GIVEN: POINT SETS

$$P = \{p_i\}_{i=1:n}$$

$$Q = \{q_i\}_{i=1:n}$$

+ CORRESPONDENCE, i.e., $p_i \leftrightarrow q_i \quad \forall \; i$

GOAL. FIND RIGID TRANSFORM (R, t) SUCH THAT

$$E(R,t) = \sum \| p_i - Rq_i - t \|^2$$

IS MINIMIZED, i.e., BEST ALIGNING RIGID TRANSFORM FOR THE GIVEN CORRESPONDENCE.

$$E(R,t) = \sum_{i=1}^{n} \| p_i - Rq_i - t \|^2 \quad (1)$$

SUCH THAT $\underbrace{RR^T = I \quad (2) \quad + \quad \det(R) = 1 \quad (3)}_{\text{RIGID TRANSF}}$

$$\min_{R,t} E(R,t) \Rightarrow \left. \begin{aligned} \frac{\partial E}{\partial R} &= 0 \quad 4a \\ \frac{\partial E}{\partial t} &= 0 \quad 4b \end{aligned} \right.$$

FROM (4b)

$$\frac{\partial E}{\partial t} = \sum_i 2 (p_i - Rq_i - t) = 0$$

$$\Rightarrow \frac{\sum p_i}{n} - \frac{\sum_i Rq_i}{n} - \frac{\sum t}{n} = 0$$

$$\Rightarrow \boxed{\bar{p} = R\bar{q} + t} \quad (5)$$

$$\left. \begin{aligned} \bar{p} &= \frac{\sum p_i}{n} \\ \bar{q} &= \frac{\sum q_i}{n} \end{aligned} \right\}$$ MEAN OF POINT SETS

USING (5) IN (1) WE GET,

$$E(R,t) = \sum_{i=1}^{n} \| p_i - Rq_i + R\bar{q} - \bar{p} \|^2$$

$$= \sum_{i=1}^{n} \| (p_i - \bar{p}) - R(q_i - \bar{q}) \|^2$$

WHERE

$$\left\{ \begin{aligned} \tilde{p}_i &= p_i - \bar{p} \\ \tilde{q}_i &= q_i - \bar{q} \end{aligned} \right. \quad \forall \; i$$

$$= \sum_i \| \tilde{p}_i - R\tilde{q}_i \|^2$$

$$= E(R)$$ $\Big\}$ ONLY DEPENDS ON R

EQUATION ① REDUCES TO,

$$E(R) = \sum_i \|\tilde{p}_i - R\tilde{q}_i\|^2$$

$$\boxed{\min_R \ E(R) \quad s.t. \quad RR^T = I, \quad \det R = 1}$$ ⑥

$$E(R) = \sum_i (\tilde{p}_i - R\tilde{q}_i)^T (\tilde{p}_i - R\tilde{q}_i)$$

RECALL

$\|x\|^2 = x^T x$

when x is A COLUMN VECTOR

$$= \sum_i \tilde{p}_i^T \tilde{p}_i - \sum_i \tilde{p}_i^T R\tilde{q}_i - \sum_i \tilde{q}_i^T R^T \tilde{p}_i + \sum_i \tilde{q}_i^T \underbrace{R^T R}_{\text{IDENTITY USING ②}} \tilde{q}_i$$

IF $x^T . y$ IS A SCALAR

THEN, $x^T y = y^T x$

$\tilde{q}_i^T R^T \tilde{p}_i = (R\tilde{q}_i)^T \tilde{p}_i = \tilde{p}_i^T R\tilde{q}_i$

$$= \underbrace{\sum_i \tilde{p}_i^T \tilde{p}_i + \sum \tilde{q}_i^T \tilde{q}_i}_{\text{INDEPENDENT OF } R} - 2\sum \tilde{p}_i^T R\tilde{q}_i$$

$$\Rightarrow \min_R E(R) = \min -2\sum \tilde{p}_i^T R\tilde{q}_i = \max_R \sum \tilde{p}_i^T R\tilde{q}_i$$

EQUIVALENT TO

SO, EQUATION ⑥ REDUCES TO

$$\boxed{\max \sum_i \tilde{p}_i^T R\tilde{q}_i \quad s.t. \quad RR^T = I, \quad \det R = 1}$$ ⑦

$$\sum_i \tilde{p}_i^T R\tilde{q}_i = \text{TRACE}\left\{ \underbrace{\begin{bmatrix} - \tilde{p}_1^T - \\ - \tilde{p}_2^T - \\ \vdots \\ - \tilde{p}_n^T - \end{bmatrix}_{n\times 3}}_{\tilde{P}^T} [R]_{3\times 3} \underbrace{\begin{bmatrix} \tilde{q}_1 & \tilde{q}_2 & \dots & \tilde{q}_n \end{bmatrix}_{3\times n}}_{\tilde{Q} \text{ matrix}} \right\}$$

$$= \text{TRACE}\left(\tilde{P}^T R \tilde{Q}\right)$$

TRACE OF A MATRIX = SUM OF DIAGONALS

SO, EQUATION (7) REDUCES TO

$$\boxed{\max_{R} \text{TRACE}\left(\tilde{P}^T R \tilde{Q}\right) \quad \text{s.t.} \quad RR^T = I,\ \det(R) = 1} \qquad (8)$$

$$\text{TRACE}(AB) = \text{trace}(BA) \quad - (9)$$

$$\Rightarrow \text{trace}\left(\tilde{P}^T R \tilde{Q}\right) = \text{trace}\ \tilde{R} Q \tilde{P}^T$$

LET $\tilde{Q}\tilde{P}^T = \underbrace{U \Sigma V^T}_{\text{SINGULAR VALUE DECOMPOSITION}}$

$$\left\{ \text{SVD}\left(\tilde{Q}\tilde{P}^T\right) = U \Sigma V^T \right.$$

$$\uparrow$$

$$\text{SVD}\left(\sum_i q_i \tilde{p}_i^T\right)$$

$$\Rightarrow \text{trace}\left(\tilde{P}^T R \tilde{Q}\right) = \text{trace}\left(R U \Sigma V^T\right) \quad \{\text{USING } (9)$$

$$= \text{trace}\left(\Sigma \underbrace{V^T R U}_{\text{ORTHONORMAL SINCE}}\right)$$

$$V^T V = U^T U = R^T R = I$$

SAY $M = V^T R U = \begin{vmatrix} m_1 & m_2 & m_3 \end{vmatrix}$

SINCE $M^T M = I \Rightarrow 1 = m_i^T m_i = \sum_{i=1}^{3} m_{ij}^2 \Rightarrow |m_{ij}| \leq 1$ (10)

~~max~~ EQUATION (8) REDUCES TO

$$\max_R \text{trace}(\Sigma M) = \max_R \text{trace} \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_3 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$= \sum_{i=1}^{3} \sigma_i\ m_{ii}$$

$$\leq \sum_{i=1}^{3} \sigma_i \quad (11) \qquad \left\{ \begin{array}{l} \text{USING } (10) \\ \& \ \cancel{\sigma\sigma}\ \sigma_i \geq 0 \end{array} \right.$$

SO $\max_R \text{tr}(\Sigma M) \leq \sum_i \sigma_i$

& EQUALITY HOLDS WHEN $m_{ii} = 1$ { FOLLOWS FROM (10) & (11)

so $\max_R tr(\tilde{P}^T R \tilde{Q}) \Rightarrow m_{ii} = 1 \quad i = 1:3$

$$\Rightarrow M = I$$

$$\Rightarrow V^T R U = I$$

$$\Rightarrow V V^T R U U^T = V U^T$$

$VV^T = I$
$UU^T = I$

$$\Rightarrow \boxed{R = VU^T}$$

IF $det(VU^T) = 1$, THEN $\boxed{\begin{array}{l} R = VU^T \\ t = \bar{p} - R\bar{q} \end{array}}$ (USING 5

IF $det(VU^T) = -1$, THEN $\boxed{\begin{array}{l} R = V \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} U^T \\ t = \bar{p} - R\bar{q} \end{array}}$

FINAL SOLUTION.

$$\rightarrow \begin{cases} \bar{p} = \dfrac{\sum p_i}{n} \\ \bar{q} = \dfrac{\sum q_i}{n} \end{cases}$$

$$\rightarrow SVD\left(\sum_i \tilde{q}_i \tilde{p}_i^T\right) = SVD(\tilde{Q}\tilde{P}^T) = U \Sigma V^T$$

$$\rightarrow \boxed{\begin{array}{l} R = V \begin{bmatrix} 1 & & \\ & 1 & \\ & & det(VU^T) \end{bmatrix} U^T \\ t = \bar{p} - R\bar{q} \end{array}}$$