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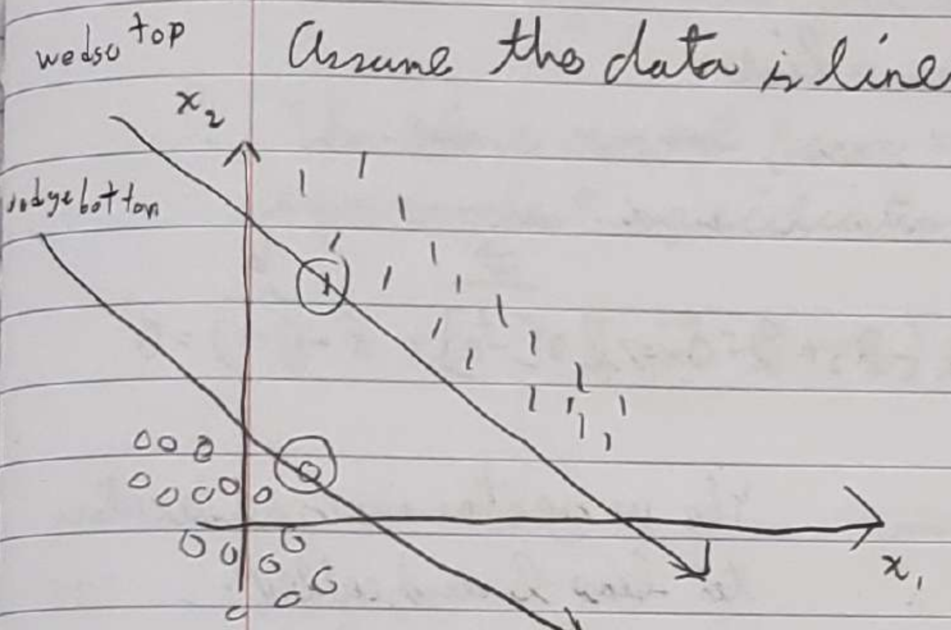
$\mathcal{Y} = \{0, 1\}$, $p+1=3$, $\mathcal{H} = \{ \mathbb{1}_{\vec{w} \cdot \vec{x} \geq 0} : \vec{w} \in \mathbb{R}^3 \}$

Assume the data is linearly separable so it looks like:



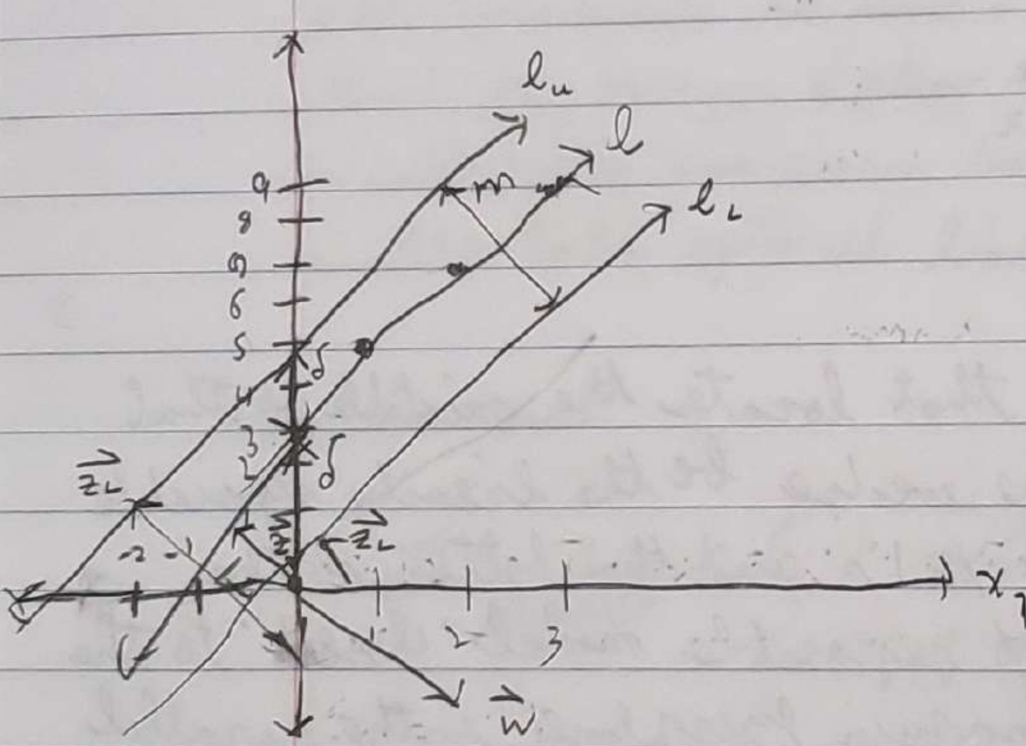
We need an algorithm that locates the middle of that wedge. Let the top of the wedge be the linearly separable model "closest" to the $y=1$'s and the bottom of the wedge be the linearly separable model "closest" to the $y=0$'s. The "max margin hyperplane" is the parallel line in the center of the top and bottom.

Note: there are two critical observations (the circled points). Since observations are x-vectors, these critical observations are called "support vectors" and hence the final model is called a "support vector machine" (SVM). "Machine" is a fancy word meaning "complex model"; so "machine learning" just means "learning complex models". To find the SVM...

First rewrite $\mathcal{H} = \{ \mathbb{I}\ \vec{w}\cdot\vec{x} - b \geq 0 : \vec{w} \in \mathbb{R}^p, b \in \mathbb{R} \}$

Note $\underbrace{\vec{w}\cdot\vec{x} - b = 0}$ defines a line
Hesse normal
form.

$$l : x_2 = 2x_1 + 3 \Rightarrow l : 2x_1 - x_2 + 3 = 0 \Rightarrow l : \overbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}^{\vec{w}} \cdot \vec{x} - \overbrace{(-3)}^{b} = 0$$



The w vector is perpendicular to line l and called the 'normal vector'

Let $\vec{w}_0 := \dfrac{\vec{w}}{\|\vec{w}\|}$

The direction of the w vector with unit length

$\vec{z} = \alpha \vec{w}_0$, $\vec{z} \in l$

$$\vec{w}\cdot\vec{z} - b = 0$$

$$\Downarrow$$

$$\vec{w}\cdot(\alpha\,\vec{w}_0) - b = 0$$

$$\Rightarrow \frac{\alpha}{\|\vec{w}\|}\|\vec{w}\|^2 - b = 0$$

$$\Rightarrow \alpha = \frac{b}{\|\vec{w}\|} \Rightarrow \vec{z} = \frac{b}{\|\vec{w}\|}\vec{w}_0$$

Let $m > 0$ be the perpendicular distance between l_u and l_L and let $\delta > 0$ be the distance between l_u and l (and l_L and l) on the x_2 axis

$l_u : \vec{w}\cdot\vec{x} - (b+\delta) = 0$, $\vec{z}_u = \dfrac{b+\delta}{\|\vec{w}\|}\vec{w}_0$

$l_L : \vec{w}\cdot\vec{x} - (b-\delta) = 0$, $\vec{z}_L = \dfrac{b-\delta}{\|\vec{w}\|}\vec{w}_0$

$$\Downarrow$$

$$m = \|\vec{z}_u - \vec{z}_L\| = \left\| \frac{b+\delta}{\|\vec{w}\|}\vec{w}_0 - \frac{b-\delta}{\|\vec{w}\|}\vec{w}_0 \right\|$$

$$\downarrow$$

$$= \frac{1}{\|\vec{w}\|} 2\delta \|\vec{w}_0\| = \frac{2\delta}{\|\vec{w}\|}$$

Goal is to make m as large as possible (maximum margin)
<=> making the w vector as small as possible.

The Hesse normal form is not unique. There are infinite equivalent specification of a line:

$$\forall c \neq 0 \quad c(\vec{w} \cdot \vec{x} - b) = 0 \quad \text{Let } c = \frac{1}{\delta}$$

$$\Downarrow$$

$$m = \frac{2}{\|\vec{w}\|}$$

Now we need two conditions

(I) All $y = 1$'s are above or equal to l_u:

$$\forall i \text{ st } y_i = 1 \quad \vec{w} \cdot \vec{x}_i - (b+1) \geq 0 \Rightarrow \vec{w} \cdot \vec{x}_i - b \geq 1 \Rightarrow \tfrac{1}{2}(\vec{w} \cdot \vec{x}_i - b) \geq \tfrac{1}{2}$$

$$\Downarrow$$

$$(y_i - \tfrac{1}{2})(\vec{w} \cdot \vec{x}_i - b) \geq \tfrac{1}{2}$$

(II) All $y = 0$'s are below or equal to l_l:

$$\forall i \text{ st } y_i = 0 \quad \vec{w} \cdot \vec{x}_i - (b-1) \leq 0 \Rightarrow \vec{w} \cdot \vec{x}_i - b \leq -1 \Rightarrow \tfrac{1}{2}(\vec{w} \cdot \vec{x}_i - b) \leq -\tfrac{1}{2}$$

$$\Rightarrow -\tfrac{1}{2}(\vec{w} \cdot \vec{x}_i - b) \geq \tfrac{1}{2}$$

$$\Downarrow$$

$$(y_i - \tfrac{1}{2})(\vec{w} \cdot \vec{x}_i - b) \geq \tfrac{1}{2}$$

note how both inequalities are the same for both I and II. Thus this inequality satisfies both constraints. So all observations will be in their right places

$$\forall i \; (y_i - \tfrac{1}{2})(\vec{w} \cdot \vec{x}_i - b) \geq \tfrac{1}{2} \Rightarrow \text{line is linearly separable.}$$

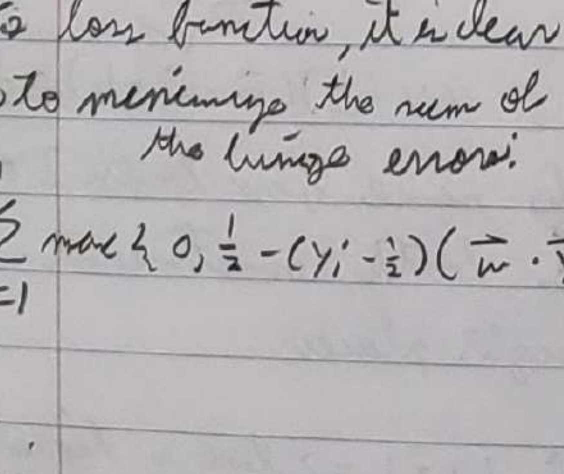
you compute the SVM by optimizing the following problem:

$$\min \|\vec{w}\| \text{ st } \forall i \; (y_i - \tfrac{1}{2})(\vec{w} \cdot \vec{x}_i - b) \geq \tfrac{1}{2} \text{ is true}$$

and return the resulting w vector and b. There is no analytical solution, you need optimization algorithm. It can be solved with quadratic programming and other procedures as well.

note: everything we did above generalizes to $p > 2$. note: most textbooks have 1's in the place of our $\frac{1}{2}$'s that's because they assumed $y = \{-1, 1\}$ but we assumed binary.

What if the data is not linearly separable? You can never satisfy that constraint ... So this whole thing doesn't work. We will use a new objective function / loss function / error-tallying function called "hinge loss":



$$H_i := \max\left\{0, \tfrac{1}{2} - \overbrace{(y_i - \tfrac{1}{2})(\vec{w} \cdot \vec{x}_i - b)}^{\text{should be} \geq \frac{1}{2}}\right\}$$

Let's say a point is d away from where it should be.

$$(y_i - \tfrac{1}{2})(\vec{w} \cdot \vec{x}_i - b) = \tfrac{1}{2} - d$$

$$H_i = \max\left\{0, \tfrac{1}{2} - (\tfrac{1}{2} - d)\right\} = \max\{0, d\} = d$$

With this loss function, it is clear we wish to minimize the sum of the hinge errors:

$$SHE := \sum_{i=1}^{n} \max\left\{0, \tfrac{1}{2} - (y_i - \tfrac{1}{2})(\vec{w} \cdot \vec{x}_i - b)\right\}$$

But we also want to maximize the margin. So we combine both consideration together into the objective function of Vapnik (1963):

$$\underset{\vec{w}, b}{\text{argmin}} \left\{ \underbrace{\frac{1}{n} SHE}_{\text{minimizing distance errors}} + \lambda \underbrace{\|\vec{w}\|^2}_{\text{maximizing the width of the wedge.}} \right\}$$

once λ is set, the computer can do the optimization to find the resulting SVM even using out of the box R packages.

What is λ? It is a "hyperparameter", "tuning parameter". It is set by you! It controls the tradeoff between these two considerations.

$$g = \mathcal{A}(\mathbb{D}, \mathcal{H}, \lambda)$$

What if you have the modeling setting where $y = \{1, 2, \ldots L\}$, a nominal categorical response with $L > 2$ levels. The model will still be a "classification model" but not a "binary classification model" and its sometimes called a "multinomial classification model". but not a "binary classification model" and its sometimes called a "multinomial classification model". What is the null model g_0?
Again, g_0 = Sample Mode $[Y]$.

Consider a model that predict on a new x_* by looking through the training data and finding the "closest" x_i vector and returning its y_i as the predicted response value. This is called a "nearest neighbor" model. Further, you may also want to find the K closest observation and return the mode of those K observations as the predicted response value (randomize ties). That's called "K nearest neighbor" (KNN) model where K is a natural number hyperparameter. There is another hyperparameter that must be specified the "distance function" $d: \mathcal{X}^2 \to \mathbb{R}_{\geq 0}$ The typical distance function is Euclidean distance squared:

$$d(\vec{x}_*, \vec{x}_i) := \sum_{j=1}^{p} (x_{i,j} - x_{*,j})^2$$

What is $\mathcal{H}$? $\mathcal{A}$?