COMP0130: Robotic Vision and Navigation

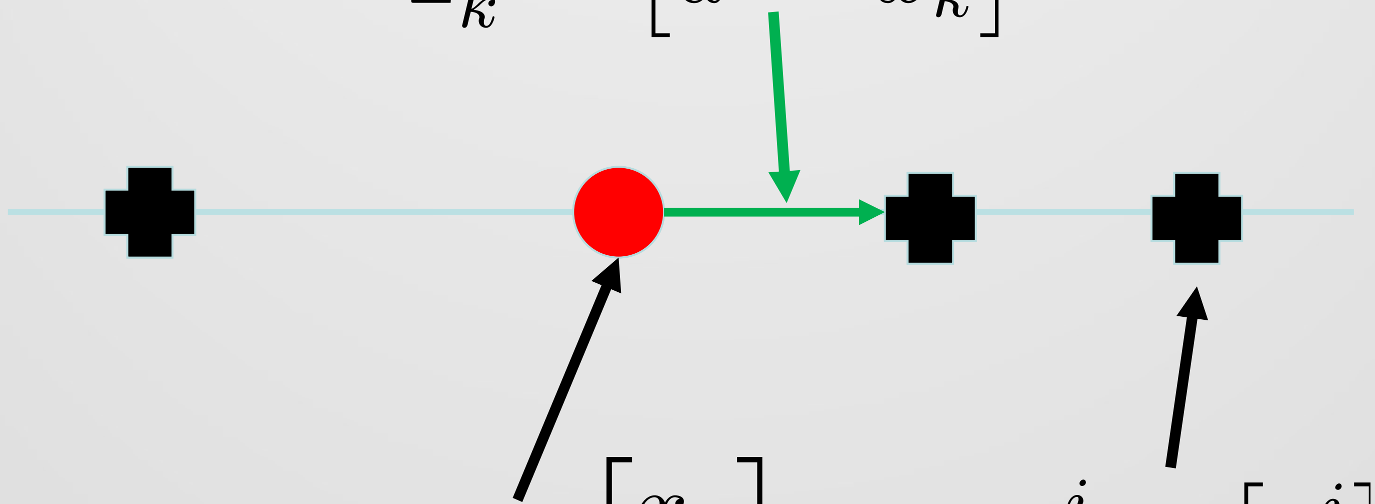
Lecture 05C:Using the Kalman Filter for SLAM

Structure

- Introduce a simple reference example (BeadSLAM) which we'll use to illustrate a lot of properties of SLAM
- Describe how to apply a Kalman filter in the BeadSLAM case
- Interpret what's going on with BeadSLAM
- Discuss some convergence properties that it – and ideally all – SLAM algorithms should exhibit

Simple Motivating Example… BeadSLAM

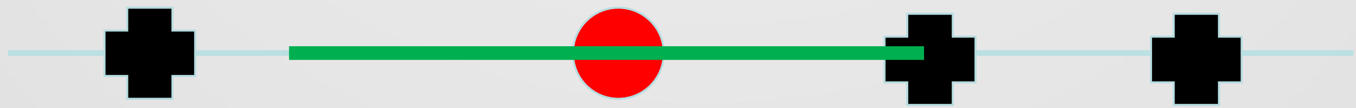
$$\mathbf{z}_k^i = \begin{bmatrix} u^i - x_k \end{bmatrix}$$



$$\mathbf{x}_k = \begin{bmatrix} x_k \\ \dot{x}_k \end{bmatrix}$$

$$\mathbf{m}^i = \begin{bmatrix} u^i \end{bmatrix}$$

BeadSLAM Partial Observation of the Map



BeadSLAM Demo

Ground Truth State Space Model

- This contains the robot state, and the state of all the landmarks,

$$\mathbf{s}_k = \begin{bmatrix} \mathbf{x}_k \\ \mathbf{m} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_k \\ \mathbf{m}^1 \\ \vdots \\ \mathbf{m}^N \end{bmatrix}$$

BeadSLAM Process Model

$$\mathbf{x}_{k+1} = \mathbf{F}\mathbf{x}_k + \mathbf{G}\mathbf{u}_k + \mathbf{v}_k$$

$$\mathbf{F} = \begin{bmatrix} 1 & \Delta T \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{v}_k \sim \mathcal{G}\left(\mathbf{v}; \mathbf{0}, \mathbf{Q}\right)$$

$$\mathbf{G} = \begin{bmatrix} \Delta T^2/2 \\ \Delta T \end{bmatrix}$$

$$\mathbf{Q} = \sigma_V^2 \begin{bmatrix} \Delta T^3/3 & \Delta T^2/2 \\ \Delta T^2/2 & \Delta T \end{bmatrix}$$

BeadSLAM Observation Model

$z = u^i - x$

$$\mathbf{z}_k^j = \mathbf{H}_k^{i_j} \mathbf{s}_k + \mathbf{w}_k^j$$

$$\mathbf{H}_k^{i_j} = \begin{bmatrix} -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$\mathbf{w}_k^j \sim \mathcal{G}\left(\mathbf{w}; 0, \mathbf{R}_k^j\right)$$

Implementing the Kalman Filter

- We have to:
  - Specify the algorithm flow
  - Specify the state space
  - Implement the steps in the algorithm

Notation for Estimates

- We will often be concerned with random variables of the form

$$\mathbf{s}_{i|j}$$

Estimate at time i

Given observations up to time step j

Notation for Timing

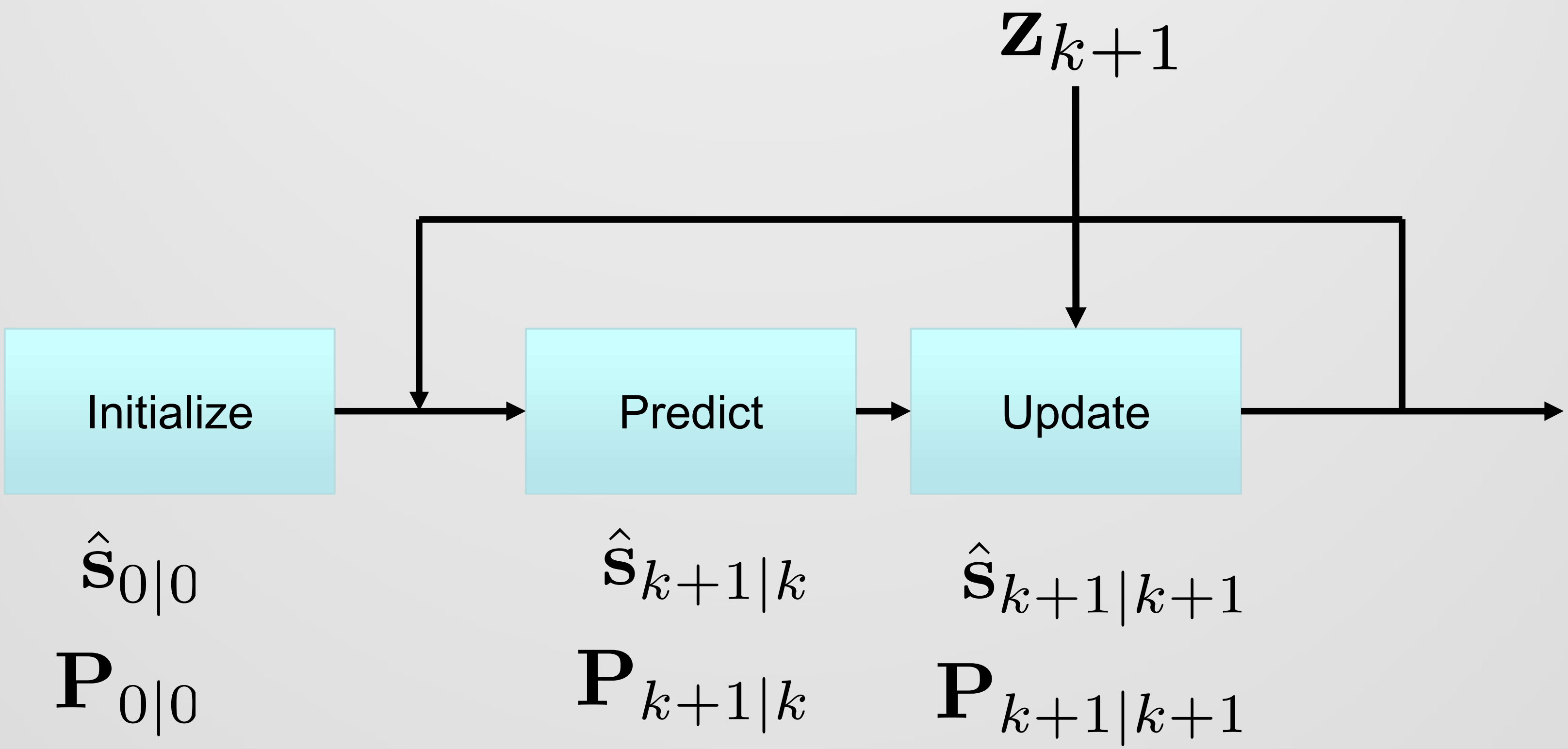
$s = \begin{bmatrix} x \\ m \end{bmatrix}$

- For the Kalman filter the mean and covariance estimates are

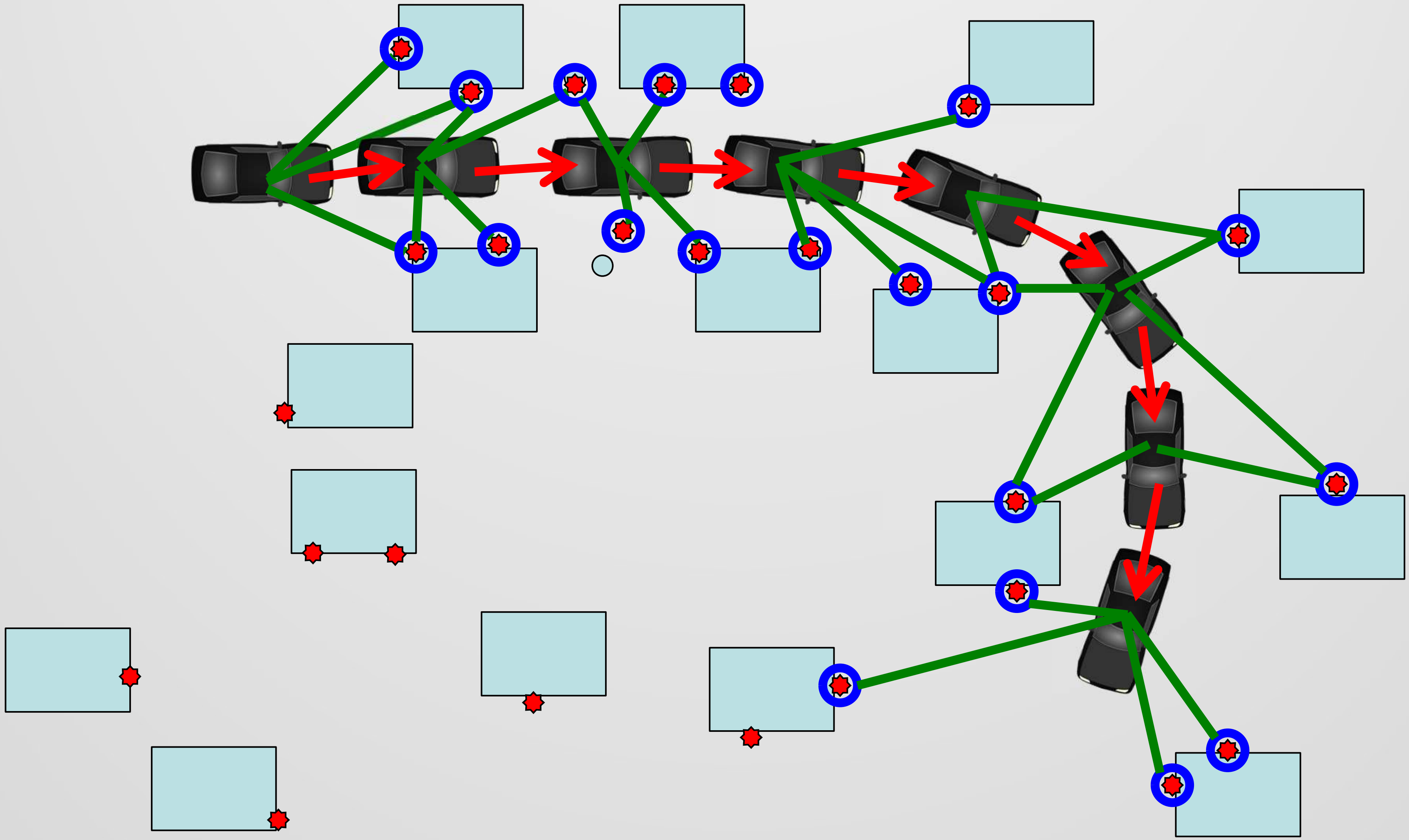
$$\hat{\mathbf{s}}_{i|j}\,, \mathbf{P}_{i|j}$$

- Note:
  - When $i>j$ this is a *predicted* value
  - When $i=j$ this is a *filtered* value
  - When $i<j$ this is a *smoothed* value

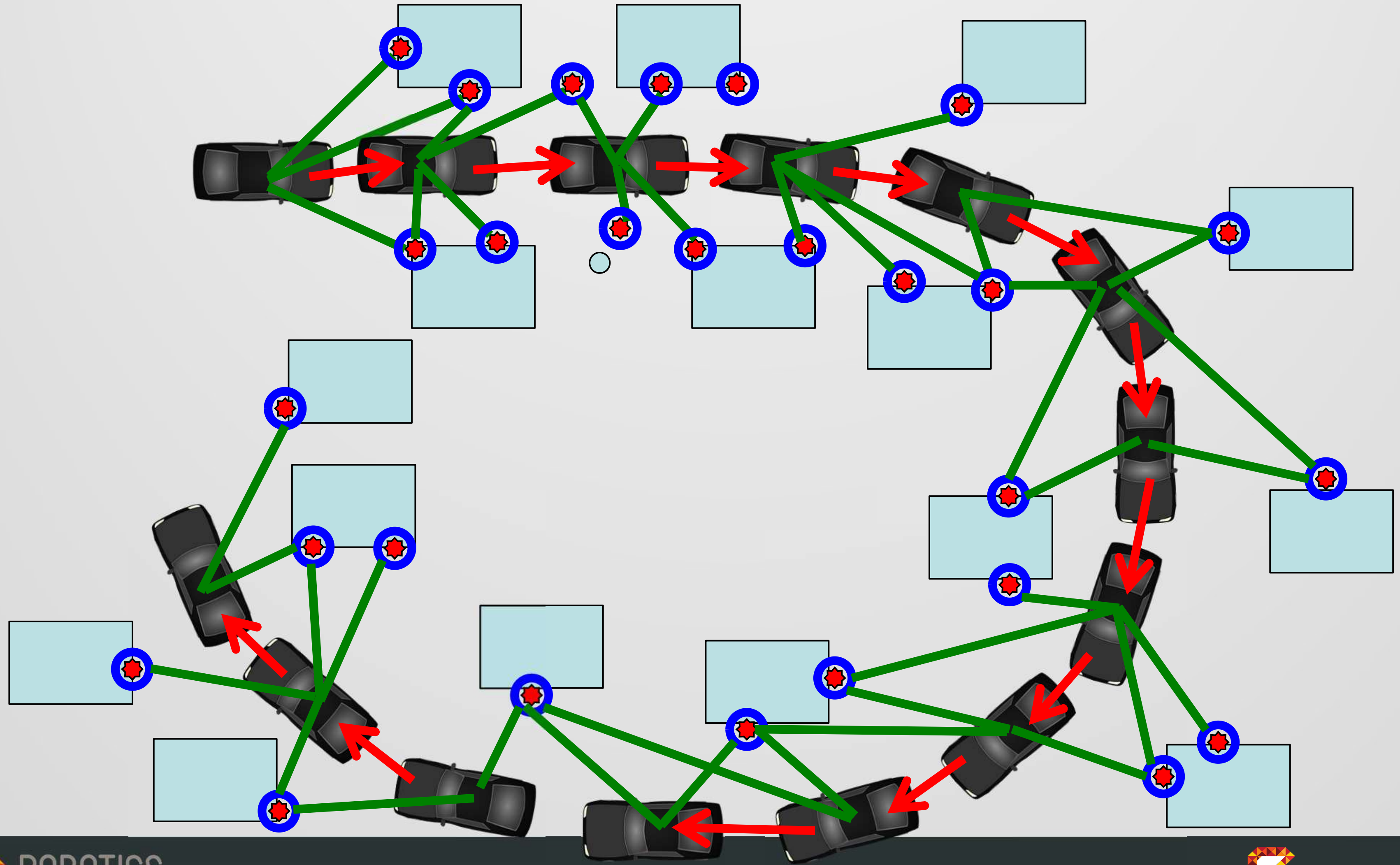
Standard Kalman Filter Structure



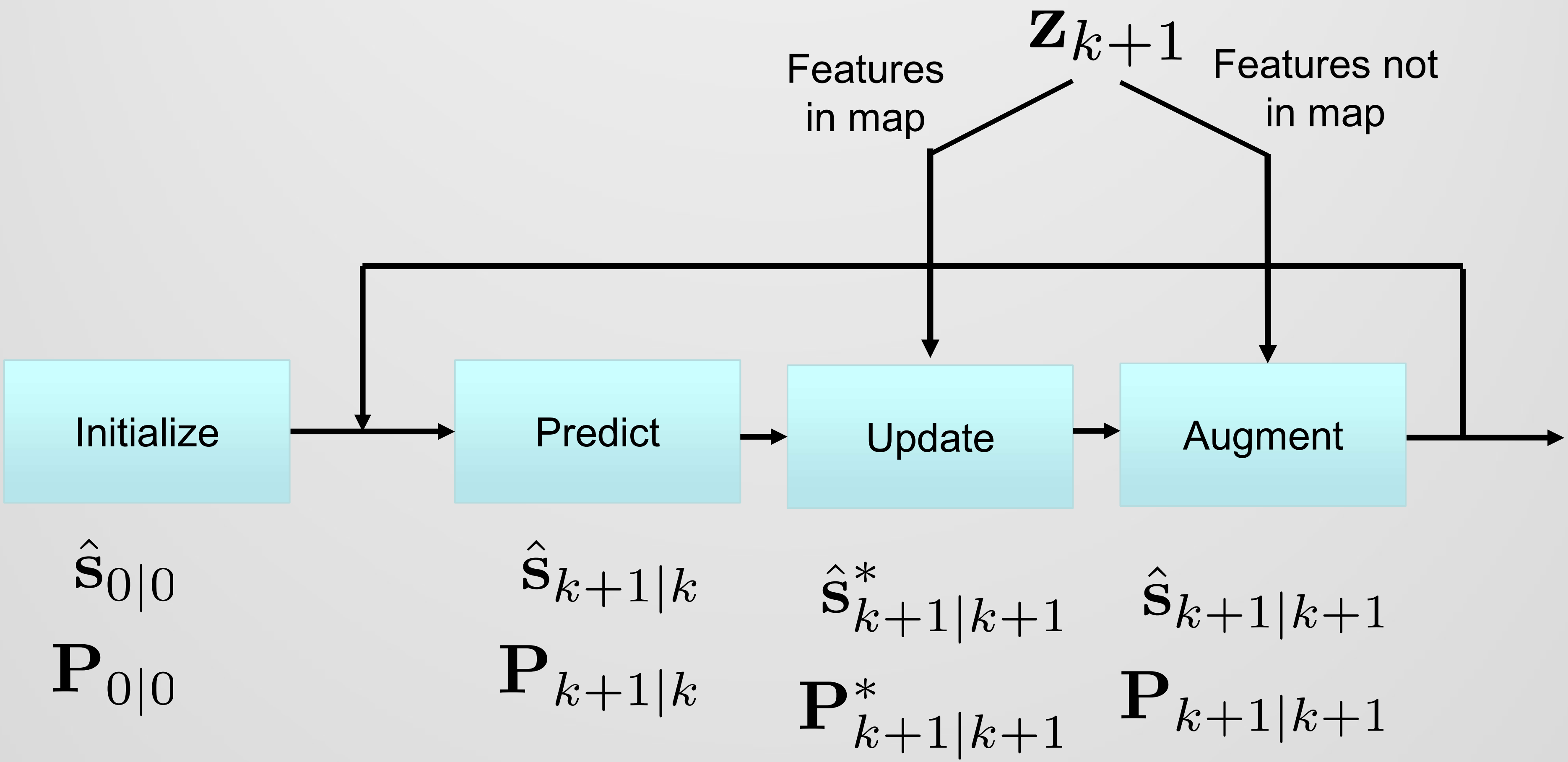
SLAM Maps Change



SLAM Maps Change



SLAM Kalman Filter Structure



Kalman Filter State

$$\hat{\mathbf{s}}_{k|k} = \begin{bmatrix} \hat{\mathbf{x}}_{k|k} \\ \hat{\mathbf{m}}_k \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}}_{k|k} \\ \hat{\mathbf{m}}^1_k \\ \vdots \\ \hat{\mathbf{m}}^{N_k}_k \end{bmatrix}$$

$\hat{x}_{i|j}$ $\hat{m}^i_j$ $\hat{m}^1$

Kalman Filter Process Model

$x_{k+1} = F x_{k} + \dots$

$$\hat{\mathbf{s}}_{k+1|k} = \mathbf{F}_s \hat{\mathbf{s}}_{k|k} + \mathbf{G}_s \mathbf{u}_k + v_k$$

$$s = \begin{bmatrix} \hat{x} \\ m \end{bmatrix} \qquad F_s = \begin{bmatrix} F & 0 \\ 0 & I \end{bmatrix} \qquad \begin{bmatrix} v_k \\ 0 \\ 0 \end{bmatrix}$$

F, V

Kalman Filter Process Model

$$\mathbf{P}_{k+1|k} = \mathbf{F}_s^\top \mathbf{P}_{k|k} \mathbf{F}_s + \mathbf{B}_s^\top \mathbf{Q}_k \mathbf{B}_s$$

$$\begin{bmatrix} F & 0 \\ 0 & I \end{bmatrix} \qquad \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$$

Kalman Filter Covariance

- The covariance is computed from

$$\mathbf{P}_{k|k} = \mathrm{E}\left[\begin{pmatrix}\mathbf{x}_k - \hat{\mathbf{x}}_{k|k} \\ \mathbf{m} - \hat{\mathbf{m}}_k\end{pmatrix}\begin{pmatrix}\mathbf{x}_k - \hat{\mathbf{x}}_{k|k} \\ \mathbf{m} - \hat{\mathbf{m}}_k\end{pmatrix}^\top\right]$$

The Structure of the Covariance Matrix

- Using the "proper" notation, the equations look

$$\mathbf{P}_{k|k} = \begin{bmatrix} \mathbf{P}^{\mathbf{xx}} & \mathbf{P}^{\mathbf{xm}^1} & \mathbf{P}^{\mathbf{xm}^2} & \ldots & \mathbf{P}^{\mathbf{xm}^{N_k}} \\ \mathbf{P}^{\mathbf{m}^1\mathbf{x}} & \mathbf{P}^{\mathbf{m}^1\mathbf{m}^1} & \mathbf{P}^{\mathbf{m}^1\mathbf{m}^2} & \ldots & \mathbf{P}^{\mathbf{m}^1\mathbf{m}^{N_k}} \\ \mathbf{P}^{\mathbf{m}^2\mathbf{x}} & \mathbf{P}^{\mathbf{m}^2\mathbf{m}^1} & \mathbf{P}^{\mathbf{m}^2\mathbf{m}^2} & \ldots & \mathbf{P}^{\mathbf{m}^2\mathbf{m}^{N_k}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}^{\mathbf{m}^{N_k}\mathbf{x}} & \mathbf{P}^{\mathbf{m}^{N_k}\mathbf{m}^1} & \mathbf{P}^{\mathbf{m}^{N_k}\mathbf{m}^2} & \ldots & \mathbf{P}^{\mathbf{m}^{N_k}\mathbf{m}^{N_k}} \end{bmatrix}_{k|k}$$

The Structure of the Covariance Matrix

- Simplified notation

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}^{\mathbf{x}\mathbf{x}} & \mathbf{P}^{\mathbf{x}1} & \mathbf{P}^{\mathbf{x}2} & \dots & \mathbf{P}^{\mathbf{x}N_k} \\ \mathbf{P}^{1\mathbf{x}} & \mathbf{P}^{11} & \mathbf{P}^{12} & \dots & \mathbf{P}^{1N_k} \\ \mathbf{P}^{2\mathbf{x}} & \mathbf{P}^{21} & \mathbf{P}^{22} & \dots & \mathbf{P}^{2N_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}^{N_k\mathbf{x}} & \mathbf{P}^{N_k1} & \mathbf{P}^{N_k2} & \dots & \mathbf{P}^{N_kN_k} \end{bmatrix}$$

The Structure of the Covariance Matrix

- We'll use the high-level structure

$$\mathrm{P} = \begin{bmatrix} \mathrm{P}^{\mathrm{xx}} & \mathrm{P}^{\mathrm{xm}} \\ \mathrm{P}^{\mathrm{mx}} & \mathrm{P}^{\mathrm{mm}} \end{bmatrix}$$

Platform Block

$$\mathbf{P^{xx}} = \begin{bmatrix} P^{xx} & P^{x\dot{x}} \\ P^{\dot{x}x} & P^{\dot{x}\dot{x}} \end{bmatrix}$$

Map Block

$$\mathbf{P}^{\mathbf{mm}} = \begin{bmatrix} \mathbf{P}^{11} & \mathbf{P}^{12} & \cdots & \mathbf{P}^{1N_k} \\ \mathbf{P}^{21} & \mathbf{P}^{22} & \cdots & \mathbf{P}^{2N_k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}^{N_k1} & \mathbf{P}^{N_k2} & \cdots & \mathbf{P}^{N_kN_k} \end{bmatrix}$$

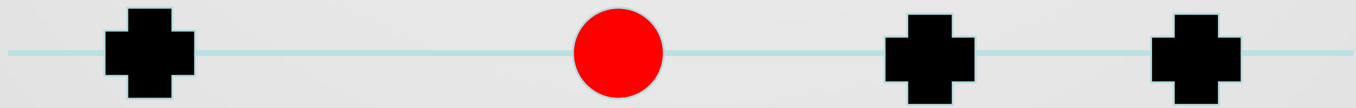
Vehicle-Map Block

$$\mathbf{P}^{\mathbf{xm}} = \begin{bmatrix} \mathbf{P}^{\mathbf{x}1} & \mathbf{P}^{\mathbf{x}2} & \ldots & \mathbf{P}^{\mathbf{x}N_k} \end{bmatrix}$$

BeadSLAM: The First Few Timesteps

Timestep	Action
0	Initialize
1	Predict and observe landmark 1
2	Predict and update landmark 1
3	Predict, observe landmark 2

Step 0: Initialize



Timestep 0: The Real World

- The real world starts off with the platform at the origin and a fully populated map

$$\mathbf{s}_0 = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{m}^1 \\ \vdots \\ \mathbf{m}^N \end{bmatrix}$$

Timestep 0: Initialise the Filter

- The filter is initialized at the origin with an empty map and no uncertainty

$$\hat{\mathbf{s}}_{0|0} = \begin{bmatrix} \hat{\mathbf{x}}_{0|0} \end{bmatrix}$$

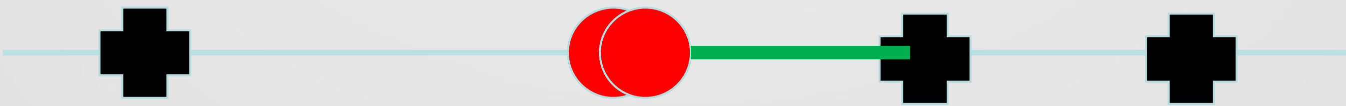
$$\hat{\mathbf{x}}_{0|0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{P}_{0|0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

BeadSLAM: The First Few Timesteps

Timestep	Action
~~0~~	~~Initialize~~
1	Predict and observe landmark 1
2	Predict and update landmark 1
3	Predict, observe landmark 2

Step 1: Predict and Observe Landmark 1



Timestep 1: Predict + Observe in Real World

- The system advances a single timestep,

$$\mathbf{s}_1 = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{m}^1 \\ \vdots \\ \mathbf{m}^N \end{bmatrix} \qquad \mathbf{x}_1 = \mathbf{F}\mathbf{x}_0 + \mathbf{G}\mathbf{u}_1 + \mathbf{v}_1$$

- An observation of beacon 1 is taken,

$$\mathbf{z}_1^1 = m^1 - x_1 + \mathbf{w}_1^1$$

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Timestep 1: Predict the State

$$\hat{\mathbf{s}}_{1|0} = \mathbf{F}_{s}\hat{\mathbf{s}}_{0|0} + \mathbf{G}_{s}\mathbf{u}_1$$

$$\hat{s}_{0|0} = \left[\hat{x}_{0|0}\right]$$

$$\hat{s}_{1|0} = F\hat{x}_{0|0} + Gu_1$$

Timestep 1: Predict the Covariance

$$\mathbf{P}_{1|0} = \mathbf{F}_s \mathbf{P}_{0|0} \mathbf{F}_s^\top + \mathbf{B}_s \mathbf{Q}_1 \mathbf{B}_s^\top$$

$$P_{1|0} = F P_{0|0} F^T + B Q_1 B^T$$

Timestep 1: Update in Kalman Filter

- There are no landmarks in the map so this is a no-op,

$$\hat{\mathbf{s}}^*_{1|1} = \hat{\mathbf{s}}_{1|0}$$

$$\mathbf{P}^*_{1|1} = \mathbf{P}_{1|0}$$

Timestep 1: Augment in the Kalman Filter

- We now need to add a new landmark to the map
- We'll need to carry out the operations

$$\left[\hat{\mathbf{x}}^{*}_{1|1}\right] \rightarrow \begin{bmatrix} \hat{\mathbf{x}}_{1|1} \\ \hat{\mathbf{m}}^{1}_{1} \end{bmatrix}$$

$$\left[\mathbf{P}^{\mathbf{x}^{*}\mathbf{x}^{*}}\right]_{1|1} \rightarrow \begin{bmatrix} \mathbf{P}^{\mathbf{xx}} & \mathbf{P}^{\mathbf{x}1} \\ \mathbf{P}^{1\mathbf{x}} & \mathbf{P}^{11} \end{bmatrix}_{1|1}$$

Timestep 1: Augment in the Kalman Filter

- To model this augmentation step, we are going to create a new augment operator
- This will be of the form

$$\hat{\mathbf{s}}_{1|1} = \mathbf{A}_k \hat{\mathbf{s}}^*_{1|1} + \mathbf{J}_k \mathbf{z}^j_k$$

- The associated covariance will be

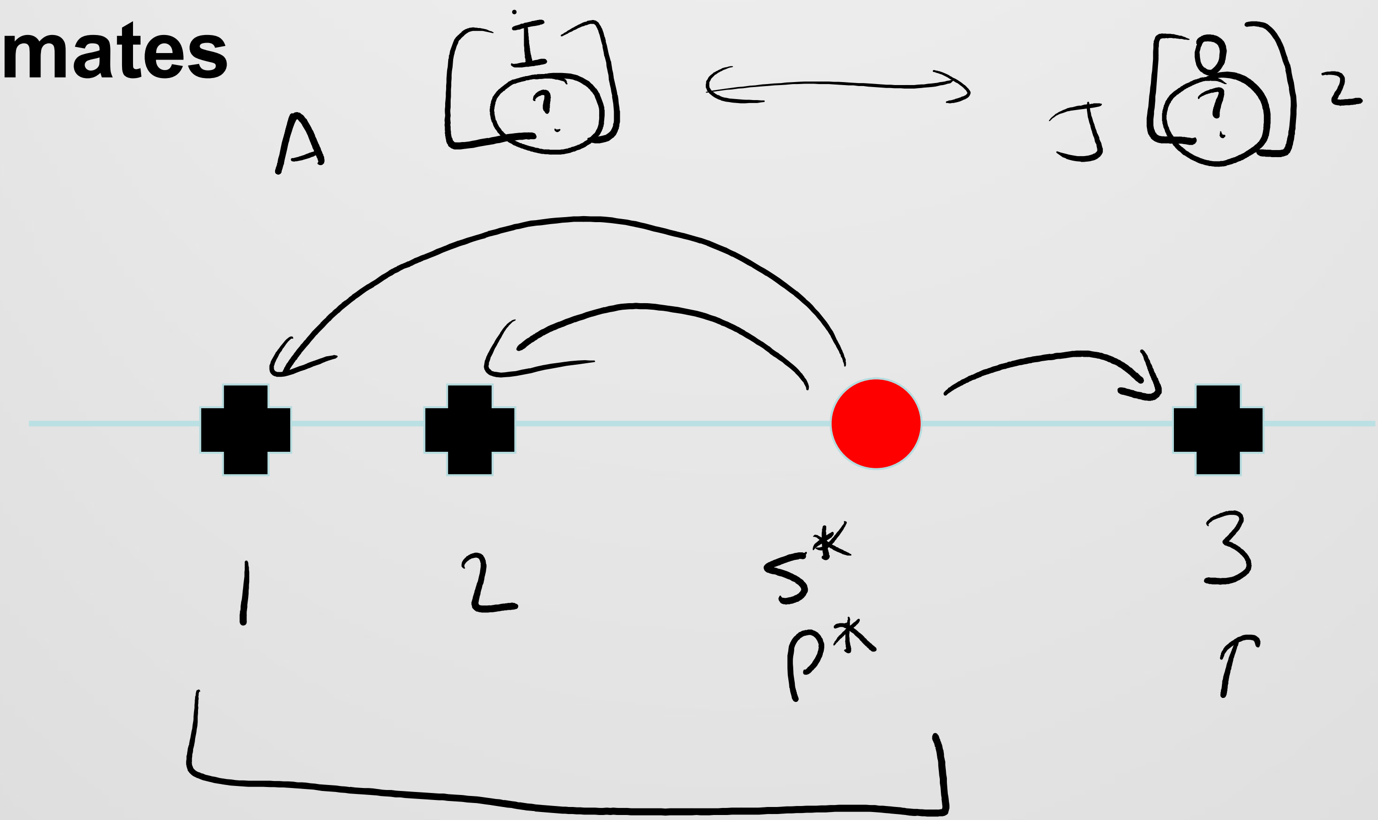
$$\mathbf{P}_{1|1} = \mathbf{A}_k \mathbf{P}^*_{1|1} \mathbf{A}^\top_k + \mathbf{J}_k \mathbf{R}^j_k \mathbf{J}^\top_k$$

Timestep 1: Augment in the Kalman Filter

$$\hat{\mathbf{s}}_{1|1} = \underline{\mathbf{A}_k}\hat{\mathbf{s}}^*_{1|1} + \underline{\mathbf{J}_k}\mathbf{z}^j_k$$

$$\begin{bmatrix} \hat{x}_{1|1} \\ \hat{m}^j_1 \end{bmatrix} \quad \begin{bmatrix} \hat{x}^*_{1|0} \end{bmatrix} + \begin{bmatrix} \\ \\ \end{bmatrix} \quad z^j_k$$

Augmentation Does Not Change Existing Estimates



Augmentation Does Not Change Existing Estimates

- We first assume that augmenting the map with a new landmark does not change any values estimated so far
- Therefore,

$$\hat{\mathbf{x}}_{1|1} = \hat{\mathbf{x}}^*_{1|1}$$

- Therefore, we need to figure out how to add the landmark

Timestep 1: Inverse Observation Model

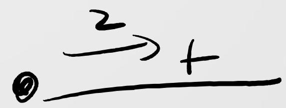
- Recall the observation model has the form

$$\mathbf{z}_k^j = \mathbf{h}\left[\mathbf{x}_k, \mathbf{m}^{i_j}, \mathbf{w}_k^j\right]$$

- We will introduce the "inverse observation model"

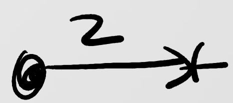
$$\mathbf{m}^{i_j} = \mathbf{g}\left[\mathbf{x}_k, \mathbf{z}_k^j, \mathbf{w}_k^j\right]$$

BeadSLAM Inverse Observation Model



- The observation model is

$$z_k^j = u^{i_j} - x_k + w_k^j$$



- The inverse observation model is

$$u^{i_j} = z_k^j + x_k - w_k^j$$

BeadSLAM Inverse Observation Model

- Therefore, the inverse observation model is

$$\mathbf{m}^{i_j} = \mathbf{C}_k \mathbf{x}_k + \mathbf{D}_k (\mathbf{z}_k^j - \mathbf{w}_k^j)$$

- The augmentation operation looks like

$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{m}^{i_j} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{C}_k \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_k \end{bmatrix} (\mathbf{z}_k^j - \mathbf{w}_k^j)$$

Computing the Mean

$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{m}^{i_j} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{C}_k \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_k \end{bmatrix} (\mathbf{z}_k^j - \mathbf{w}_k^j)$$

Timestep 1: Augmenting the Model

- Therefore, the equation for the augmented state is

$$\hat{\mathbf{s}}_{1|1} = \mathbf{A}_k \hat{\mathbf{s}}^*_{1|1} + \mathbf{J}_k \mathbf{z}^j_k$$

$$\mathbf{A}_k = \begin{bmatrix} \mathbf{I} \\ \mathbf{C}_k \end{bmatrix}, \ \mathbf{J}_k = \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_k \end{bmatrix}$$

Timestep 1: Augment as Linear Operation

- For BeadSLAM,

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \mathbf{J}_k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$z = u - x$

$u = x + z$

Timestep 1: Augment

- For BeadSLAM,

$$\hat{\mathbf{s}}_{1|1} = \begin{bmatrix} \hat{x}^*_{1|1} \\ \hat{\dot{x}}^*_{1|1} \\ \hat{x}^*_{1|1} + z^1_1 \end{bmatrix}$$

Timestep 1: Augmented Covariance

- The covariance matrix is

$$\mathbf{P}_{1|1} = \mathbf{A}_1 \mathbf{P}^*_{1|1} \mathbf{A}_1^\top + \mathbf{J}_1 \mathbf{R}_1^1 \mathbf{J}_1^\top$$

$$= \begin{bmatrix} P^{x^* x^*} & P^{x^* \dot{x}^*} & P^{x^* x^*} \\ P^{\dot{x}^* x^*} & P^{\dot{x}^* \dot{x}^*} & P^{\dot{x}^* x^*} \\ P^{x^* x^*} & P^{x^* \dot{x}^*} & P^{x^* x^*} + R_1^1 \end{bmatrix}$$

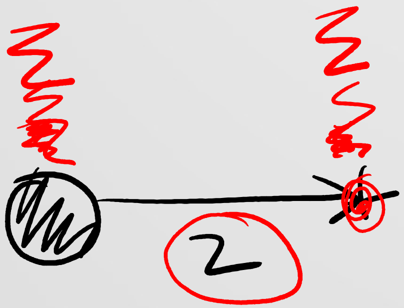
Interpreting the Covariance Matrix

P^{xx}

$$\begin{bmatrix} P^{x^*x^*} & P^{x^*\dot{x}^*} & P^{x^*x^*} \\ P^{\dot{x}^*x^*} & P^{\dot{x}^*\dot{x}^*} & P^{\dot{x}^*x^*} \\ P^{x^*x^*} & P^{x^*\dot{x}^*} & P^{x^*x^*} + R_1^1 \end{bmatrix}$$

P^{xx} P^{xm}

P^{mx} P^{mm}



BeadSLAM: The First Few Timesteps

Timestep	Action
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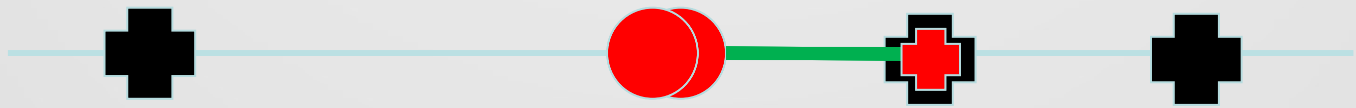
Step 2: Predict and Observe Landmark 1



Step 2: Predict + Observe in Real System

$$\mathbf{s}_2 = \begin{bmatrix} \mathbf{x}_2 \\ \mathbf{m}^1 \\ \vdots \\ \mathbf{m}^N \end{bmatrix} \qquad \mathbf{x}_2 = \mathbf{F}\mathbf{x}_1\mathbf{G}\mathbf{u}_2 + \mathbf{v}_2$$

$$\mathbf{z}_2^1 = m^1 - x_2 + \mathbf{w}_2^1$$

Step 2: Mean Predict

$$\hat{\mathbf{s}}_{2|1} = \mathbf{F}_s \hat{\mathbf{s}}_{1|1} + \mathbf{G}_s \mathbf{u}_2$$

$$\mathbf{F}_s = \begin{bmatrix} \mathbf{F} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\hat{\mathbf{s}}_{2|1} = \begin{bmatrix} \hat{\mathbf{x}}_{2|1} \\ \hat{\mathbf{m}}_1^1 \end{bmatrix} = \begin{bmatrix} \mathbf{F}\hat{\mathbf{x}}_{1|1} + \mathbf{G}\mathbf{u}_2 \\ \hat{\mathbf{m}}_1^1 \end{bmatrix}$$

Step 2: Covariance Prediction

$$\mathbf{P}_{2|1} = \mathbf{F}_s \mathbf{P}_{1|1} \mathbf{F}_s^\top + \mathbf{B}_s \mathbf{Q}_2 \mathbf{B}_s^\top$$

$$= \begin{bmatrix} \mathbf{P}^{\mathbf{xx}} & \mathbf{P}^{\mathbf{xm}^1} \\ \mathbf{P}^{\mathbf{m}^1\mathbf{x}} & \mathbf{P}^{\mathbf{m}^1\mathbf{m}^1} \end{bmatrix}_{2|1}$$

Step 2: Covariance Prediction

$$\mathbf{P}_{2|1} = \mathbf{F}_s \mathbf{P}_{1|1} \mathbf{F}_s^\top + \mathbf{B}_s \mathbf{Q}_2 \mathbf{B}_s^\top$$

$$P_{21} = \begin{bmatrix} F & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P^{xx} & P^{xm} \\ P^{mx} & P^{mm} \end{bmatrix} \begin{bmatrix} F^T & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} BQB^T & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} F & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P^{xx} F^T & P^{xm} \\ P^{mx} F^T & P^{mm} \end{bmatrix} + \begin{bmatrix} BQB^T & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} FP^{xx}F^T & FP^{xm} \\ P^{mx}F^T & P^{mn} \end{bmatrix} + \begin{bmatrix} BQB^T & 0 \\ 0 & 0 \end{bmatrix}$$

UCL

Step 2: Covariance Prediction

$\rightarrow +$

2

P^{xx}

$$
\begin{bmatrix}
\mathbf{F}\mathbf{P}^{\mathbf{x}\mathbf{x}}\mathbf{F}^\top + \mathbf{Q} & \mathbf{F}\mathbf{P}^{\mathbf{x}1} & \mathbf{F}\mathbf{P}^{\mathbf{x}2} & \dots & \mathbf{F}\mathbf{P}^{\mathbf{x}N_k} \\
\mathbf{P}^{1\mathbf{x}}\mathbf{F}^\top & \mathbf{P}^{11} & \mathbf{P}^{12} & \dots & \mathbf{P}^{1N_k} \\
\mathbf{P}^{2\mathbf{x}}\mathbf{F}^\top & \mathbf{P}^{21} & \mathbf{P}^{22} & \dots & \mathbf{P}^{2N_k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{P}^{N_k\mathbf{x}}\mathbf{F}^\top & \mathbf{P}^{N_k1} & \mathbf{P}^{N_k2} & \dots & \mathbf{P}^{N_kN_k}
\end{bmatrix}
$$

P^{mm}

Step 2: Covariance Prediction

$$\mathbf{P}^{\mathbf{xx}}_{2|1} = \mathbf{F}\mathbf{P}^{\mathbf{xx}}_{1|1}\mathbf{F}^{\top} + \mathbf{Q}_2$$

$$\mathbf{P}^{\mathbf{mm}}_{2|1} = \mathbf{P}^{\mathbf{mm}}_{1|1}$$

$$\mathbf{P}^{\mathbf{xm}}_{2|1} = \mathbf{F}\mathbf{P}^{\mathbf{xm}}_{1|1}$$

Step 2: Update

- We update using the standard Kalman filter equations
- The update can be written in several ways, but I prefer

$$\hat{\mathbf{s}}^{*}_{k+1|k+1} = \hat{\mathbf{s}}_{k+1|k} + \mathbf{W}_{k+1}\boldsymbol{\nu}_{k+1|k}$$

$$\mathbf{P}^{*}_{k+1|k+1} = \mathbf{P}_{k+1|k} - \mathbf{W}_{k+1}\mathbf{S}_{k+1|k}\mathbf{W}^{\top}_{k+1}$$

$(\underline{I} - WH)P$

Terms in the Kalman Filter Update

$$\boldsymbol{\nu}_{k+1|k} = \mathbf{z}_k - \mathbf{H}_s \hat{\mathbf{s}}_{k+1|k}$$

$$\mathbf{C}_{k+1|k} = \mathbf{P}_{k+1|k} \mathbf{H}_s^\top$$

$$\mathbf{S}_{k+1|k} = \mathbf{H}_s \mathbf{C}_{k+1|k} + \mathbf{R}$$

$$\mathbf{W}_{k+1|k} = \mathbf{C}_{k+1|k} \mathbf{S}_{k+1|k}^{-1}$$

Kalman Filter Update Properties

- The covariance does not increase during an update
- The states which are updated are controlled by the structure of $\mathbf{C}_{k+1|k}$

Step 2: Kalman Filter Update

$$\hat{\mathbf{s}}^*_{k+1|k+1} = \hat{\mathbf{s}}_{k+1|k} + \mathbf{W}_{k+1}\boldsymbol{\nu}_{k+1|k}$$

z

$W = C S^{-1}$

$$\hat{S}_{k+1|k} + C S^{-1} \begin{bmatrix} \nu \end{bmatrix}$$

$$C = \begin{bmatrix} C^{xz} \\ C^{x2} \\ C^{m'2} \end{bmatrix}$$

$$C = \begin{bmatrix} C^{x2} \\ C^{x2} \\ 0 \end{bmatrix}$$

Kalman Filter Update Properties

$$\mathbf{C}_{2|1} = \mathbf{P}_{2|1}\mathbf{H}_s^\top$$

$$= \begin{bmatrix} P_{xx} & P_{x\dot{x}} & P_{xm^1} \\ P_{\dot{x}x} & P_{\dot{x}\dot{x}} & P_{\dot{x}m^1} \\ P_{m^1x} & P_{m^1\dot{x}} & P_{m^1m^1} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} P_{xm^1} - P_{xx} \\ P_{\dot{x}m^1} - P_{\dot{x}x} \\ P_{m^1m^1} - P_{m^1x} \end{bmatrix} \begin{matrix} \Leftarrow \\ \Leftarrow \\ \Leftarrow \end{matrix}$$

Kalman Filter Update Properties

$$P^{m_1' m_1'} - P^{m_1' x}$$

$$P_1^{xx} + R_1' - (F P_1^{xx})$$

$$= (I - F) P_1^{xx} + R_1' \neq 0$$

$$F \neq I$$

Step 2: Prediction and Update

$FPF^T + Q$

- The covariance in the vehicle *increases*, the landmarks *do not*

FP^{xm}

- The cross correlation between the vehicle and beacon change as a result of the process model and process noise
- In the update, both the vehicle and beacon covariances decline

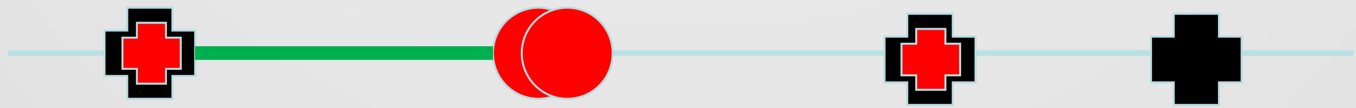
ROBOTICS
Innovation + Application

UCL ENGINEERING
Change the world

BeadSLAM: The First Few Timesteps

Timestep	Action
~~0~~	~~Initialize~~
~~1~~	~~Predict and observe landmark 1~~
~~2~~	~~Predict and update landmark 1~~
3	Predict, observe landmark 2

Step 3: Predict and Observe Landmark 2



Step 3: Predict and Observe Landmark 2

- The predict step is the same as in the last step
- We do not observe landmark 1 and so the predicted estimate is the partially updated estimate
- However, we now need to see what happens when we augment a map with another landmark

Step 3: Augmentation

- We use the same kind of augmentation operator again,

x, m_1, m_2

x, m_1

$$\hat{\mathbf{s}}_{3|3} = \underline{\mathbf{A}_3}\hat{\mathbf{s}}^*_{3|3} + \underline{\mathbf{J}_3}\mathbf{z}^1_3$$

$$\mathbf{P}_{3|3} = \mathbf{A}_3\mathbf{P}^*_{3|3}\mathbf{A}_3^\top + \mathbf{J}_3\mathbf{R}^1_3\mathbf{J}_3^\top$$

UCL

Step 3: Augmentation

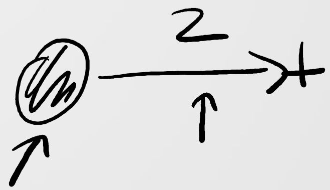
$$\hat{\mathbf{s}}_{3|3} = \begin{bmatrix} \hat{x}^*_{3|3} \\ \hat{\dot{x}}^*_{3|3} \\ \hat{m}^1_3 \\ \hat{m}^2_3 \end{bmatrix} = \begin{bmatrix} \hat{x}^*_{3|3} \\ \hat{\dot{x}}^*_{3|3} \\ \hat{m}^1_2 \\ \hat{x}^*_{3|3} + z^1_3 \end{bmatrix}$$

Step 3: Augmentation Linear Operation

$$\mathbf{A}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \mathbf{J}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$x + 2$

Step 3: Augmentation

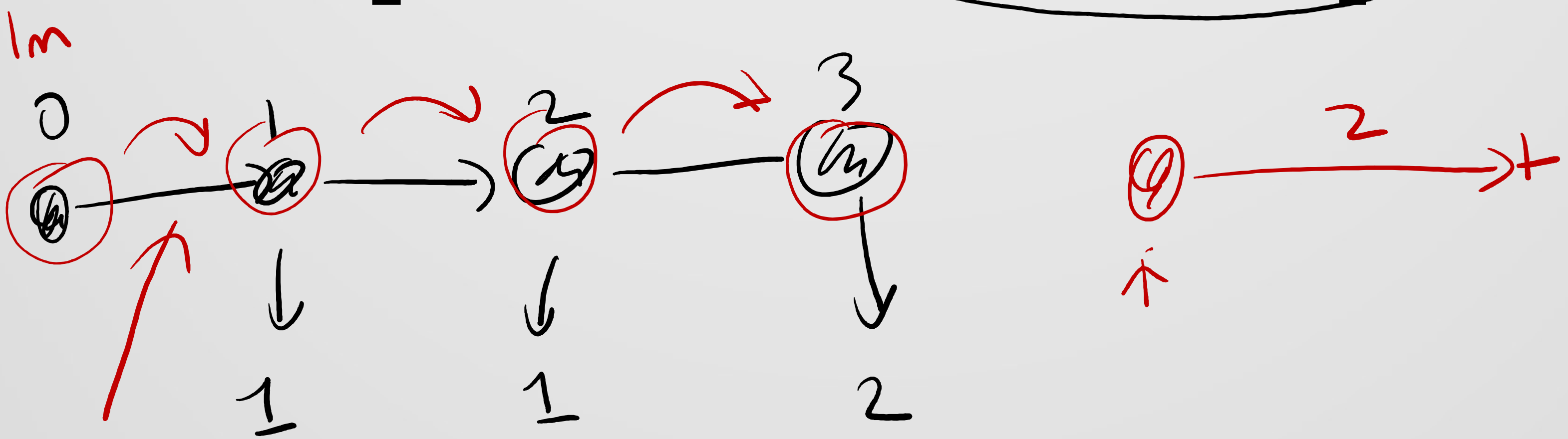


$$\mathbf{P}_{3|3} = \mathbf{A}_3 \mathbf{P}^*_{3|3} \mathbf{A}_3^\top + \mathbf{J}_3 \mathbf{R}^1_3 \mathbf{J}_3^\top$$

$$= \begin{bmatrix} P^{x^* x^*} & P^{x^* \dot{x}^*} & P^{x^* m^{1*}} & P^{x^* x^*} \\ P^{\dot{x}^* x^*} & P^{\dot{x}^* \dot{x}^*} & P^{\dot{x}^* m^{1*}} & P^{\dot{x}^* x^*} \\ P^{m^{1*} x^*} & P^{m^{1} * \dot{x}^*} & P^{m^{1*} m^{1*}} & P^{m^{1*} x^*} \\ P^{x^* x^*} & P^{x^* \dot{x}^*} & P^{x^* m^{1*}} & P^{x^* x^*} + R^1_3 \end{bmatrix}$$

Step 3: Augmentation

$$\begin{bmatrix} P^{m^{1*}m^{1*}} & P^{m^{1*}x^*} \\ P^{x^*m^{1*}} & P^{x^*x^*} + R_3^1 \end{bmatrix}$$



$X_{k+1} = F x_k + G u_k + v_k$

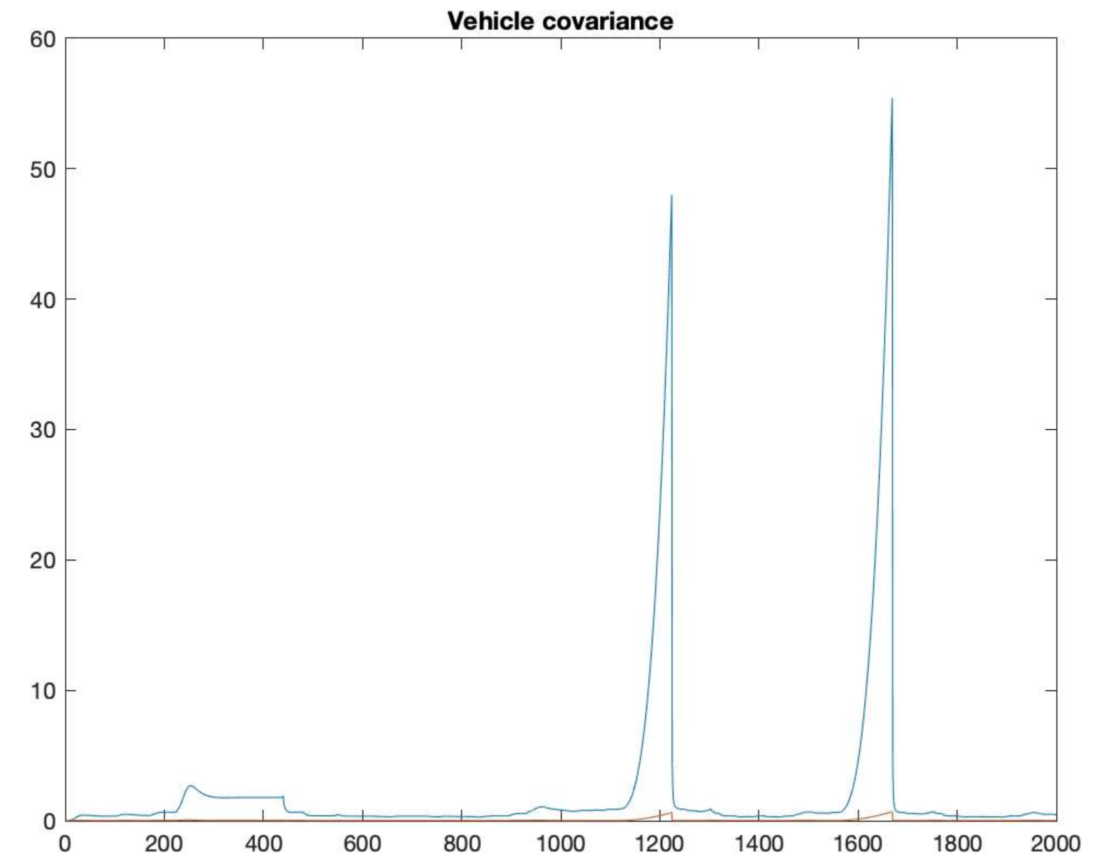
Analysing BeadSLAM

- We analyse BeadSLAM by looking at the covariance history only

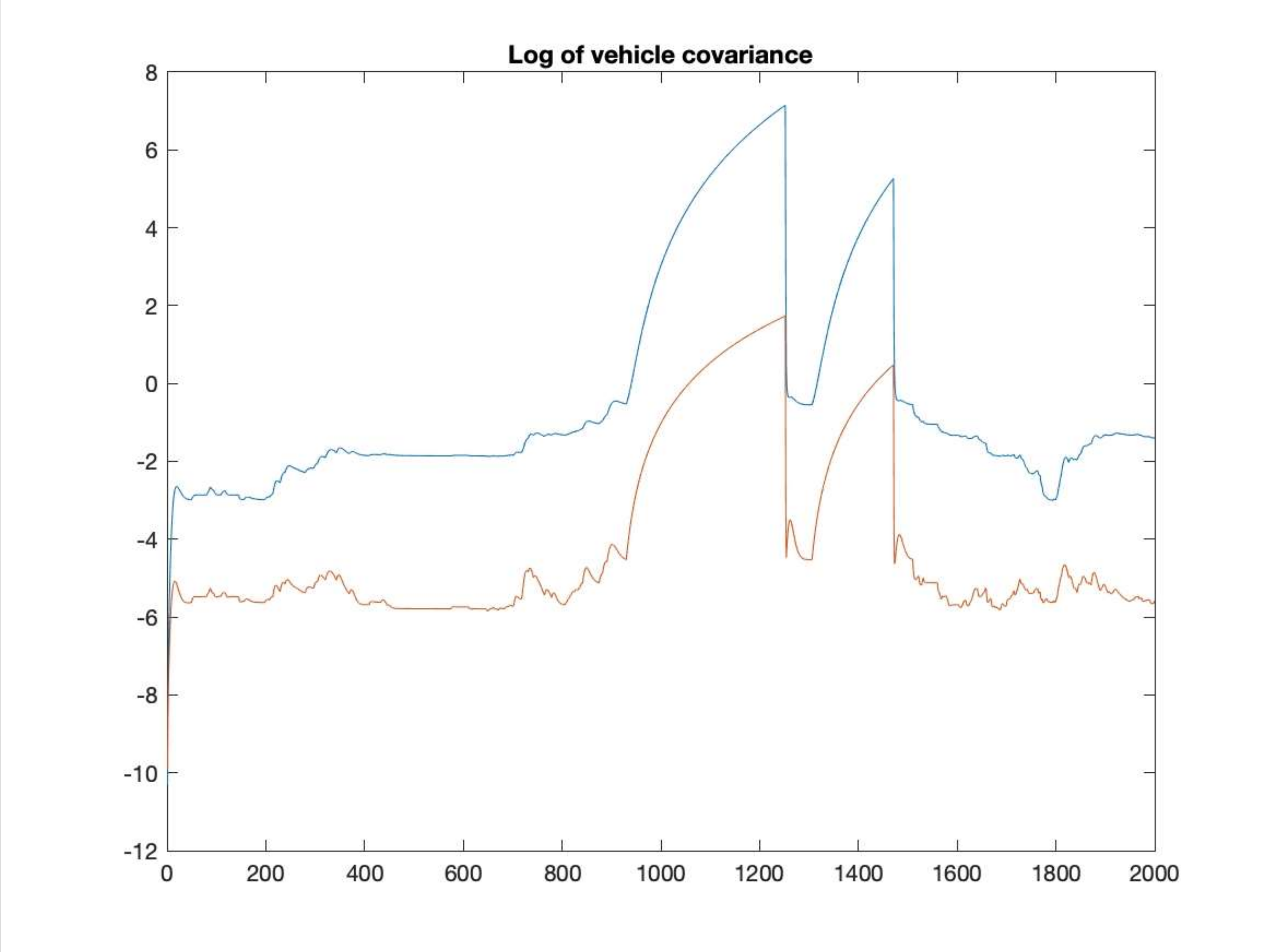
P^{xx} P^{xm} P^{mm}

- We look at the following:
  - The vehicle position covariance
  - The vehicle position / landmark correlations
  - The landmark position covariances

BeadSLAM Vehicle Covariance



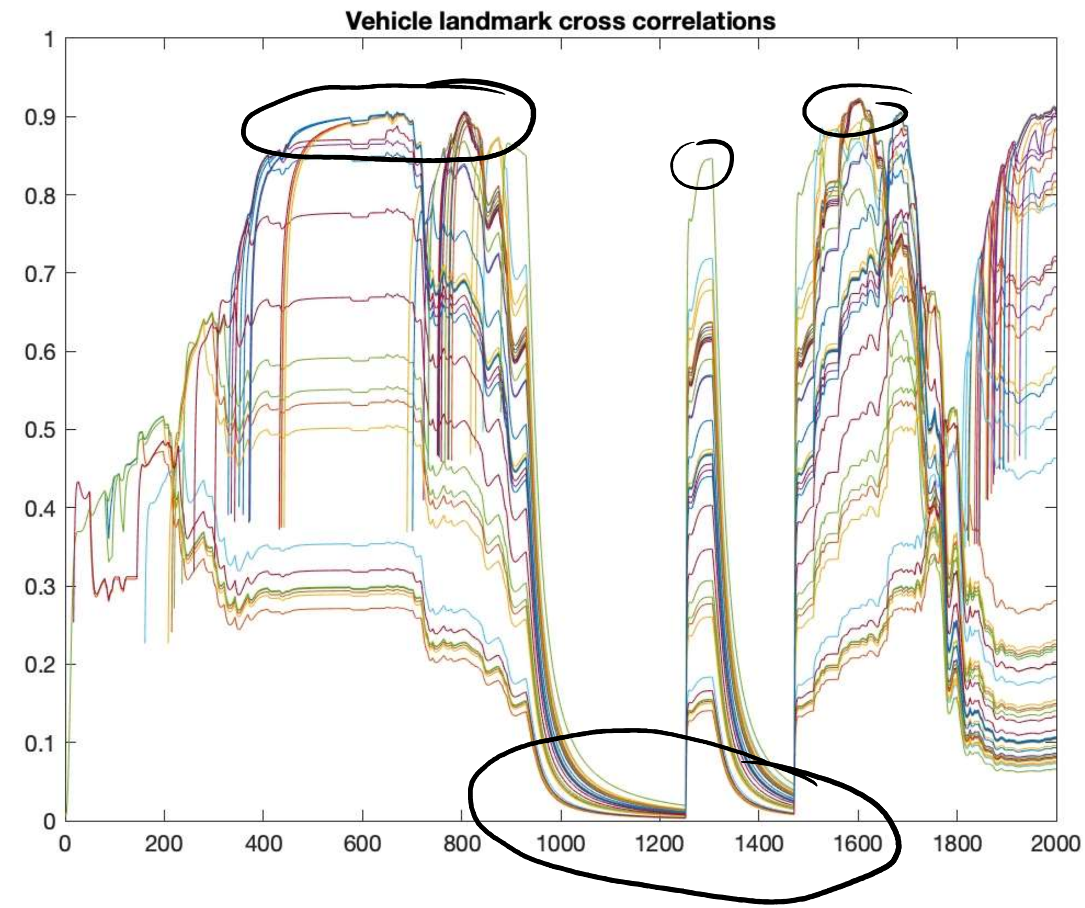
BeadSLAM Log Vehicle Covariance



Landmark Vehicle Cross Correlation

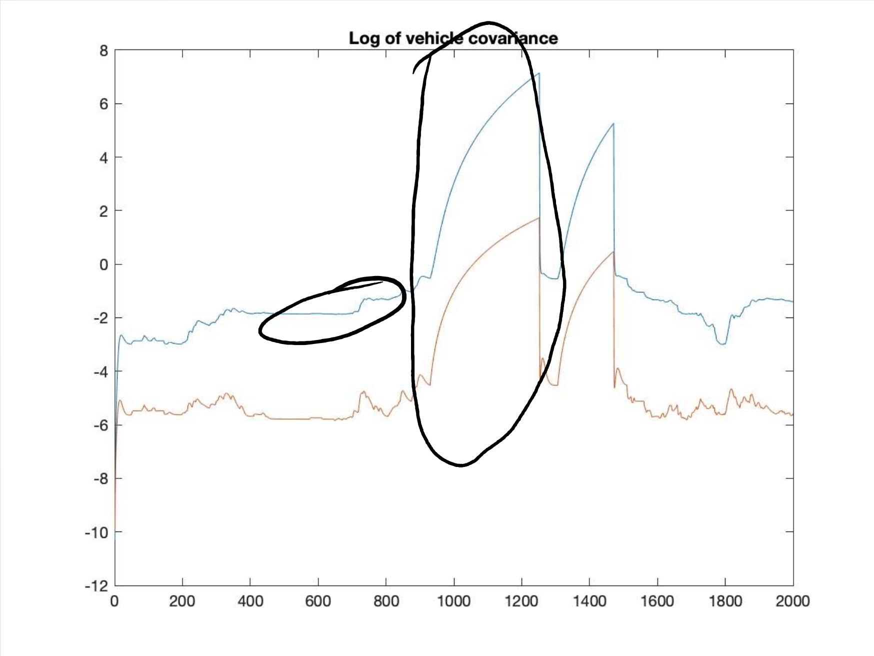
P^{xm}

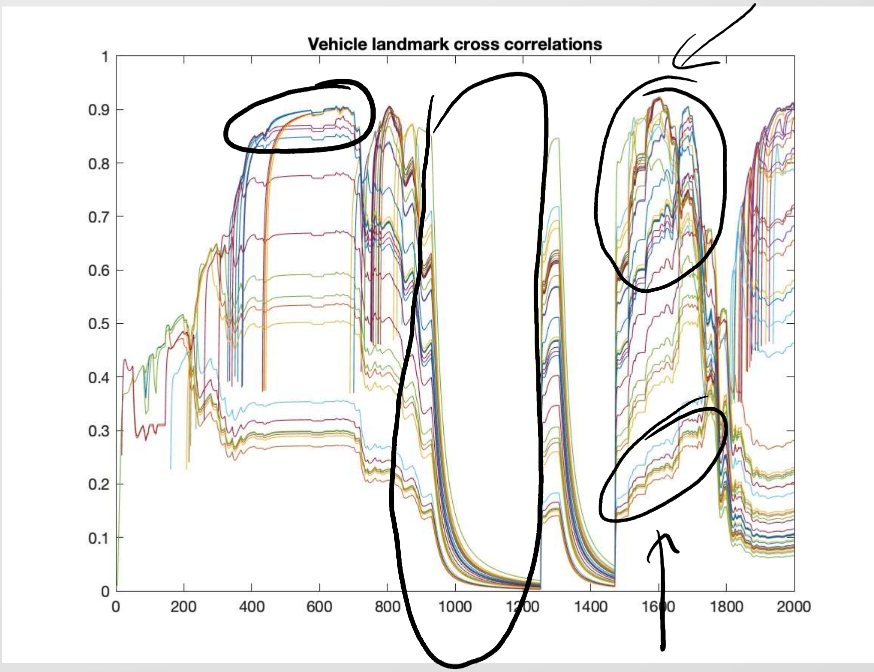
$\frac{P^{mm}}{P^{xx}}$



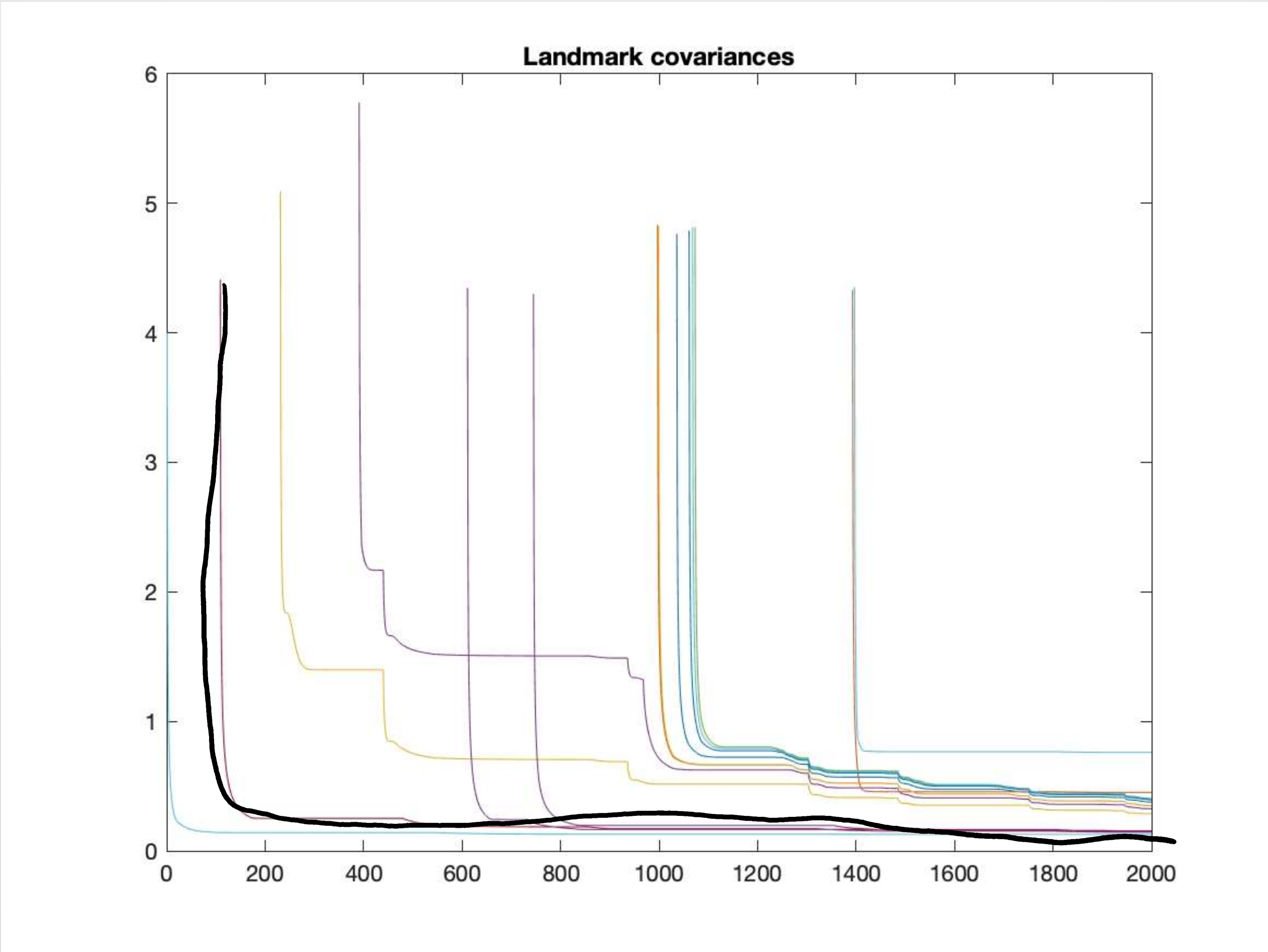
Landmark Vehicle Cross Correlation

$$\frac{pmm}{pxx}$$





Landmark Covariances



Landmark Convergence

2

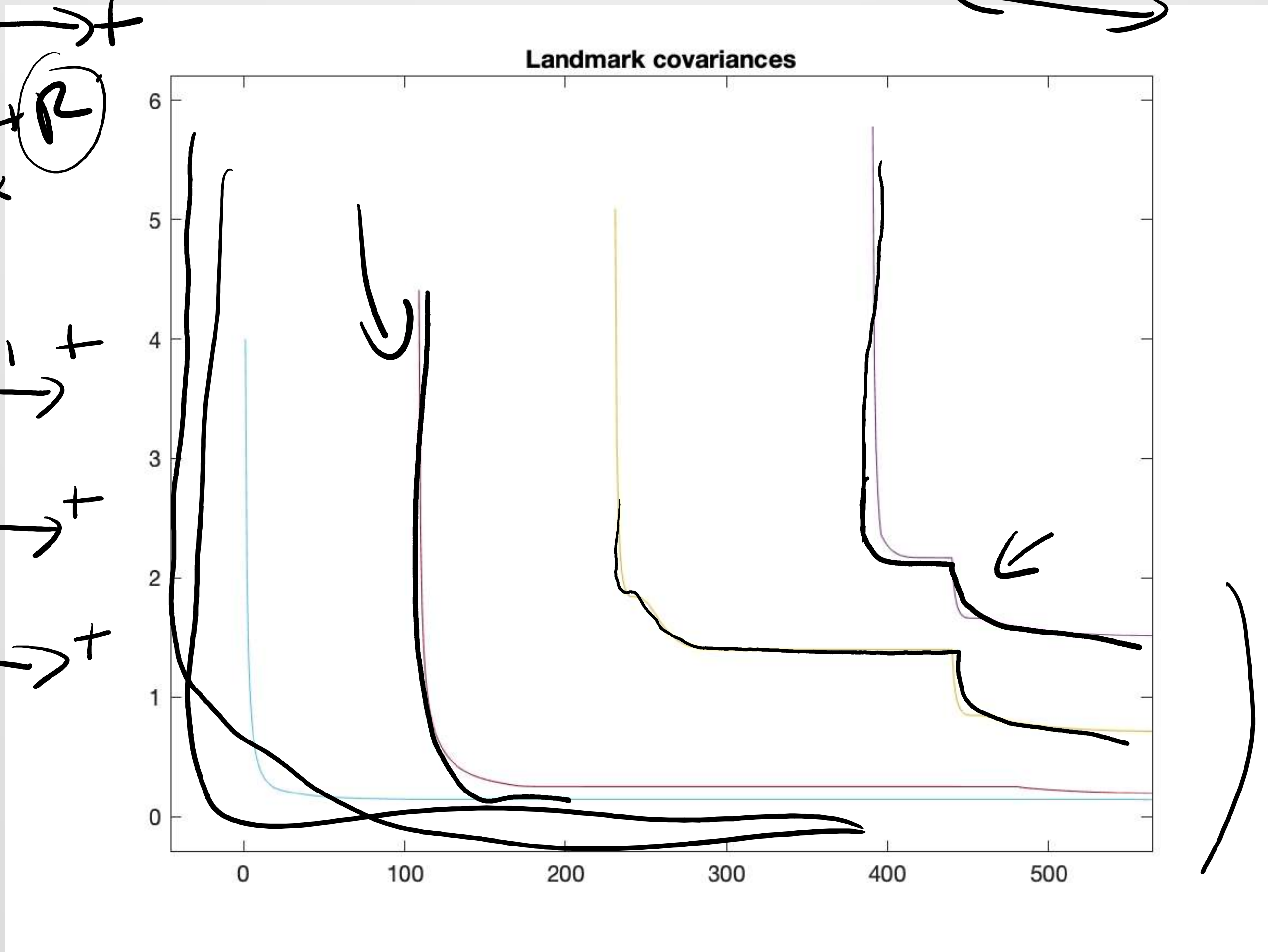
$\uparrow P^{xx} + R$

$R \gg P^{xx}$

z_1

z_2

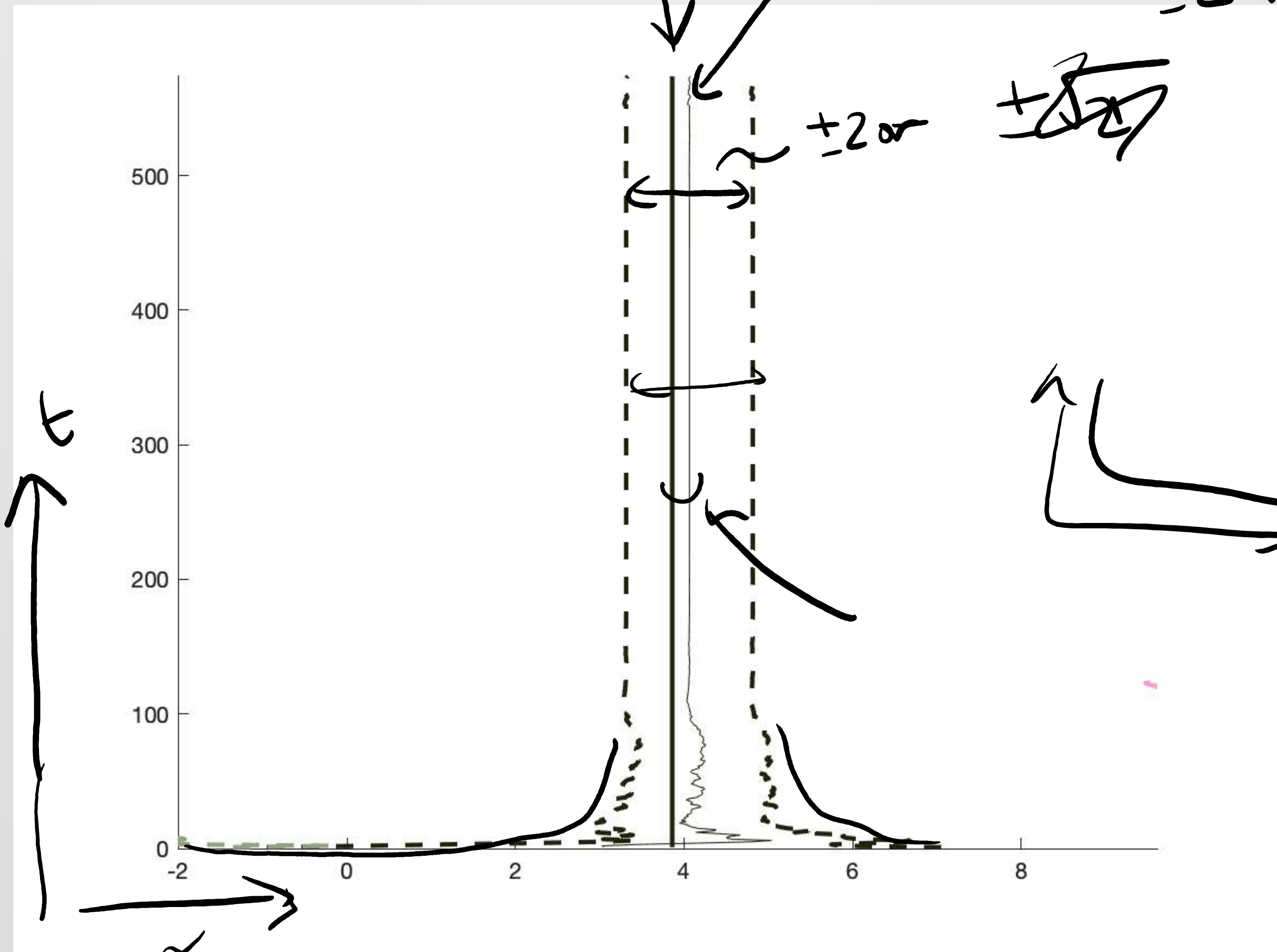
z_3



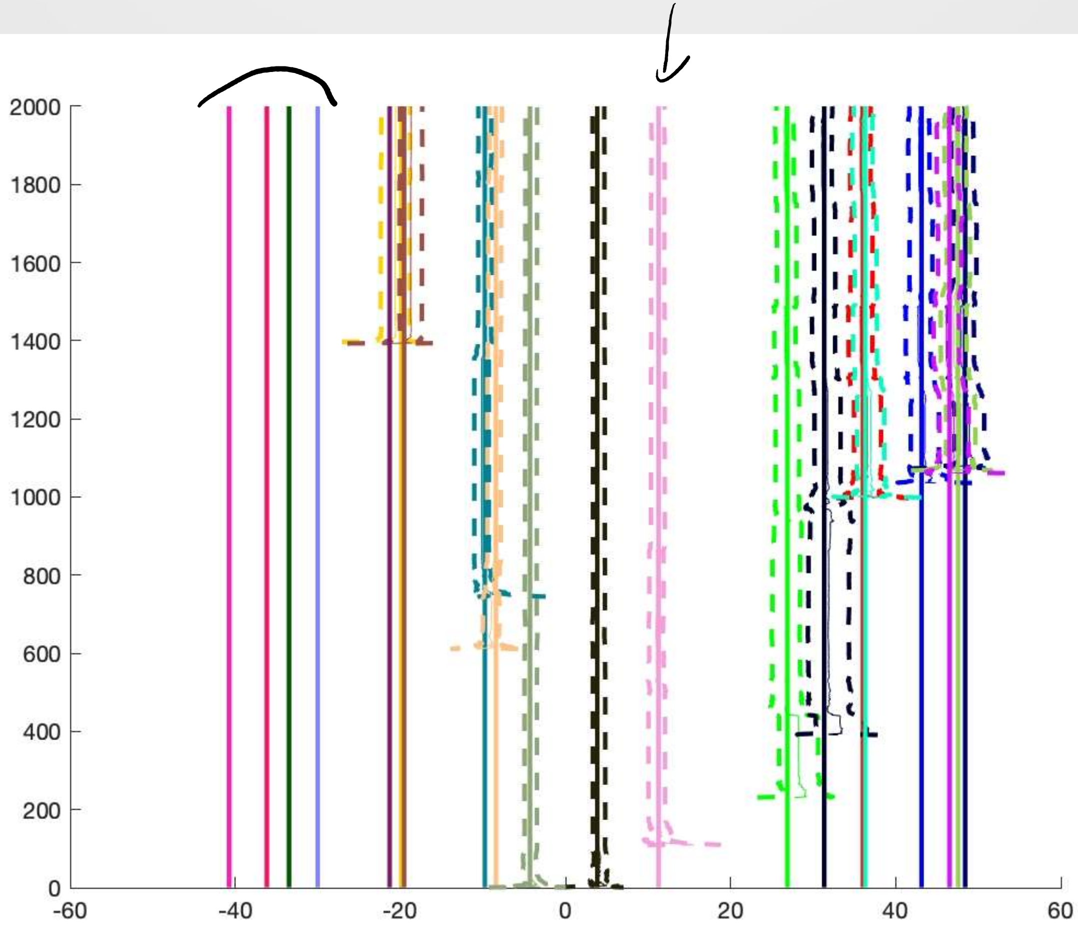
Waterfall Plot

$x,\ m,\ m^2$

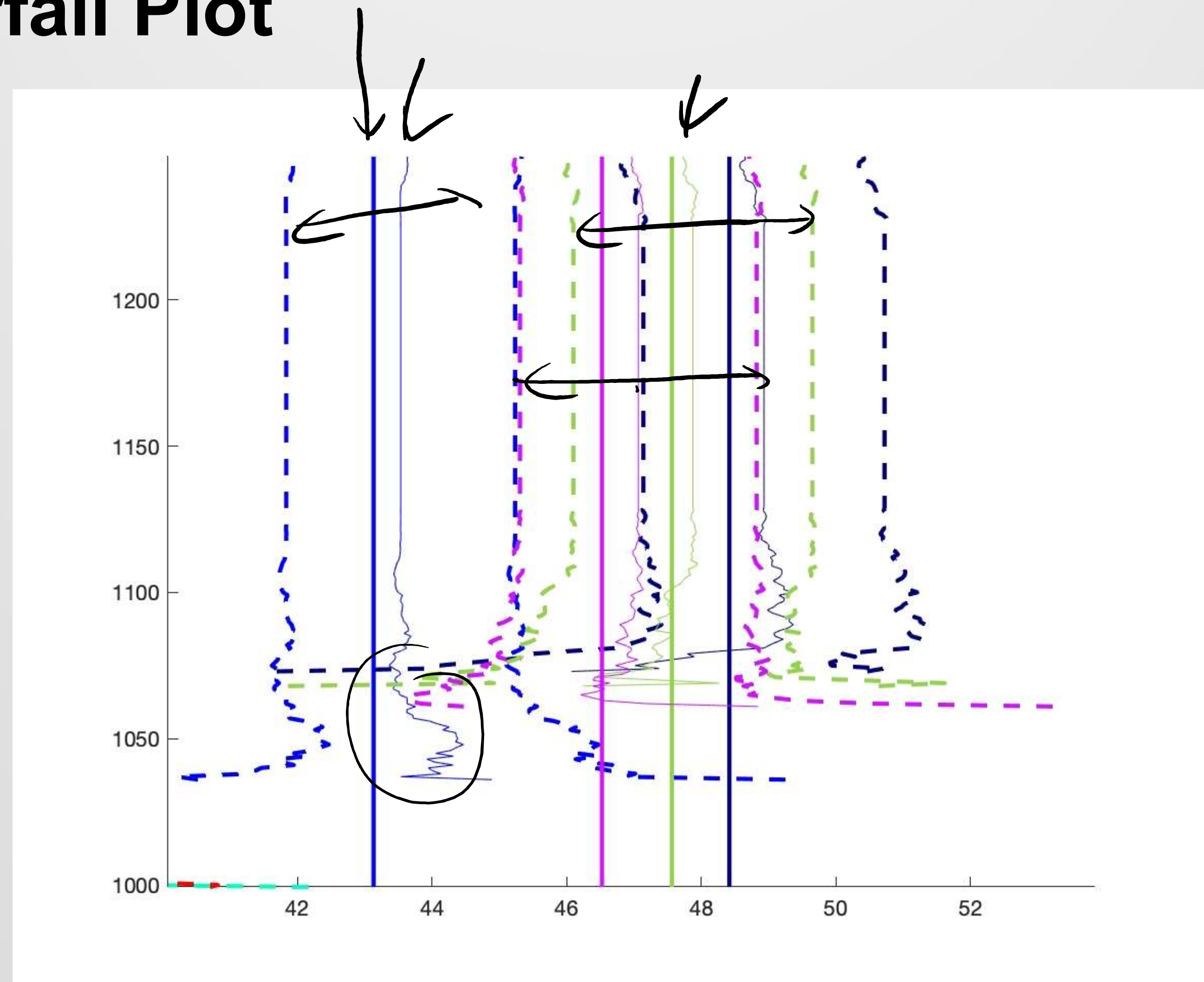
$\pm 2\sqrt{p}$



Waterfall Plot



Waterfall Plot



Theoretical Properties of SLAM Algorithms

P^{xx}
P^{xm}
P^{mm}

1. The determinant of any submatrix of the map covariance matrix decreases monotonically as observations are successively made.

Map Covariance Block

P^{mm}

$$\begin{bmatrix} \mathbf{P}_{\mathbf{m}^1\mathbf{m}^1} & \mathbf{P}_{\mathbf{m}^1\mathbf{m}^2} & \cdots & \mathbf{P}_{\mathbf{m}^1\mathbf{m}^{N_k}} \\ \mathbf{P}_{\mathbf{m}^2\mathbf{m}^1} & \mathbf{P}_{\mathbf{m}^2\mathbf{m}^2} & \cdots & \mathbf{P}_{\mathbf{m}^2\mathbf{m}^{N_k}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}_{\mathbf{m}^{N_k}\mathbf{m}^1} & \mathbf{P}_{\mathbf{m}^{N_k}\mathbf{m}^2} & \cdots & \mathbf{P}_{\mathbf{m}^{N_k}\mathbf{m}^{N_k}} \end{bmatrix}$$

Map Covariance Block

$$\begin{bmatrix} \mathbf{P}_{\mathbf{m}^1\mathbf{m}^1} & \mathbf{P}_{\mathbf{m}^1\mathbf{m}^2} & \cdots & \mathbf{P}_{\mathbf{m}^1\mathbf{m}^{N_k}} \\ \mathbf{P}_{\mathbf{m}^2\mathbf{m}^1} & \mathbf{P}_{\mathbf{m}^2\mathbf{m}^2} & \cdots & \mathbf{P}_{\mathbf{m}^2\mathbf{m}^{N_k}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}_{\mathbf{m}^{N_k}\mathbf{m}^1} & \mathbf{P}_{\mathbf{m}^{N_k}\mathbf{m}^2} & \cdots & \mathbf{P}_{\mathbf{m}^{N_k}\mathbf{m}^{N_k}} \end{bmatrix}$$

Map Covariance Block

$$\begin{bmatrix} \mathbf{P}_{\mathbf{m}^1\mathbf{m}^1} & \mathbf{P}_{\mathbf{m}^1\mathbf{m}^2} & \cdots & \mathbf{P}_{\mathbf{m}^1\mathbf{m}^{N_k}} \\ \mathbf{P}_{\mathbf{m}^2\mathbf{m}^1} & \boxed{\mathbf{P}_{\mathbf{m}^2\mathbf{m}^2}} & \cdots & \boxed{\mathbf{P}_{\mathbf{m}^2\mathbf{m}^{N_k}}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}_{\mathbf{m}^{N_k}\mathbf{m}^1} & \boxed{\mathbf{P}_{\mathbf{m}^{N_k}\mathbf{m}^2}} & \cdots & \boxed{\mathbf{P}_{\mathbf{m}^{N_k}\mathbf{m}^{N_k}}} \end{bmatrix}$$

Theoretical Properties of SLAM Algorithms

1. The determinant of any submatrix of the map covariance matrix decreases monotonically as observations are successively made.
2. In the limit as the number of observations increases, the landmark estimates become fully correlated.

Full Correlation of the Map Block

$$\begin{bmatrix} \mathbf{P}_{\mathbf{m}^1\mathbf{m}^1} & \mathbf{P}_{\mathbf{m}^1\mathbf{m}^2} & \cdots & \mathbf{P}_{\mathbf{m}^1\mathbf{m}^{N_k}} \\ \mathbf{P}_{\mathbf{m}^2\mathbf{m}^1} & \mathbf{P}_{\mathbf{m}^2\mathbf{m}^2} & \cdots & \mathbf{P}_{\mathbf{m}^2\mathbf{m}^{N_k}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}_{\mathbf{m}^{N_k}\mathbf{m}^1} & \mathbf{P}_{\mathbf{m}^{N_k}\mathbf{m}^2} & \cdots & \mathbf{P}_{\mathbf{m}^{N_k}\mathbf{m}^{N_k}} \end{bmatrix}$$

Full Correlation of the Map Block

$$\begin{bmatrix} \mathbf{P}_{\#\#} & \mathbf{P}_{\#\#} & \cdots & \mathbf{P}_{\#\#} \\ \mathbf{P}_{\#\#} & \mathbf{P}_{\#\#} & \cdots & \mathbf{P}_{\#\#} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}_{\#\#} & \mathbf{P}_{\#\#} & \cdots & \mathbf{P}_{\#\#} \end{bmatrix}$$

Demonstrating the Convergence

- We constructed a scenario with 3 landmarks
- These landmarks are observed all the time
- The filter was run for different numbers of timesteps
- The final covariance matrix was output

Covariance Matrices (2k, 4k, 16k steps)

Pxx

pxm

pmm

0.2551	0.0108	0.0907	0.0907	0.0907
0.0108	0.0015	-0.0000	-0.0000	0.0000
0.0907	-0.0000	0.0921	0.0901	0.0901
0.0907	-0.0000	0.0901	0.0921	0.0901
0.0907	0.0000	0.0901	0.0901	0.0921

0.2551	0.0108	0.0907	0.0907	0.0907
0.0108	0.0015	-0.0000	-0.0000	0.0000
0.0907	-0.0000	0.0914	0.0904	0.0904
0.0907	-0.0000	0.0904	0.0914	0.0904
0.0907	0.0000	0.0904	0.0904	0.0914

0.2551	0.0108	0.0907	0.0907	0.0907
0.0108	0.0015	-0.0000	0.0000	-0.0000
0.0907	-0.0000	0.0909	0.0906	0.0906
0.0907	0.0000	0.0906	0.0909	0.0906
0.0907	-0.0000	0.0906	0.0906	0.0909

Theoretical Properties of SLAM Algorithms

1. The determinant of any submatrix of the map covariance matrix decreases monotonically as observations are successively made.
2. In the limit as the number of observations increases, the landmark estimates become fully correlated.
3. In the limit, the covariance associated with any single landmark location estimate is determined only by the initial covariance in the vehicle location estimate

Explanation of the Convergence Result

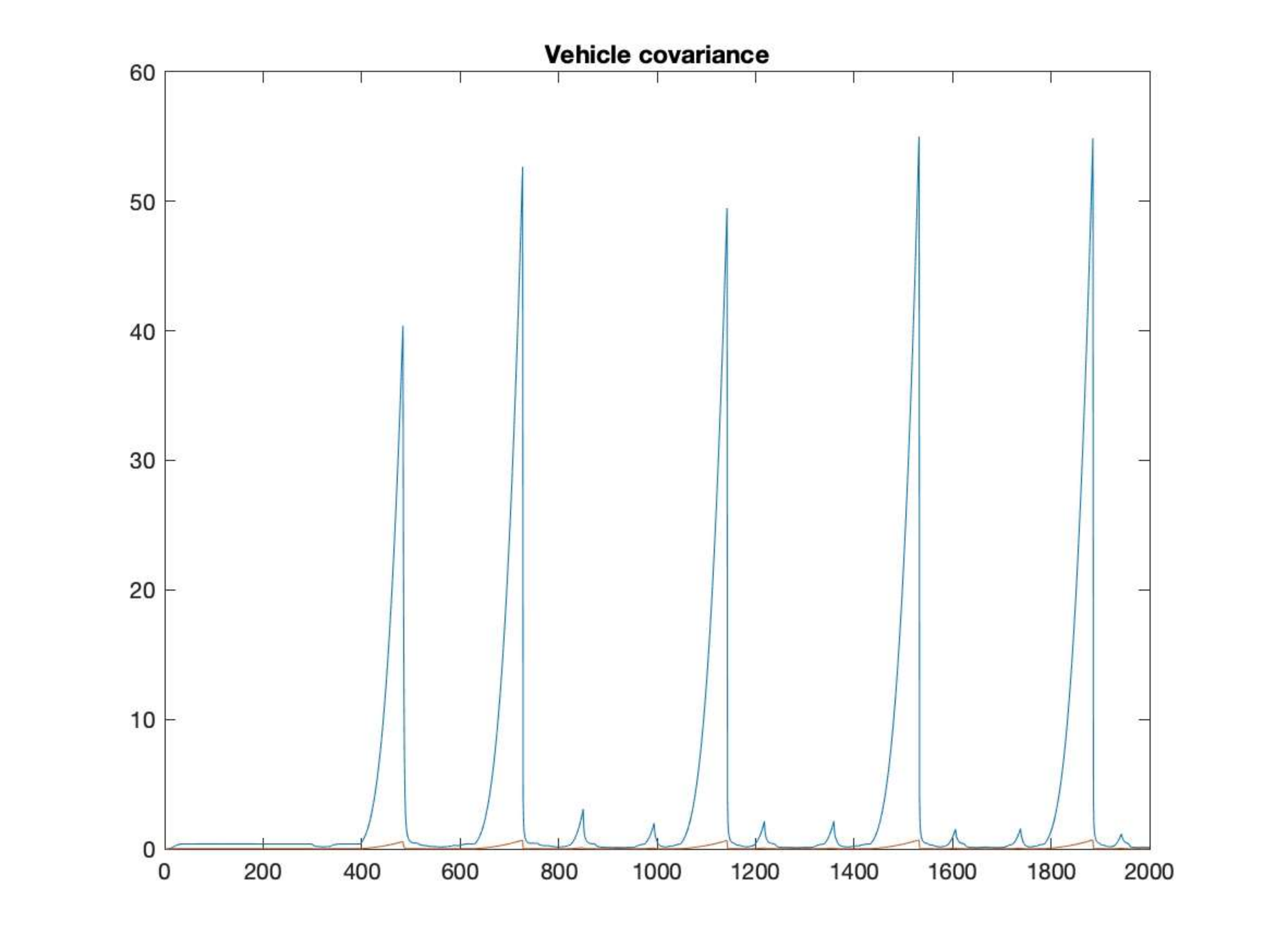
- The convergence result means that all the beacons become ”locked” relative to one another
  - The covariance of the relative distance between the beacons becomes zero everywhere
- The reason is that all the independent process and observation noises all get filtered out
- The vehicle covariance still fluctuates, however, depending upon where the vehicle is and what it can see

Removing Correlations

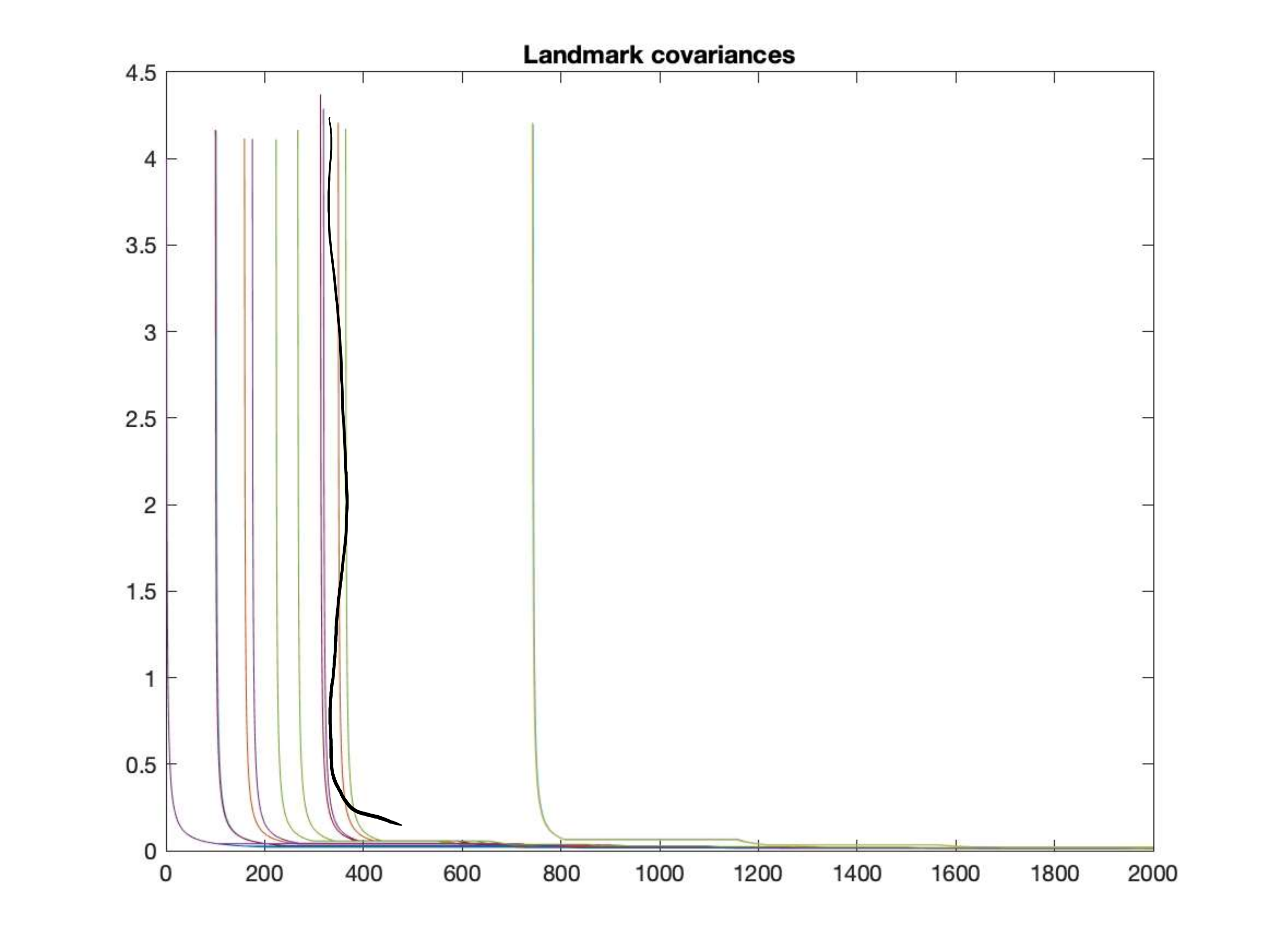
- What happens if we get rid of the correlations?
- In this case, we approximate the system covariance as

$$\mathbf{P}_{k|k} = \begin{bmatrix} \mathbf{P_{xx}} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{P_{m^1m^1}} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P_{m^2m^2}} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{P}_{\mathbf{m}^{n_k}\mathbf{m}^{n_k}} \end{bmatrix}$$

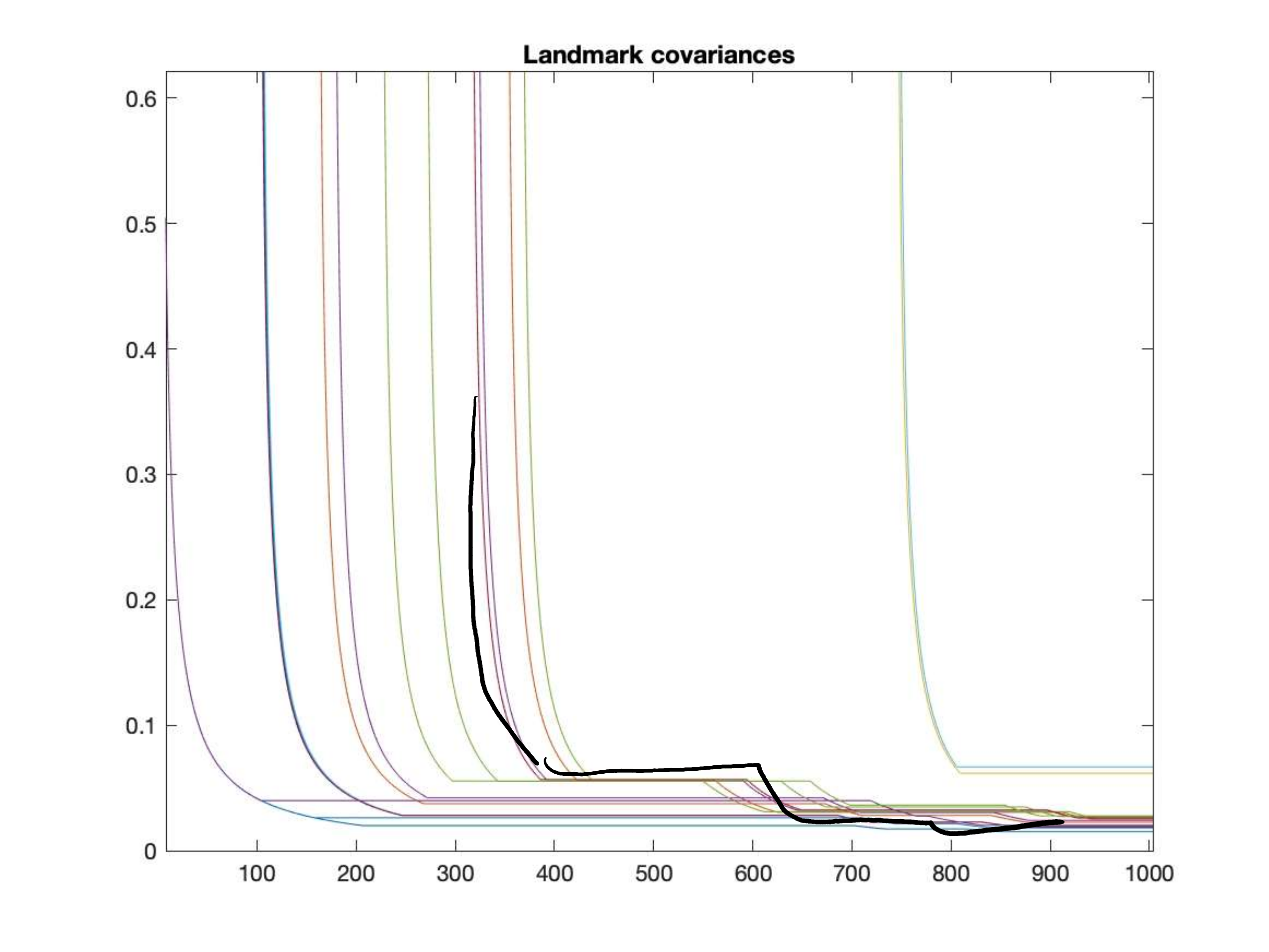
Vehicle Covariance



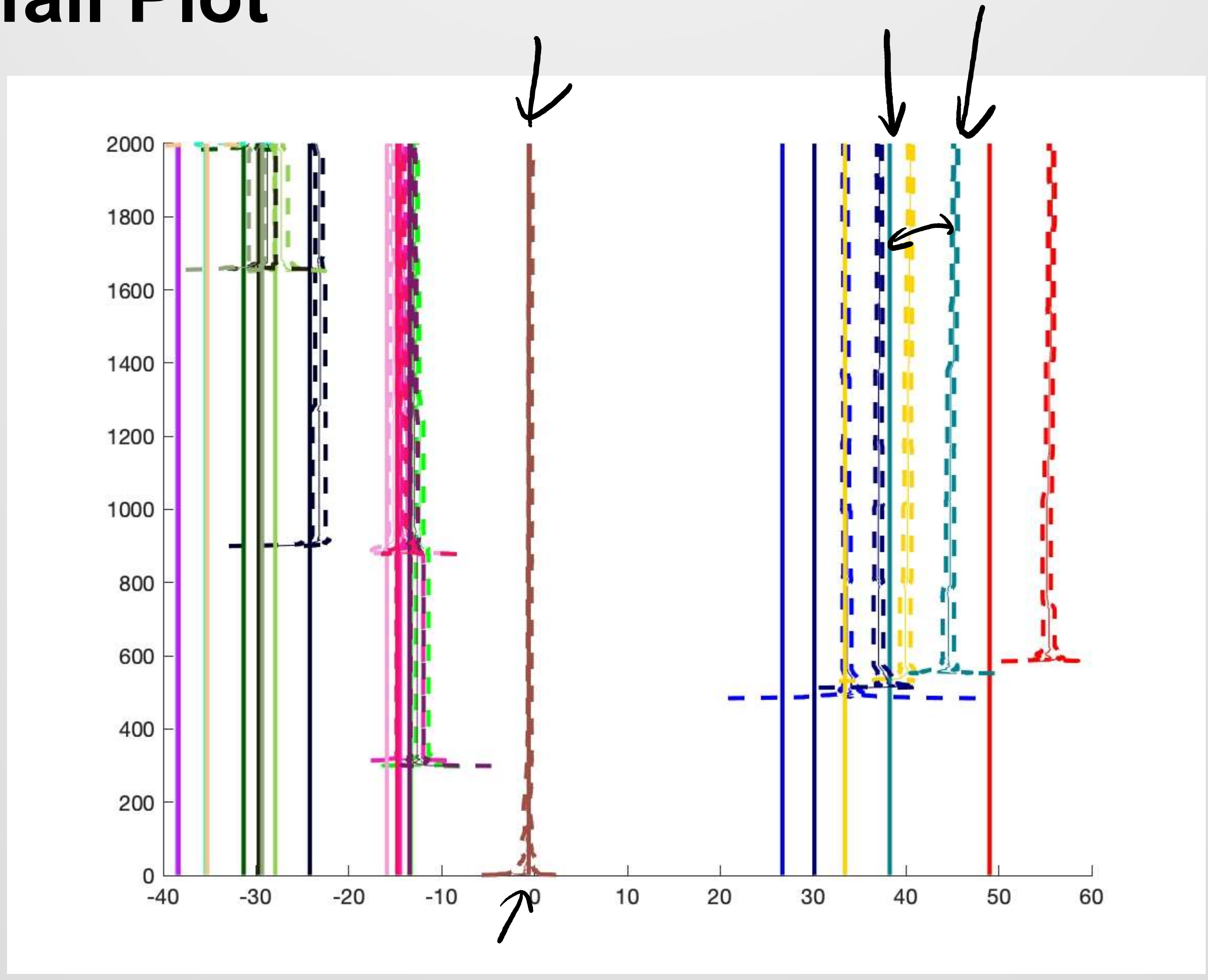
Landmark Covariances



Landmark Covariances (Zoomed)



Waterfall Plot



Waterfall Plot