COMP0130: Robotic Vision and Navigation

Lecture 06C: Examples of Graphs

Structure

- Motivation
- 2D Particle Linear Observation Example
- Interpreting the Distribution in the Graph
- 2D Particle Nonlinear Observation Example

Motivation

- We have shown that we can represent estimates using the joint density

$$f\left(\mathbf{x}_{0:k} \middle| \mathbb{I}_k\right) \propto \underline{f\left(\mathbf{x}_0\right)} \prod_{i=1}^{k} \underline{f\left(\mathbf{x}_i \middle| \mathbf{x}_{i-1}, \mathbf{u}_i\right)}$$

$$\times \prod_{i=1}^{k} \underline{L\left(\mathbf{x}_i ; \mathbf{z}_i\right)}$$

Motivation

- This equation is *exact* (up to proportionality):
  - It is the full functional form of the distribution
  - There is no integration or marginalization
- So what does it look like to implement?
- And what do the probability distributions in the graph mean?

2D Particle Linear Observation Example

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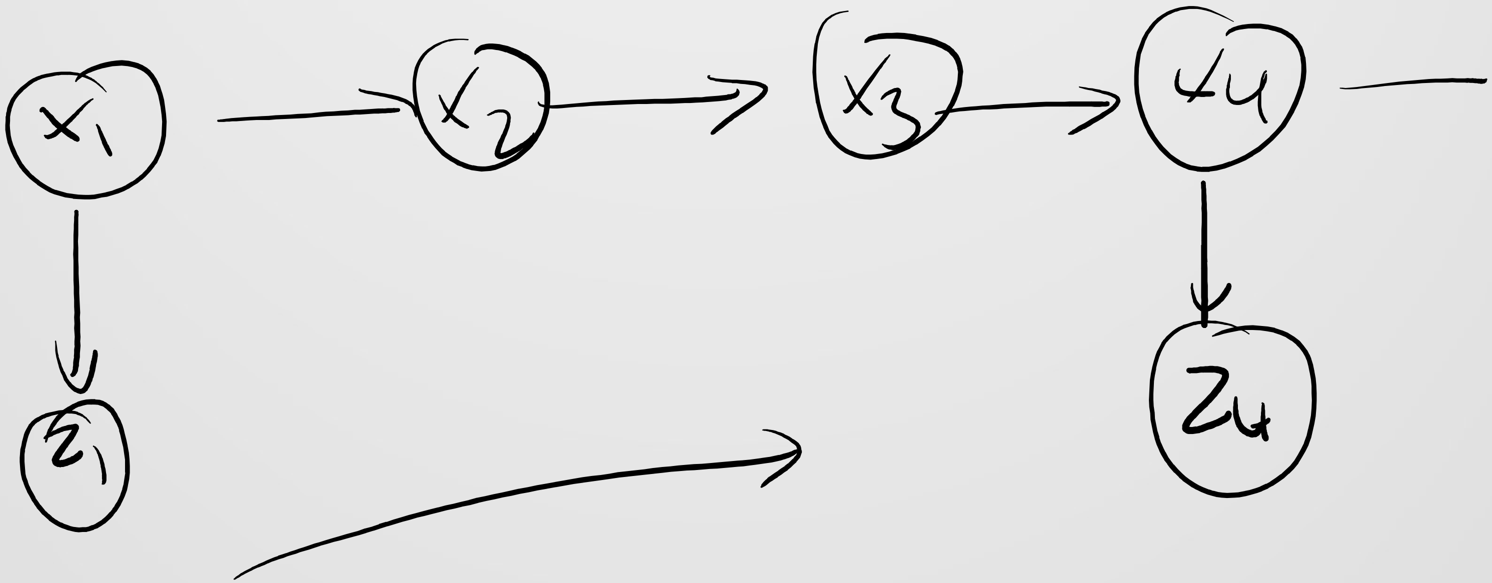
Linear Example

- We are tracking a particle in 2D:
  - The target state is position and velocity in 2D
  - The movement is piecewise constant velocity
  - The observations are the platform position
  - Observation noise is linear and additive
  - Observations are only available once every 50 timesteps

Graph with Infrequent Observations



Linear System

- Once we have the equations, we substitute

$$f\left(\mathbf{x}_{0:k}|\mathbb{I}_k\right) \propto f\left(\mathbf{x}_0\right) \prod_{i=1}^{k} f\left(\mathbf{x}_i|\mathbf{x}_{i-1}, \mathbf{u}_i\right)$$

$$\times \prod_{\cancel{i=1}\; i \in Z}^{k} L\left(\mathbf{x}_i; \mathbf{z}_i\right) \qquad Z\{1, 5, 10, \ldots\}$$

Linear System

- State vector is:

$$\mathbf{x}_k = \begin{bmatrix} x_k & \dot{x}_k & y_k & \dot{y}_k \end{bmatrix}^\top$$

- Process model:

$$\mathbf{x}_k = \mathbf{F}\mathbf{x}_{k-1} + \mathbf{v}_k$$

$$\mathbf{F} = \begin{bmatrix} 1 & \Delta T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Q} = q \begin{bmatrix} \Delta T^3/3 & \Delta T^2/2 & 0 & 0 \\ \Delta T^2/2 & \Delta T & 0 & 0 \\ 0 & 0 & \Delta T^3/3 & \Delta T^2/2 \\ 0 & 0 & \Delta T^2/2 & \Delta T \end{bmatrix}$$

- Observation vector:

$$\mathbf{z}_k = \begin{bmatrix} x_k & y_k \end{bmatrix}^\top$$

- Observation model:

$$\mathbf{z}_k = \mathbf{H}\mathbf{x}_k + \mathbf{w}_k$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{R} = \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Linear System

- We need to define the state transition probabilities and the measurement likelihoods

- As explained in the last lecture, we get these from the process and observation models

State Transition Probabilities

- Recall that the state transition probability

$$f\left(\mathbf{x}_k \middle| \mathbf{x}_{k-1}, \mathbf{u}_k\right) = \underline{f_{\mathbf{v}}\left(\mathbf{v}_k = \mathbf{e}\left[\mathbf{x}_k, \mathbf{x}_{k-1}, \mathbf{u}_k\right]\right)}$$

- From the process model,

$$\mathbf{e}\left[\mathbf{x}_k, \mathbf{x}_{k-1}, \cancel{\mathbf{u}_k}\right] = \underline{\mathbf{x}_k - \mathbf{F}\mathbf{x}_{k-1}}$$

$$x_k = F x_{k-1} + v_k \qquad v_k = \boxed{x_k - F x_{k-1}}$$

State Transition Probabilities

- We have modelled the process as a zero-mean Gaussian with covariance $\boldsymbol{Q}_k$
- Therefore, the state transition probability is

$$f\left(\mathbf{x}_k | \mathbf{x}_{k-1}, \mathbf{u}_k\right) \propto \exp\left\{-\frac{1}{2}\underbrace{\left(\mathbf{x}_k - \mathbf{F}\mathbf{x}_{k-1}\right)}_{v_k}{}^\top \mathbf{Q}_k^{-1}\left(\mathbf{x}_k - \mathbf{F}\mathbf{x}_{k-1}\right)\right\}$$

$$\underline{E[v_k] = 0}$$

Measurement Likelihood Function

- Recall the measurement likelihood function is

$$f(\mathbf{z}_k|\mathbf{x}_k) = f_{\mathbf{w}}\left(\mathbf{w}_k = \mathbf{l}\left[\mathbf{x}_k, \mathbf{z}_k\right]\right)$$

- From the observation model,

$$\mathbf{l}\left[\mathbf{x}_k, \mathbf{z}_k\right] = \mathbf{z}_k - \mathbf{H}\mathbf{x}_k$$

$$z_k = H x_k + v_k \qquad w_k = z_k - H x_k$$

UCL

Measurement Likelihood Function

- Again, using the Gaussians, we get

$$L(\mathbf{x}_k; \mathbf{z}_k) \propto \exp\left\{-\frac{1}{2}\overbrace{(\mathbf{z}_k - \mathbf{H}\mathbf{x}_k)}^{e_2}{}^{\top}\mathbf{R}_k^{-1}\underbrace{(\mathbf{z}_k - \mathbf{H}\mathbf{x}_k)}\right\}$$

Using the Probabilities and Likelihoods

- Substitute for the Gaussians in this expression

$(x_{k+1} - F x_k)^T Q^{-1} \ldots$

$$f\left(\mathbf{x}_{0:k} \middle| \mathbb{I}_k\right) \propto f\left(\mathbf{x}_0\right) \prod_{i=1}^{k} \underline{f\left(\mathbf{x}_i \middle| \mathbf{x}_{i-1}, \mathbf{u}_i\right)}$$

$x_2 - F x_1$ $x_3 - F x_2$ $x_4 - F x_3$

1 2 3 4

$$\times \prod_{i=1}^{k} L\left(\mathbf{x}_i ; \mathbf{z}_i\right)$$

$z_1 - H x$ $z_i - H x_2$ $\ldots$

$\Gamma\,(z - Hx)^T R^{-1} \ldots$

Interpreting the Distribution in the Graph

- *Motivation*
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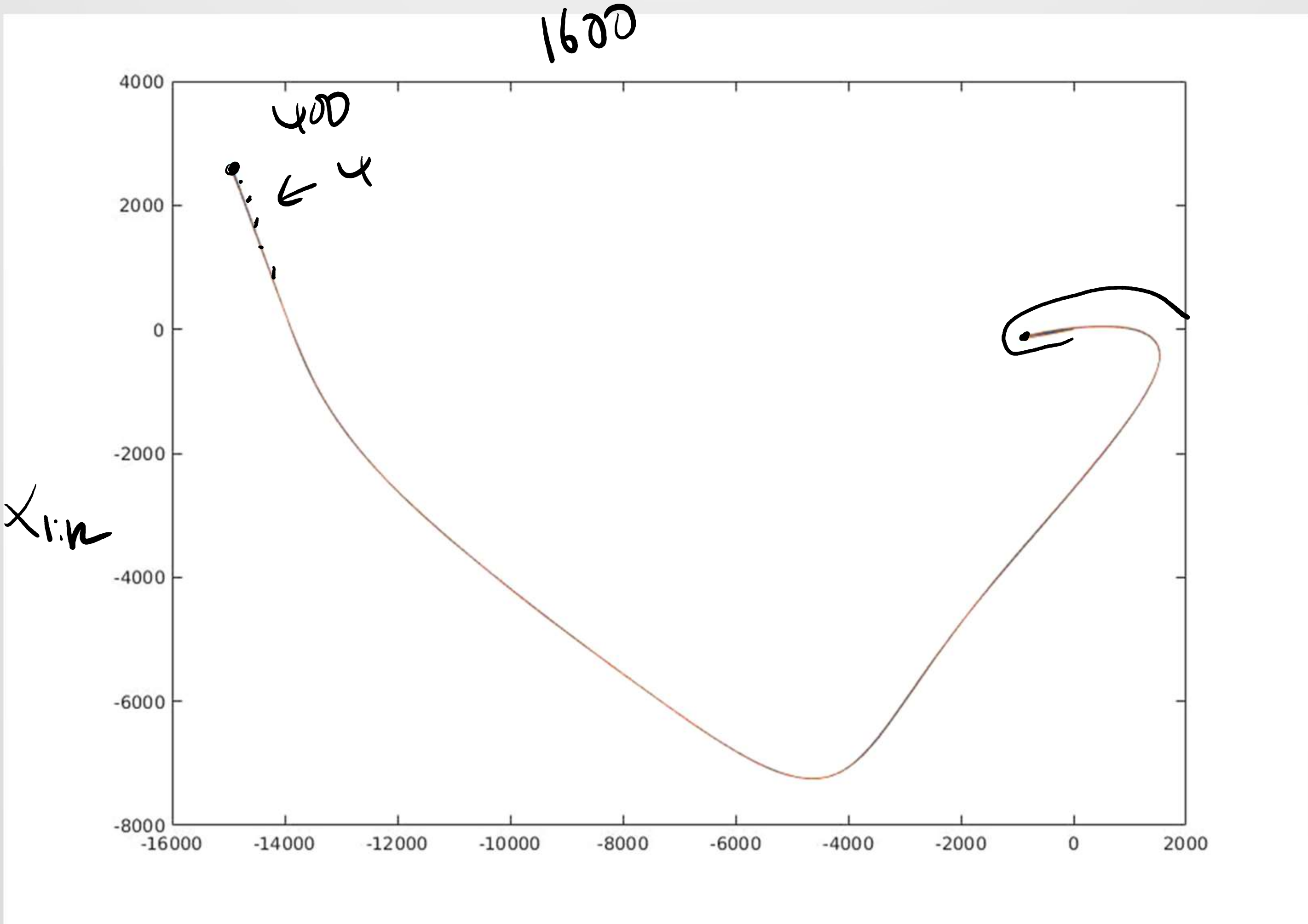
Interpreting the Distribution in the Graph

- Recall that the graph stores the entire history of the platform as the state vector

- Therefore, a single instance describe the *entire run*

$$x_{1:n} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

A Single Target State

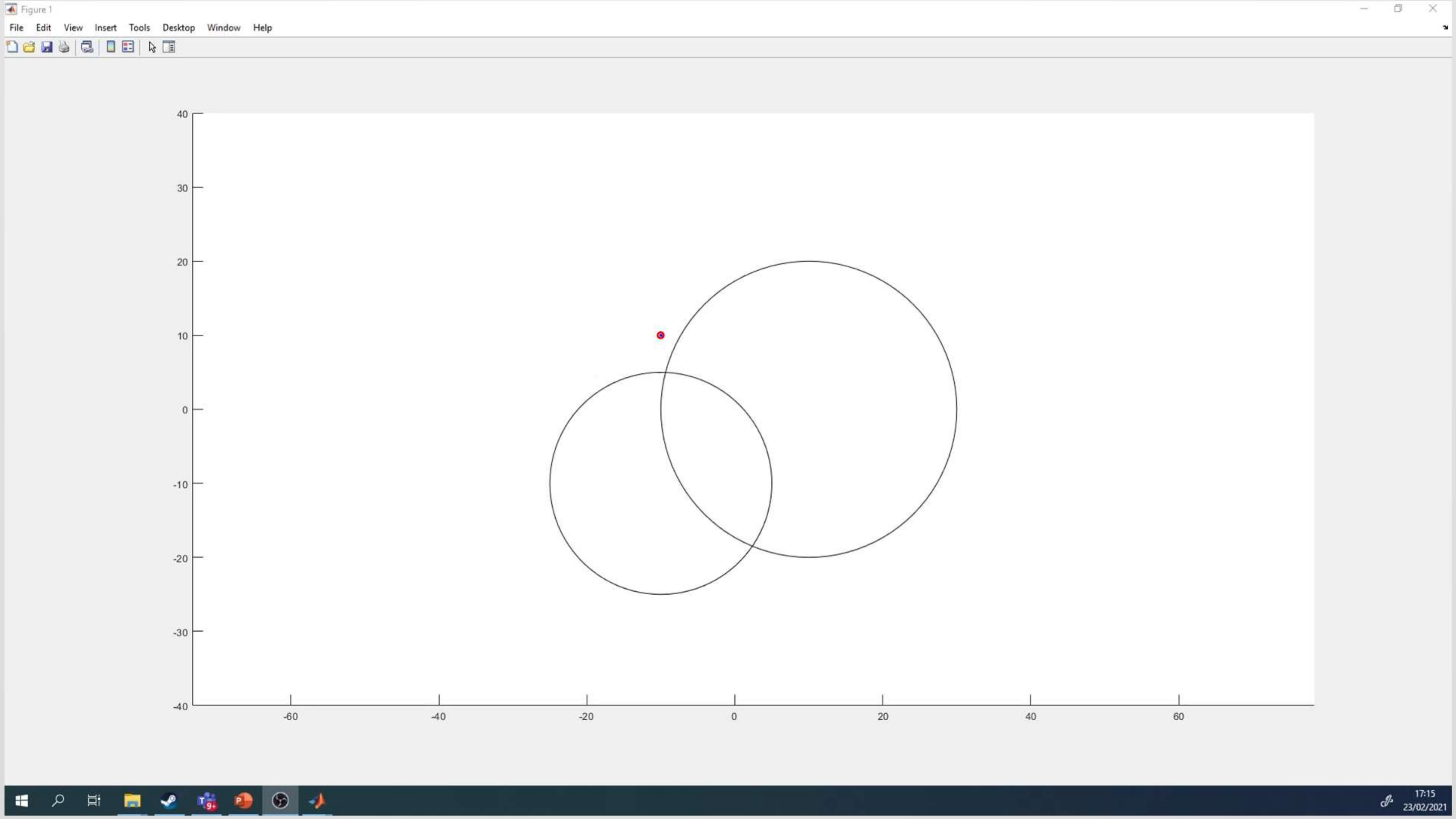


Interpreting Uncertainty Using Sampling

- We'd like to interpret the uncertainty in the state estimate caused by the graph
- At the moment we can't use moments because we don't have the normalization constant
- However, instead we can use sampling techniques
- Sampling generates a lot of particles to represent a distribution

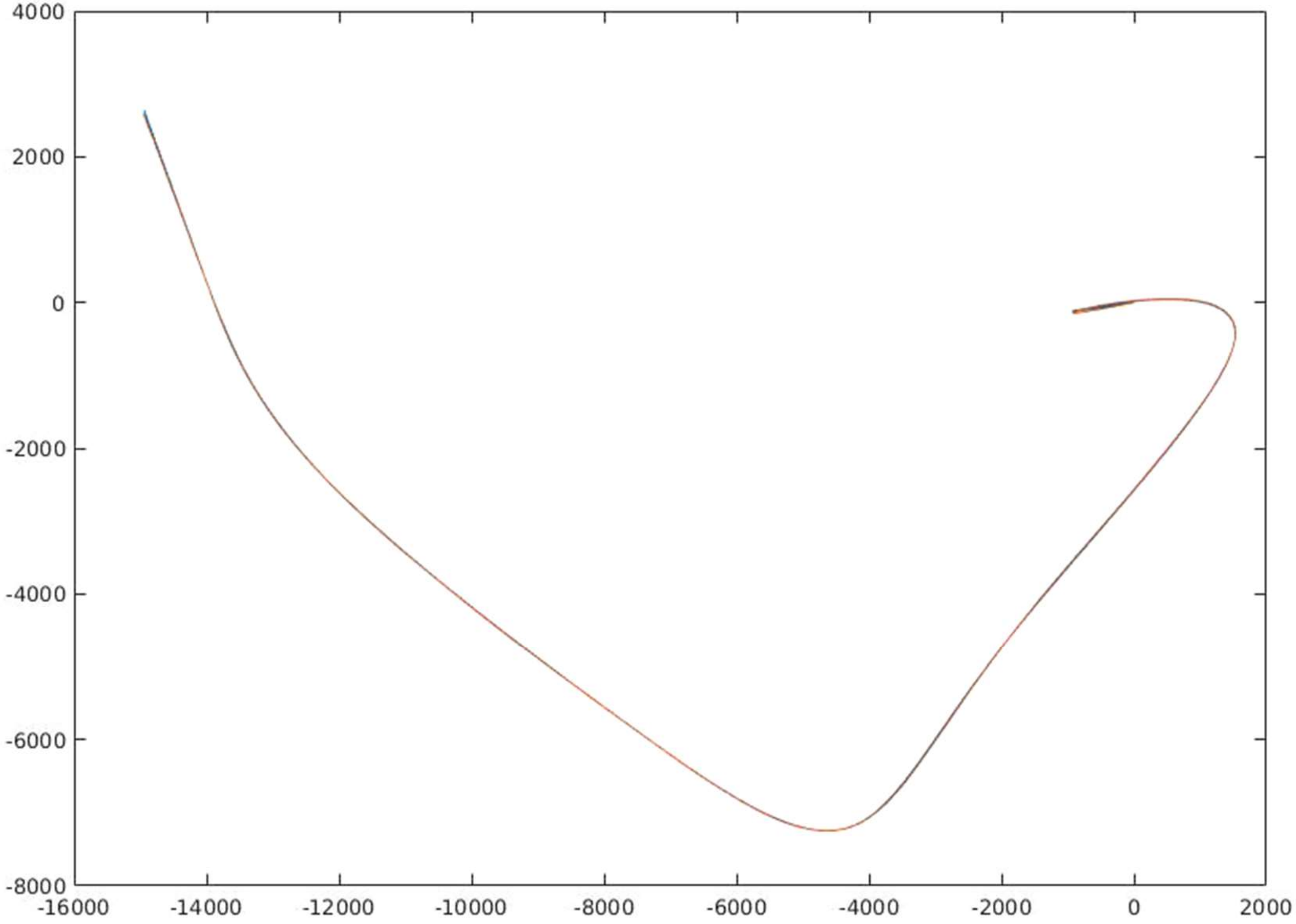
Sampling and Linear Measurements

Sampling and Very Nonlinear Measurements



Sampling from the Graph

- To get a sense of what's actually stored in the graph, we'd like to draw samples of it and plot them
- It turns out that we can sample from a graph without normalizing it
- We use an approach called Riemannian Hamiltonian Monte Carlo
- This is out of scope of the module, but the code for it is provided as part of the lab model answers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2000
0
-2000
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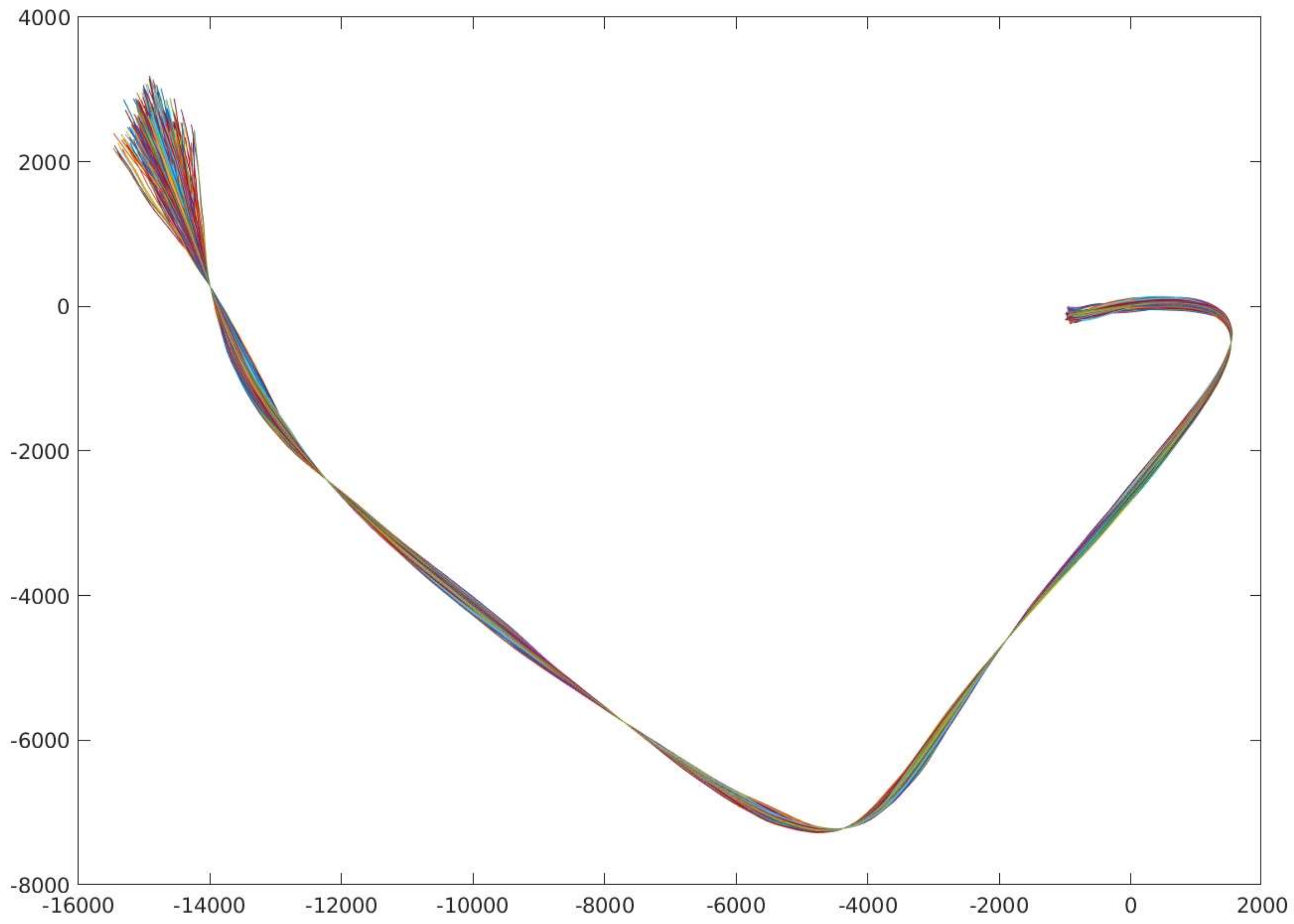
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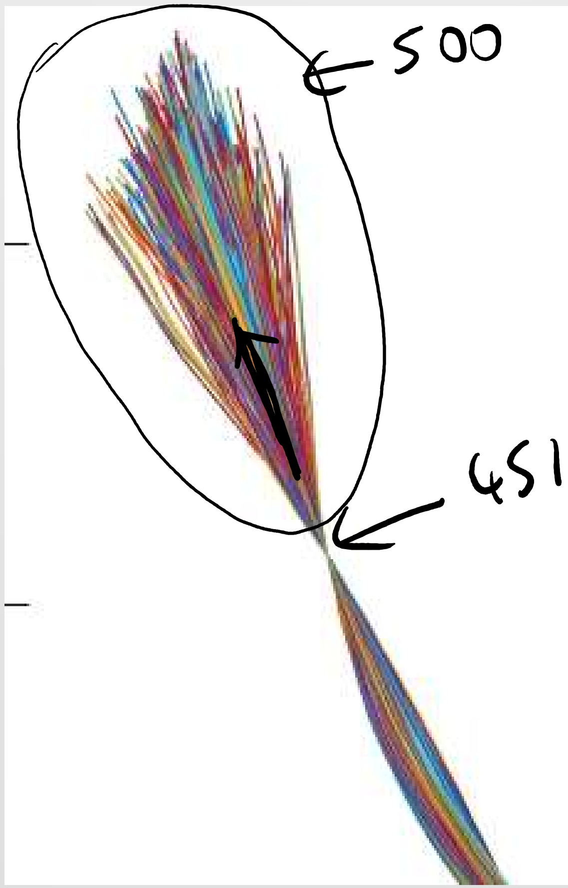
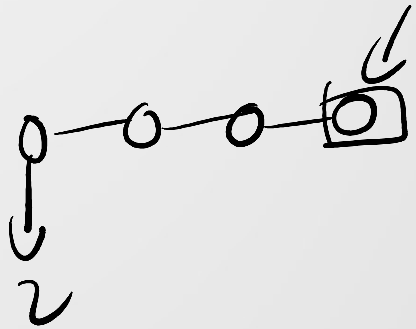
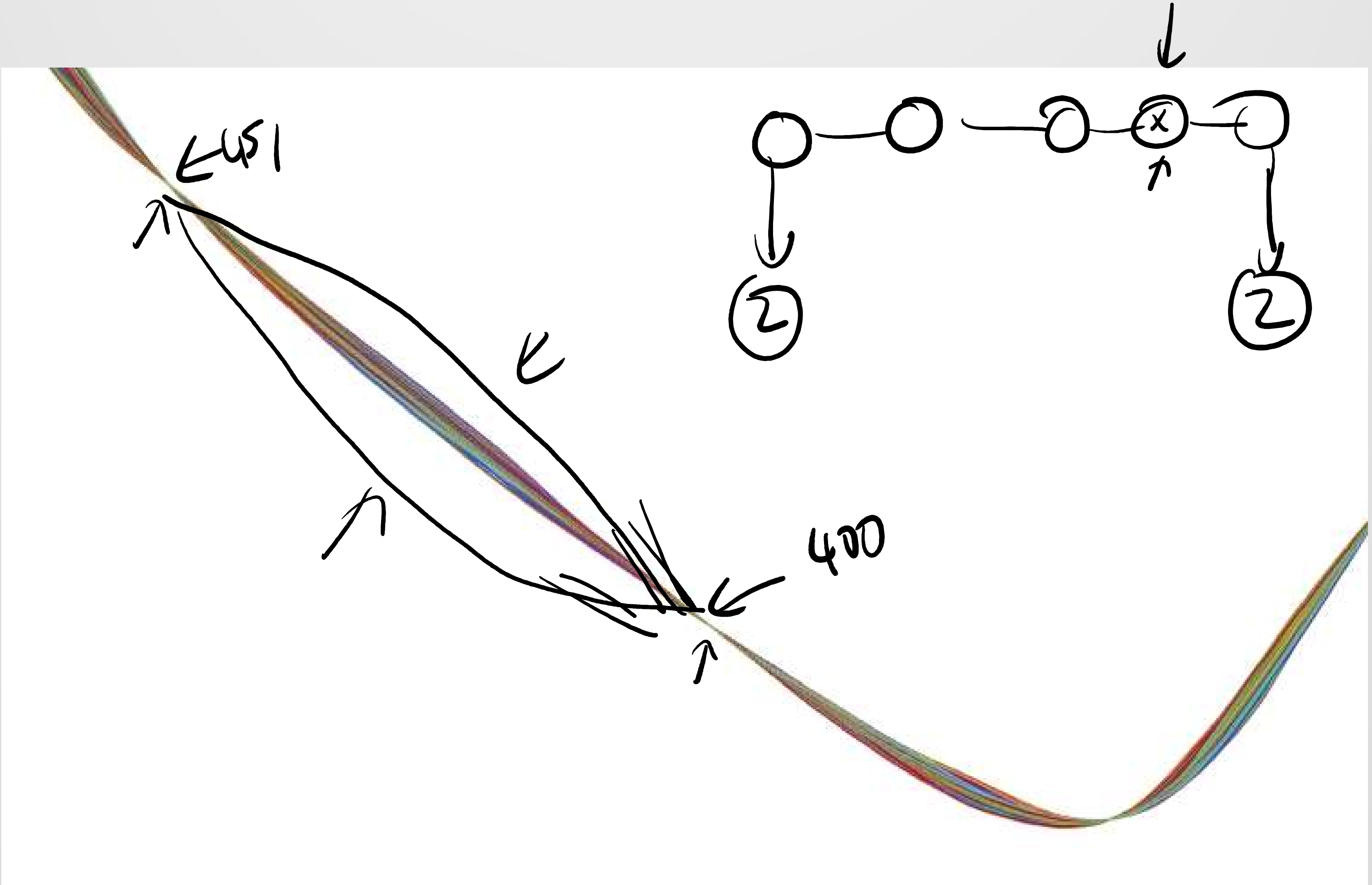
Plume





Regular "Pulses"



2D Particle Nonlinear Observation Example

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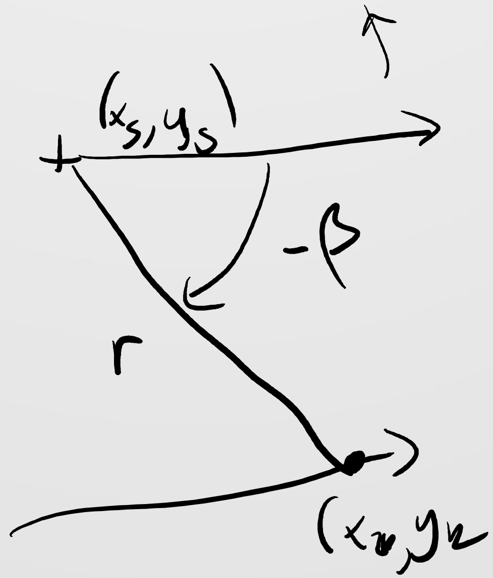
2D Particle Nonlinear Observation Example

- We now change the example by replacing the sensor with a sensor which measures range and bearing from a sensor is located at (x_s, y_s)

- The sensors are corrupted by additive Gaussian noise

Setup



2D Particle Nonlinear Observation Example

- The observation models are now

$$r_k = \sqrt{(x_k - x_s)^2 + (y_k - y_s)^2} + w_k^r$$

$$\beta_k = \tan^{-1}\left(\frac{y_k - y_s}{x_k - x_s}\right) + w_k^\beta$$

Measurement Likelihood Function

- Recall that

$$f\left(\mathbf{z}_k \middle| \mathbf{x}_k\right) = f_{\mathbf{w}}\left(\mathbf{w}_k = \mathbf{l}\left[\mathbf{x}_k, \mathbf{z}_k\right]\right)$$

- This time, however, the equations are nonlinear and so

$$\mathbf{l}\left[\mathbf{x}_k, \mathbf{z}_k\right] = \begin{bmatrix} l_r\left[\mathbf{x}_k, \mathbf{z}_k\right] \\ l_\beta\left[\mathbf{x}_k, \mathbf{z}_k\right] \end{bmatrix}$$

Measurement Likelihood Function

- We need to make the measurement noise the subject of the formula
- Therefore

$z_k = r_k = \sqrt{\Delta x^2 + \Delta y^2} + w$

$w = r_k - \sqrt{\Delta x^2 + \Delta y^2}$

$$l_r\left[\mathbf{x}_k, \mathbf{z}_k\right] = r_k - \sqrt{(x_k - x_s)^2 + (y_k - y_s)^2}$$

w

$$l_\beta\left[\mathbf{x}_k, \mathbf{z}_k\right] = \beta_k - \tan^{-1}\left(\frac{y_k - y_s}{x_k - x_s}\right)$$

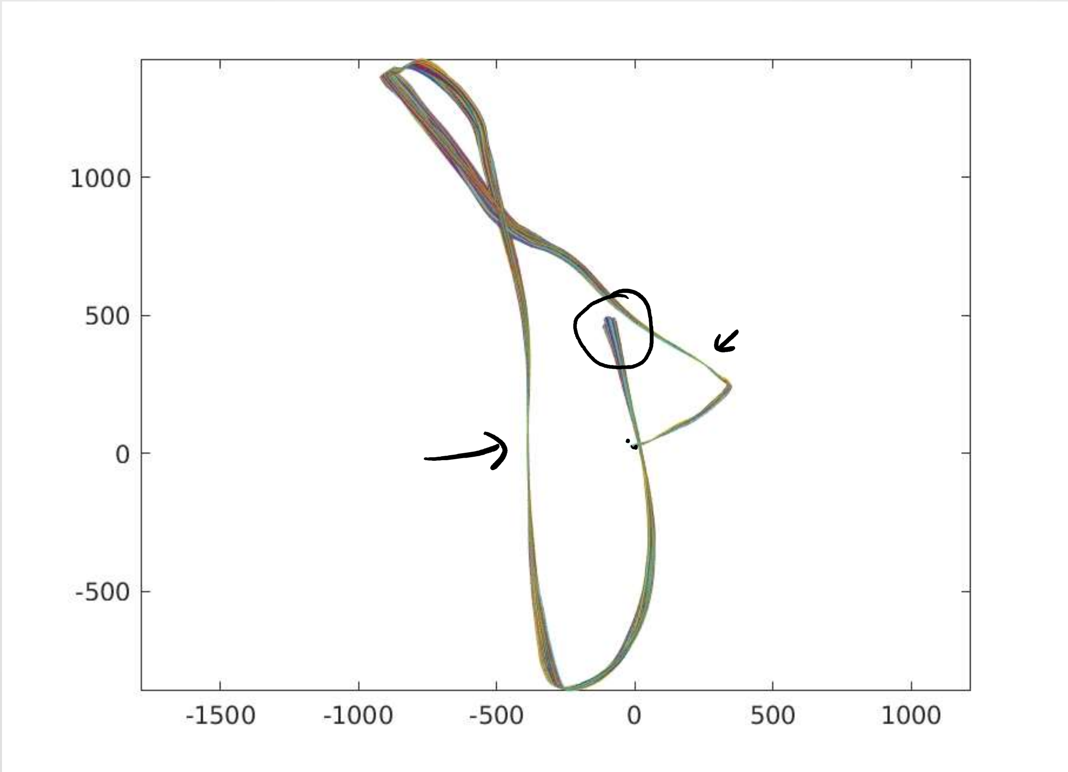
$w \sim G(0, R_w)$

Measurement Likelihood Function

- Using the Gaussian assumptions,

$$L(\mathbf{x}_k;\mathbf{z}_k) \propto \exp\left\{-\frac{1}{2}\mathbf{l}\left[\mathbf{x}_k,\mathbf{z}_k\right]^\top \mathbf{R}_k^{-1}\mathbf{l}\left[\mathbf{x}_k,\mathbf{z}_k\right]\right\}$$

Polar Measurement Results



(r, β)

$\sigma_r \sim 1m$

$\sigma_\beta \sim 5^\circ$

Cartoon Version of Vehicle Prediction

- The control input is the wheel speed and front wheel steer angle,

$$\mathbf{u}_k = \begin{bmatrix} s_k & \delta_k \end{bmatrix}^\top$$

- The new pose is computed from

$$\begin{aligned} x_k &= x_{k-1} + s_k \Delta T \cos\left(\psi_{k-1} + \delta_k\right) \\ y_k &= y_{k-1} + s_k \Delta T \sin\left(\psi_{k-1} + \delta_k\right) \\ \psi_k &= \psi_{k-1} + \frac{s_k \Delta T \sin \delta_k}{B} \end{aligned}$$

Cartoon Version of Vehicle Prediction