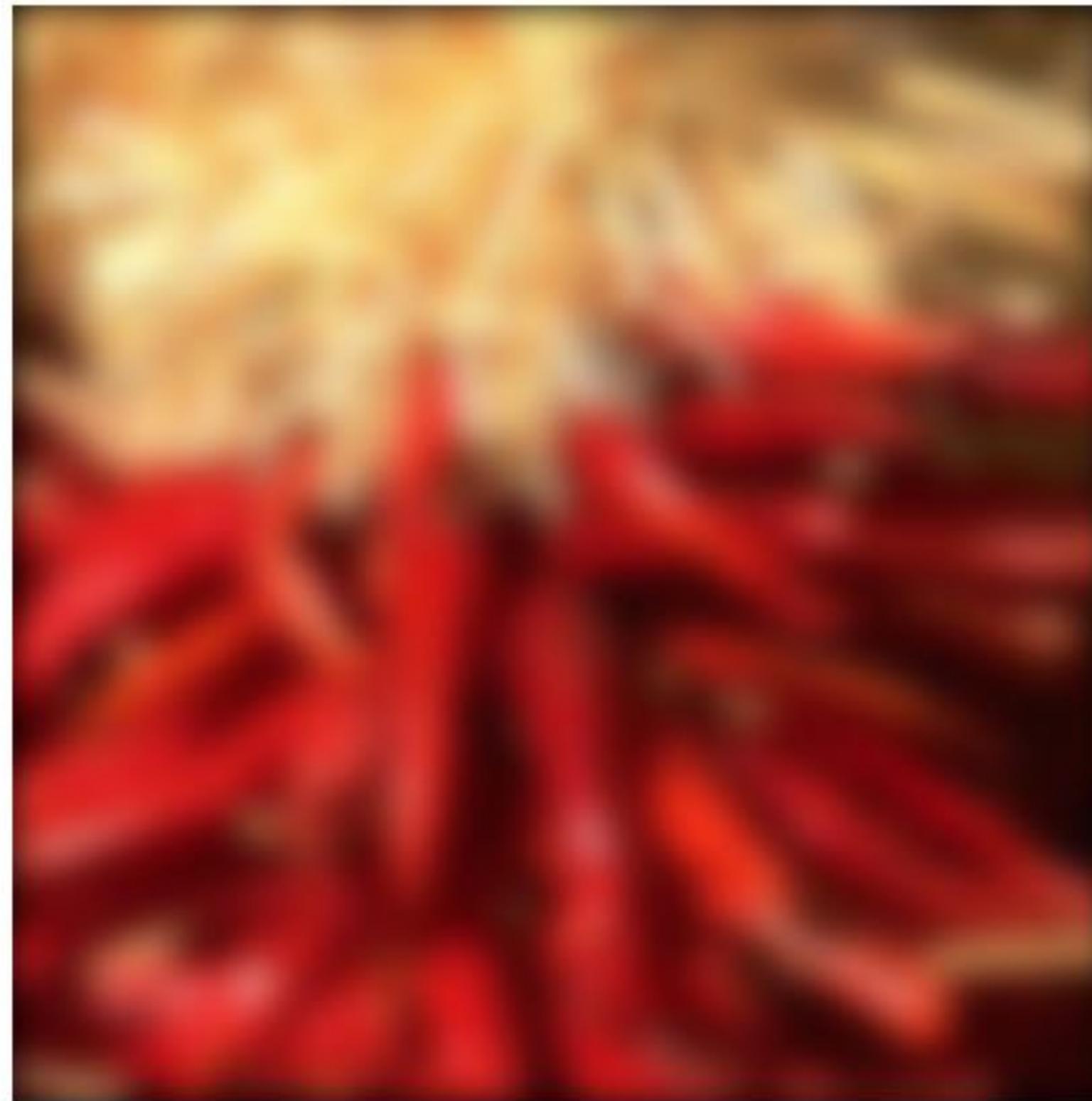
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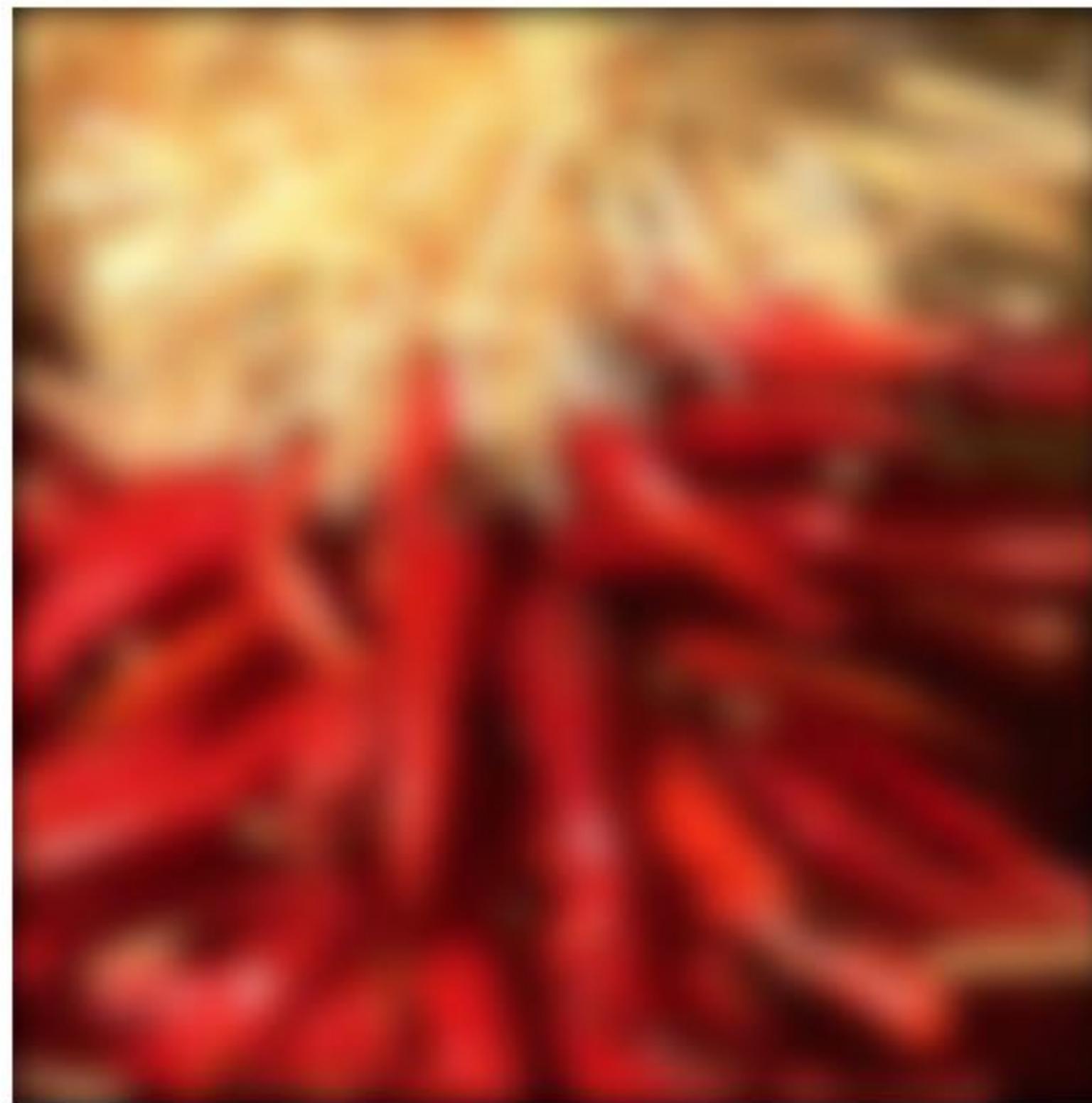
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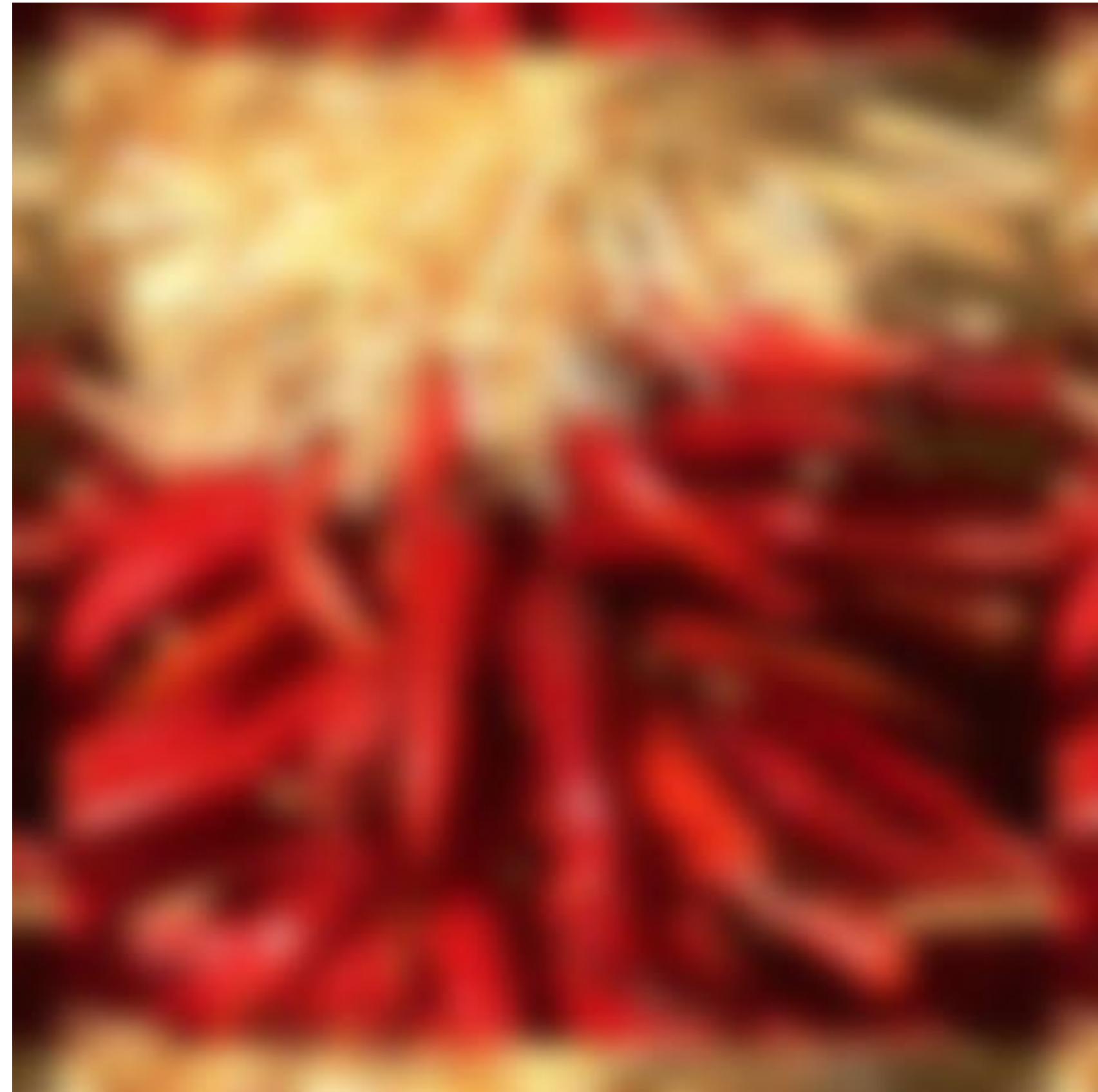
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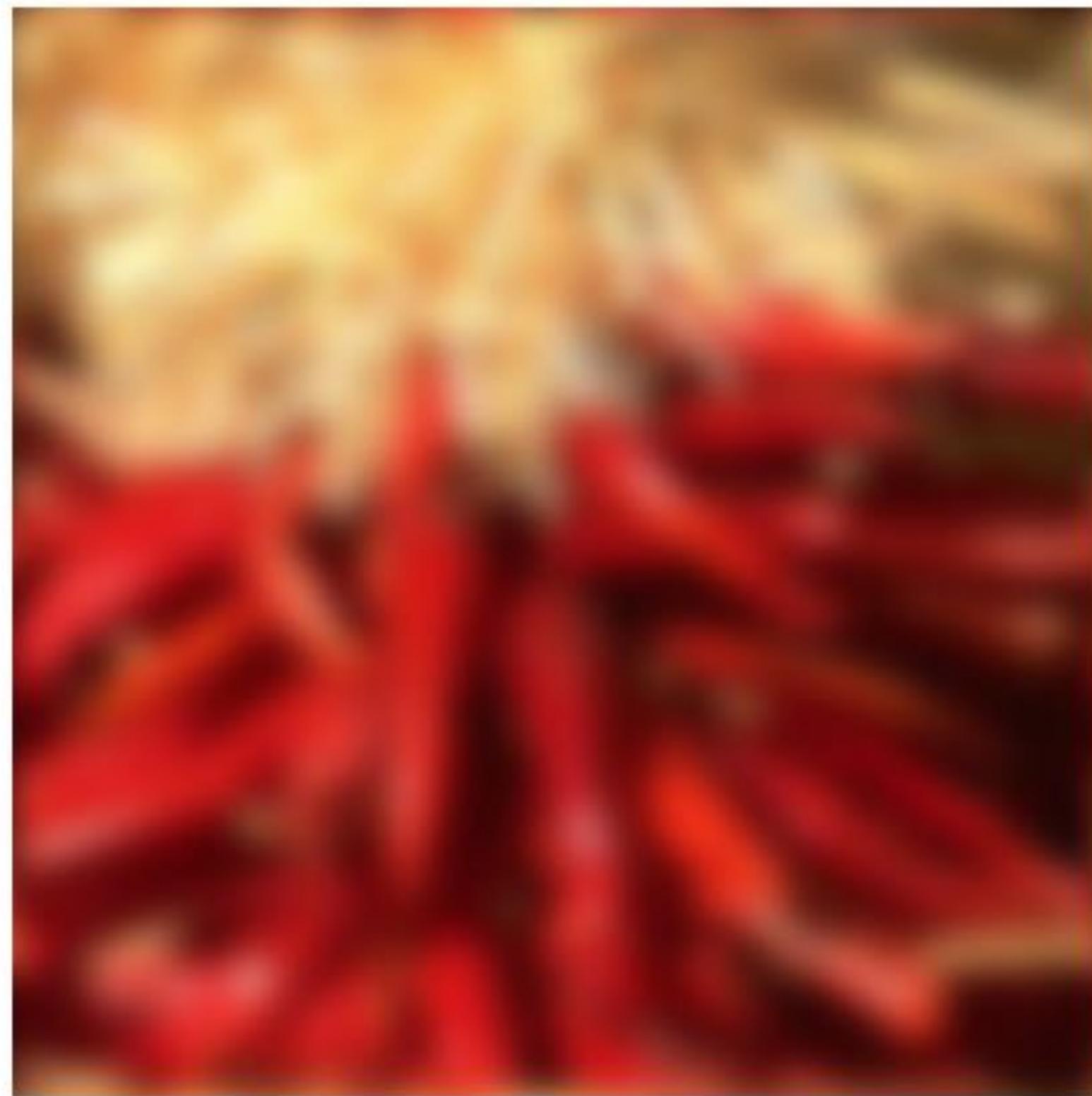
‘Boring’ Details

- When filter window falls off the edge of the image?
 - need to extrapolate
 - methods:
 - clip filter (black)
 - wrap around
 - copy edge
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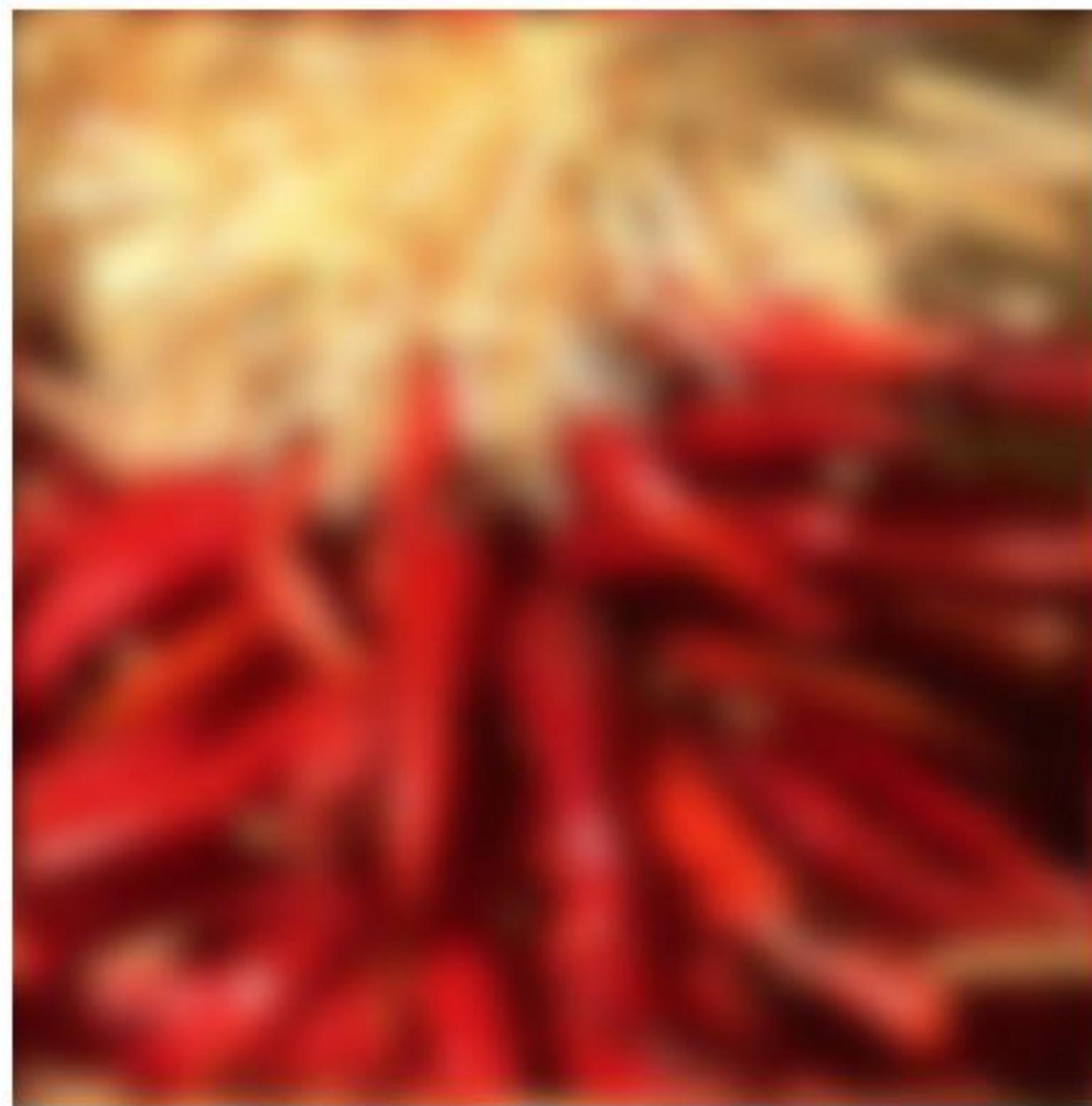
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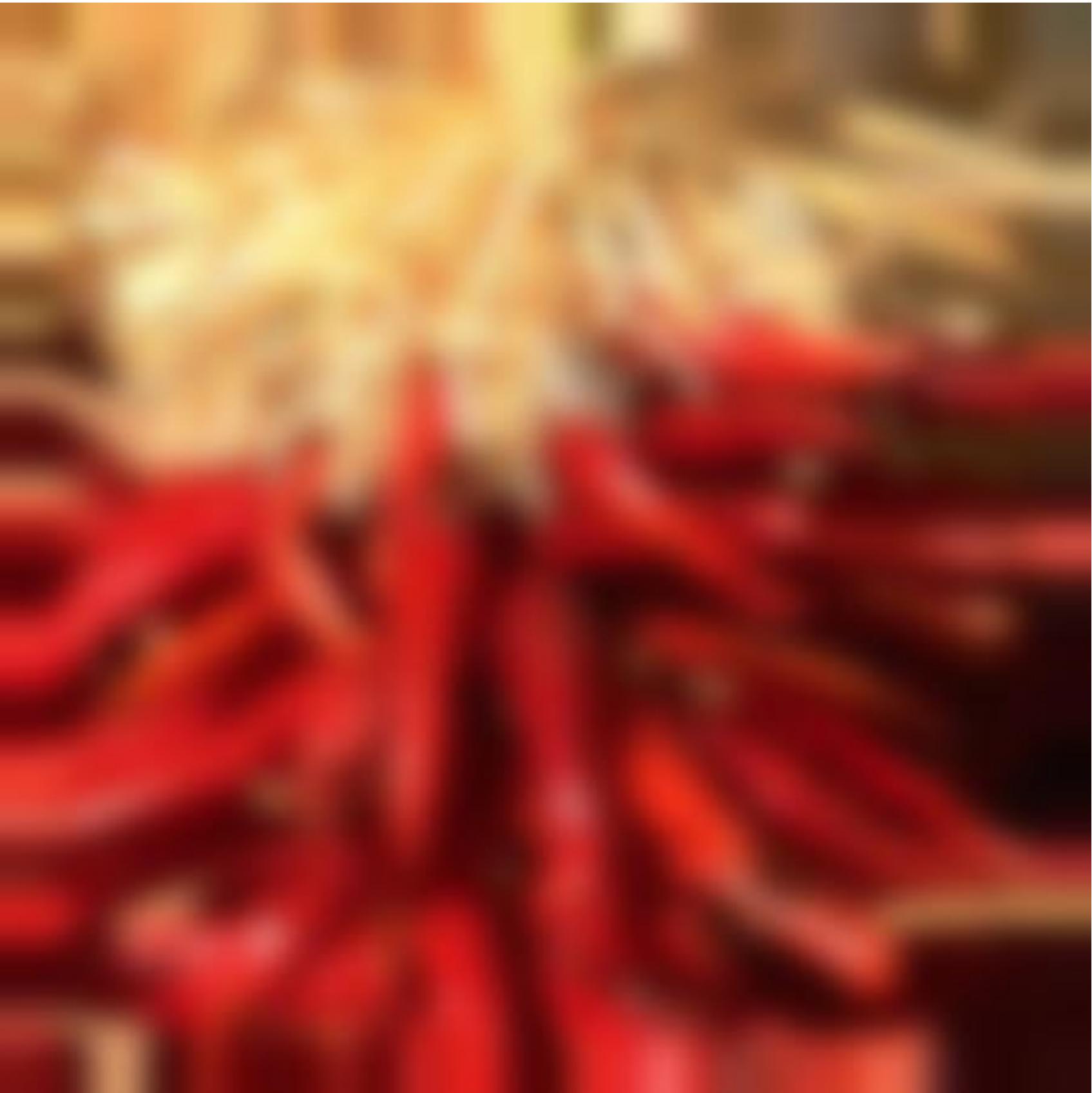
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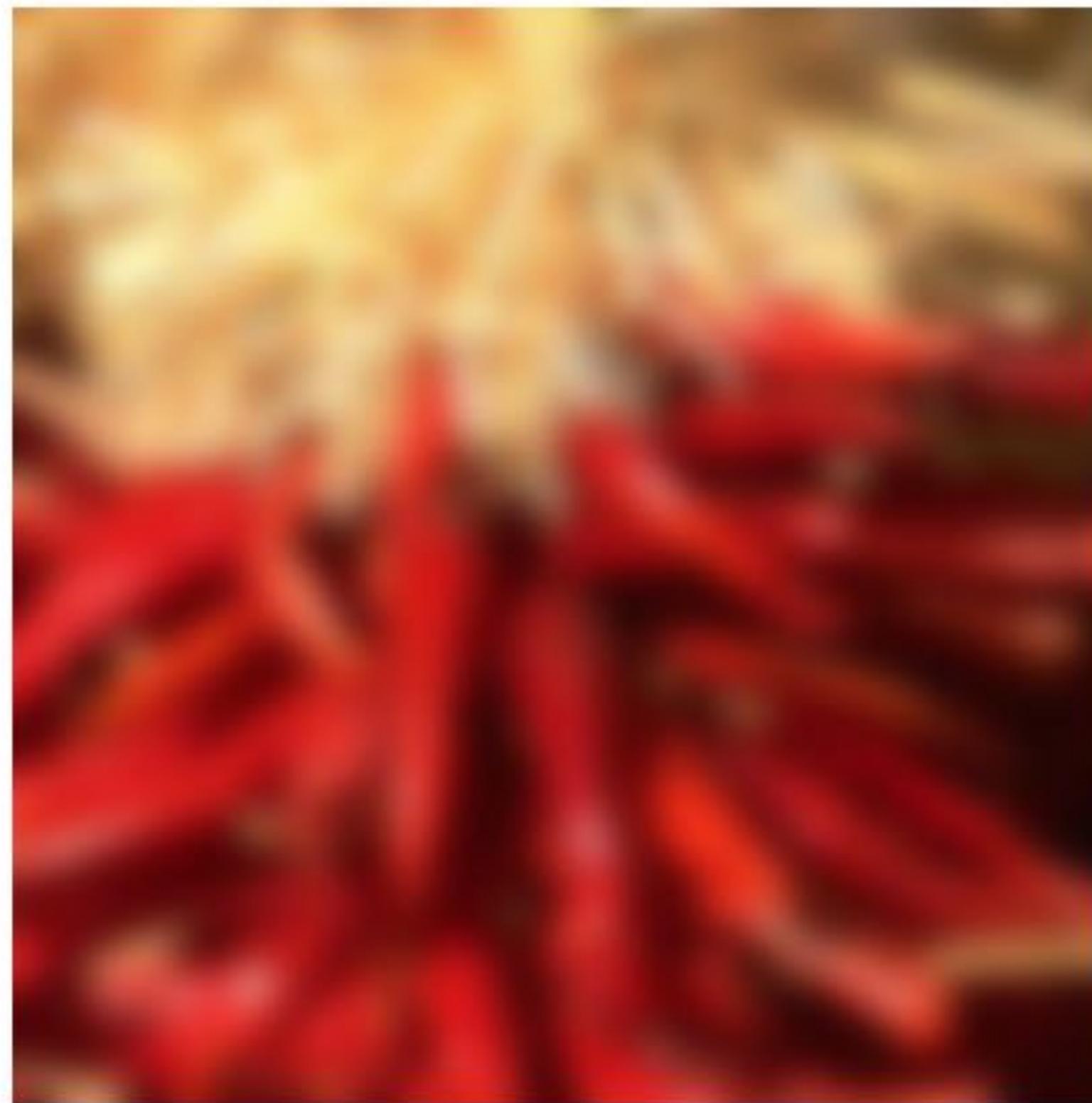
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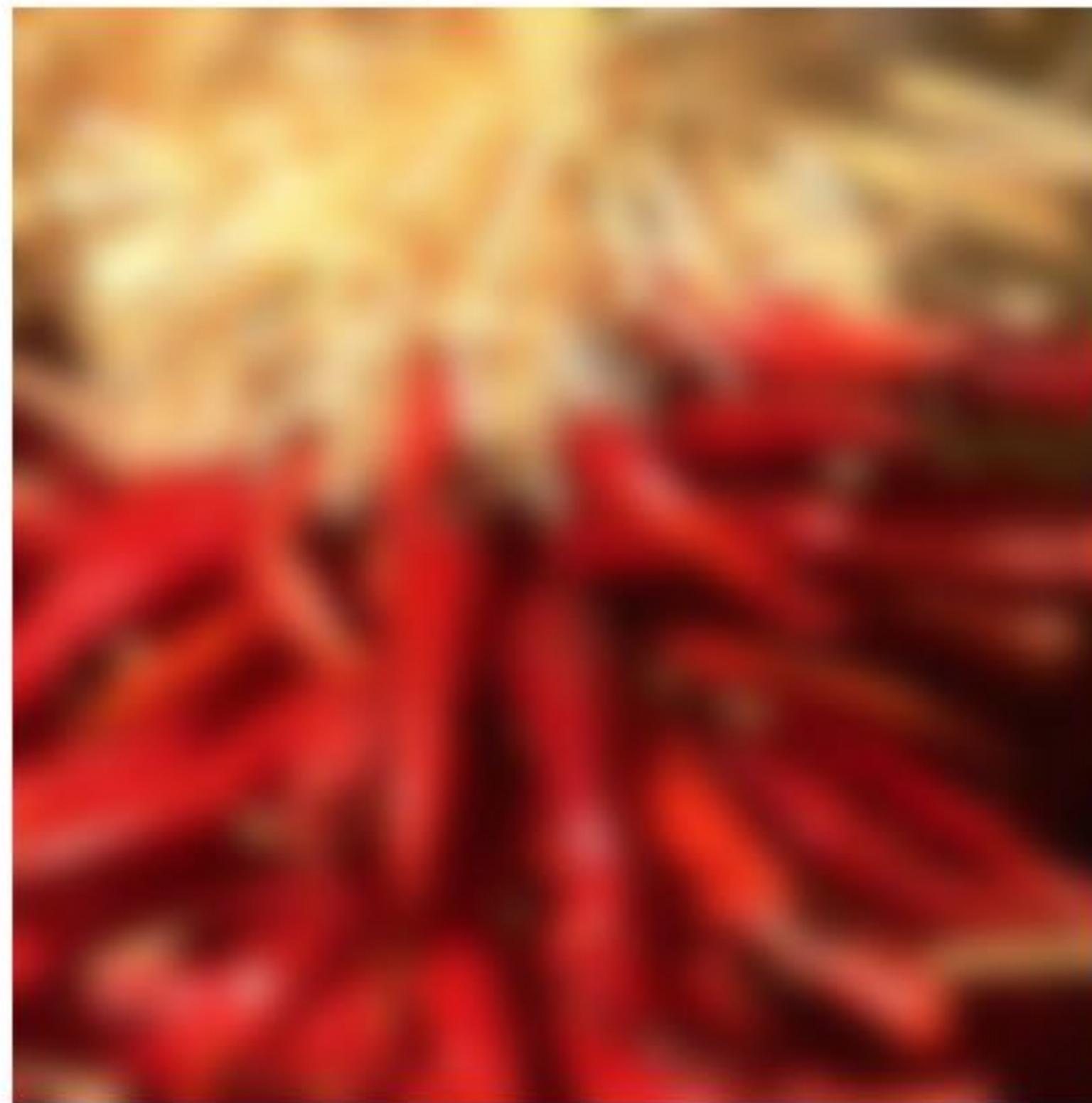
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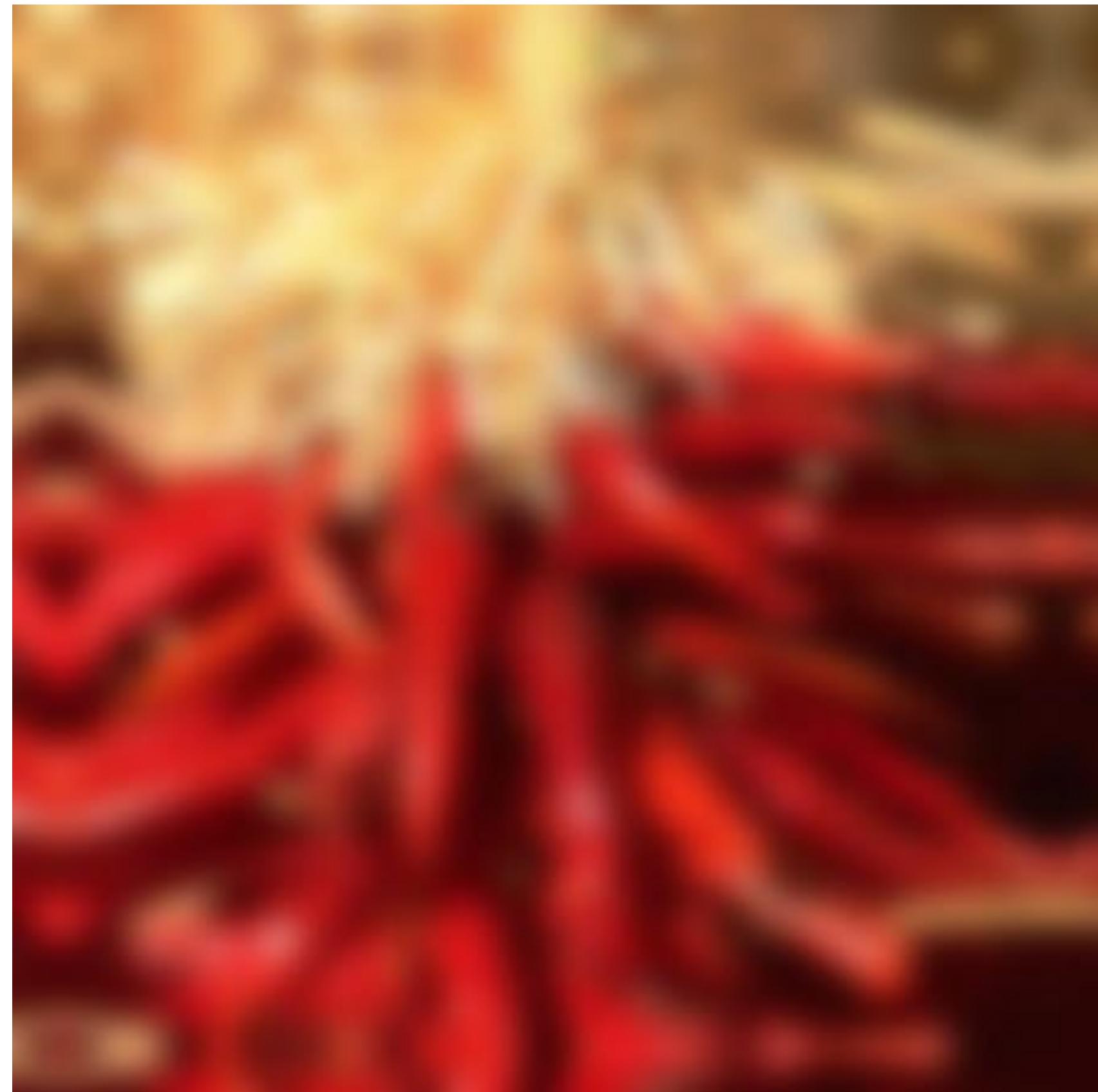
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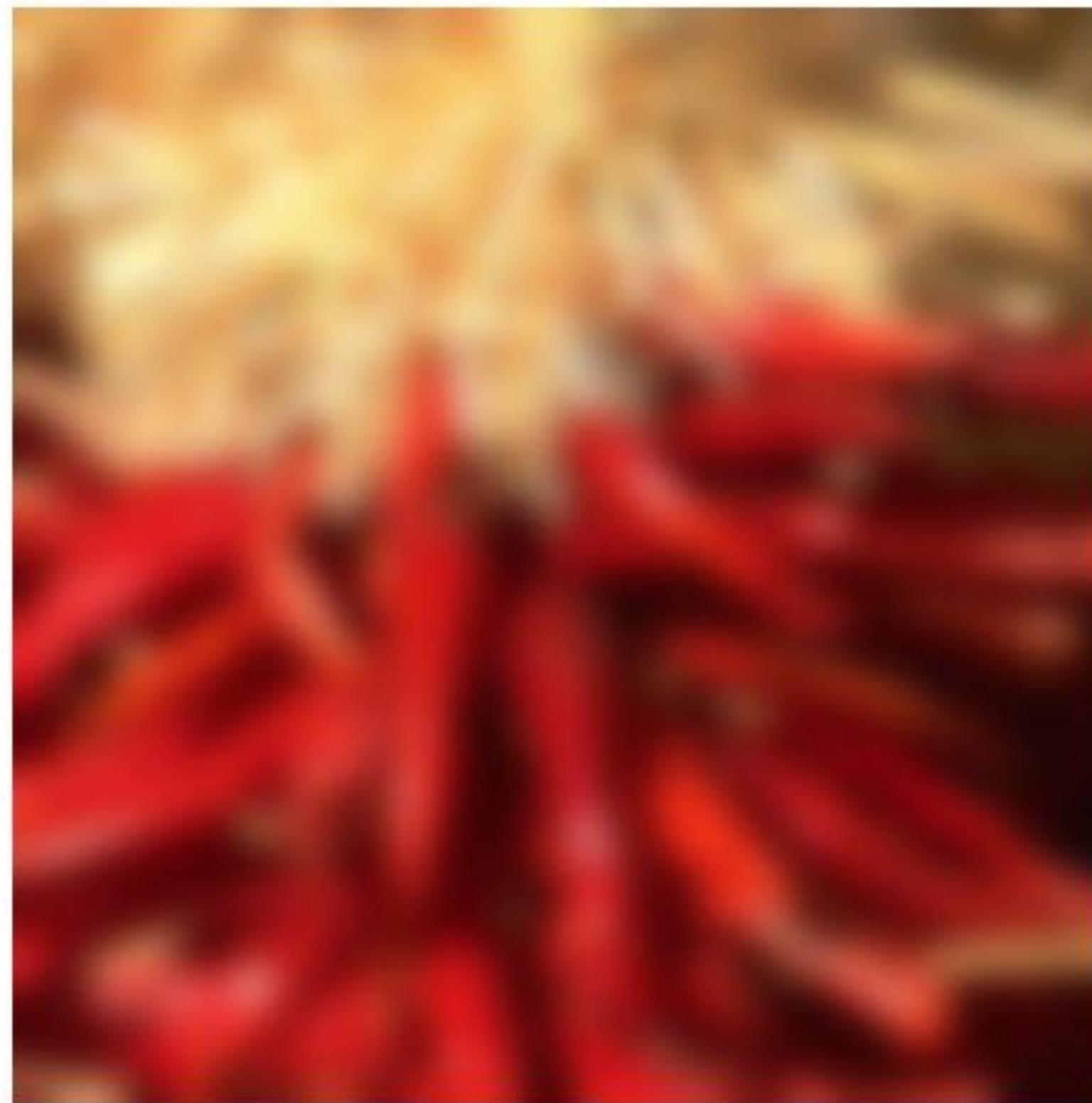
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Smoothing Kernels (Low-pass filters)

Mean filter:

$$\frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Smoothing Kernels (Low-pass filters)

Mean filter:

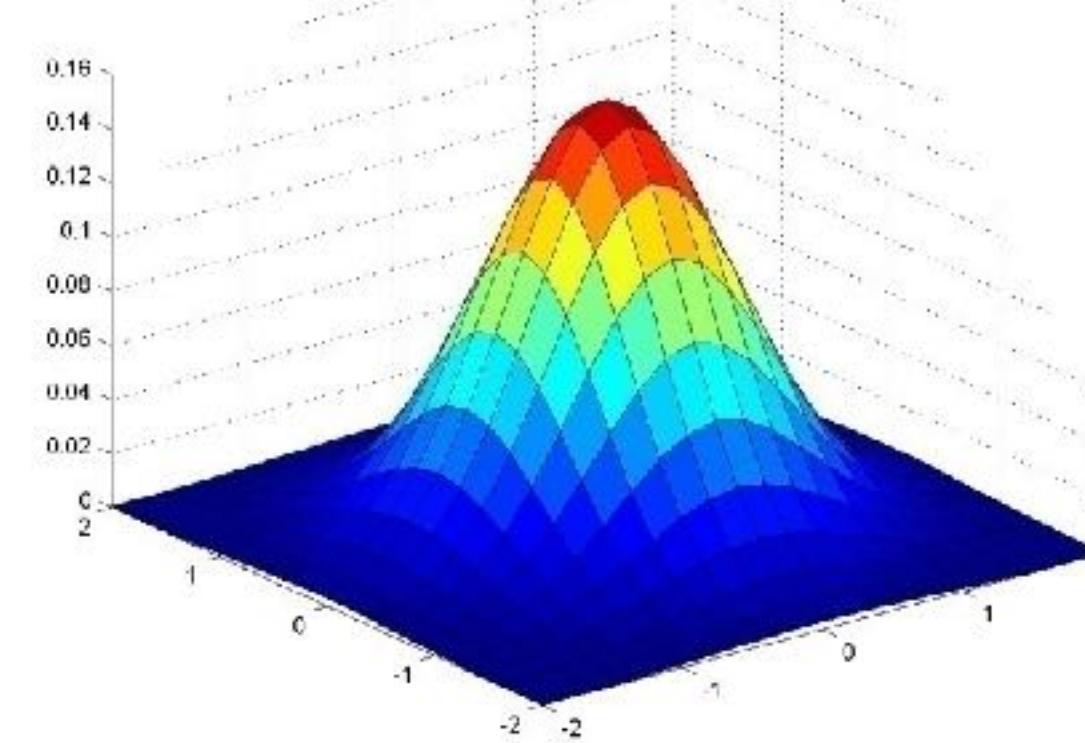
$$\frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Weighted smoothing filters:

$$\frac{1}{10} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\frac{1}{16} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Gaussian Kernel



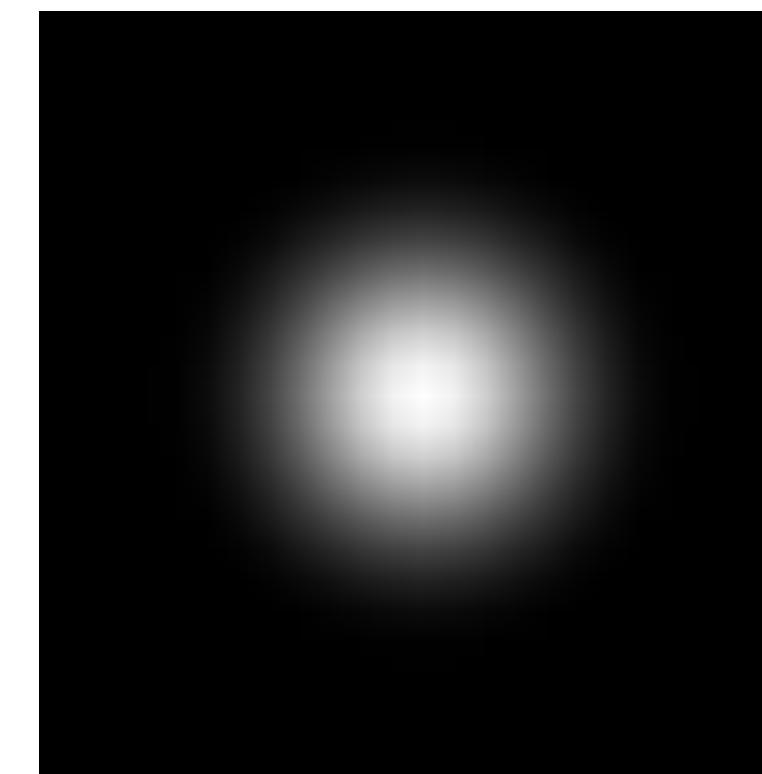
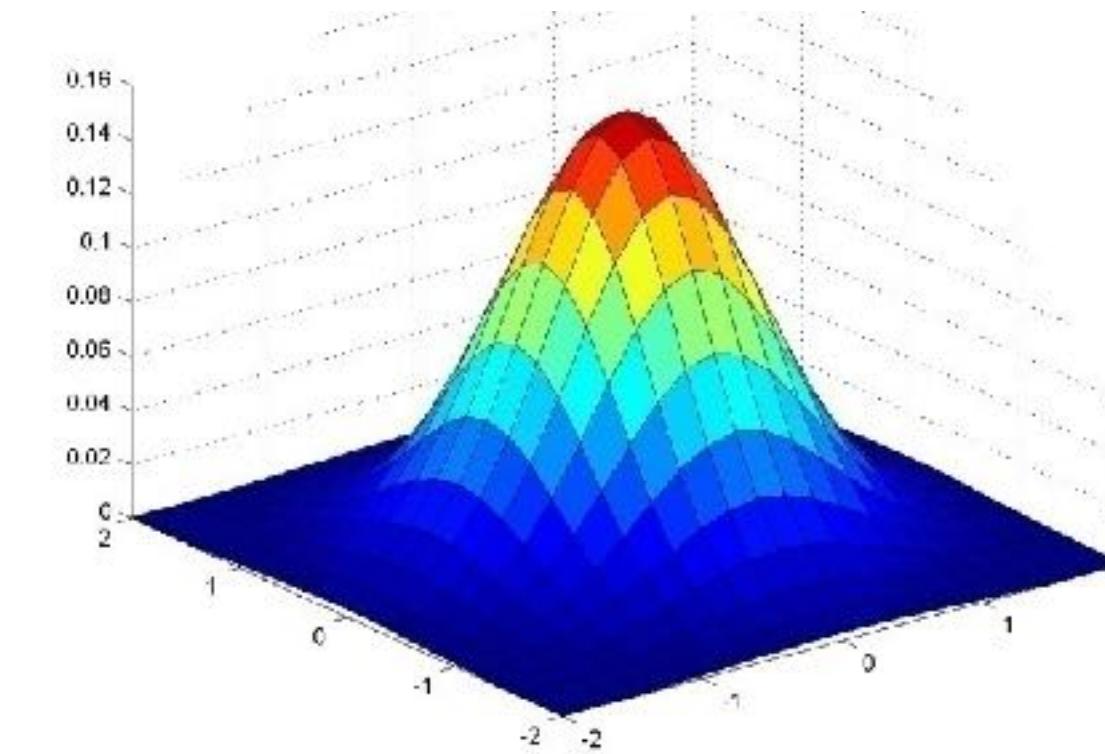
| | | | | |
|-------|-------|-------|-------|-------|
| 0.003 | 0.013 | 0.022 | 0.013 | 0.003 |
| 0.013 | 0.059 | 0.097 | 0.059 | 0.013 |
| 0.022 | 0.097 | 0.159 | 0.097 | 0.022 |
| 0.013 | 0.059 | 0.097 | 0.059 | 0.013 |
| 0.003 | 0.013 | 0.022 | 0.013 | 0.003 |

$$G_{\sigma} := \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

$5 \times 5, \sigma = 1$

Gaussian Kernel

- Weight contributions of neighbouring pixels related to nearness



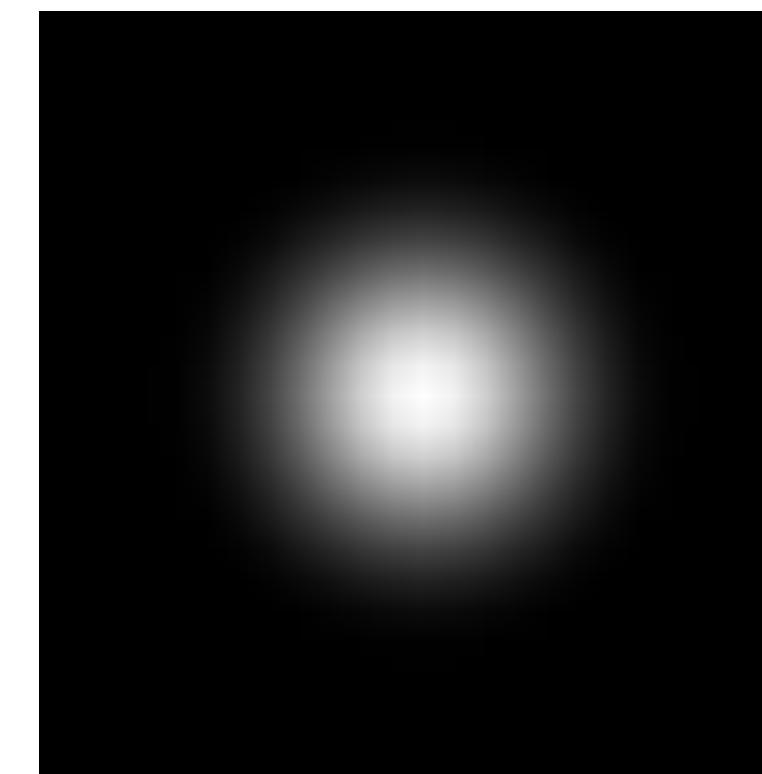
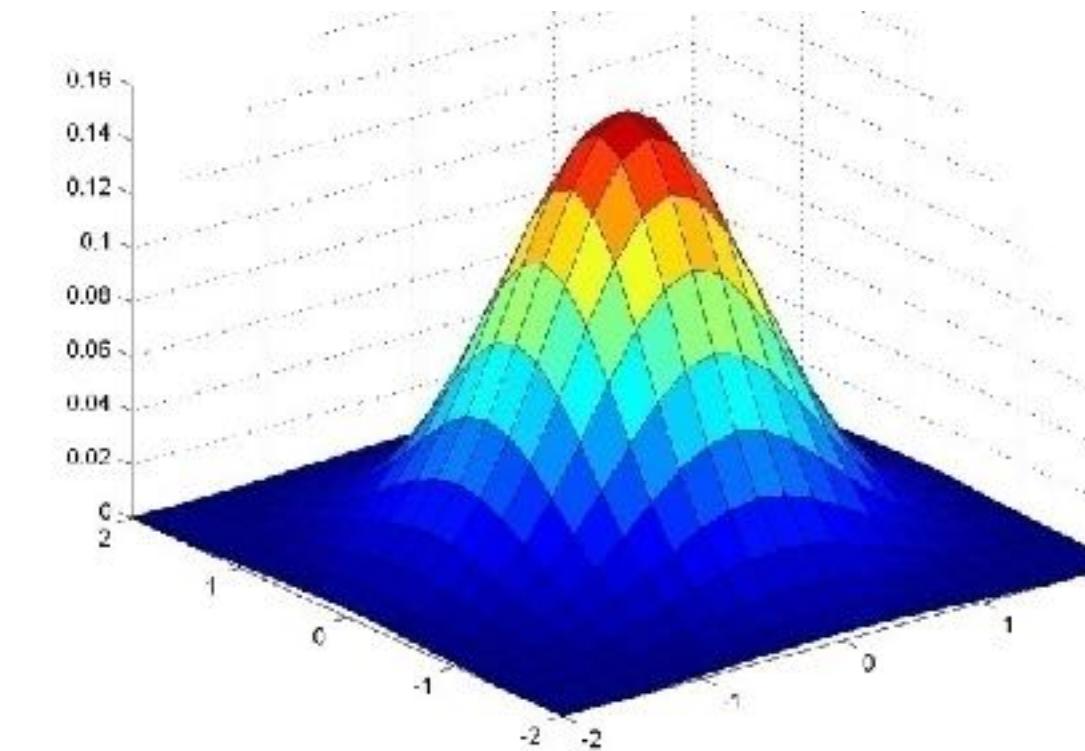
| | | | | |
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$5 \times 5, \sigma = 1$

$$G_{\sigma} := \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

- Constant factor in the front makes volume sum to 1
 - (subject to rounding errors, so better to normalize by sum of matrix)

Review of Sigma (σ)

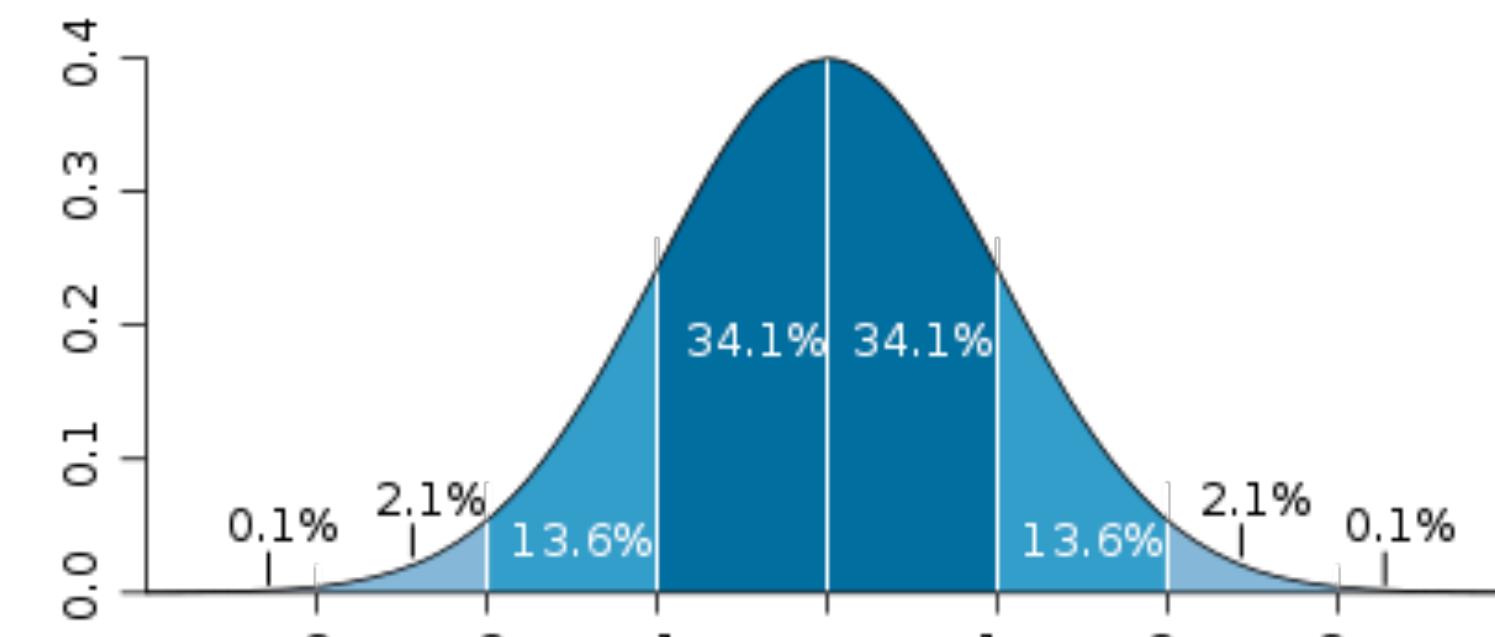
$\sigma = \text{sigma} = \text{standard deviation} = \sqrt{(\text{variance})}$

Data, $\mathbf{d} = [d_1 \quad d_2 \quad d_3 \quad \dots \quad d_n]$

$$\text{Mean} := \mu = \frac{\sum_i \mathbf{d}_i}{n}$$

$$\text{Variance} := \sum_i (\mathbf{d}_i - \mu)^2 / n$$

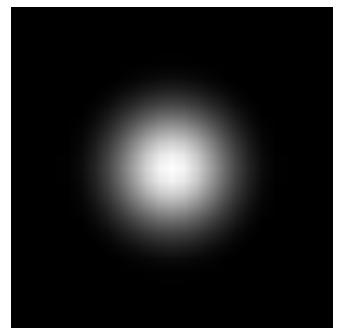
$$\text{StdDeviation} := \sigma = \sqrt{\text{Variance}}$$



[Figure](#) by Petter Strandmark

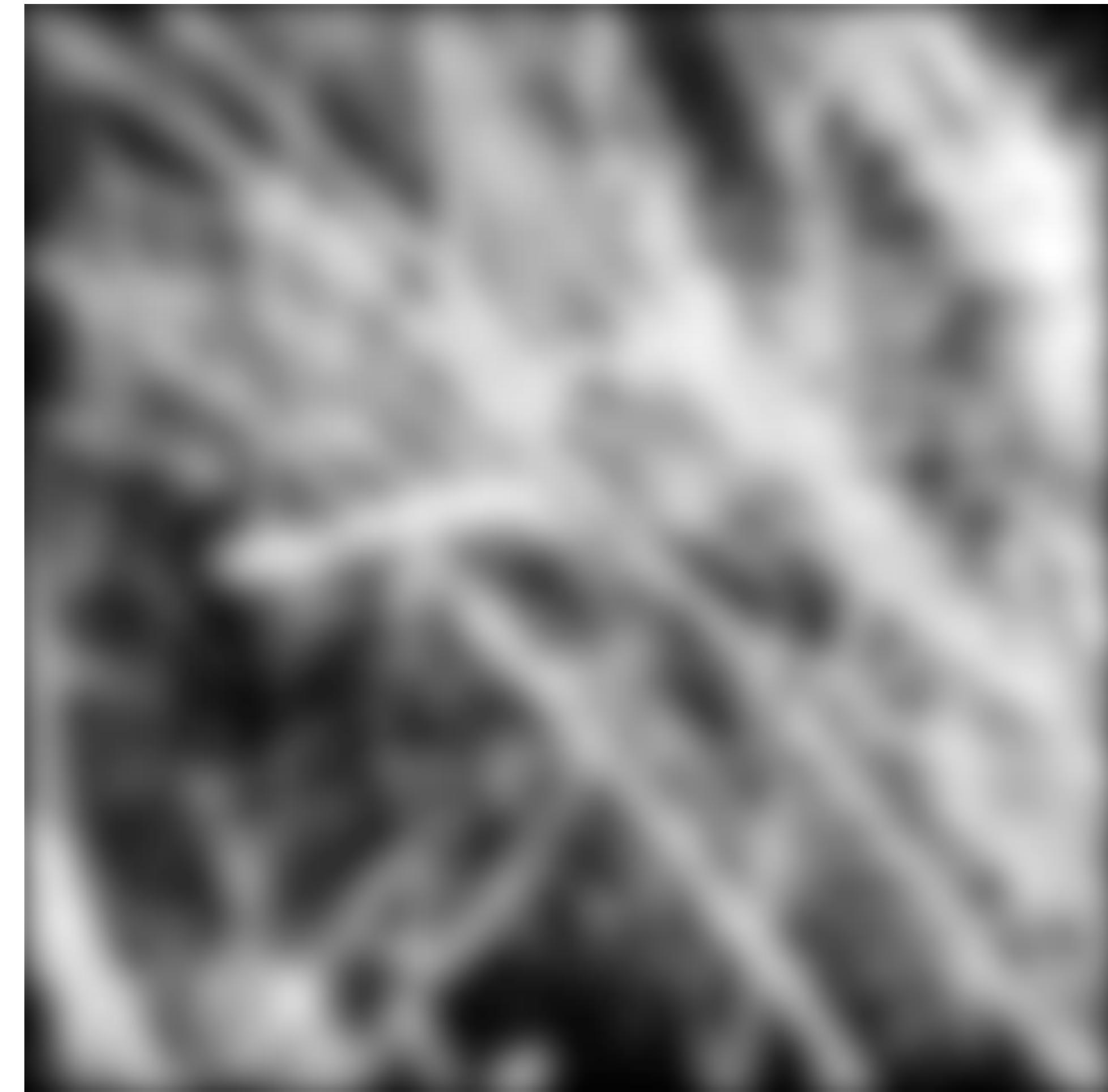
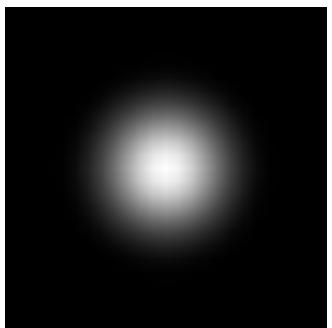
mean (d) , var (d) , std (d)

Smoothing with Gaussian (Blur)



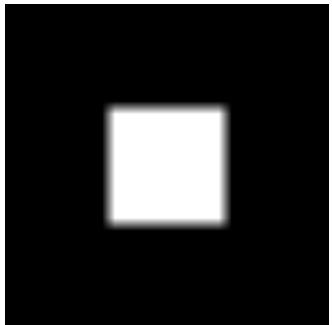
Slide credit: D.A. Forsyth

Smoothing with Gaussian (Blur)



Slide credit: D.A. Forsyth

Smoothing with Box Filter



Slide credit: D.A. Forsyth

Smoothing with Box Filter



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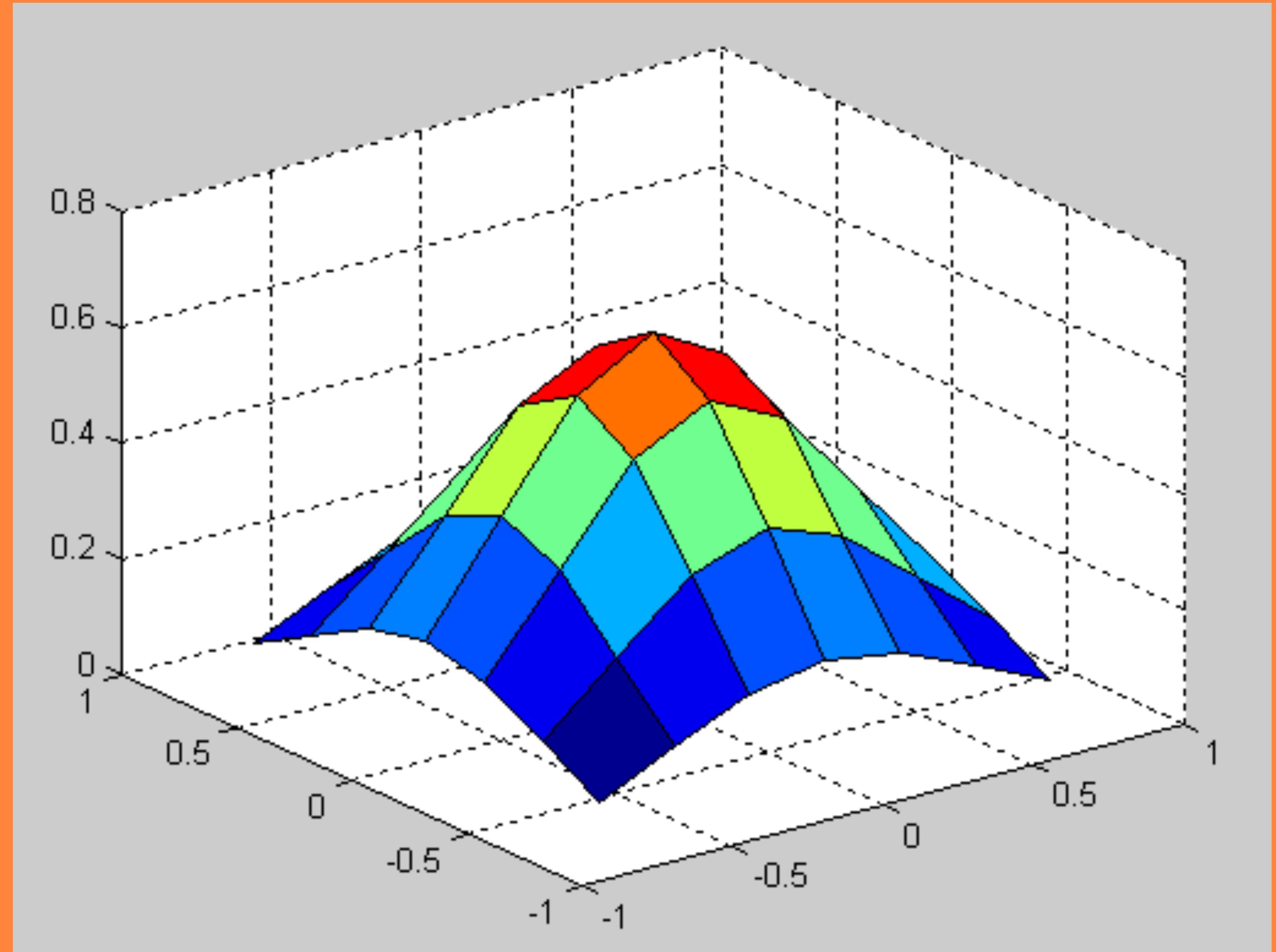
X =

| | | | | | | |
|---------|---------|---------|---|--------|--------|--------|
| -0.7500 | -0.5000 | -0.2500 | 0 | 0.2500 | 0.5000 | 0.7500 |
| -0.7500 | -0.5000 | -0.2500 | 0 | 0.2500 | 0.5000 | 0.7500 |
| -0.7500 | -0.5000 | -0.2500 | 0 | 0.2500 | 0.5000 | 0.7500 |
| -0.7500 | -0.5000 | -0.2500 | 0 | 0.2500 | 0.5000 | 0.7500 |
| -0.7500 | -0.5000 | -0.2500 | 0 | 0.2500 | 0.5000 | 0.7500 |
| -0.7500 | -0.5000 | -0.2500 | 0 | 0.2500 | 0.5000 | 0.7500 |
| -0.7500 | -0.5000 | -0.2500 | 0 | 0.2500 | 0.5000 | 0.7500 |

Y =

| | | | | | | |
|---------|---------|---------|---------|---------|---------|---------|
| -0.7500 | -0.7500 | -0.7500 | -0.7500 | -0.7500 | -0.7500 | -0.7500 |
| -0.5000 | -0.5000 | -0.5000 | -0.5000 | -0.5000 | -0.5000 | -0.5000 |
| -0.2500 | -0.2500 | -0.2500 | -0.2500 | -0.2500 | -0.2500 | -0.2500 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.2500 | 0.2500 | 0.2500 | 0.2500 | 0.2500 | 0.2500 | 0.2500 |
| 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 |
| 0.7500 | 0.7500 | 0.7500 | 0.7500 | 0.7500 | 0.7500 | 0.7500 |

```
>> Z = (1/(2*pi*sigma.^2)) * exp(-(X.^2 + Y.^2)/(2*sigma.^2));
>> surf(X,Y,Z)    % shown with sigma = 0.5
>>
>> % Probably want Z = Z / sum(Z(:));
```



Separable Kernels

- **Separable** filters can be written as $K(m,n) = f(m)g(n)$
- For a rectangular neighbourhood with size $(2M+1)\times(2N+1)$,

$$I'(m, n) = f \star (g \star I(N(m, n)))$$

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$$I'(m, n) = \sum_{i=-M}^{M} f(i) I''(m - i, n)$$

$$I''(m, n) = \sum_{j=-N}^{N} g(j) I(m, n - j)$$

Gaussian Smoothing Kernels

$$\begin{aligned} g(x, y) &= \frac{1}{2\pi\sigma^2} \exp\left(\frac{-(x^2 + y^2)}{2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-y^2}{2\sigma^2}\right) \\ &= g(x)g(y) \end{aligned}$$

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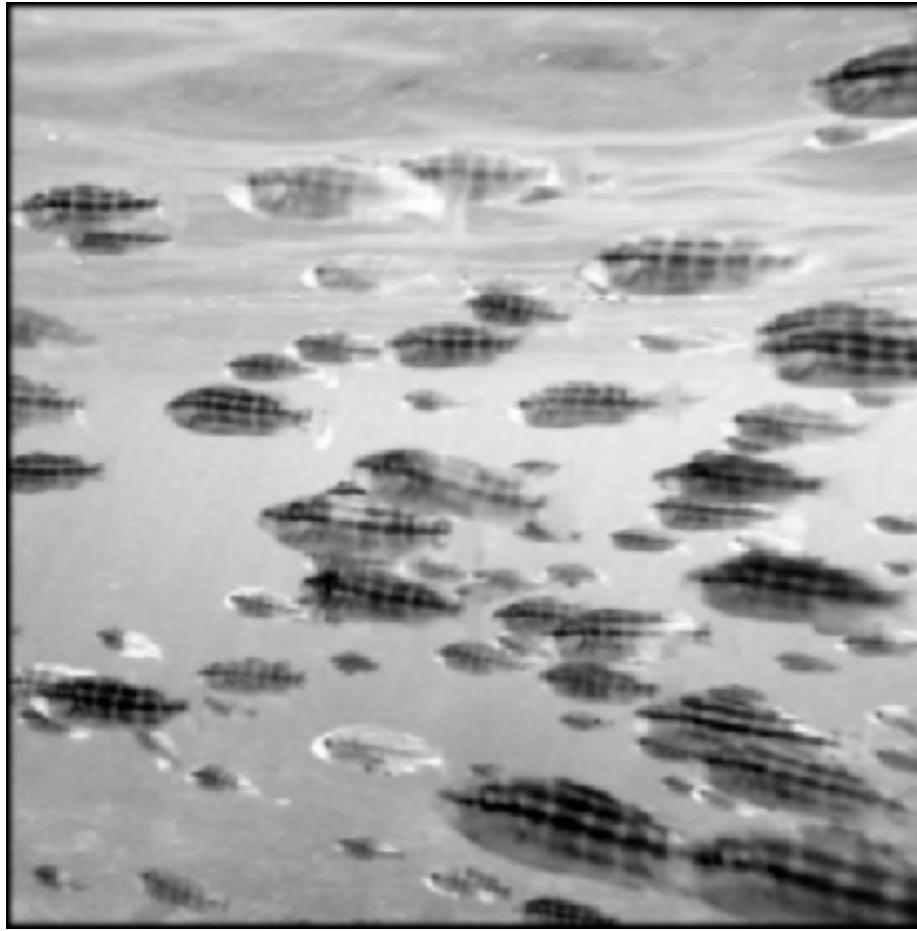
Separable!

To see how svd() can be used to obtain the separated kernels of a rank 1 kernel.

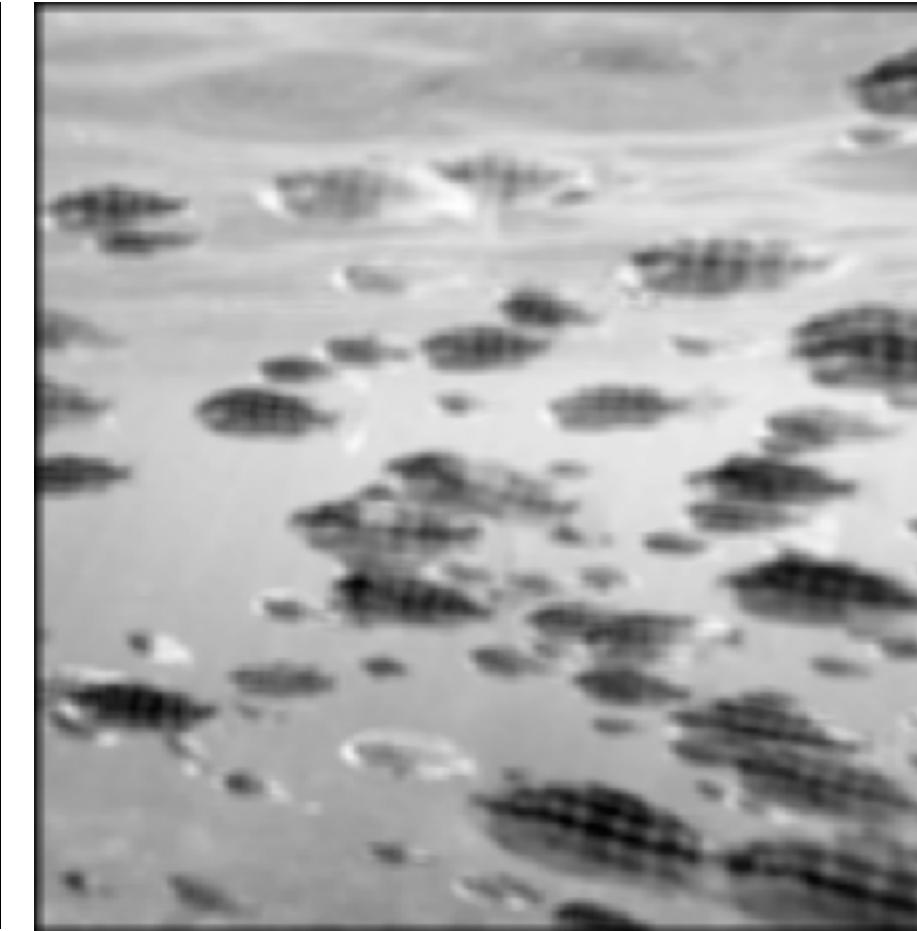
Gaussian Smoothing Kernels

- Amount of smoothing depends on σ and window size.
- Width $> 3\sigma$

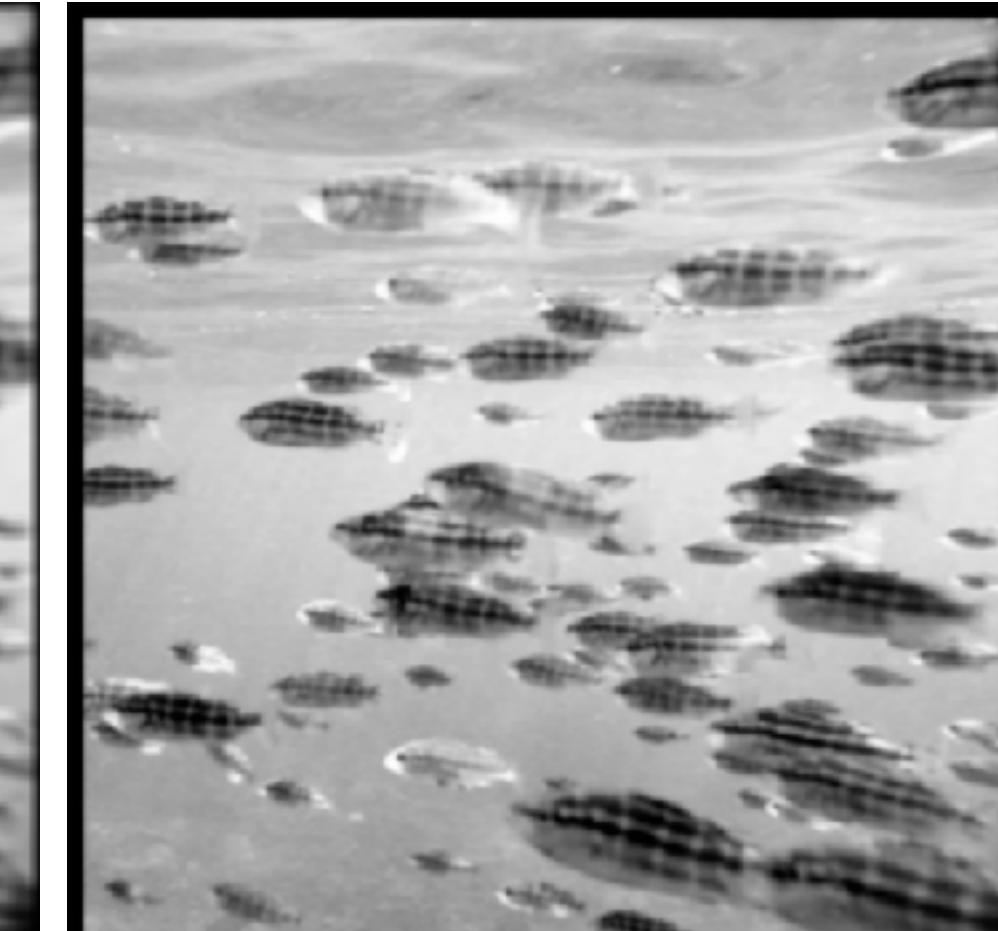
7x7; $\sigma = 1.$



7x7; $\sigma = 9.$



19x19; $\sigma = 1.$



19x19; $\sigma = 9.$



Scale Space

Scale Space

- Convolution of a Gaussian with standard deviation σ with itself, is a Gaussian with standard deviation $\sigma\sqrt{2}$.

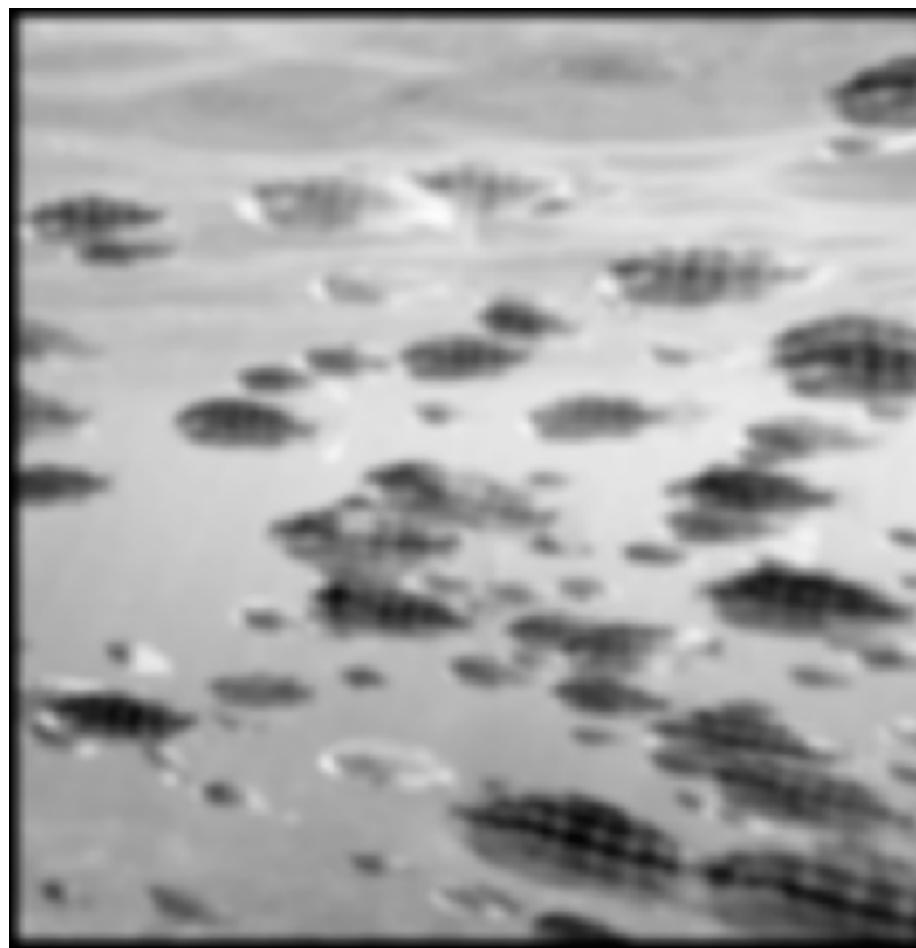
Scale Space

- Convolution of a Gaussian with standard deviation σ with itself, is a Gaussian with standard deviation $\sigma\sqrt{2}$.
- Repeated convolution by a Gaussian filter produces the ***scale space*** of an image.

Scale Space Example

11x11; $\sigma = 3.$

1



2



3



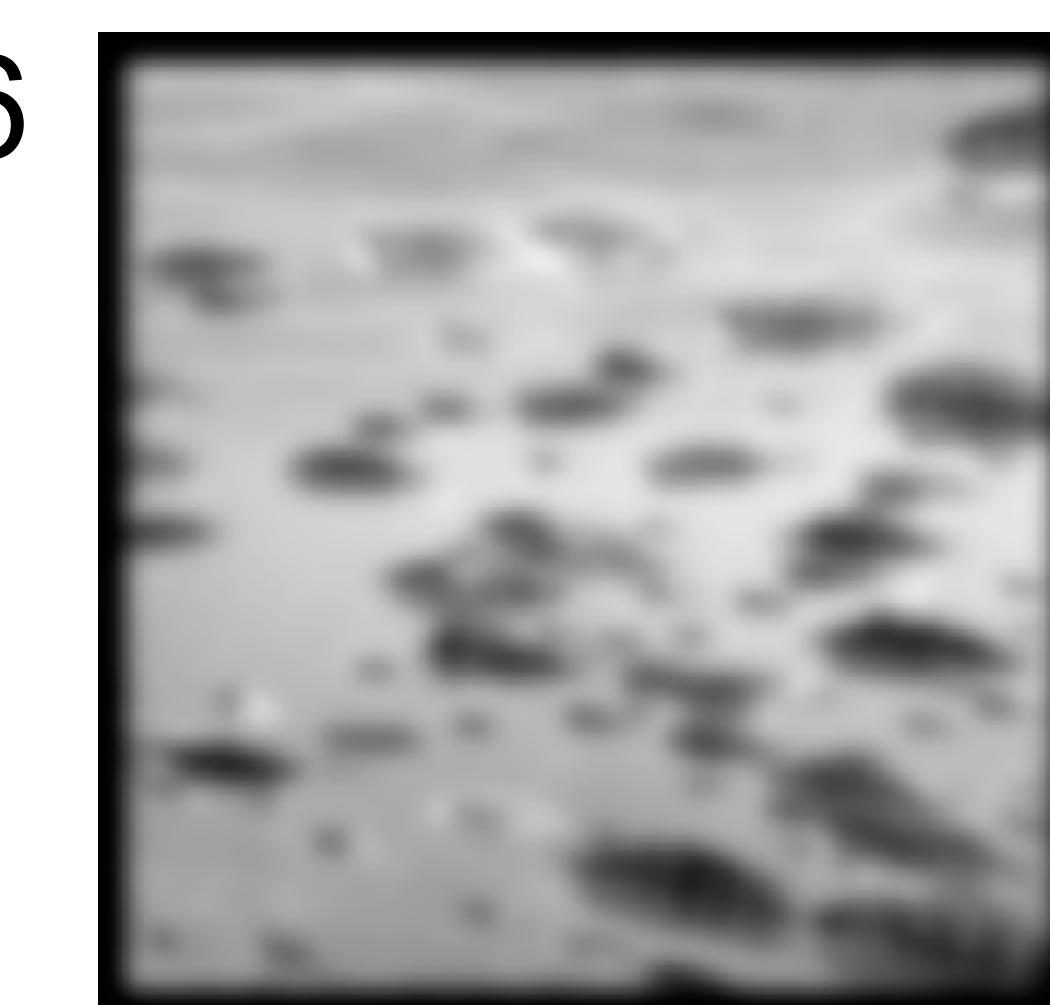
4



5



6



Gaussian Smoothing Kernel Top-5

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3. Still one lobe in frequency domain
 - No corruption from high frequencies

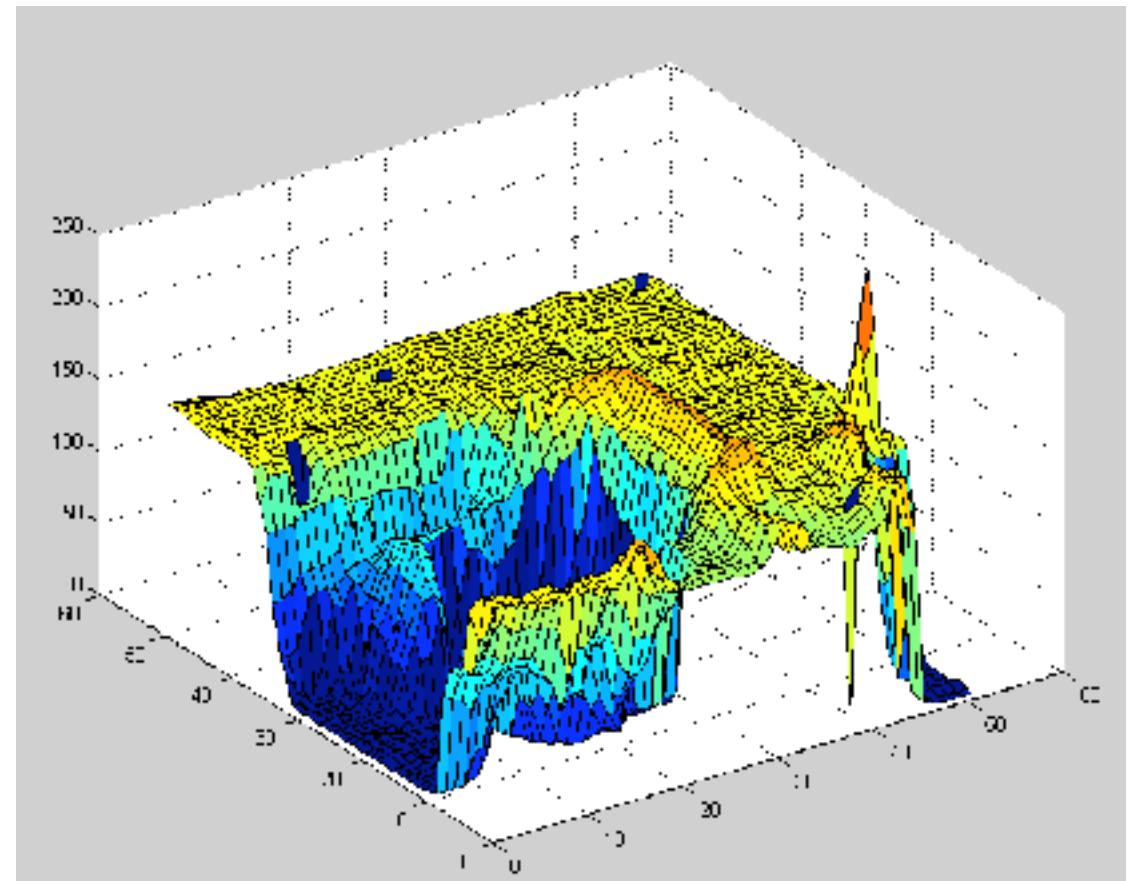
Gaussian Smoothing Kernel Top-5

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Gaussian Smoothing Kernel Top-5

1. Rotationally symmetric
2. Has a single lobe/mode
 - Neighbour's influence decreases monotonically
3. Still one lobe in frequency domain
 - No corruption from high frequencies
4. Simple relationship to σ
5. Easy to implement efficiently

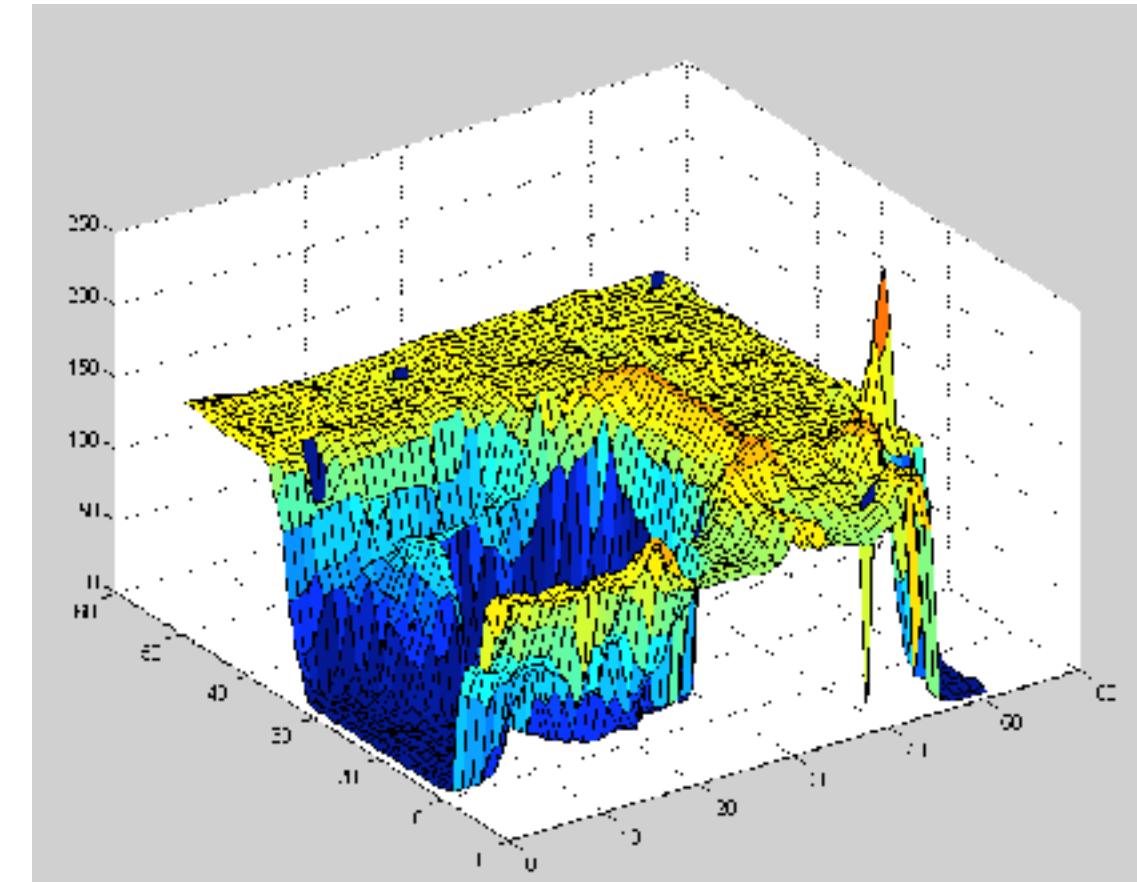
Smoothing vs. Sharpening



$z = I'$ s
intensity

Smoothing vs. Sharpening

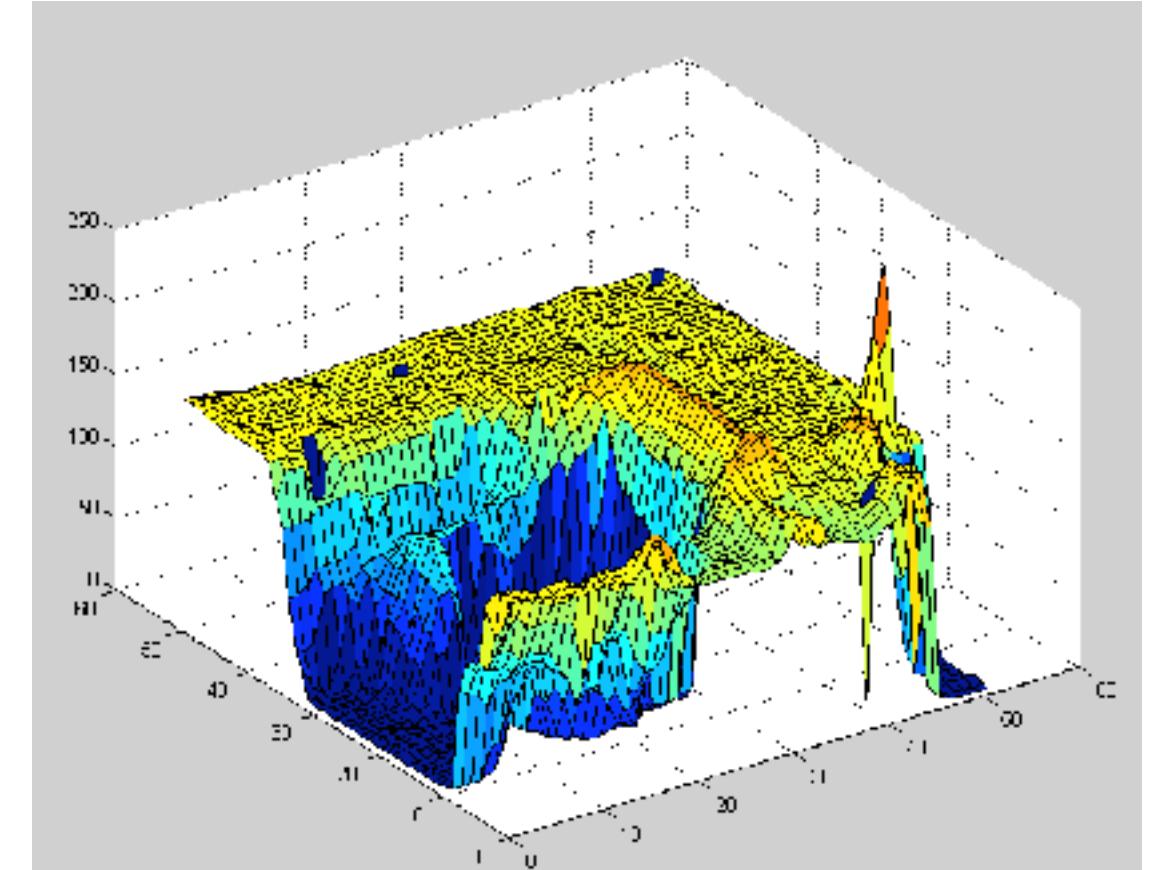
- Smoothing \leftrightarrow Integration, i.e.,
 - Summing
 - Sand-papering the height field's slope



$z = I$'s
intensity

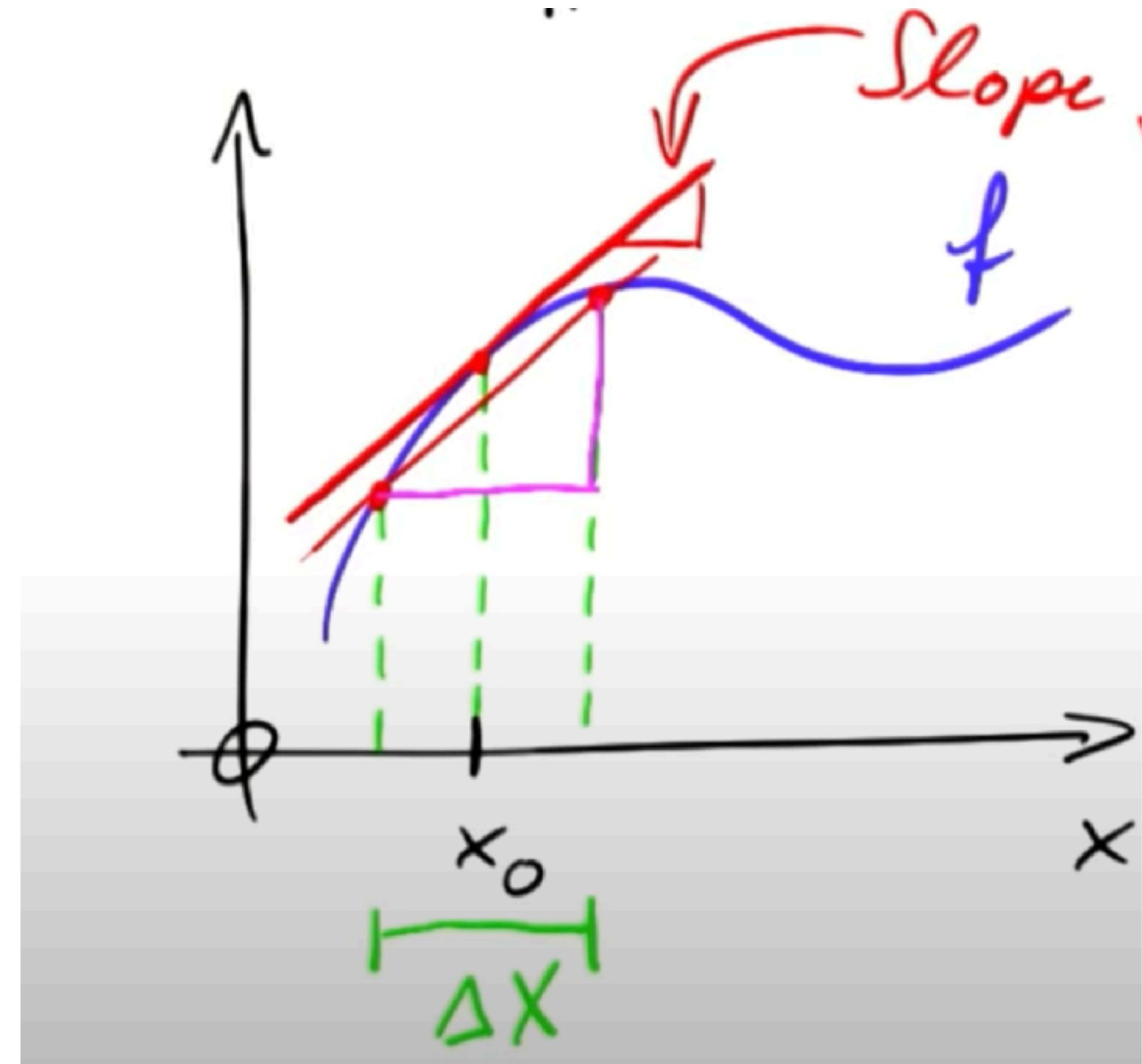
Smoothing vs. Sharpening

- Smoothing \leftrightarrow Integration, i.e.,
 - Summing
 - Sand-papering the height field's slope
- Sharpening \leftrightarrow Differentiation, i.e.,
 - Subtracting
 - Accentuating the height field's slope



$z = I'$'s
intensity

1st and 2nd Derivative



Differentiation and Convolution

Differentiation and Convolution

- Recall, for 2D function, $f(x,y)$:

$$\frac{\partial f}{\partial x} = \lim_{\epsilon \rightarrow 0} \left(\frac{f(x + \epsilon, y) - f(x, y)}{\epsilon} \right)$$

Differentiation and Convolution

- Recall, for 2D function, $f(x,y)$:

$$\frac{\partial f}{\partial x} = \lim_{\epsilon \rightarrow 0} \left(\frac{f(x + \epsilon, y) - f(x, y)}{\epsilon} \right)$$

- This is linear and shift invariant,
so the result of a convolution

| | |
|----|---|
| -1 | 1 |
|----|---|

Differentiation and Convolution

- Recall, for 2D function, $f(x,y)$:

$$\frac{\partial f}{\partial x} = \lim_{\epsilon \rightarrow 0} \left(\frac{f(x + \epsilon, y) - f(x, y)}{\epsilon} \right)$$

- This is linear and shift invariant,
so the result of a convolution
- We could approximate this as (which is a convolution)

| | |
|----|---|
| -1 | 1 |
|----|---|

$$\frac{\partial f}{\partial x} \approx \frac{f(x_{n+1}, y) - f(x_n, y)}{\Delta x}$$

1st and 2nd Derivative

The derivative of a function f at a point x is defined by the limit

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Approximation of the derivative when h is small

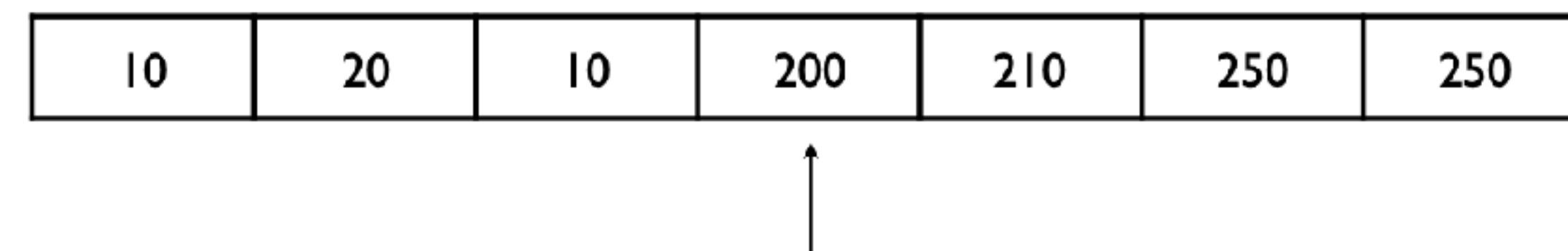
This definition is based on the ‘forward difference’ but ...

1st and 2nd Derivative

it turns out that using the ‘central difference’ is more accurate

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + 0.5h) - f(x - 0.5h)}{h}$$

How do we compute the derivative of a discrete signal?



1st and 2nd Derivative

it turns out that using the ‘central difference’ is more accurate

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + 0.5h) - f(x - 0.5h)}{h}$$

How do we compute the derivative of a discrete signal?

| | | | | | | |
|----|----|----|-----|-----|-----|-----|
| 10 | 20 | 10 | 200 | 210 | 250 | 250 |
|----|----|----|-----|-----|-----|-----|

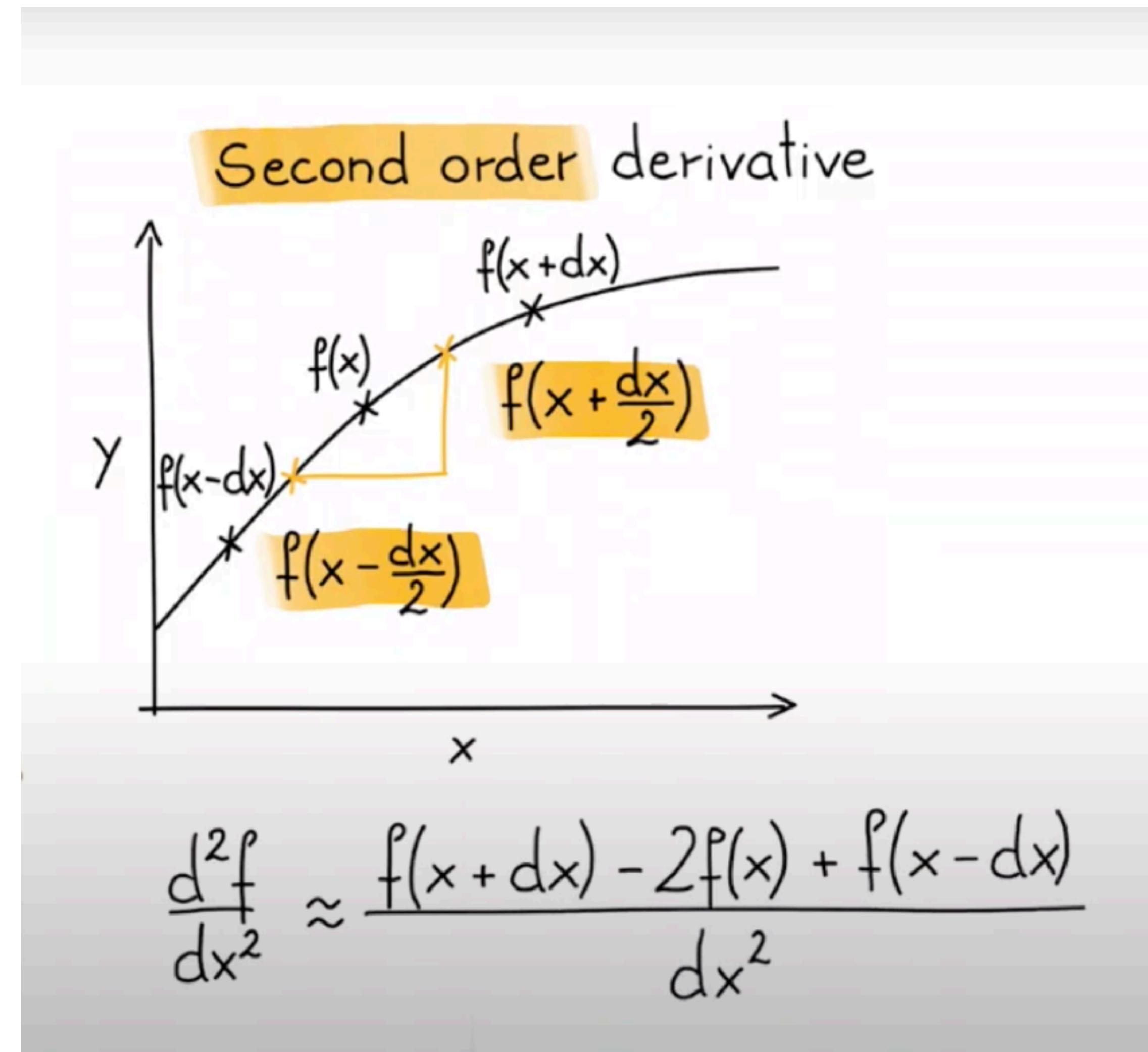


$$f'(x) = \frac{f(x+1) - f(x-1)}{2} = \frac{210 - 10}{2} = 100$$

| | | |
|----|---|---|
| -1 | 0 | 1 |
|----|---|---|

1D derivative filter

2nd Derivative



1st and 2nd Derivative

(Look ahead)

$$\frac{\partial f}{\partial x} = f(x + 1) - f(x)$$

(Look ahead and look back)

$$\frac{\partial^2 f}{\partial x^2} = f(x + 1) + f(x - 1) - 2f(x)$$

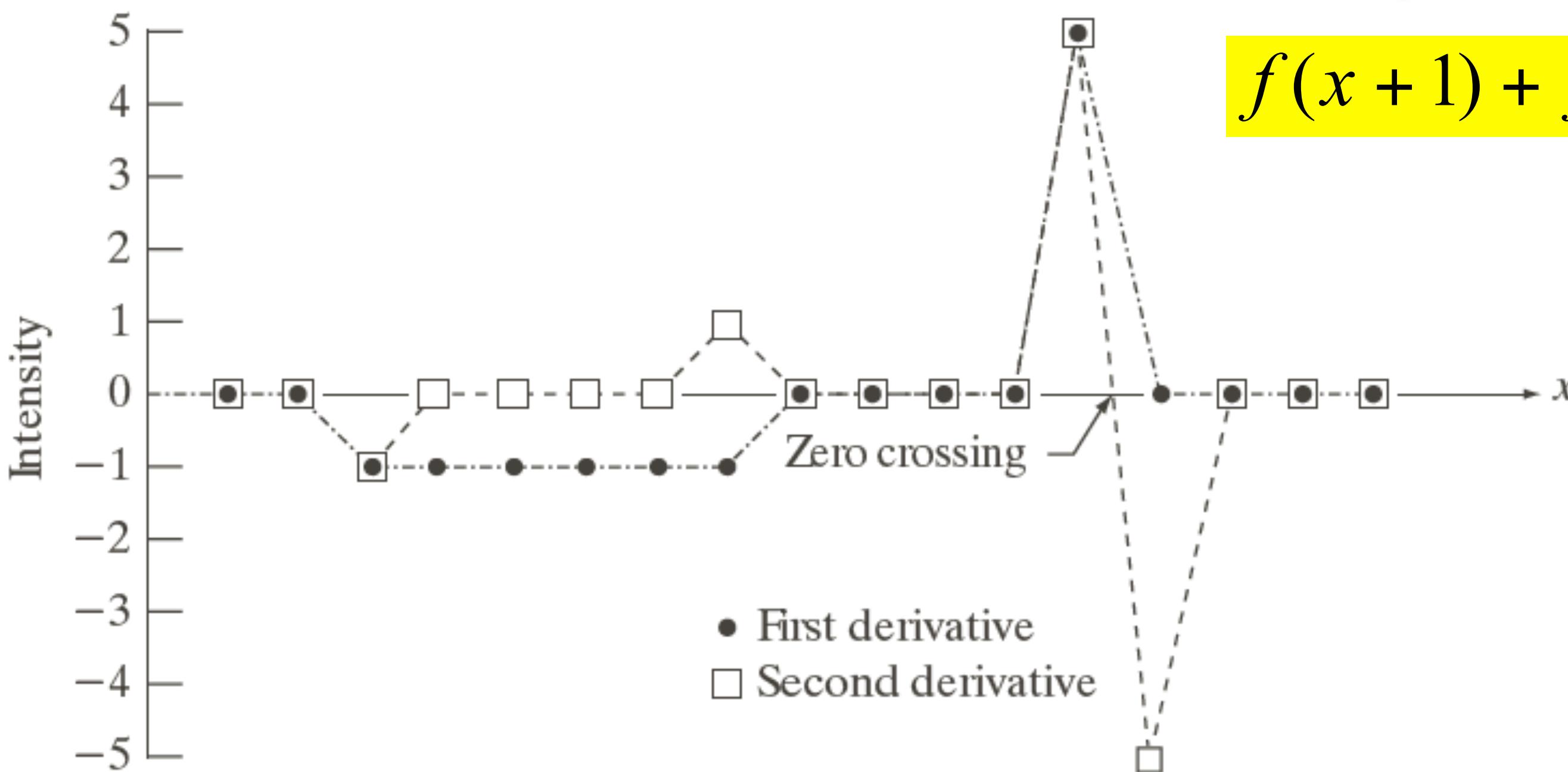
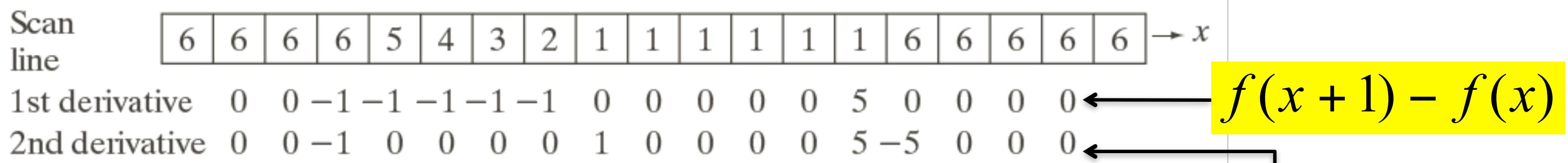
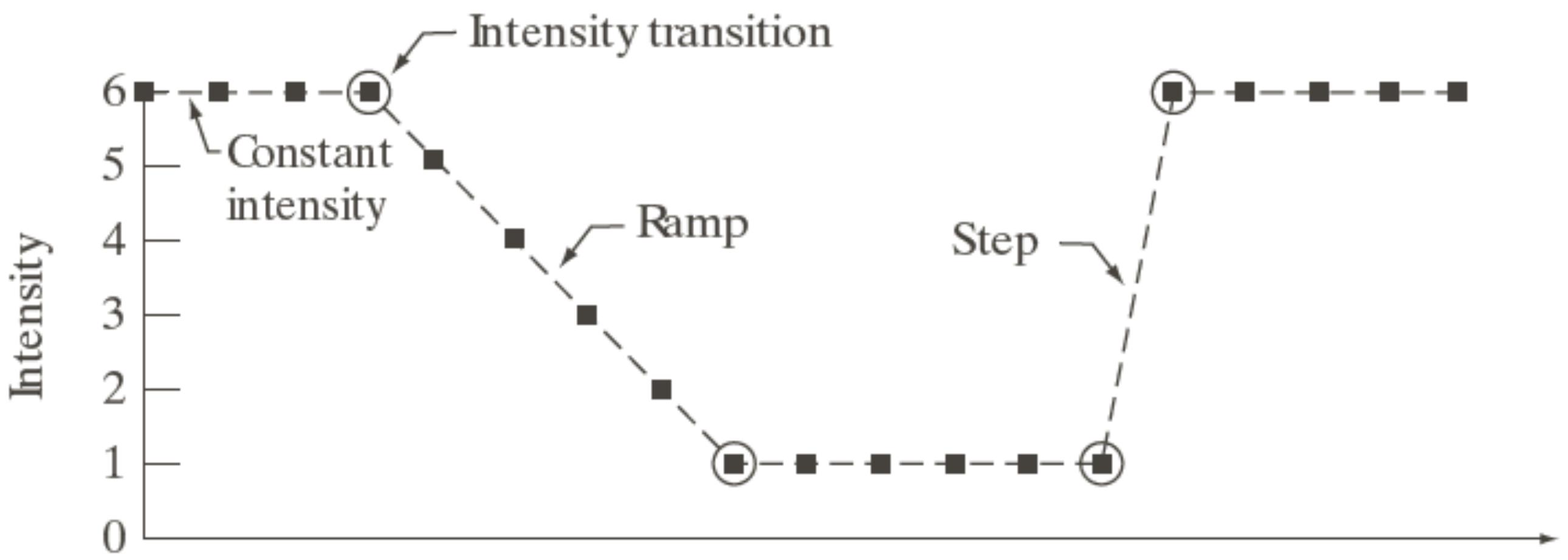


Figure from [DIP](#) by Gonzalez+Woods, 2008

Vertical Gradients from Finite Differences



Differential Filters

Prewitt operator:

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Sobel operator:

$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Differential Filters

Decomposing the Sobel filter

$$\begin{array}{|c|c|c|} \hline 1 & 0 & -1 \\ \hline 2 & 0 & -2 \\ \hline 1 & 0 & -1 \\ \hline \end{array} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 0 & -1 \\ \hline \end{array}$$

Sobel

1st Derivative: From 1D to 2D

$$\frac{\partial f}{\partial x} = f(x+1) - f(x)$$

in 1D

2D:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} g_x \\ g_y \end{bmatrix}$$

- Gradient is anisotropic
- $mag(\nabla f) = \sqrt{g_x^2 + g_y^2}$ is isotropic

- Orientation of gradient: $\alpha(x, y) = \tan^{-1} \left[\frac{g_y}{g_x} \right]$

Imagine g_x and g_y as vectors that form two sides of a right triangle.
The result of vector-addition points in the direction of the gradient!

2nd Derivative: From 1D to 2D

$$\frac{\partial^2 f}{\partial x^2} = f(x+1) + f(x-1) - 2f(x) \quad \text{in 1D}$$

2D: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

$$\left. \begin{array}{l} \frac{\partial^2 f}{\partial x^2} = f(x+1, y) + f(x-1, y) - 2f(x, y) \\ \frac{\partial^2 f}{\partial y^2} = f(x, y+1) + f(x, y-1) - 2f(x, y) \end{array} \right\}$$

| | | |
|---|----|---|
| 0 | 1 | 0 |
| 1 | -4 | 1 |
| 0 | 1 | 0 |

| | | |
|----|----|----|
| -1 | -1 | -1 |
| -1 | 8 | -1 |
| -1 | -1 | -1 |

Incorporating diagonals,
isotropic in increments of 45°

Laplacian: use this (negative) version
of kernel for convenience

Image Sharpening

Image Sharpening

- Also known as *enhancement*

Image Sharpening

- Also known as *enhancement*
- Increases the high frequency components to enhance edges

Image Sharpening

- Also known as *enhancement*
- Increases the high frequency components to enhance edges
- $I' = I + \alpha(k^*I)$, where k is a high-pass filter kernel and α is a scalar in $[0,1]$

Image Sharpening: Example 1



$$I' = I + \alpha \nabla^2 I = I + \alpha(K * I)$$

| | | |
|----|----|----|
| -1 | -1 | -1 |
| -1 | 8 | -1 |
| -1 | -1 | -1 |

Figure from NASA, obtained on [DIP](#)

Image Sharpening: Example 1

$$I' = I + \alpha \nabla^2 I = I + \alpha \underline{(K * I)}$$

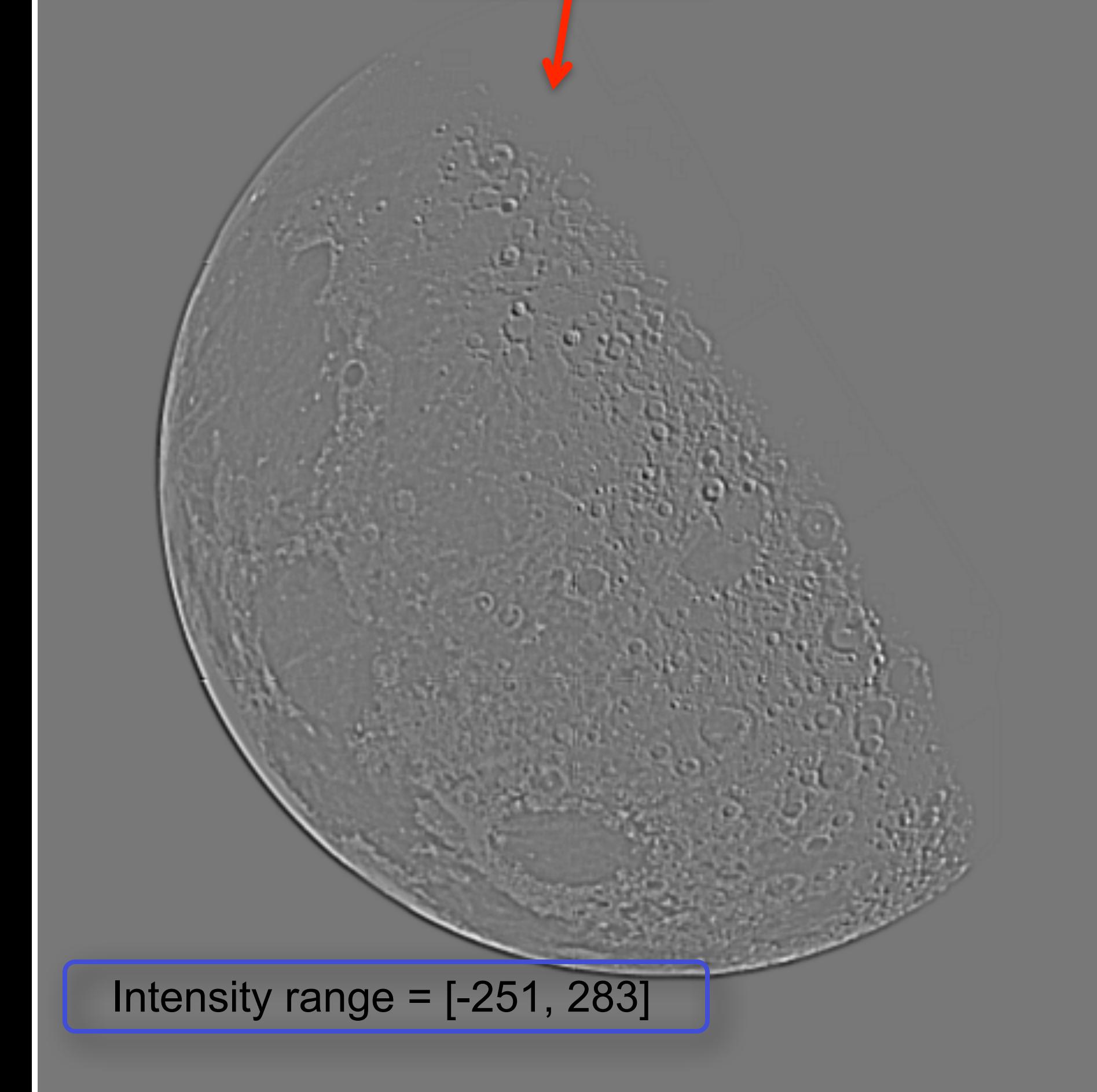
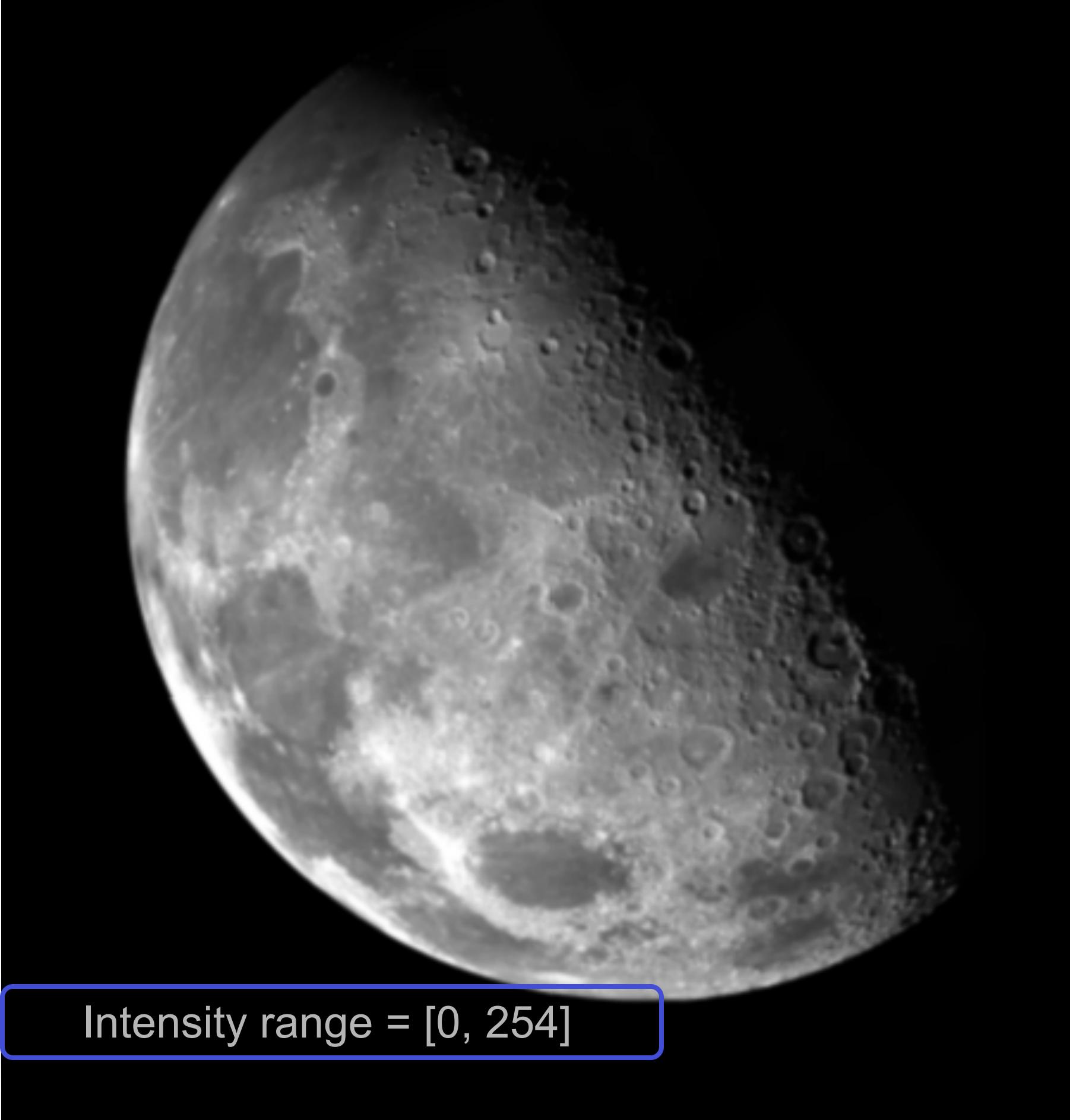


Image Sharpening: Example 1

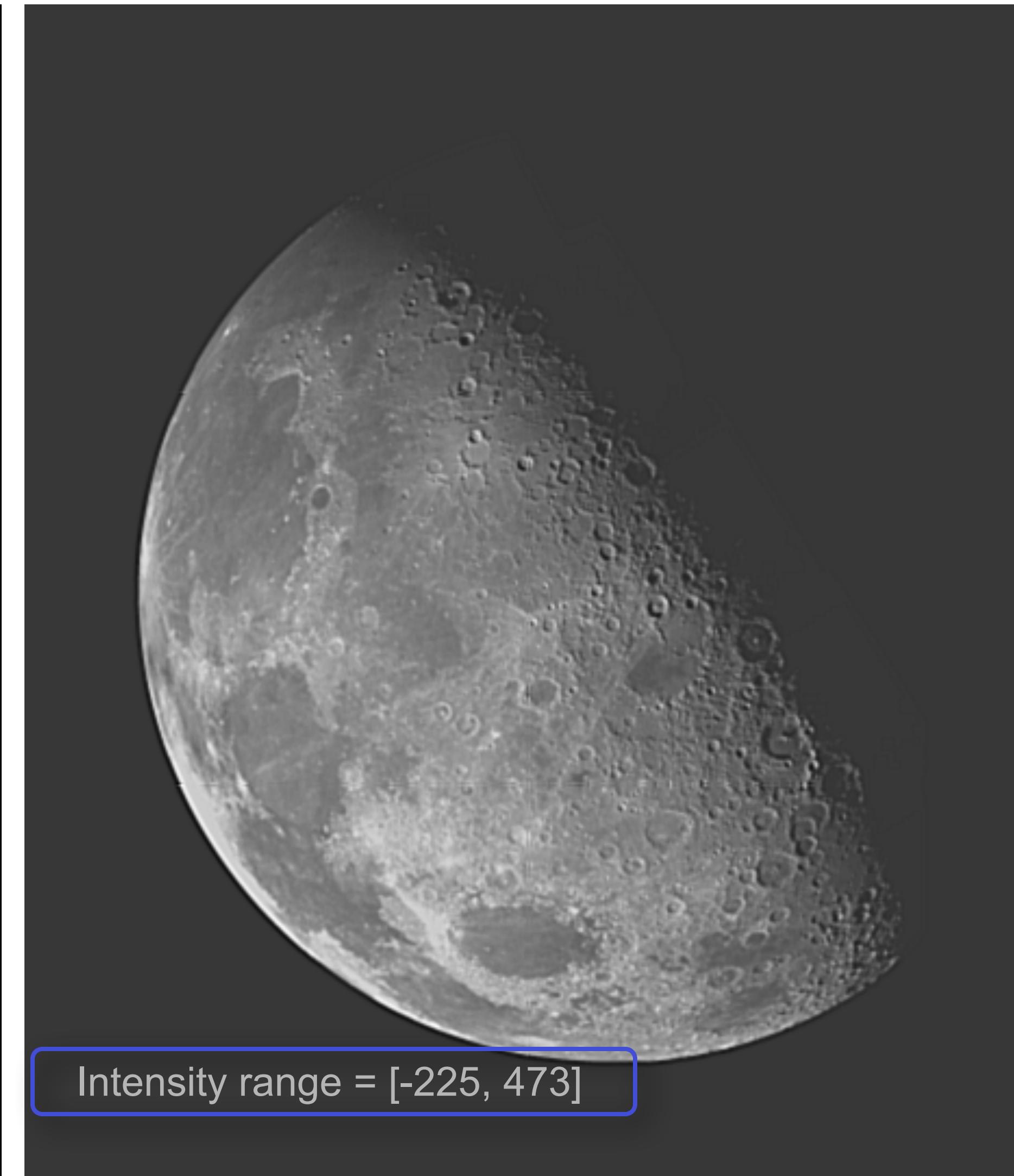


Image Sharpening: Example 1

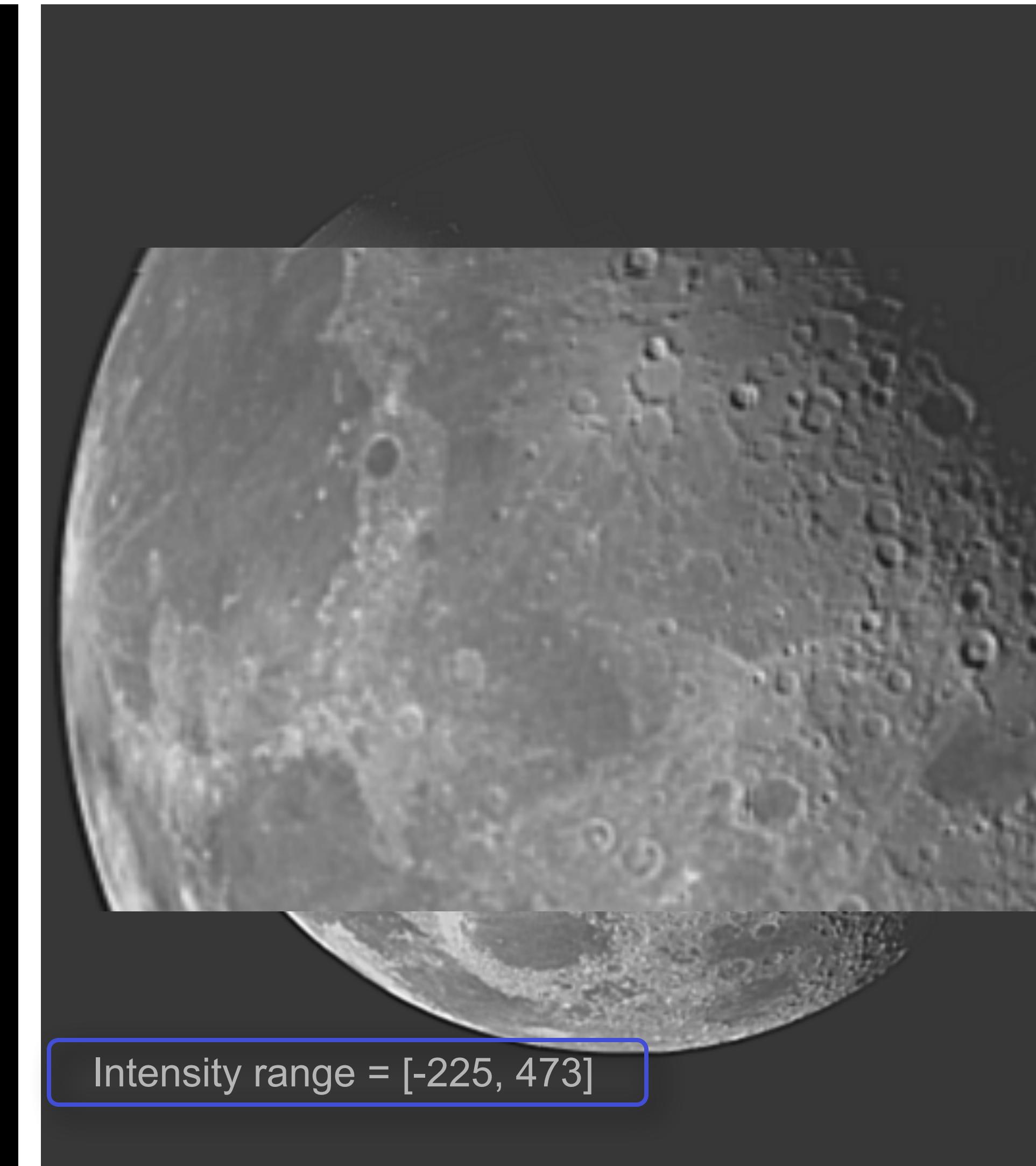


Image Sharpening: Example 2

$$I' = I + \alpha(I - K * I) \Leftarrow \text{"Unsharp Mask"}$$

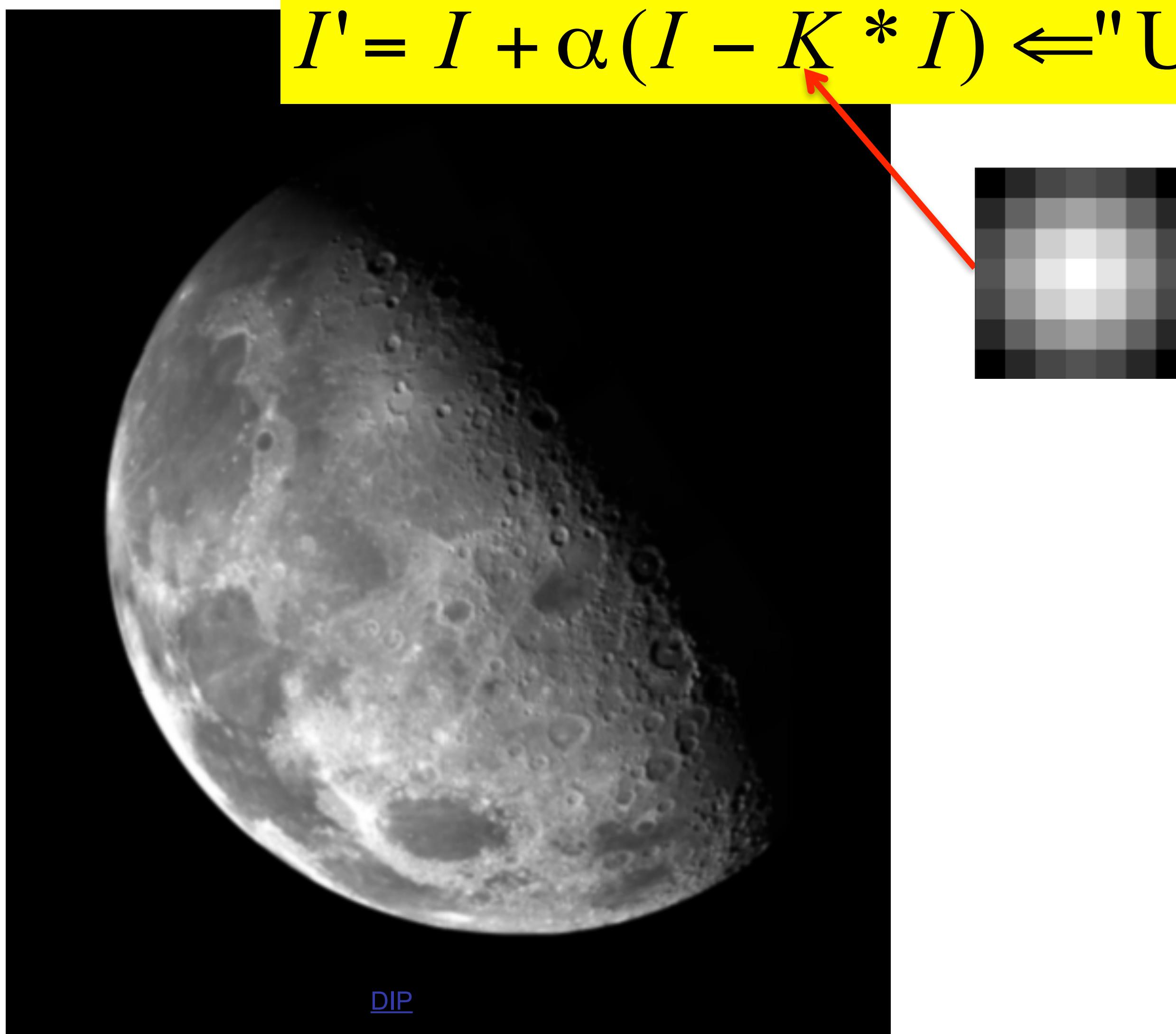
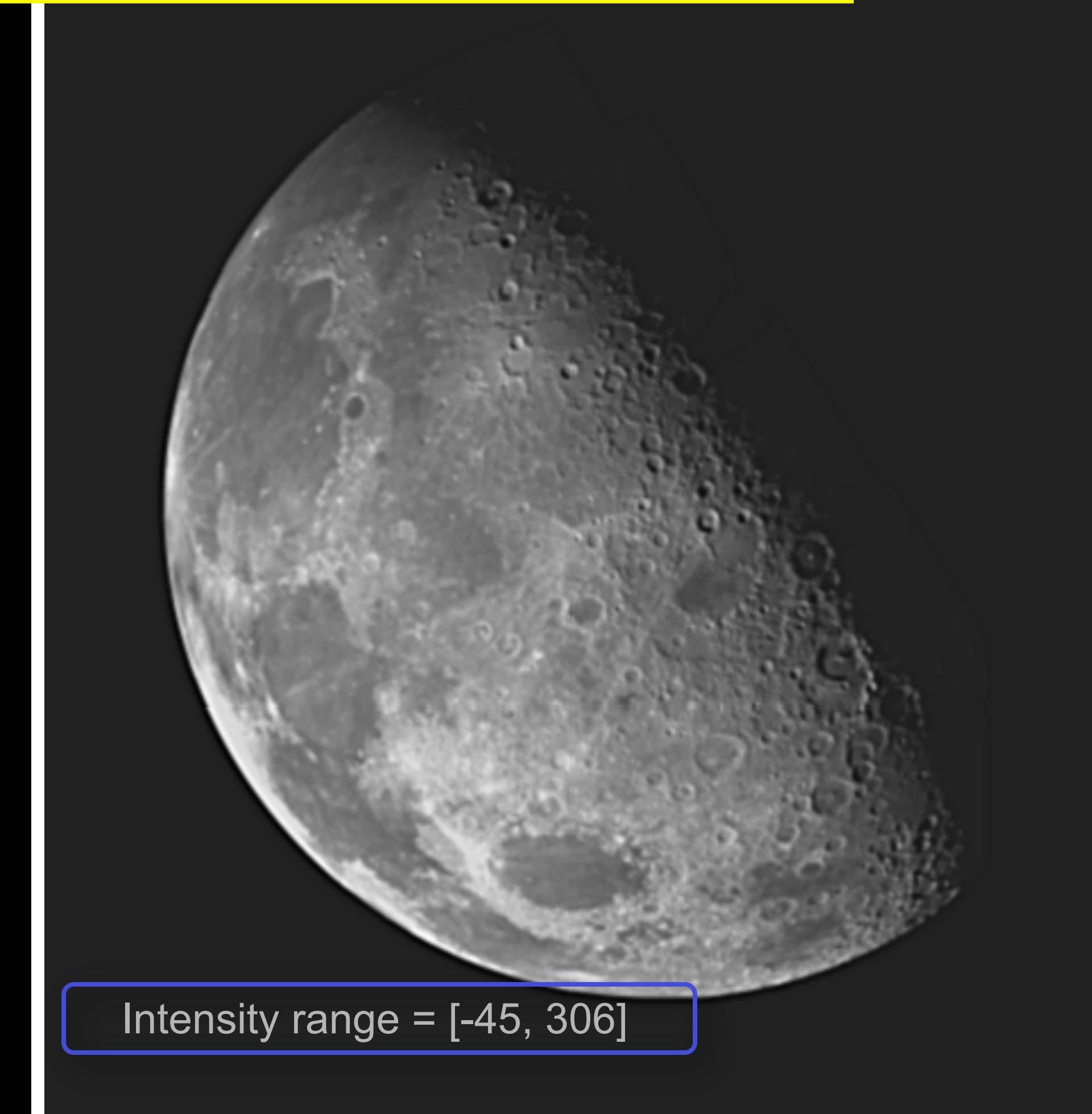


Image Sharpening: Example 2

$$I' = 2I - K * I \Leftarrow \text{"Unsharp Mask"}$$

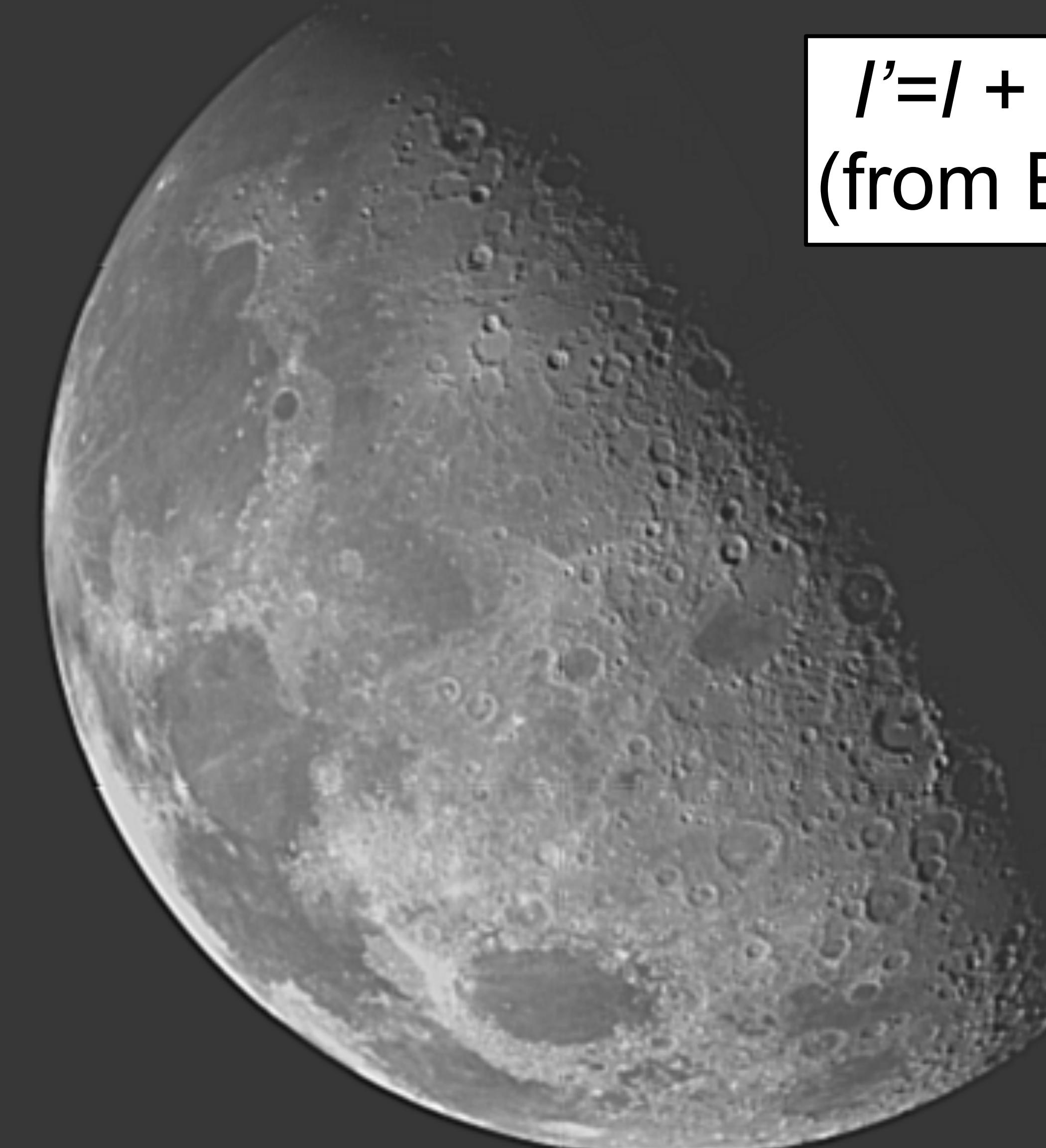






I' = Unsharp Mask
(from Example 2)

$I' = I + \text{Laplacian}$
(from Example 1)



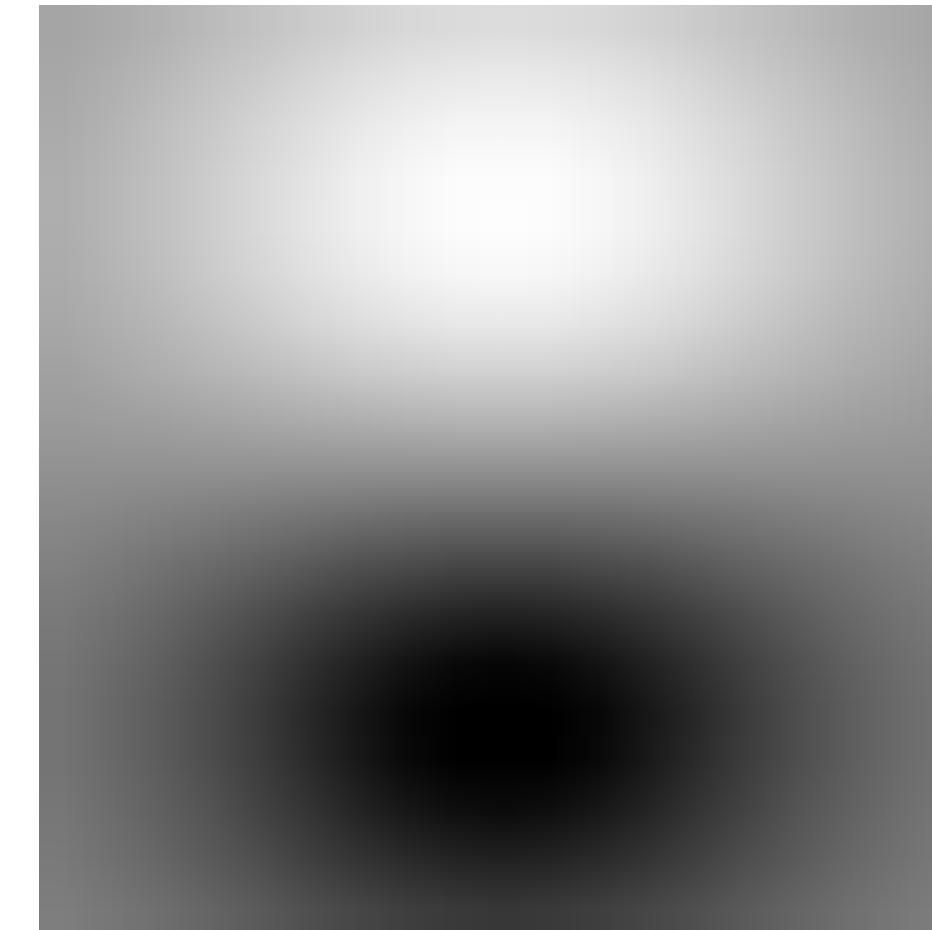
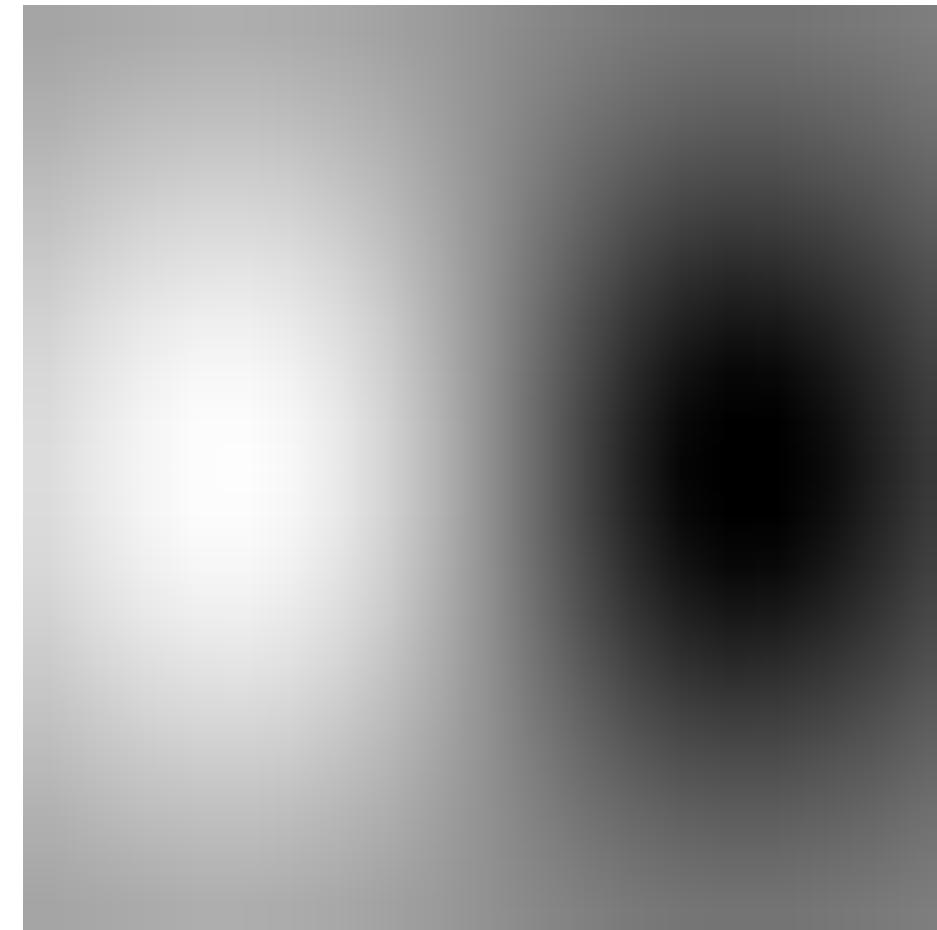
Filters are Templates

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- Filter at some point can be seen as taking a dot-product between the image and some vector

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- Filter at some point can be seen as taking a dot-product between the image and some vector
- Filtering the image is a set of dot products
 - filters look like the effects they are intended to find
 - filters find effects they look like



Scaled Representations

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- Find *correlations/convolutions* at all scales
 - e.g., when finding hands or faces, we don't know what size they will be in a particular image
 - Template size is constant, but image size changes

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 - look at coarse scales, then refine with finer scales
 - much less cost, but may miss best match
- Examining all *levels of detail*
 - Find edges with different amounts of blur
 - Find textures with different spatial frequencies (*levels of detail*)

Fourier Domain

Fourier Domain

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- Convolutions are simpler in the Fourier Domain

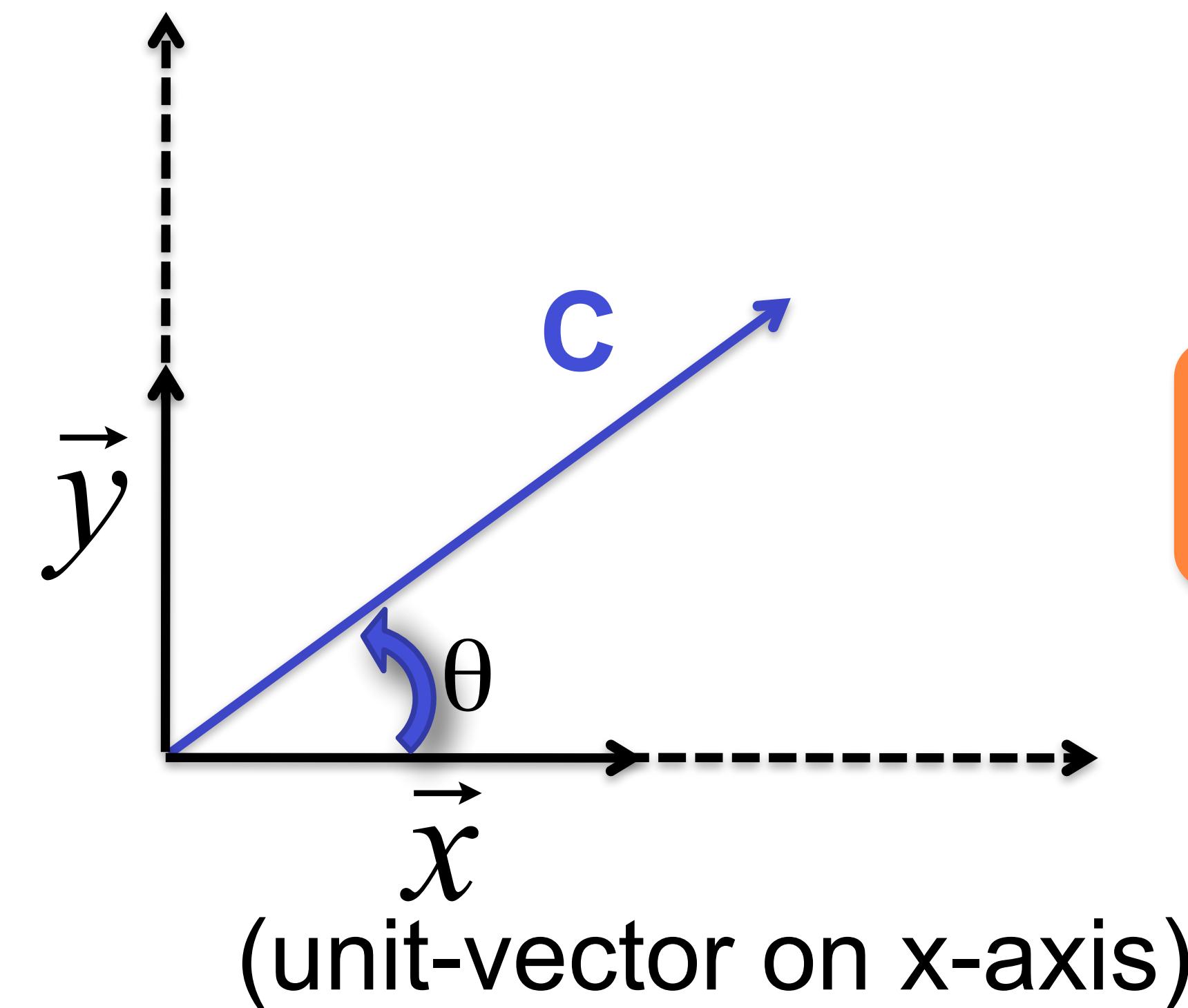
Jean Baptiste Joseph Fourier (1768-1830)

Fourier Series:

Any periodic function
can be rewritten as a
weighted sum of sines
and cosines of different
frequencies.

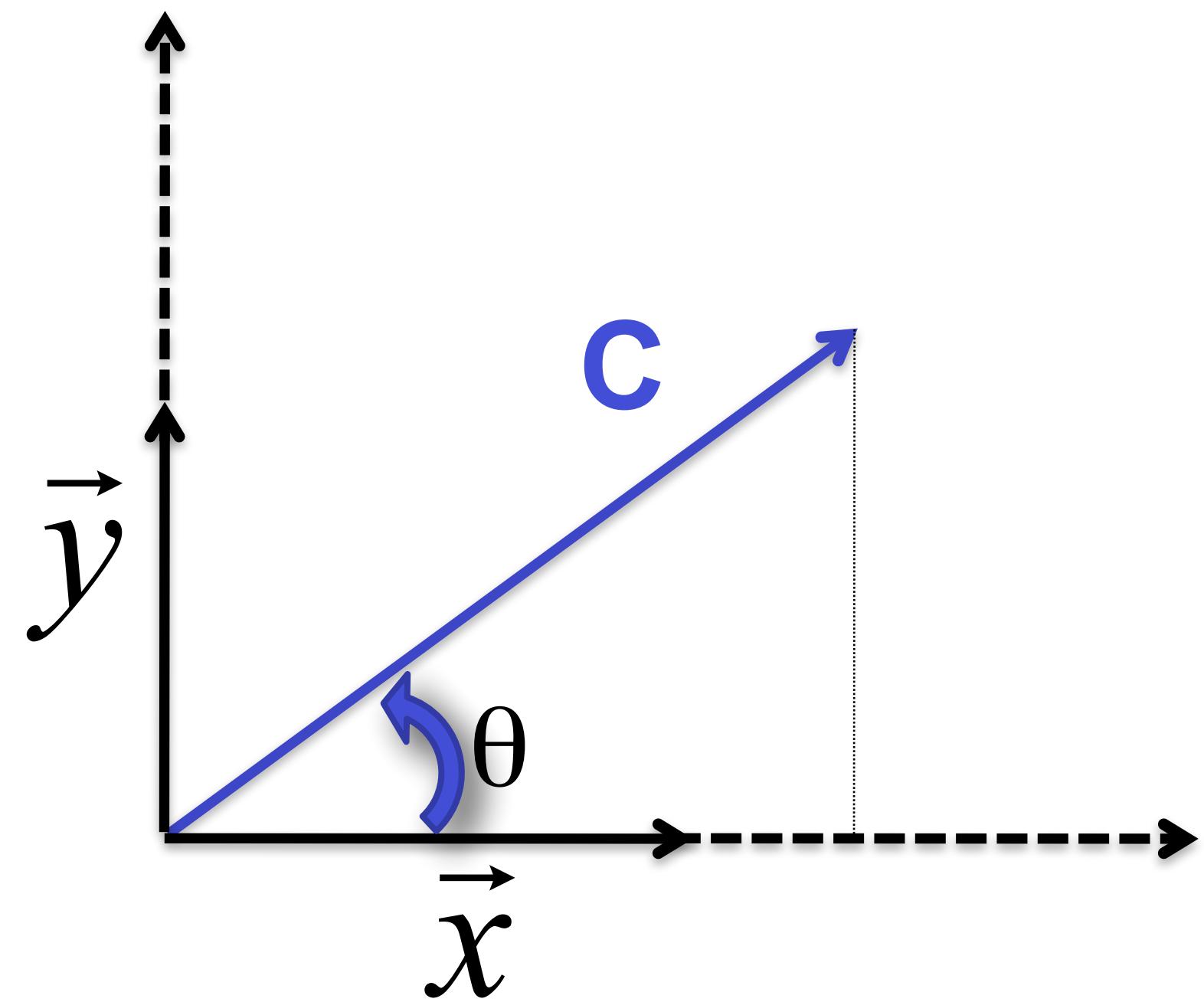


Geometry Review

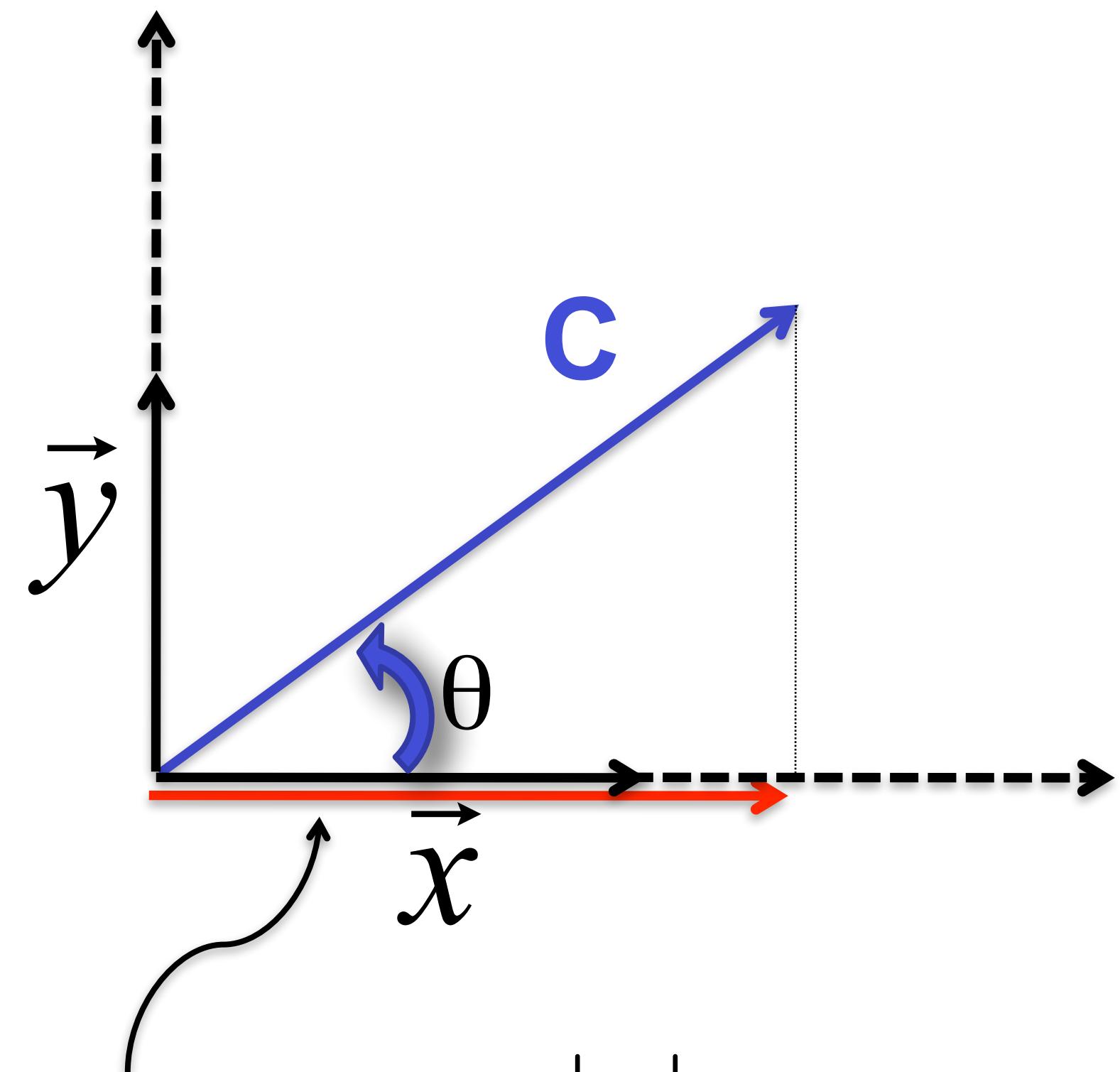


Matlab:
 $\theta = \text{atan2}(y, x)$

Geometry Review

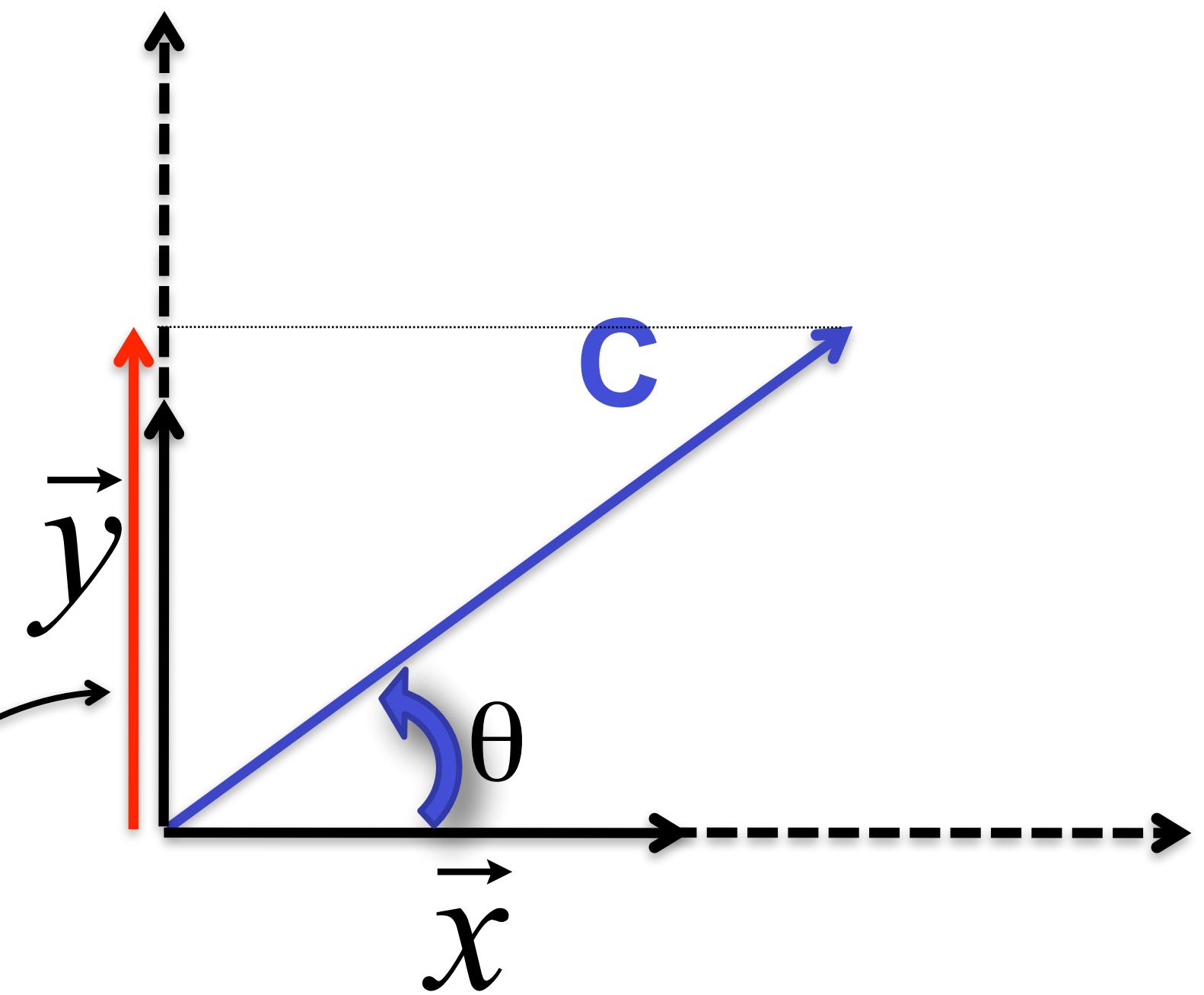


Geometry Review



$$x - \text{component} = |C| \vec{x} \cos \theta$$

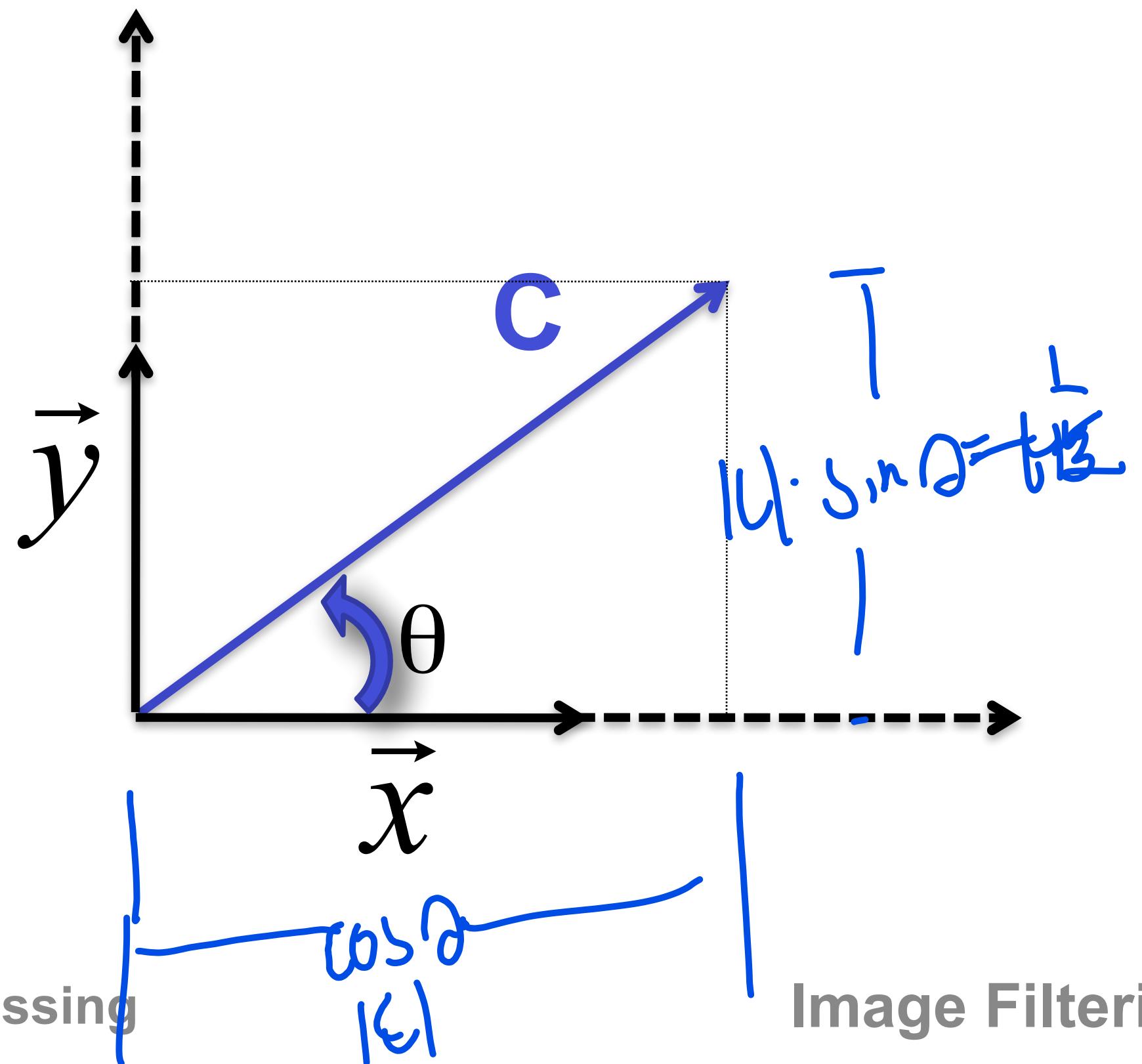
Geometry Review



$$y - \text{component} = |C| \vec{y} \sin \theta$$

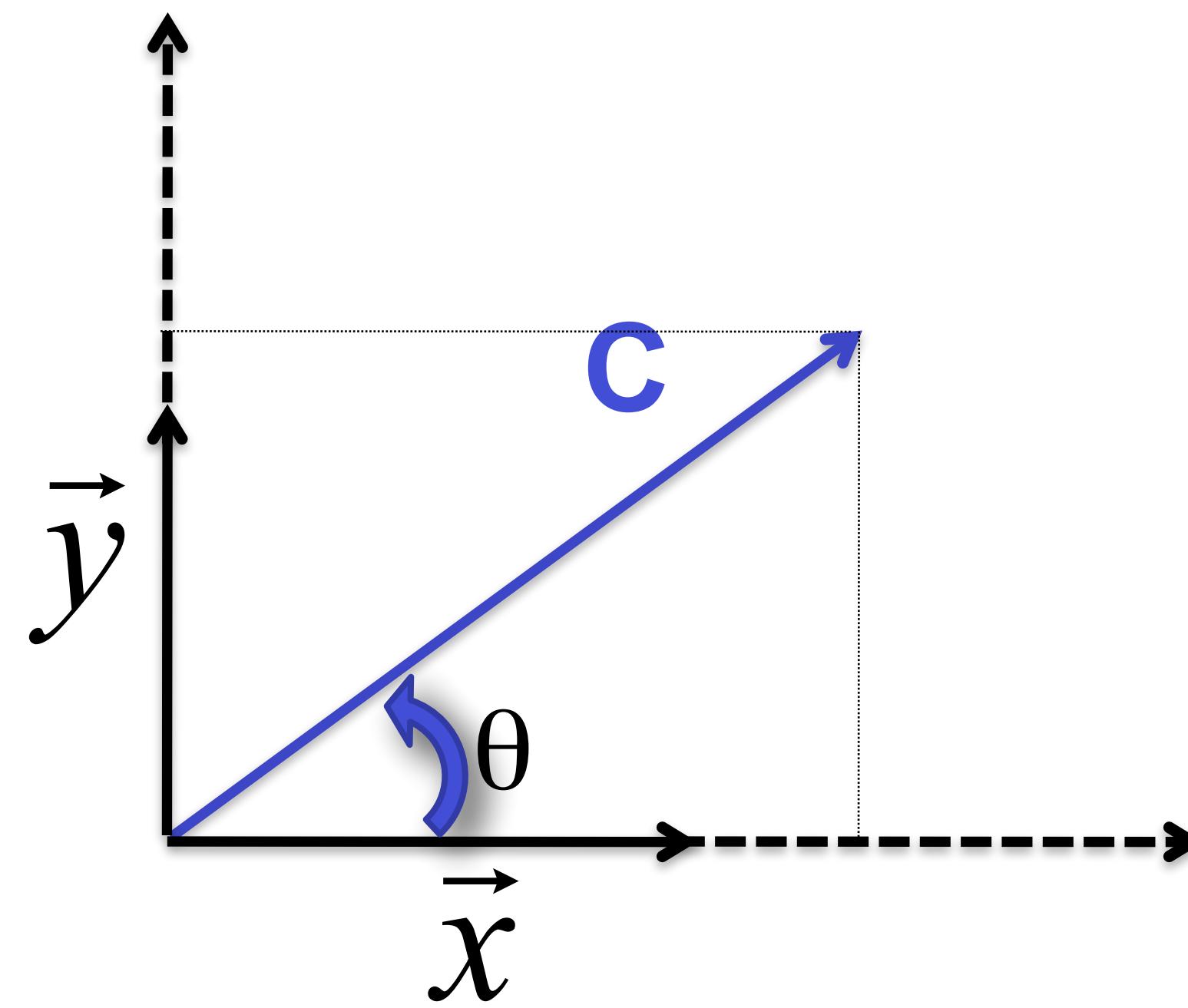
Geometry Review

$$C = |C|(\vec{x} \cos \theta + \vec{y} \sin \theta)$$



Geometry Review

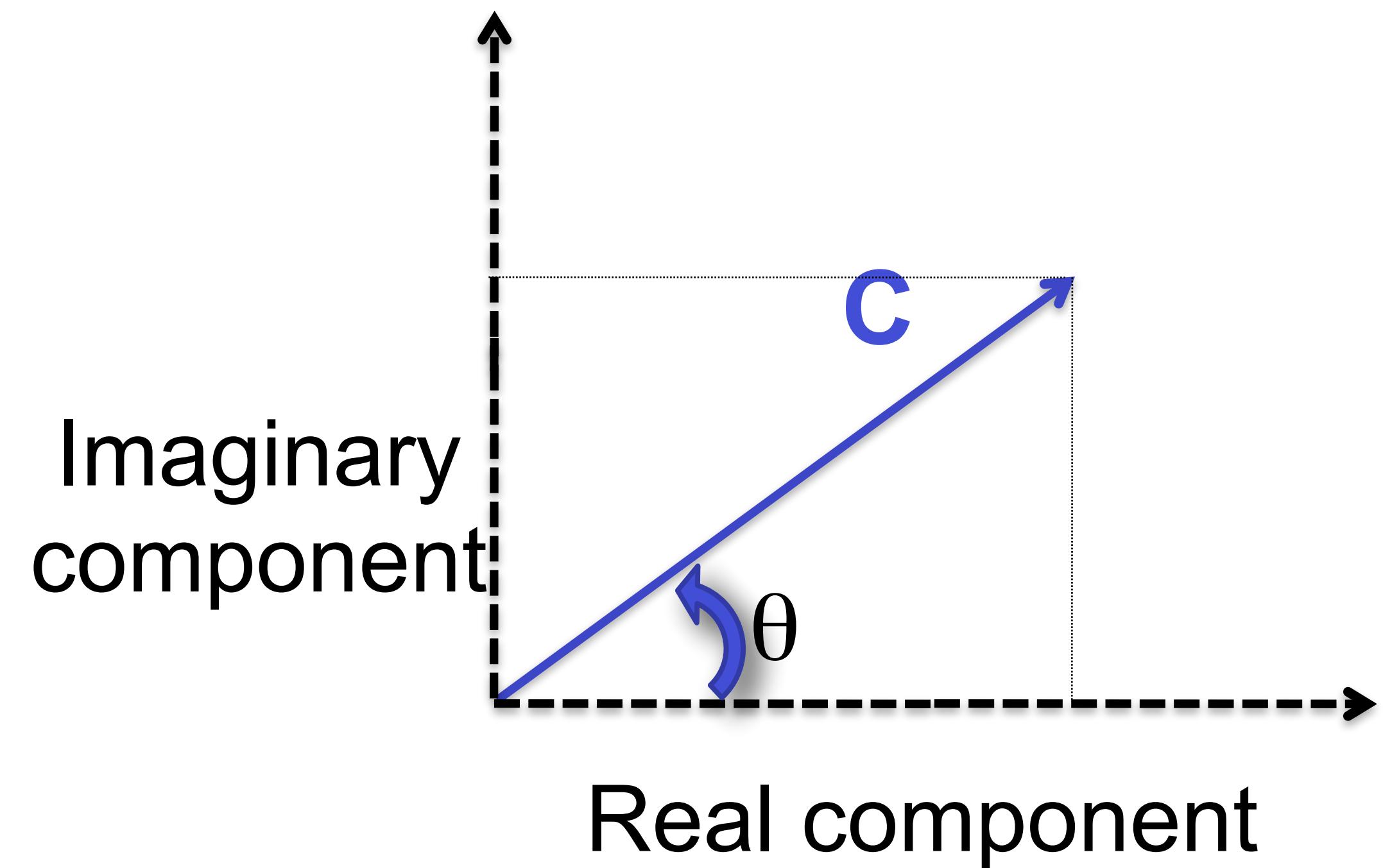
$$C = |C|(\vec{x} \cos \theta + \vec{y} \sin \theta)$$



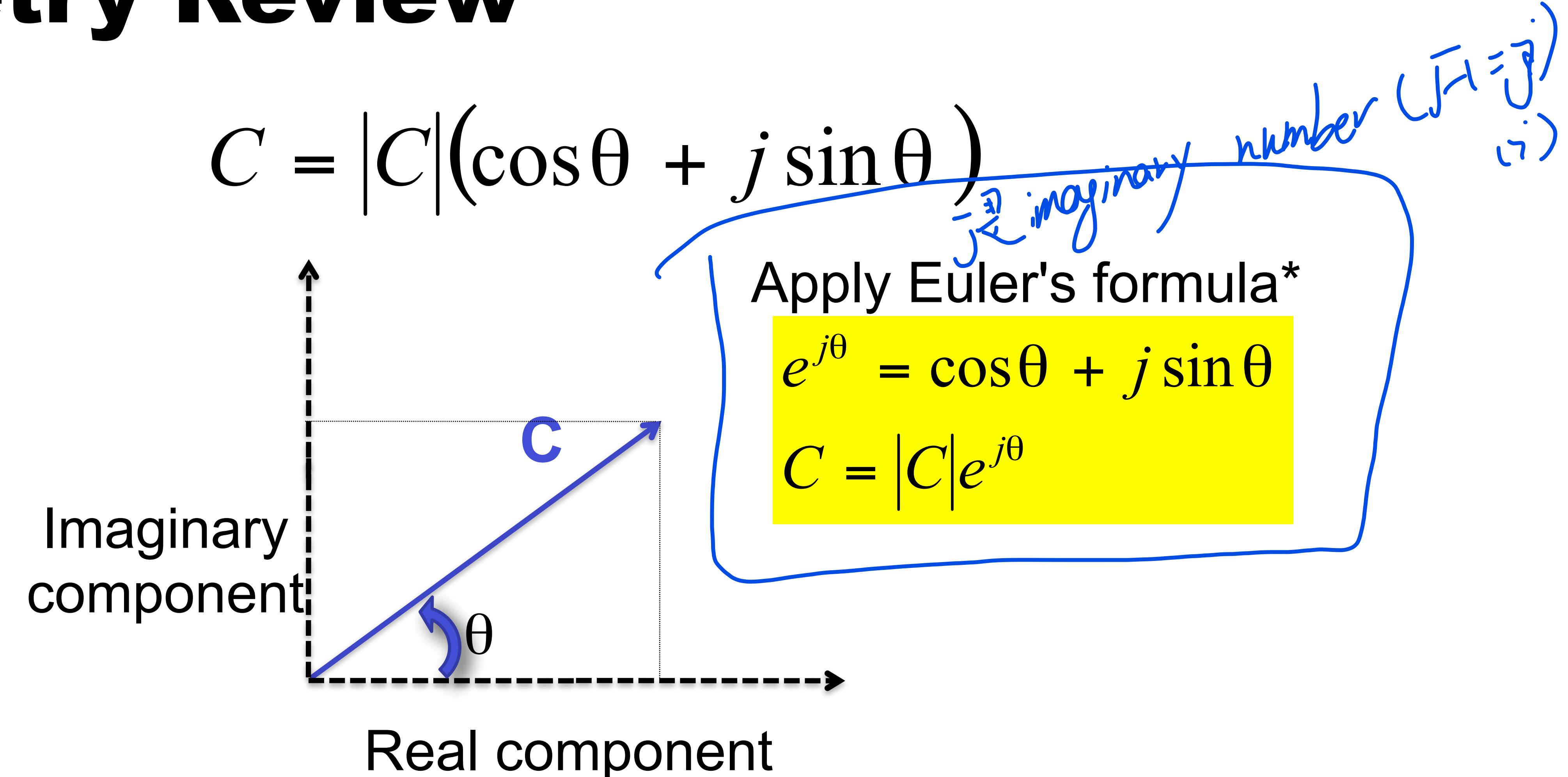
Now, use these
polar coordinates to
represent imaginary
numbers!

Geometry Review

$$C = |C|(\cos \theta + j \sin \theta)$$

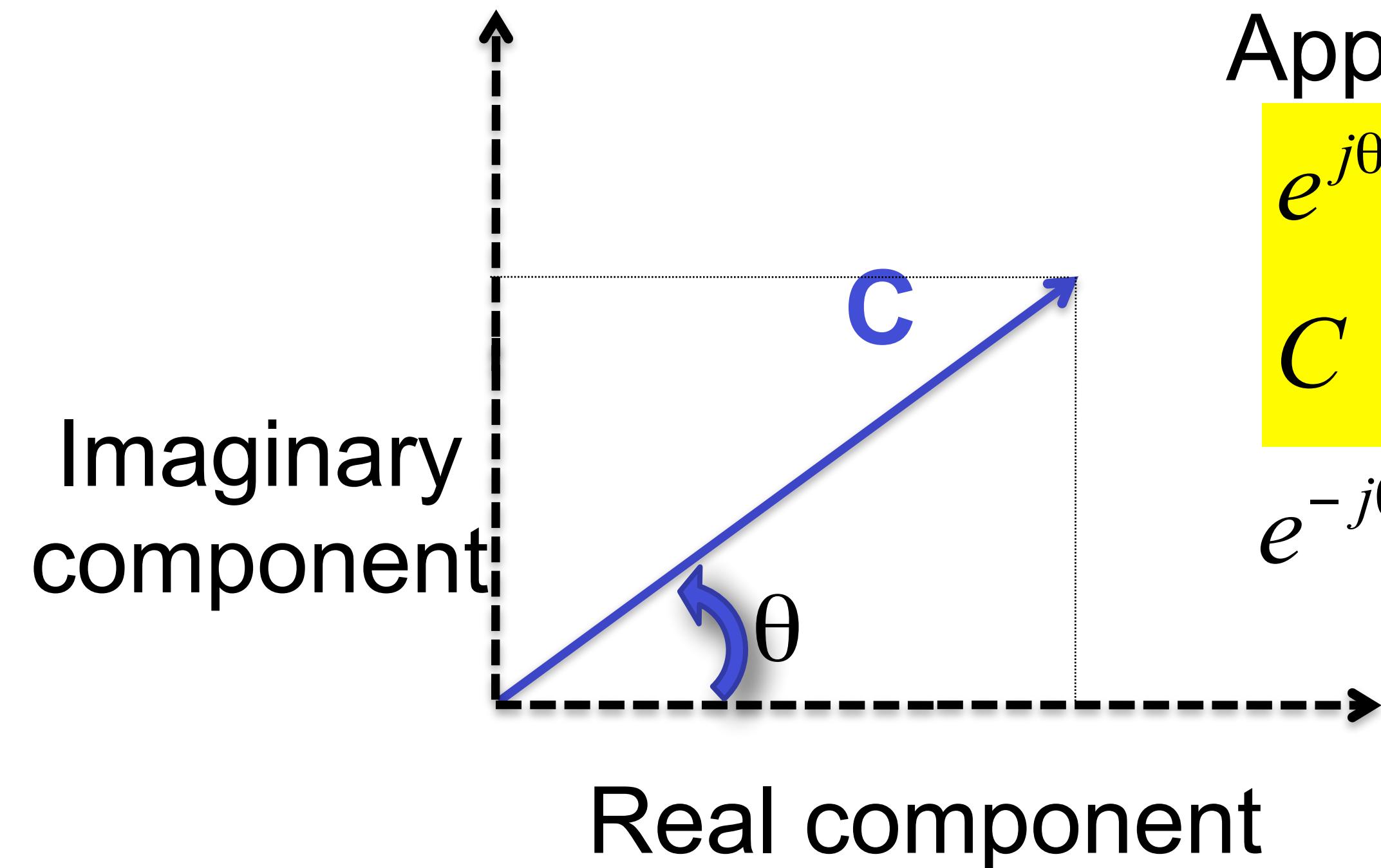


Geometry Review



Geometry Review

$$C = |C|(\cos \theta + j \sin \theta)$$



Apply Euler's formula*

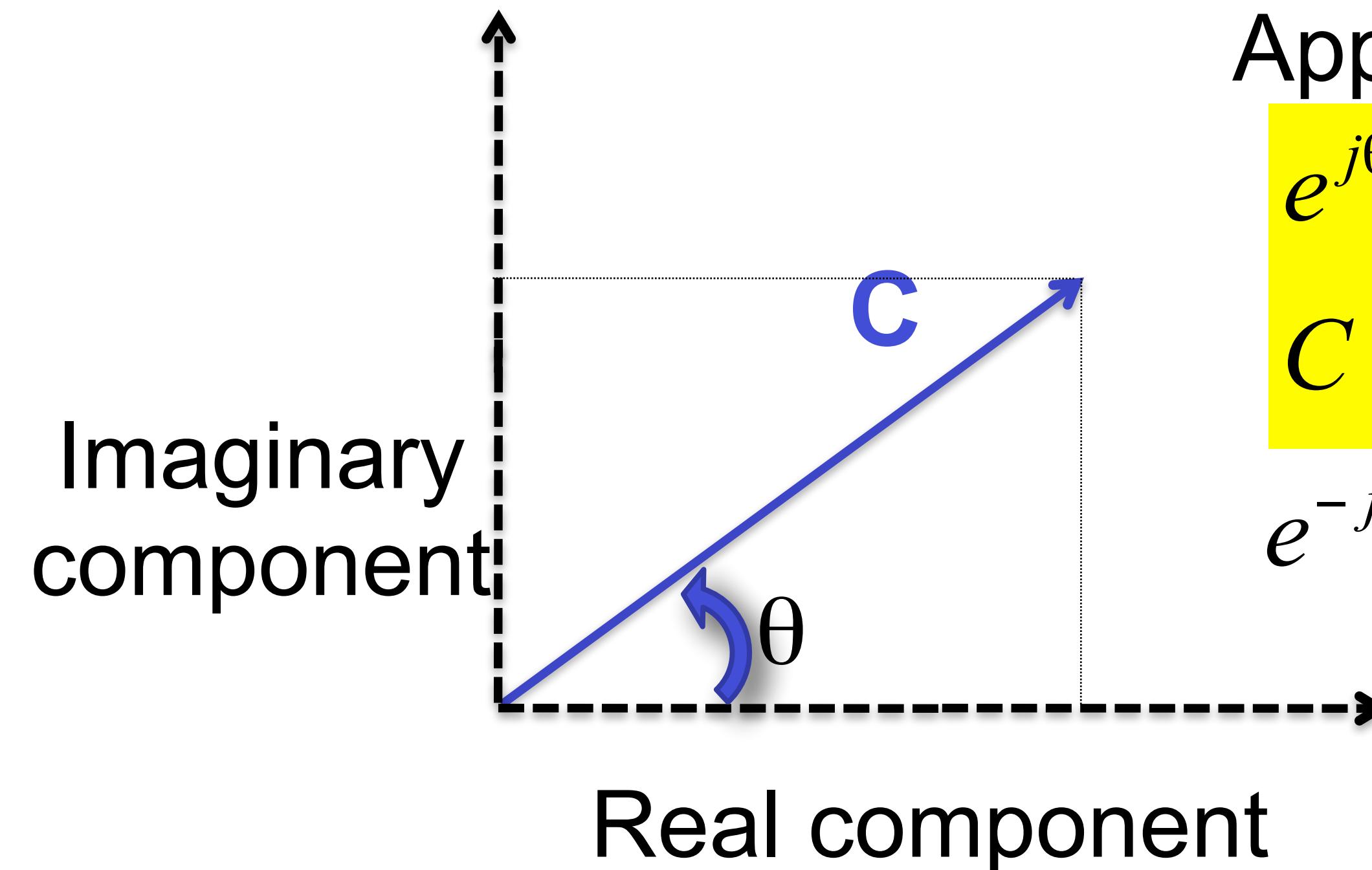
$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$C = |C|e^{j\theta}$$

$$e^{-j\theta} = \cos \theta - j \sin \theta$$

Geometry Review

$$C = |C|(\cos \theta + j \sin \theta)$$



Apply Euler's formula*

$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$C = |C|e^{j\theta}$$

$$e^{-j\theta} = \cos \theta - j \sin \theta$$

Now let's represent all points using this polar notation. If the points came from a periodic function (of space or time, using variable t here), we could model them using a Fourier Series!

Fourier Transform (Continuous, 1D)

$$\mathcal{F}\{f(t)\} = F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

The way to convert back from Fourier domain is the Inverse Fourier Transform (more later):

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

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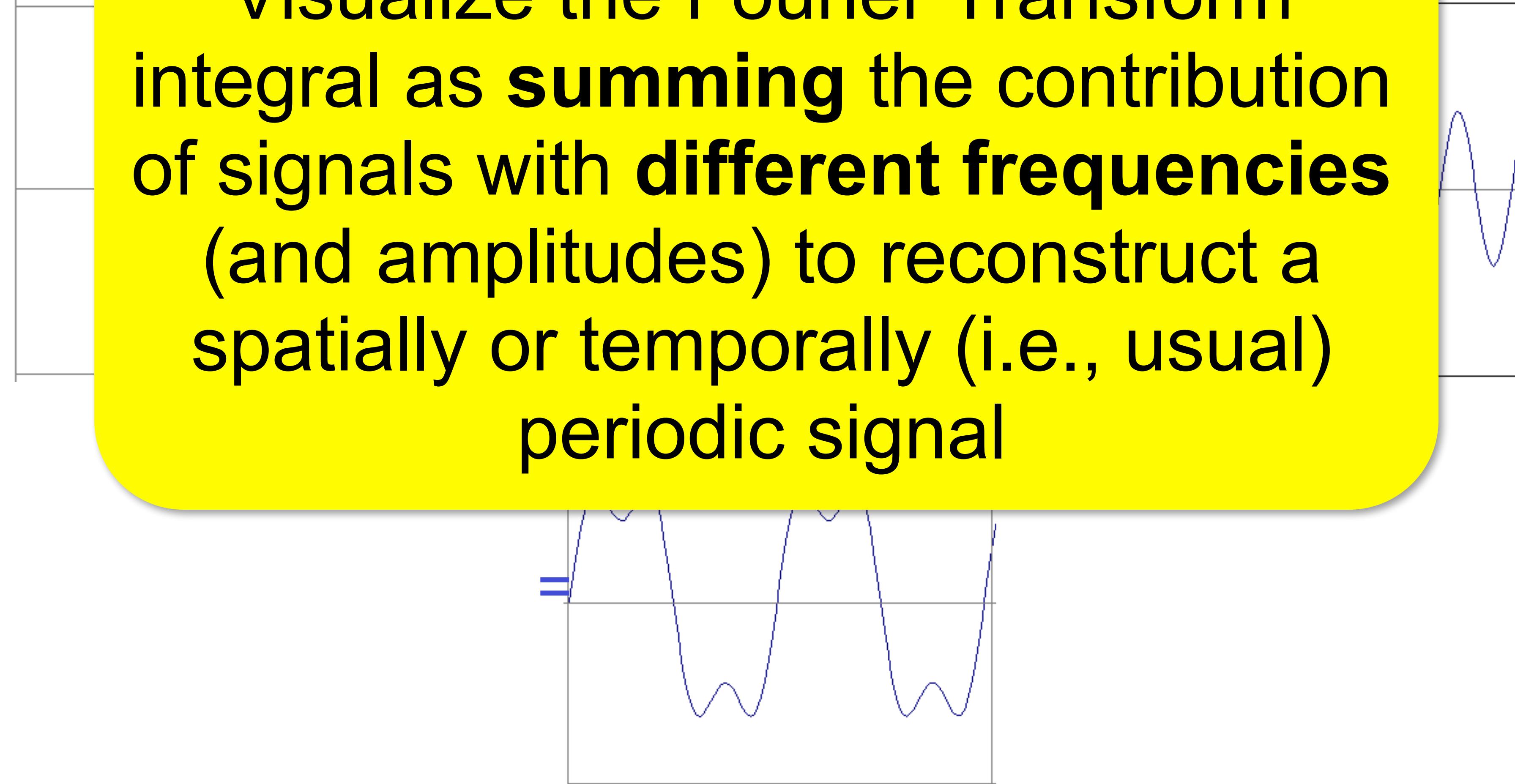
In Frequency domain, think of μ as cycles/sec (Hz)
(according to units of t)

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Inverse Fourier Transform (more later):

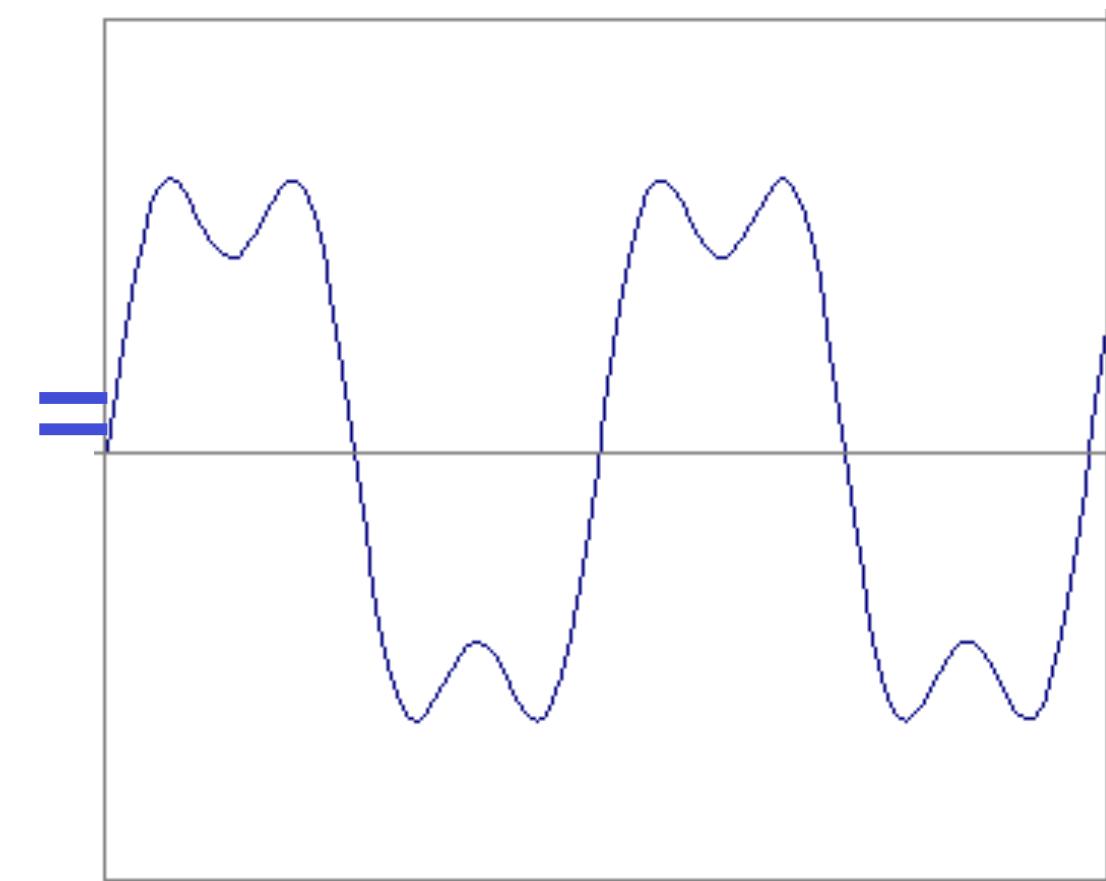
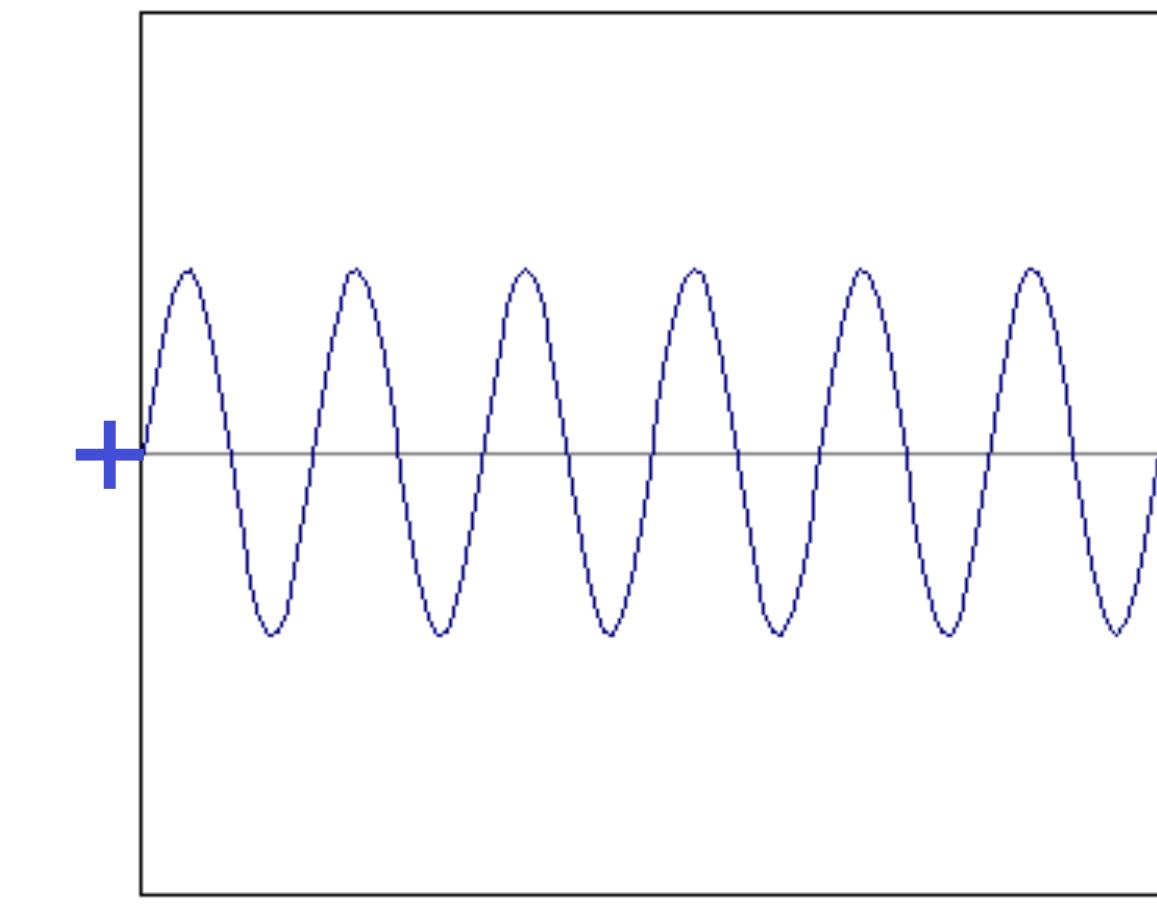
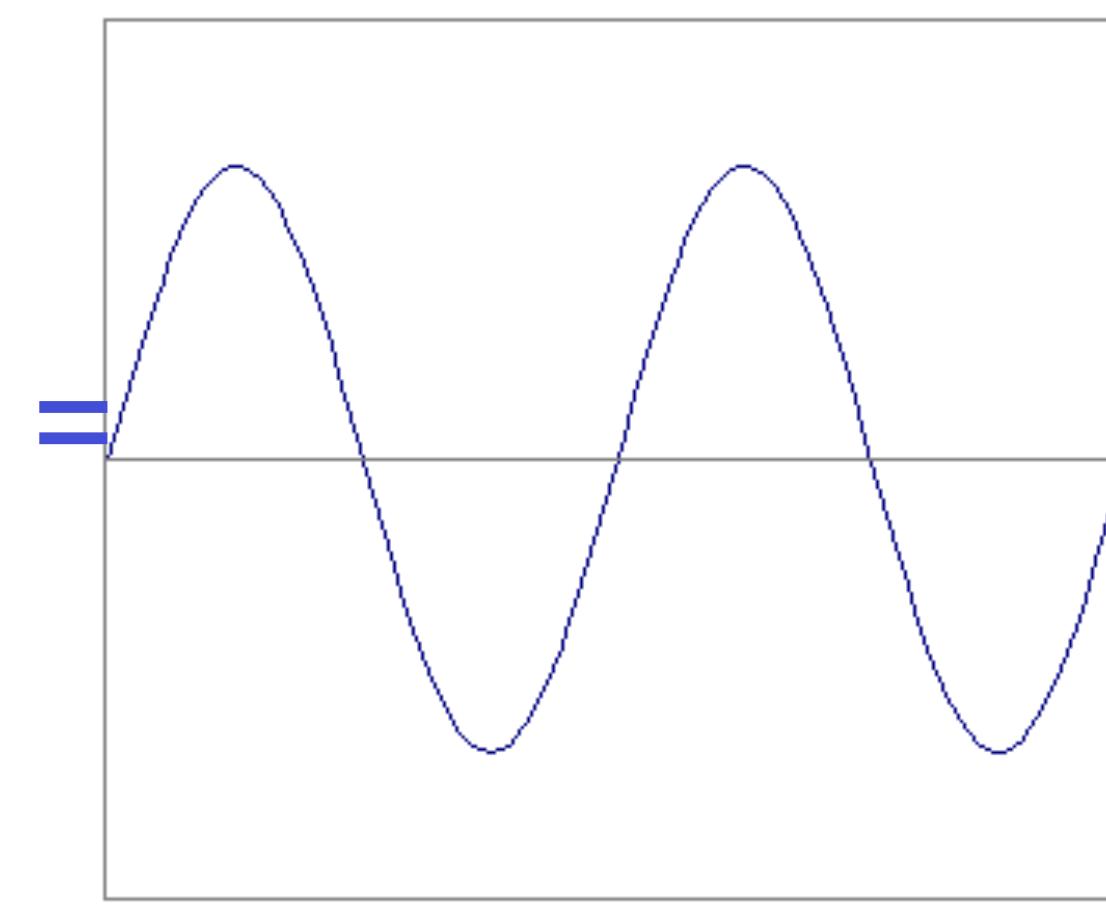
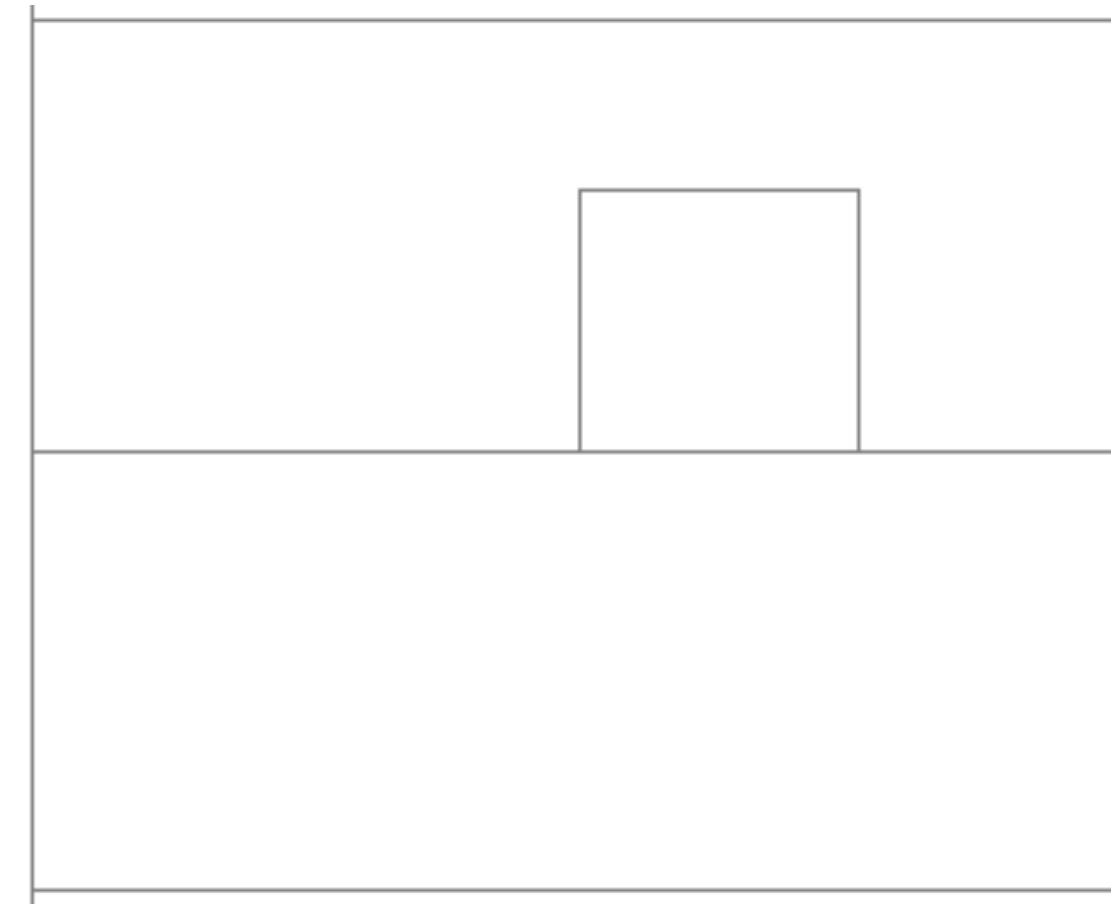
$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

Frequency Spectra

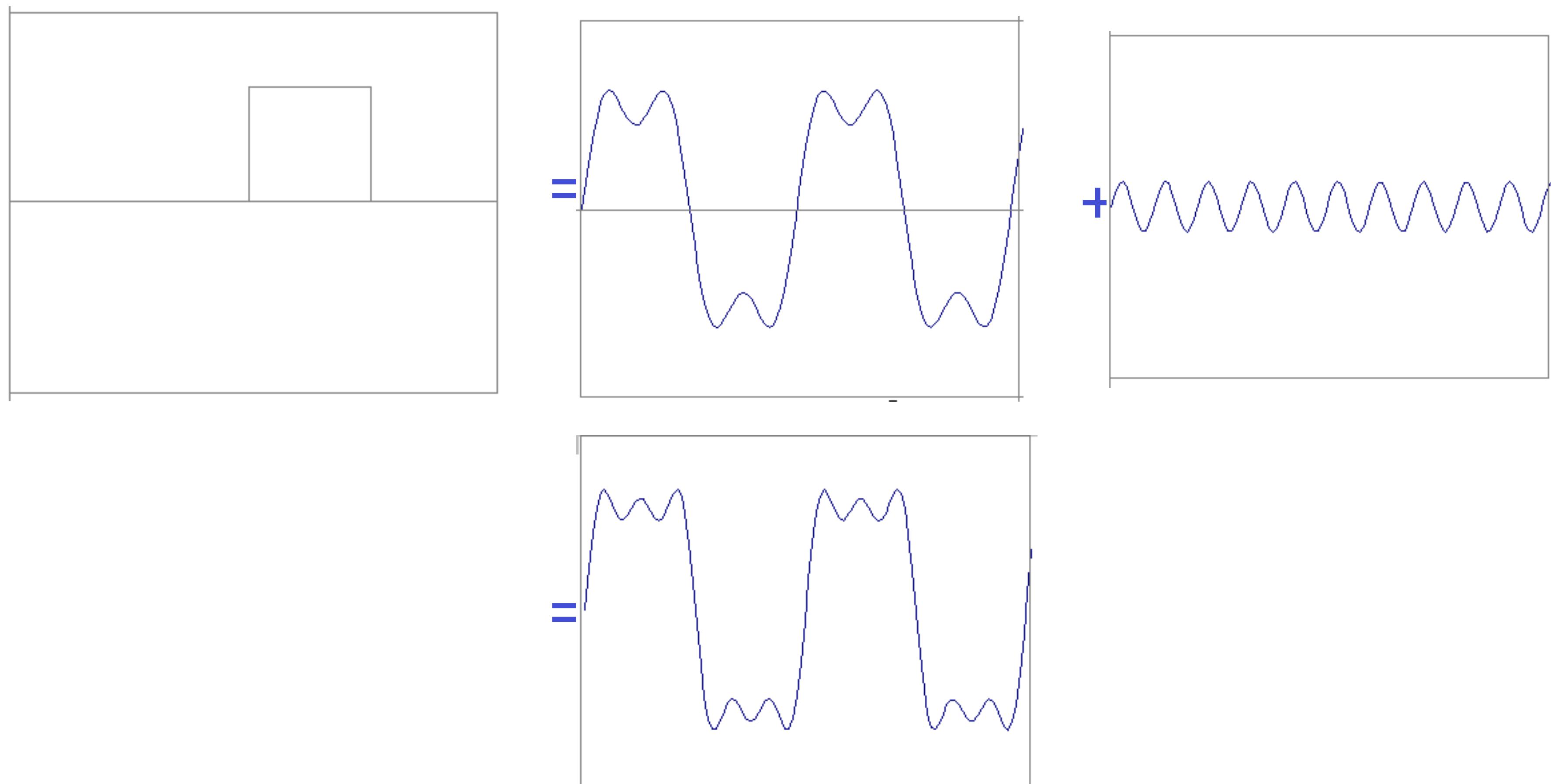
Visualize the Fourier Transform integral as **summing** the contribution of signals with **different frequencies** (and amplitudes) to reconstruct a spatially or temporally (i.e., usual) periodic signal



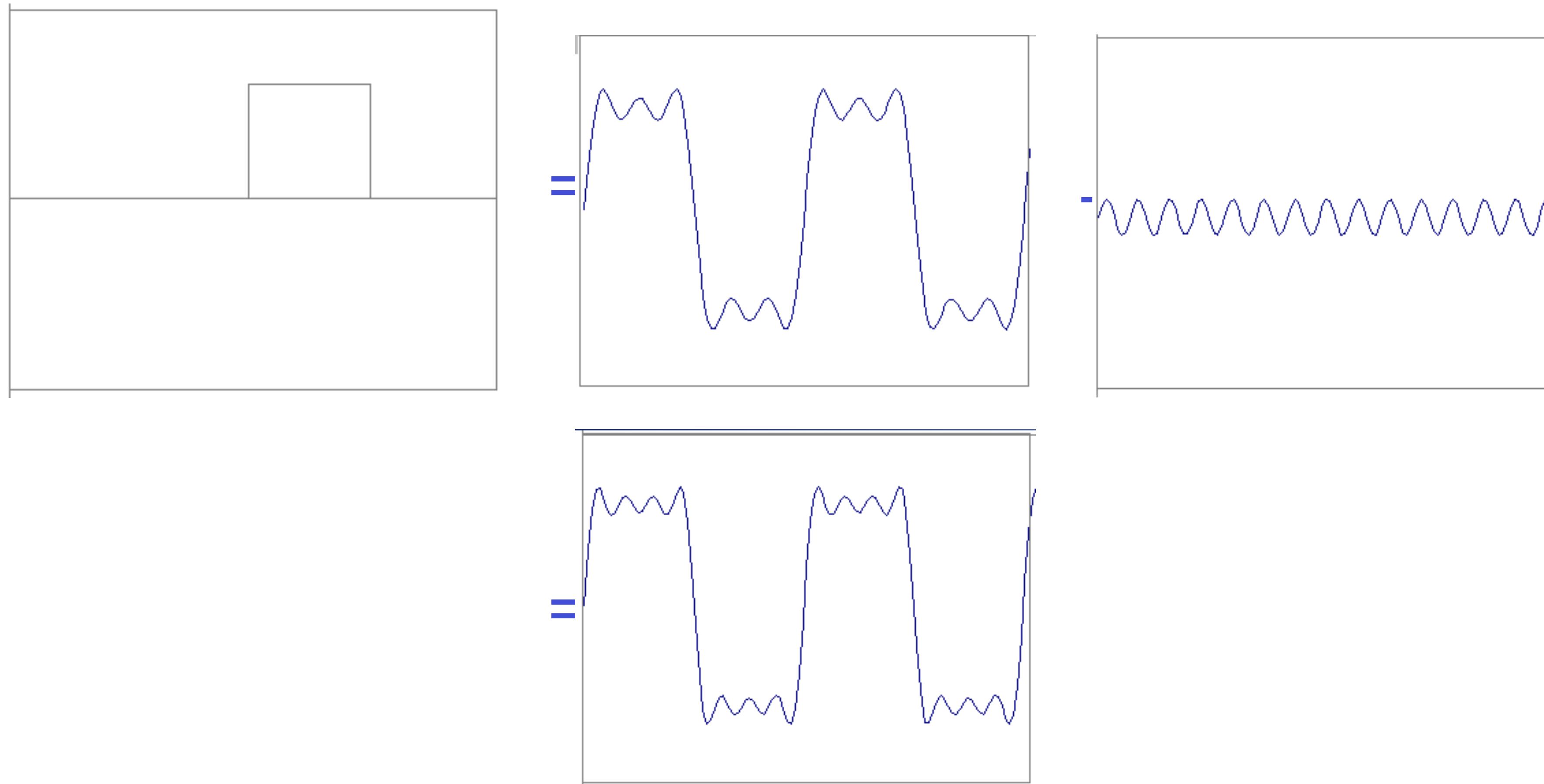
Frequency Spectra



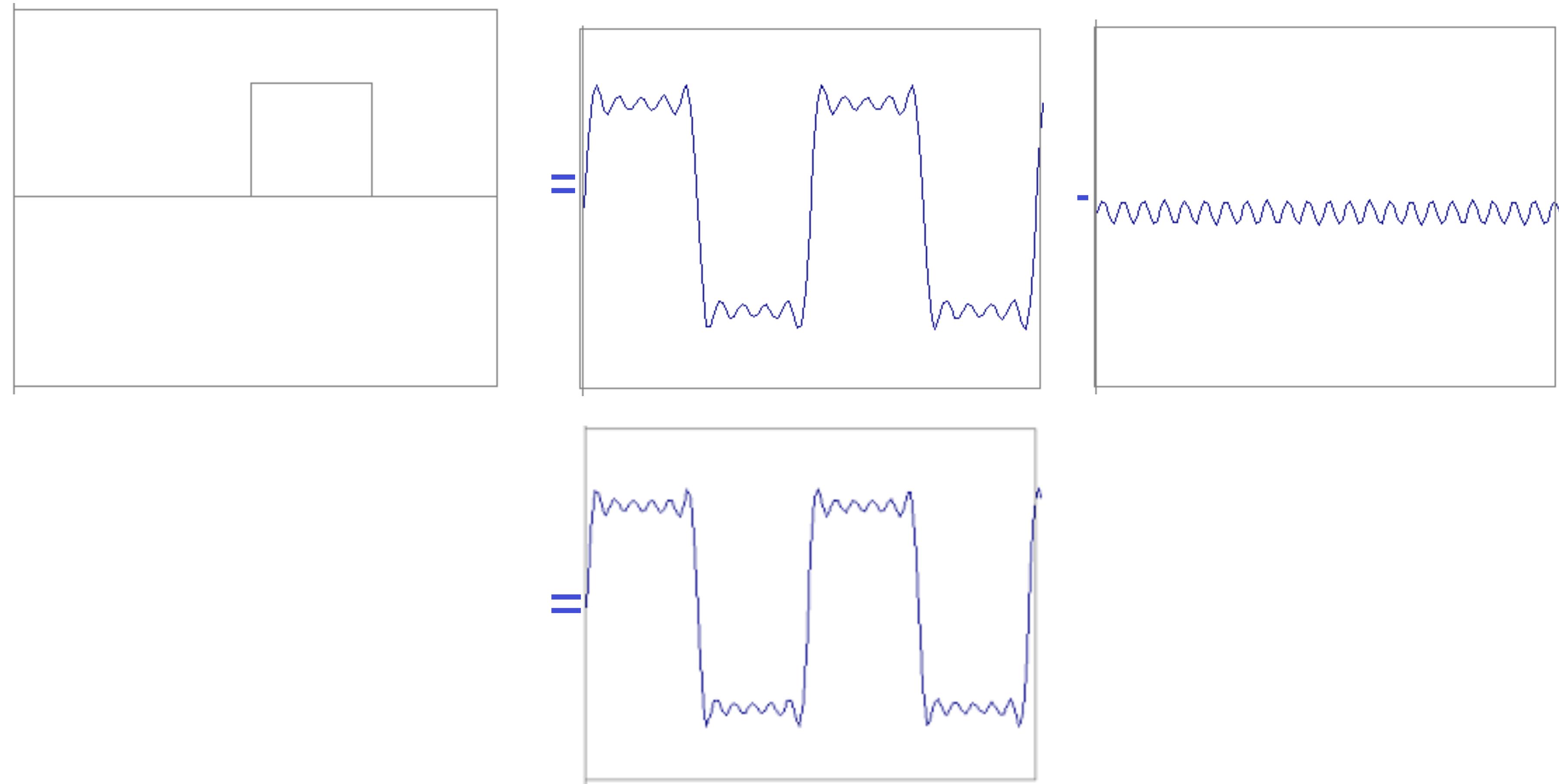
Frequency Spectra



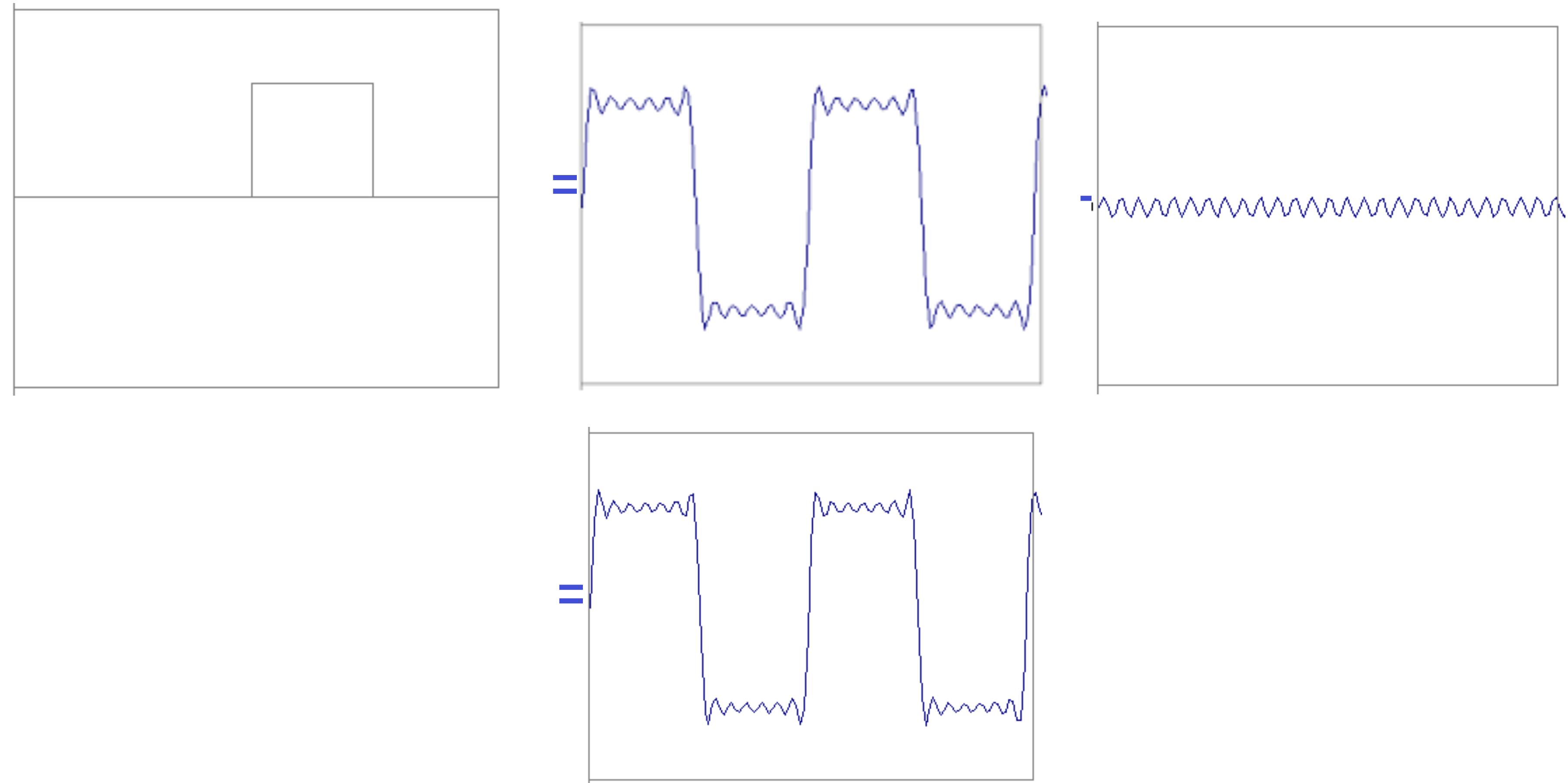
Frequency Spectra



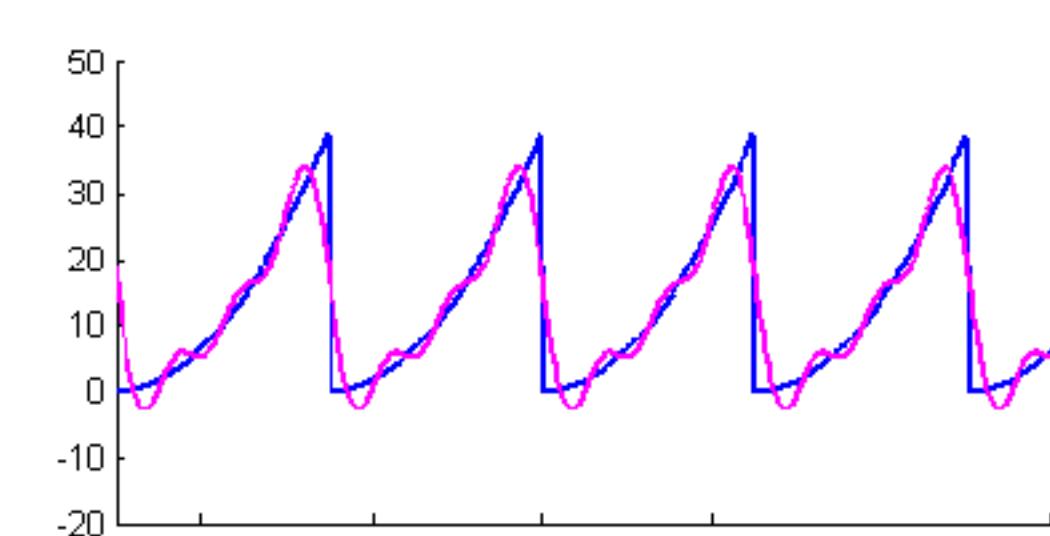
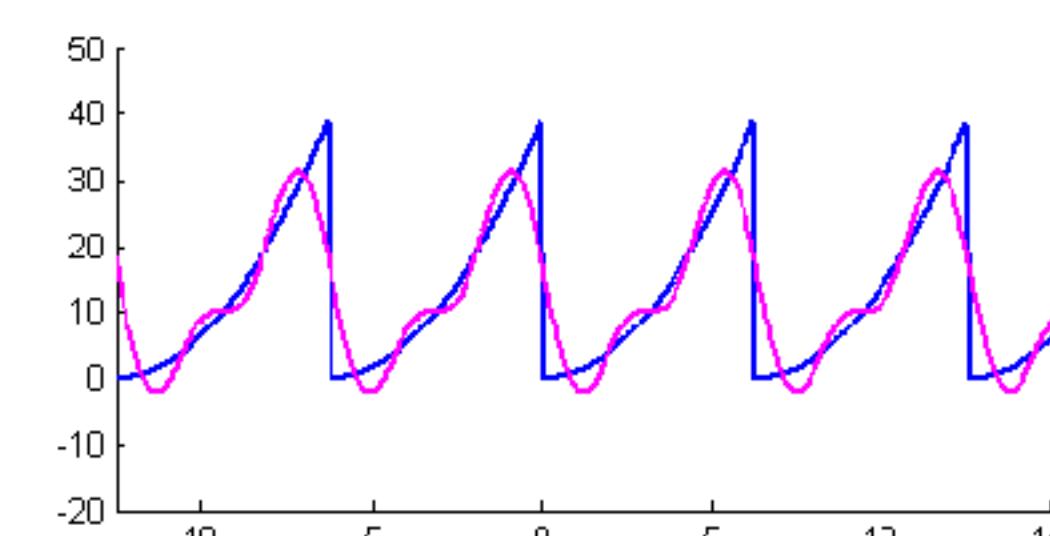
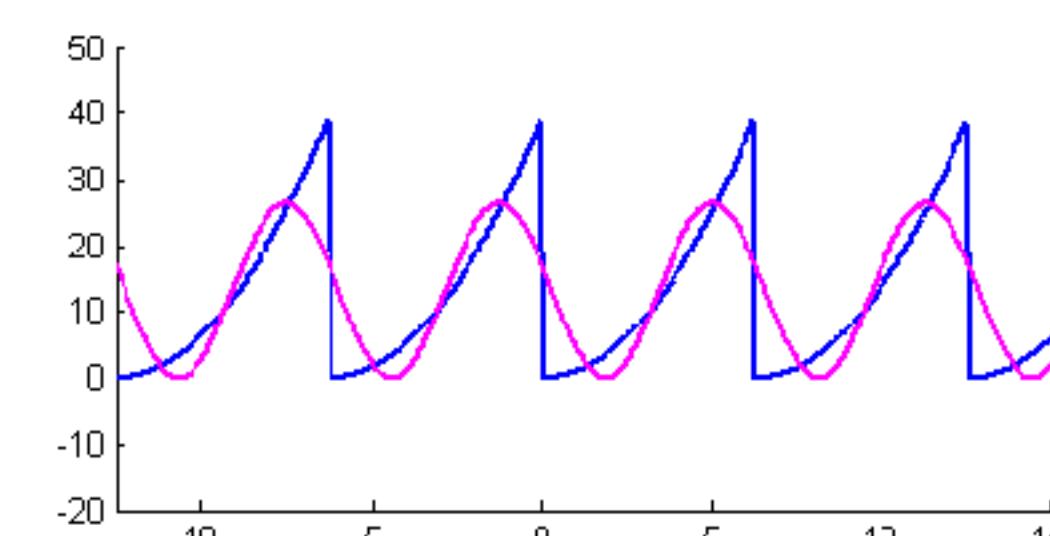
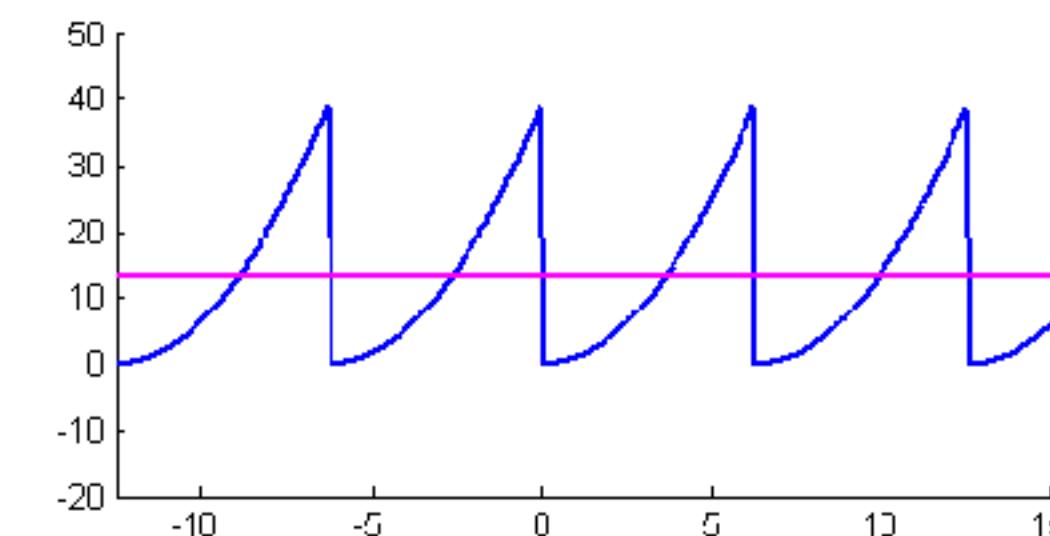
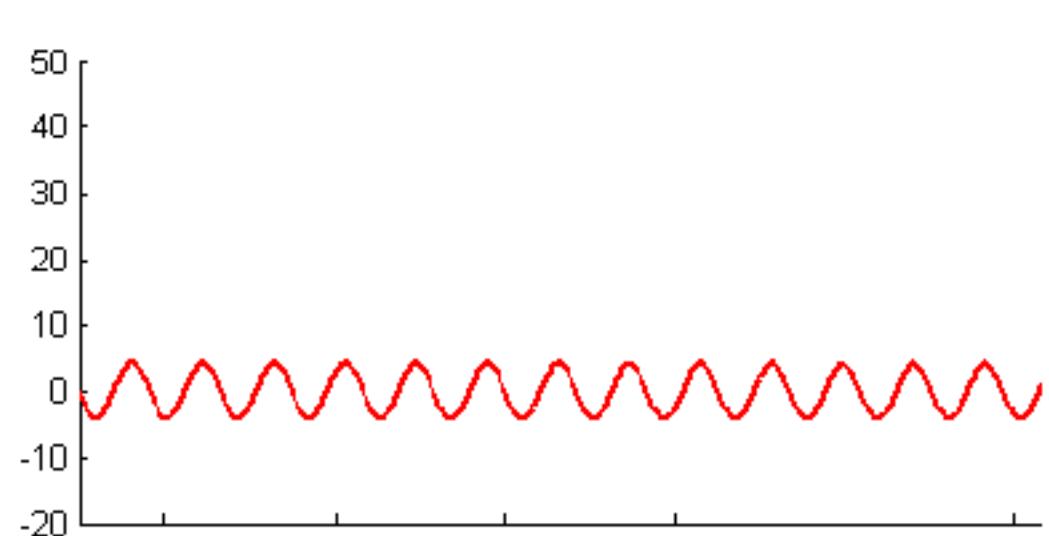
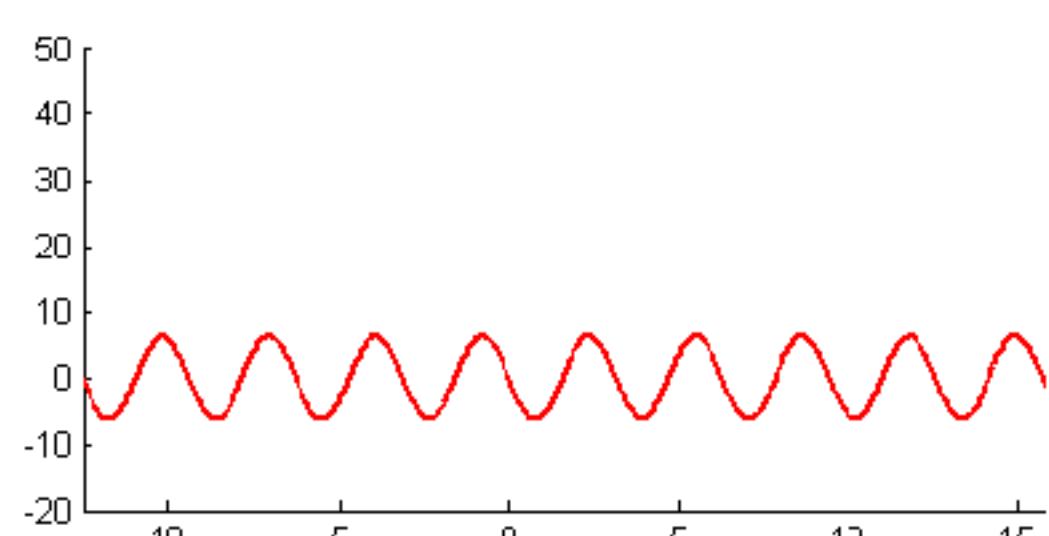
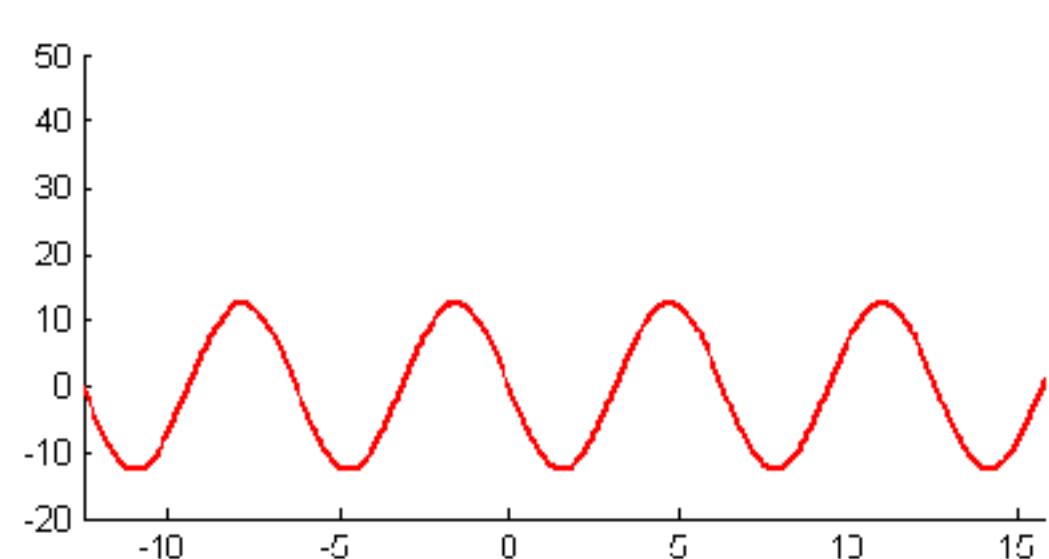
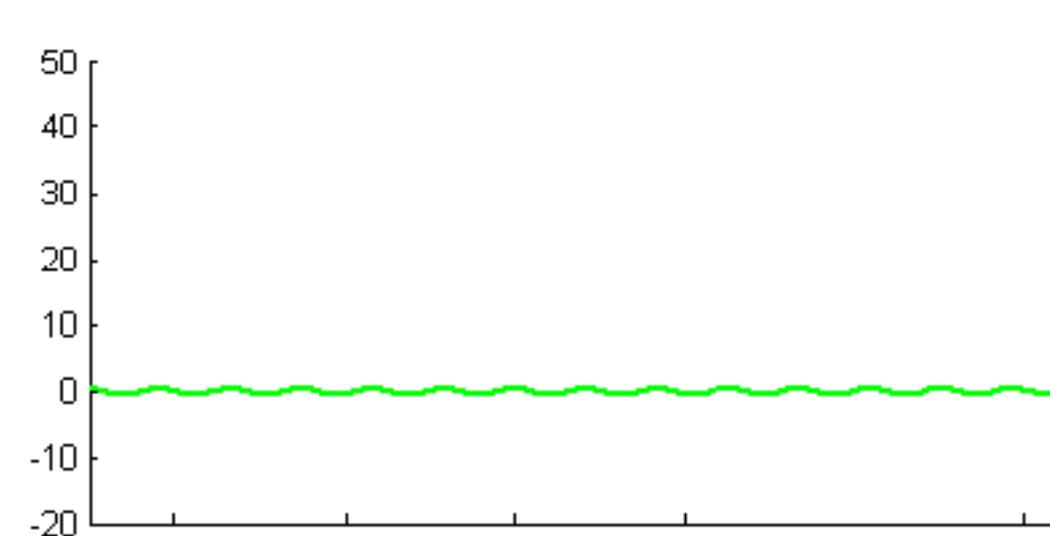
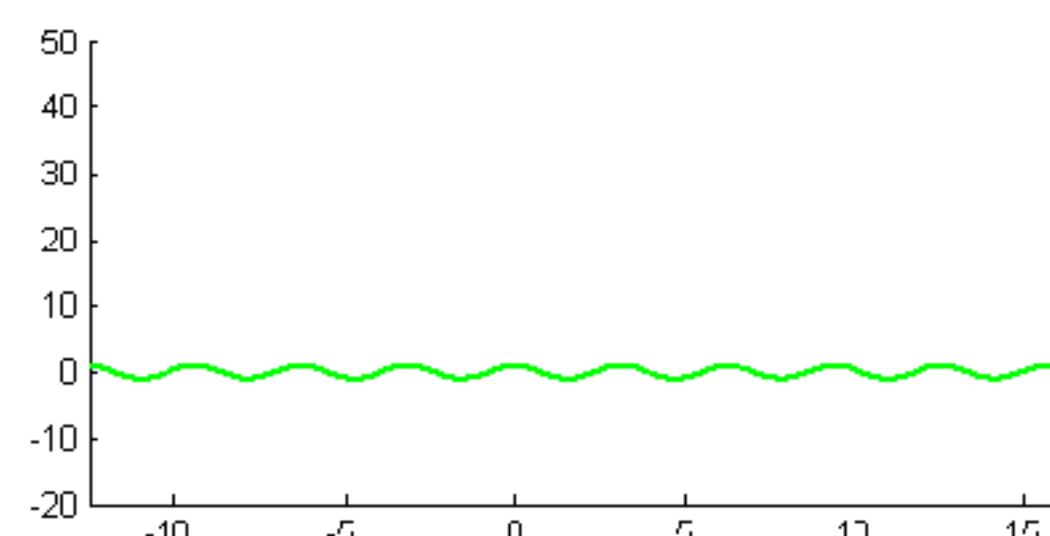
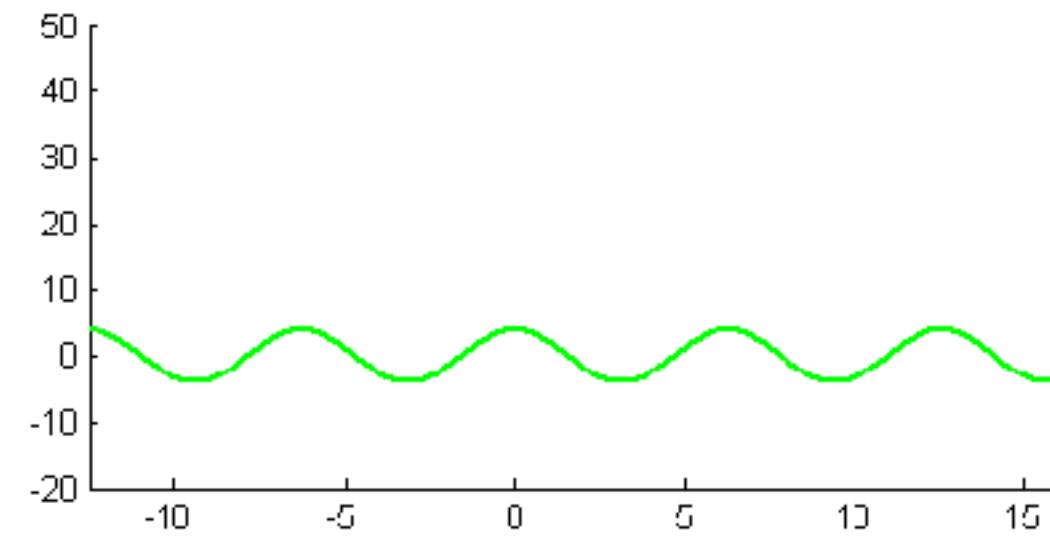
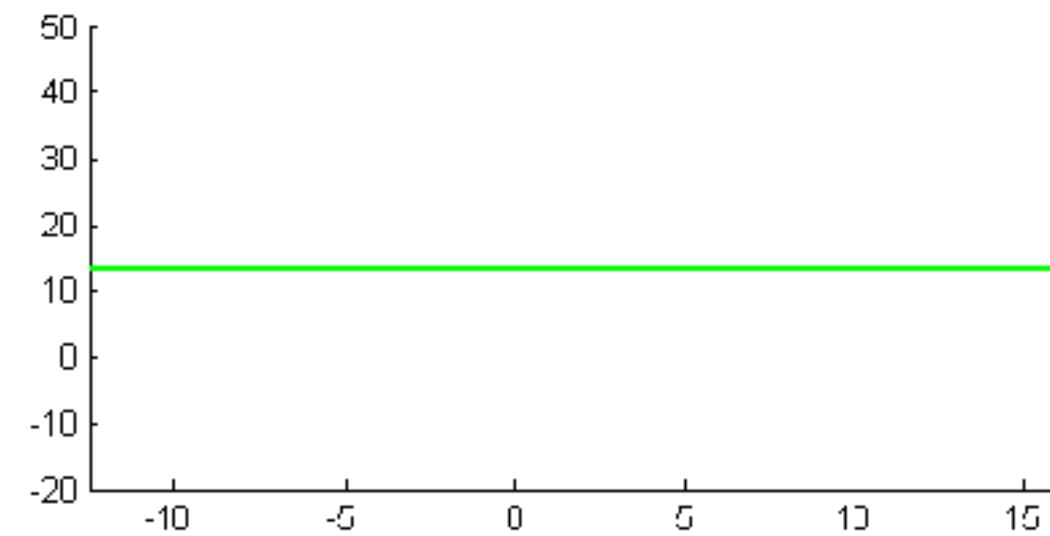
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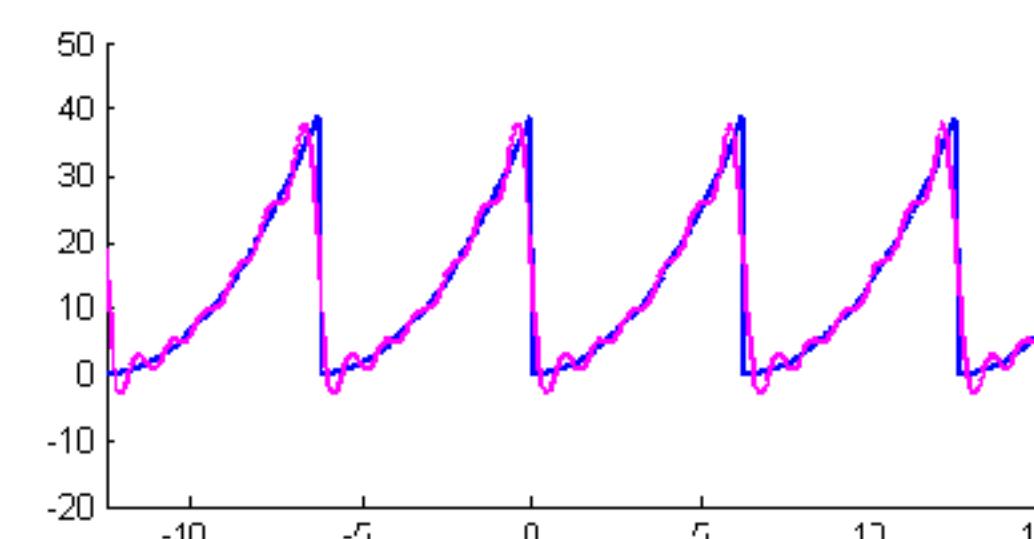
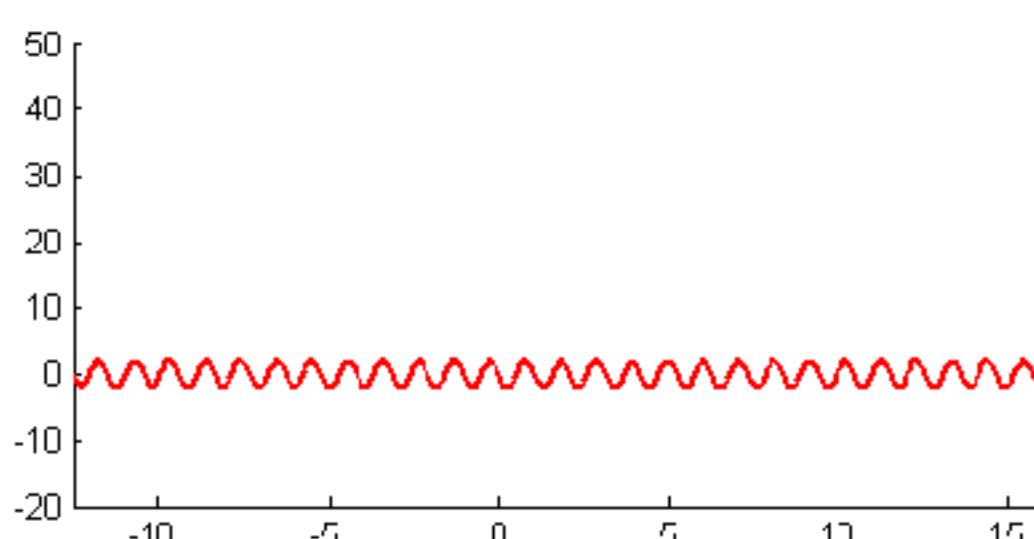
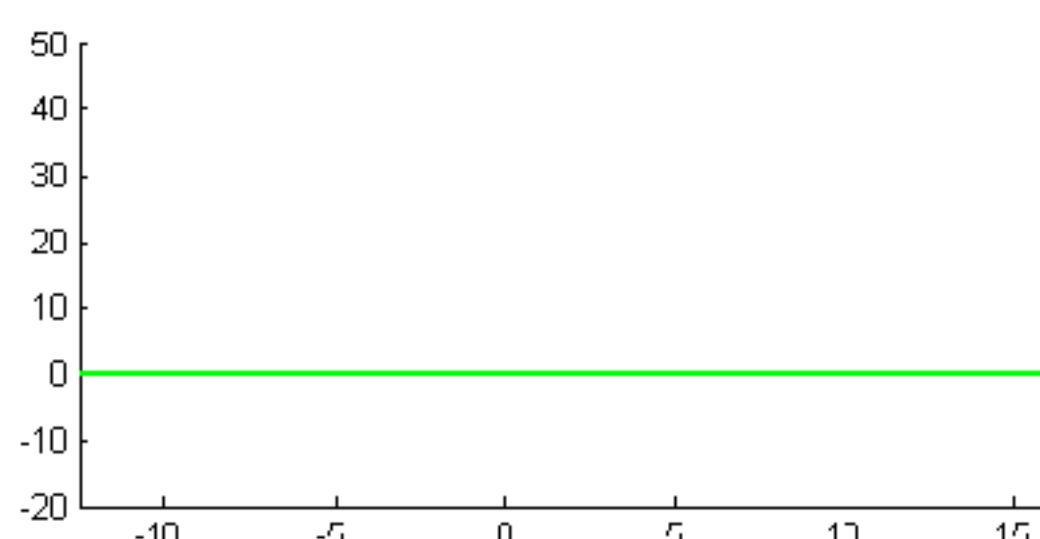
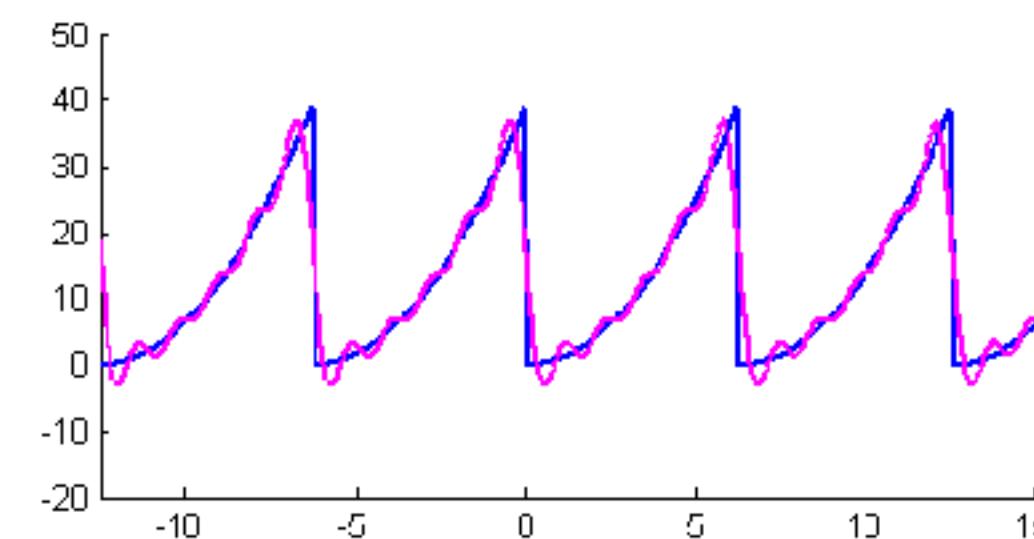
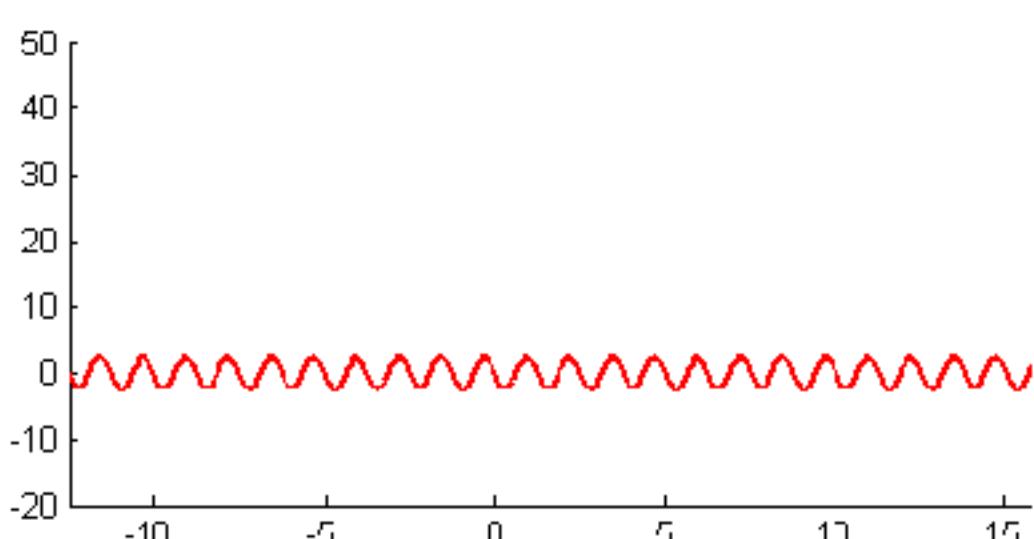
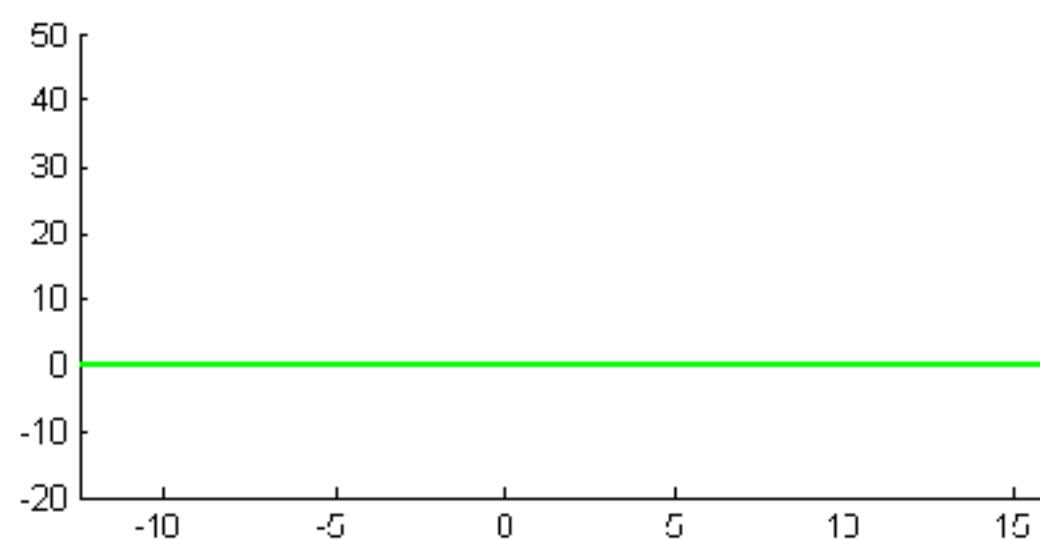
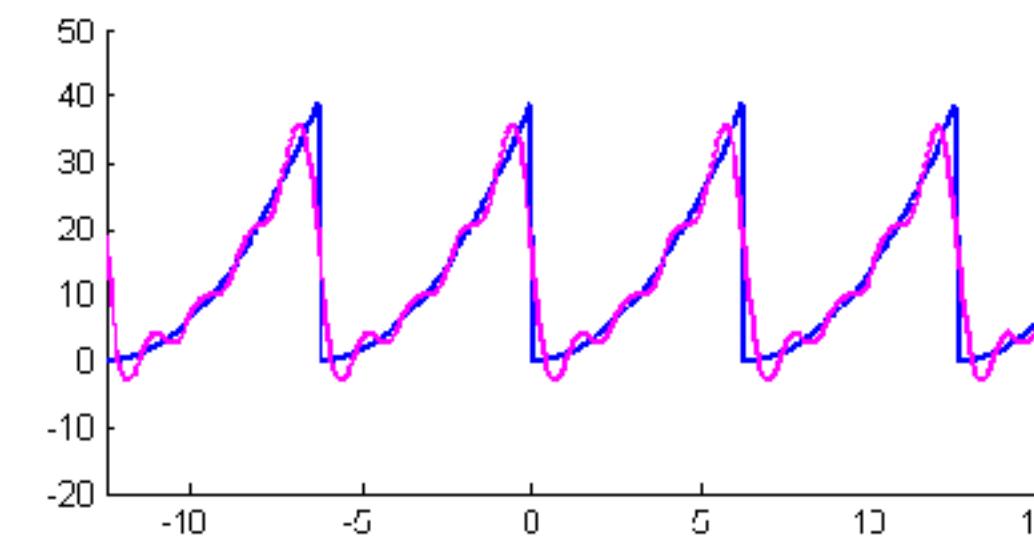
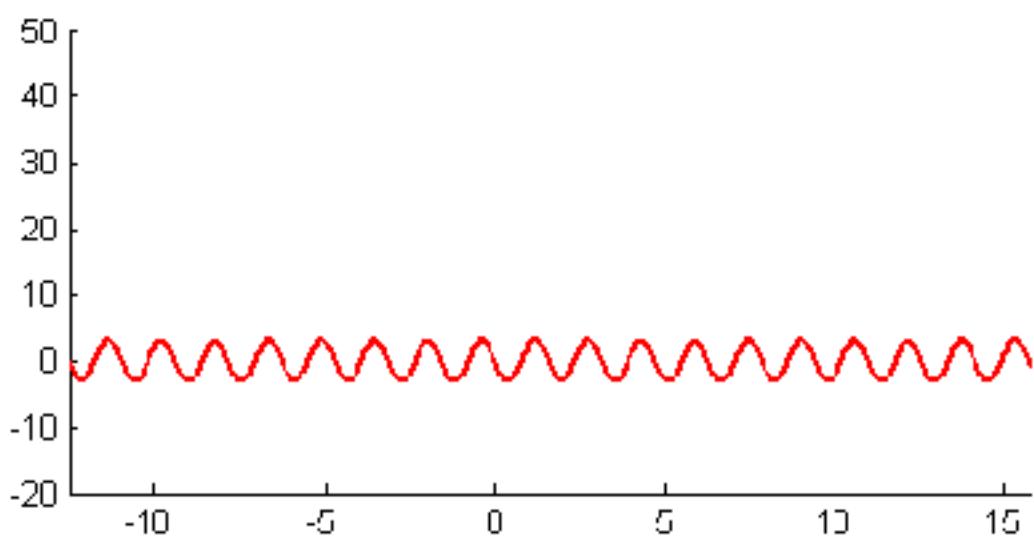
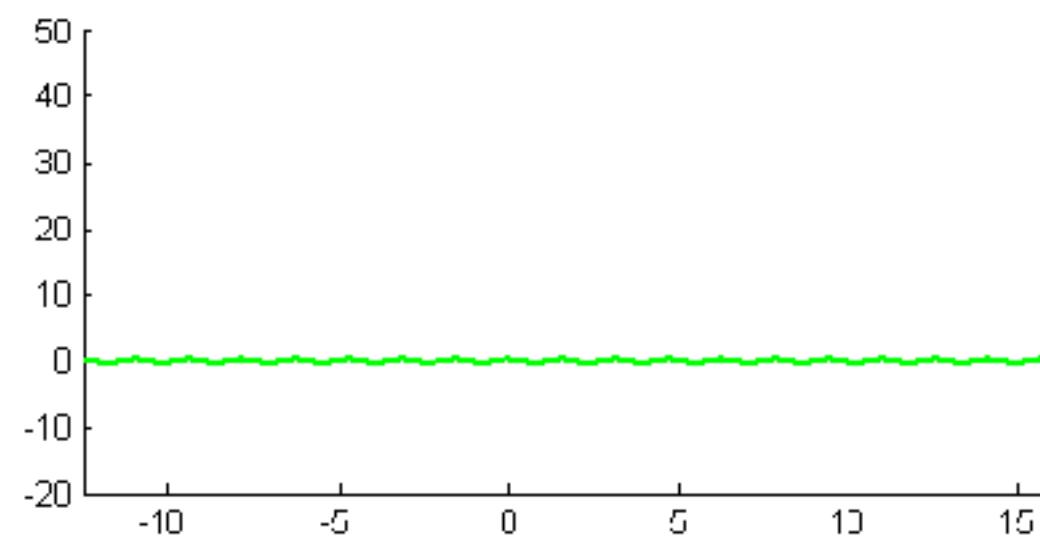
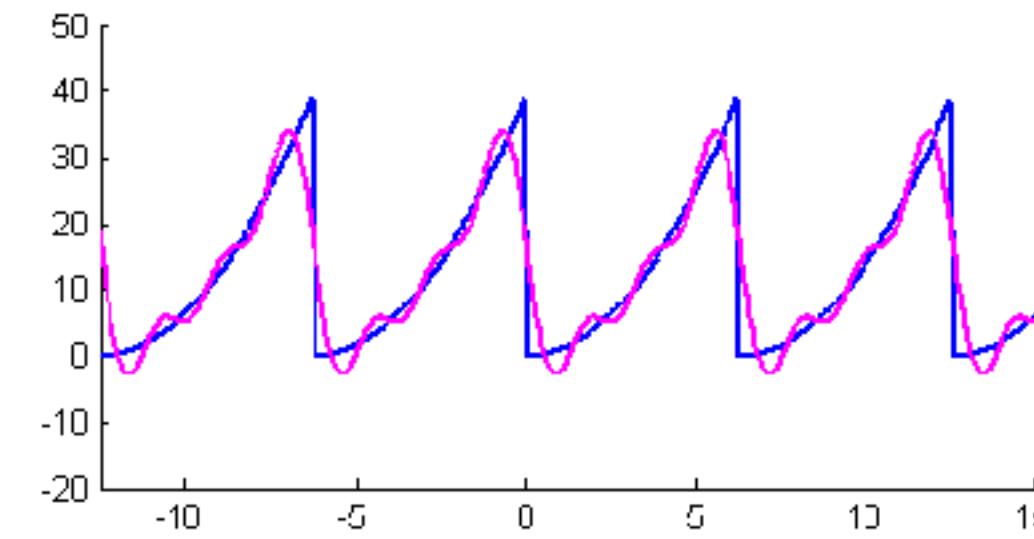
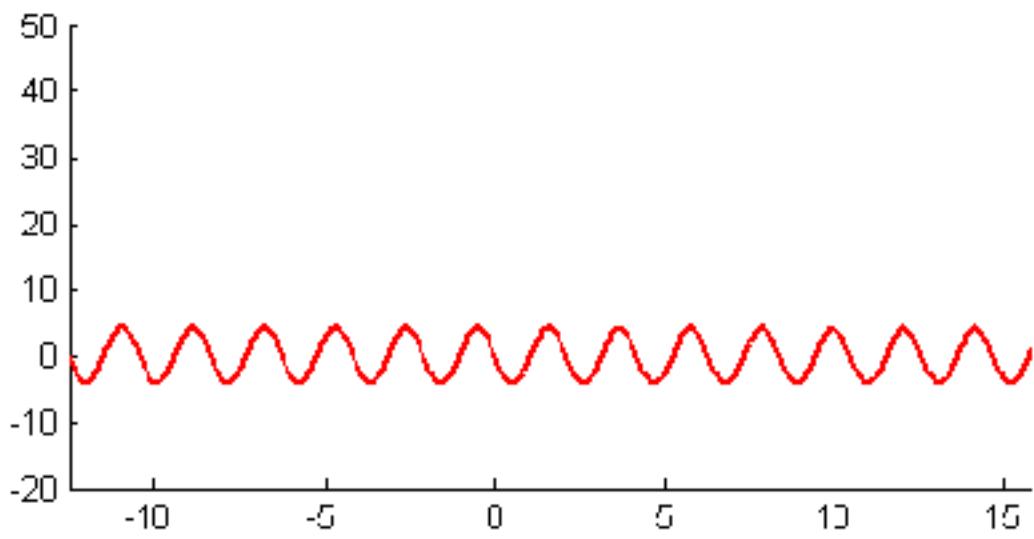
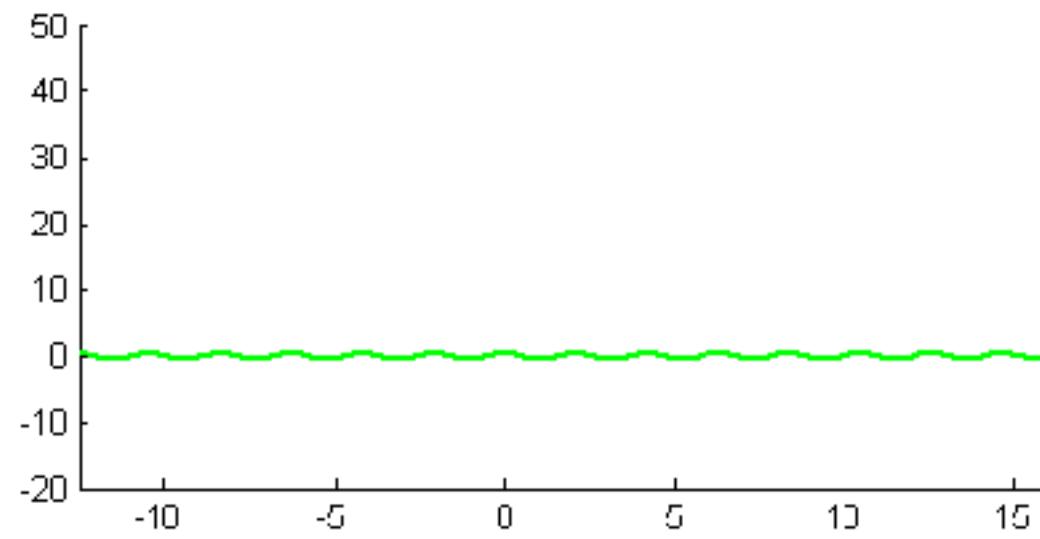


Frequency Spectra



x^2 Example





Discrete Fourier Transform (1D)

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux / M} \quad u = 0, 1, 2, \dots, M-1$$

Takes a discretized signal of length M (e.g., a row of an image, indexed by x) and returns a new (complex) function of length M that is indexed by u .

What is u ? It represents frequency of sinusoidal component

Discrete Fourier Transform (1D)

Discrete Fourier Transform (DFT)

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux / M} \quad u = 0, 1, 2, \dots, M-1$$

Discrete Fourier Transform (DFT)

$$F(u) = \sum_{x=0}^{M-1} f(x) [\cos(2\pi ux / M) - j \sin(2\pi ux / M)]$$

Discrete Fourier Transform (1D)

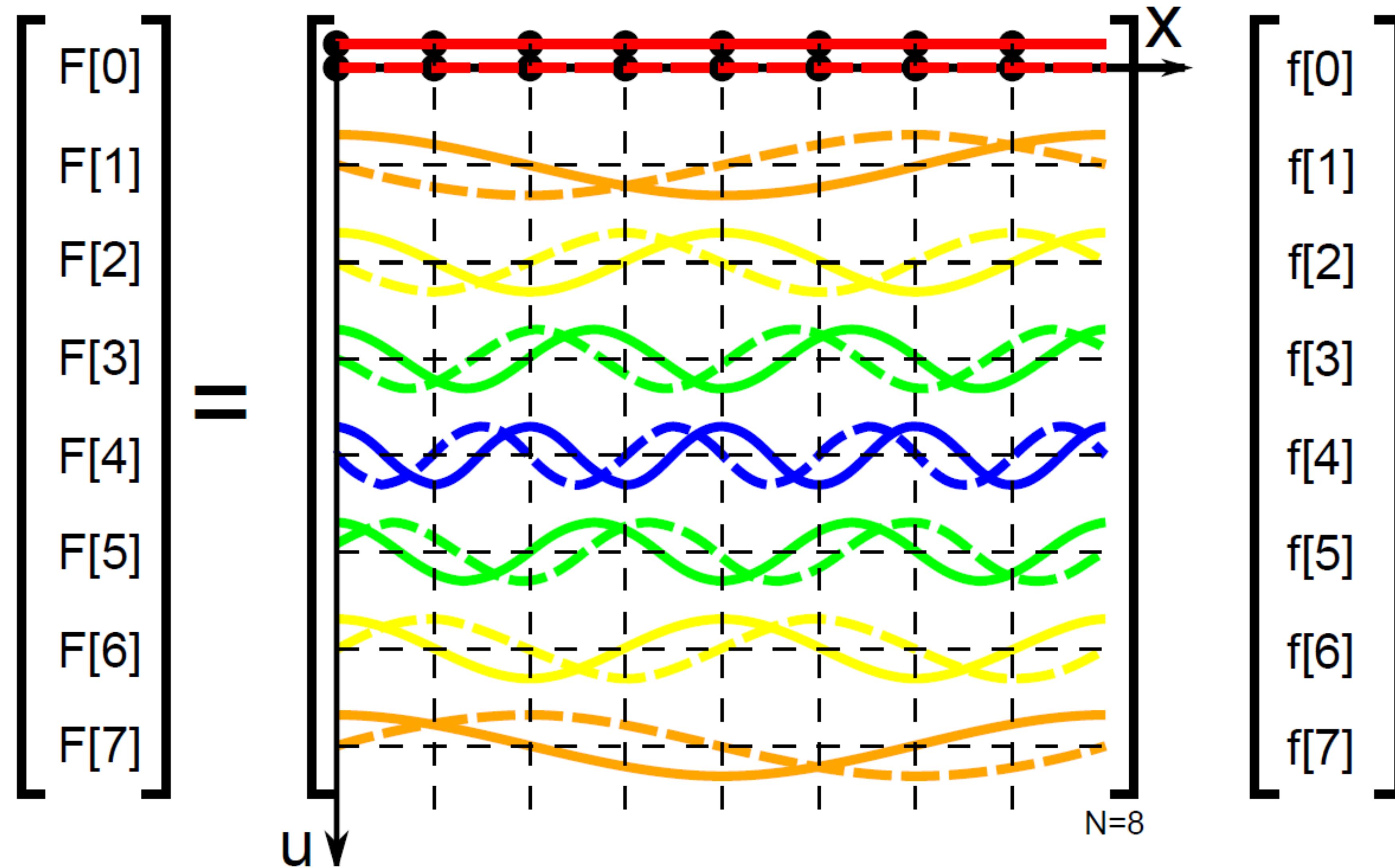
Discrete Fourier Transform (DFT)

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux / M} \quad u = 0, 1, 2, \dots, M-1$$

To understand this
apply Euler's formula* $e^{j\theta} = \cos\theta + j\sin\theta$

Discrete Fourier Transform (DFT)

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Can think of this as a change of basis: we move from space in which signal is represented as coefficients of delta functions to frequency where signal is represented as coefficients of sinusoidal basis functions

Fourier Transform (Derivation: Continuous 1D Discrete)

(again)
$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

Model the sampling of $f(t)$ as multiplication
with a sampling function:

$$\tilde{f}(t) = \sum_{x=-\infty}^{\infty} f(t) \delta(t - x\Delta t)$$

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What is $\delta(\cdot)$ and
why does this give
us a comb-filter?

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} (continued...)

$$\begin{aligned}
\tilde{F}(\mu) &= \int_{-\infty}^{\infty} \underline{\tilde{f}(t)} e^{-j2\pi\mu t} dt \\
&= \int_{-\infty}^{\infty} \sum_{x=-\infty}^{\infty} f(t) \delta(t - x\Delta t) \underline{e^{-j2\pi\mu t}} dt \\
&= \sum_{x=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \delta(t - x\Delta t) e^{-j2\pi\mu \underline{t}} dt \\
&\quad (\text{Integral is only non-zero when } \underline{t = x\Delta t}) \\
&= \sum_{x=-\infty}^{\infty} f(x) e^{-j2\pi\mu x\Delta t}
\end{aligned}$$

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&= \sum_{x=-\infty}^{\infty} f(x) e^{-j2\pi\mu \underline{x\Delta t}} \quad \left. \right\} \text{(continued...)}
\end{aligned}$$

$$\tilde{F}(\mu) = \sum_{x=-\infty}^{\infty} f(x) e^{-j2\pi\mu x \Delta t} \quad] \quad (\text{from last page})$$

$\text{freq} = \frac{1}{\Delta t}$ and we want M equally-spaced samples,
so sweep continuous frequency-space
with samples μ , where $\mu = \frac{u}{M} \frac{1}{\Delta t}$

Discrete Fourier Transform (DFT)

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi u x / M} \quad u = 0, 1, 2, \dots, M-1$$

Inverse Discrete Fourier Transform (IDFT)

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi u x / M} \quad x = 0, 1, 2, \dots, M-1$$

Discrete Fourier Transform (2D)

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi \left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

where:

$$u = 0, 1, 2, \dots, M-1$$

$$v = 0, 1, 2, \dots, N-1$$

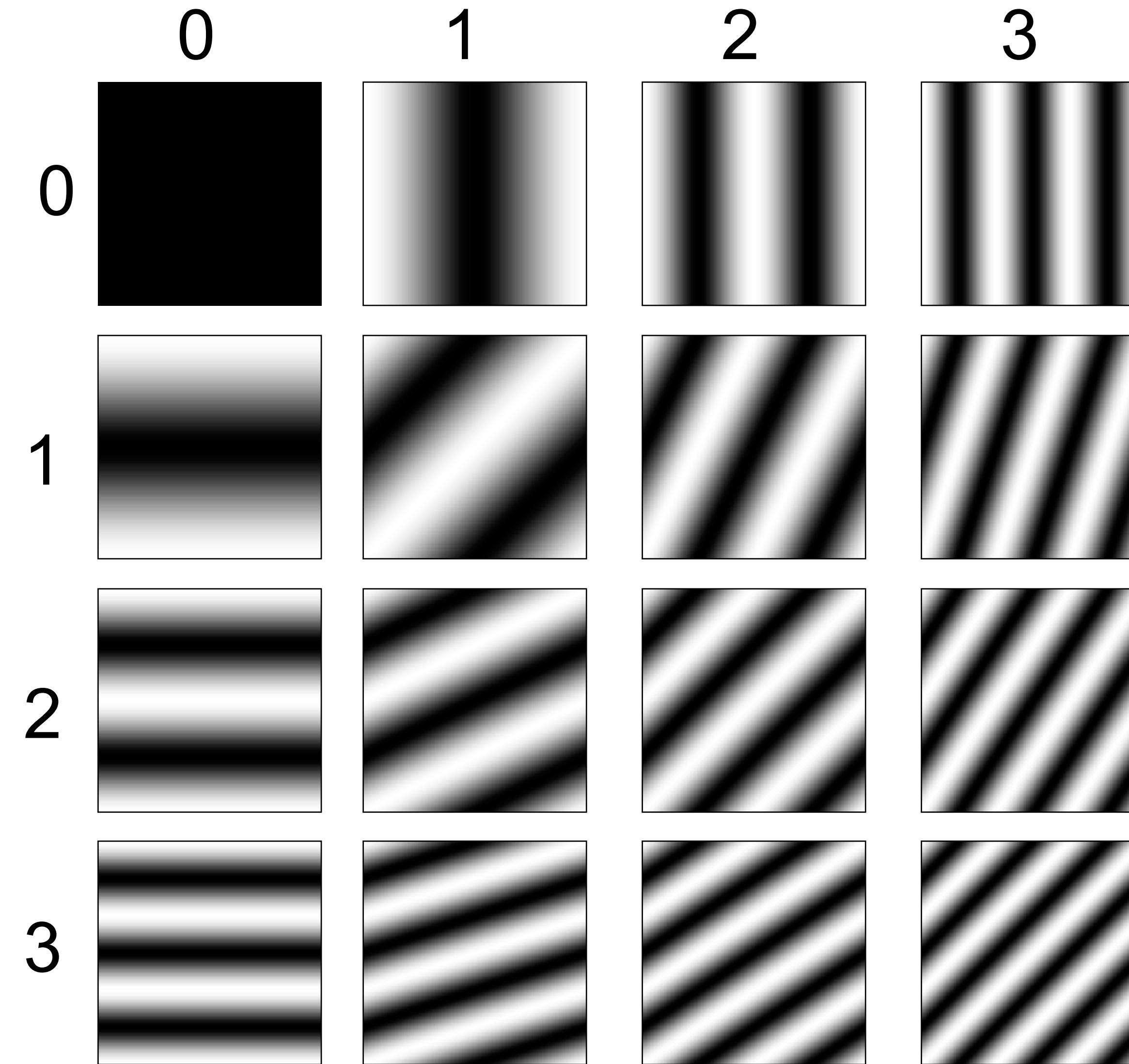
$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi \left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

where:

$$x = 0, 1, 2, \dots, M-1$$

$$y = 0, 1, 2, \dots, N-1$$

Fourier Basis Functions



2D Fourier Domain Example



Full image

2D Fourier Domain Example



Full image



First 1 basis fn

2D Fourier Domain Example



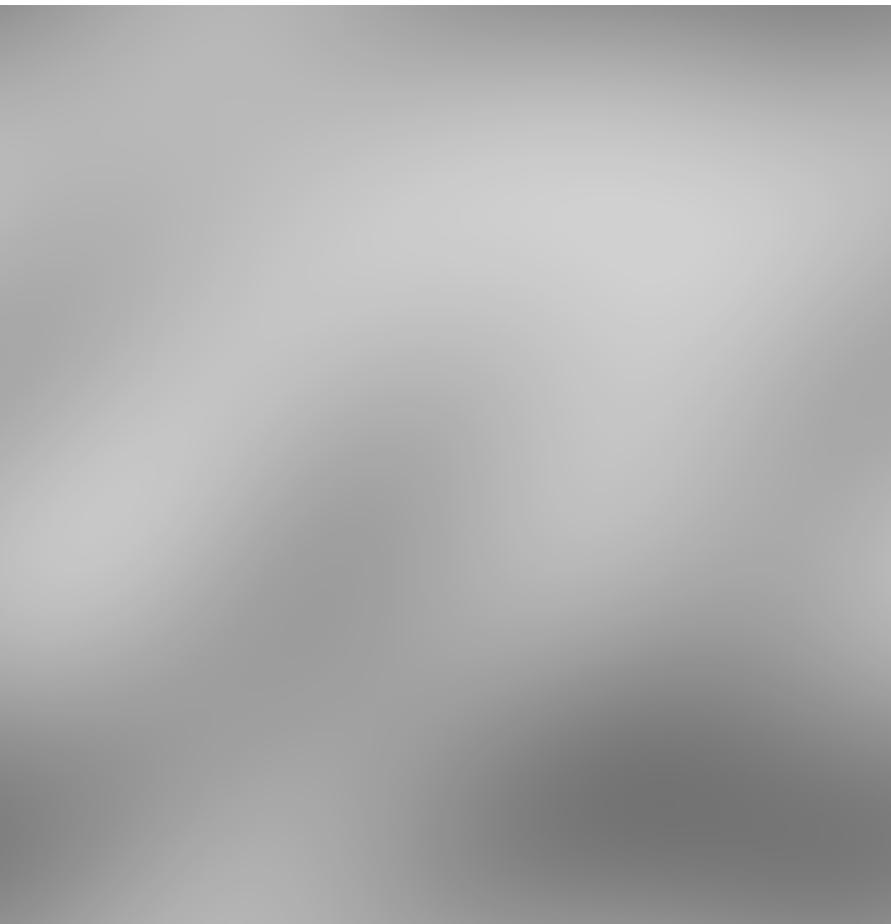
Full image



First 1 basis fn



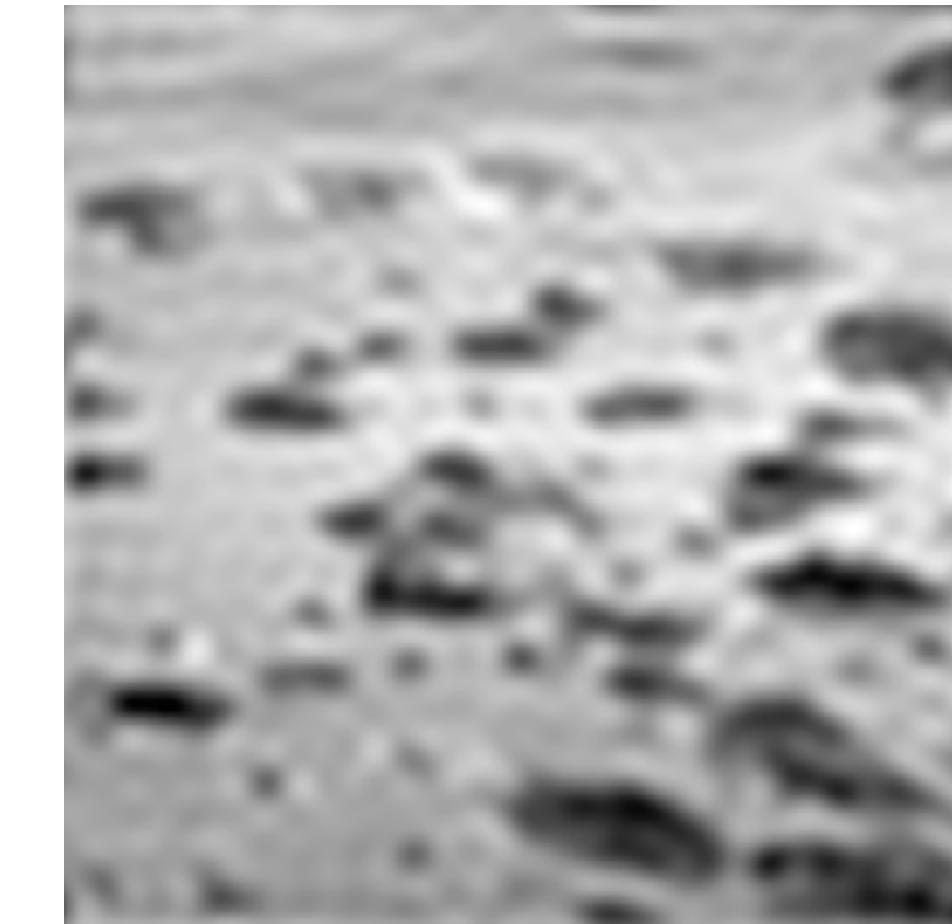
First 4 basis fns



First 9 basis fns



First 16 basis fns
Image Filtering



First 400 basis fns

Fast Fourier Transform

Fast Fourier Transform

- Naïve DFT is $O(N^4)$

Fast Fourier Transform

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- The FFT computes the DFT of an $N \times N$ image in $O(N^2 \log_2 N)$ time if $N = 2^M$

Fast Fourier Transform

- Naïve DFT is $O(N^4)$
- The FFT computes the DFT of an $N \times N$ image in $O(N^2 \log_2 N)$ time if $N = 2^M$
- FFTW = Fastest FT in the West (<http://www.fftw.org/>)

Image Spectra

Image Spectra

- F is a 2D $M \times N$ array of complex numbers.

Image Spectra

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Image Spectra

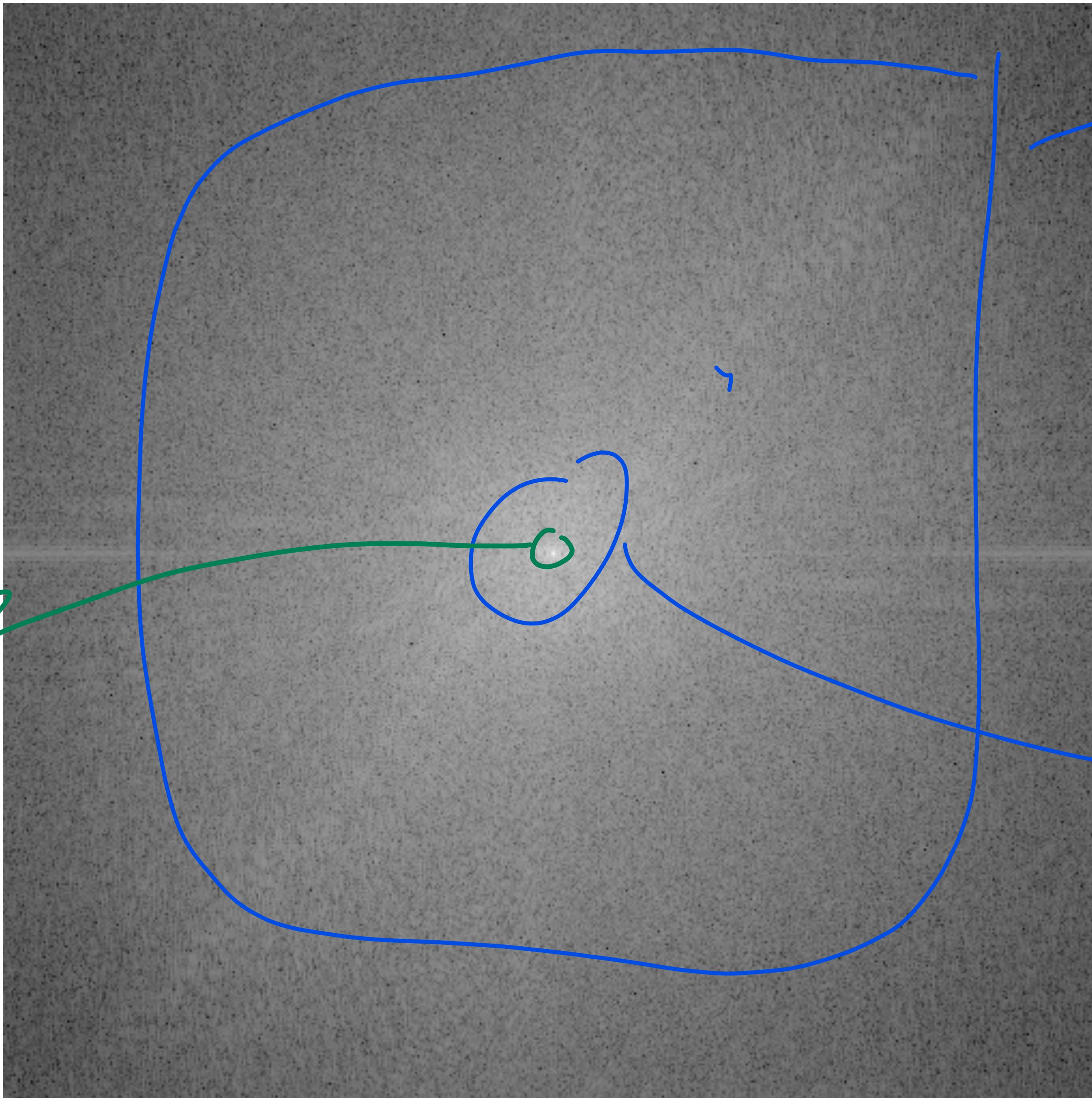
- F is a 2D $M \times N$ array of complex numbers.
- Common to plot $|F|$ and $\arg(F)$ as images
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$$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N).$$
- F has **conjugate symmetry** so
$$|F(u,v)| = |F(-u,-v)|$$



Slide by Alexei Efros

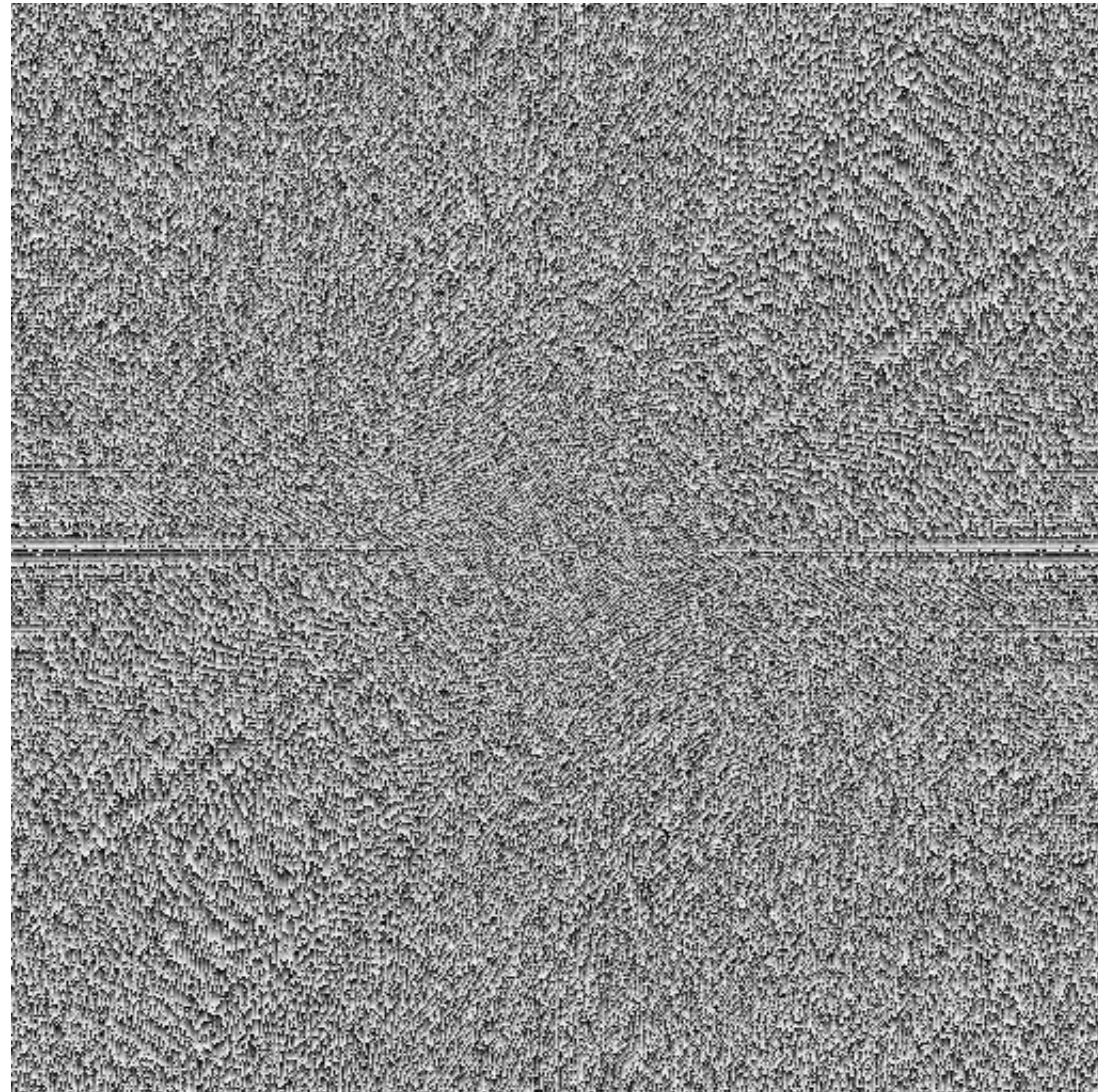
This is the magnitude transform of the cheetah pic

中间代表frequency =
无幅值
左侧代表 pixel
无幅值 代表 高频
右侧代表低频
无幅值
频率



→ 值大
frequency ↑
W W W
对左图局中 L
采样支撑
值小
frequency L

This is the
phase
transform of
the cheetah
pic

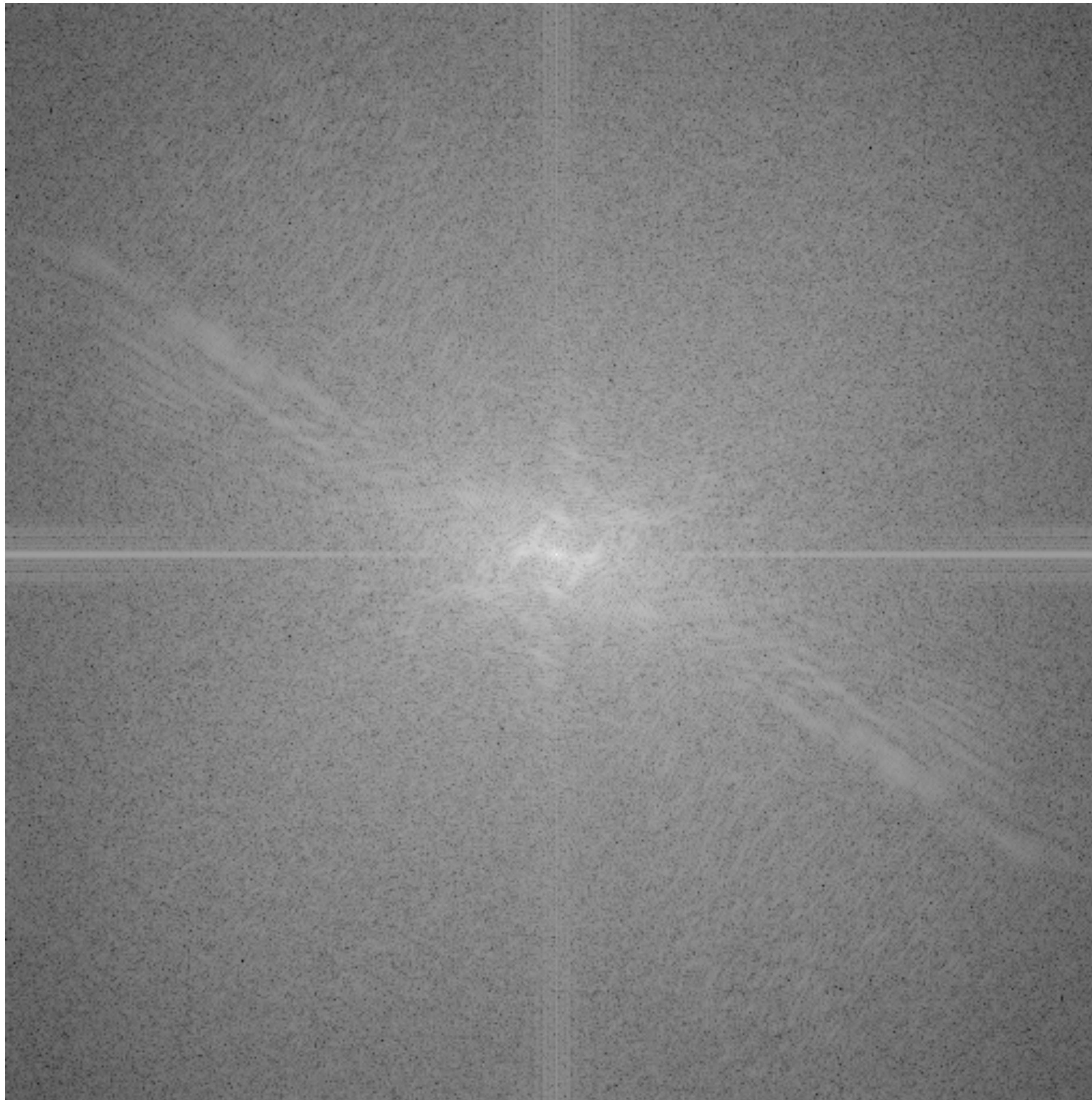


→ 底部
偏移和角度
→ 纹理

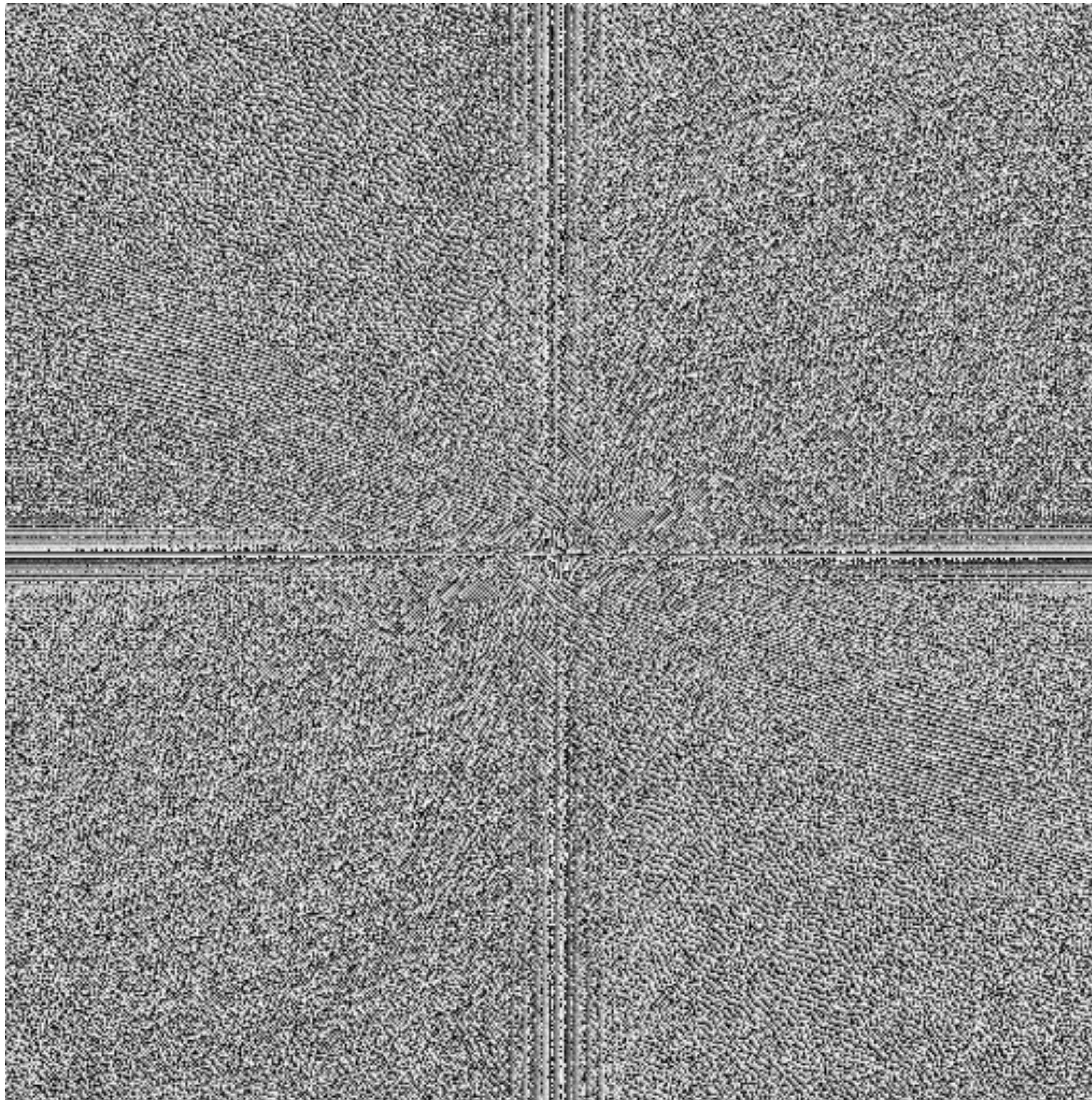


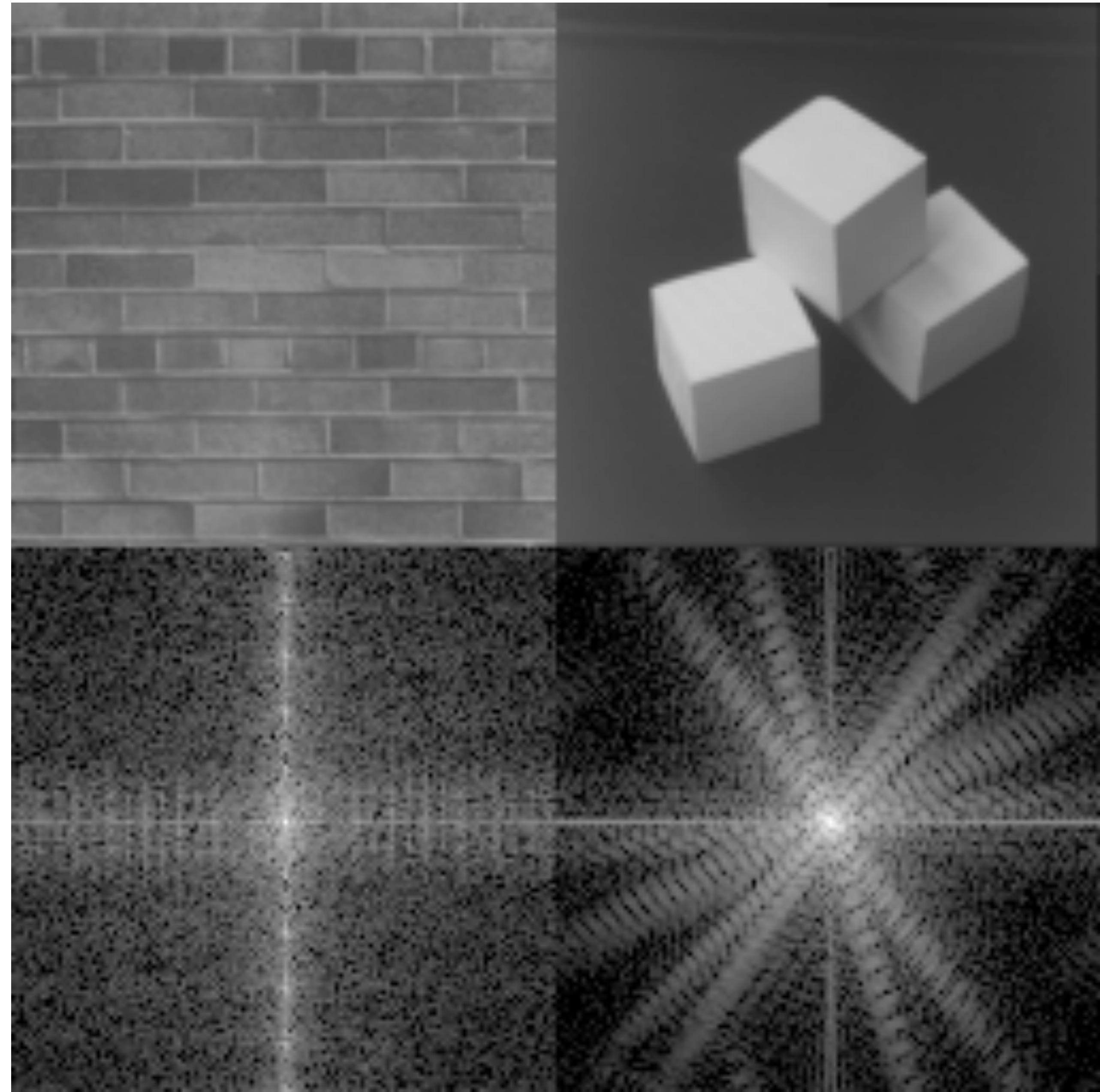
Slide by Alexei Efros

This is the
magnitude
transform of
the zebra
pic



This is the
phase
transform of
the zebra
pic



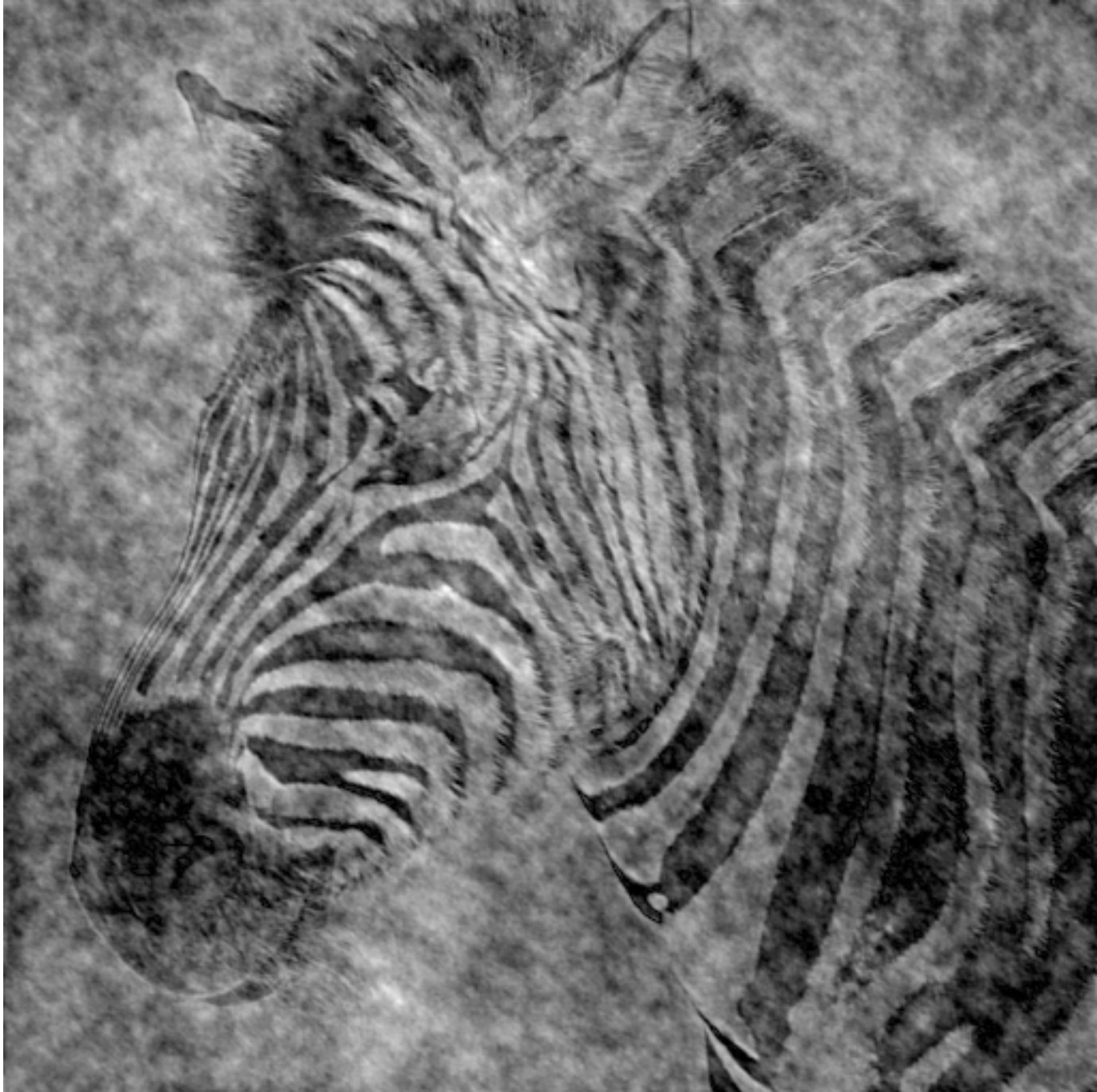


Oriented Components

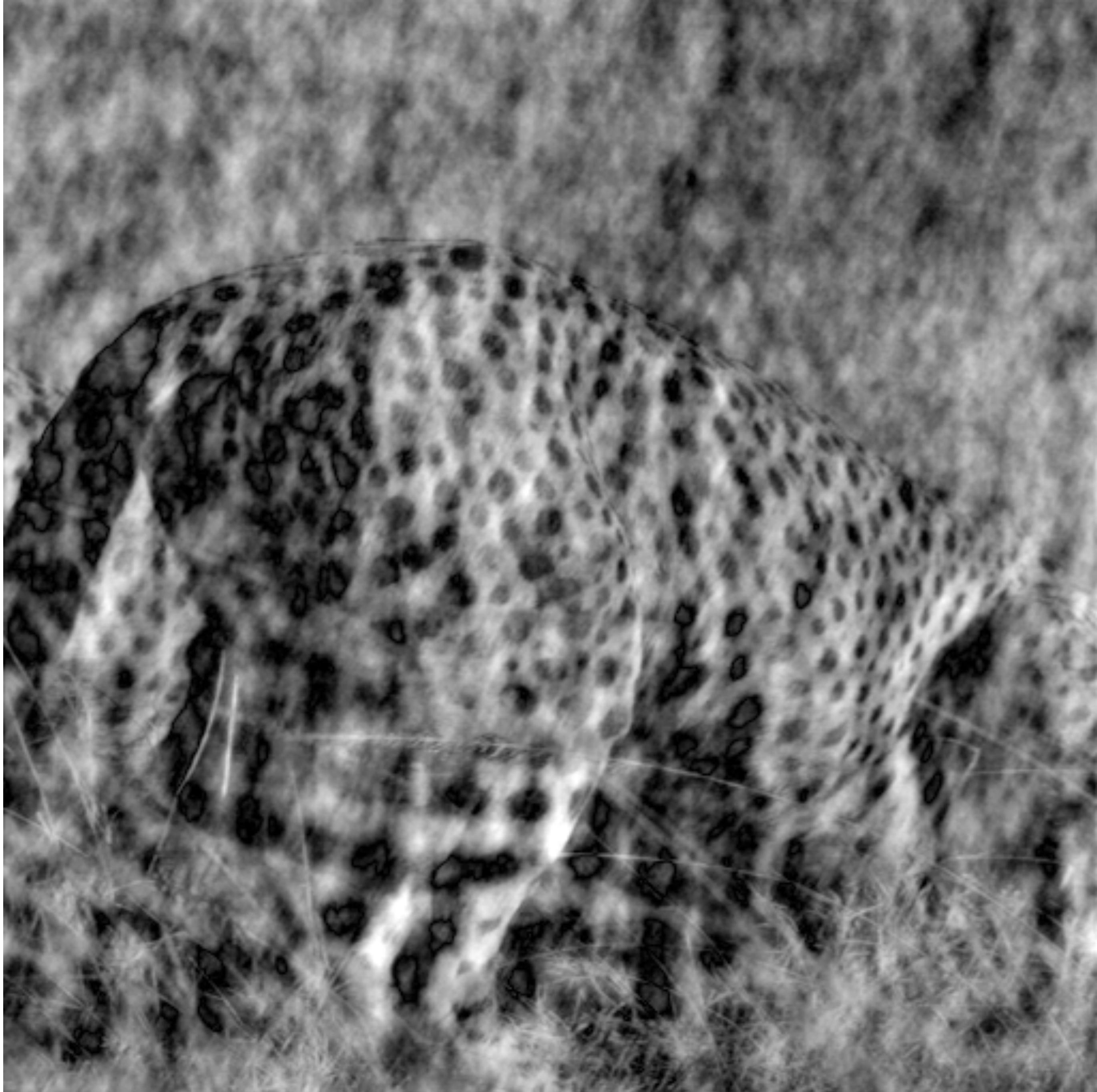
Curious things about FT on images

- The magnitude spectra of all natural images are quite similar
 - Heavy on low-frequencies,
falling off in high frequencies
- Most information in the image is carried in the phase, not the amplitude
 - Not quite clear why

Reconstruction with zebra phase,
cheetah magnitude



Reconstruction with cheetah phase,
zebra magnitude



Filtering in the Fourier Domain (why we care)

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- The Convolution Theorem states that

$$\mathfrak{F}[f * g] = \mathfrak{F}[f]\mathfrak{F}[g]$$

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- The Convolution Theorem states that

$$\mathfrak{F}[f * g] = \mathfrak{F}[f]\mathfrak{F}[g]$$

where \mathfrak{F} denotes the Fourier Transform

- To convolve image f by kernel g :
 - Compute DFTs F and G of f and g
 - Multiply F by G
 - Compute the inverse DFT of FG .

```

function g = dftfilt(f, H)
    %DFTFILT Performs frequency domain filtering.
    %
    % From DIPUM by Gonzalez, Woods, and Eddins 2004
    %
    % G = DFTFILT(F, H) filters F in the frequency domain using the
    % filter transfer function H. The output, G, is the filtered
    % image, which has the same size as F. DFTFILT automatically pads
    % H to be the same size as F. Function PADDEDSIZE can be used to
    % determine an appropriate size for H.
    %
    % DFTFILT assumes that F is real and that H is a real, uncentered
    % circularly-symmetric filter function.
    %
    % Copyright 2002-2004 R. C. Gonzalez, R. E. Woods, & S. L. Eddins
    % Digital Image Processing Using MATLAB, Prentice-Hall, 2004
    % $Revision: 1.5 $ $Date: 2003/08/25 14:28:22 $

    % Obtain the FFT of the padded input.
    F = fft2(f, size(H, 1), size(H, 2));

    % Perform filtering.
    g = real(ifft2(H.*F));

    % Crop to original size.
    g = g(1:size(f, 1), 1:size(f, 2));

```

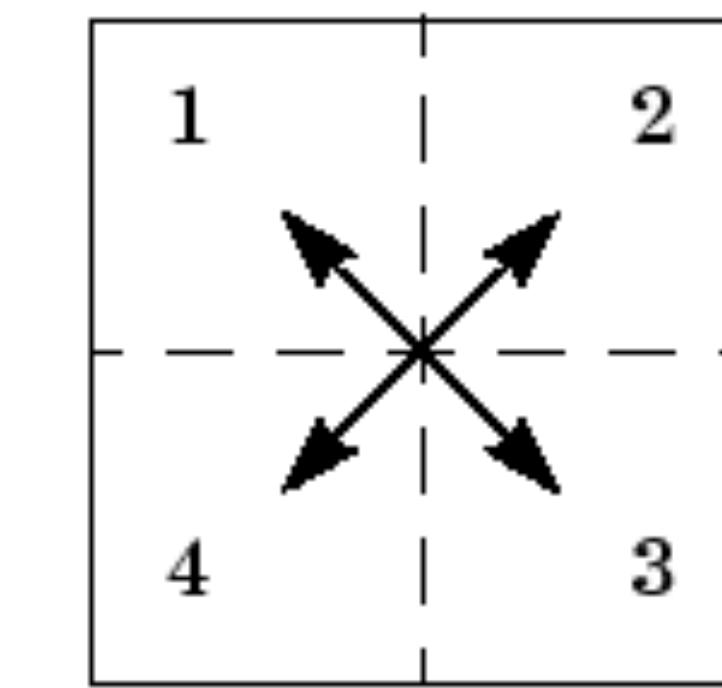
See also: `fftshift()` and `ifftshift()`

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```



(from Matlab Help)

See also: `fftshift()` and `ifftshift()`

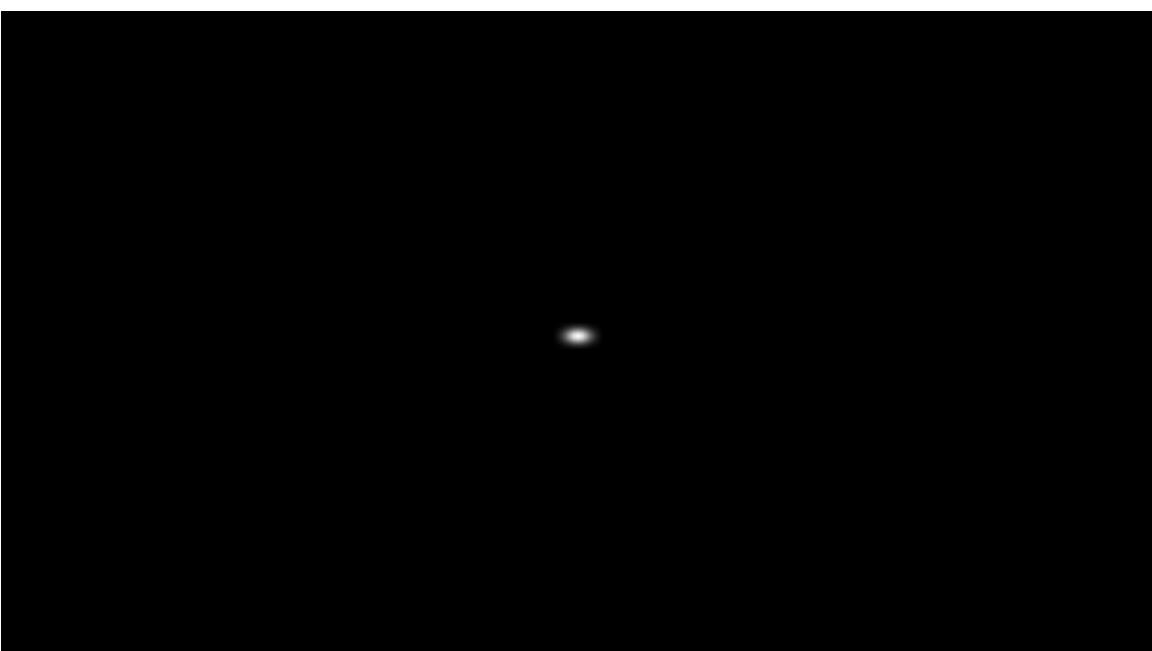
Example: 2D Convolution Theorem

$f(x,y)$



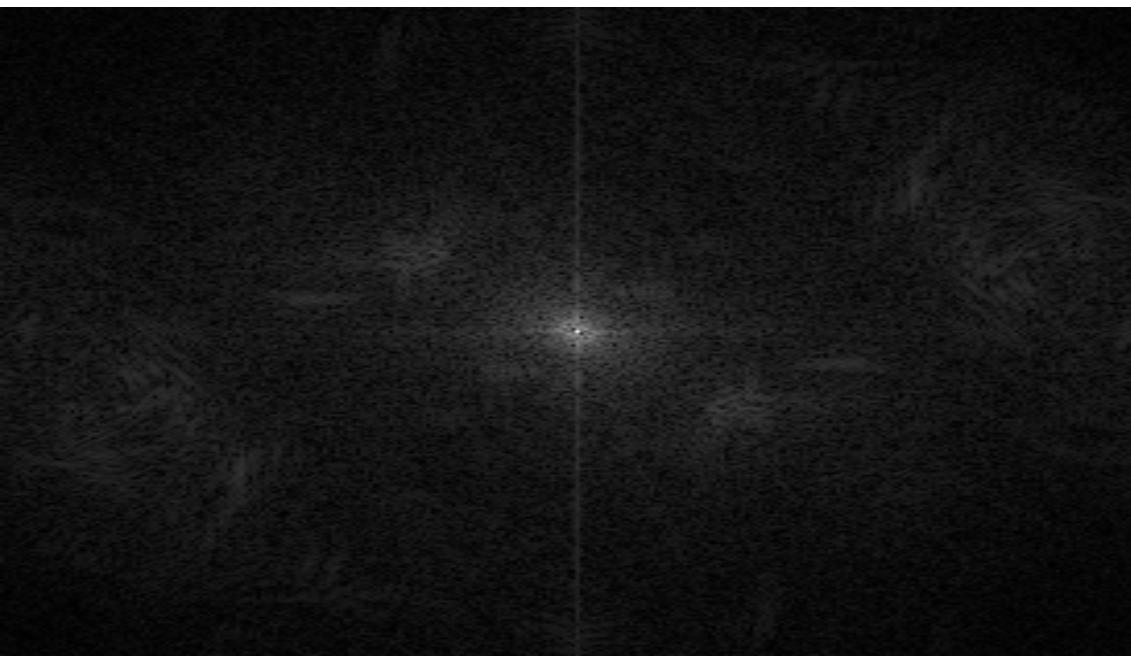
*

$g(x,y)$



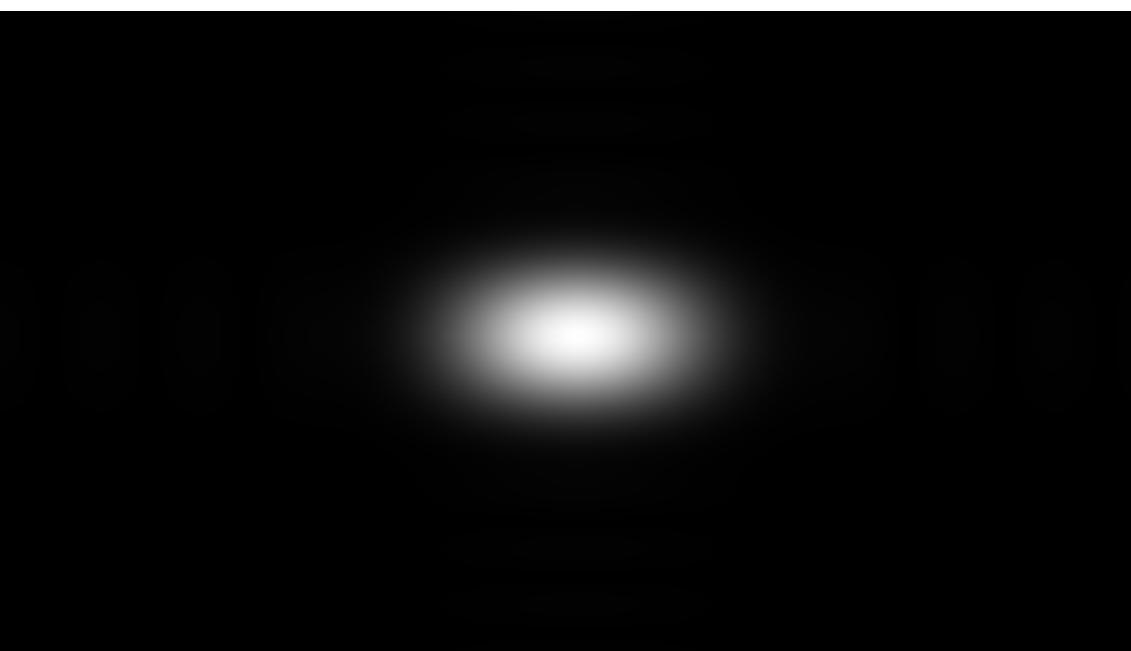
↓

$h(x,y)$



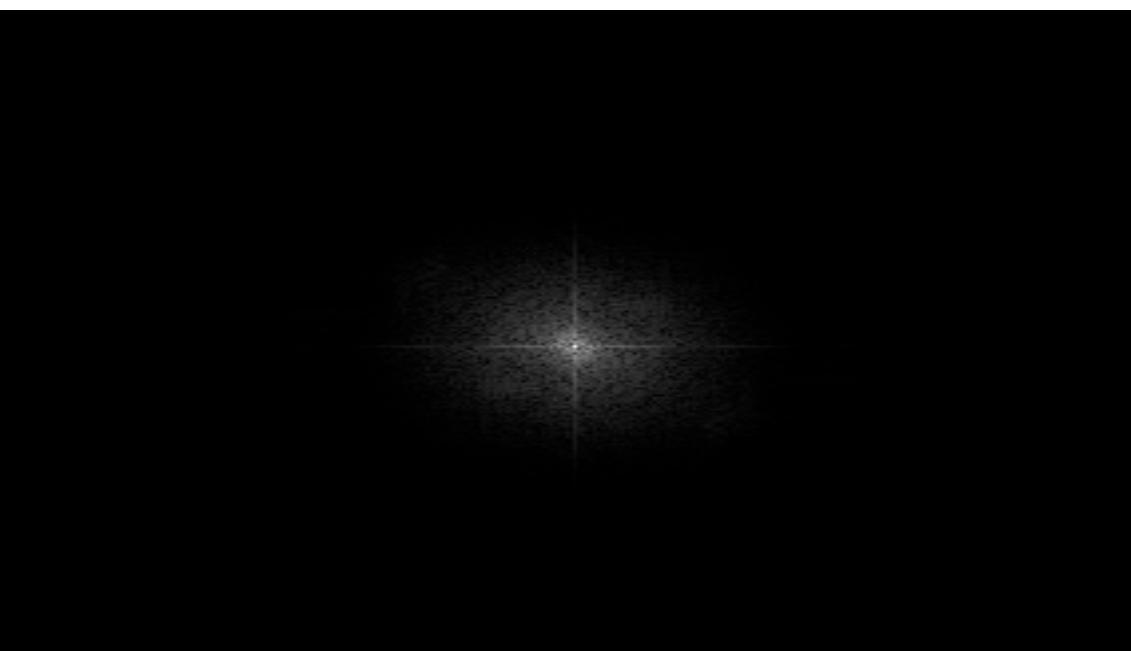
×

$|F(s_x, s_y)|$



↓

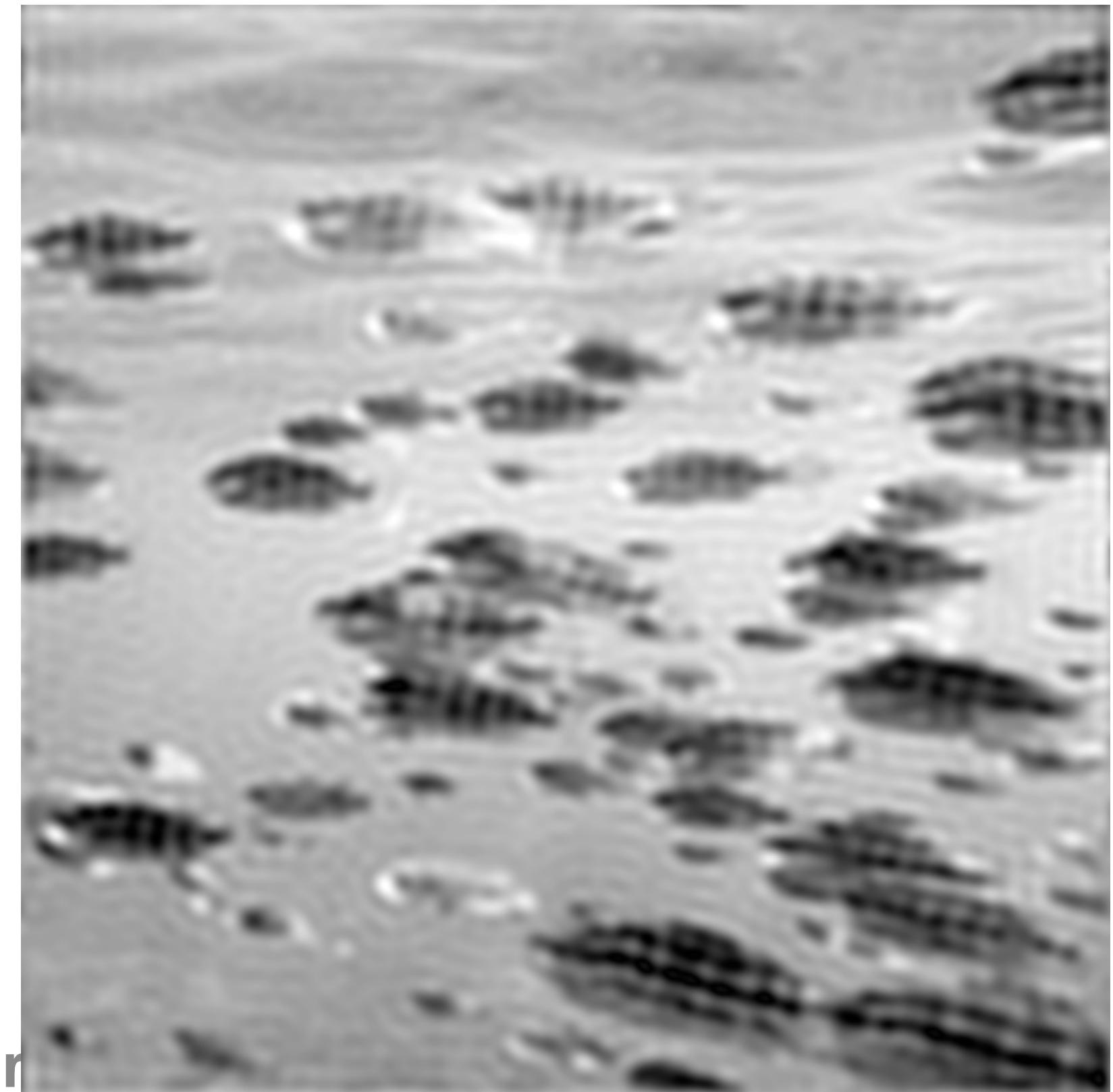
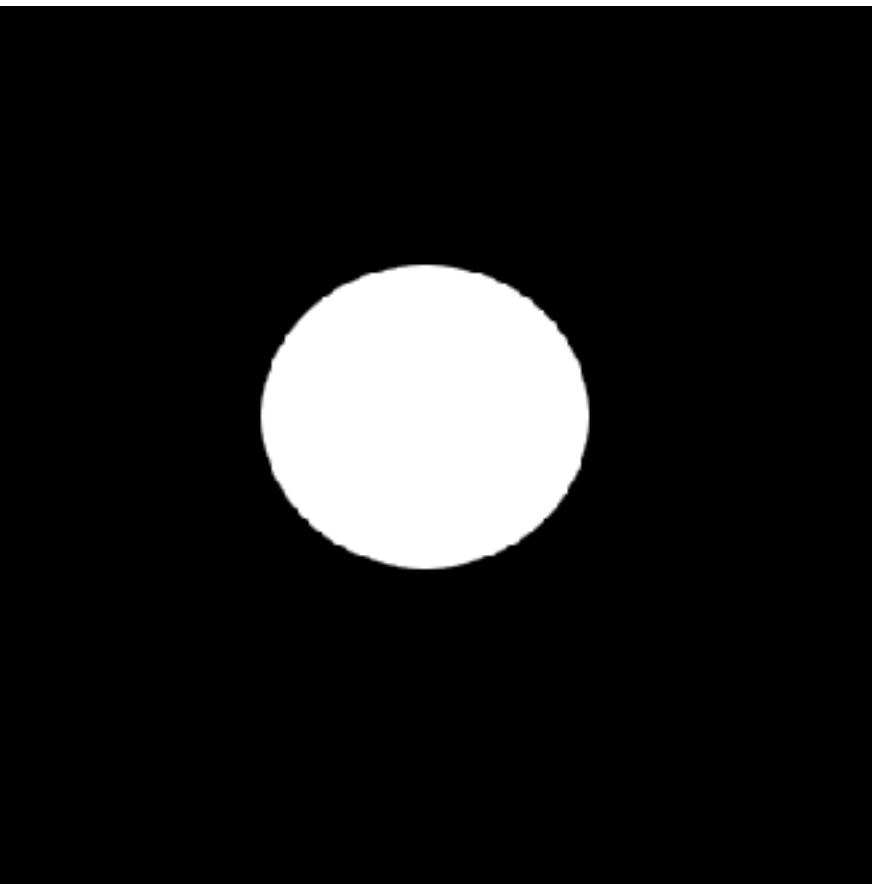
$|G(s_x, s_y)|$



$|H(s_x, s_y)|$

Ideal Low-Pass Filter

$$K(u, v) = \begin{cases} 0 & u^2 + v^2 > r^2 \\ 1 & \text{otherwise} \end{cases}$$



Ideal Low-Pass

$$K(u, v) = \begin{cases} 0 & u^2 + v^2 \\ 1 & \text{otherwise} \end{cases}$$

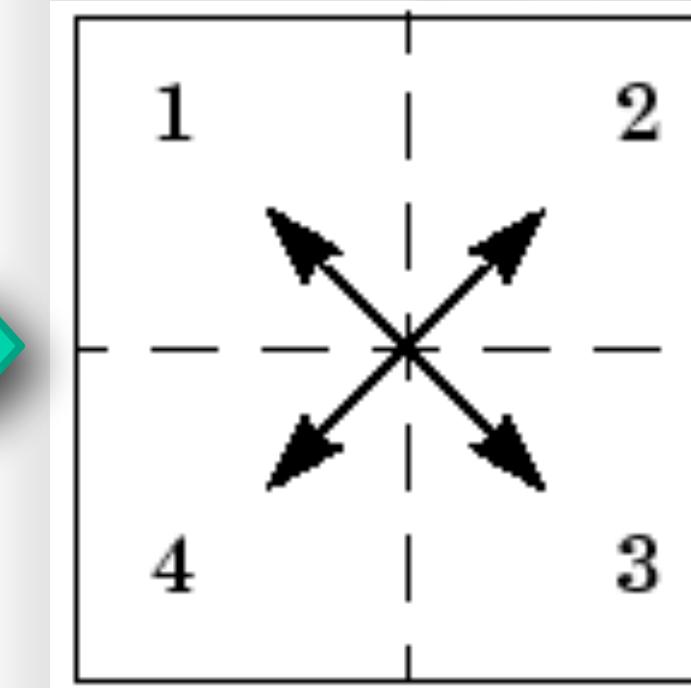
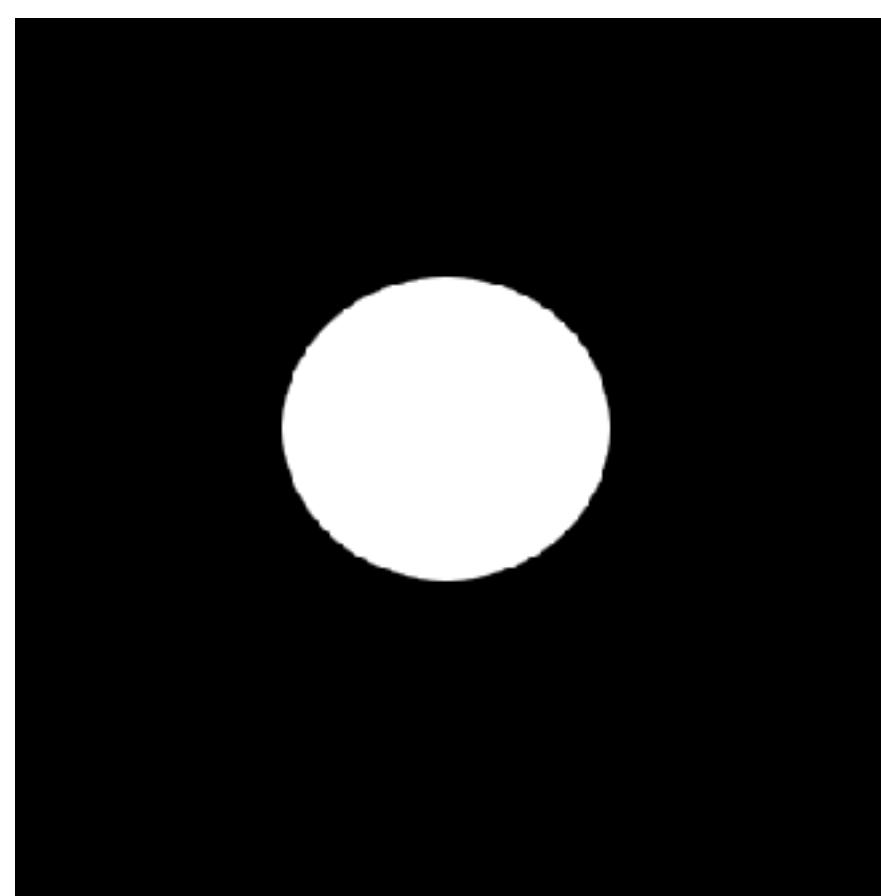
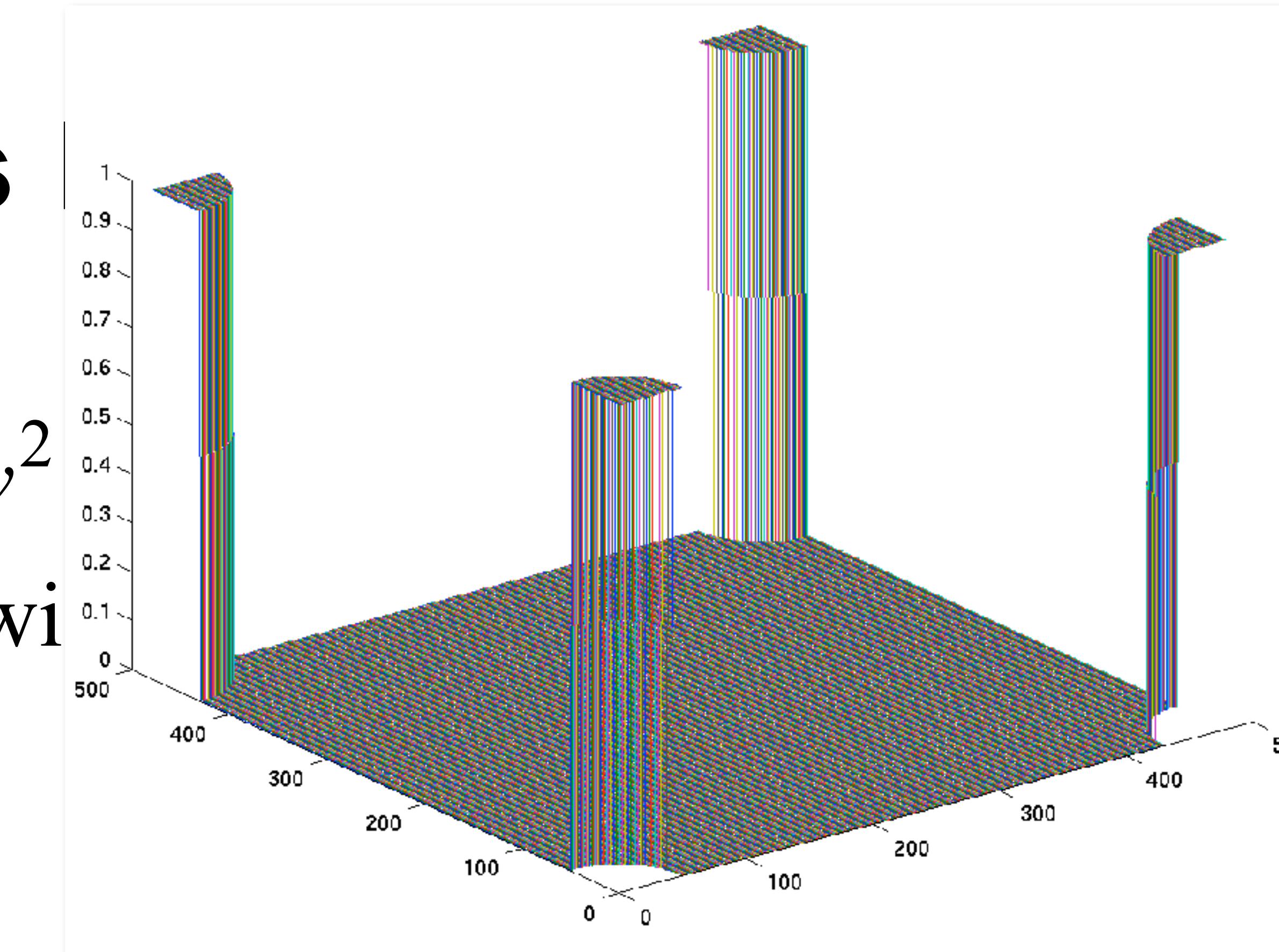
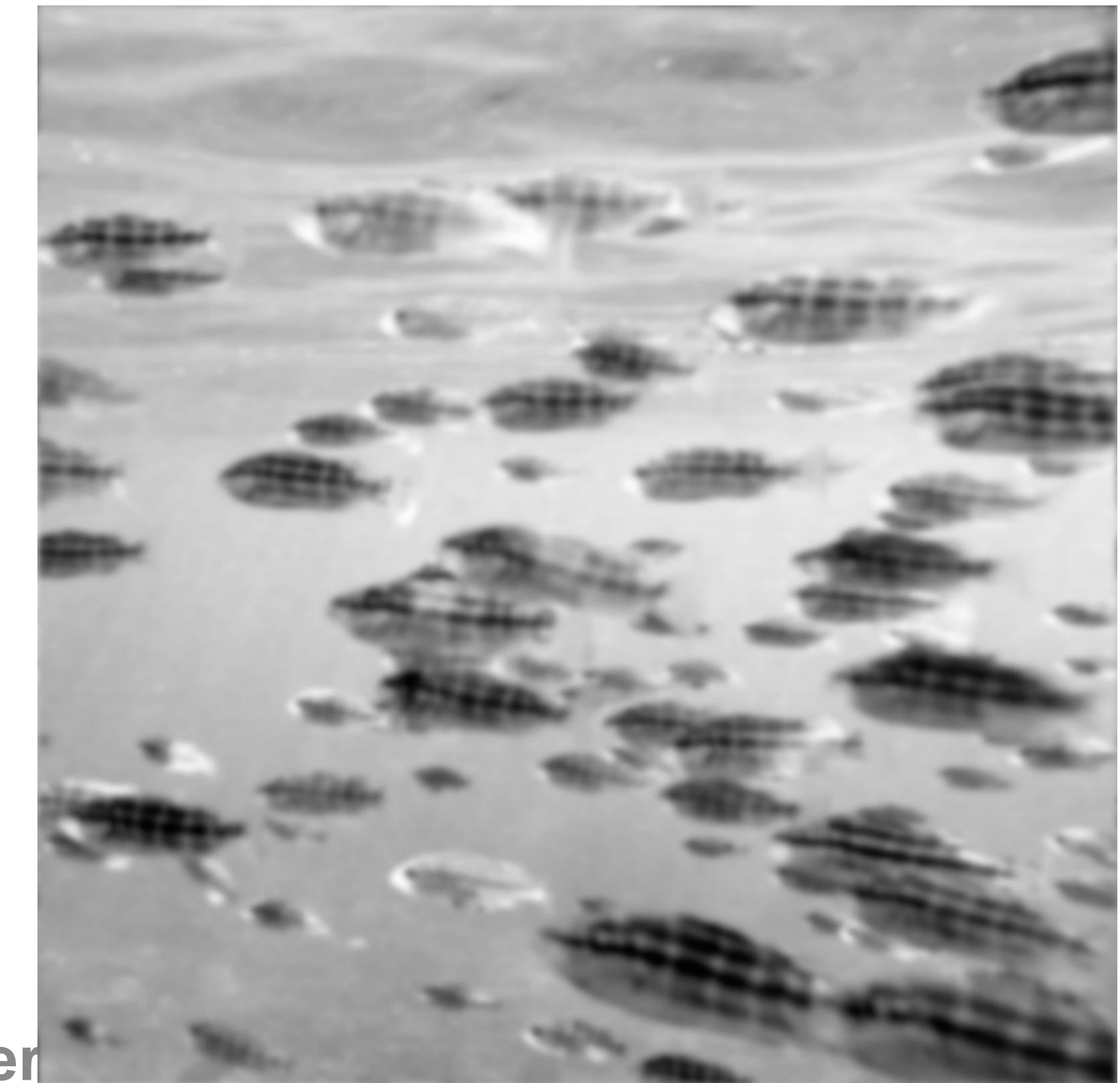
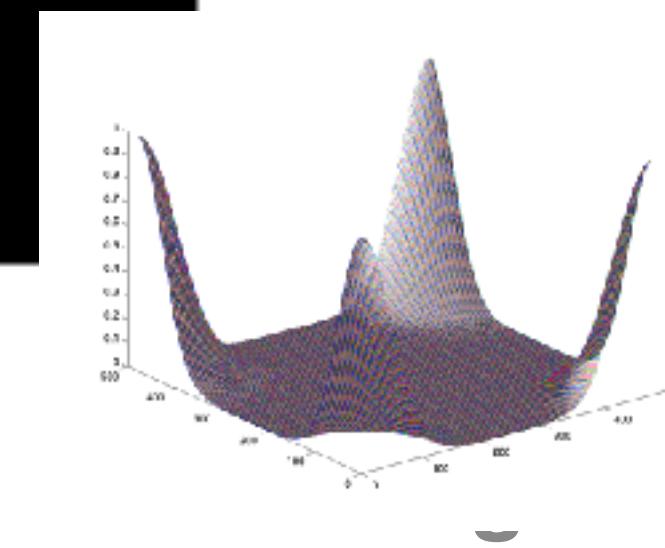
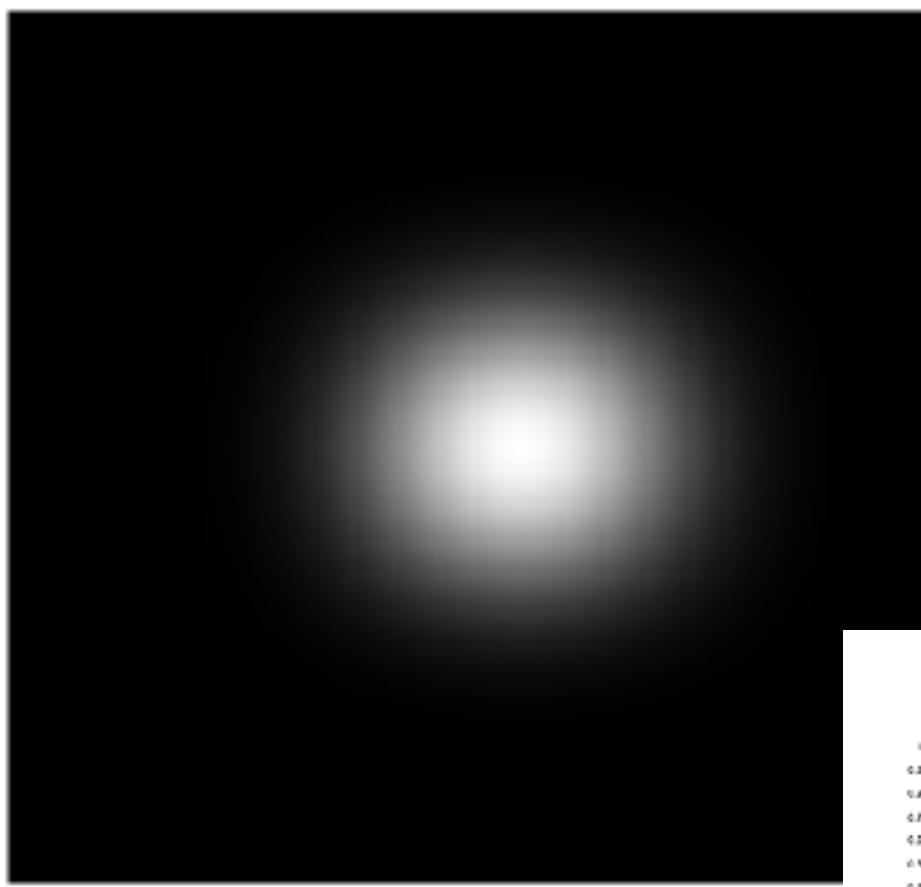


Image Filter



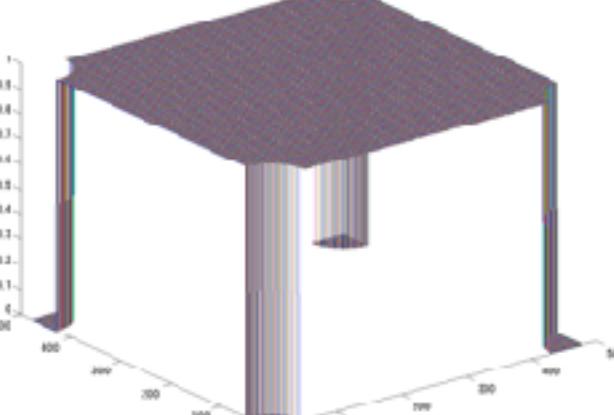
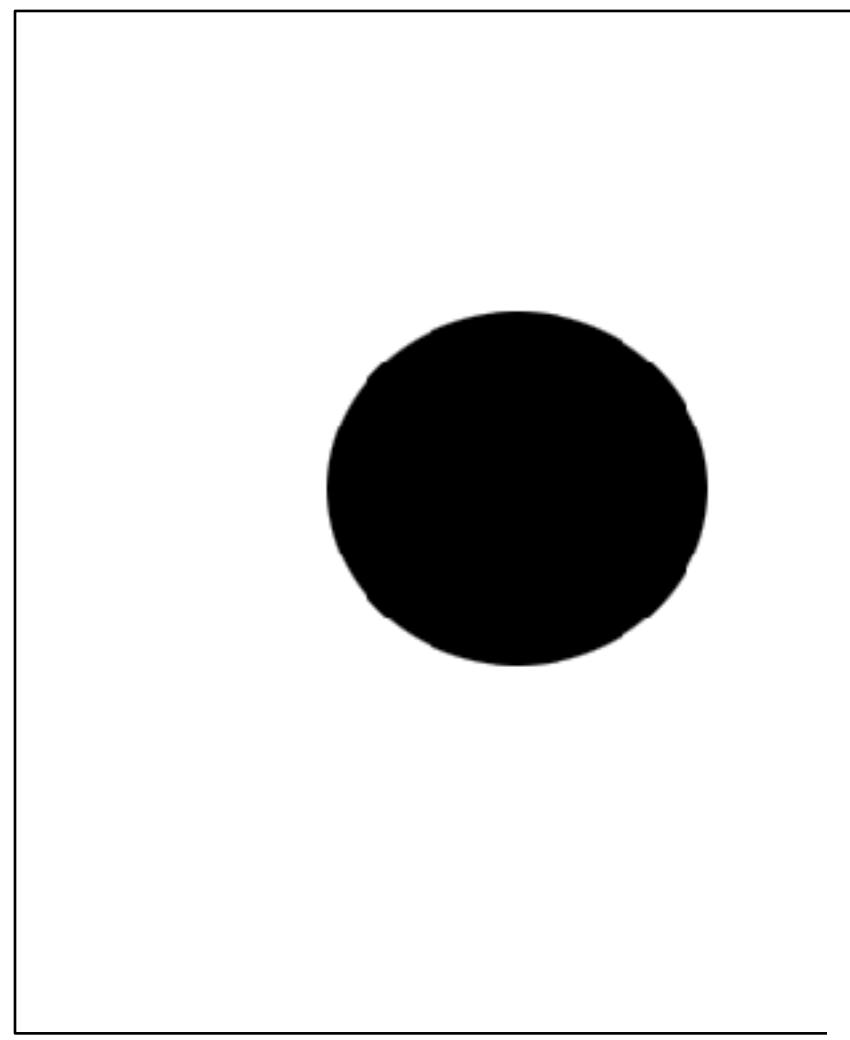
Gaussian Low-Pass Filter

$$K(u, v) = \exp\left(-\frac{u^2 + v^2}{2\sigma^2}\right)$$



Ideal High-Pass Filter

$$K(u, v) = \begin{cases} 0 & u^2 + v^2 < r^2 \\ 1 & \text{otherwise} \end{cases}$$

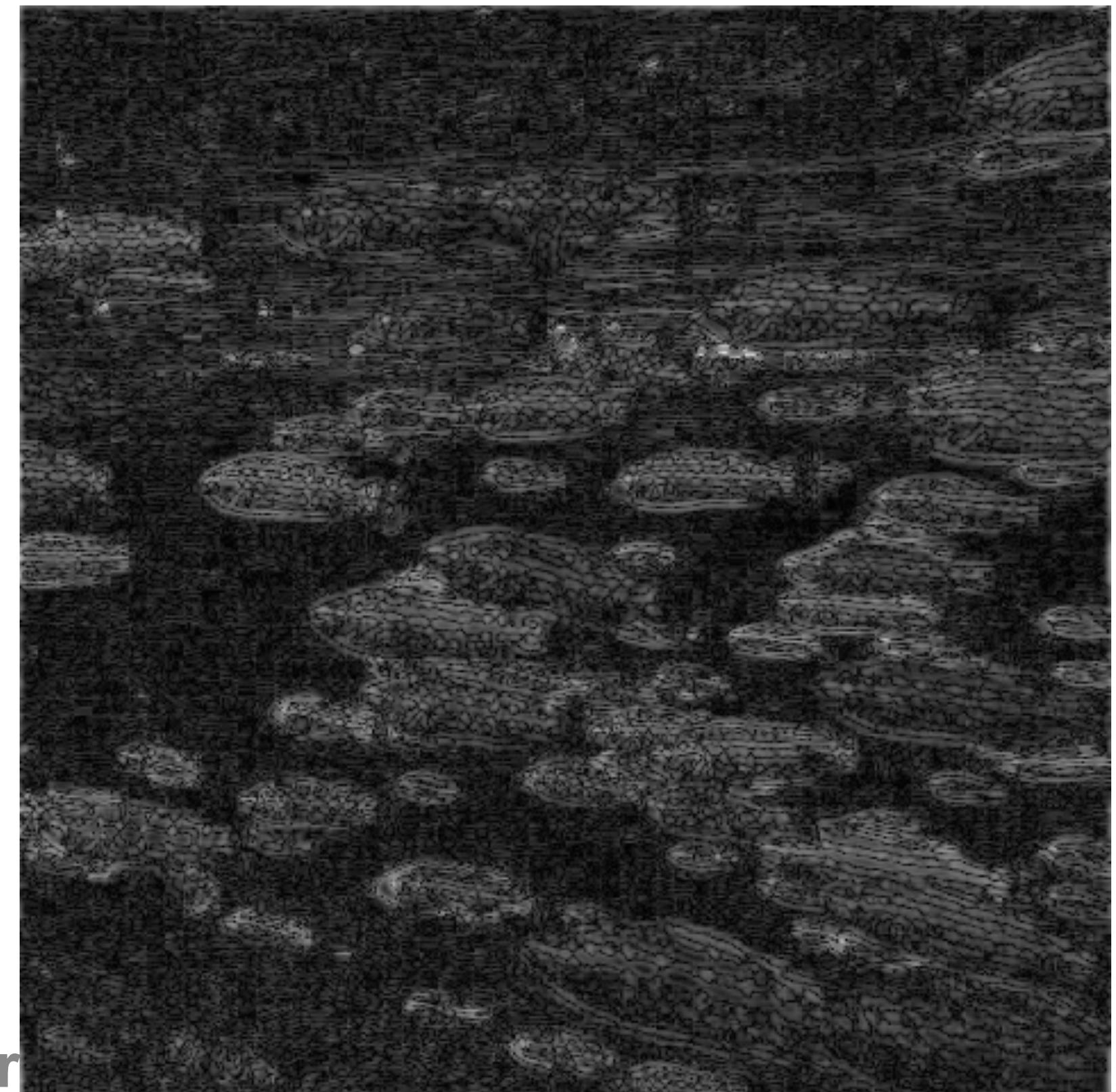
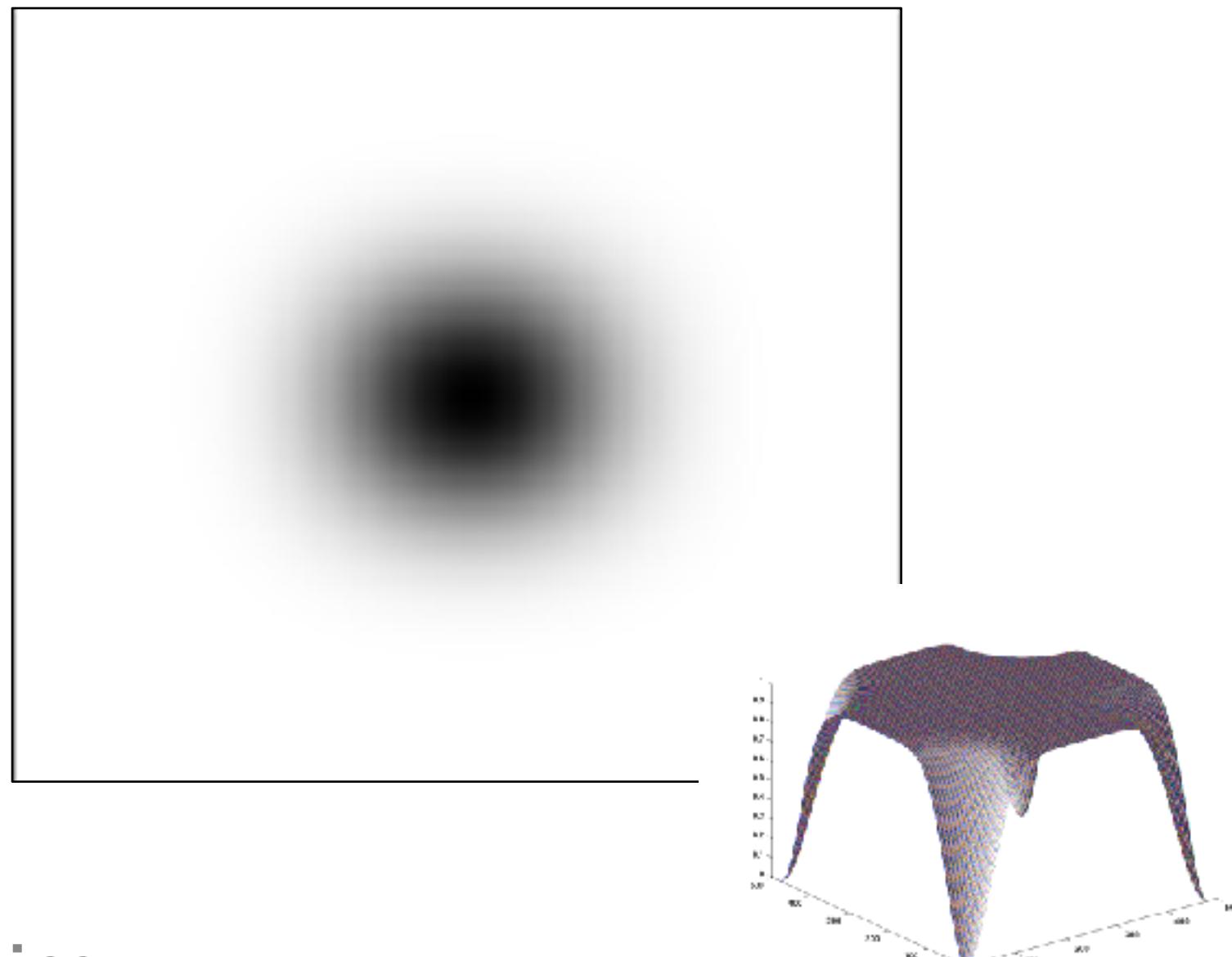


en



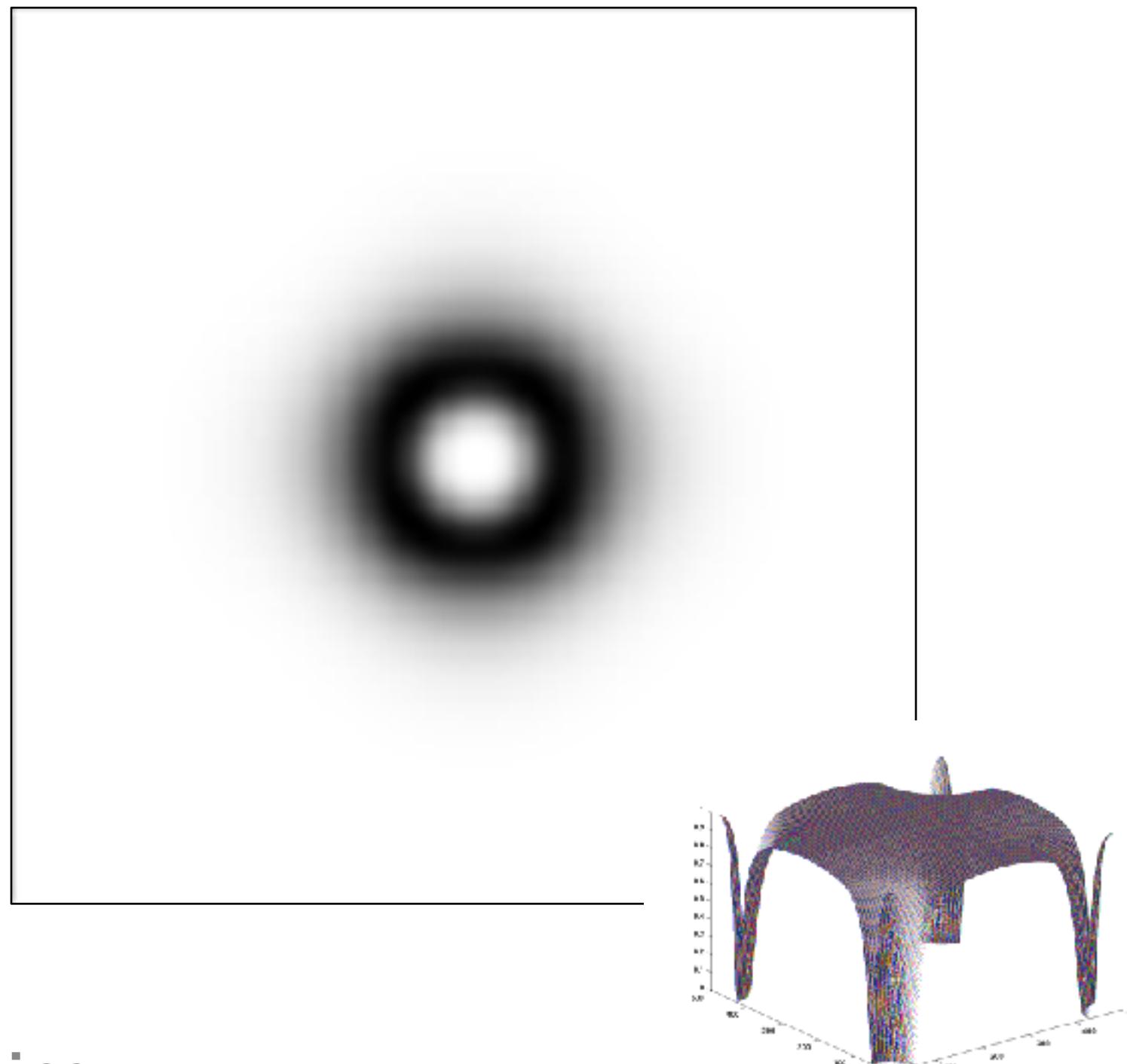
Gaussian High-Pass Filter

$$K(u, v) = 1 - \exp\left(-\frac{u^2 + v^2}{2\sigma^2}\right)$$



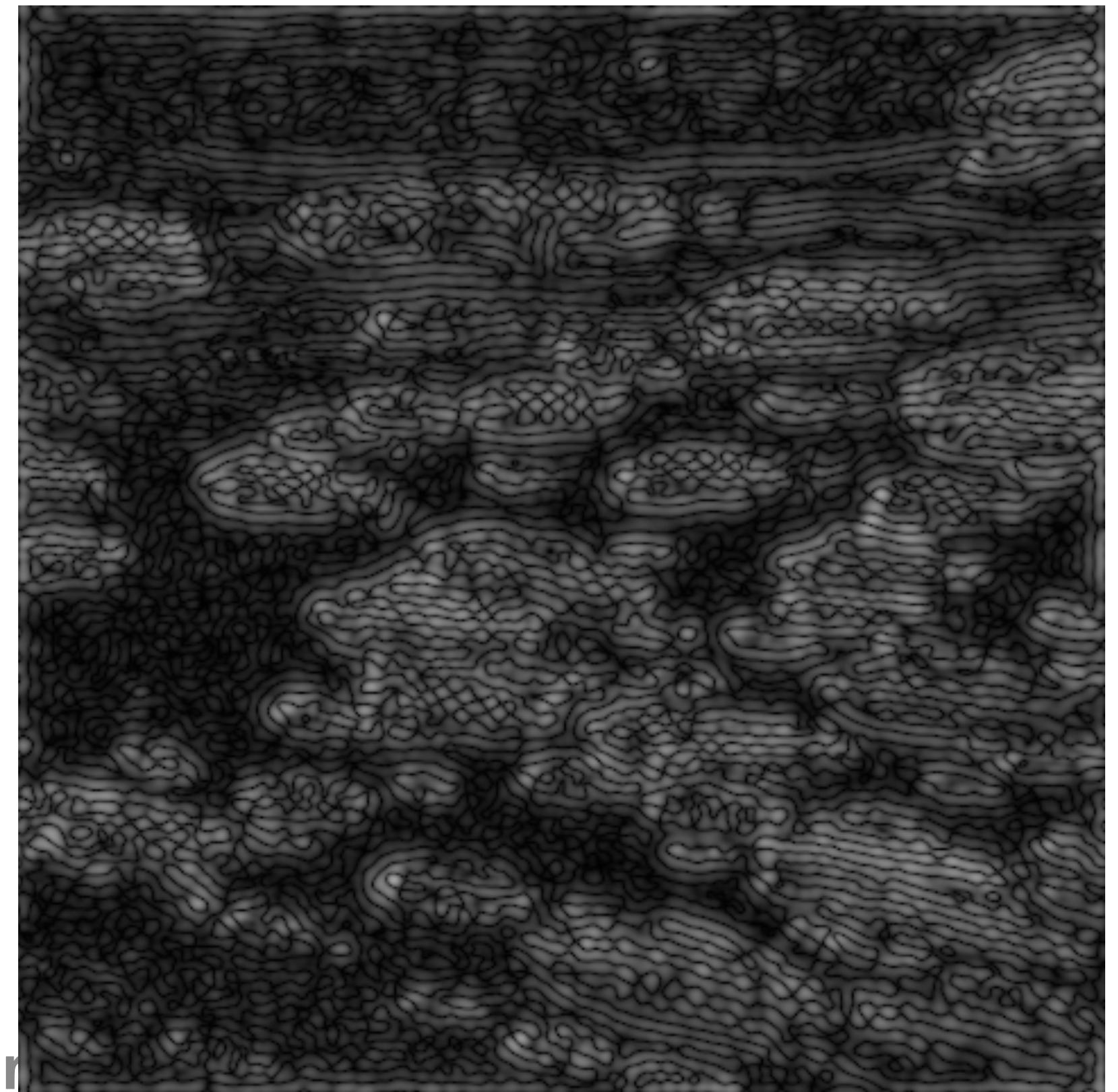
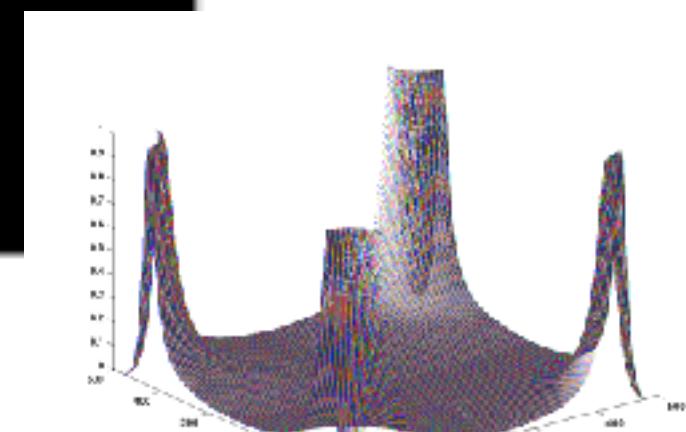
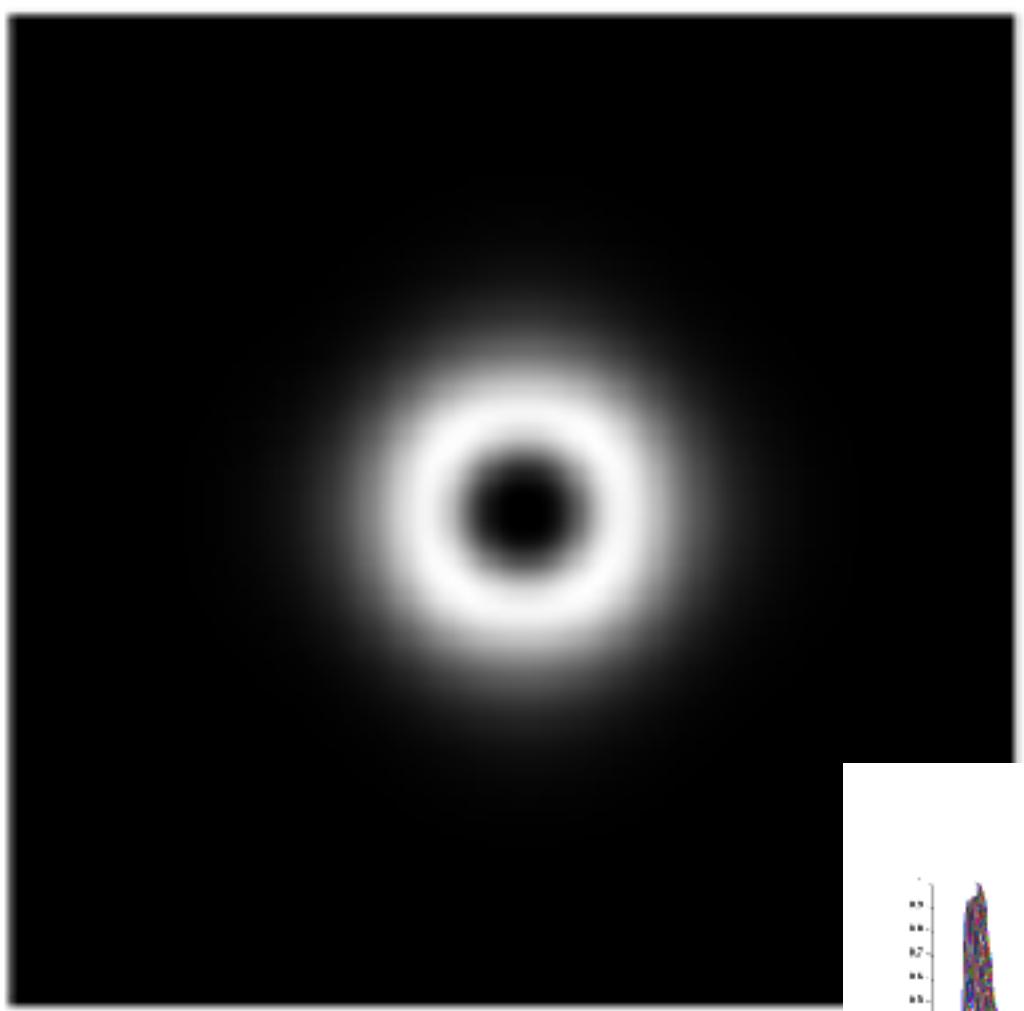
Band-Stop Filter

$$K(u, v) = \frac{1}{1 + (\Omega r / \sqrt{u^2 + v^2 - r_0^2})^{2n}}$$

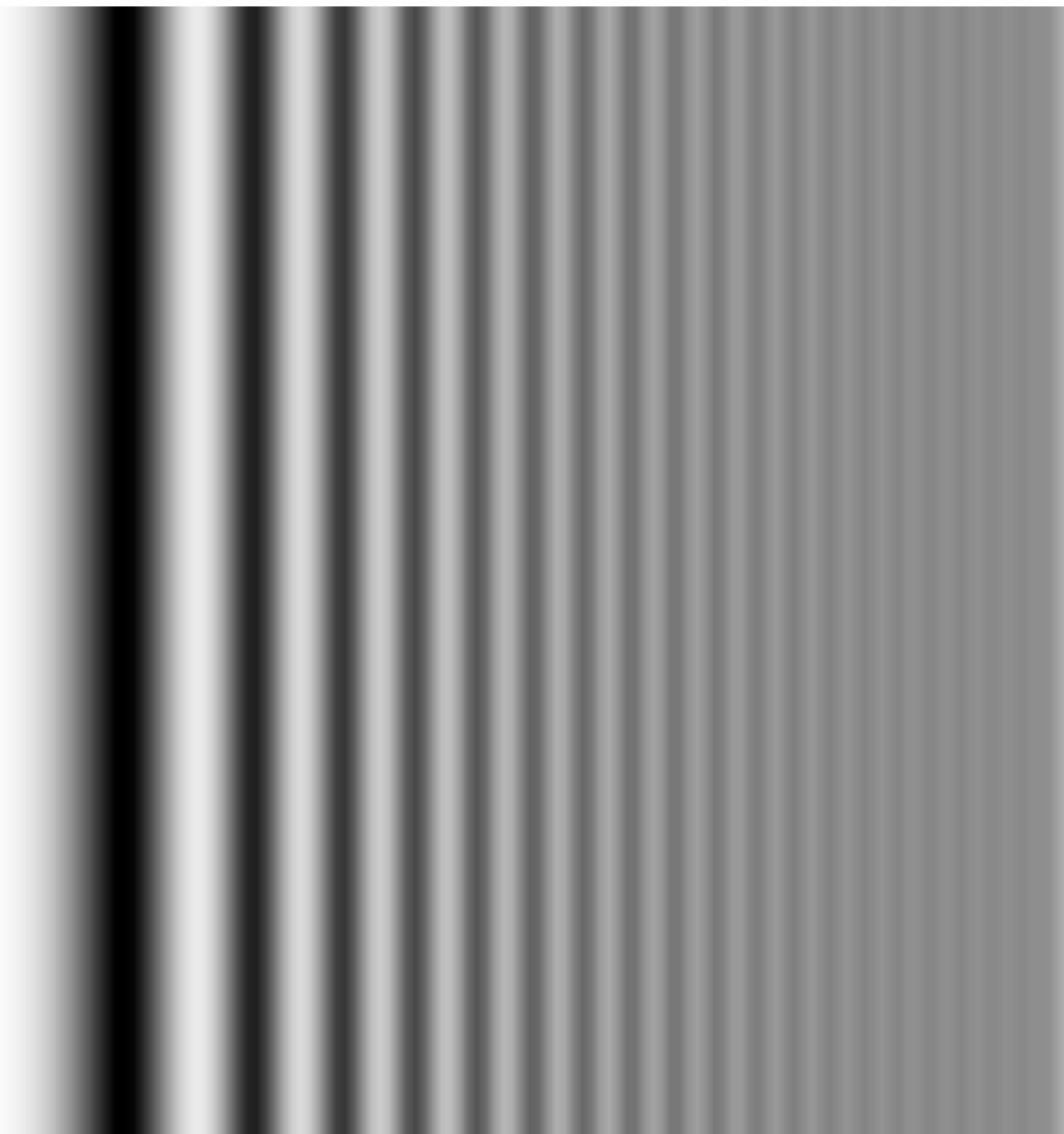


Band-Pass Filter

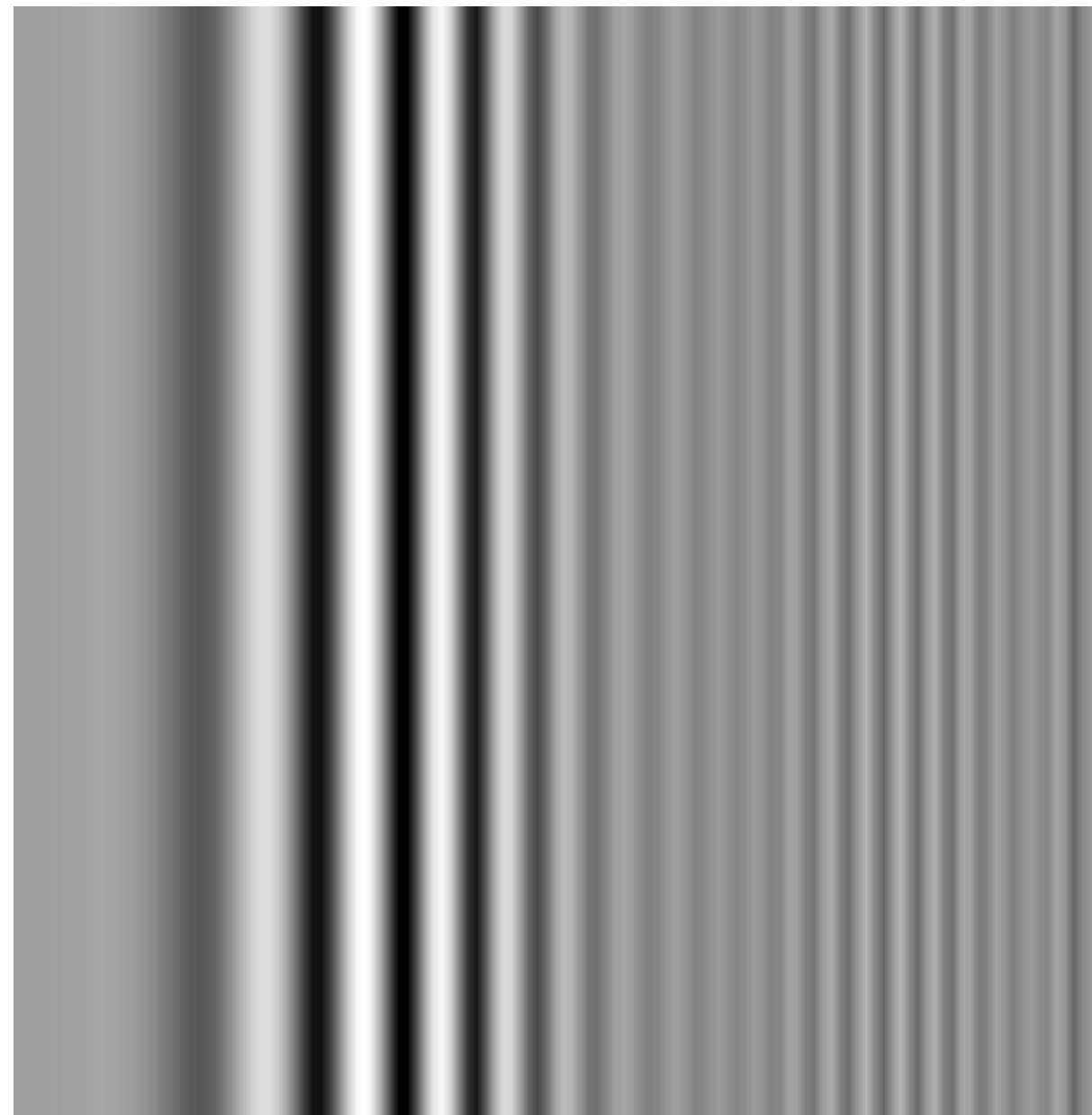
$$K(u, v) = 1 - \frac{1}{1 + (\Omega r / \sqrt{u^2 + v^2 - r_0^2})^{2n}}$$



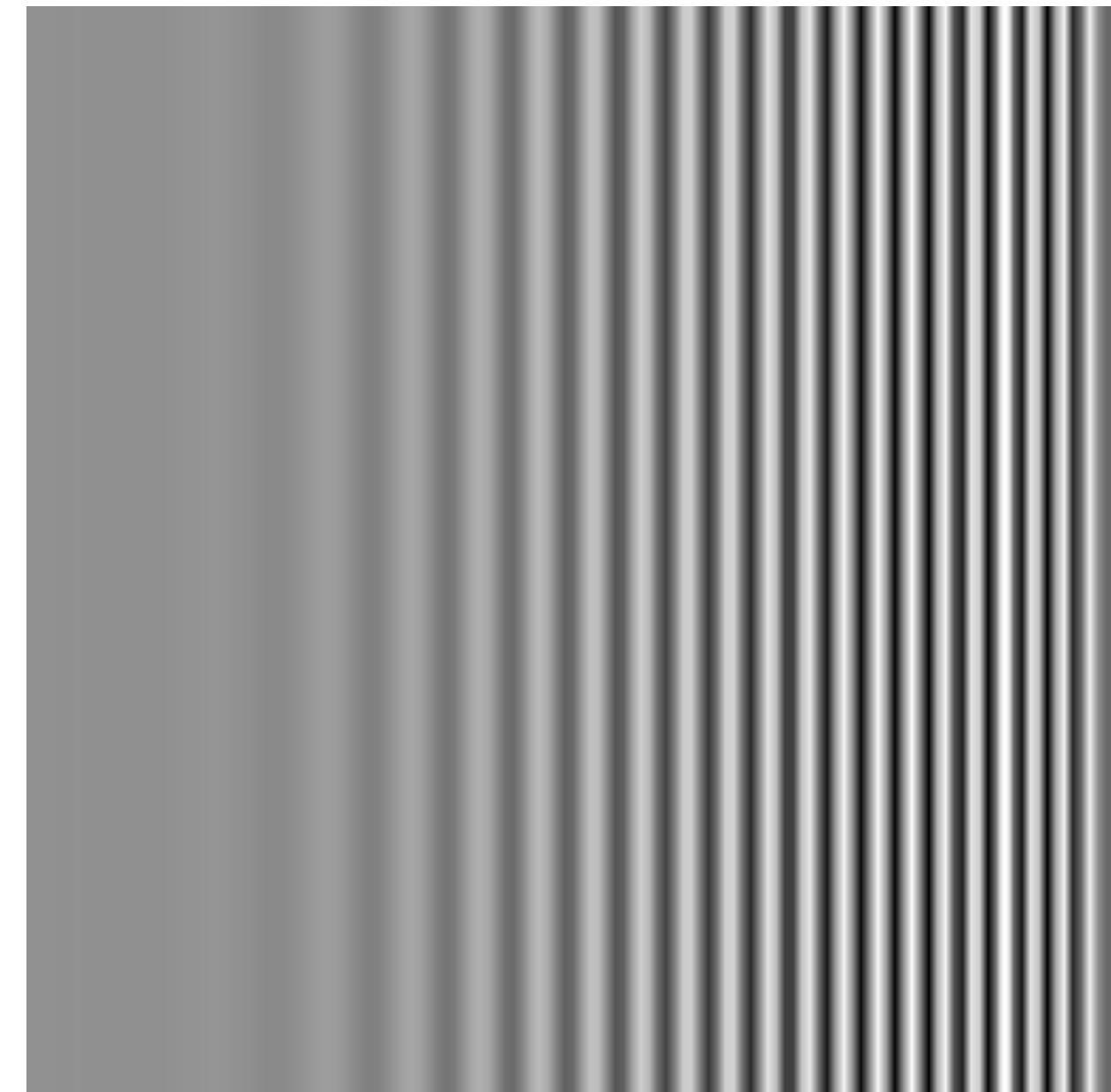
X-pass Filter



(a) Lowpass filter



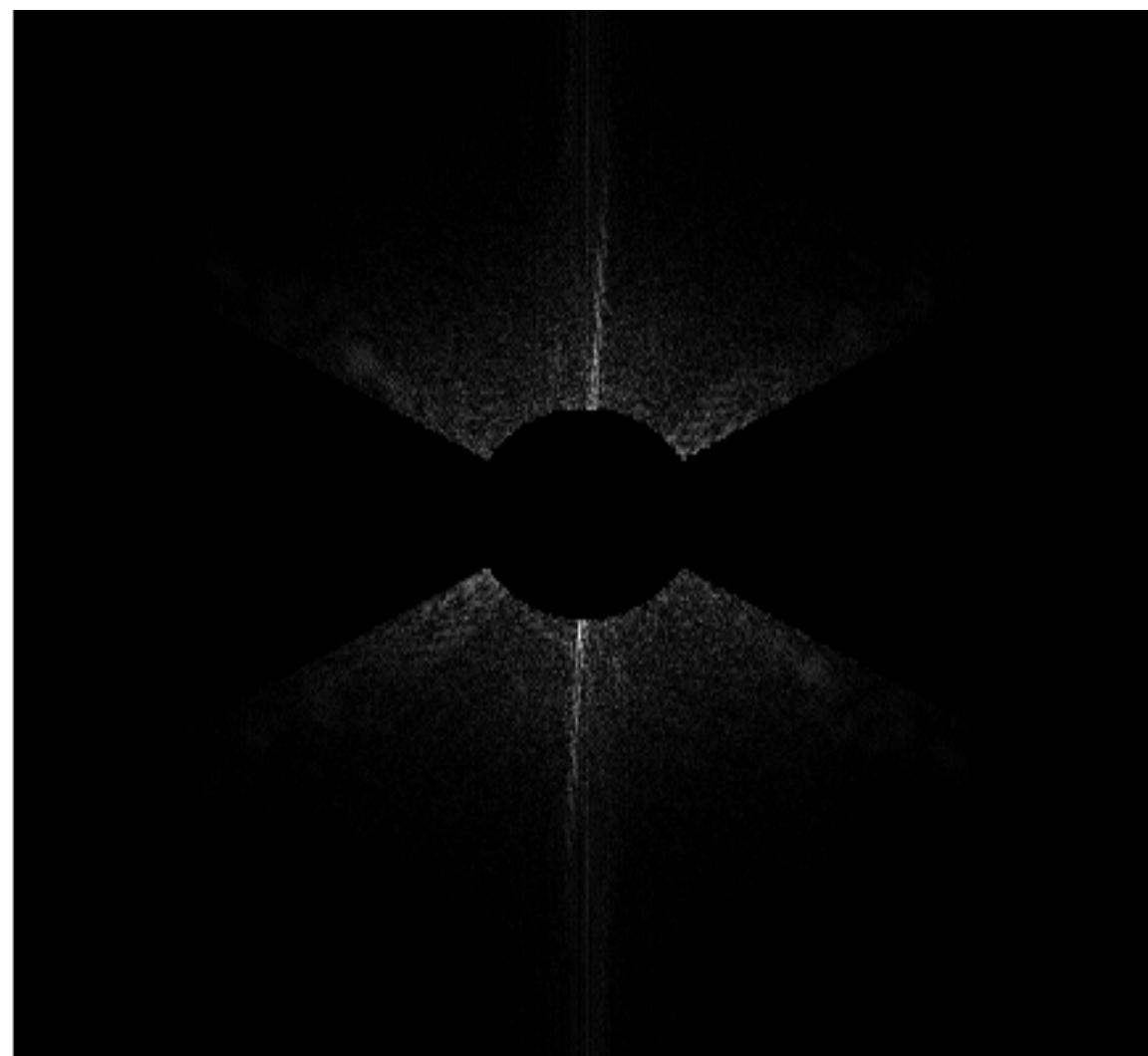
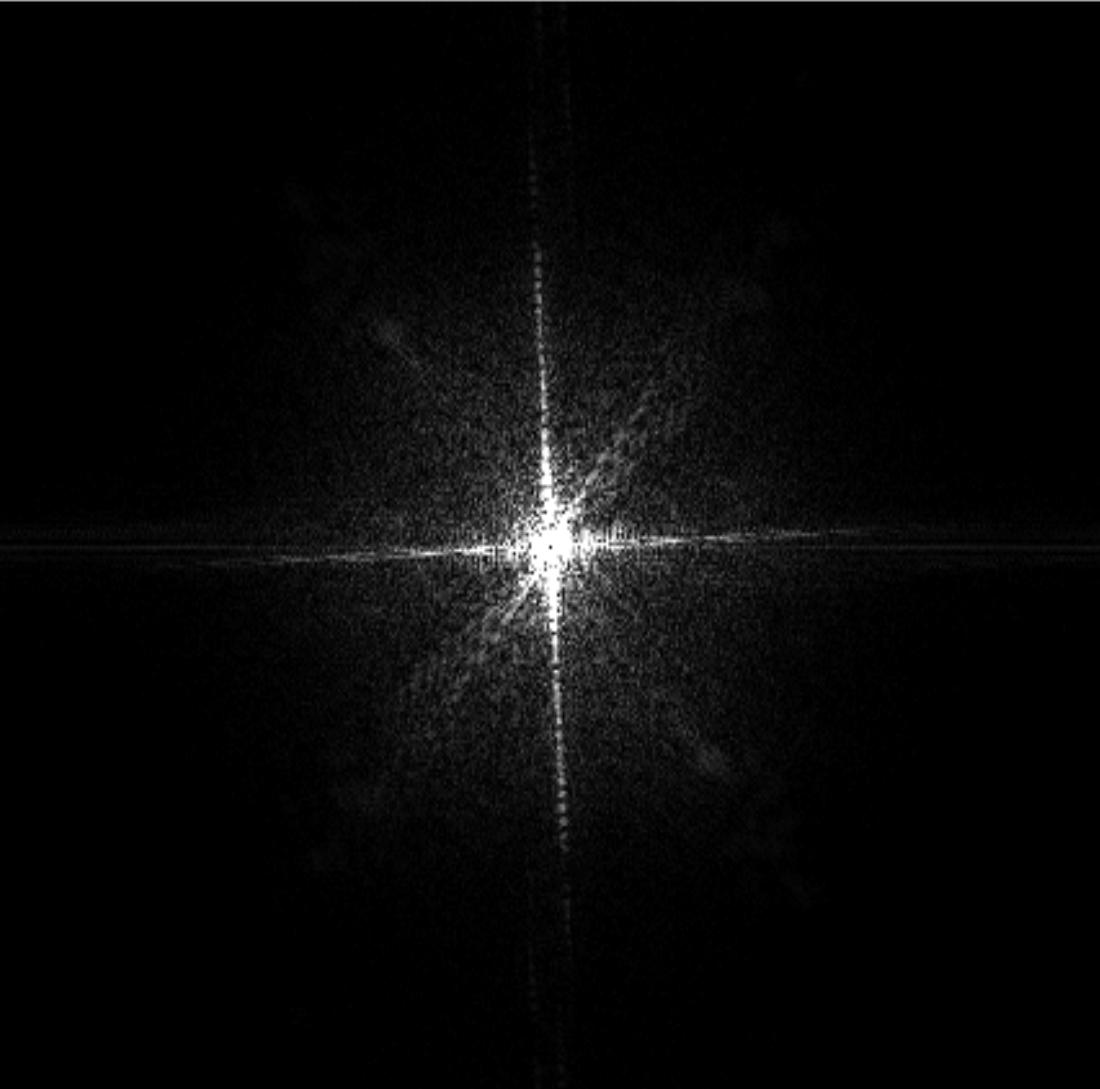
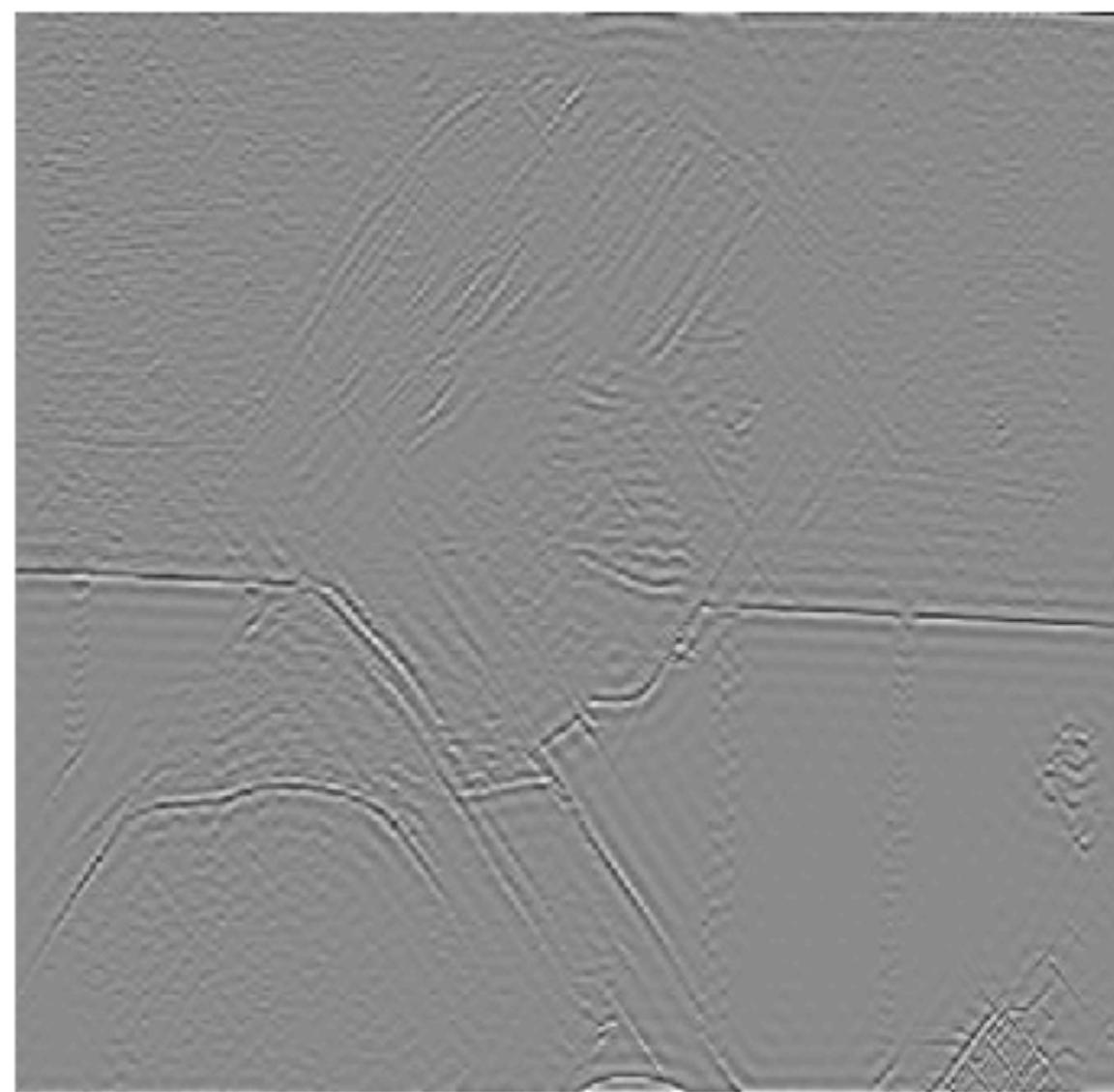
(b) Bandpass filter

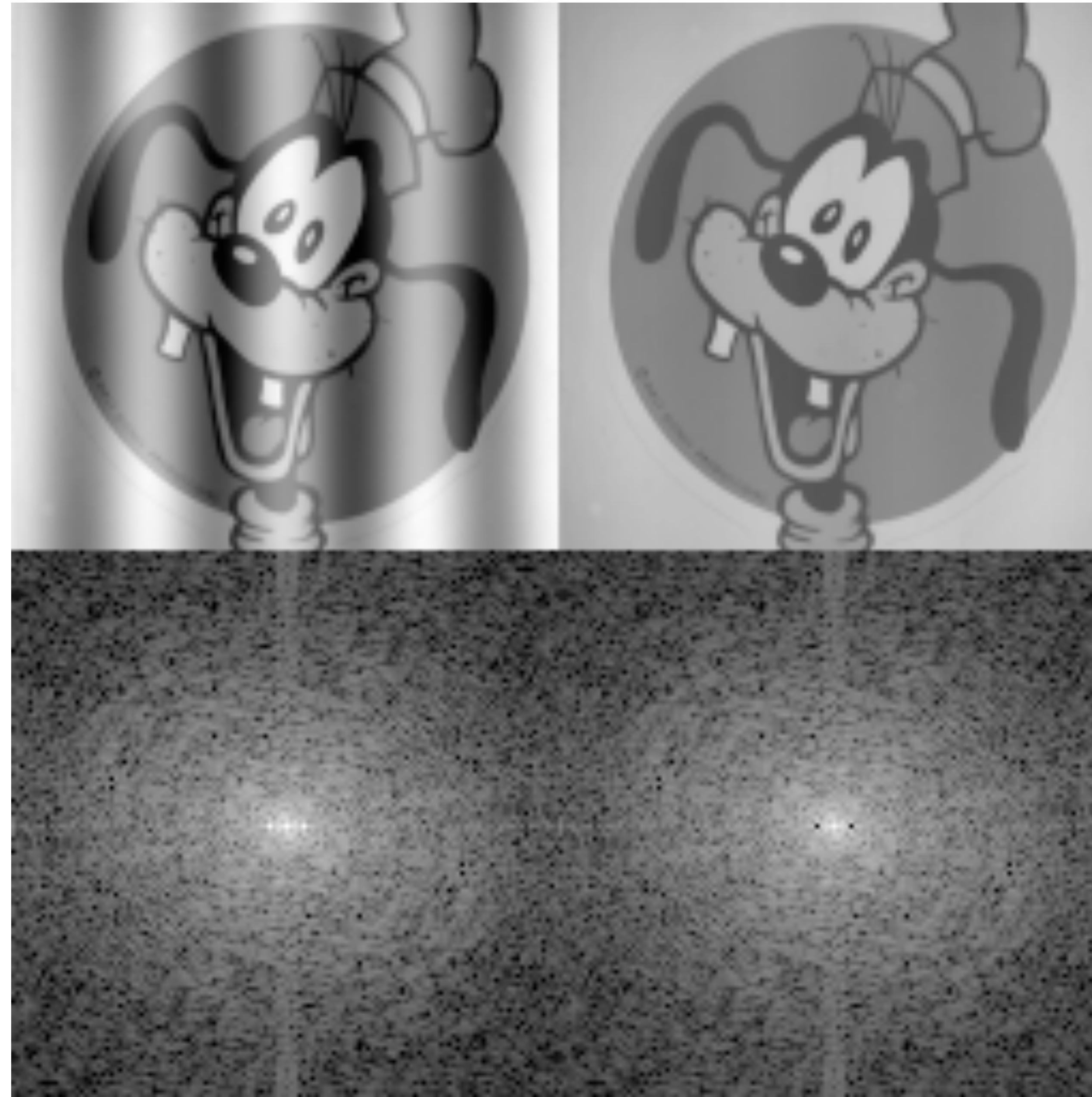


(c) Highpass filter

From Young, Gerbrands and van Vliet;
CVOnline.

Orientation Filtering





Removing Unwanted Periodic Component

Rank Filters

- Simple non-linear filters based on the ordering of the grey levels in the neighbourhood.

Max Filter

$$I'(m, n) = \max\{I(i, j) : (i, j) \in N(m, n)\}$$



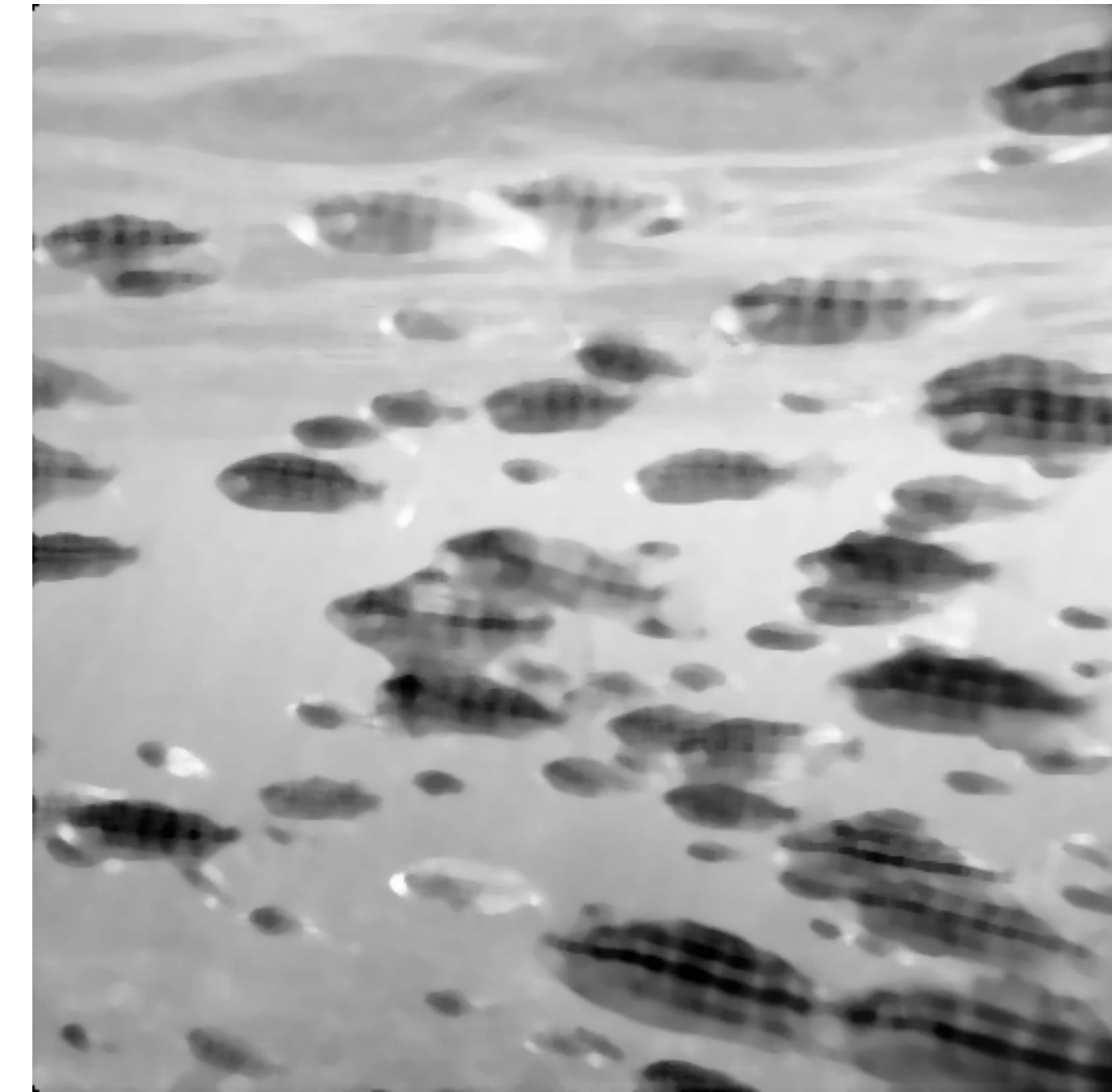
Min Filter

$$I'(m, n) = \min\{I(i, j) : (i, j) \in N(m, n)\}$$



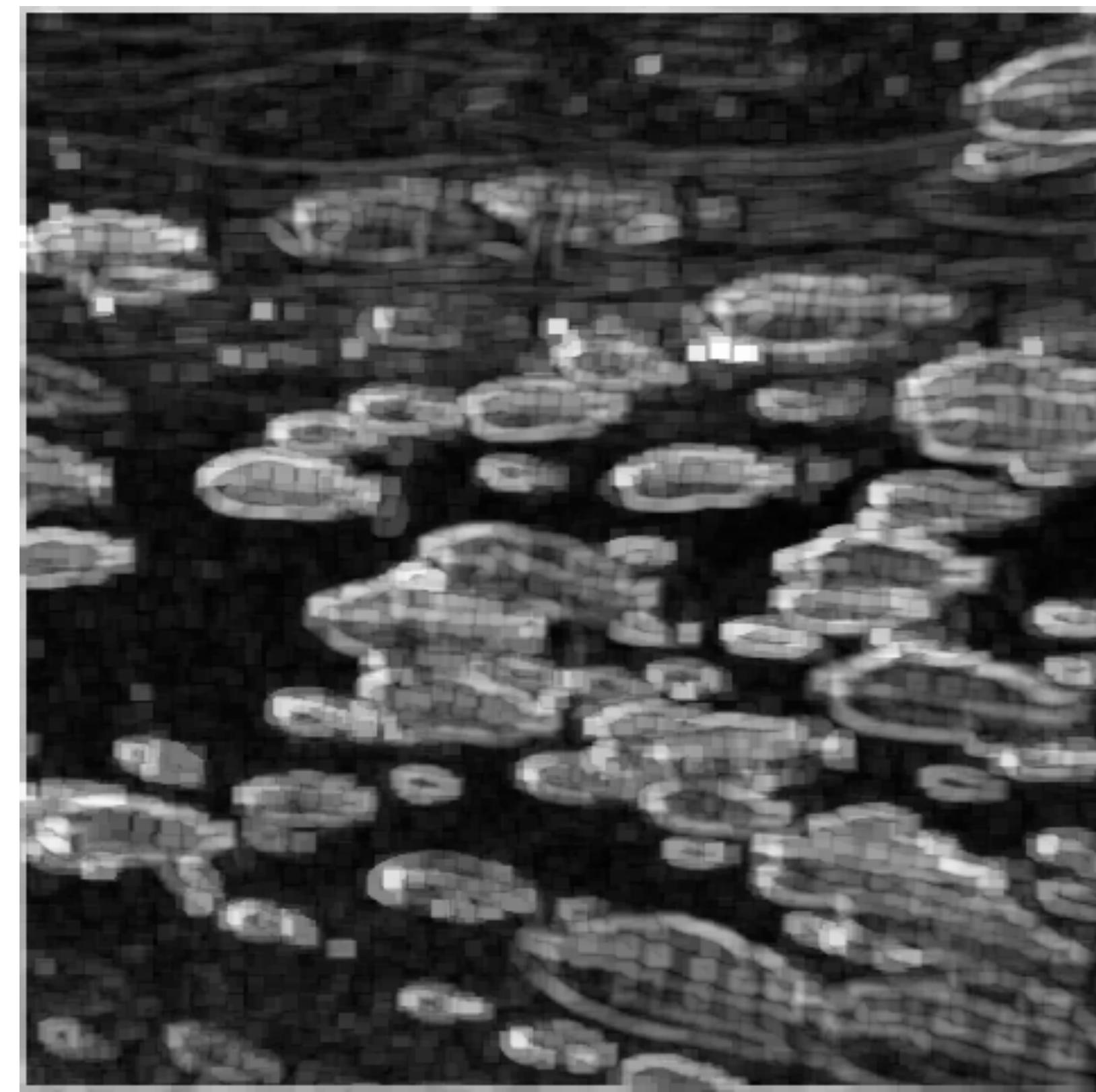
Median Filter

$$I'(m, n) = \text{median}\{I(i, j) : (i, j) \in N(m, n)\}$$



Range Filter

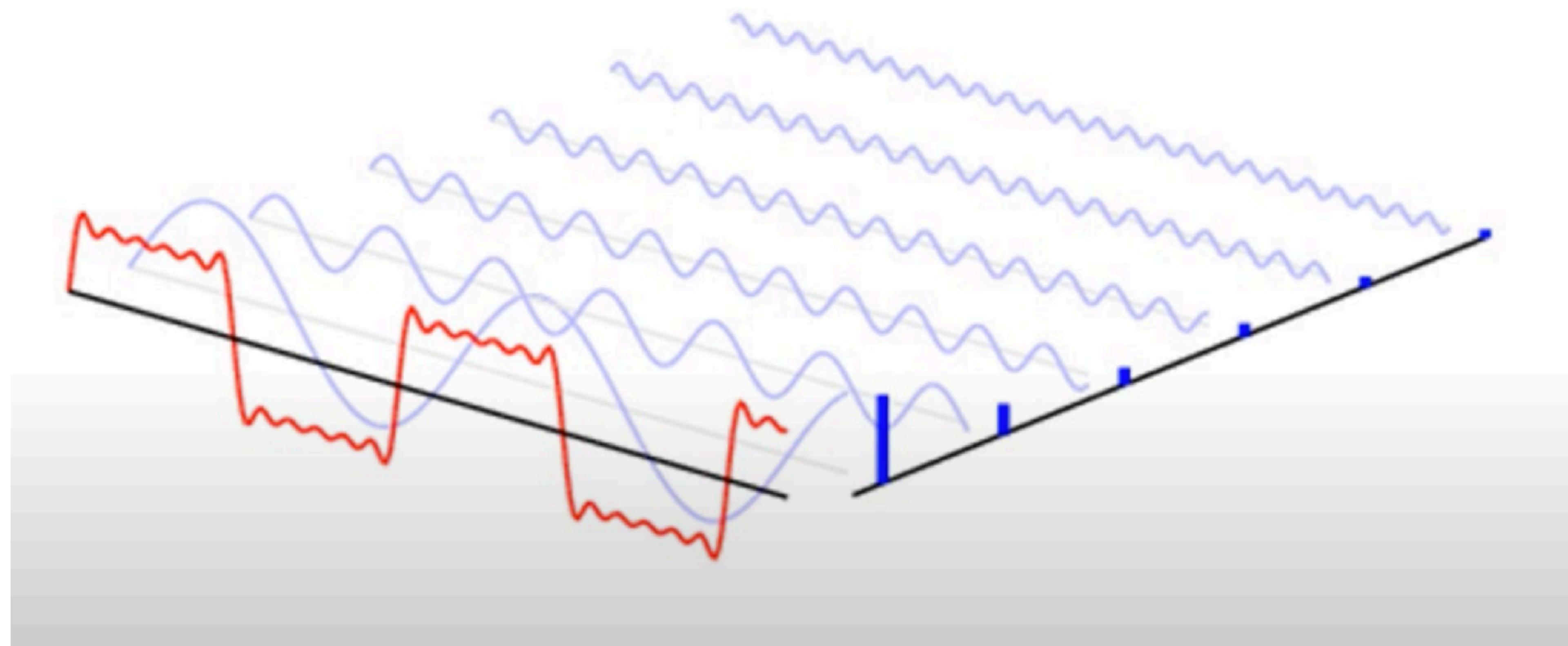
$$I'(m, n) = \max\{I(i, j) : (i, j) \in N(m, n)\} - \min\{I(i, j) : (i, j) \in N(m, n)\}$$



Summary

- **Linear filtering**
- **Convolution in the image domain**
 - Separable filters
 - Smoothing filters and Gaussian filters
 - Differential filters
- **Convolution in the Fourier domain**
- **Rank filters**

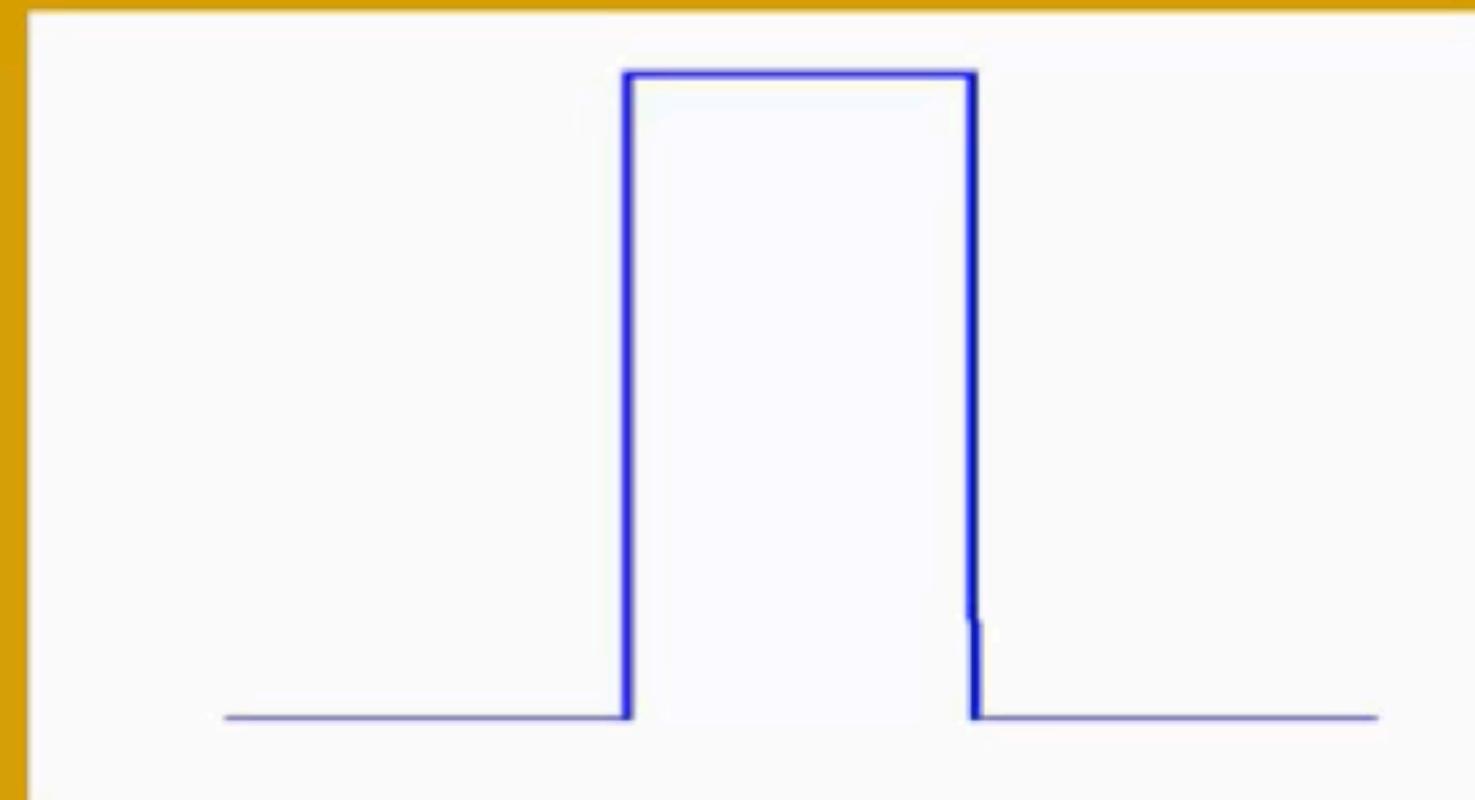
Fourier transform



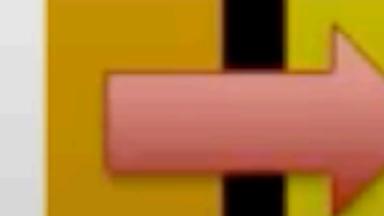
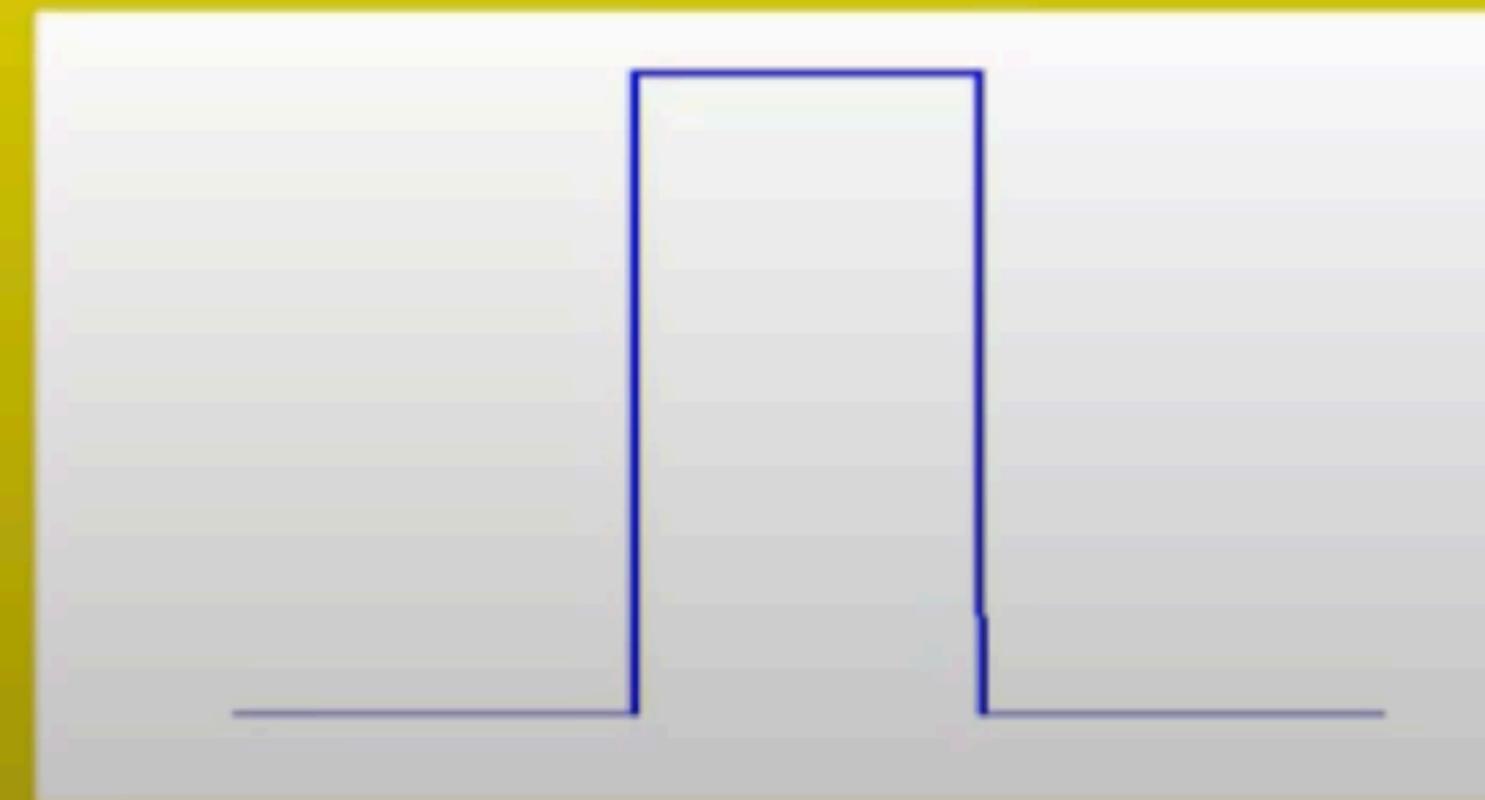
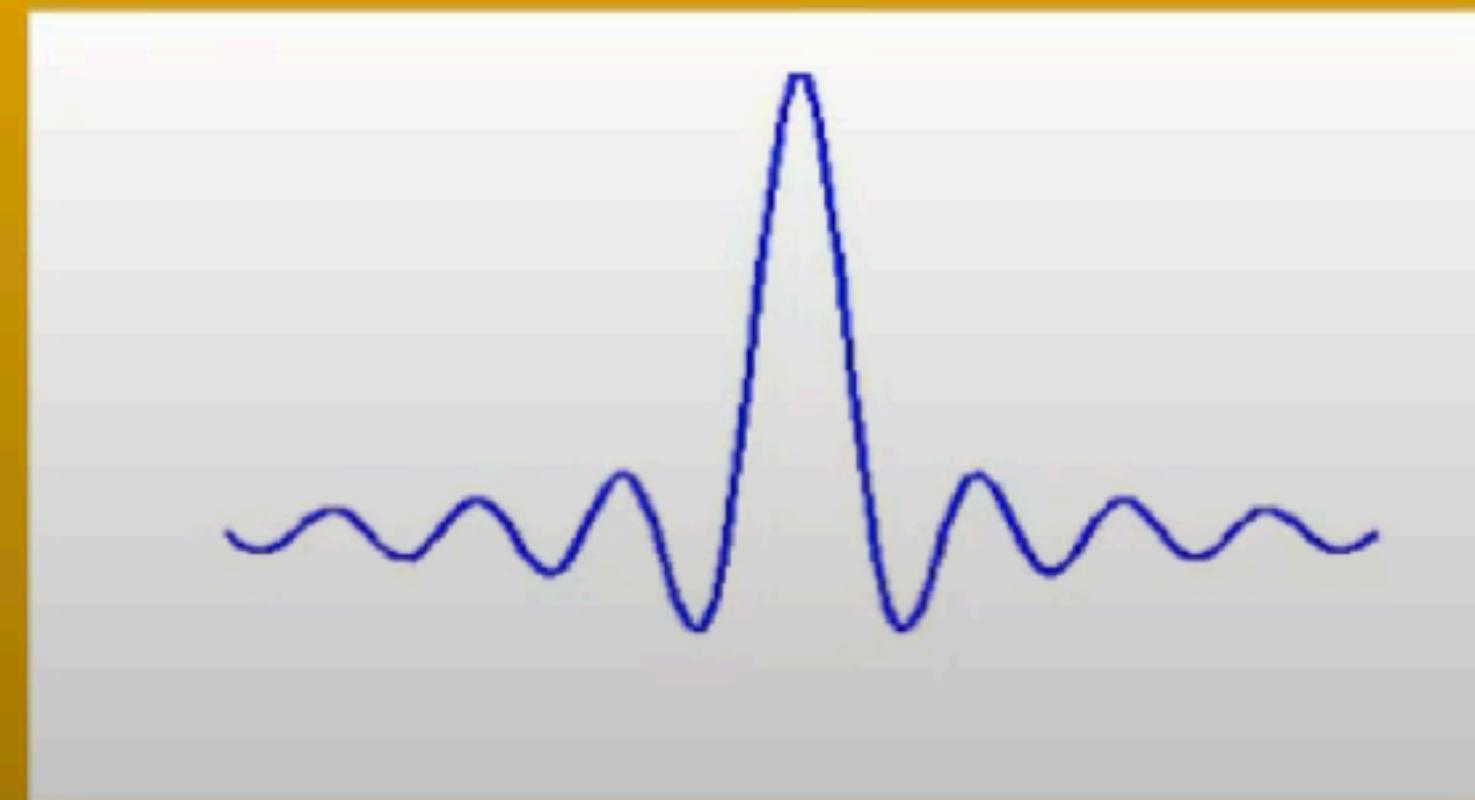
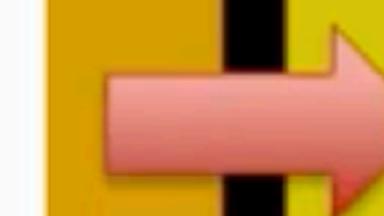
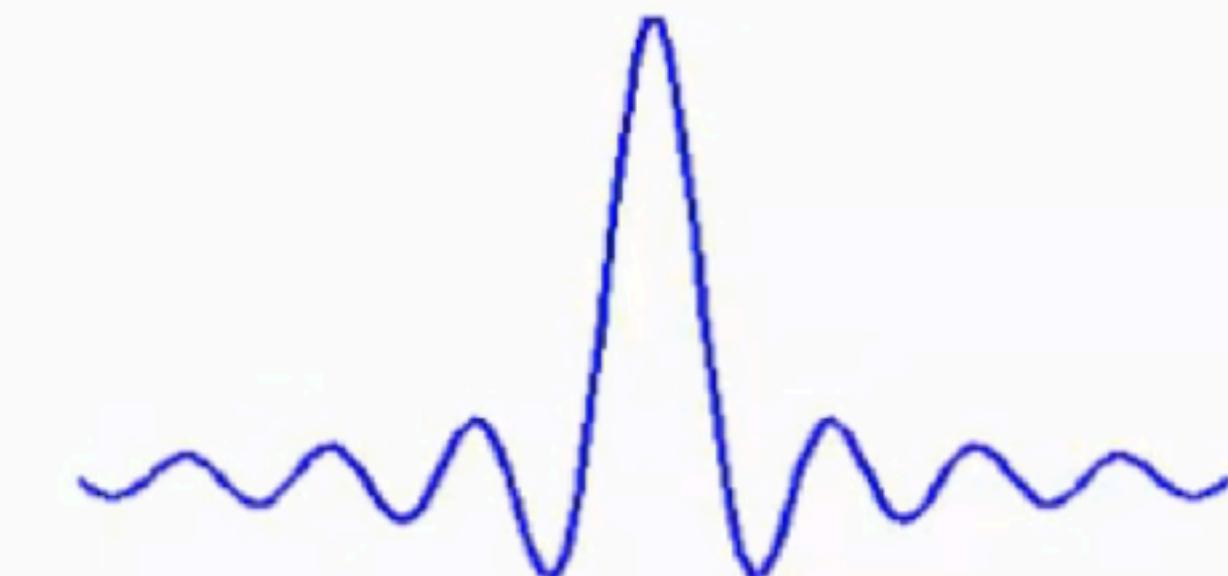
Time domain

Frequency domain

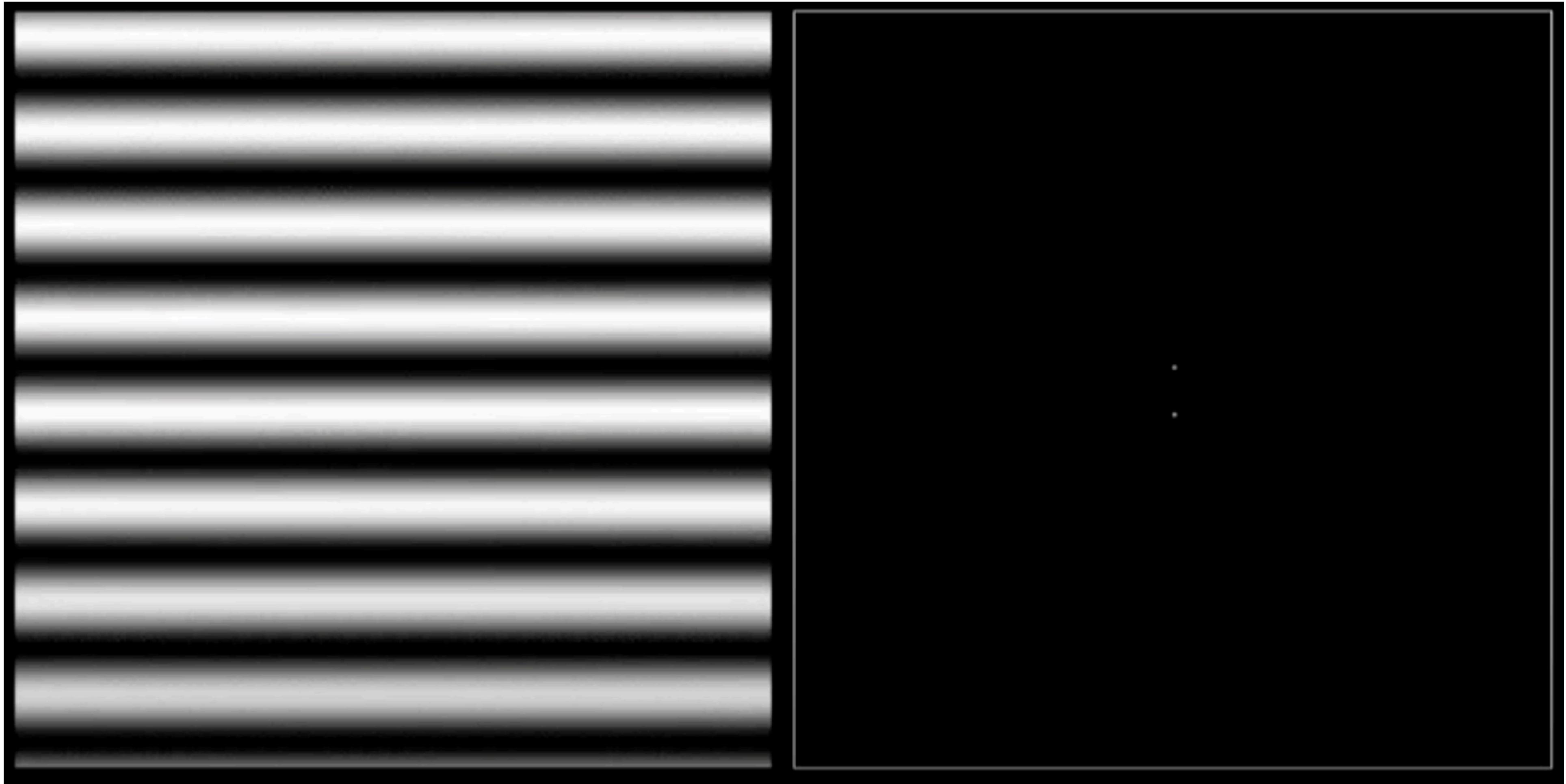
Time domain

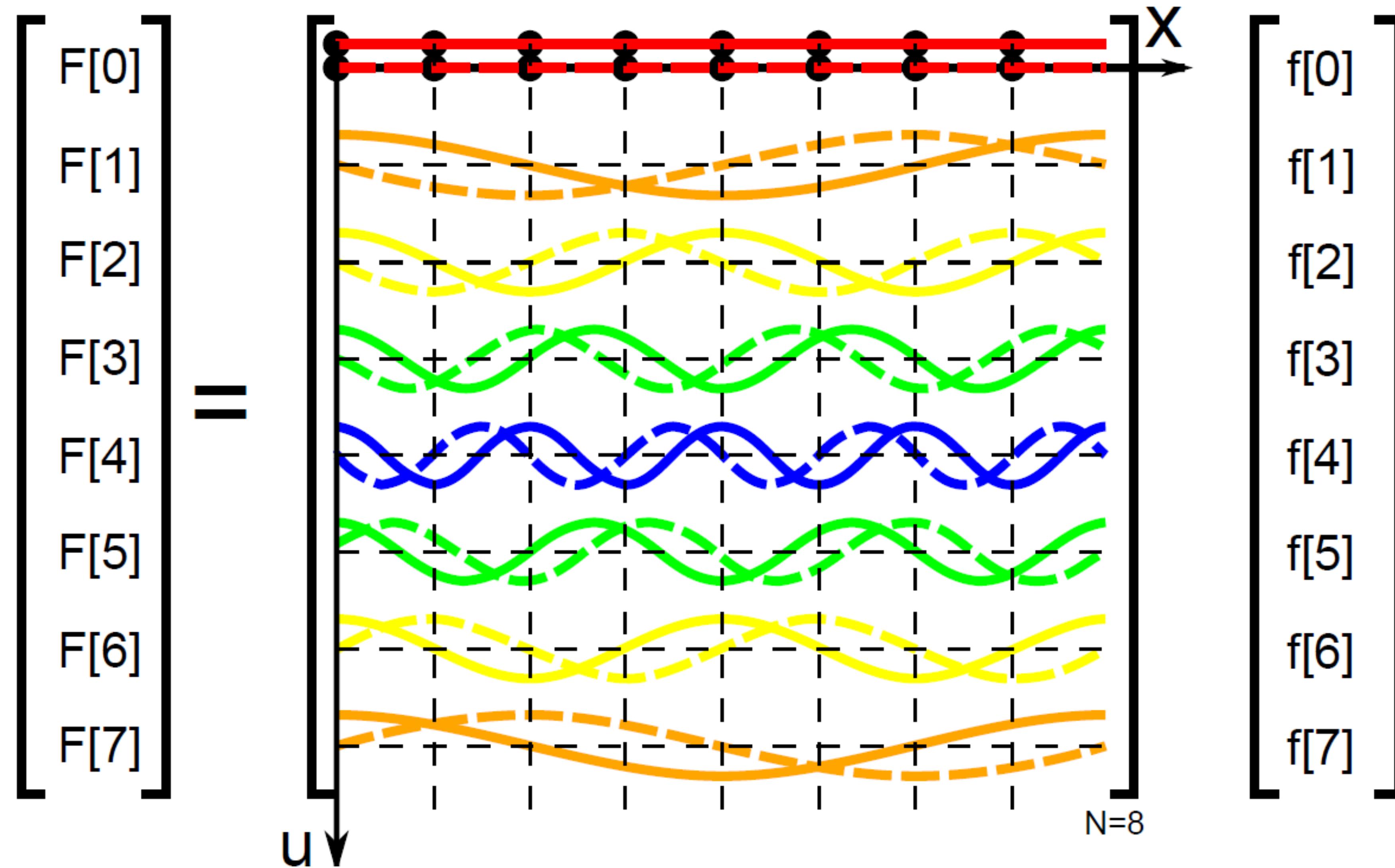


Frequency domain

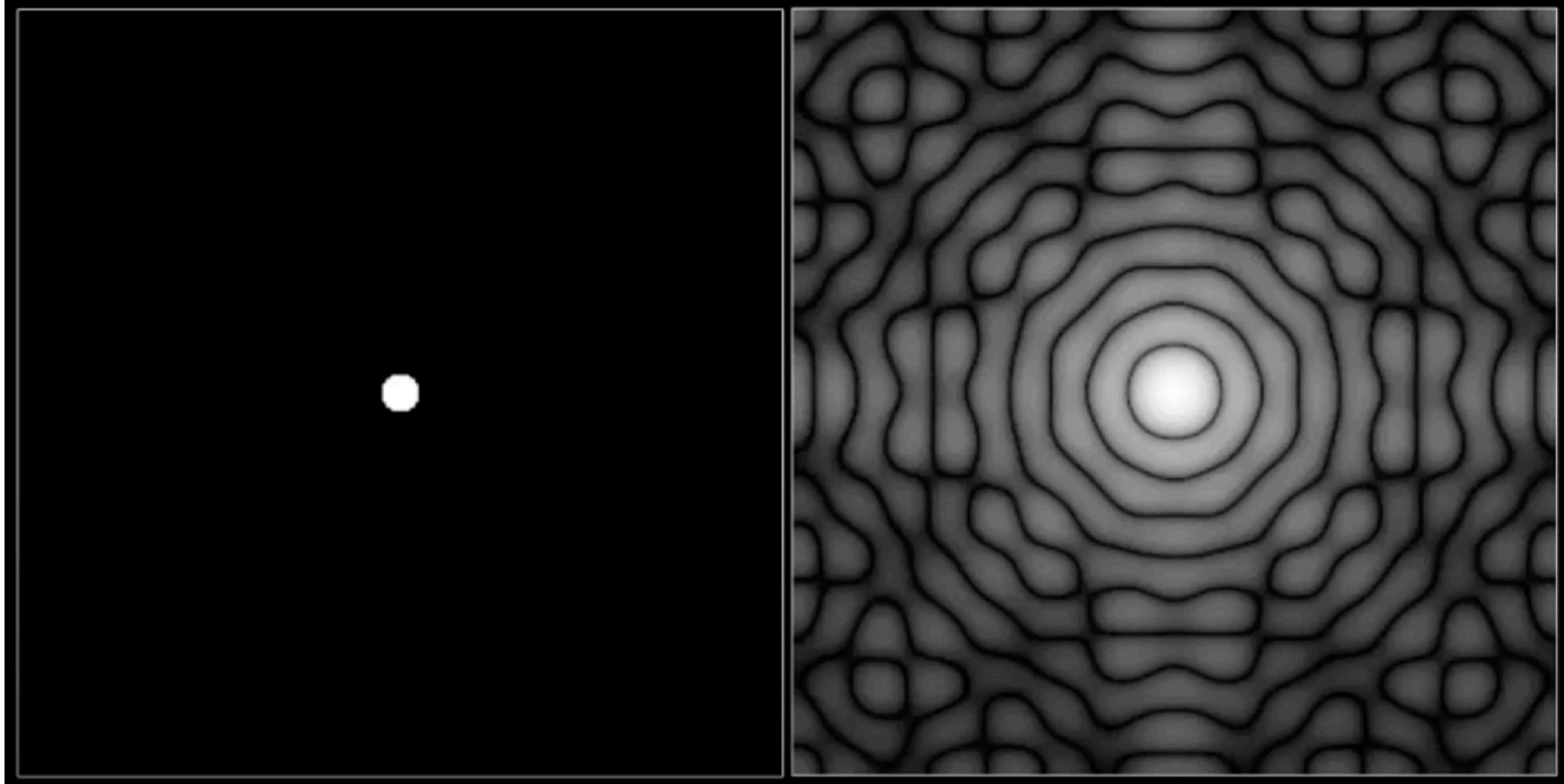


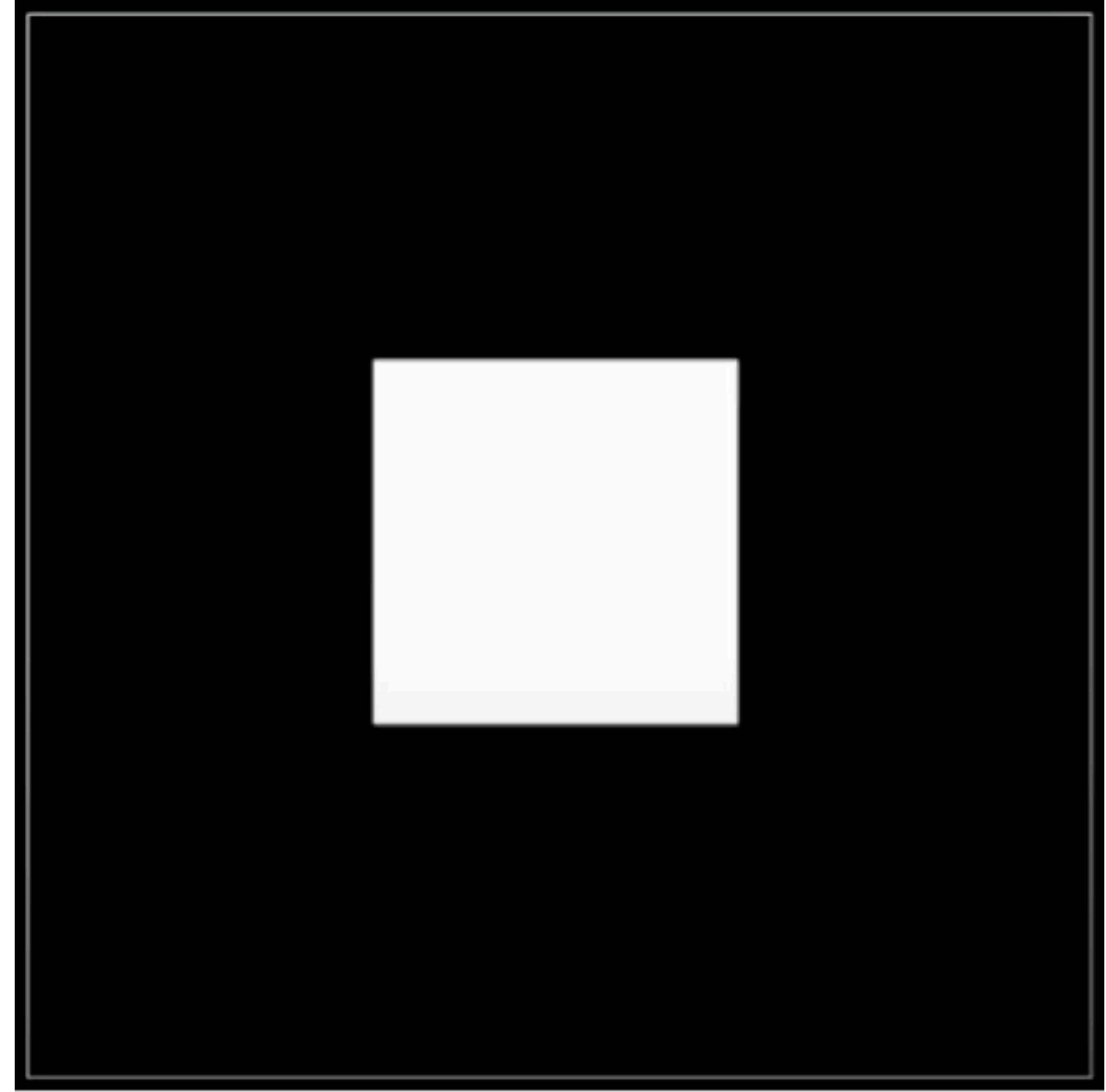


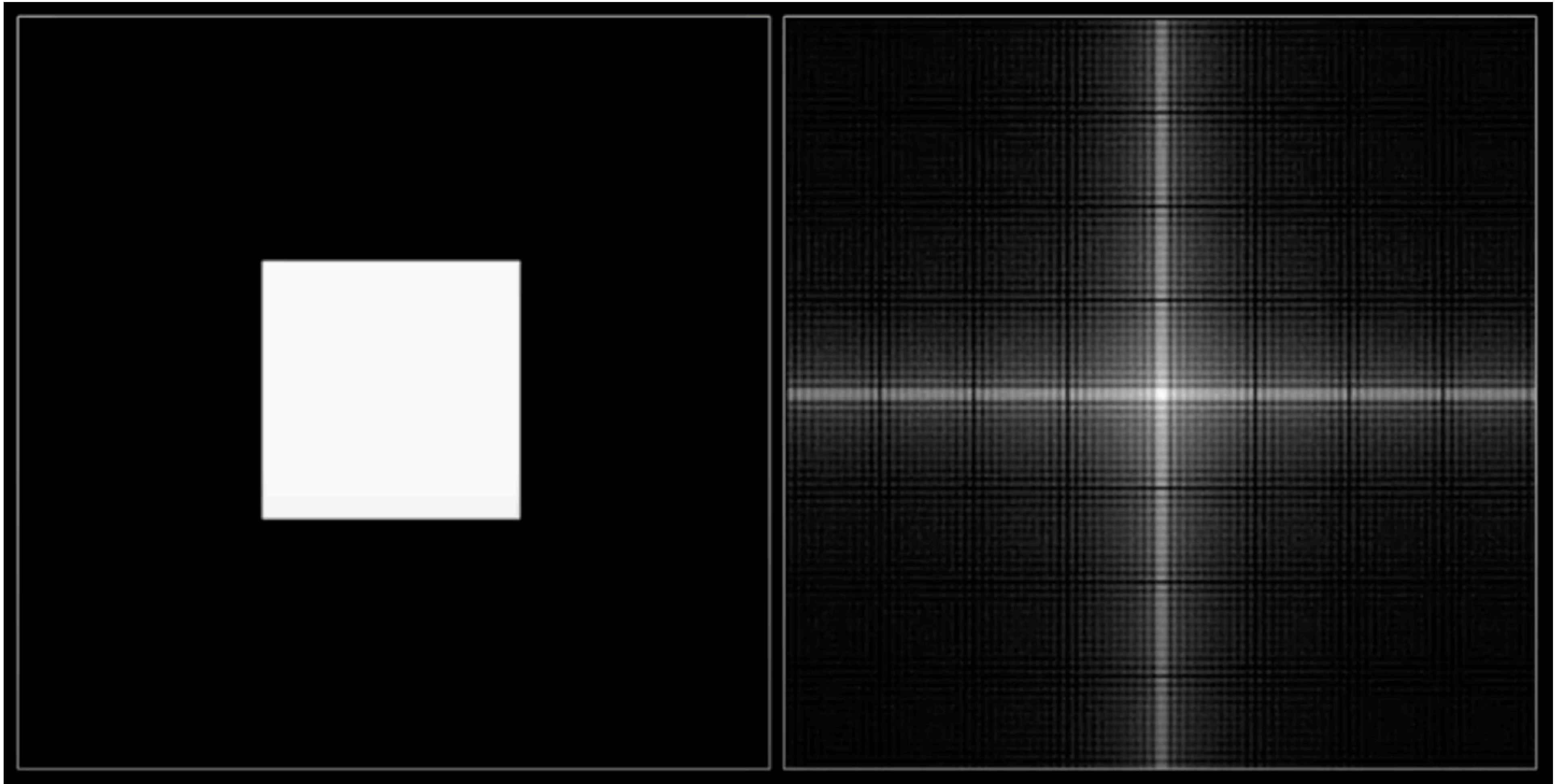




Can think of this as a change of basis: we move from space in which signal is represented as coefficients of delta functions to frequency where signal is represented as coefficients of sinusoidal basis functions







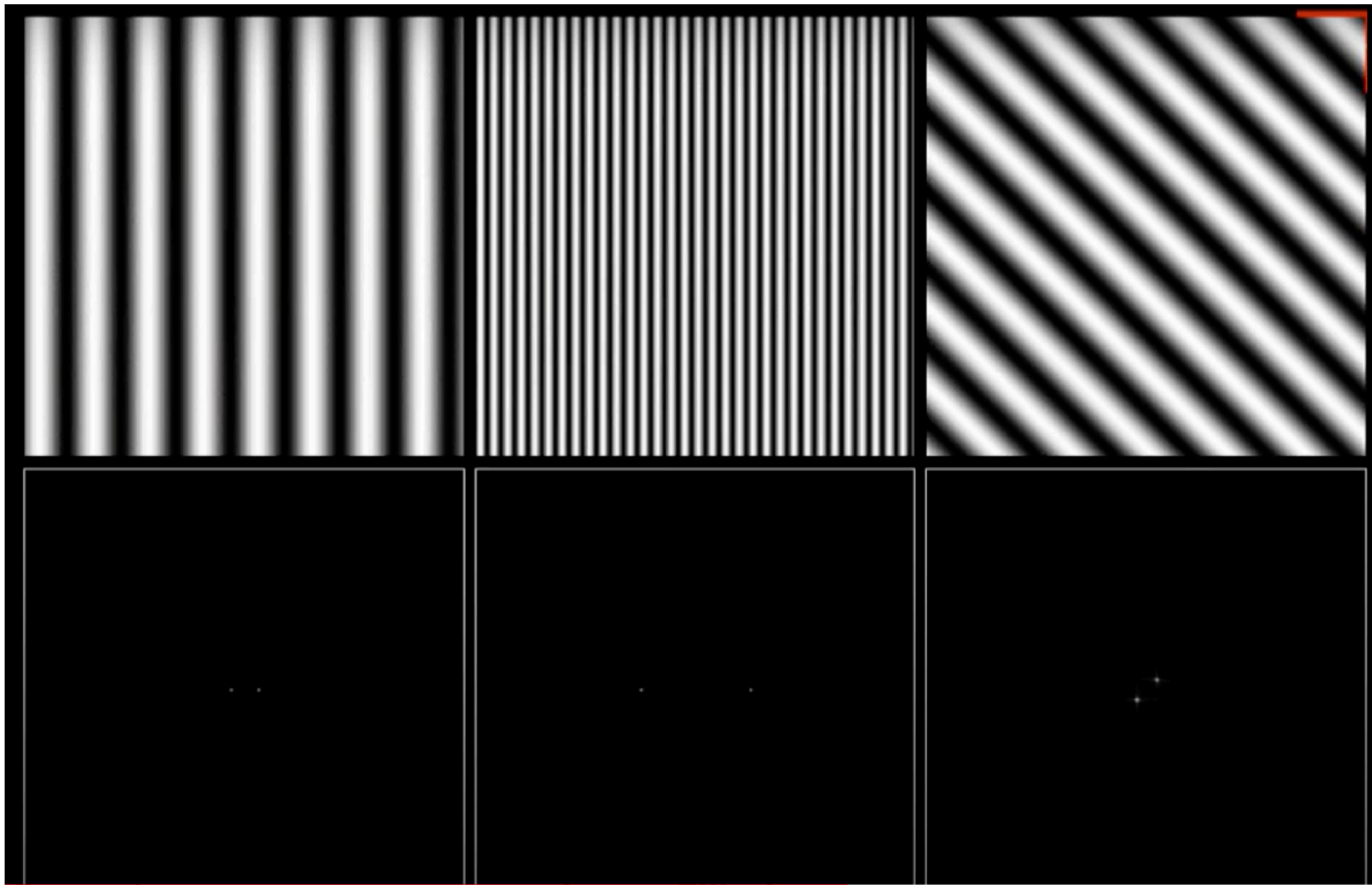




Image: 'cameraman'

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