DISCRETE FOURIER TRANSFORM in 1D

- the Fourier Transform maps a function in the time/space domain, to the temporal/spatial frequency domain.

$$\underset{\uparrow \text{ space}}{f(x)} \xrightarrow{\text{Fourier Transform}} \underset{\uparrow \text{ frequency}}{F(\nu)}$$

$$F(\nu) = \underbrace{\sum_{x=0}^{M-1} f(x)\cdot \overset{\text{Basis}}{\boxed{\cos\left(\frac{2\pi\nu}{M}x\right)}}}_{\text{Real}} - i \underbrace{\sum_{x=0}^{M-1} f(x)\cdot \overset{\text{Basis}}{\boxed{\sin\left(\frac{2\pi\nu}{M}x\right)}}}_{\text{Imaginary}}$$

where

$$\boxed{\begin{array}{l} x = 0, 1, 2, \dots, M-1 \\ \nu = 0, 1, 2, \dots, M-1 \end{array}}$$

⇒ $f(x)$ is a vector of length M.

⇒ The elements of $f(x)$ are real numbers.

⇒ $F(\nu)$ is a vector of length M.

⇒ The elements of $F(\nu)$ are COMPLEX numbers

$F(\nu) = \underset{\text{Real}}{▨} + i \underset{\text{Imaginary}}{▨}$

What do the components of the Fourier basis look like? $\nu = 0, 1, 2, 3, \dots M-1$

multiples of $\frac{2\pi}{M}$

harmonics

For instance, $\nu = 3$ means 3 cycles over the period M

COSINE BASIS

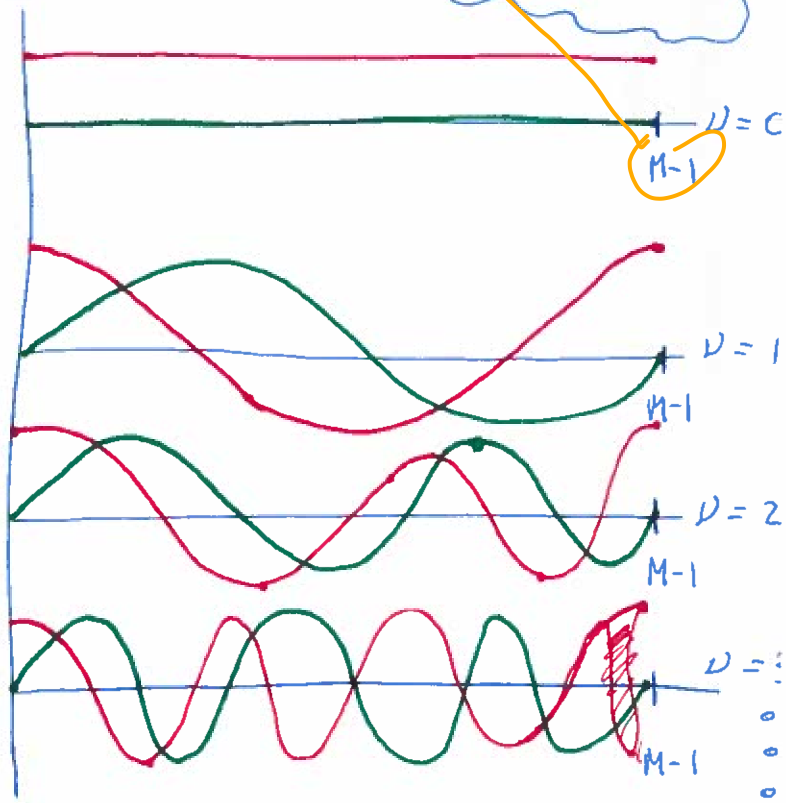
$$\begin{bmatrix} 1 \\ \cos \frac{2\pi}{M}x \\ \cos \frac{4\pi}{M}x \\ \cos \frac{6\pi}{M}x \\ \vdots \\ \cos \frac{(M-1)\pi}{M}x \end{bmatrix} \begin{array}{l} \nu = 0 \\ \nu = 1 \\ \nu = 2 \\ \nu = 3 \\ \vdots \\ \nu = M-1 \end{array}$$

SINE BASIS

$$\begin{bmatrix} 0 \\ \sin 2\pi/_M\, x \\ \sin 4\pi/_M\, x \\ \sin \frac{6\pi}{M} x \\ \vdots \\ \sin \frac{(M-1)\pi}{M} x \end{bmatrix} \begin{array}{l} \nu = 0 \\ \nu = 1 \\ \nu = 2 \\ \nu = 3 \\ \vdots \\ \nu = M-1 \end{array}$$

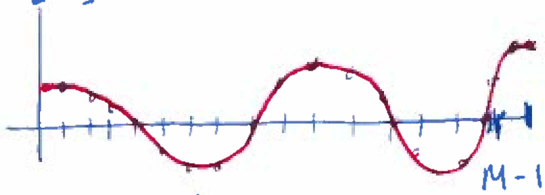
cos

sin



$\Longrightarrow$ Let us calculate $F[\nu]$ for $\nu = 2$ { F(2) is a COMPLEX number! }

• Real part of F[2]

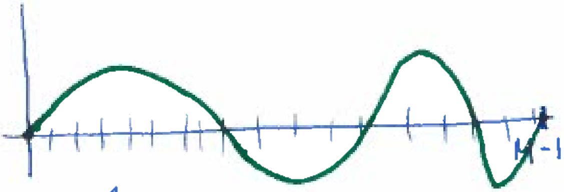
Real(F[2]) =  $\cdot \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(M-1) \end{bmatrix} = \underbrace{Re(F(2))}_{\text{Scalar}}$

M samples $\cos \frac{4\pi}{M} x$ $\boxed{\nu = 2}$ 2 cycles

M components

dot product

• Imaginary part of F[2]

Im(F[2])  $\cdot \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(M-1) \end{bmatrix} = \underbrace{Im(F(2))}_{\text{Scalar}}$

M samples $\sin \frac{4\pi}{T} x$ $\boxed{\nu = 2}$ 2cycles

M components

$\boxed{\text{Example}}$ Calculate one element of F(ν), ν = 2

$F[2] = Re(F[2]) - i\, Im(F[2])$

Let us take the second multiple $\boxed{\nu = 2}$

$F[2] = \underbrace{[\;\;]}_{\cos\frac{4\pi}{T}x \;\; 1\times M} \cdot \underbrace{[\;\;]}_{f(x)\;\; M\times 1} - i \underbrace{[\;\;]}_{\sin\frac{4\pi}{T}x \;\; 1\times M} \cdot \underbrace{[\;\;]}_{f(x)\;\; M\times 1} = \overbrace{\underbrace{\square + i \square}_{\text{complex number}}}^{F(2)}$

Re Im

$\frac{2\pi}{M}\nu \xrightarrow{\nu=2} \frac{4\pi}{M}$

dot product

如果这个frequency占比多 这样就大 (=how much frequency part in that signal)

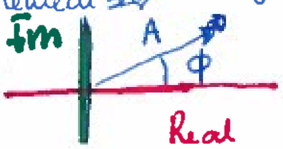
dot product

in this case, f(x) is a 1D image.

We do this for every frequency $\underbrace{\nu = 0, 1, 2, 3, 4, \ldots, M-1}_{M \text{ values}}$

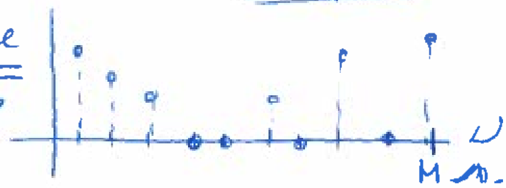
to build F(ν). F(ν) is a vector of length M where each component is a complex number

Remember



A magnitude 幅度

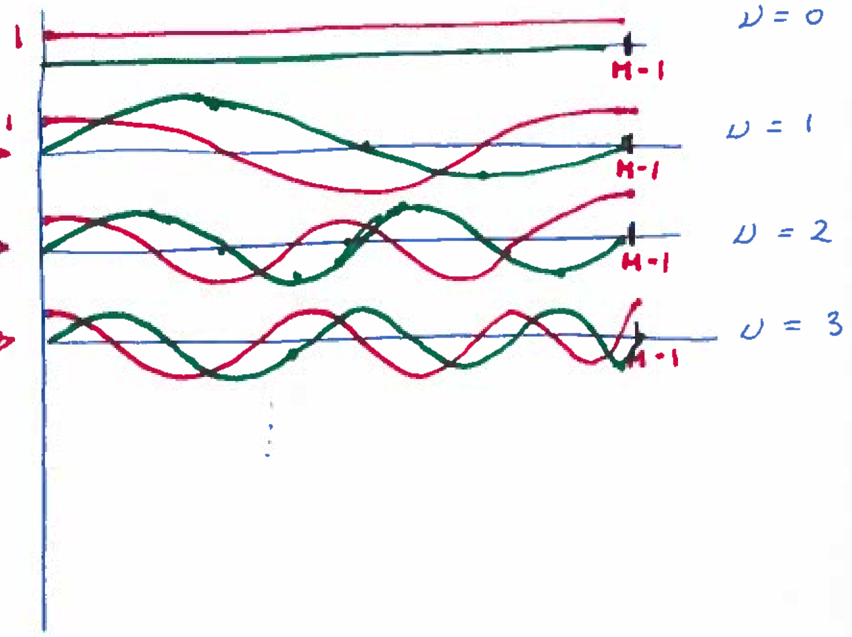
phase 相位 φ



$$F[\nu] = \underbrace{\sum_{x=0}^{M-1} f(x) \underbrace{\cos\left(\frac{2\pi\nu}{N}x\right)}_{\text{basis}}}_{\text{Real}} - i \underbrace{\sum_{x=0}^{M-1} f(x) \underbrace{\sin\left(\frac{2\pi\nu}{N}x\right)}_{\text{basis}}}_{\text{Imaginary}}$$

$\nu = 0, 1, 2, \ldots, M-1$ harmonics

$$\begin{bmatrix} 1 \\ \cos\frac{2\pi}{M}x \\ \cos\frac{4\pi}{M}x \\ \cos\frac{6\pi}{M}x \\ \vdots \\ \cos\frac{(M-1)\pi}{M}x \end{bmatrix}$$



Cosine Basis

$$\begin{bmatrix} 0 \\ \sin\frac{2\pi}{M}x \\ \sin\frac{4\pi}{M}x \\ \sin\frac{6\pi}{M}x \\ \vdots \\ \sin\frac{(M-1)\pi}{M}x \end{bmatrix}$$

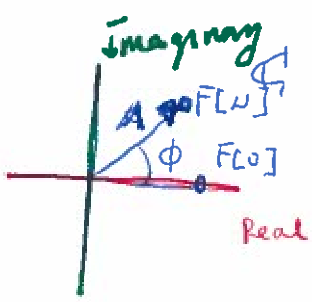
$\nu=0$, $\nu=1$, $\nu=2$, $\nu=3$

Sine Basis

$$F[\nu] = \begin{bmatrix} F[0] \\ F[1] \\ \vdots \\ F[M-1] \end{bmatrix} = \begin{bmatrix} \text{Real } F[0] \\ \text{Real } F[1] \\ \vdots \\ \text{Real } F[M-1] \end{bmatrix} + i \begin{bmatrix} \text{Im}(F[0]) \\ \text{Im } F[1] \\ \vdots \\ \text{Im } F[M-1] \end{bmatrix}$$

Real Imag.

$\in \mathbb{C}$

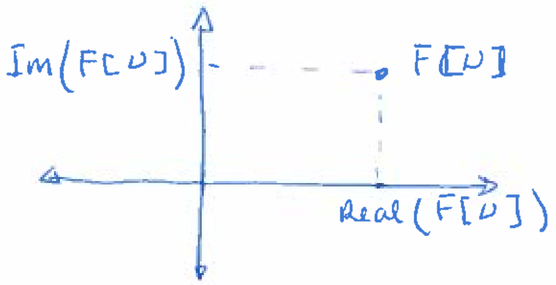


- The discrete Fourier Transform can be seen as a coordinate transformation in a finite-dimensional vector space.

- Each point in the Fourier domain contains two pieces of information:
  - the amplitude
  - the phase

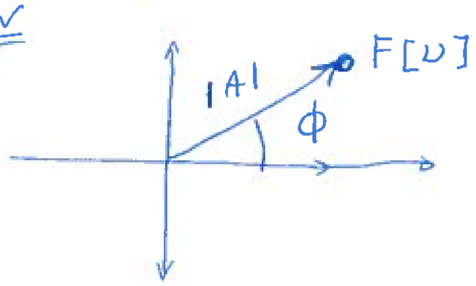
  We see the image from another point of view.

- Complex numbers can be represented in two ways:



$F[u] = Real + i\ Im$

or

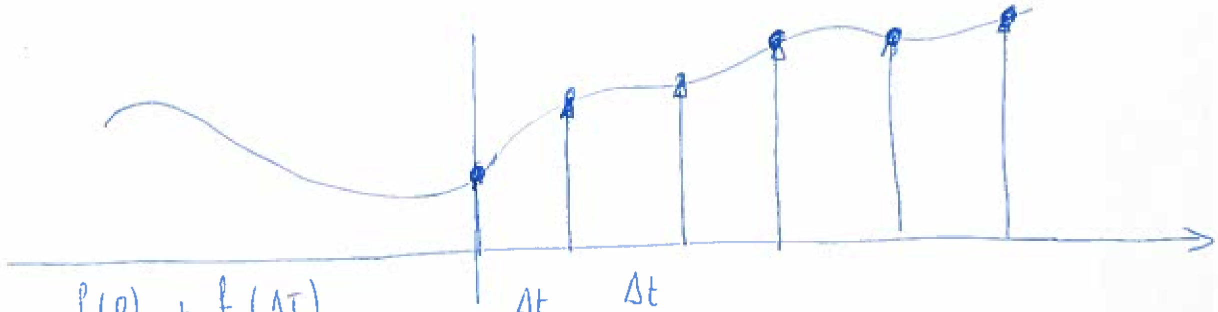


F[u] → Amplitude: |A|, Phase: ϕ

- The "power spectrum" is defined as the squared amplitude of the Fourier components.

- An important detail: The complex valued DFT of a real valued discrete signal is symmetric, i.e. it can be fully determined by the values in one half-space. The other half space is obtained by mirroring at the centre: $M/2$
  so, $F[u] = [F[0]\ F[1]\ F[2], \dots F[M/2], F[M/2 - 1], \dots F[2]\ F[1]]$

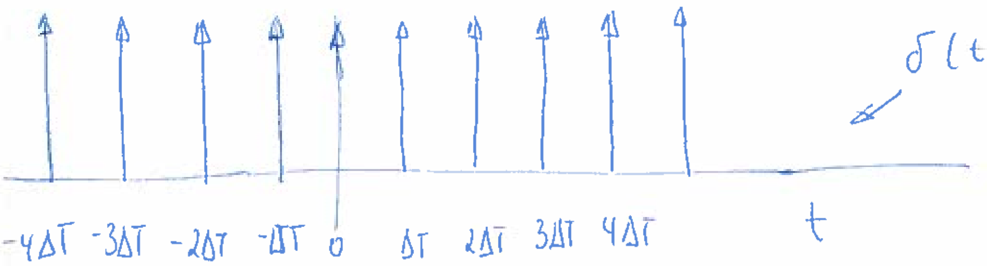
$$\tilde{f}(t) = \sum_{x=-\infty}^{\infty} f(t)\,\delta(t - x\Delta t)$$



$f(0) + f(\Delta T)$

$$\tilde{f}(t) = f(t)\,\delta(t) + f(\Delta t)\,\delta(t - \Delta t) + f(2\Delta t)\,\delta(t - 2\Delta t)\;.$$

$\delta(t - x\Delta t)$



- Impulse train
- Sampling function

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t)\, e^{-i2\pi\mu t}\, dt$$

$$= \int_{-\infty}^{\infty} \sum_{x=-\infty}^{\infty} f(t)\,\delta(t - x\Delta t)\, e^{-i2\pi\mu t}\, dt$$

$$= \sum_{x=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} f(t)\,\delta(t - x\Delta t)\, e^{-i2\pi\mu t}\, dt}_{f(}$$

only non-zero when $t = x\Delta t$

$$= \sum_{x=-\infty}^{\infty} f(x)\, e^{-i2\pi\mu x\Delta t}$$

M

$$\mu = \frac{u}{M\Delta t}$$