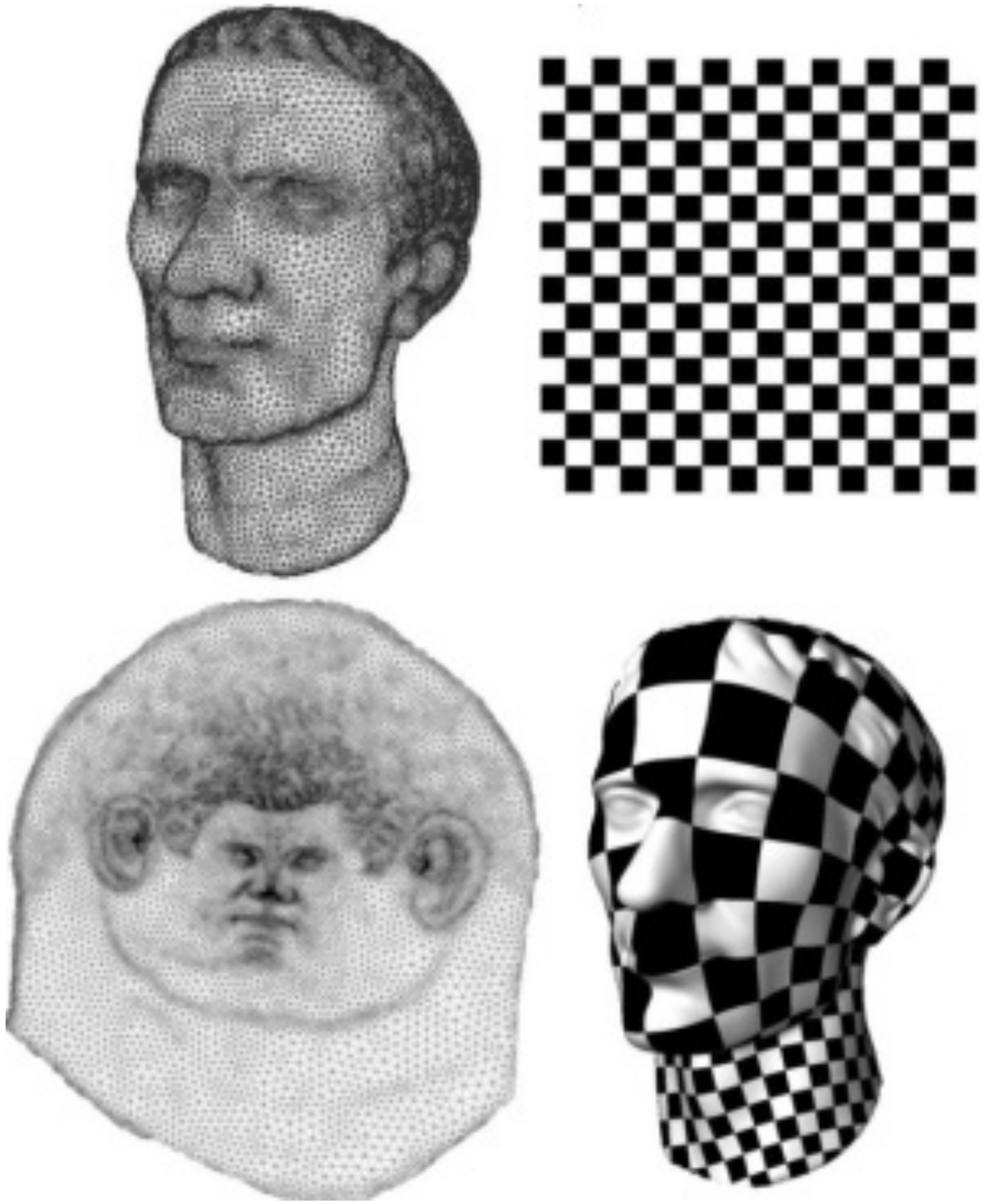
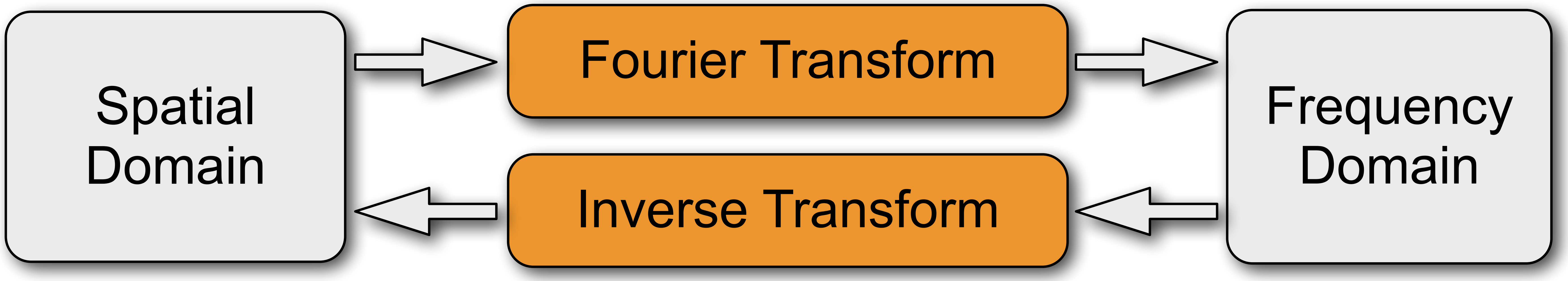
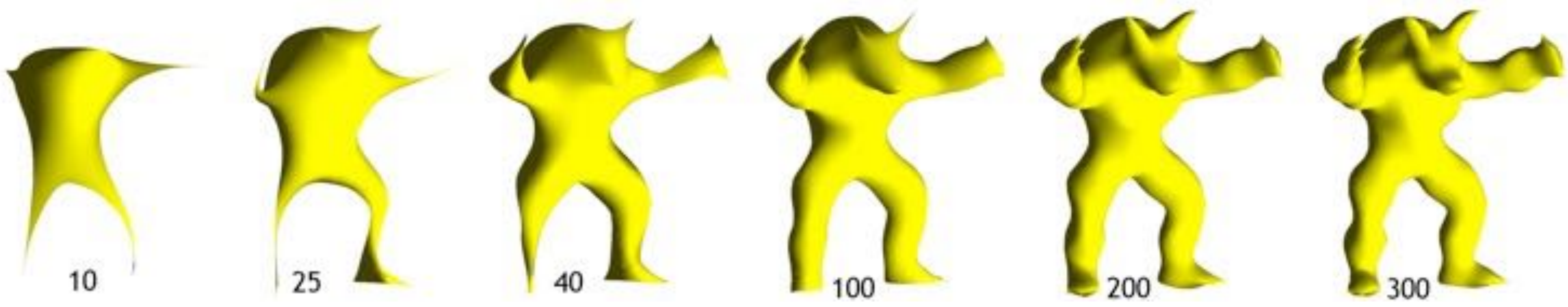
Mesh Parameterization



Spectral Analysis

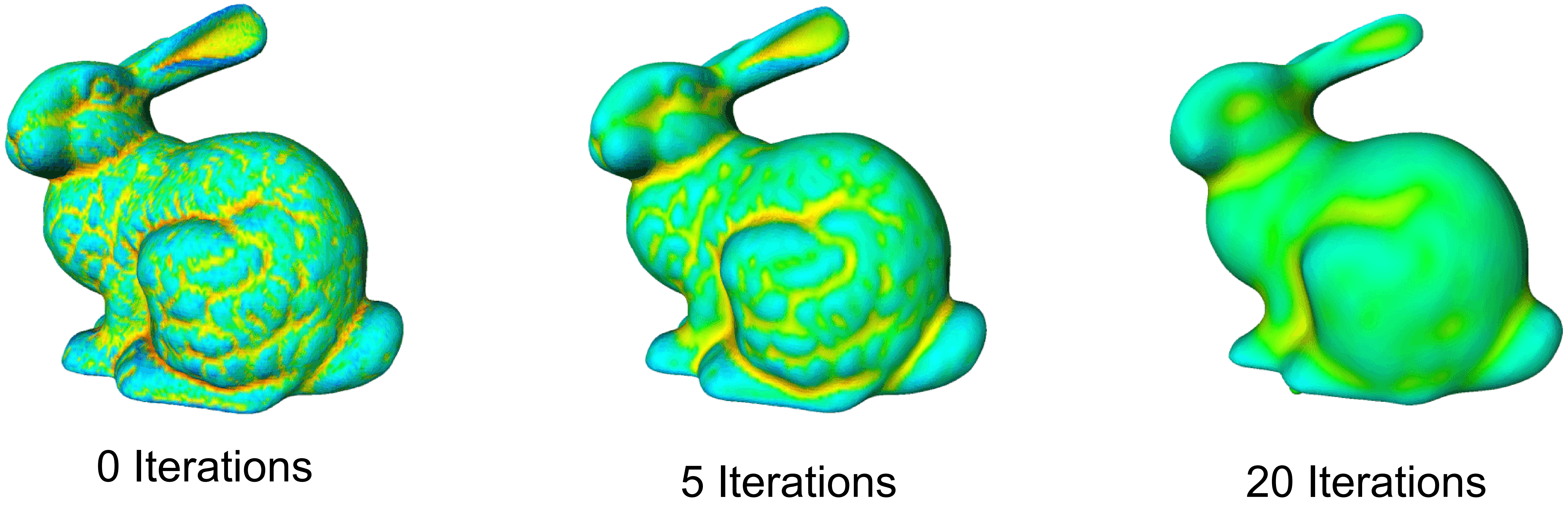


- Discrete Laplace-Beltrami matrix **L**
  - Eigenvectors are “natural vibrations”
  - Eigenvalues are “natural frequencies”



Diffusion Flow

- Iterate (explicit model) $\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda \Delta \mathbf{p}_i$



Implicit versus Explicit

Membrane Surfaces

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

- Membrane energy (surface area)

$$\int_\Omega \|\mathbf{p}_u\|^2 + \|\mathbf{p}_v\|^2 \, \mathrm{d}u\mathrm{d}v \;\rightarrow\; \min$$

- Variational calculus

$$\Delta \mathbf{p} = 0$$

Thin-Plate Surfaces

- Surface parameterization

$$\mathbf{p} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

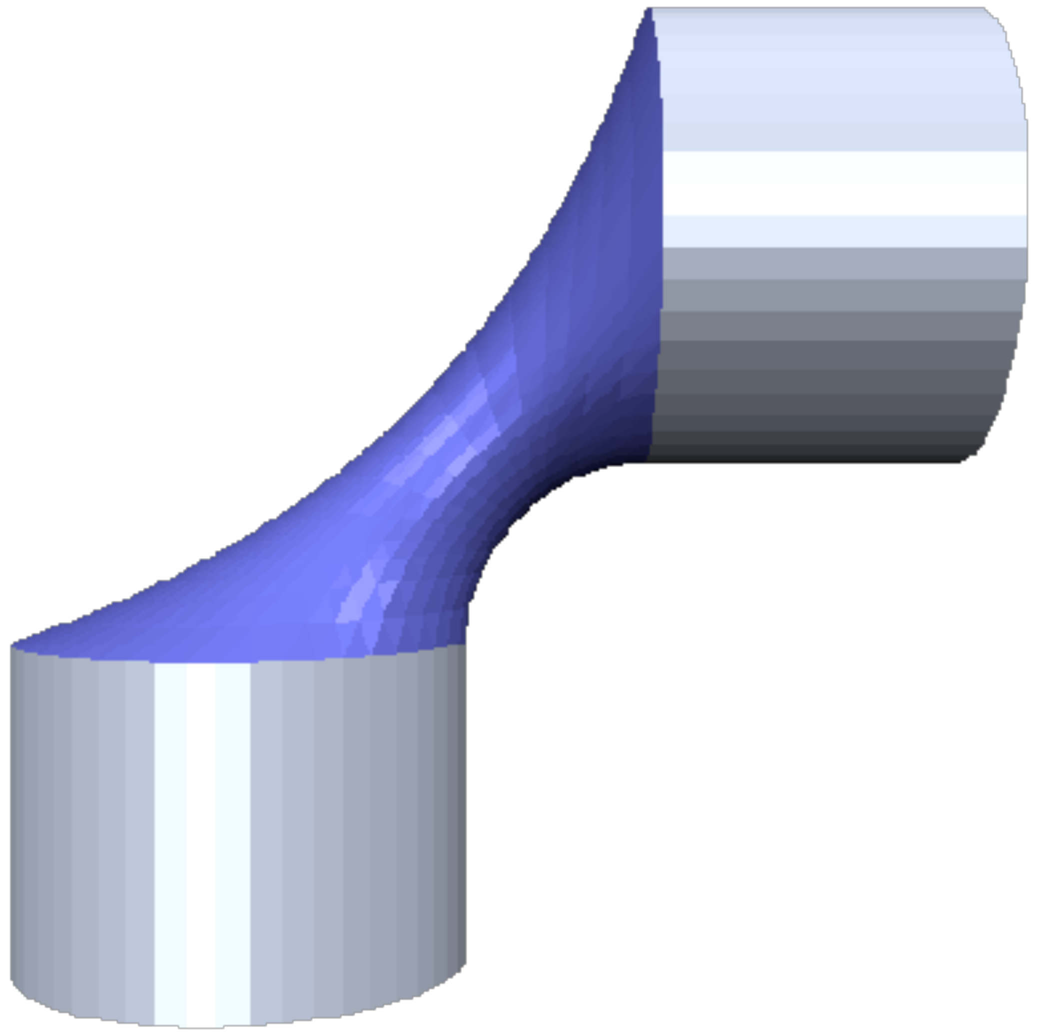
- Thin-plate energy (curvature)

$$\int_\Omega \|\mathbf{p}_{uu}\|^2 + 2\,\|\mathbf{p}_{uv}\|^2 + \|\mathbf{p}_{vv}\|^2 \,\mathrm{d}u\mathrm{d}v \;\rightarrow\; \min$$

- Variational calculus

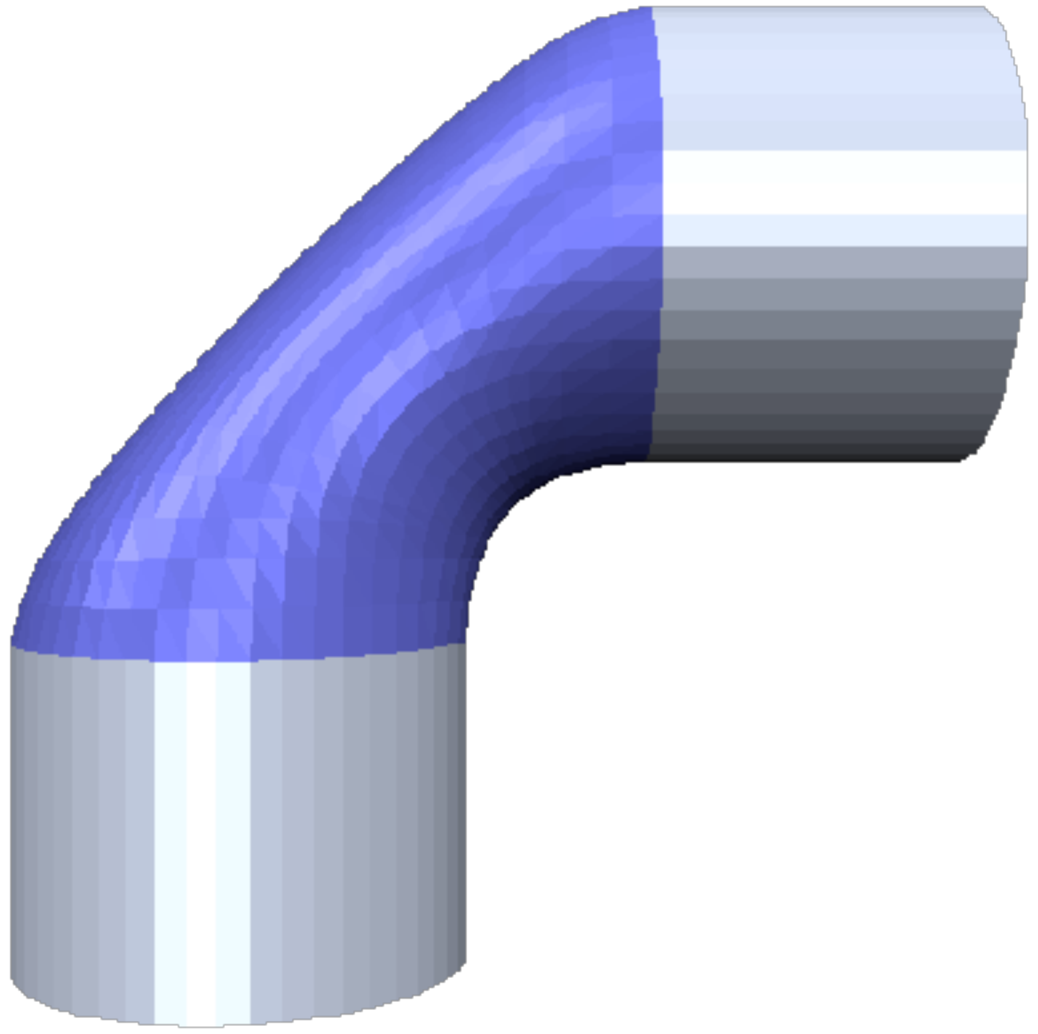
$$\Delta^2 \mathbf{p} = 0$$

Energy Functionals



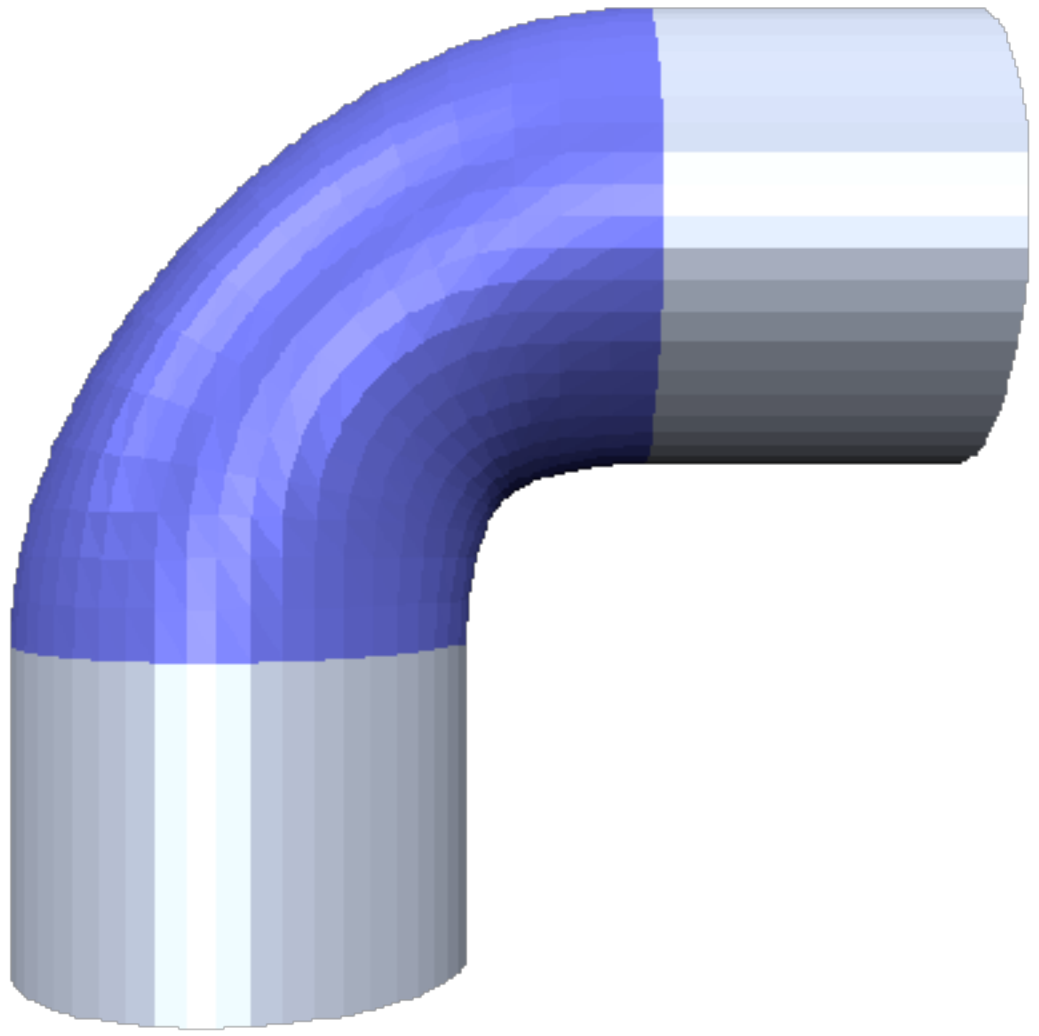
Membrane (area)

$$\Delta_{\mathcal{S}}\mathbf{p} = 0$$



Thin Plate (curvature)

$$\Delta^2_{\mathcal{S}}\mathbf{p} = 0$$



$$\Delta^3_{\mathcal{S}}\mathbf{p} = 0$$

Analysis

- Minimizer surfaces satisfy Euler-Lagrange PDE

$$\Delta_{\mathcal{S}}^{k}\mathbf{p} = 0$$

- They are stationary surfaces of Laplacian flows

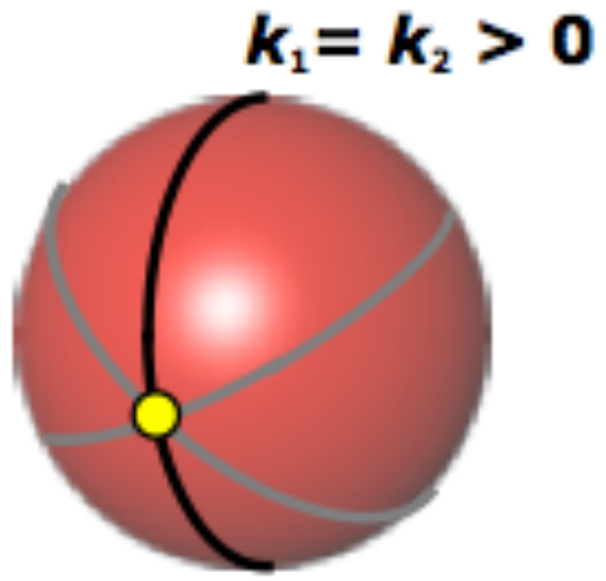
$$\frac{\partial \mathbf{p}}{\partial t} = \Delta_{\mathcal{S}}^{k}\mathbf{p}$$

- Explicit flow integration corresponds to iterative solution of linear system

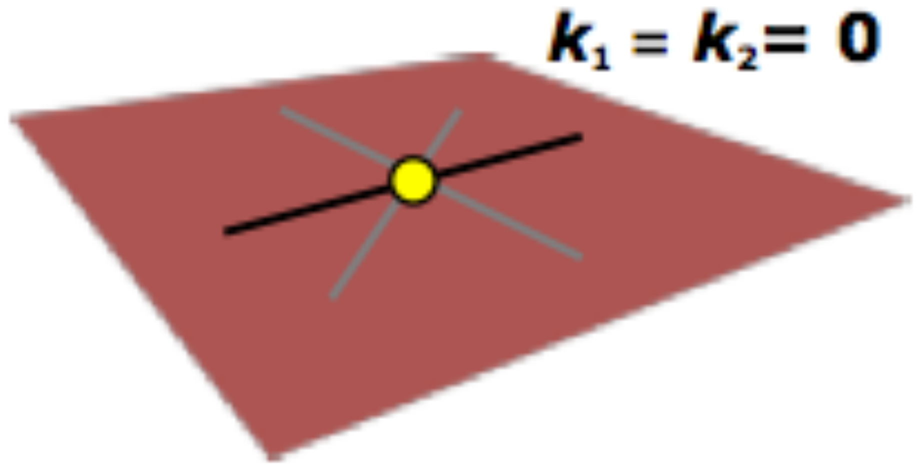
Classification (using K)

Isotropic

Equal in all directions



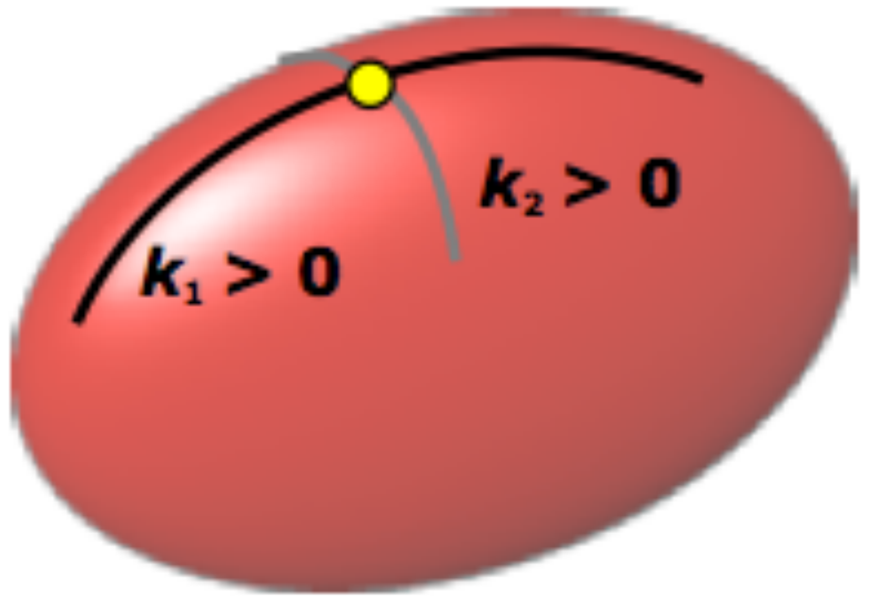
spherical



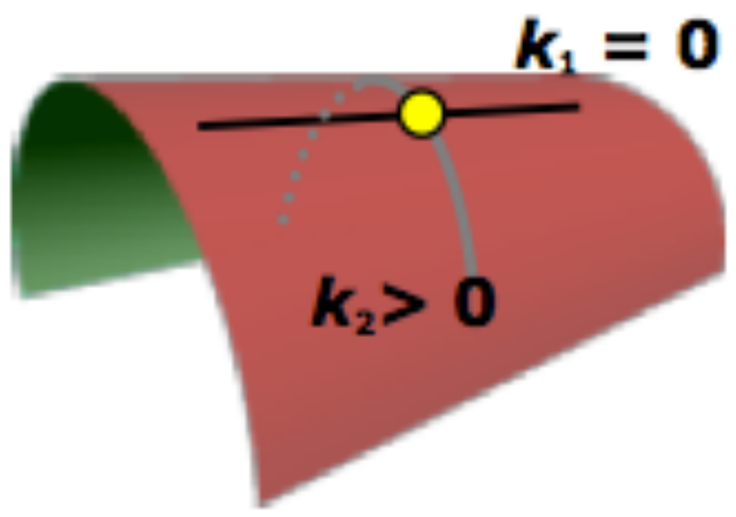
planar

Anisotropic

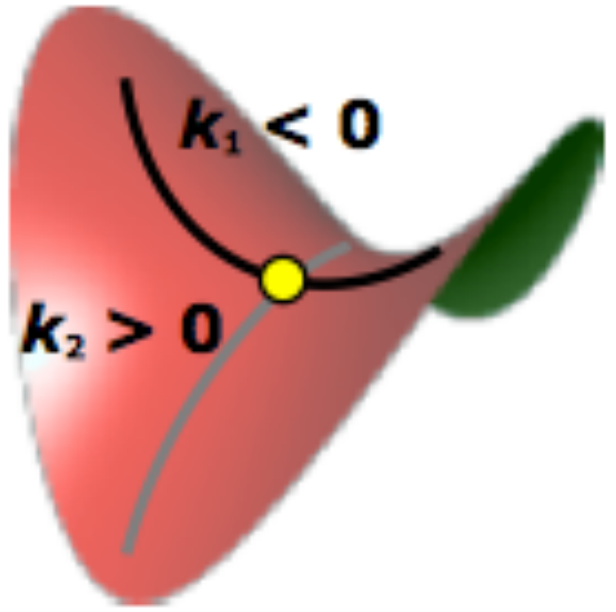
Distinct principal directions



elliptic
$K > 0$

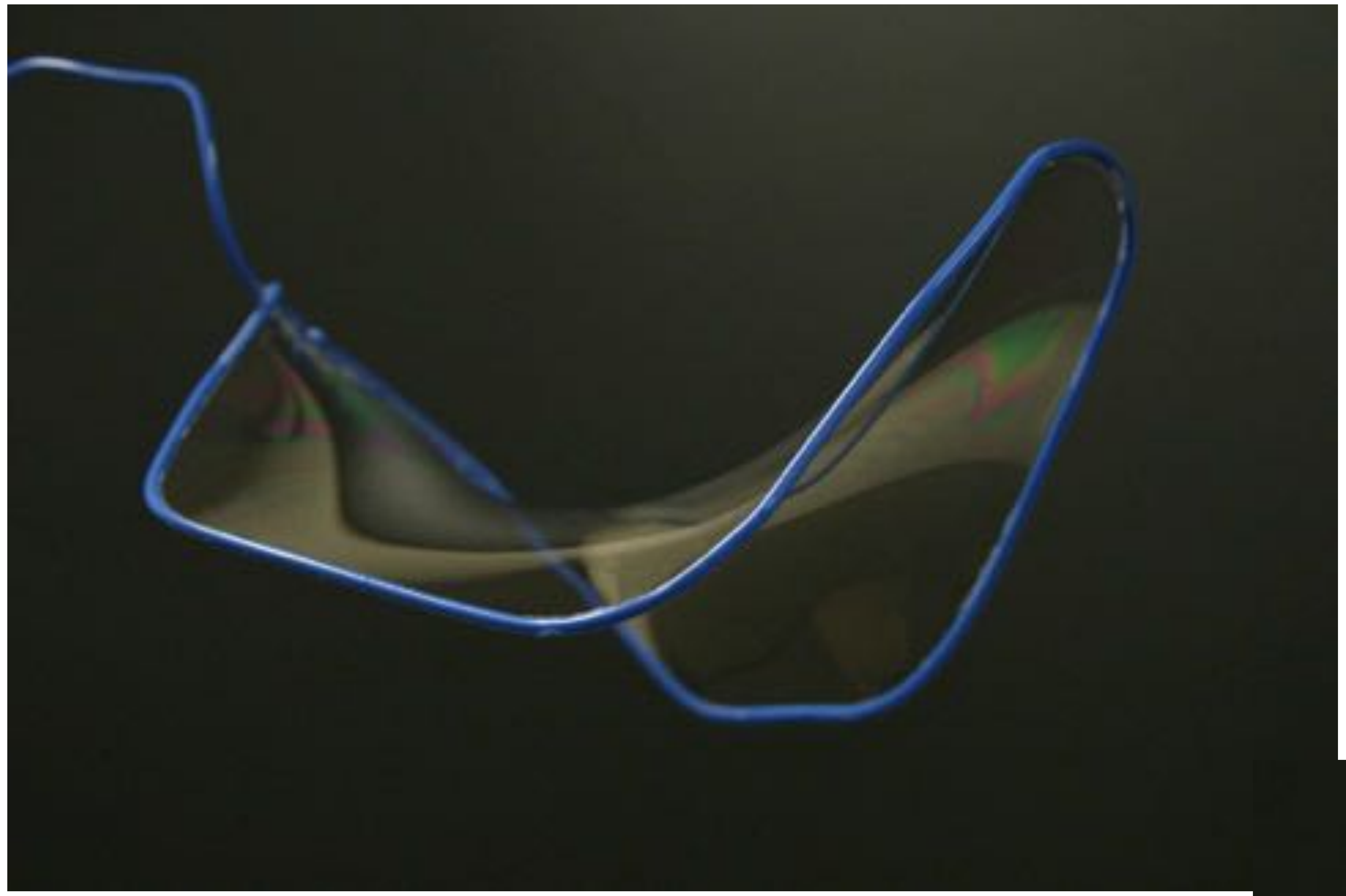


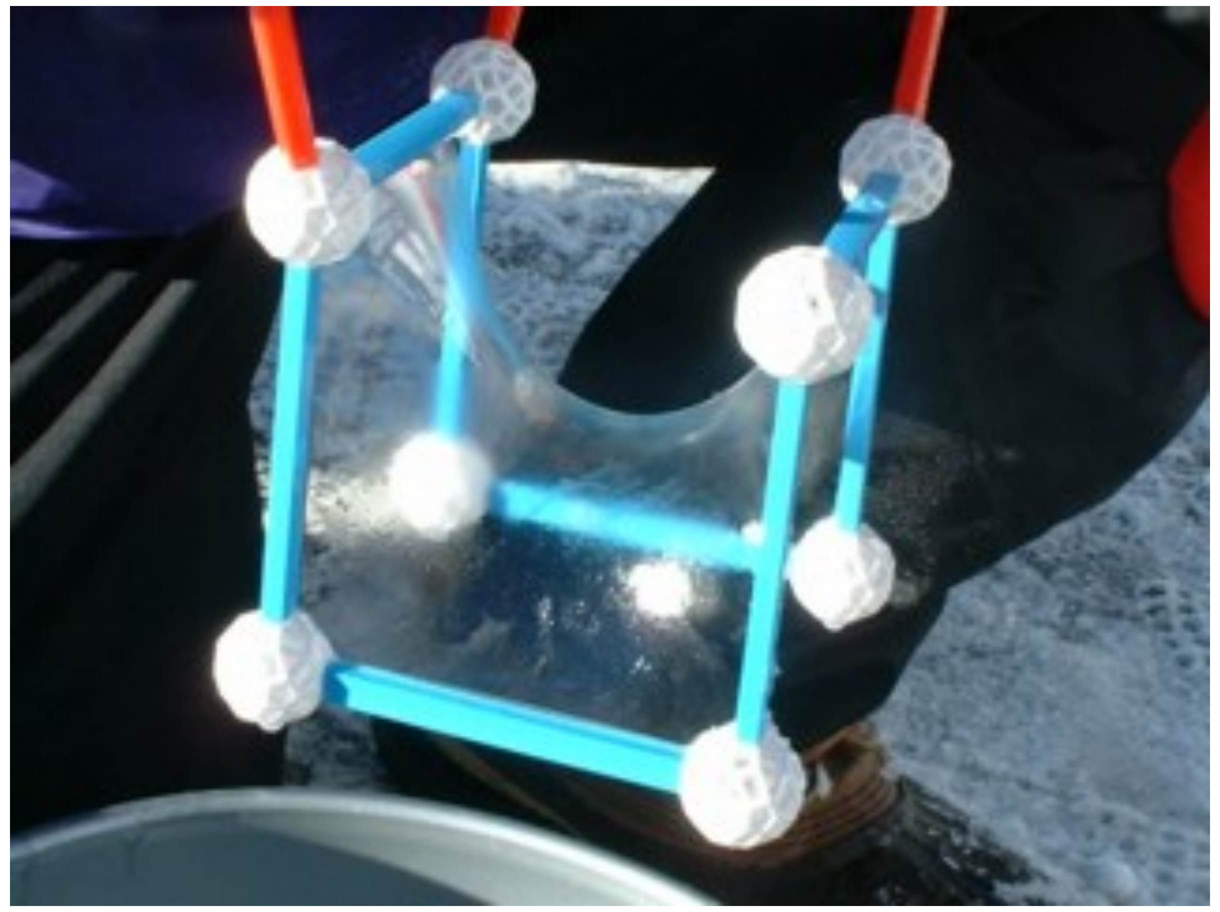
parabolic
$K = 0$
developable



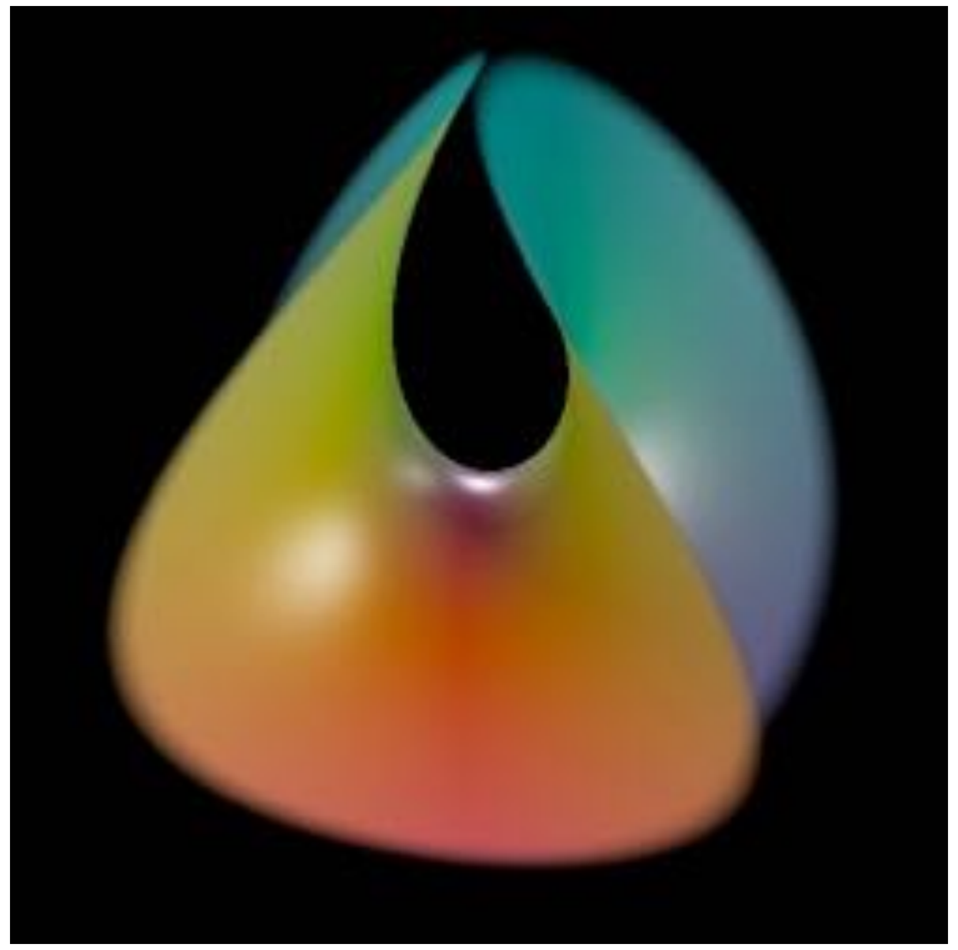
hyperbolic
$K < 0$

Soap Films

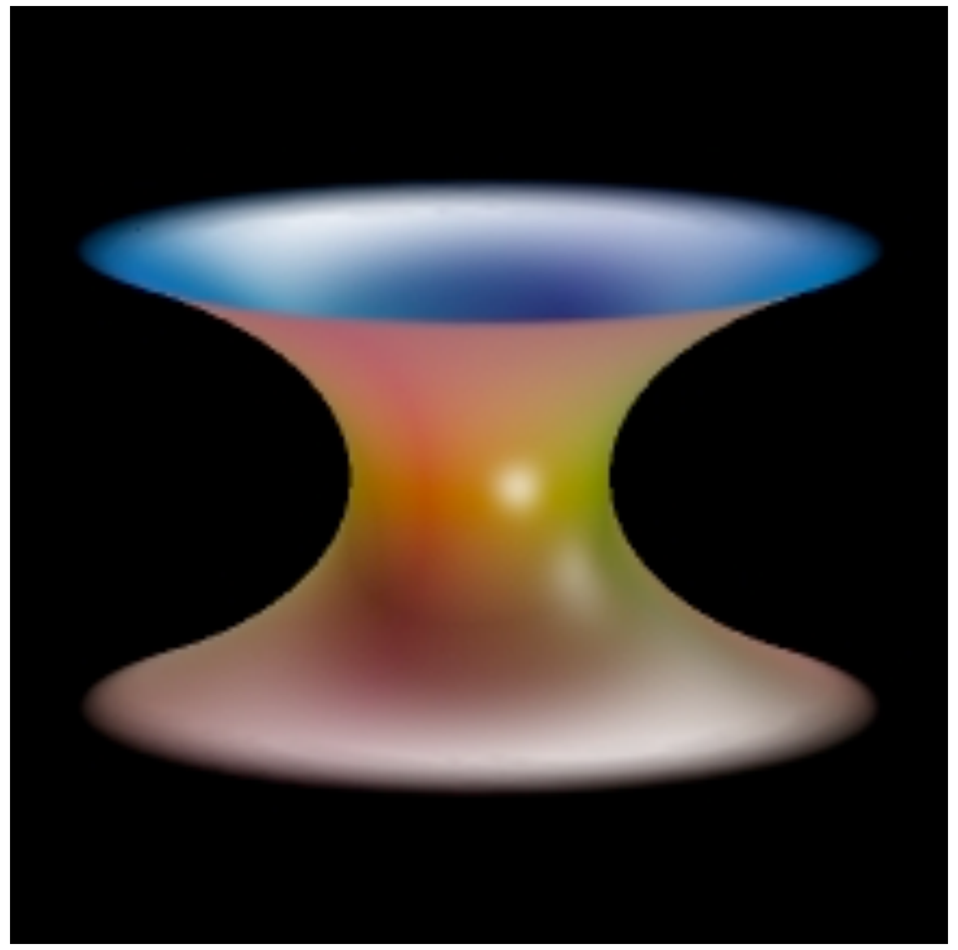




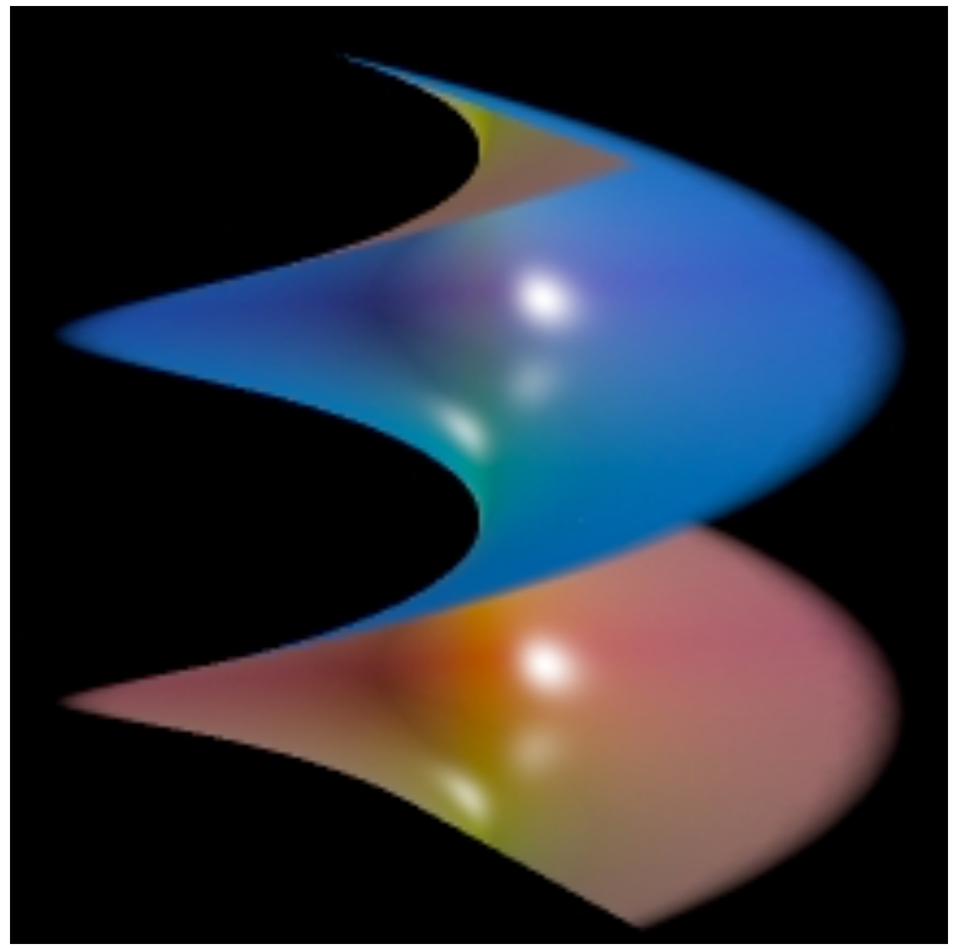
Minimal Surfaces



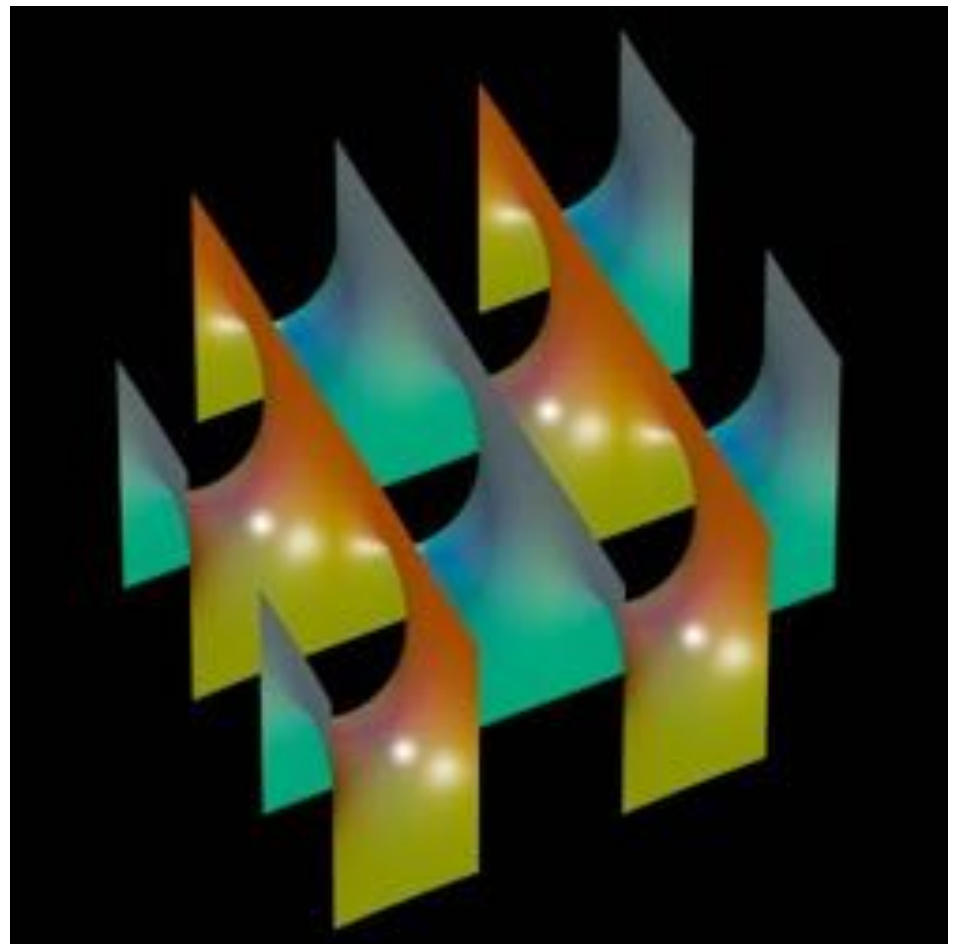
Enneper's Surface



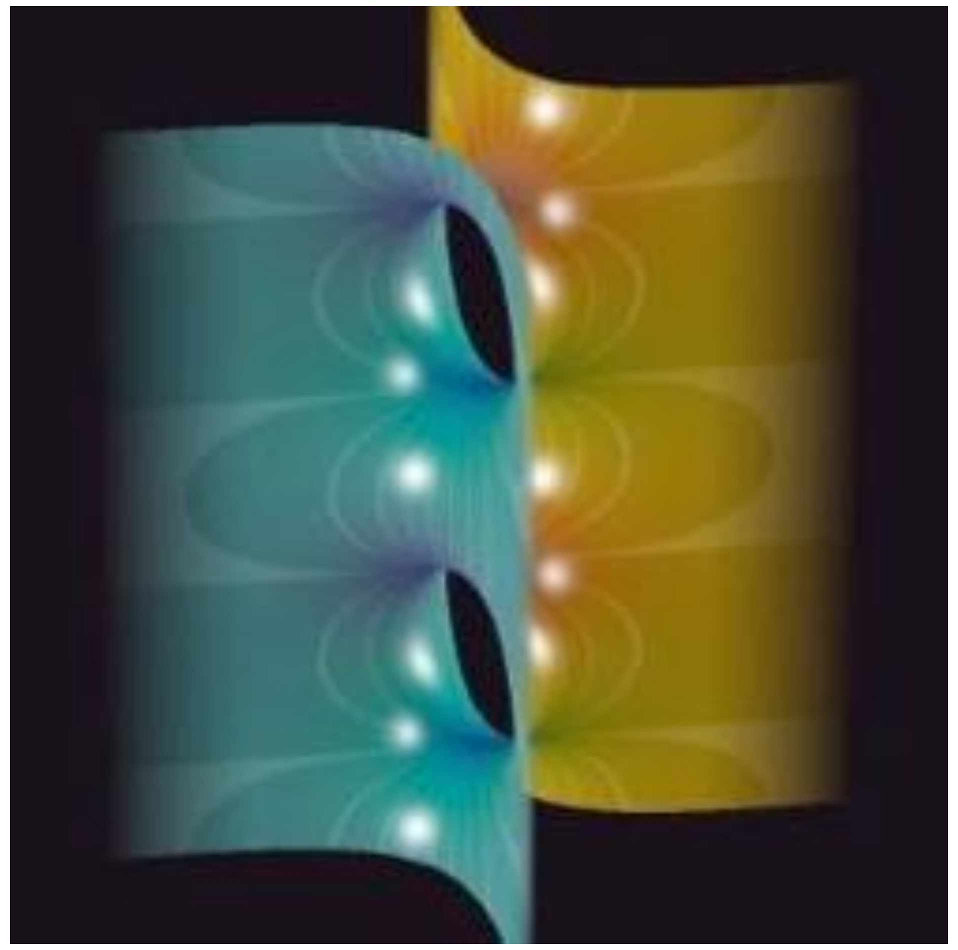
Catenoid



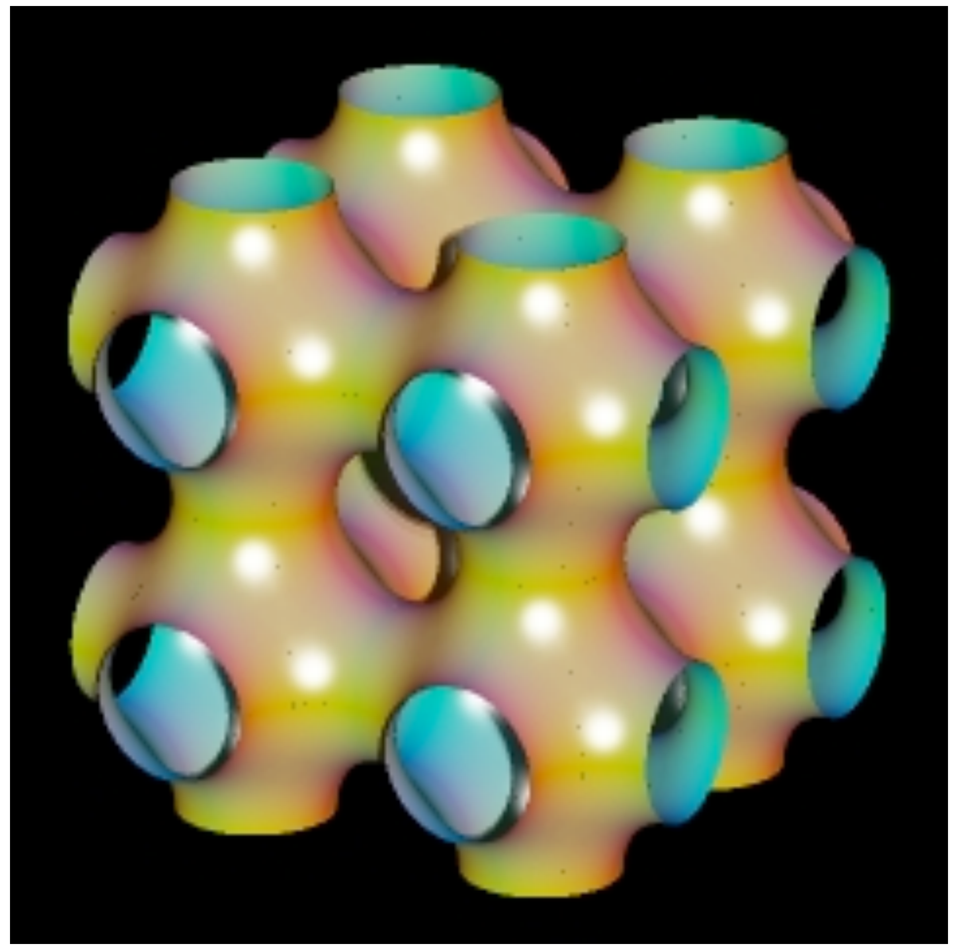
Helicoid



Scherk's First Surface



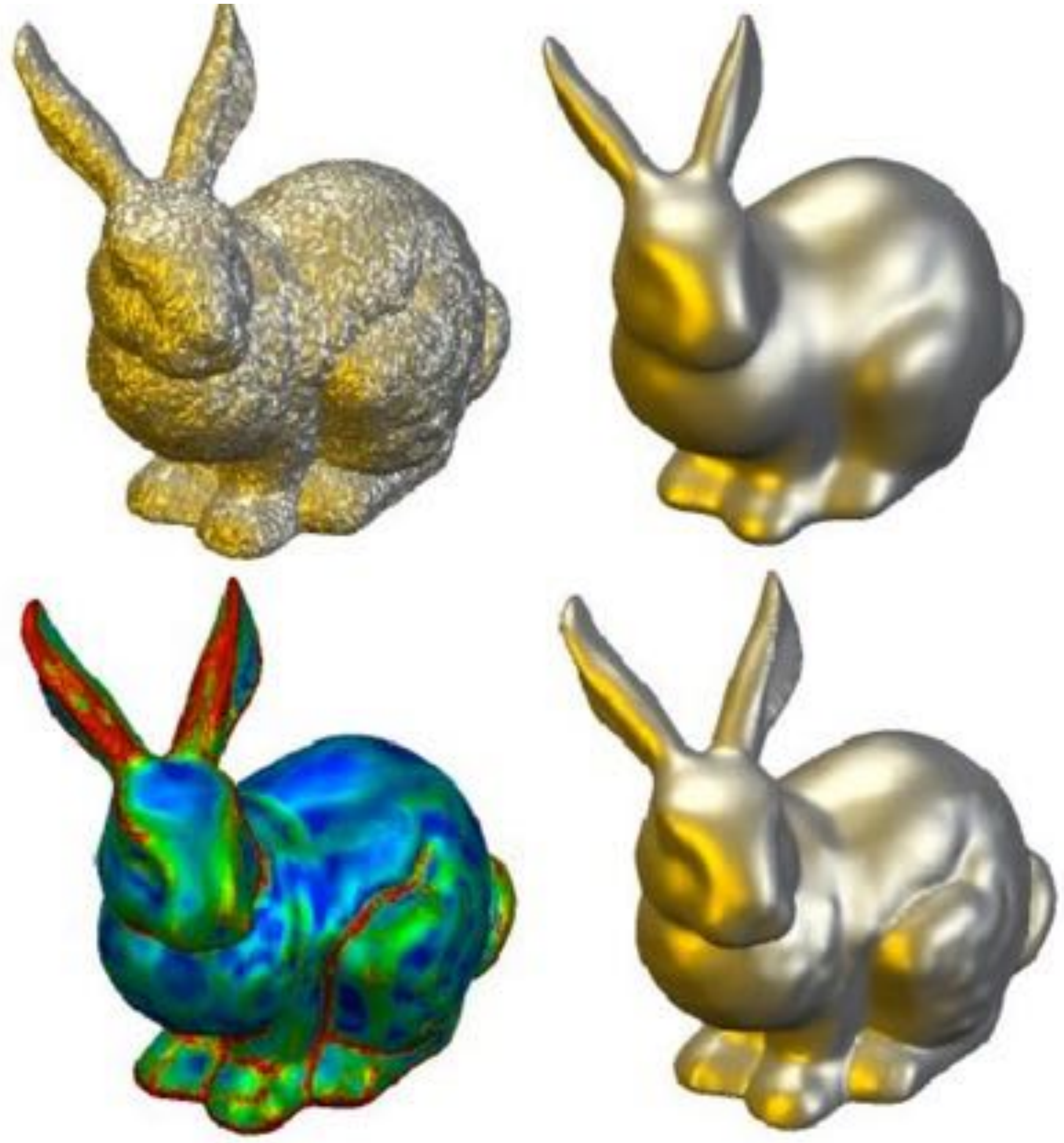
Scherk's Second Surface



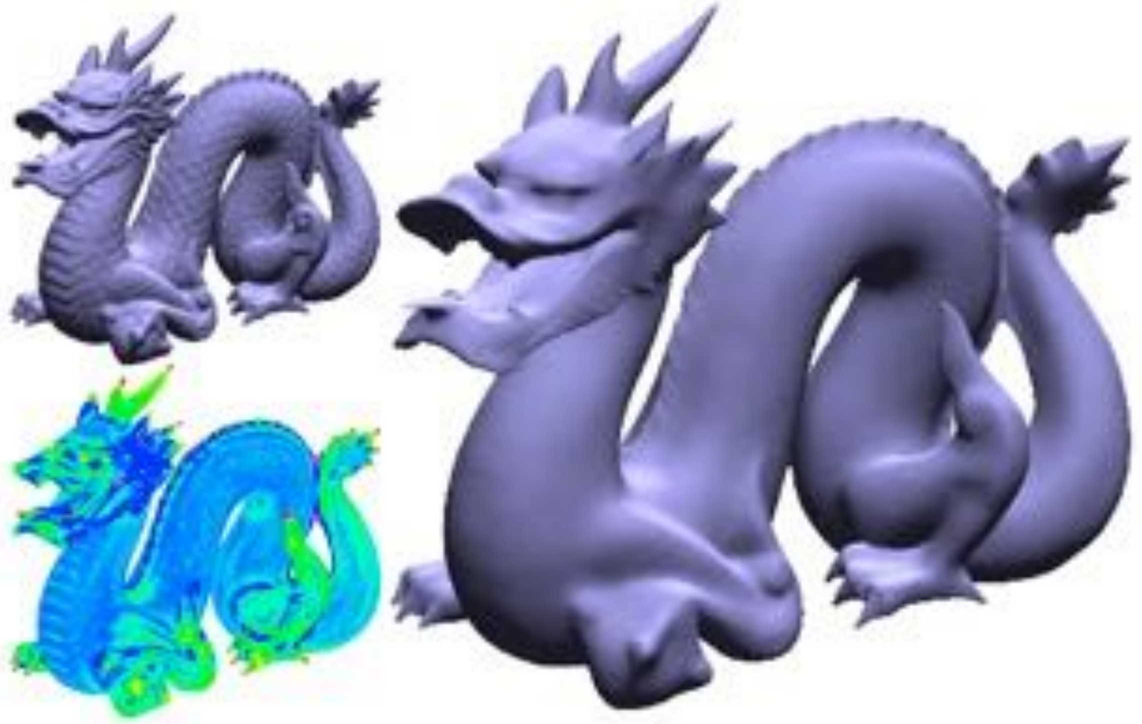
Schwarz P Surface

source: http://www.msri.org/about/sgp/jim/geom/minimal/library/index.html

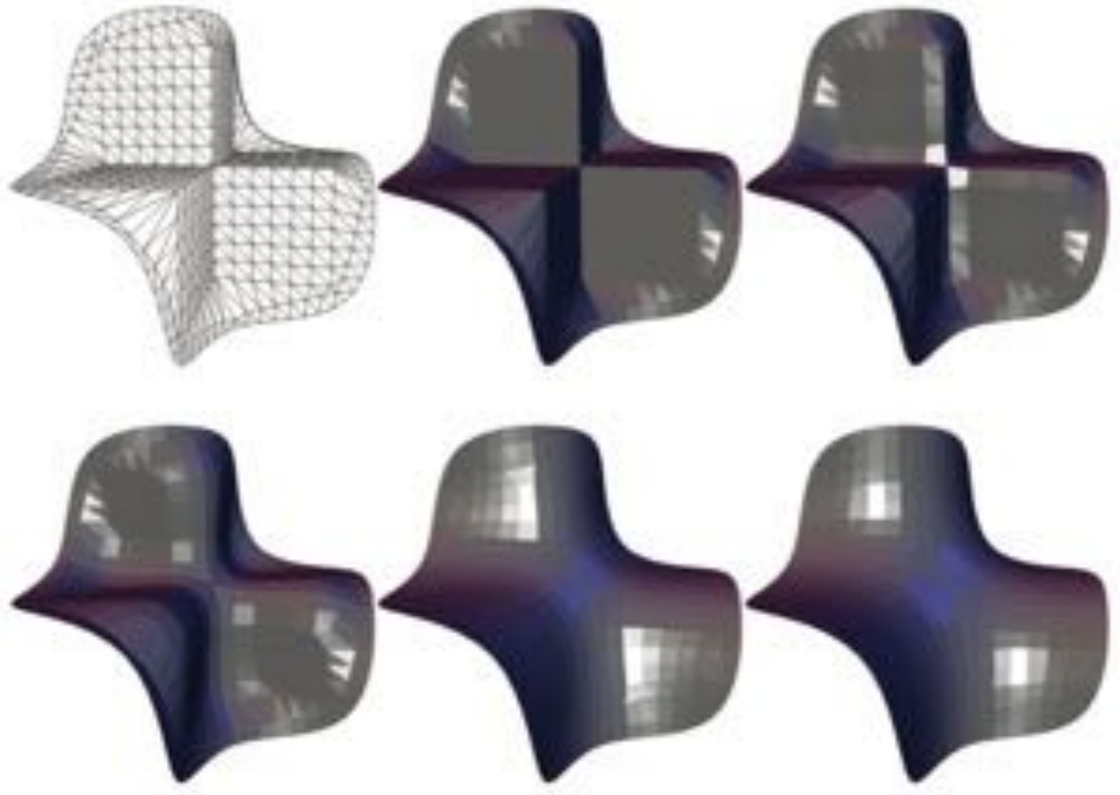
Advanced Methods



U. Clarenz, U. Diewald, and M. Rumpf.
Nonlinear anisotropic diffusion in surface processing
Proceedings of IEEE Visualization 2000

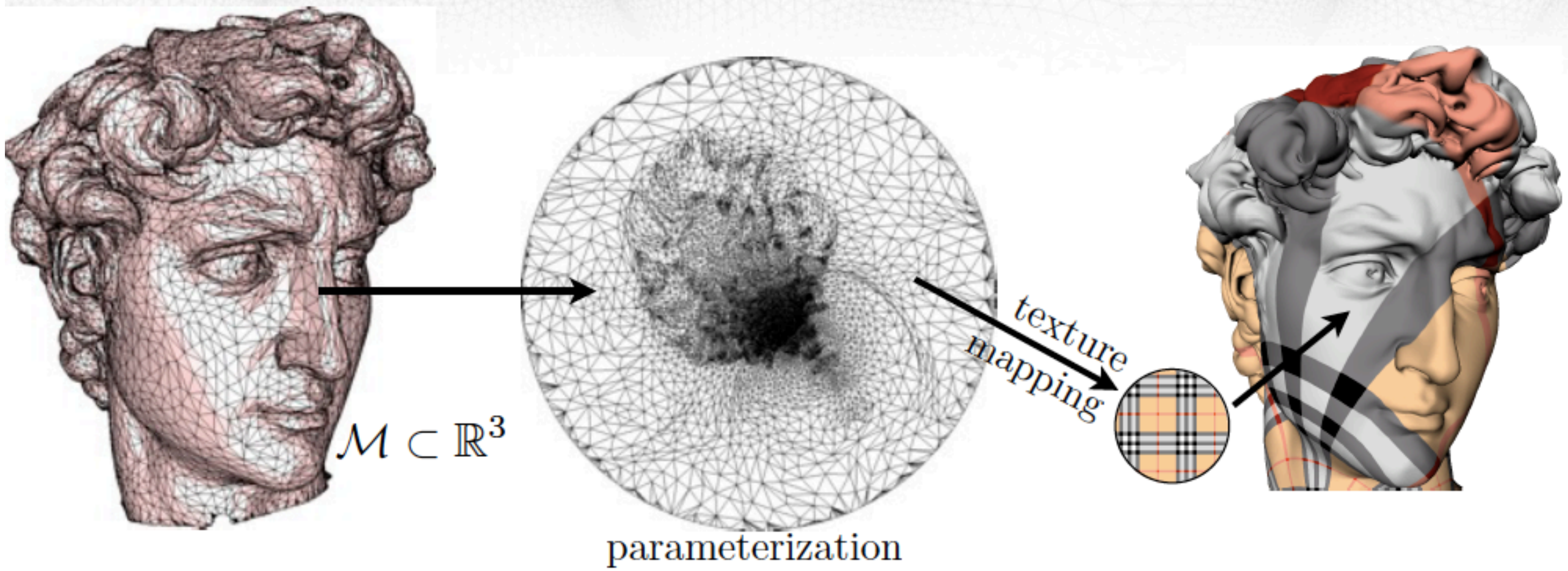


T. Jones, F. Durand, M. Desbrun
Non-Iterative Feature-Preserving Mesh Smoothing
ACM Siggraph 2003

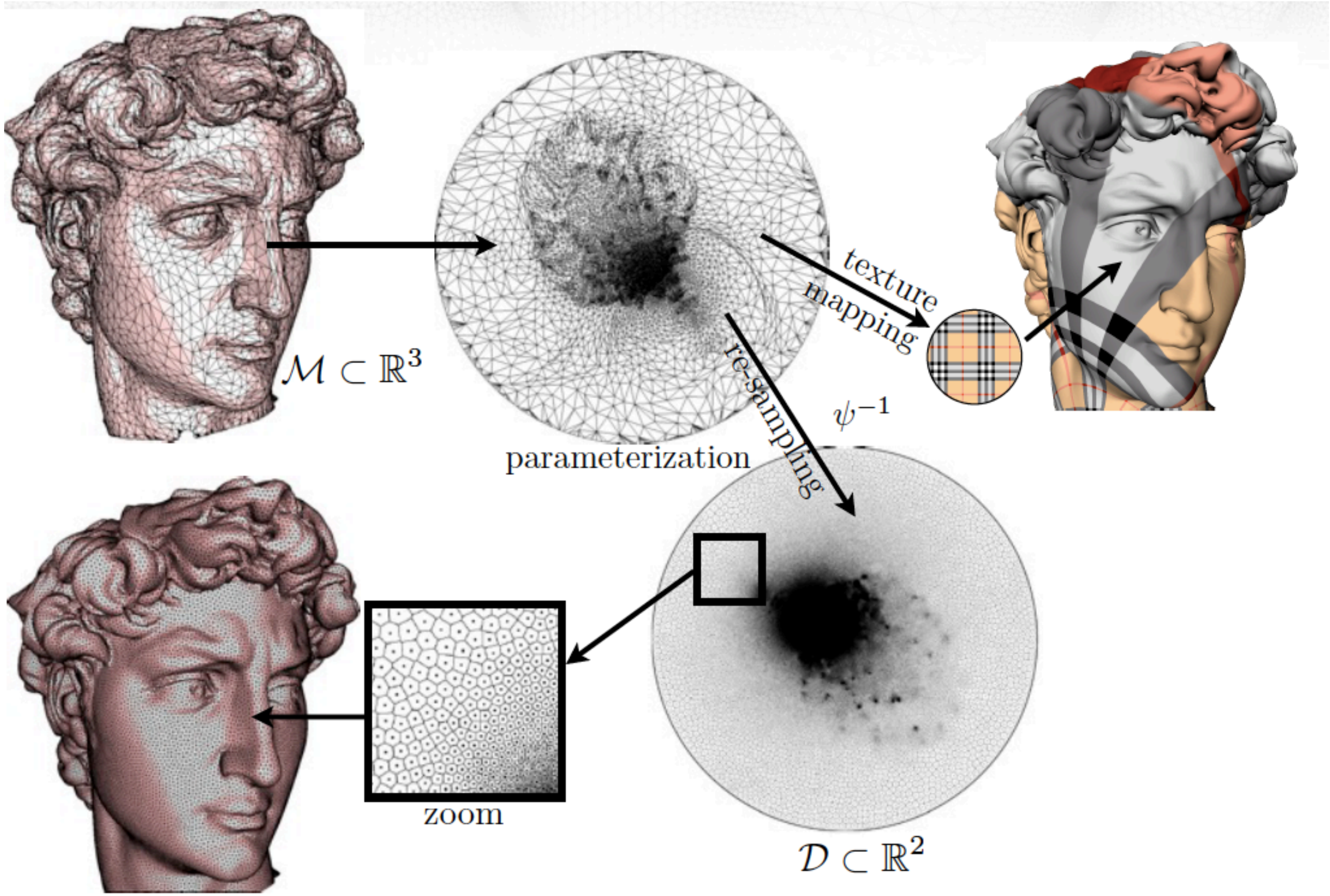


A. Bobenko, P. Schroeder
Discrete Willmore Flow, *SGP 2005*

Texture Mapping

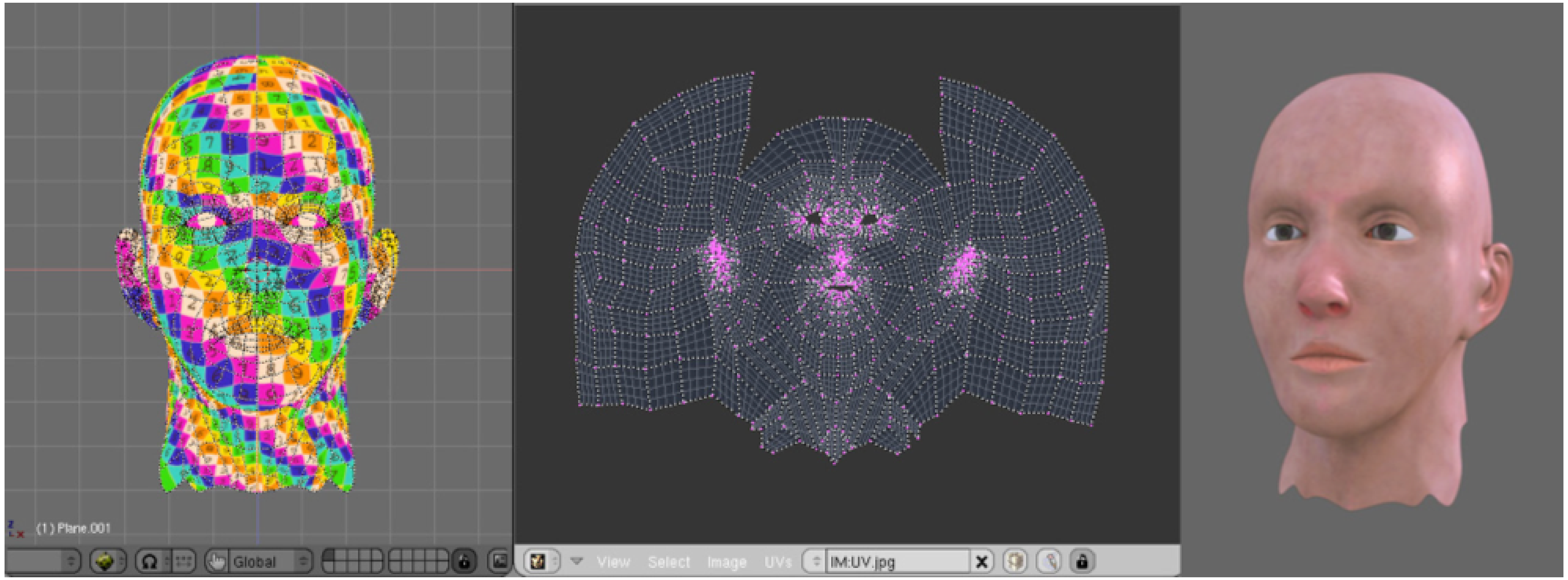


Remeshing



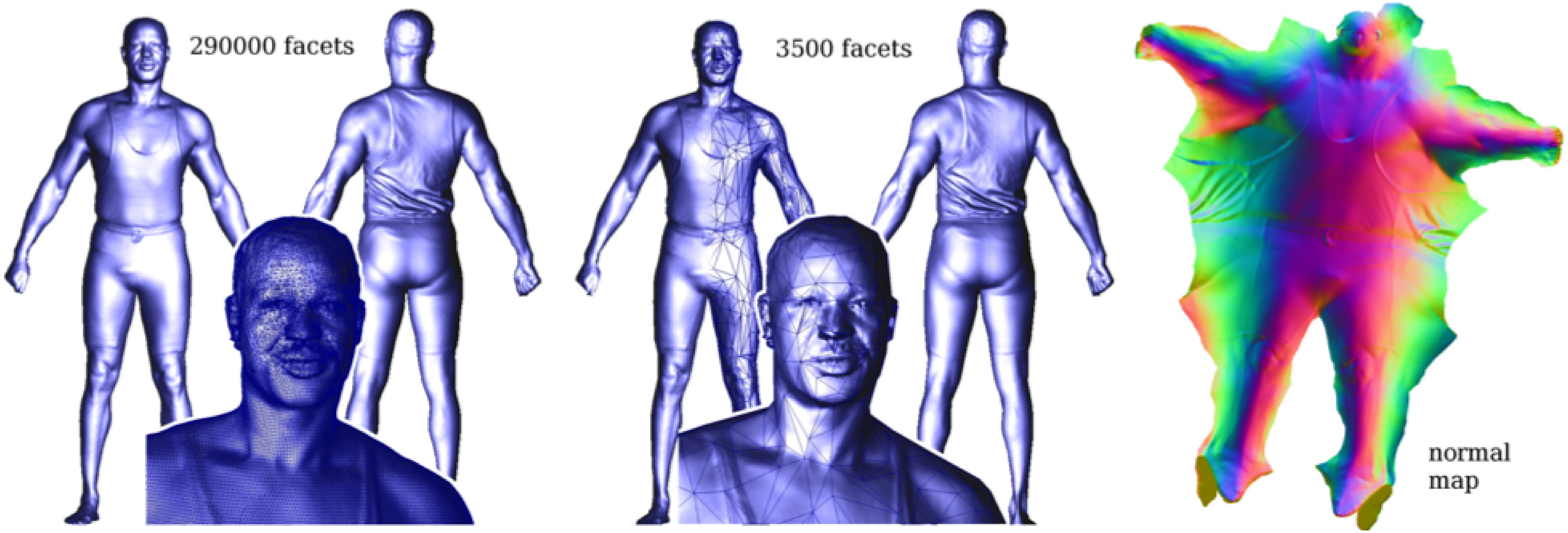
$\mathcal{M} \subset \mathbb{R}^3$
parameterization
texture mapping
re-sampling
ψ^{-1}
zoom
$\mathcal{D} \subset \mathbb{R}^2$

Motivation: Texture Mapping

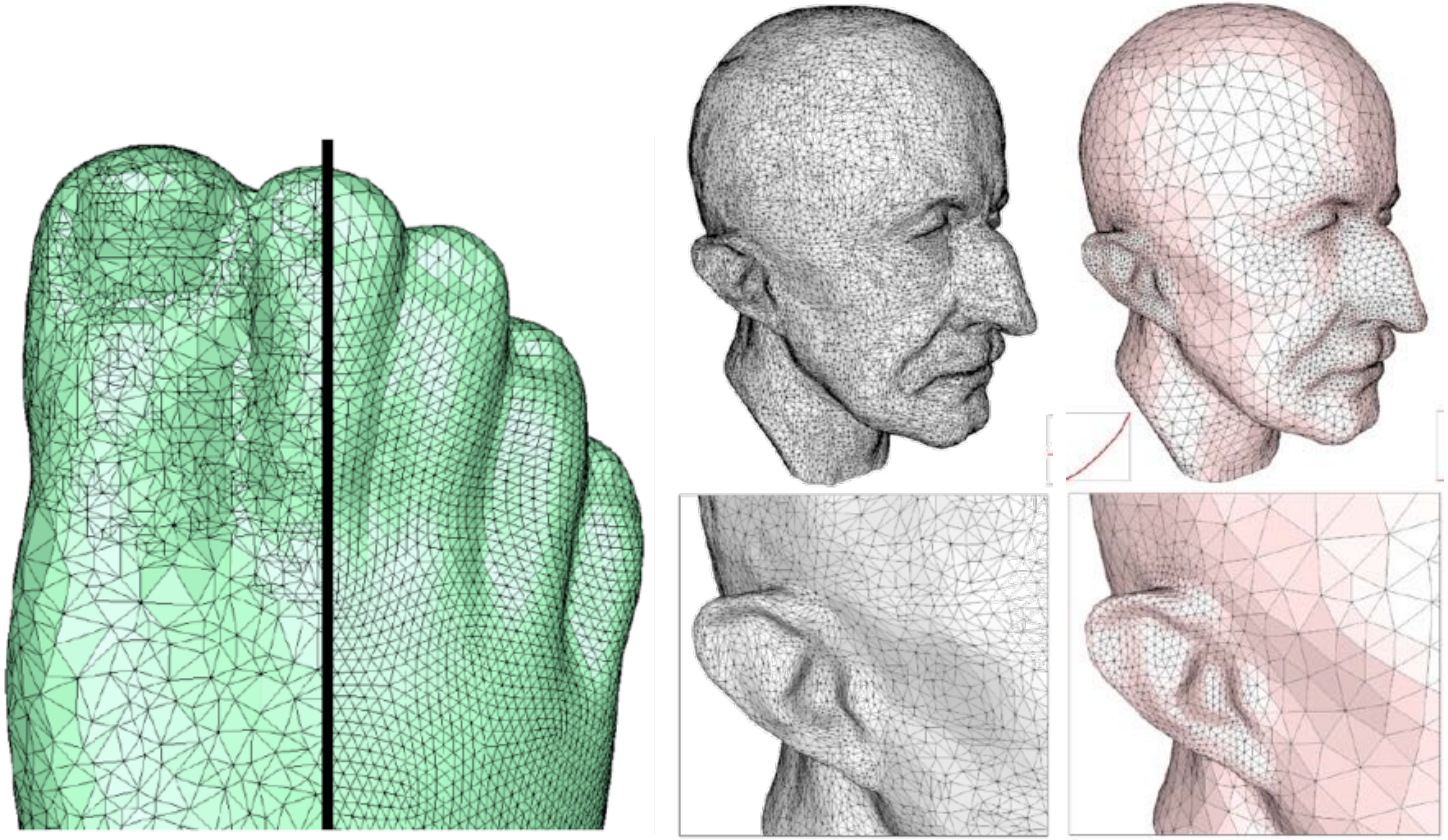


Levy et al.: *Least squares conformal maps for automatic texture atlas generation*, SIGGRAPH 2002.

Motivation: Normal Mapping

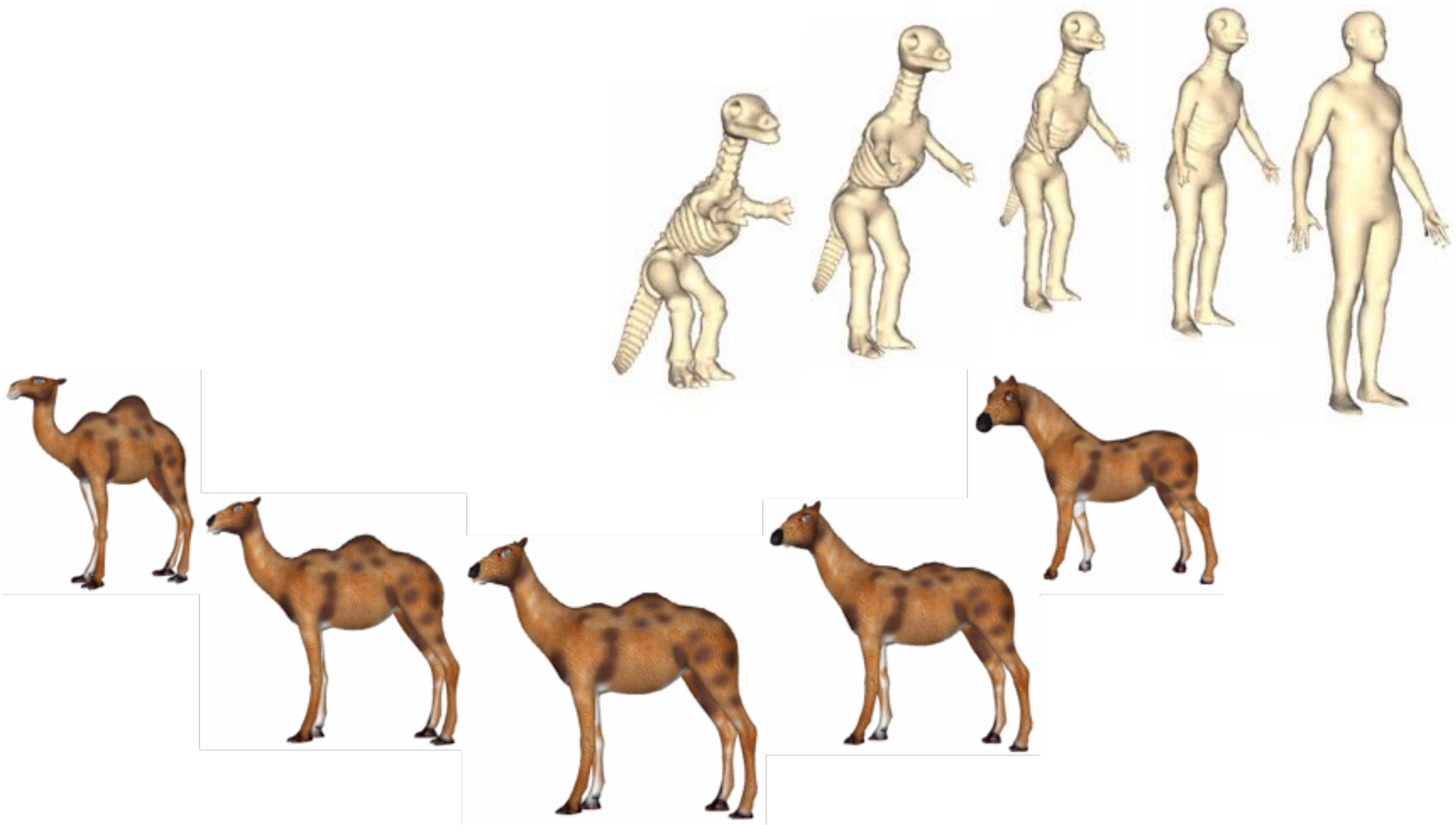


Motivation: Remeshing



Alliez et al.: *Interactive Geometry Remeshing*, SIGGRAPH 2002.

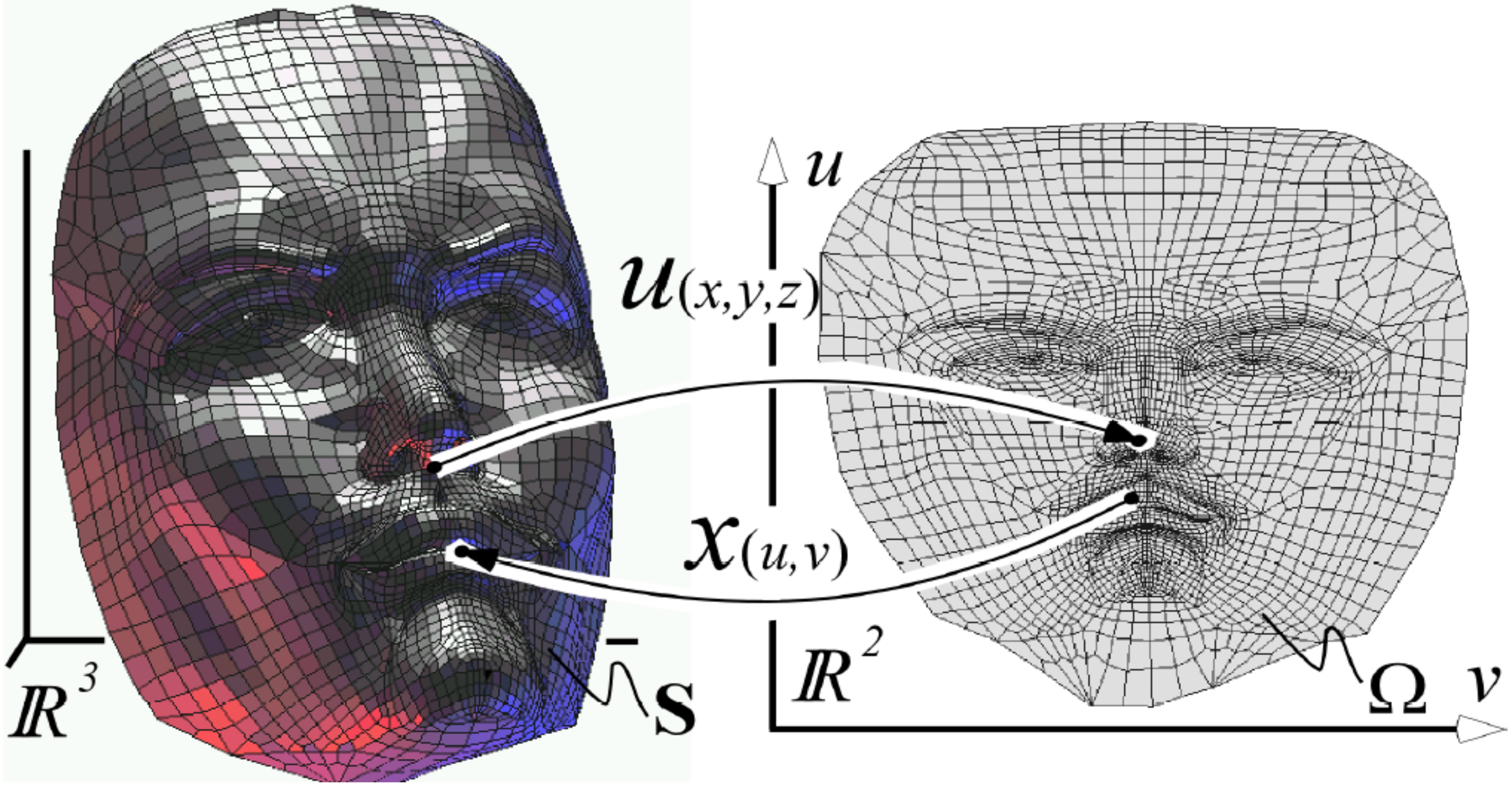
Motivation: Shape Interpolation



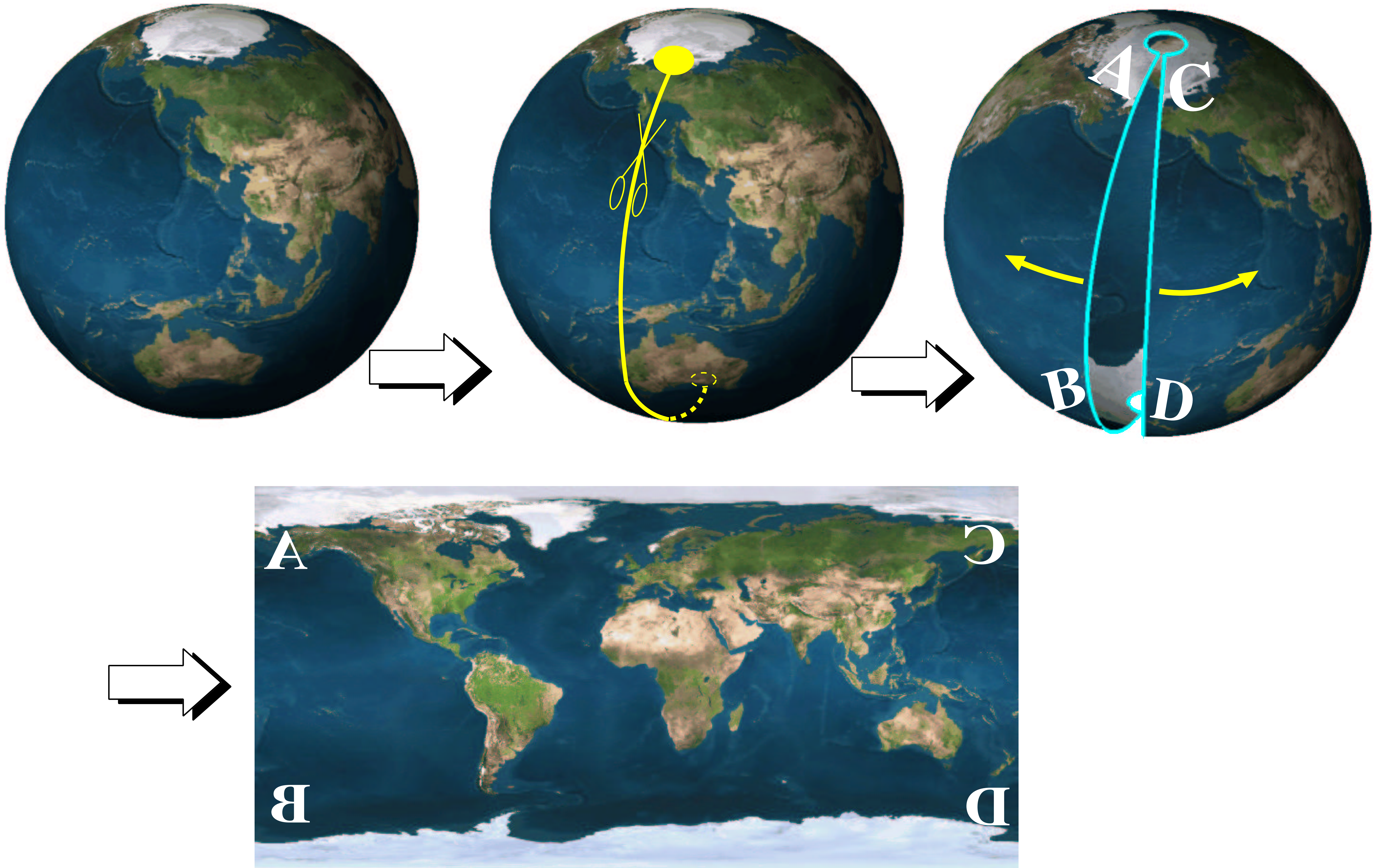
Kraevoy, Sheffer: Cross-Parameterization and Compatible Remeshing of 3D Models, SIGGRAPH, 2004

Mesh Parameterization

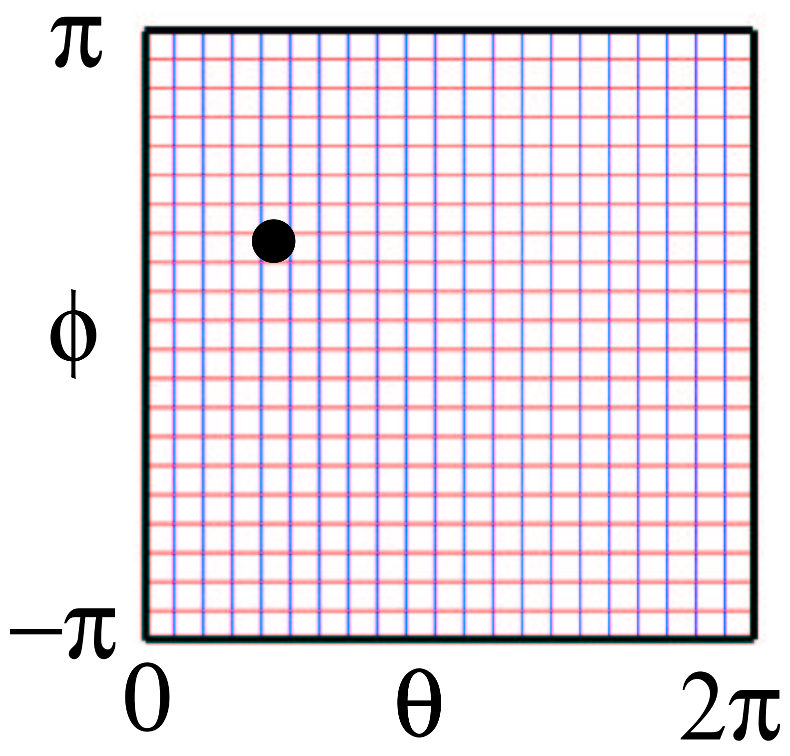
Find a one-to-one mapping between given surface mesh and 2D parameter domain

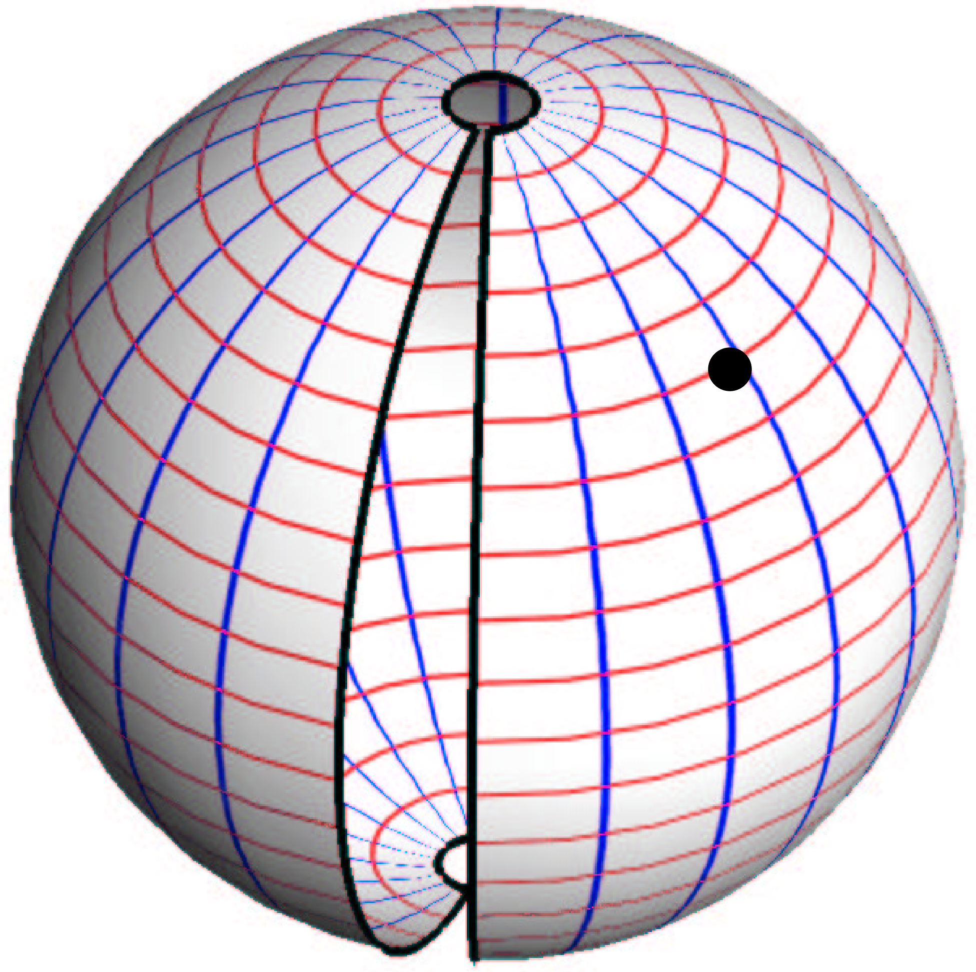


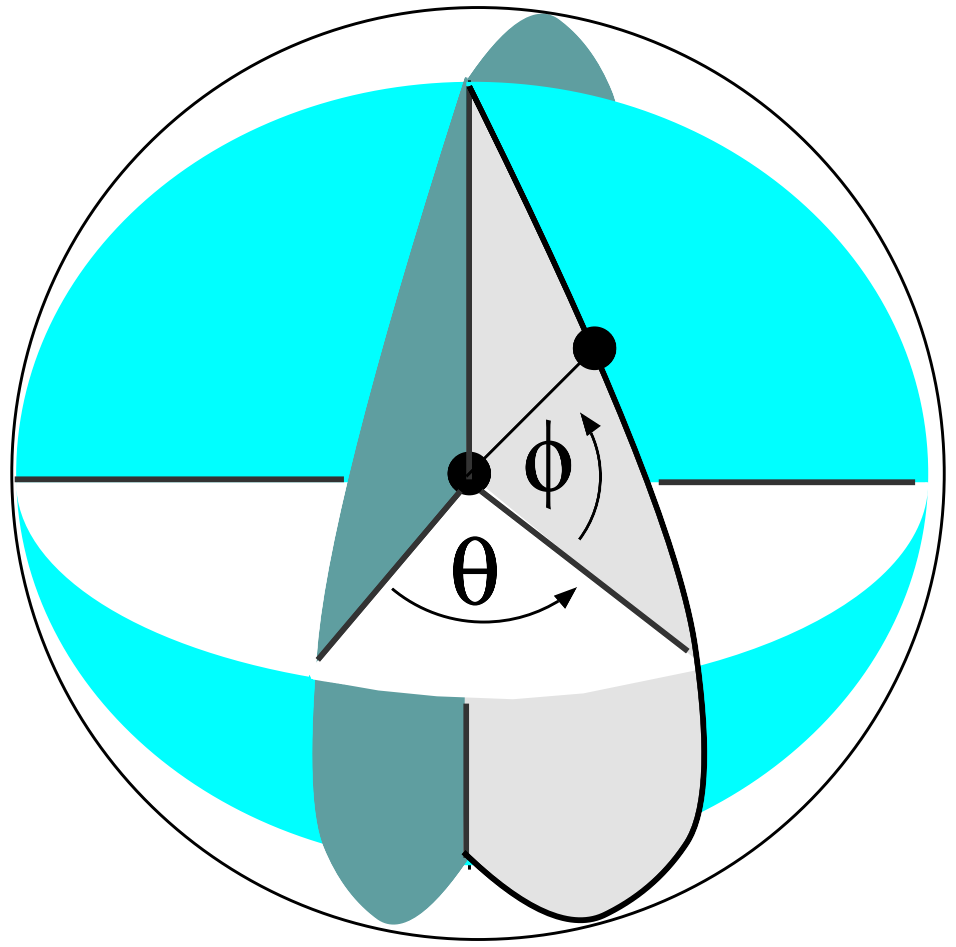
Unfolding the World



Spherical Coordinates



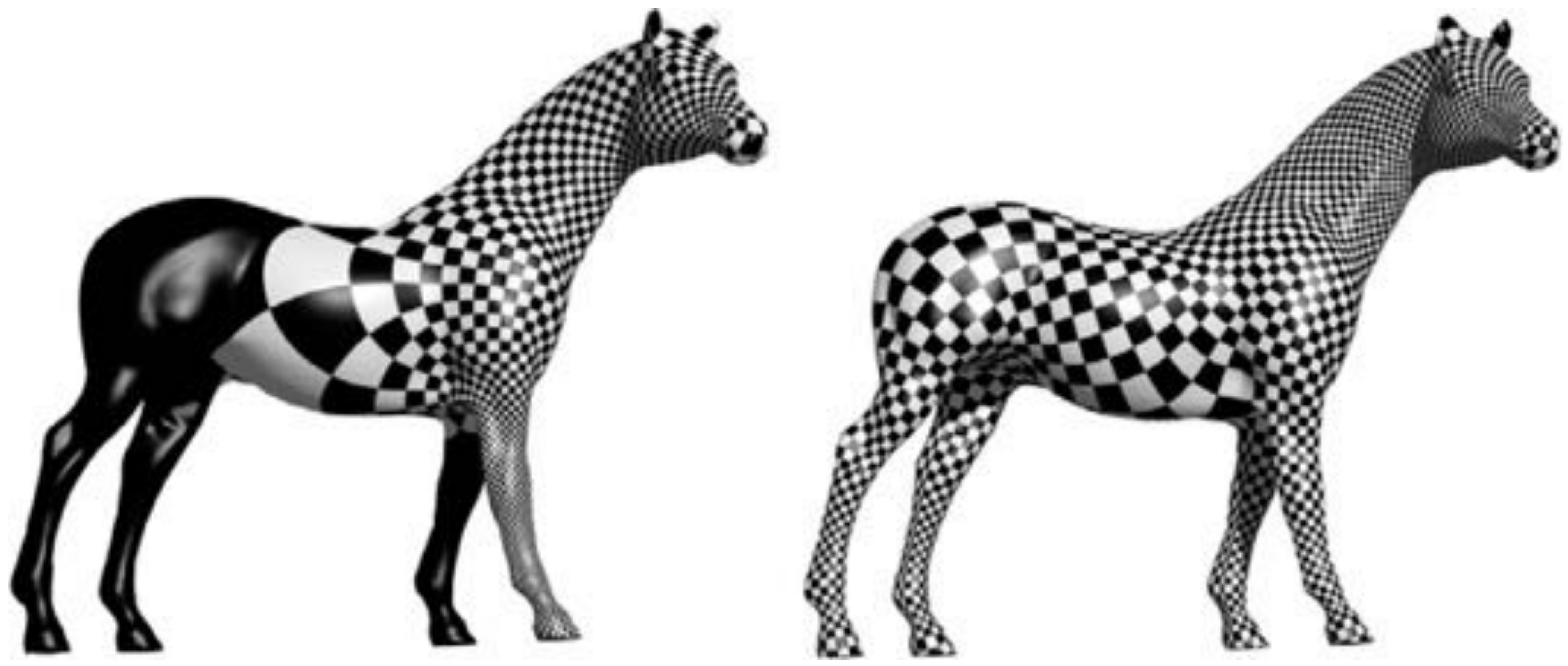




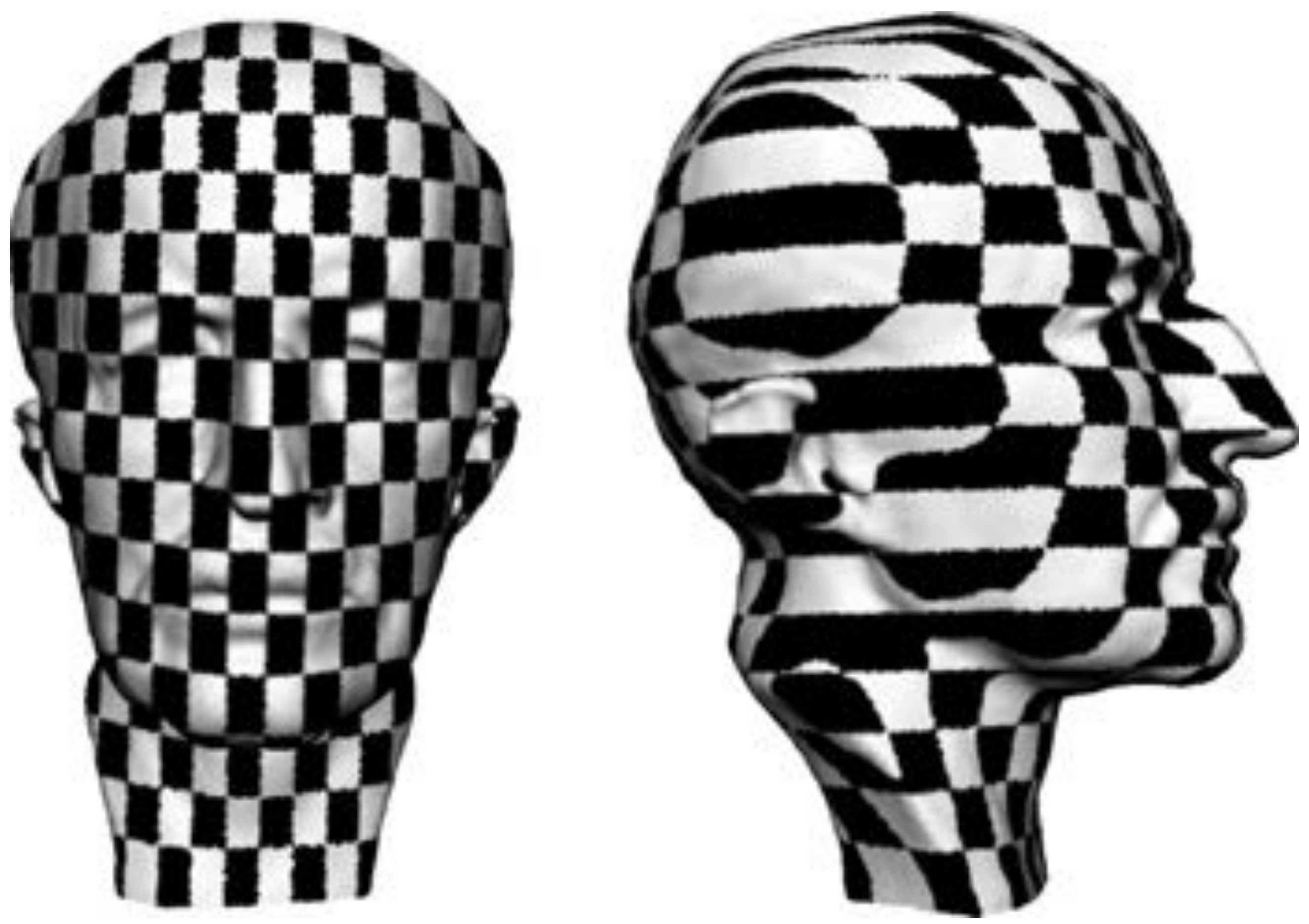
$$\begin{bmatrix} \theta \\ \phi \end{bmatrix} \mapsto \begin{bmatrix} \sin\theta \sin\phi \\ \cos\theta \ \sin\phi \\ \cos\phi \end{bmatrix}$$

Desirable Properties

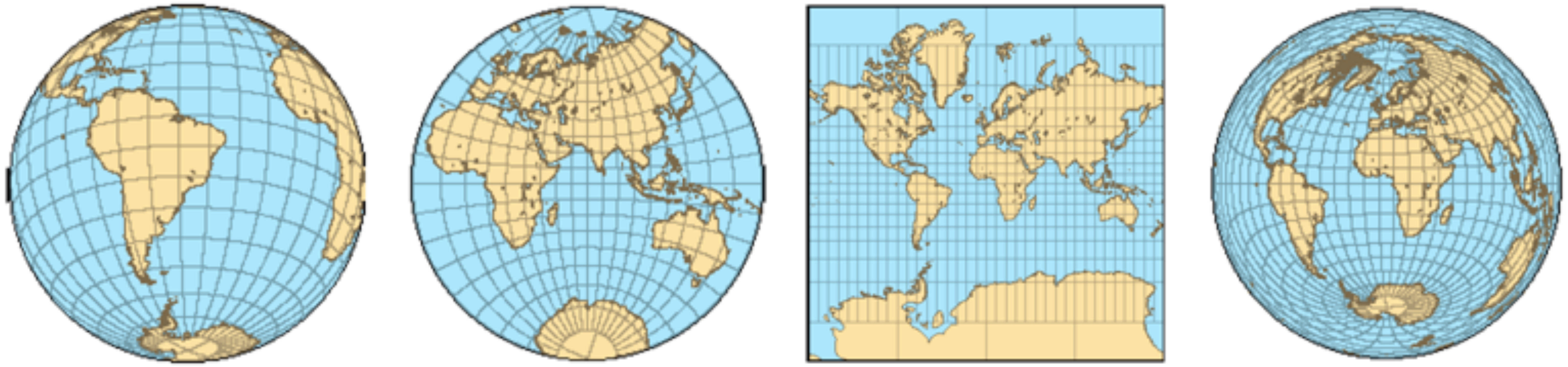
- Low distortion



- Bijective mapping



Cartography



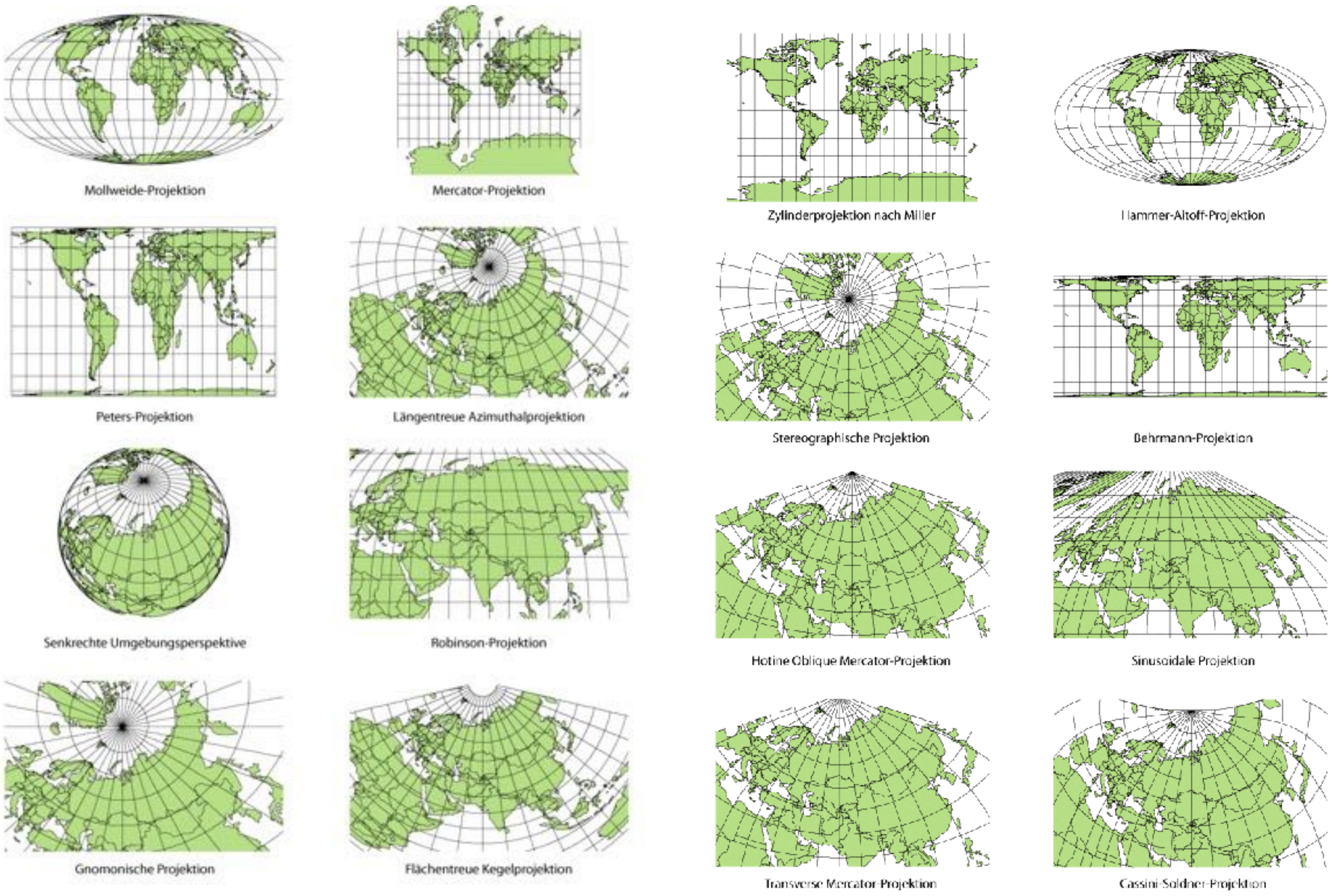
orthographic | stereographic | Mercator | Lambert

stereographic ↑ preserves angles = conformal

Lambert ↑ preserves area = equiareal

Floater, Hormann: *Surface Parameterization: A Tutorial and Survey*, Advances in Multiresolution for Geometric Modeling, 2005

More Maps



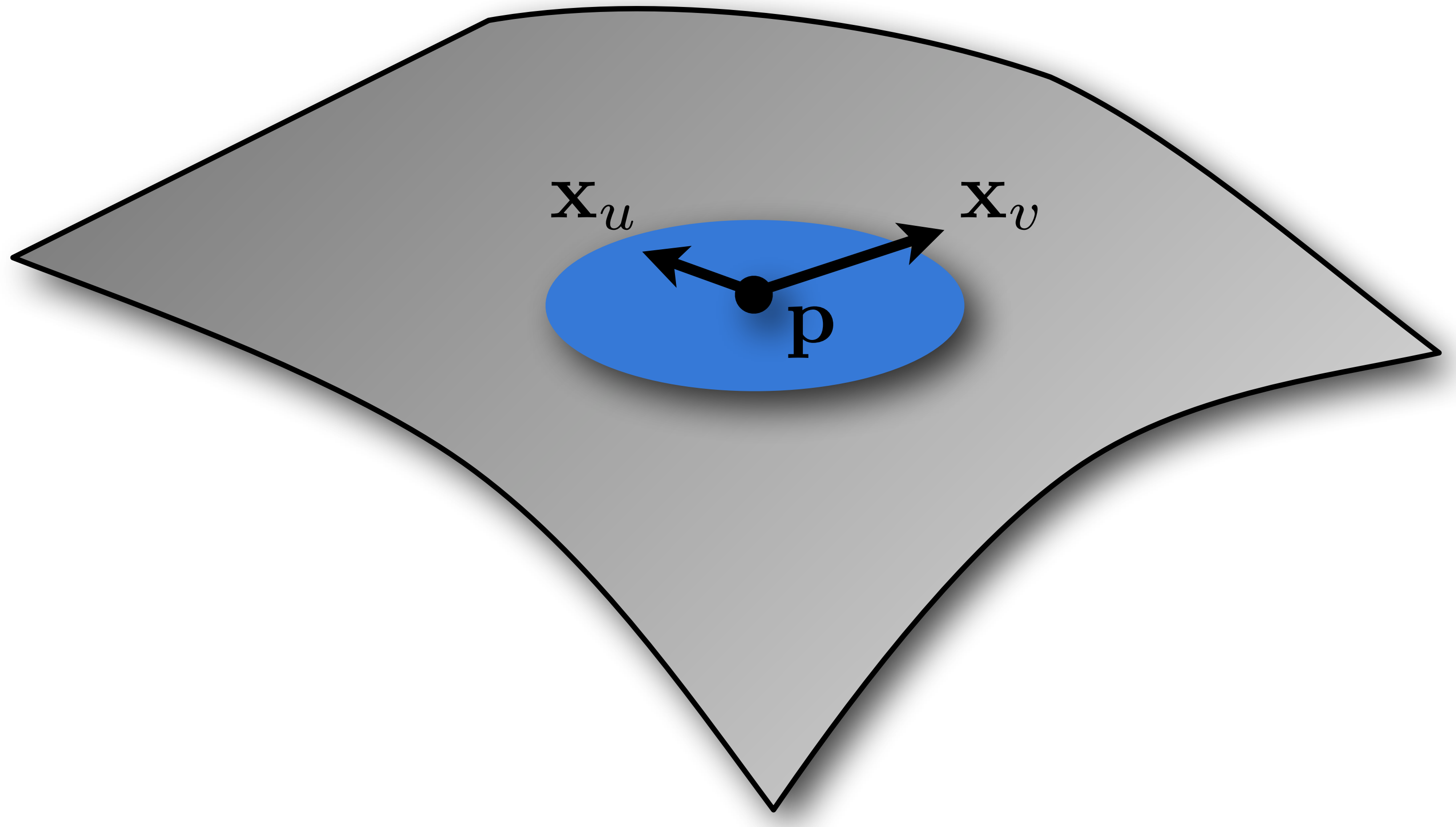
Mollweide-Projektion
Mercator-Projektion
Zylinderprojektion nach Miller
Hammer-Altoff-Projektion
Peters-Projektion
Längentreue Azimuthalprojektion
Stereographische Projektion
Behrmann-Projektion
Senkrechte Umgebungsperspektive
Robinson-Projektion
Hotine Oblique Mercator-Projektion
Sinusoidale Projektion
Gnomonische Projektion
Flächentreue Kegelprojektion
Transverse Mercator-Projektion
Cassini-Soldner-Projektion

Differential Geometry Revisited

- Parametric surface representation

$$\mathbf{x} : \Omega \subset \mathbb{R}^2 \quad \rightarrow \quad \mathcal{S} \subset \mathbb{R}^3$$

$$(u, v) \quad \mapsto \quad \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$



- *Regular* if
  - Coordinate functions *x, y, z* are smooth
  - Tangents are linearly independent

$$\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$

Definitions

- A regular parameterization $\mathbf{x} : \Omega \to \mathcal{S}$ is
  - ***conformal*** *(angle preserving)*, if the **angle** of every pair of intersecting curves on $\mathcal{S}$ is the same as that of the corresponding pre-images in Ω.
  - ***equiareal*** *(area preserving)* if every part of Ω is mapped onto a part of $\mathcal{S}$ with the same **area**.
  - ***isometric*** *(length preserving)*, if the **length** of any arc on $\mathcal{S}$ is the same as that of its pre-image in Ω.

ABF (white board)

Angle Based Flattening (ABF)

$$F(\alpha) := \sum_{i=1}^{P} \sum_{j=1}^{3} w_i^j (\alpha_i^j - \phi_i^j)^2$$

$$\phi_i^{j(k)} = \begin{cases} \beta_i^{j(k)} \frac{2\pi}{\sum_i \beta_i^{j(k)}} & \text{interior vertex} \\ \beta_i^{j(k)} & \text{exterior vertex} \end{cases}$$

$$\beta_i^j \geq \epsilon_1 > 0$$

Angle Based Flattening (ABF)

$$F(\alpha) := \sum_{i=1}^{P} \sum_{j=1}^{3} w_i^j (\alpha_i^j - \phi_i^j)^2$$

$$\alpha_i^j \geq \epsilon_2 > 0$$

$$\sum_{j=1}^{3} \alpha_i^j - \pi = 0$$

$$\sum_i \alpha_i^{j(k)} - 2\pi = 0$$

$$\frac{\Pi_i \sin\left(\alpha_i^{j(k)+1}\right)}{\Pi_i \sin\left(\alpha_i^{j(k)-1}\right)} - 1 = 0$$

Tutte's Embedding

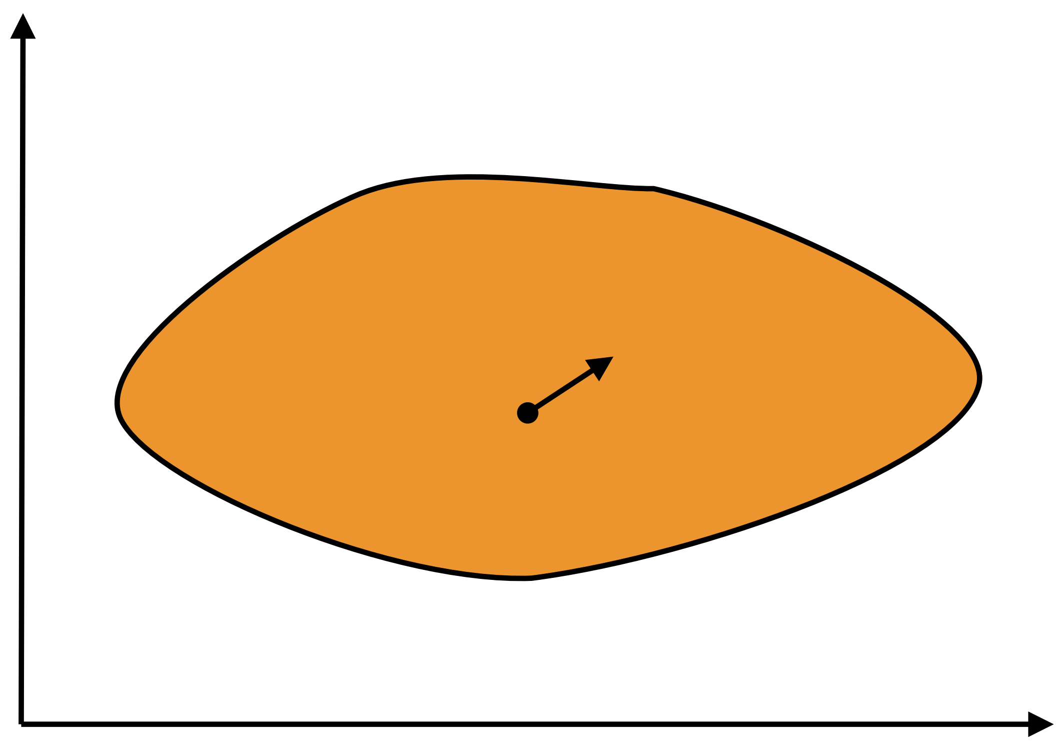
$$\Delta \mathbf{u} = 0$$

$$M^{-1} C \mathbf{u} = 0$$

$$C \mathbf{u} = 0$$

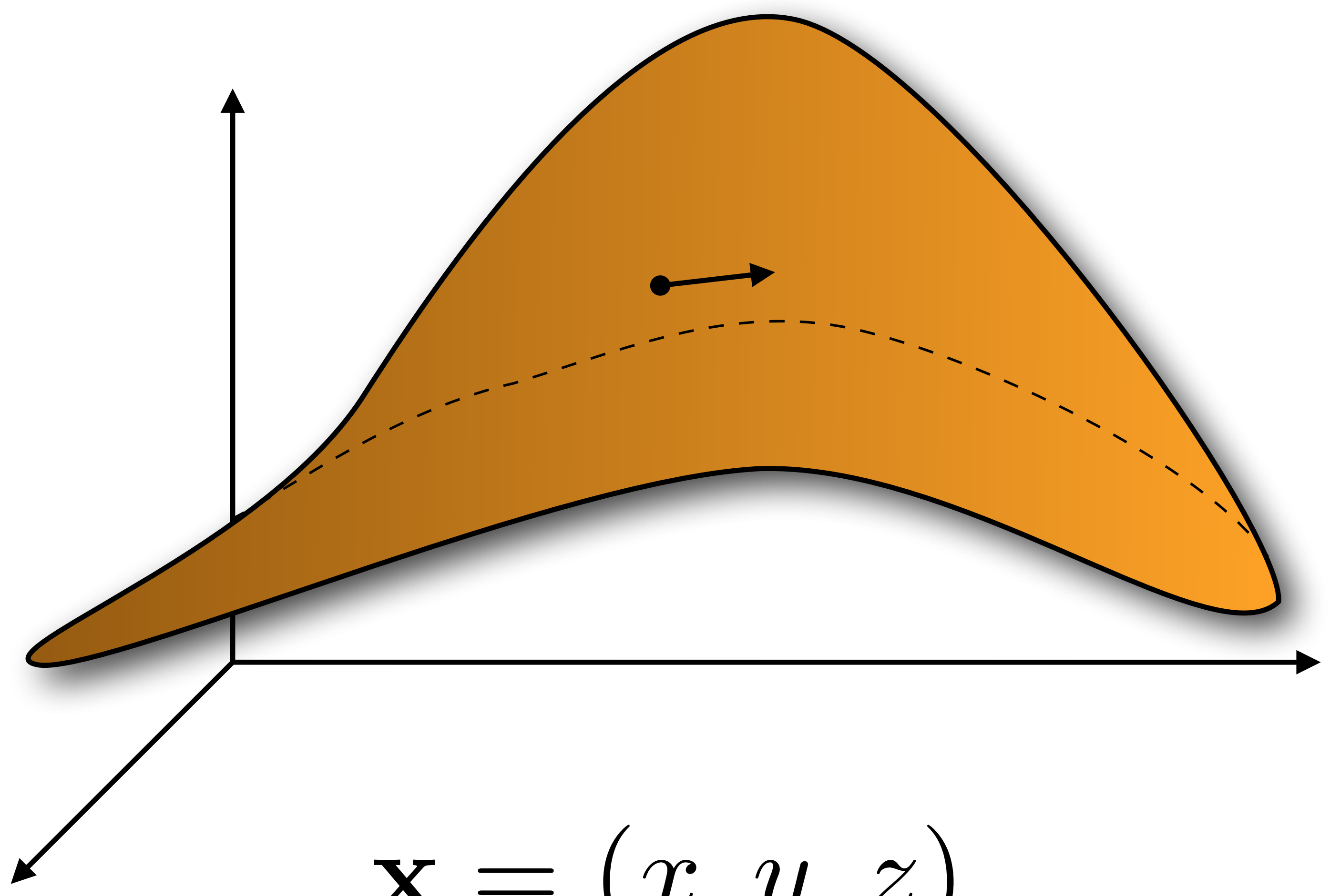
Subject to boundary points being on a convex polygon

Distortion Analysis: Jacobian



$\mathbf{u} = (u, v)$

$$\mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$



$\mathbf{x} = (x, y, z)$

$$\mathrm{d}\mathbf{x} = \mathbf{J}\,\mathrm{d}\mathbf{u} \qquad \mathbf{J} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix} = [\mathbf{x}_u \quad \mathbf{x}_v]$$

$$\|\mathrm{d}\mathbf{x}\|^2 = (\mathrm{d}\mathbf{u})^T \mathbf{J}^T \mathbf{J}\,\mathrm{d}\mathbf{u} = (\mathrm{d}\mathbf{u})^T \mathbf{I}\,\mathrm{d}\mathbf{u}$$

First Fundamental Form

- Characterizes the surface locally

$$\mathbf{I} = \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix}$$

- Allows to measure on the surface
  - **Angles (conformal)** $\cos\theta = \left(\mathrm{d}\mathbf{u}_1^T \,\mathbf{I}\, \mathrm{d}\mathbf{u}_2\right) / \left(\|\mathrm{d}\mathbf{u}_1\| \cdot \|\,\mathrm{d}\mathbf{u}_2\|\right)$
  - **Length (isometric)** $\mathrm{d}s^2 = \mathrm{d}\mathbf{u}^T \,\mathbf{I}\, \mathrm{d}\mathbf{u}$
  - **Area (equiareal)** $\mathrm{d}A = \det(\mathbf{I})\, \mathrm{d}u\, \mathrm{d}v$

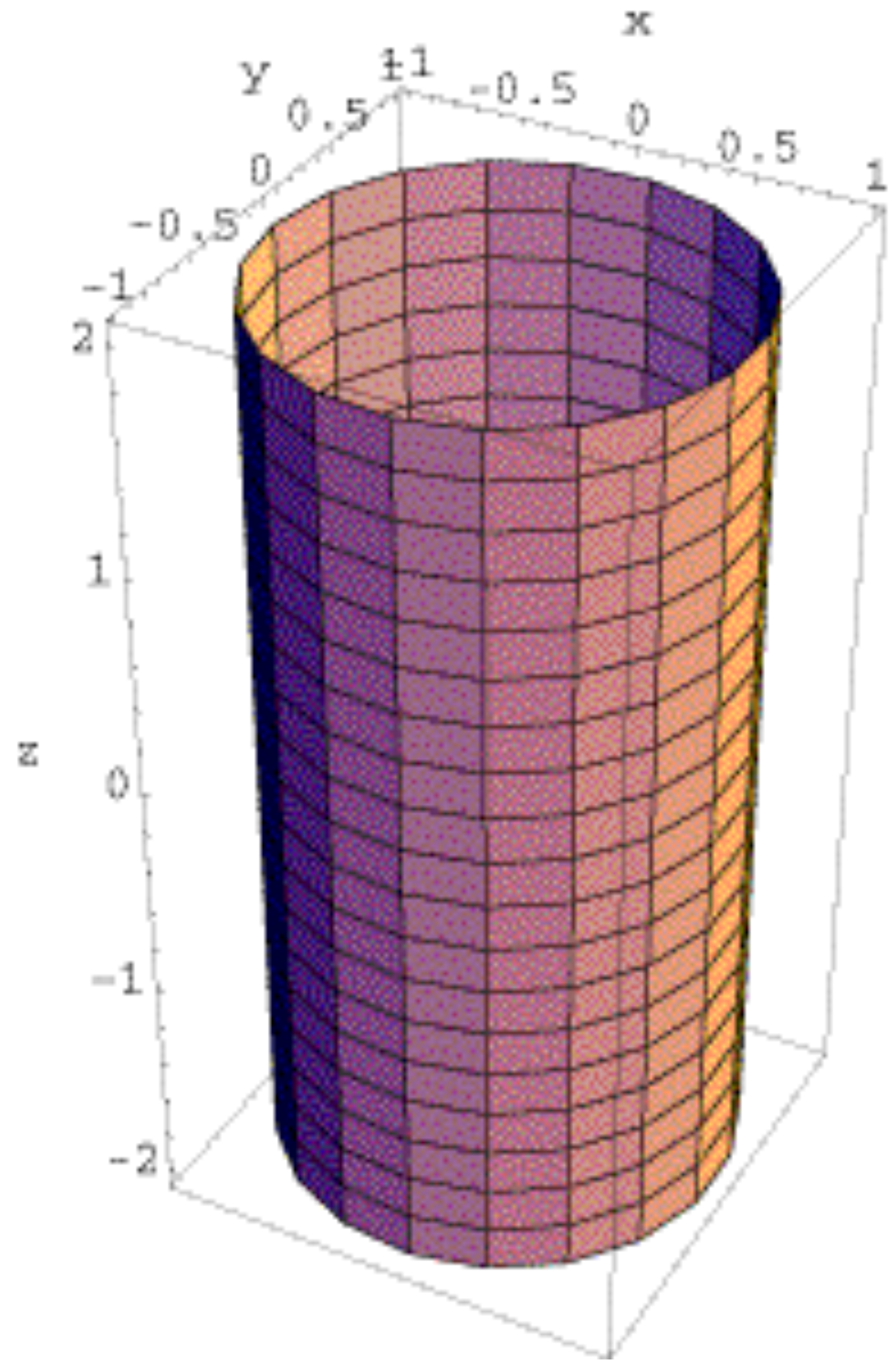
Isometric Maps

- A regular parameterization $\mathbf{x}(u,v)$ is ***isometric***, iff its first fundamental form is the identity:

$$\mathbf{I}(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- A surface has an isometric parameterization iff it has zero Gaussian curvature.

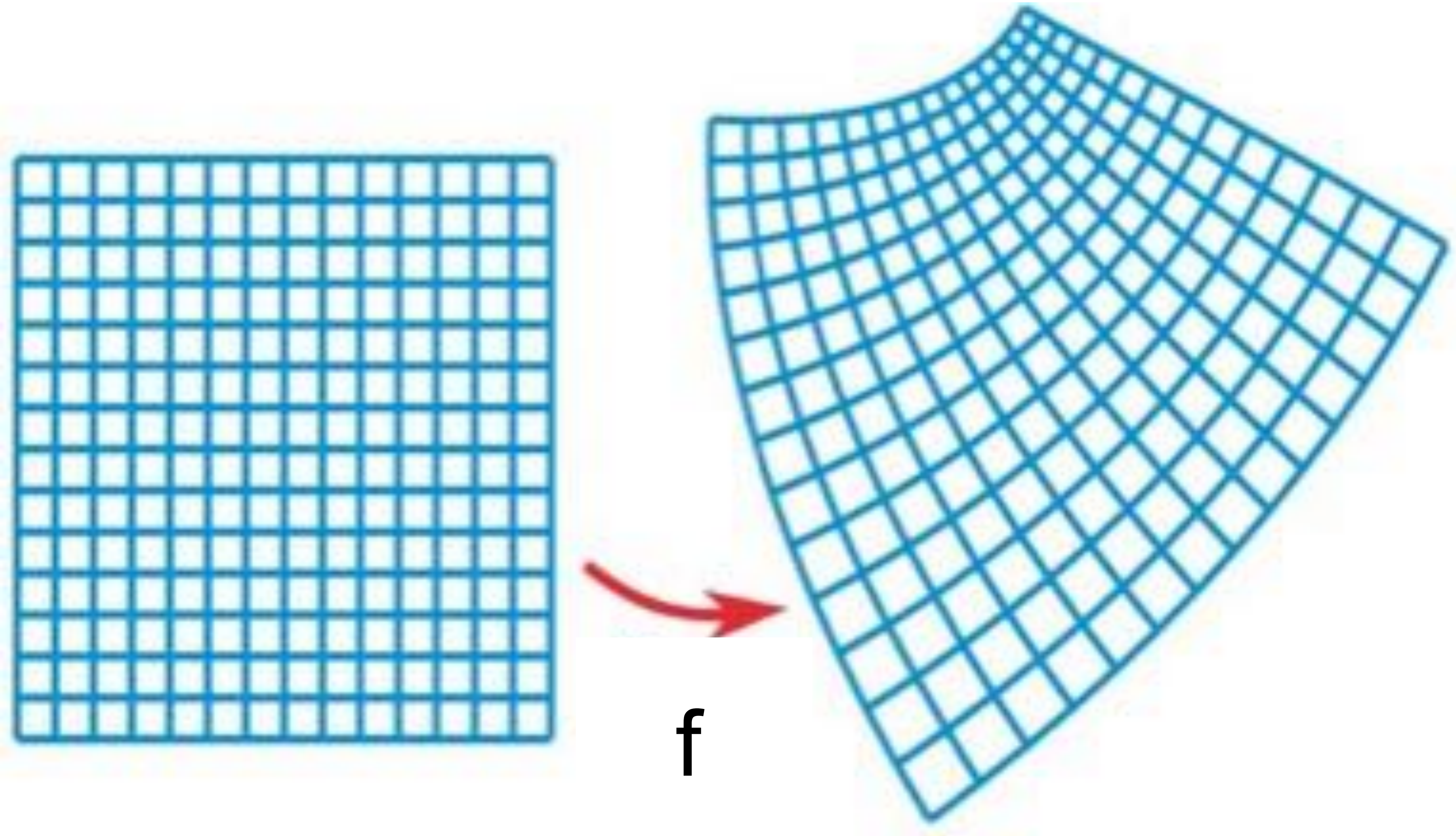
Cylinder

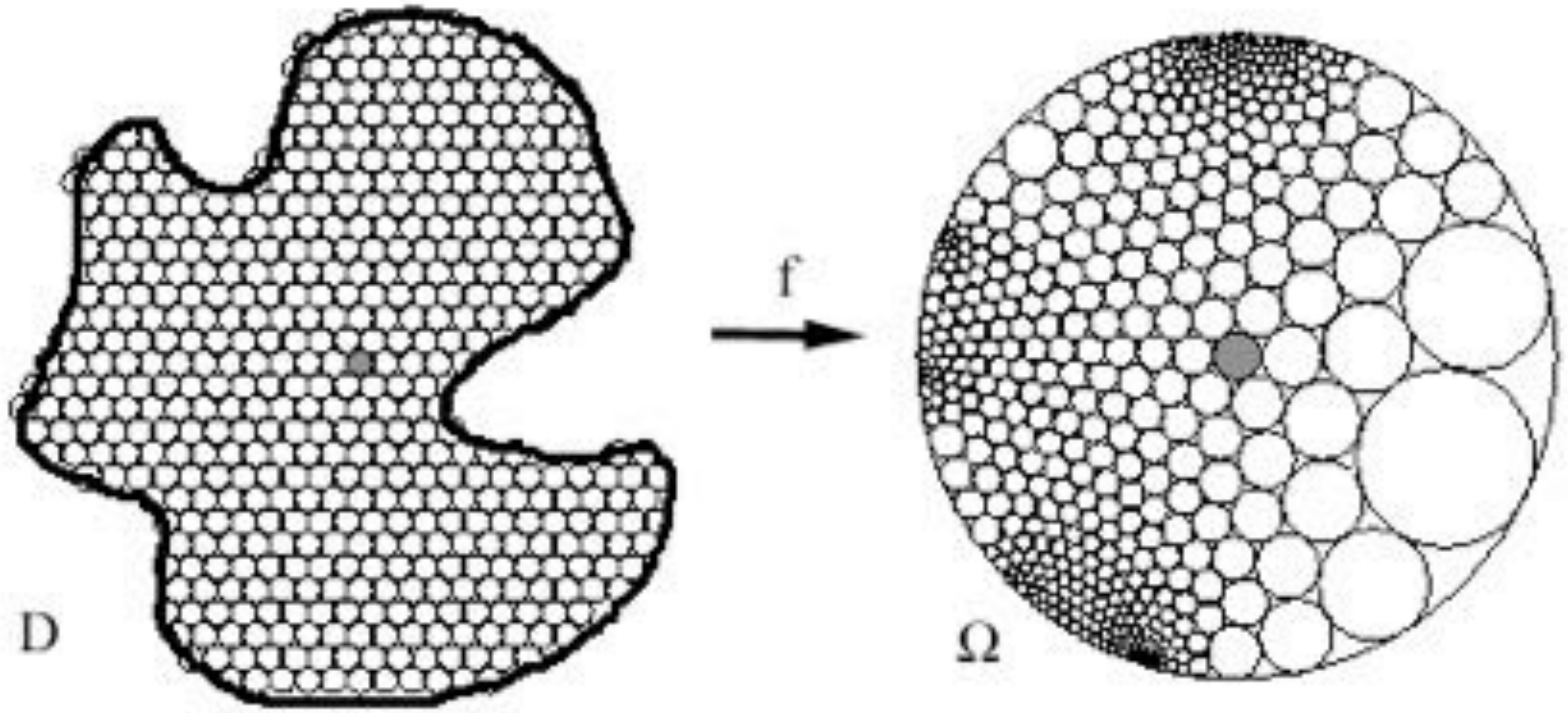


Conformal Maps

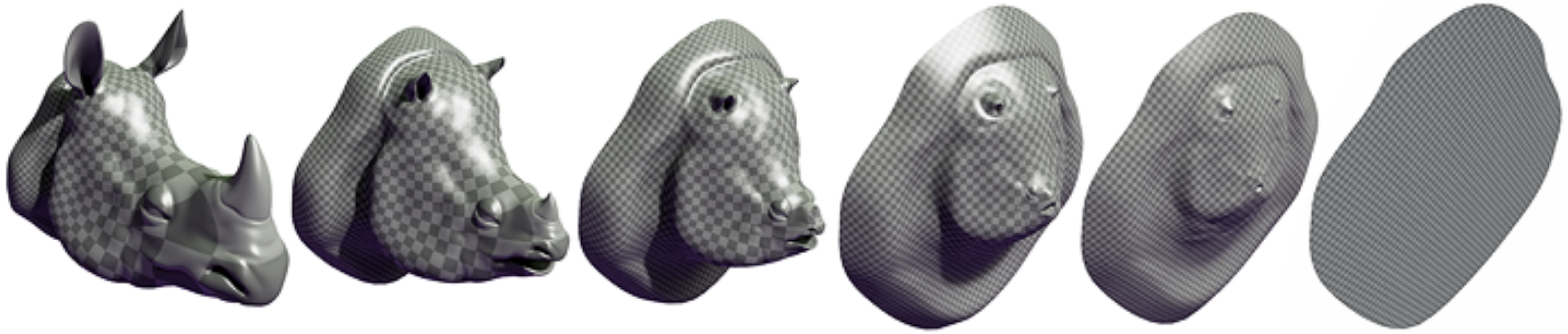
- A regular parameterization $\mathbf{x}(u,v)$ is ***conformal***, iff its first fundamental form is a scalar multiple of the identity:

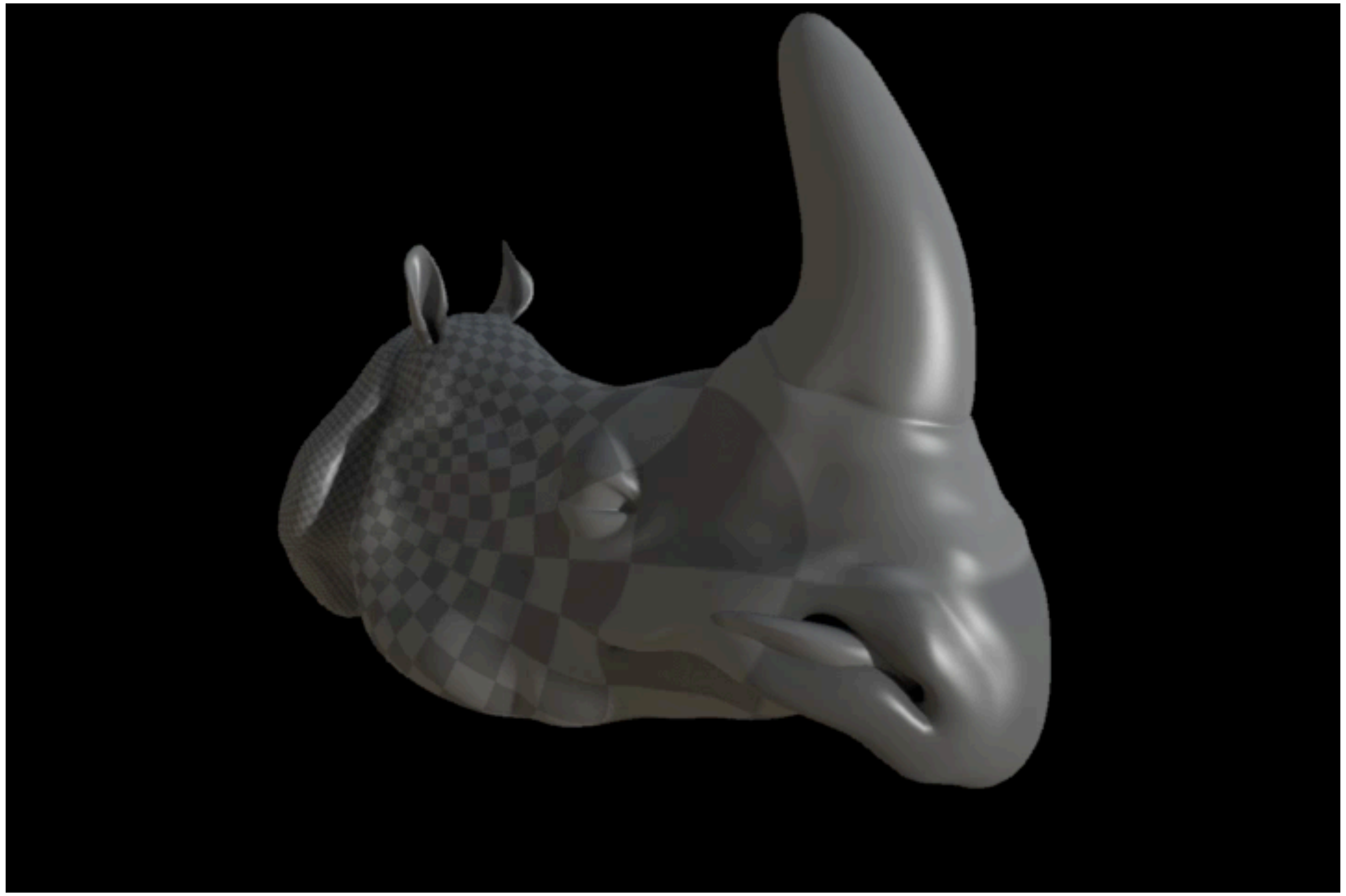
$$\mathbf{I}(u,v) \;=\; s(u,v) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$





Conformal Flow



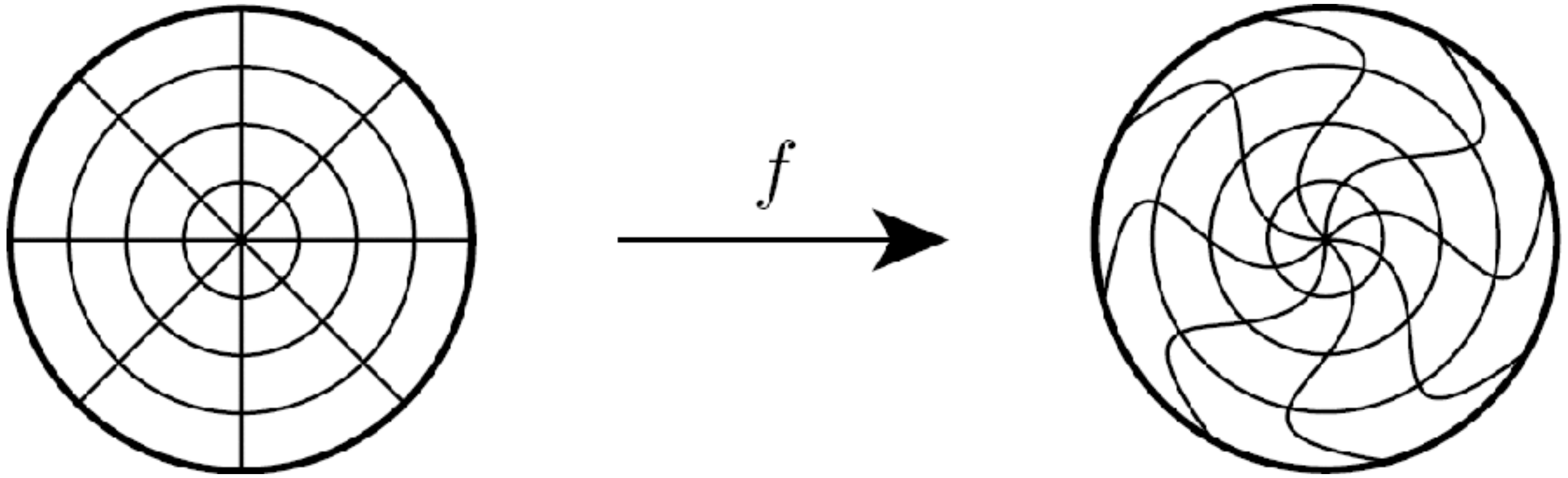


Crane et al., Spin Transformations of Discrete Surfaces, ACM Siggraph 2011

Equiareal Maps

- A regular parameterization **x**(*u*,*v*) is ***equiareal***, iff the determinant of its first fundamental form is 1:

$$\det(\mathbf{I}(u,v)) \;=\; 1$$



Relationships

- An isometric parameterization is conformal and equiareal, and vice versa:

  isometric $\Leftrightarrow$ conformal + equiareal

- Isometric is ideal, but rare.
  In practice, people try to compute:
  – Conformal (always exist)
  – Equiareal
  – Some balance between the two

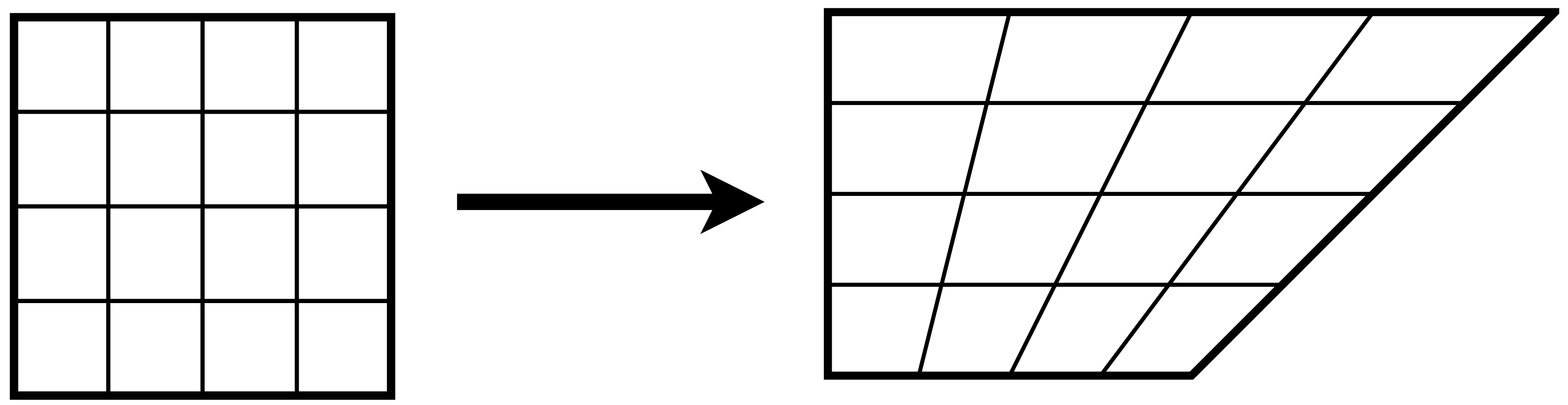
Harmonic Maps

- A regular parameterization $\mathbf{x}(u,v)$ is ***harmonic***, if and only if it satisfies

$$\Delta \mathbf{x}(u,v) = 0$$

- isometric ⇒ conformal ⇒ harmonic

- Easier to compute than conformal, but does not preserve angles



Harmonic Maps

- A harmonic map minimizes the Dirichlet energy

$$\int_\Omega \|\nabla \mathbf{x}\|^2 \;=\; \int_\Omega \|\mathbf{x}_u\|^2 + \|\mathbf{x}_v\|^2 \;\mathrm{d}u\,\mathrm{d}v$$

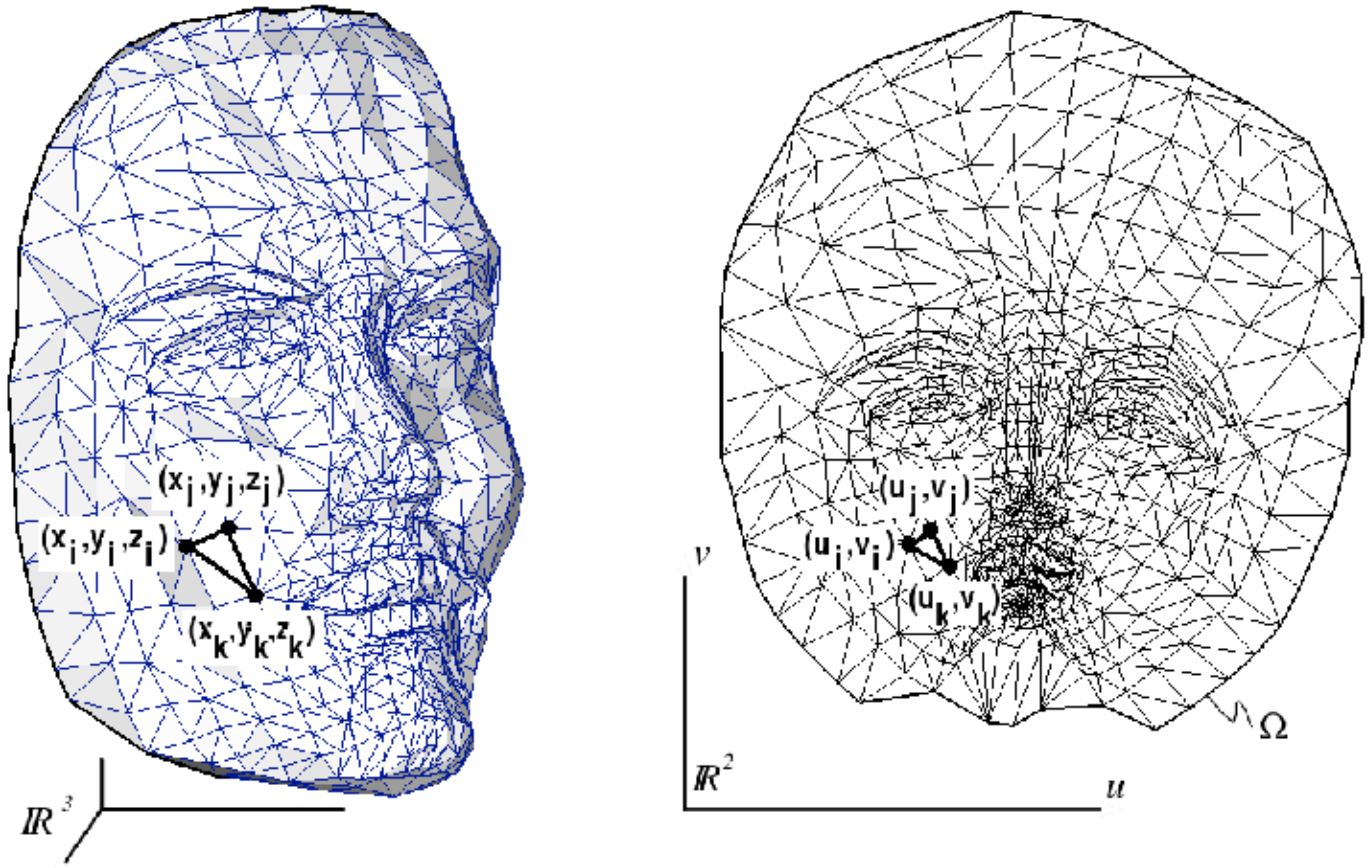
- Variational calculus then tells us that

$$\Delta \mathbf{x}(u,v) \;=\; 0$$

- If $\mathbf{x} : \Omega \rightarrow S$ is ***harmonic*** and maps the boundary $\partial\Omega$ of a ***convex*** region $\Omega \subset \mathbf{R}^2$ homeomorphically onto the boundary ∂S, then $\mathbf{x}$ is one-to-one (i.e., bijective).

Discrete Harmonic Maps

- Piecewise linear mapping of a discrete 3D triangle mesh onto a planar 2D polygon
- Slightly different situation: Given a 3D mesh, compute the inverse parameterization $\mathbf{u} : S \to \Omega$



$$\Delta \mathbf{x}(u, v) = 0$$

Discrete Harmonic Maps

1. Map the boundary ∂S homeomorphically to some **(convex) polygon** $\partial\Omega$ in the parameter **plane**
2. Minimize the Dirichlet energy of **u** by solving the corresponding Euler-Lagrange PDE

$$\Delta_{\mathcal{S}}\,\mathbf{u} \;=\; 0$$

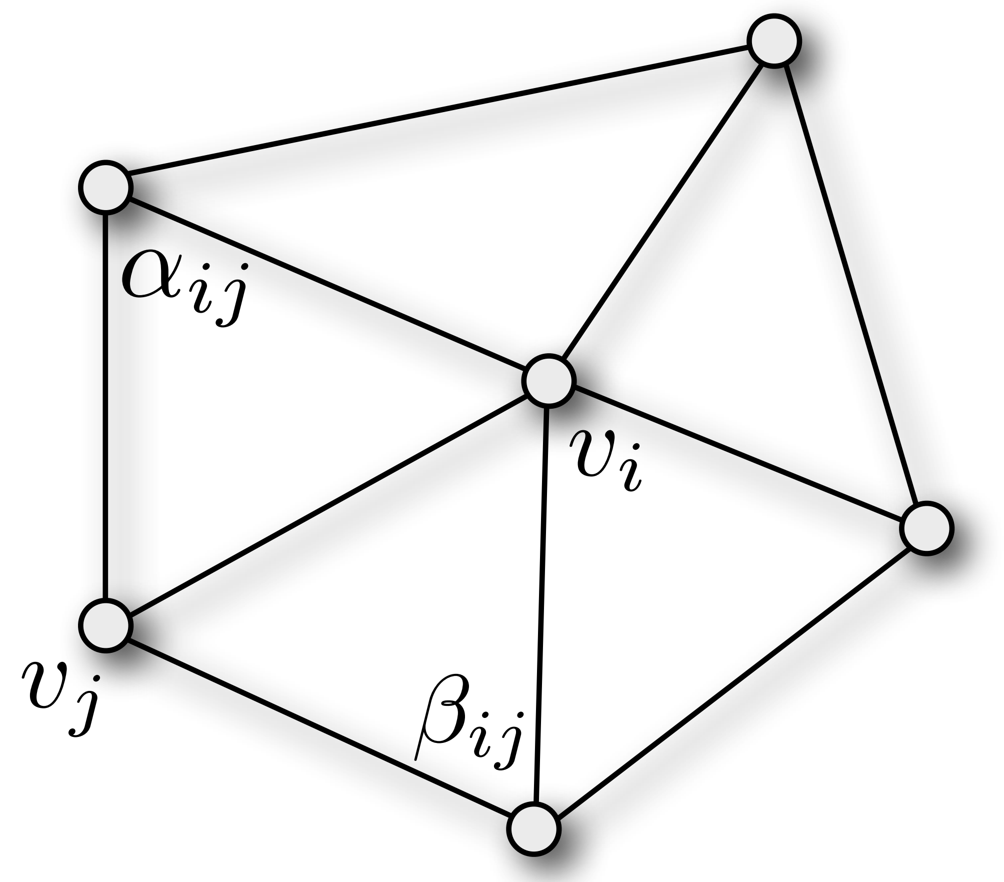
   - Requires discretization of Laplace-Beltrami
   - *Compare to surface smoothing*

Discrete Harmonic Maps

- System of linear equations

$$\forall v_i \in \mathcal{S} \ : \quad \sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} \left(\mathbf{u}(v_j) - \mathbf{u}(v_i) \right)$$

$$w_{ij} = \cot\alpha_{ij} + \cot\beta_{ij}$$



- Properties of system matrix:
  - Symmetric + positive definite → unique solution
  - Sparse → efficient solvers

Discrete Harmonic Maps

- But…
  - Does same theory hold for ***discrete*** harmonic maps as for harmonic maps?
  - In other words, is it possible for triangles to flip or become degenerate?

Convex Combination Maps

- If the linear equations are satisfied

$$\sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} \left(\mathbf{u}(v_j) - \mathbf{u}(v_i)\right)$$

and if the weights satisfy

$$w_{ij} > 0 \quad \wedge \quad \sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} = 1$$

then we get a ***convex combination*** *mapping*.

Convex Combination Maps

- Each $\mathbf{u}(v_i)$ is a convex combination of $\mathbf{u}(v_j)$

$$\mathbf{u}(v_i) = \sum_{v_j \in \mathcal{N}_1(v_i)} w_{ij} \mathbf{u}(v_j)$$

- If $\mathbf{u} : S \rightarrow \Omega$ is a convex combination map that maps the boundary ∂S homeomorphically to the boundary $\partial\Omega$ of a ***convex*** region $\Omega \subset \mathbf{R}^2$, then ***u*** *is one-to-one*.

Convex Combination Maps

1. Uniform barycentric weights

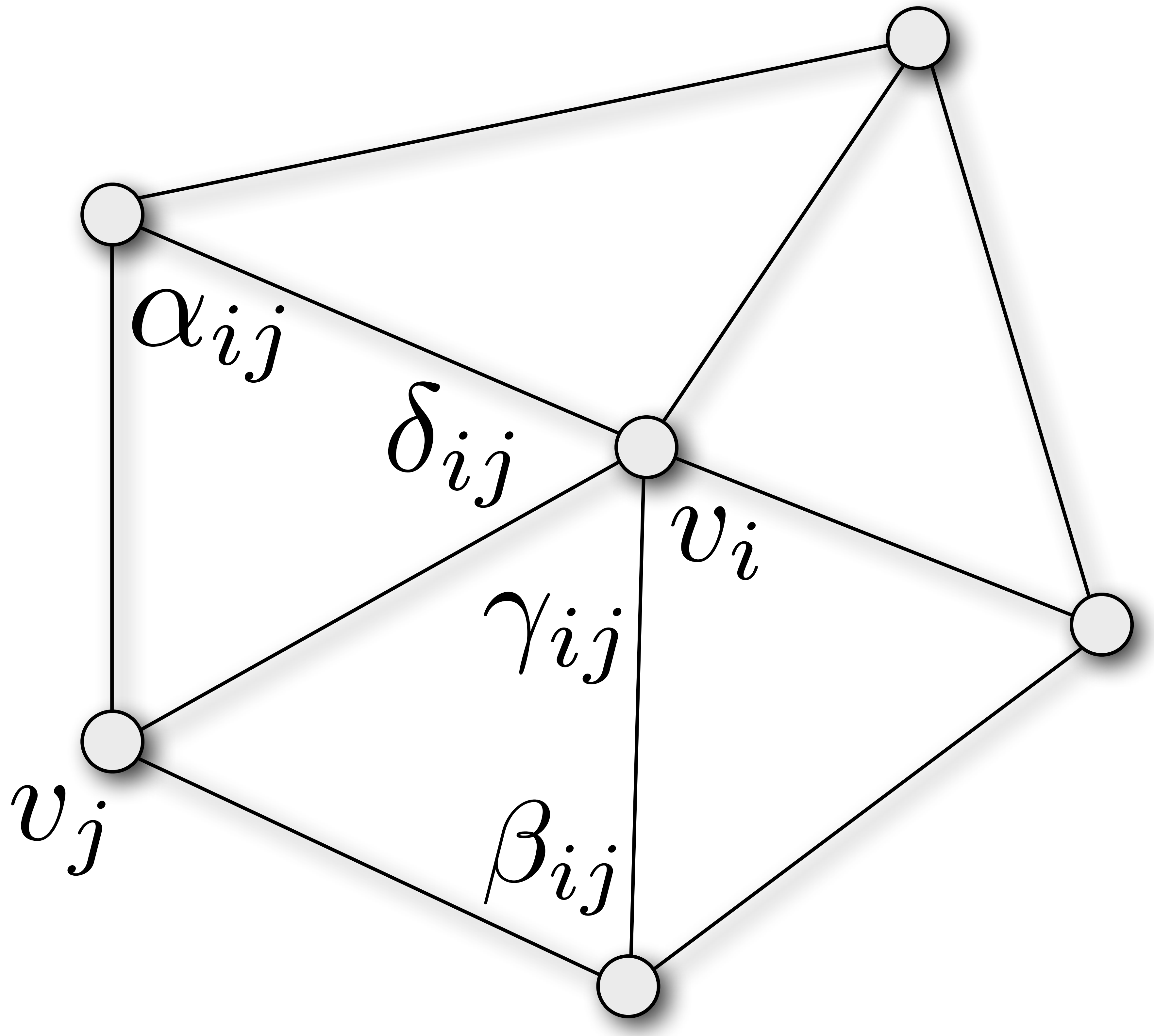
$$w_{ij} \; = \; 1/\text{valence}(v_i)$$

2. Cotangent weights (> 0 if $\alpha_{ij} + \beta_{ij} < \pi$)

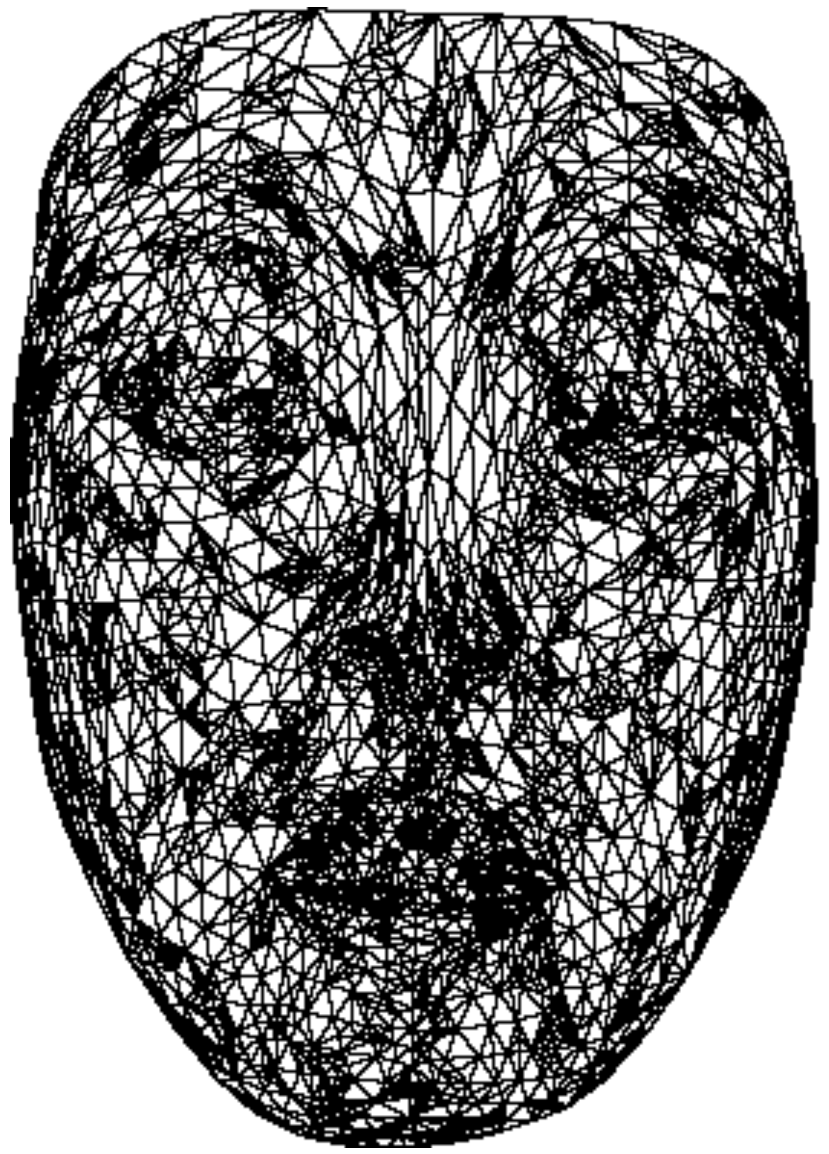
$$w_{ij} \; = \; \cot(\alpha_{ij}) + \cot(\beta_{ij})$$

3. Mean value weights

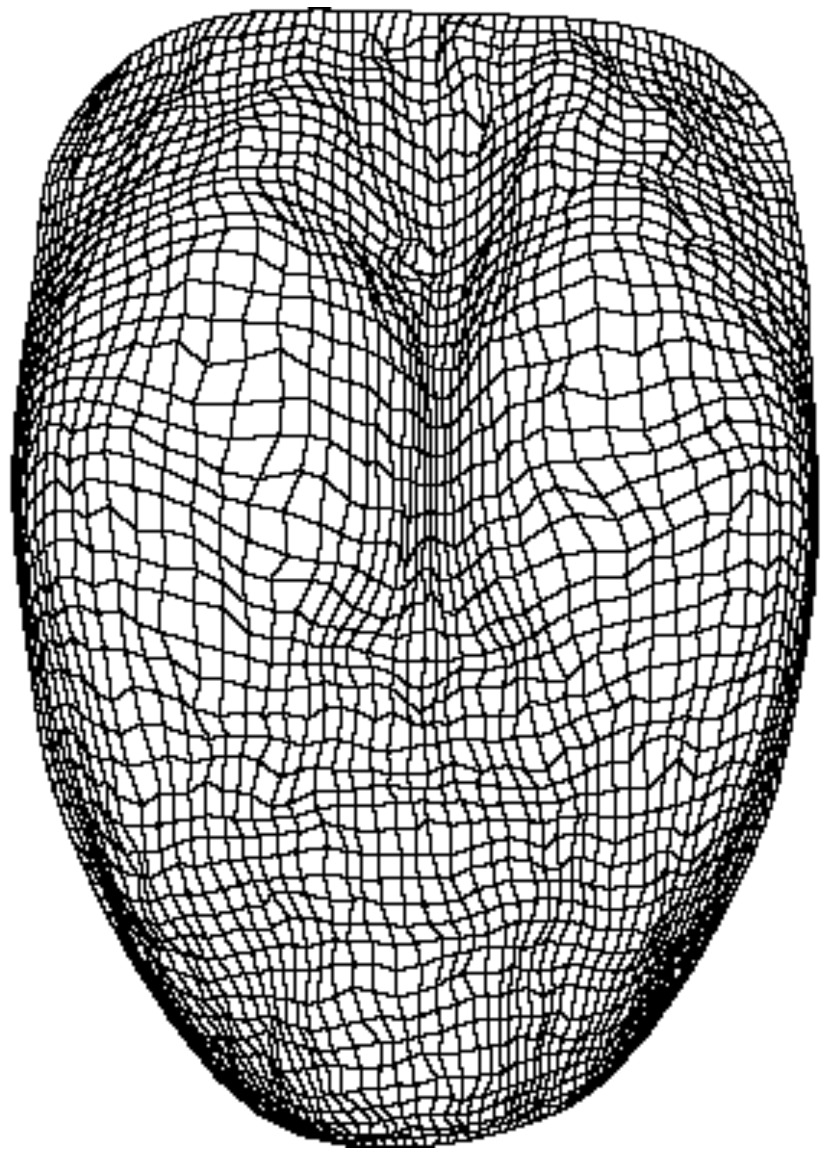
$$w_{ij} \; = \; \frac{\tan(\delta_{ij}/2) + \tan(\gamma_{ij}/2)}{\|\mathbf{p}_j - \mathbf{p}_i\|}$$



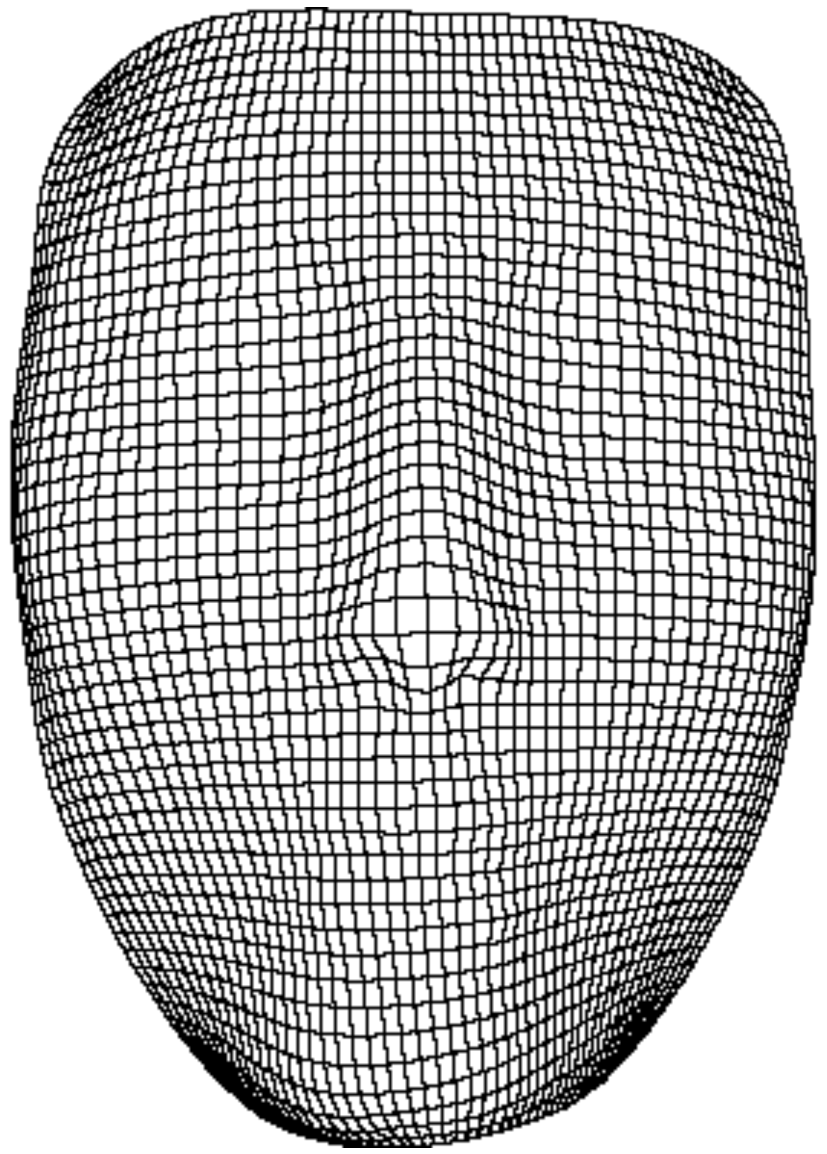
Convex Combination Maps



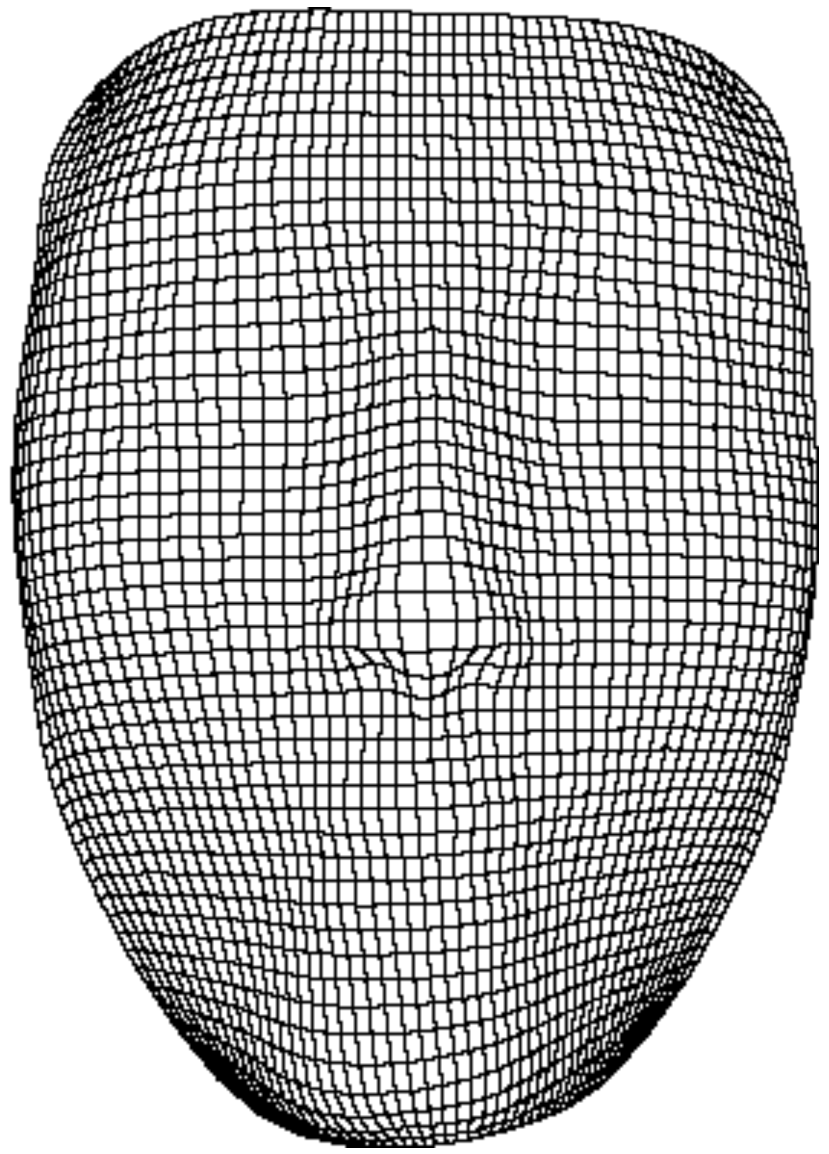
original mesh



uniform weights



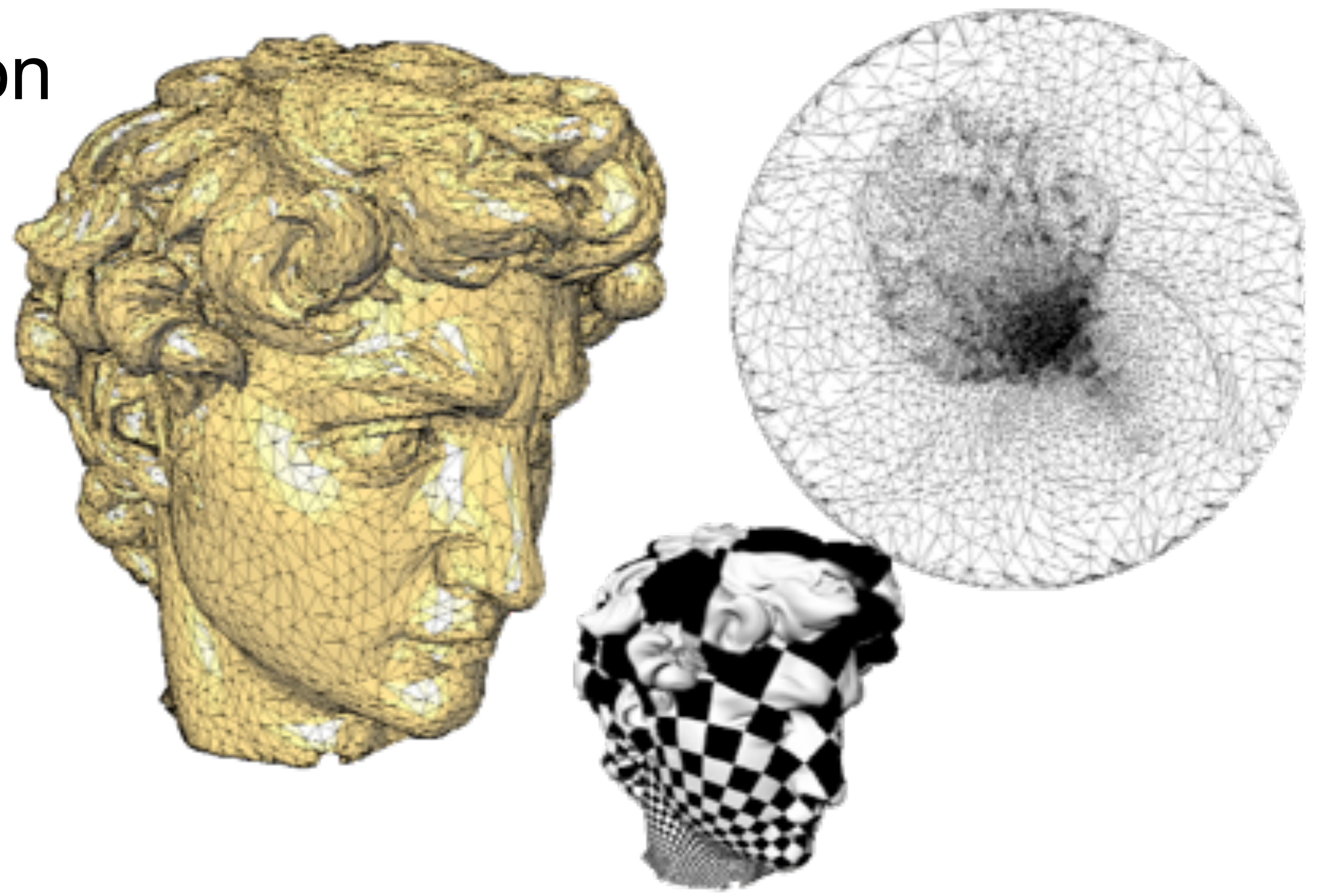
cotan weights



mean value

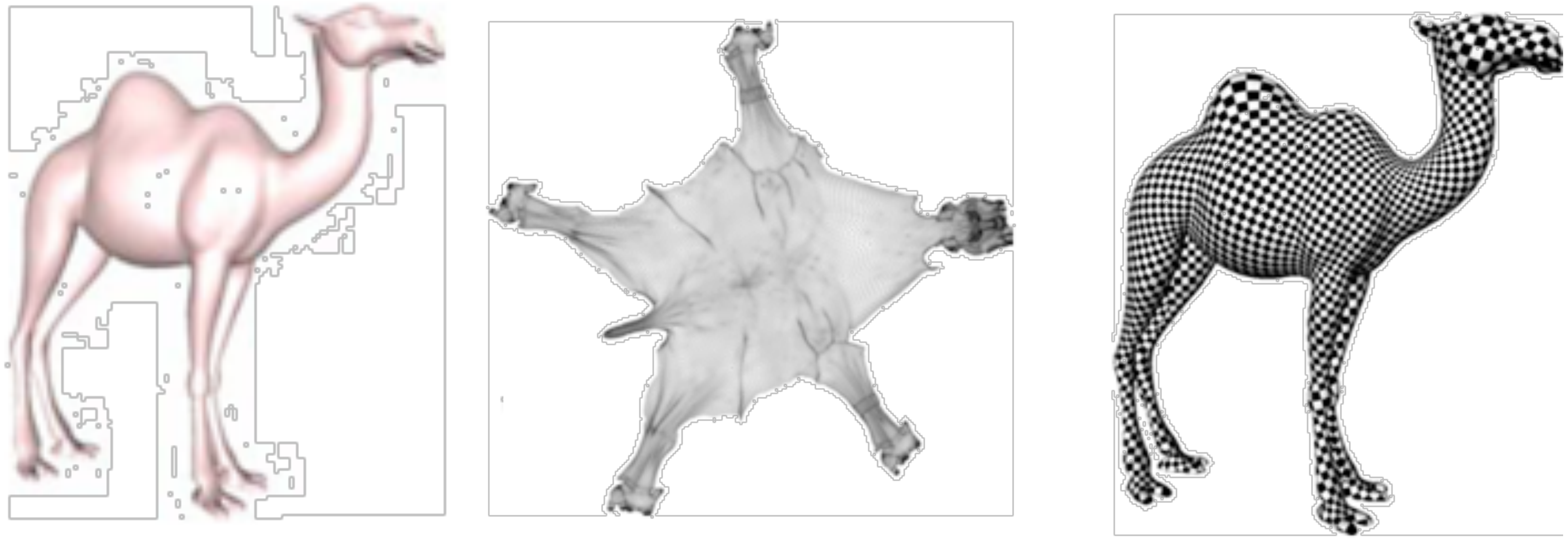
Fixing the Boundary

- Choose a simple **convex** shape
  - Triangle, square, circle
- Distribute points on boundary
  - Use **chord length (arc-length)** parameterization
- Fixed boundary can create high distortion

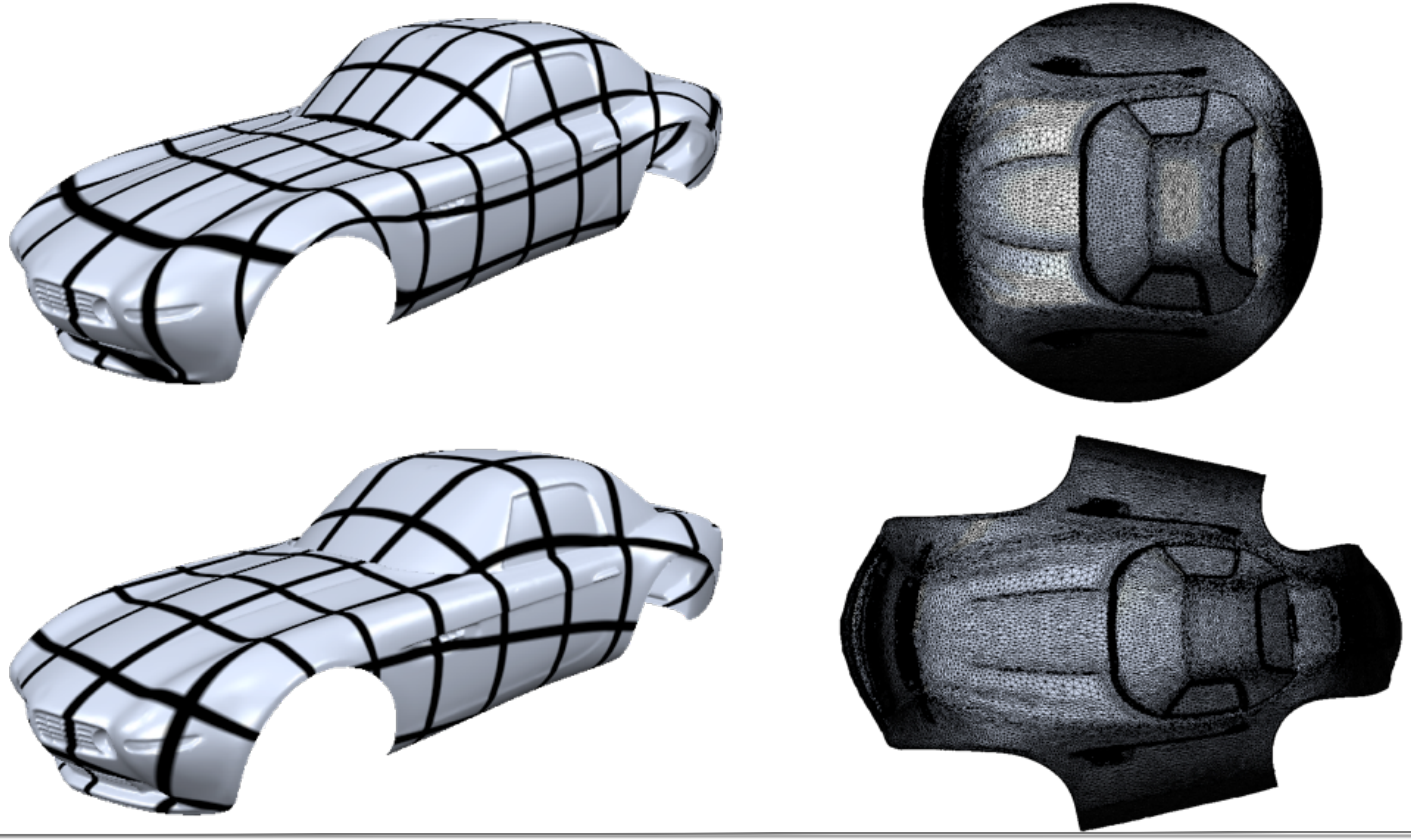


Open Boundary Mappings

- Include boundary vertices in the optimization
- Produces mappings with lower distortion

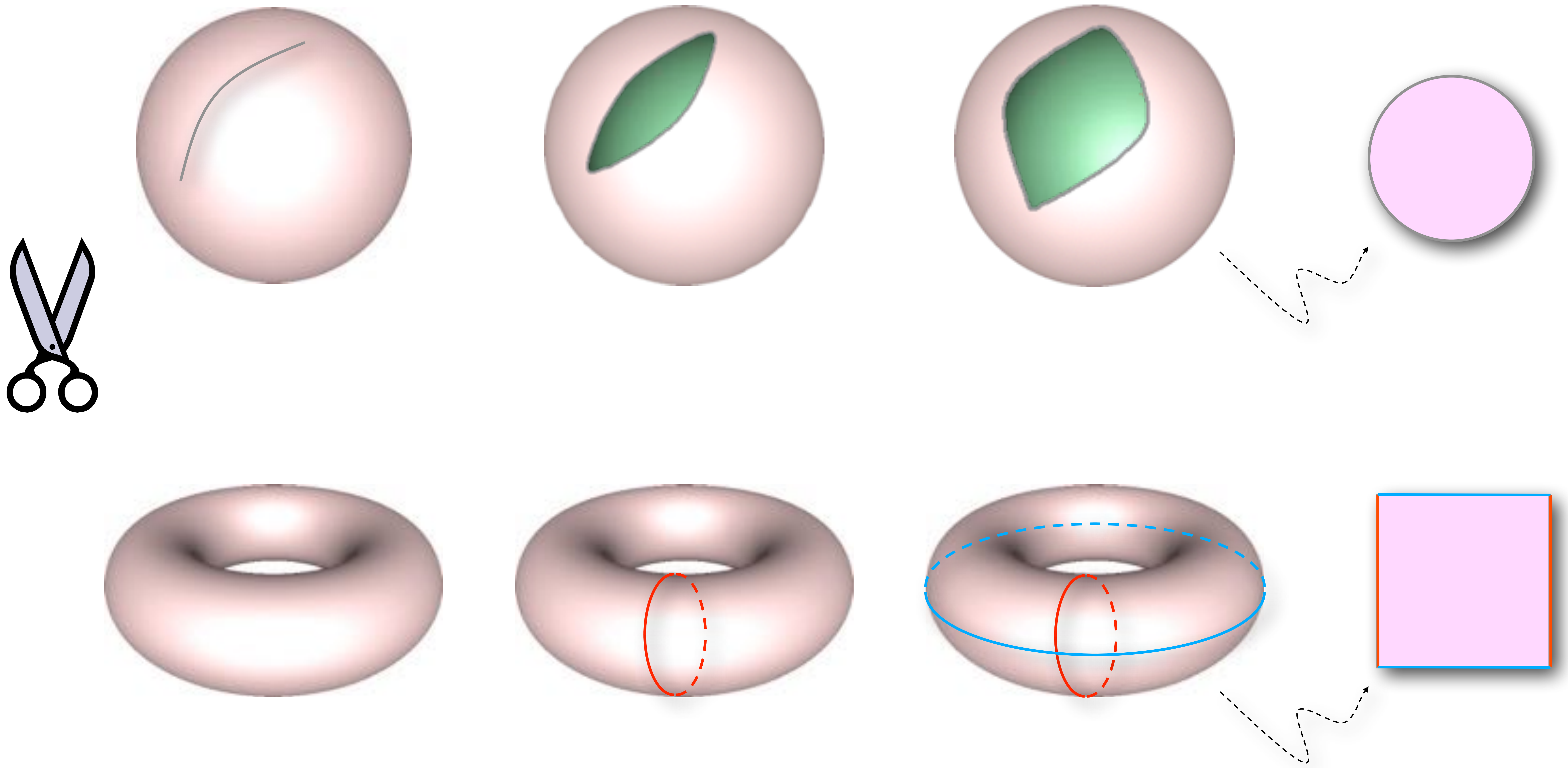


Open Boundary Mappings

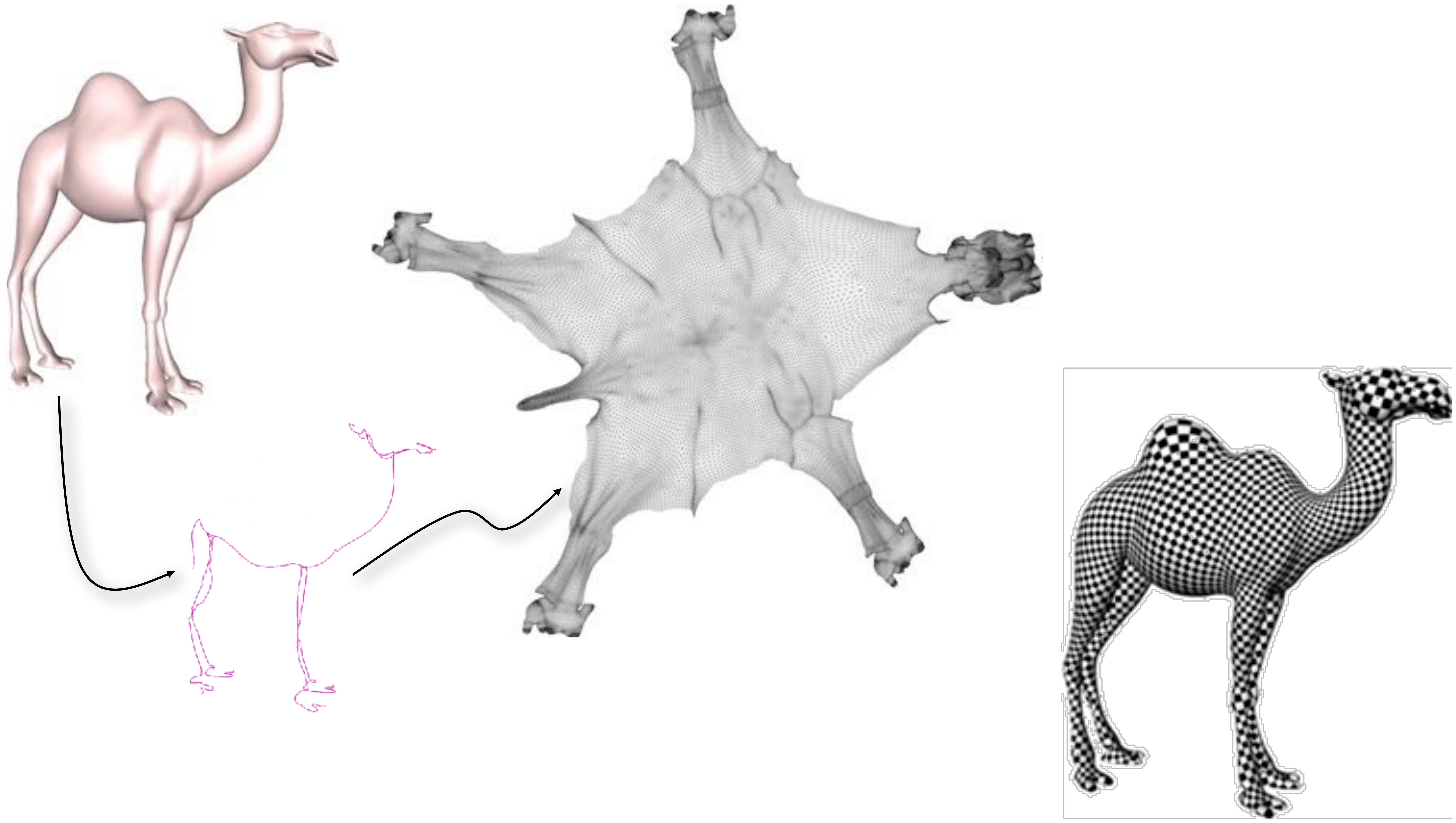


Need disk-like topology

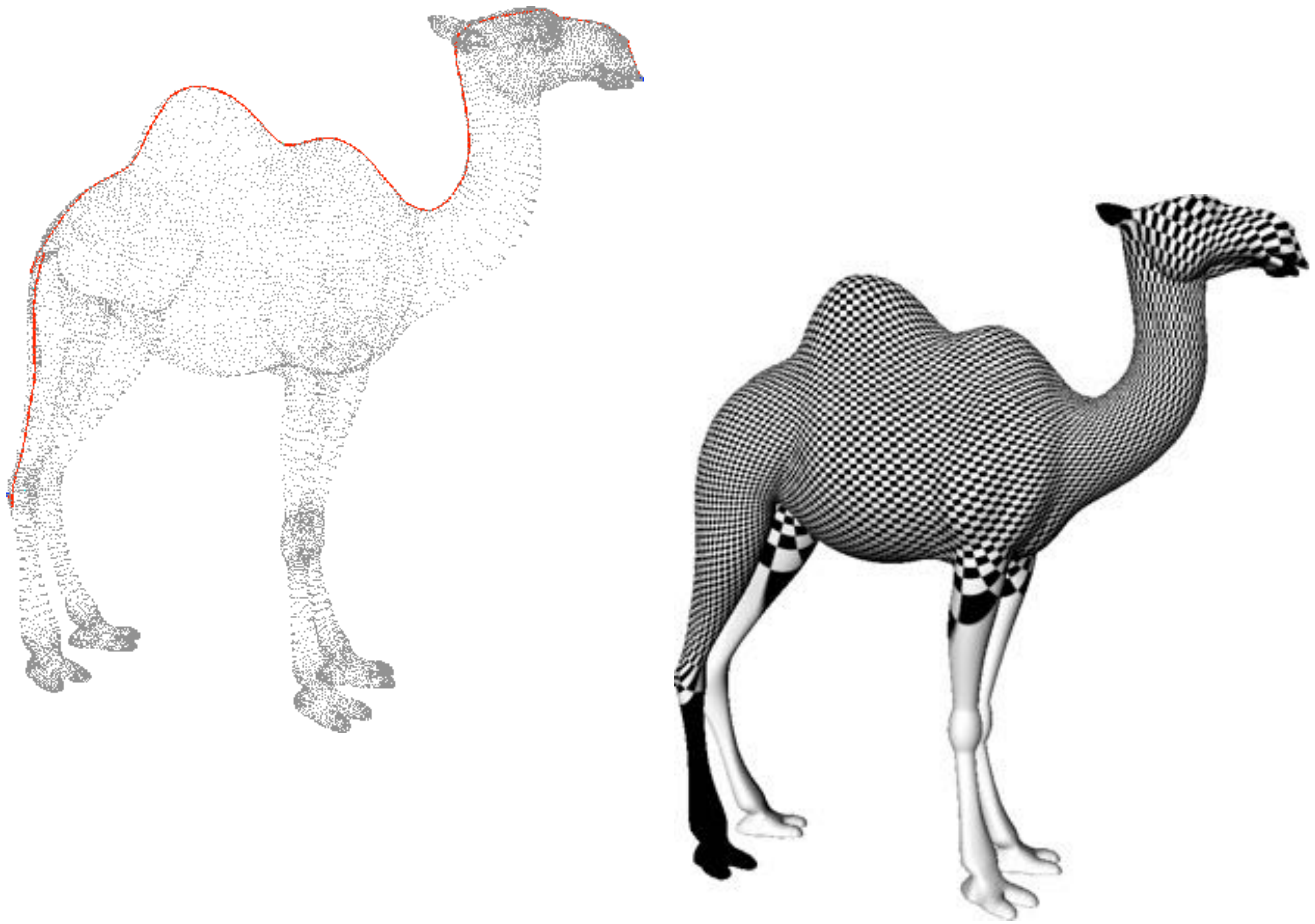
- Introduce cuts on the mesh



Smart Cut, Free Boundary

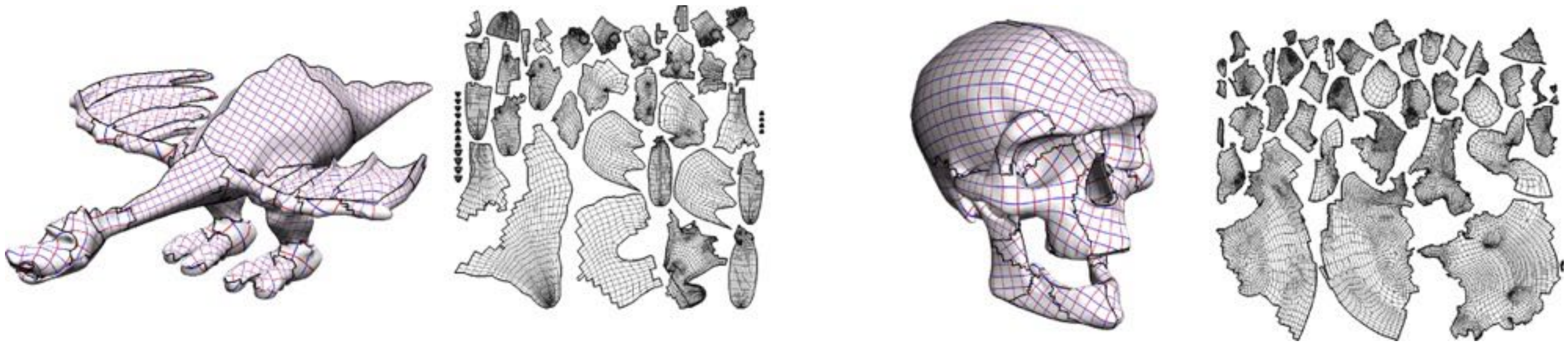


Naive Cut, Numerical Problems



Texture Atlas Generation

- Split model into number of patches (atlas)
  - because higher genus models cannot be mapped onto plane and/or
  - because distortion will be too high eventually



Levy, Petitjean, Ray, Maillot: *Least Squares Conformal Maps for Automatic Texture Atlas Generation*, SIGGRAPH, 2002

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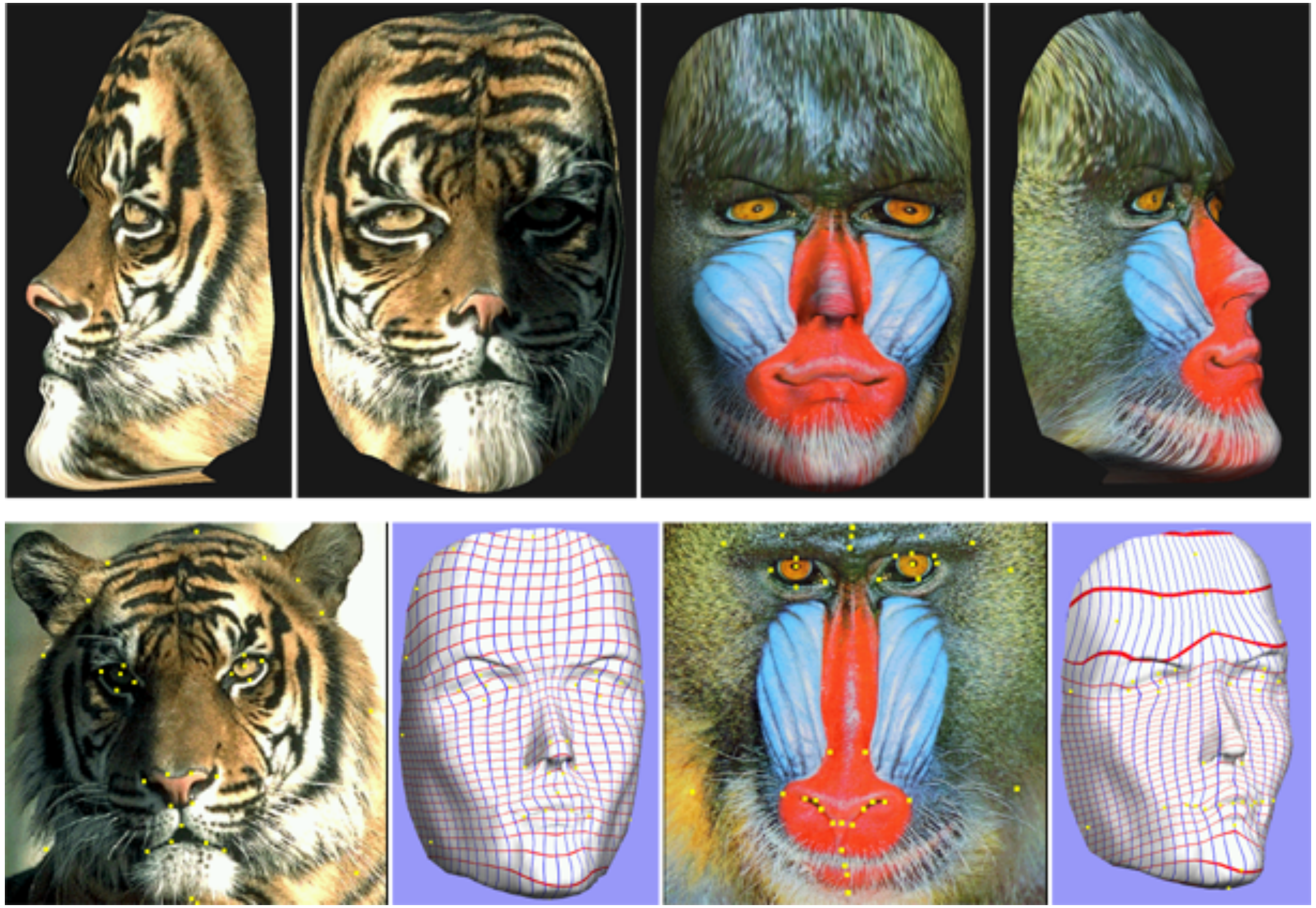
Constrained Parameterizations



Levy: *Constraint Texture Mapping*, SIGGRAPH 2001.

Literature

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- Hormann et al.: *Mesh Parameterization, Theory and Practice*, Siggraph 2007 Course Notes
  - http://www2.in.tu- clausthal.de/~hormann/parameterization/ index.html
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