Lee Introduction To Manifolds Exercises

Robert

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1 Exercise 2.4

Definition 1. Let (M, d) be a metric space

Then, $B_r^{(d)}(x) = \{ y \in M : d(y, x) < r \}$ is the **open ball** of radius r around x

Definition 2. Let (M, d) be a metric space.

Then, $B_r^{(d)}[x] = \{ y \in M : d(y, x) \leq r \}$ is the **closed ball** of radius r around x

Definition 3. Let (M, d) be a metric space. Then, we call $\mathcal{T}_d$ the metric topology generated by d (the set of all possible unions of open sets in M) [a]

[a] It is important to note that in metric spaces, all open sets are unions of open balls

1.1 Exercise 2.4 (a)

Theorem 1.1.1. Let M be a metric space with metrics d and d'. Then, the metric topologies generated by d, denoted as $\mathcal{T}_d$ and d', denoted as $\mathcal{T}_{d'}$ are equivalent, $\mathcal{T}_d = \mathcal{T}_{d'}$ iff for every $x \in M$ and every $r > 0$ there exists $r_1, r_2 > 0$ such that $B_{r_1}^{(d')}(x) \subseteq B_r^{(d)}(x)$ and $B_{r_2}^{(d)}(x) \subseteq B_r^{(d')}(x)$

Proof. ($\Rightarrow$) Suppose $\mathcal{T}_d = \mathcal{T}_{d'}$. Let $x \in M$ and $r > 0$ be arbitrary. Since $\mathcal{T}_d = \mathcal{T}_{d'}$, we have that $B_r^{(d)}(x) \in \mathcal{T}_{d'}$. So, $B_r^{(d)}(x)$ is the union of open balls in $\mathcal{T}_{d'}$. But this means we must have some $r_1 > 0$ such that $B_{r_1}^{(d')}(x) \subseteq B_r^{(d)}(x)$. It's trivial to see that $B_r^{(d')}(x) \in \mathcal{T}_d$ which again means it is the union of open balls in $\mathcal{T}_d$, thus implying we must have some $r_2 > 0$ such that $B_{r_2}^{(d)}(x) \subseteq B_r^{(d')}(x)$. This completes the proof of the forward direction.

($\Leftarrow$) Suppose that for every $x \in M$ and every $r > 0$ there exists r_1, r_2 such that $B_{r_1}^{(d')}(x) \subseteq B_r^{(d)}(x)$ and $B_{r_2}^{(d)}(x) \subseteq B_r^{(d')}(x)$. Suppose for contradiction that

$\mathcal{T}_d \neq \mathcal{T}_{d'}$. So, either we have an $O^{(d)}$ such that $O^{(d)} \notin \mathcal{T}_{d'}$ or we have $O^{(d')}$ such that $O^{(d')} \notin \mathcal{T}_d$. Let's do case 1 first, that $O^{(d)} \notin \mathcal{T}_{d'}$. Since $O^{(d)}$ is not open in $\mathcal{T}_{d'}$, there is an $x_0 \in O^{(d)}$ such that

$$\forall j > 0, \quad B_j^{(d')}(x_0) \nsubseteq O^{(d)} \tag{1}$$

Since $O^{(d)}$ is open in $\mathcal{T}_d$, for this particular x_0, we have $r_0 > 0$ such that $B_{r_0}^{(d)}(x_0) \subseteq O^{(d)}$. By hypothesis, we now have an $r_1 > 0$ such that $B_{r_1}^{(d')}(x_0) \subseteq B_{r_0}^{(d)}(x_0)$. But now, we have that $B_{r_1}^{(d')}(x_0) \subseteq B_{r_0}^{(d)}(x_0) \subseteq O^{(d)}$, but this contradicts Equation (1). The argument for the other case is similar and left out because it is tedious. $\square$

1.2 Exercise 2.4 (b)

Claim 1.2.1. Let (M, d) be a metric space, let $c > 0$ and define $d'(x, y) = c \cdot d(x, y)$. Then, $\mathcal{T}_d = \mathcal{T}_d'$

Proof. Let $O \in \mathcal{T}_d$. Let $\Lambda \subseteq \mathbb{R} \times M$ be the radius-point pairs such that $O = \bigcup_{(r,x)\in\Lambda} B_r^{(d)}(x)$. Let $(r, x) \in \Lambda$ be arbitrary. Expanding the definition of $B_r^{(d)}(x)$, we get that

$$\begin{aligned} B_r^{(d)}(x) &= \{\, y \in M : d(y, x) < r \,\} \\ &= \{\, y \in M : c \cdot d(y, x) < cr \,\} \\ &= \{\, y \in M : d'(y, x) < cr \,\} \\ &= B_{cr}^{(d')}(x) \end{aligned}$$

So, $O = \bigcup_{(r,x)\in\Lambda} B_{cr}^{(d')}(x) \in \mathcal{T}_{d'}$. This shows that $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$. The argument $\mathcal{T}_d \supseteq \mathcal{T}_{d'}$ is similar and tedious and so skipped. $\square$

Alternative proof by using Theorem 1.1.1

Proof. Since $d'(x, y) = c \cdot d(x, y)$, we have $\frac{d'(x,y)}{c} = d(x, y)$ Pick $r_1 = \frac{r}{c}$. Then $B_{r_1}^{(d')}(x) \subseteq B_r^{(d)}(x)$ Now, for $r_2 = cr$, it is immediately obvious that $B_{r_2}^{(d)}(x) \subseteq B_r^{(d')}(x)$ [1] $\square$

1.3 Exercise 2.4 (c)

This is exercise B.1 but I included it here because it's useful.

Lemma 1.3.1. For $x \in \mathbb{R}^n$ such that $x = (x_1, x_2, \ldots, x_n)$, we have

$$\max\{\, |x_1|, |x_2|, \ldots, |x_n| \,\} \leq |x| \leq \sqrt{n} \max\{\, |x_1|, |x_2|, \ldots, |x_n| \,\}$$

[1]the balls are actually equal but whatever

Proof. Suppose $|x_i| = \max\{\,|x_1|, |x_2|, \ldots, |x_n|\,\}$ for $1 \leq i \leq n$. Then, $|x| = \sqrt{{x_1}^2 + {x_2}^2 + \cdots + {x_i}^2 + \cdots + {x_n}^2} \geq \sqrt{{x_i}^2} = |x_i|$. For the next bit, we have that $|x| = \sqrt{{x_1}^2 + {x_2}^2 + \cdots + {x_i}^2 + \cdots + {x_n}^2} \leq \sqrt{n \cdot {x_i}^2} = \sqrt{n} \cdot |x_i|$ $\square$

Claim 1.3.2. Let $(\mathbb{R}^n, d)$ be Euclidean n-space with the Euclidean metric $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$

Let $d'(x, y) = \max\{\,|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\,\}$ be defined for $\mathbb{R}^n$. Then, $\mathcal{T}_d = \mathcal{T}_{d'}$

Note: This proof is incomplete since I am confused by my justification

Proof. We will make use of Theorem 1.1.1. Let $x \in \mathbb{R}^n$ and $r > 0$. Then, choose $r_1 = r$. Now, let $y \in B^{(d')}_{r_1}(x)$. So, from Lemma 1.3.1 we have $d'(y, x) < r_1 = r \leq d(y, x)$. So, $y \in B^{(d)}_r(x)$ and thus $B^{(d')}_{r_1}(x) \subseteq B^{(d)}_r(x)$. Choose $r_2 = r\sqrt{n}$. Let $y \in B^{(d)}_{r_2}(x)$. Now

$$\begin{aligned}
&d'(y, x) < r \\
\Longrightarrow\ &\sqrt{n} \cdot d'(y, x) < r\sqrt{n} \\
\Longrightarrow\ &d(x, y) \leq \sqrt{n} \cdot d'(y, x) < r\sqrt{n} \\
\Longrightarrow\ &\frac{d(x, y)}{\sqrt{n}} \leq d'(y, x) < r
\end{aligned}$$

Again, by Lemma 1.3.1, $d(y, x) < r_2 \leq d'(y, x) < r$ Thus $B^{(d)}_{r_2}(x) \subseteq B^{(d')}_r(x)$ $\square$

2 Exercise 2.5

Claim 2.0.1. Let X be a topological space and Y be an open subset of X. Then, the collection of all open subsets of X in Y is a topology on Y.

Proof. Let $(X, \mathcal{T})$ denote the topological space of X. Let $\mathcal{O}_Y$ denote the collection of all open subsets of X in Y. To do so, we just need to check the properties of a topology on Y, namely, that $\varnothing, Y \in \mathcal{O}_Y$, if $O_1, O_2 \subseteq Y$ are open, then $O_1 \cap O_2$ is open in Y (aka, $O_1 \cap O_2 \in \mathcal{O}_Y$), and the arbitrary union $\bigcup_{\lambda \in \Lambda} O_\lambda$ is also open in Y.

Firstly, since $\varnothing$ and Y are obviously open in Y by definition, $\varnothing, Y \in \mathcal{O}_Y$. Now, let $O_1, O_2 \subseteq Y$ be arbitrary open subsets of Y. Since Y is open in X, any subset of Y must also be open in X, so this means that $O_1, O_2 \in \mathcal{T}$. And since $\mathcal{T}$ is a topology, $O_1 \cap O_2 \in \mathcal{T}$. Of course, $O_1 \cap O_2 \subseteq Y$, so $O_1 \cap O_2 \in \mathcal{O}_Y$. (The inductive case is trivial)

Now, let Λ be some indexing set such that for each $\lambda \in \Lambda$, O_λ is open in Y. Of course, this means that $\bigcup_{\lambda \in \Lambda} O_\lambda \subseteq Y$ and so the arbitrary union is open in Y. And since $\bigcup_{\lambda \in \Lambda} O_\lambda \in \mathcal{T}$, $\bigcup_{\lambda \in \Lambda} O_\lambda \in \mathcal{O}_Y$.

Since $\mathcal{O}_Y$ is demonstrated to have the properties of a topology on Y, the proof is complete. $\square$

3 Exercise 2.6

Claim 3.0.1. Let X be a set, and let $\{\mathcal{T}_\alpha\}_{\alpha\in A}$ be a collection of topologies on X. Then, $\mathcal{T} = \bigcap_{\alpha\in A}\mathcal{T}_\alpha$ is a topology on X.

Proof. Obviously, $\varnothing, X \in \mathcal{T}$.

Let $O_1, O_2 \in \mathcal{T}$. So, $\forall \alpha \in A\, O_1, O_2 \in \mathcal{T}_\alpha$. But since each $\mathcal{T}_\alpha$ is a topology, we know that $O_1 \cap O_2 \in \mathcal{T}_\alpha$. So obviously, $O_1 \cap O_2 \in \mathcal{T}$.

Let Λ be an indexing set such that $O_\lambda \in \mathcal{T}$ for $\lambda \in \Lambda$. Now, this means each $O_\lambda \in \mathcal{T}_\alpha$. But of course, since each $\mathcal{T}_\alpha$ is a topology, $\bigcup_{\lambda\in\Lambda} O_\lambda \in \mathcal{T}_\alpha$. And since $\mathcal{T}_\alpha$ was arbitrary, $\bigcup_{\lambda\in\Lambda} O_\lambda \in \mathcal{T}$. $\square$

4 Exercise 2.9

Prove proposition 2.8

> **Definition 4.** Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$. Then $V(x)$ is a neighborhood of x meaning that $V(x) \in \mathcal{T}$ (or that it is an open subset of X)

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Let $A^c = X \setminus A$

Proposition 4.0.1. Textbook Proposition 2.8a $x \in \operatorname{Int} A$ iff there exists $V(x) \subseteq A$

Proof. ($\Rightarrow$) Let $x \in \operatorname{Int} A$. Then, by definition of $\operatorname{Int} A$, we have some $C \subseteq X$ such that $C \subseteq A$ and $C \in \mathcal{T}$ (C is open). Then C will serve as $V(x)$.

($\Leftarrow$) trivial $\square$

Proposition 4.0.2. Textbook Proposition 2.8b $x \in \operatorname{Ext} A$ iff there exists $V(x) \subseteq A^c$

Proof. $(\Rightarrow)$

$$\begin{aligned}
\overline{A}^c &= \left(\bigcap_{\lambda\in\Lambda} B_\lambda\right)^c && \text{Each } B_\lambda \text{ is closed}\\
&= \bigcup_{\lambda\in\Lambda} (B_\lambda)^c && \text{And now each } (B_\lambda)^c \text{ is open}
\end{aligned}$$

The rest is trivial

$(\Leftarrow)$ trivial

□

Proposition 4.0.3. Textbook Proposition 2.8c $x \in \partial A$ iff for every $V(x)$, some $y_1 \in A \cap V(x)$ and some $y_2 \in A^c \cap V(x)$

Proof. $(\Rightarrow)$. Let $x \in \partial A$. Let $V(x) \in \mathcal{T}$ be arbitrary.

Here, note that C_λ are open in X and are subsets of A. And that B_γ are closed in X and contain A

$$\begin{aligned}
\partial A &= (\operatorname{Int} A \cup \operatorname{Ext} A)^c\\
&= \left(\bigcup_{\lambda\in\Lambda} C_\lambda \cup \overline{A}^c\right)^c\\
&= \left(\bigcup_{\lambda\in\Lambda} C_\lambda\right)^c \cap \overline{A}\\
&= \bigcap_{\lambda\in\Lambda} (C_\lambda)^c \cap \overline{A}\\
&= \bigcap_{\lambda\in\Lambda} (C_\lambda)^c \cap \bigcap_{\gamma\in\Gamma} B_\gamma
\end{aligned}$$

□

Proposition 4.0.4. Textbook Proposition 2.8d $x \in \overline{A}$ iff every $V(x)$ has $y \in A \cap V(x)$

Proof. $(\Rightarrow)$ Let $x \in \overline{A}$. Let $\mathcal{F} = \{ B_\lambda : \lambda \in \Lambda \}$ be a collection of every closed set in X that contains A, so $\overline{A} = \bigcap \mathcal{F}$. Suppose for contradiction that $V(x) \in \mathcal{T}$ is a neighborhood that contains no points of A, or in other words, $V(x) \cap A = \varnothing$. Since $V(x)$ is open and disjoint from A, $A \subseteq X \setminus V(x)$. So, $X \setminus V(x)$ is a closed set that contains A. Thus, $X \setminus V(x) \in \mathcal{F}$. But since $x \in \overline{A} = \bigcap \mathcal{F}$, $x \in X \setminus V(x)$. And by hypothesis, $x \in V(x)$. This is obviously absurd.

$(\Leftarrow)$ Suppose x is such that every $V(x)$ has a $y \in A \cap V(x)$. Let $\mathcal{F} = \{ B_\lambda : \lambda \in \Lambda \}$ be a collection of every closed set in X that contains A, so $\overline{A} = \bigcap \mathcal{F}$. □

Proposition 4.0.5. Textbook Proposition 2.8e $\overline{A} = A \cup \partial A = \operatorname{Int} A \cup \partial A$

Proof.

$$\partial A = X \setminus \operatorname{Int} A \cap$$

□

Proposition 4.0.6. Textbook Proposition 2.8f $\operatorname{Int} A, \operatorname{Ext} A \in \mathcal{T}$ and $\overline{A}^c, (\partial A)^c \in \mathcal{T}$. In other words, the interior and exterior are open sets, while the closure and the boundary and closed sets.

Proof. $\operatorname{Int} A$ is the union of open sets which is an open set.

$\overline{A}$ is the intersection of closed sets, which is a closed set.

$\operatorname{Ext} A = X \setminus \overline{A}$ is an open set because $\overline{A}$ is closed.

$\partial A = X \setminus (\operatorname{Int} A \cup \operatorname{Ext} A)$ is closed because $\operatorname{Int} A \cup \operatorname{Ext} A$ is open. □

Proposition 4.0.7. Textbook Proposition 2.8g The following are equivalent:

- A is open in X
- $A = \operatorname{Int} A$
- A contains none of its boundary points (aka, $\partial A \cap A = \varnothing$)
- For all $x \in A$, there exists $V(x) \subseteq A$

Proof. Suppose A is open in X. Then $A \in \mathcal{T}$. Obviously, $\operatorname{Int} A \subseteq A$. $A \subseteq \operatorname{Int} A$ follows from the definition of $\operatorname{Int} A$, namely, since, $A \subseteq A$ and A is open in X.

Suppose $A = \operatorname{Int} A$. Then by definition of $\partial A = X \setminus (\operatorname{Int} A \cup \operatorname{Ext} A)$, $\partial A \cap A = \varnothing$ immediately follows.

Suppose that $\partial A \cap A = \varnothing$. Let $x \in A$ be arbitrary. This means that $x \notin \partial A$, so $x \in \operatorname{Int} A \cup \operatorname{Ext} A$. Of course, $x \notin \operatorname{Ext} A$. So, $x \in \operatorname{Int} A$. So, $\operatorname{Int} A$ can be a neighborhood of x, and obviously since $\operatorname{Int} A \subseteq A$ we are done.

Suppose that every point of A has a neighborhood contained in A. Then, we know $\bigcup_{x \in A} V(x) \subseteq A$. Now, let $x \in A$. Then we have some $V(x)$, so $V(x) \subseteq \bigcup_{y \in A} V(y)$. And so $\bigcup_{y \in A} V(y) = A$. Since the union of open sets is open, A is open. □

Proposition 4.0.8. Textbook Proposition 2.8h The following are equivalent:

- A is closed in X
- $A = \overline{A}$
- A contains all of its boundary points (aka, $\partial A \subseteq A$)

- For all $x \in A^c$, there exists $V(x) \subseteq A^c$

Proof. Observe that A is closed iff A^c is open, and apply Proposition 4.0.7 $\square$

5 Exercise 2.9

Proposition 5.0.1. Book Exercise 2.10 Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then, A is closed iff it contains all of its limit points

Proof. $(\Rightarrow)$ Suppose A is closed. Let p be a limit point of A. Suppose for contradiction that $p \notin A$, so $p \in A^c$. Since A^c is open, p must have some neighborhood in A^c, call it V. Since p is a limit point, we must have some point $a_0 \in A$ such that $a_0 \in V$ and $a_0 \neq p$. But then, $V \subseteq A^c$, so $a_0 \in A$ and $a_0 \in A^c$, which is absurd.

Suppose A contains all of its limit points. We shall use Proposition 4.0.4 and Proposition 4.0.8 to show that $A = \overline{A}$ and thus A is closed.

$\square$