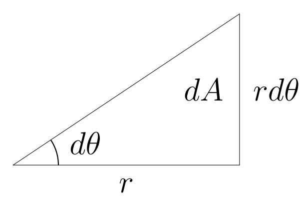
Kepler's Second Law Derivation

Definition

A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

Derivation

For this line to sweep out equal area during equal time, the rate at which the area changes, $\frac{dA}{dt}$ must stay constant. One "infinitely" small slice of the area of a planet's elliptical orbit may be represented as a triangle



with base r, height $rd\theta$, and area dA (recall that arc length is $rd\theta$). Recall the equation for the area of a triangle,

$$A = \frac{1}{2}bh.$$

Substituing the givens from our infinitely small slice above, we get

$$dA = \frac{1}{2}(r)(rd\theta).$$

Divide both sides by dt (keep in mind that r is constant)

$$\frac{dA}{dt} = \frac{1}{2}(r)(r\frac{d\theta}{dt}),$$

which simplies to

$$\frac{dA}{dt} = \frac{1}{2}rv_t.$$

Recall that angular momentum is

$$L = mrv_t$$

and is a conserved quantity. Substiting L into our area gets

$$\frac{dA}{dt} = \frac{L}{2m}.$$

We can then rearrange and solve the differential equation

$$\int dA = \int \frac{L}{2m}\, dt$$

which shows us that

$$A = \frac{L}{2m}T.$$

This shows us a directly proportional relationship between the area A and time T.