1 Kepler's Second Law Derivation

1.1 Review of Calculus

1.1.1 Limits and Derivatives

Limits describe the behavior of a function near a point. For example, $\lim_{x \to 2} x + 4 = 6$ since $2 + 4 = 6$. Special limits like

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

represent the derivative, or instantaneous slope, of a function at a point. Similarly, you may take for granted that the following equation is true:

$$\lim_{t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}.$$

1.1.2 Integration

The power rule of integration for any function x is as follows:

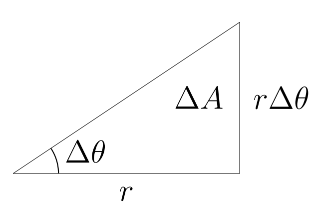
$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$

1.2 Definition

A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

1.3 Derivation

For this line to sweep out equal area during equal time, the rate at which the area changes, $\frac{dA}{dt}$ must stay constant. One small slice of the area of a planet's elliptical orbit may be represented as a triangle



with base r, height $r\Delta\theta$, and area ΔA (recall that arc length is $r\Delta\theta$). Recall the equation for the area of a triangle,

$$A = \frac{1}{2}bh.$$

Substituing the givens from our slice above, we get

$$\Delta A = \frac{1}{2}(r)(r\Delta\theta).$$

Divide both sides by Δt

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}(r)(r\frac{\Delta\theta}{\Delta t}).$$

Take the limit of boths sides as $\Delta t \to 0$ to find an "infinitely" small change of area over an "infinitely" small change of time (keep in mind that r is constant for our slice and that area A is a function of time).

$$\lim_{\Delta t\to 0}(\frac{\Delta A}{\Delta t}) = \lim_{\Delta t\to 0}(\frac{1}{2}(r)(r\frac{\Delta\theta}{\Delta t})),$$

which simplifies to

$$\frac{dA}{dt} = \frac{1}{2}(r)(r\frac{d\theta}{dt})$$

and further simplies to

$$\frac{dA}{dt} = \frac{1}{2}rv_t.$$

Recall that angular momentum is

$$L = mrv_t$$

and is a conserved quantity. Substiting L into our area gets

$$\frac{dA}{dt} = \frac{L}{2m}.$$

We can then rearrange and solve the differential equation

$$\int dA = \int \frac{L}{2m}\, dt,$$

which shows us that

$$A = \frac{L}{2m} T,$$

or in Lagrange's notation,

$$A(t) = \frac{L}{2m} t.$$

This shows us a directly proportional relationship between the area A and time T. Note that $\frac{L}{2m}$ is constant.