Lee Introduction To Manifolds Exercises

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1 Exercise 2.4

Definition 1.1. Let (X, d) be a metric space.Md

Then, $B_r\,(r)\,dx = \{\, y \in M : d(y, x) < r \,\}$ is the **open ball** of radius r around x

Definition 1.2. Let (M, d) be a metric space.

Then, $B_r^{(d)}\,[x] = \{\, y \in M : d(y, x) \leq r \,\}$ is the **closed ball** of radius r around x

Definition 1.3. Let (M, d) be a metric space. Then, we call $\mathcal{T}_d$ the metric topology generated by d (the set of all possible unions of open sets in M) [a]

[a] It is important to note that in metric spaces, all open sets are unions of open balls

1.1 Exercise 2.4 (a)

Theorem 1.1. Let M be a metric space with metrics d and d'. Then, the metric topologies generated by d, denoted as $\mathcal{T}_d$ and d', denoted as $\mathcal{T}_{d'}$ are equivalent, $\mathcal{T}_d = \mathcal{T}_{d'}$ iff for every $x \in M$ and every $r > 0$ there exists $r_1, r_2 > 0$ such that $B_r\,(r_1)\,d'x \subseteq B_r\,(r)\,dx$ and $B_r\,(r_2)\,dx \subseteq B_r\,(r)\,d'x$

Proof. $(\Rightarrow)$ Suppose $\mathcal{T}_d = \mathcal{T}_{d'}$. Let $x \in M$ and $r > 0$ be arbitrary. Since $\mathcal{T}_d = \mathcal{T}_{d'}$, we have that $B_r\,(r)\,dx \in \mathcal{T}_{d'}$. So, $B_r\,(r)\,dx$ is the union of open balls in $\mathcal{T}_{d'}$. But this means we must have some $r_1 > 0$ such that $B_r\,(r_1)\,d'x \subseteq B_r\,(r)\,dx$. It's trivial to see that $B_r\,(r)\,d'x \in \mathcal{T}_d$ which again means it is the union of open balls in $\mathcal{T}_d$, thus implying we must have some $r_2 > 0$ such that $B_r\,(r_2)\,dx \subseteq B_r\,(r)\,d'x$. This completes the proof of the forward direction.

$(\Leftarrow)$ Suppose that for every $x \in M$ and every $r > 0$ there exists r_1, r_2 such that $B_r\,(r_1)\,d'x \subseteq B_r\,(r)\,dx$ and $B_r\,(r_2)\,dx \subseteq B_r\,(r)\,d'x$. Suppose for contradiction that $\mathcal{T}_d \neq \mathcal{T}_{d'}$. So, either we have an $O^{(d)}$ such that $O^{(d)} \notin \mathcal{T}_{d'}$ or we have $O^{(d')}$ such that $O^{(d')} \notin \mathcal{T}_d$. Let's do case 1 first, that $O^{(d)} \notin \mathcal{T}_{d'}$. Since $O^{(d)}$ is not open in $\mathcal{T}_{d'}$, there is an $x_0 \in O^{(d)}$ such that

$$\forall j > 0, \quad B_r\left(j\right) d' x_0 \nsubseteq O^{(d)} \tag{1}$$

Since $O^{(d)}$ is open in $\mathcal{T}_d$, for this particular x_0, we have $r_0 > 0$ such that $B_r\left(r_0\right) d x_0 \subseteq O^{(d)}$. By hypothesis, we now have an $r_1 > 0$ such that $B_r\left(r_1\right) d' x_0 \subseteq B_r\left(r_0\right) d x_0$. But now, we have that $B_r\left(r_1\right) d' x_0 \subseteq B_r\left(r_0\right) d x_0 \subseteq O^{(d)}$, but this contradicts **??**. The argument for the other case is similar and left out because it is tedious. □

1.2 Exercise 2.4 (b)

Claim 1.2.1. *Let (M, d) be a metric space, let $c > 0$ and define $d'(x, y) = c \cdot d(x, y)$. Then, $\mathcal{T}_d = \mathcal{T}_d'$*

Proof. Let $O \in \mathcal{T}_d$. Let $\Lambda \subseteq \mathbb{R} \times M$ be the radius-point pairs such that $O = \bigcup_{(r,x)\in\Lambda} B_r\left(r\right) dx$. Let $(r, x) \in \Lambda$ be arbitrary. Expanding the definition of $B_r\left(r\right) dx$, we get that

$$\begin{aligned} B_r\left(r\right) dx &= \{\, y \in M : d(y, x) < r \,\} \\ &= \{\, y \in M : c \cdot d(y, x) < cr \,\} \\ &= \{\, y \in M : d'(y, x) < cr \,\} \\ &= B_r\left(cr\right) d'x \end{aligned}$$

So, $O = \bigcup_{(r,x)\in\Lambda} B_r\left(cr\right) d'x \in \mathcal{T}_{d'}$. This shows that $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$. The argument $\mathcal{T}_d \supseteq \mathcal{T}_{d'}$ is similar and tedious and so skipped. □

Alternative proof by using **??**

Proof. Since $d'(x, y) = c \cdot d(x, y)$, we have $\frac{d'(x,y)}{c} = d(x, y)$ Pick $r_1 = \frac{r}{c}$. Then $B_r\left(r_1\right) d'x \subseteq B_r\left(r\right) dx$ Now, for $r_2 = cr$, it is immediately obvious that $B_r\left(r_2\right) dx \subseteq B_r\left(r\right) d'x$ [1] □

1.3 Exercise 2.4 (c)

This is exercise B.1 but I included it here because it's useful.

Lemma 1.3.1 (Textbook Exercise B.1)**.** *For $x \in \mathbb{R}^n$ such that $x = (x_1, x_2, \ldots, x_n)$, we have*

$$\max\{\, |x_1|, |x_2|, \ldots, |x_n| \,\} \leq |x| \leq \sqrt{n} \max\{\, |x_1|, |x_2|, \ldots, |x_n| \,\}$$

[1] the balls are actually equal but whatever

Proof. Suppose $|x_i| = \max\{\,|x_1|, |x_2|, \ldots, |x_n|\,\}$ for $1 \leq i \leq n$. Then, $|x| = \sqrt{x_1{}^2 + x_2{}^2 + \cdots + x_i{}^2 + \cdots + x_n{}^2} \geq \sqrt{x_i{}^2} = |x_i|$. For the next bit, we have that $|x| = \sqrt{x_1{}^2 + x_2{}^2 + \cdots + x_i{}^2 + \cdots + x_n{}^2} \leq \sqrt{n \cdot x_i{}^2} = \sqrt{n} \cdot |x_i|$ $\square$

Claim 1.3.2. *Let $(\mathbb{R}^n, d)$ be Euclidean n-space with the Euclidean metric $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$*

Let $d'(x,y) = \max\{\,|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\,\}$ be defined for $\mathbb{R}^n$. Then, $\mathcal{T}_d = \mathcal{T}_{d'}$

Note: This proof is incomplete since I am confused by my justification

Proof. We will make use of **??**. Let $x \in \mathbb{R}^n$ and $r > 0$. Then, choose $r_1 = r$. Now, let $y \in B_r\,(r_1)\,d'x$. So, from **??** we have $d'(y,x) < r_1 = r \leq d(y,x)$. So, $y \in B_r\,(r)\,dx$ and thus $B_r\,(r_1)\,d'x \subseteq B_r\,(r)\,dx$. Choose $r_2 = r\sqrt{n}$. Let $y \in B_r\,(r_2)\,dx$. Now

$$\begin{aligned} & d'(y,x) < r \\ \implies & \sqrt{n} \cdot d'(y,x) < r\sqrt{n} \\ \implies & d(x,y) \leq \sqrt{n} \cdot d'(y,x) < r\sqrt{n} \\ \implies & \frac{d(x,y)}{\sqrt{n}} \leq d'(y,x) < r \end{aligned}$$

Again, by **??**, $d(y,x) < r_2 \leq d'(y,x) < r$ Thus $B_r\,(r_2)\,dx \subseteq B_r\,(r)\,d'x$ $\square$

2 Exercise 2.5

Claim 2.0.1. *Let X be a topological space and Y be an open subset of X. Then, the collection of all open subsets of X in Y is a topology on Y.*

Proof. Let $(X, \mathcal{T})$ denote the topological space of X. Let $\mathcal{O}_Y$ denote the collection of all open subsets of X in Y. To do so, we just need to check the properties of a topology on Y, namely, that $\varnothing, Y \in \mathcal{O}_Y$, if $O_1, O_2 \subseteq Y$ are open, then $O_1 \cap O_2$ is open in Y (aka, $O_1 \cap O_2 \in \mathcal{O}_Y$), and the arbitrary union $\bigcup_{\lambda \in \Lambda} O_\lambda$ is also open in Y.

Firstly, since $\varnothing$ and Y are obviously open in Y by definition, $\varnothing, Y \in \mathcal{O}_Y$. Now, let $O_1, O_2 \subseteq Y$ be arbitrary open subsets of Y. Since Y is open in X, any subset of Y must also be open in X, so this means that $O_1, O_2 \in \mathcal{T}$. And since $\mathcal{T}$ is a topology, $O_1 \cap O_2 \in \mathcal{T}$. Of course, $O_1 \cap O_2 \subseteq Y$, so $O_1 \cap O_2 \in \mathcal{O}_Y$. (The inductive case is trivial)

Now, let Λ be some indexing set such that for each $\lambda \in \Lambda$, O_λ is open in Y. Of course, this means that $\bigcup_{\lambda \in \Lambda} O_\lambda \subseteq Y$ and so the arbitrary union is open in Y. And since $\bigcup_{\lambda \in \Lambda} O_\lambda \in \mathcal{T}$, $\bigcup_{\lambda \in \Lambda} O_\lambda \in \mathcal{O}_Y$.

Since $\mathcal{O}_Y$ is demonstrated to have the properties of a topology on Y, the proof is complete. $\square$

3 Exercise 2.6

Claim 3.0.1. *Let X be a set, and let $\{\mathcal{T}_\alpha\}_{\alpha\in A}$ be a collection of topologies on X. Then, $\mathcal{T} = \bigcap_{\alpha\in A}\mathcal{T}_\alpha$ is a topology on X.*

Proof. Obviously, $\varnothing, X \in \mathcal{T}$.

Let $O_1, O_2 \in \mathcal{T}$. So, $\forall\alpha \in A\, O_1, O_2 \in \mathcal{T}_\alpha$. But since each $\mathcal{T}_\alpha$ is a topology, we know that $O_1 \cap O_2 \in \mathcal{T}_\alpha$. So obviously, $O_1 \cap O_2 \in \mathcal{T}$.

Let Λ be an indexing set such that $O_\lambda \in \mathcal{T}$ for $\lambda \in \Lambda$. Now, this means each $O_\lambda \in \mathcal{T}_\alpha$. But of course, since each $\mathcal{T}_\alpha$ is a topology, $\bigcup_{\lambda\in\Lambda} O_\lambda \in \mathcal{T}_\alpha$. And since $\mathcal{T}_\alpha$ was arbitrary, $\bigcup_{\lambda\in\Lambda} O_\lambda \in \mathcal{T}$. □

4 Exercise 2.9

Prove proposition 2.8

> **Notation 4.1.** Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$. Then U_x is a neighborhood of x meaning that $U_x \in \mathcal{T}$ (or that it is an open subset of X)

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Let $A^c = X \setminus A$

Proposition 4.0.1 (Textbook Proposition 2.8a)**.** $x \in \operatorname{Int} A$ *iff there exists* $U_x \subseteq A$

Proof. $(\Rightarrow)$ Let $x \in \operatorname{Int} A$. Then, by definition of $\operatorname{Int} A$, we have some $C \subseteq X$ such that $C \subseteq A$ and $C \in \mathcal{T}$ (C is open). Then C will serve as U_x.

$(\Leftarrow)$ trivial □

Proposition 4.0.2 (Textbook Proposition 2.8b)**.** $x \in \operatorname{Ext} A$ *iff there exists* $U_x \subseteq A^c$

Proof. $(\Rightarrow)$

$$\begin{aligned}
\overline{A}^c &= \left(\bigcap_{\lambda\in\Lambda} B_\lambda\right)^c && \text{Each } B_\lambda \text{ is closed} \\
&= \bigcup_{\lambda\in\Lambda} (B_\lambda)^c && \text{And now each } (B_\lambda)^c \text{ is open}
\end{aligned}$$

The rest is trivial

$(\Leftarrow)$ trivial

□

Proposition 4.0.3 (Textbook Proposition 2.8c)**.** *$x \in \partial A$ iff for every U_x, some $y_1 \in A \cap U_x$ and some $y_2 \in A^c \cap U_x$*

Proof. **THIS PROOF IS INCOMPLETE** ($\Rightarrow$) Let $x \in \partial A$. Let $U_x \in \mathcal{T}$ be arbitrary.

Here, note that C_λ are open in X and are subsets of A. And that B_γ are closed in X and contain A

$$\begin{aligned}
\partial A &= (\operatorname{Int} A \cup \operatorname{Ext} A)^c \\
&= \left(\bigcup_{\lambda \in \Lambda} C_\lambda \cup \overline{A}^c \right)^c \\
&= \left(\bigcup_{\lambda \in \Lambda} C_\lambda \right)^c \cap \overline{A} \\
&= \bigcap_{\lambda \in \Lambda} (C_\lambda)^c \cap \overline{A} \\
&= \bigcap_{\lambda \in \Lambda} (C_\lambda)^c \cap \bigcap_{\gamma \in \Gamma} B_\gamma
\end{aligned}$$

□

Proposition 4.0.4 (Textbook Proposition 2.8d)**.** *$x \in \overline{A}$ iff every U_x has $y \in A \cap U_x$*

Proof. ($\Rightarrow$) Let $x \in \overline{A}$. Let $\mathcal{F} = \{ B_\lambda : \lambda \in \Lambda \}$ be a collection of every closed set in X that contains A, so $\overline{A} = \bigcap \mathcal{F}$. Suppose for contradiction that $U_x \in \mathcal{T}$ is a neighborhood that contains no points of A, or in other words, $U_x \cap A = \varnothing$. Since U_x is open and disjoint from A, $A \subseteq X \setminus U_x$. So, $X \setminus U_x$ is a closed set that contains A. Thus, $X \setminus U_x \in \mathcal{F}$. But since $x \in \overline{A} = \bigcap \mathcal{F}$, $x \in X \setminus U_x$. And by hypothesis, $x \in U_x$. This is obviously absurd.

($\Leftarrow$) Suppose x is such that every U_x has a $y \in A \cap U_x$. Let $\mathcal{F} = \{ B_\lambda : \lambda \in \Lambda \}$ be a collection of every closed set in X that contains A, so $\overline{A} = \bigcap \mathcal{F}$.

THIS IS INCOMPLETE (the converse direction) □

Proposition 4.0.5 (Textbook Proposition 2.8e)**.** $\overline{A} = A \cup \partial A = \operatorname{Int} A \cup \partial A$

Proof. **THIS IS INCOMPLETE**

$$\partial A = X \setminus \operatorname{Int} A \cap$$

□

Proposition 4.0.6 (Textbook Proposition 2.8f). *The interior and exterior are open sets, while the closure and the boundary and closed sets.* $\operatorname{Int} A, \operatorname{Ext} A \in \mathcal{T}$ *and* $\overline{A}^c, (\partial A)^c \in \mathcal{T}$.

Proof. $\operatorname{Int} A$ is the union of open sets which is an open set.

$\overline{A}$ is the intersection of closed sets, which is a closed set.

$\operatorname{Ext} A = X \setminus \overline{A}$ is an open set because $\overline{A}$ is closed.

$\partial A = X \setminus (\operatorname{Int} A \cup \operatorname{Ext} A)$ is closed because $\operatorname{Int} A \cup \operatorname{Ext} A$ is open. $\square$

Proposition 4.0.7 (Textbook Proposition 2.8g). *The following are equivalent:*

1. *A is open in X*
2. $A = \operatorname{Int} A$
3. *A contains none of its boundary points (aka, $\partial A \cap A = \varnothing$)*
4. *For all $x \in A$, there exists $U_x \subseteq A$*

Proof. We will prove 1 implies 2, 2 implies 3, 3 implies 4 and 4 implies 1 (in that order)

Suppose A is open in X. Then $A \in \mathcal{T}$. Obviously, $\operatorname{Int} A \subseteq A$. $A \subseteq \operatorname{Int} A$ follows from the definition of $\operatorname{Int} A$, namely, since, $A \subseteq A$ and A is open in X.

Suppose $A = \operatorname{Int} A$. Then by definition of $\partial A = X \setminus (\operatorname{Int} A \cup \operatorname{Ext} A)$, $\partial A \cap A = \varnothing$ immediately follows.

Suppose that $\partial A \cap A = \varnothing$. Let $x \in A$ be arbitrary. This means that $x \notin \partial A$, so $x \in \operatorname{Int} A \cup \operatorname{Ext} A$. Of course, $x \notin \operatorname{Ext} A$. So, $x \in \operatorname{Int} A$. So, $\operatorname{Int} A$ can be a neighborhood of x, and obviously since $\operatorname{Int} A \subseteq A$ we are done.

Suppose that every point of A has a neighborhood contained in A. Then, we know $\bigcup_{x \in A} U_x \subseteq A$. Now, let $x \in A$. Then we have some U_x, so $U_x \subseteq \bigcup_{y \in A} U_y$. And so $\bigcup_{y \in A} U_y = A$. Since the union of open sets is open, A is open. $\square$

Proposition 4.0.8 (Textbook Proposition 2.8h). *The following are equivalent:*

- *A is closed in X*
- $A = \overline{A}$
- *A contains all of its boundary points (aka, $\partial A \subseteq A$)*
- *For all $x \in A^c$, there exists $U_x \subseteq A^c$*

Proof. Observe that A is closed iff A^c is open, and apply **??** $\square$

5 Exercise 2.10

Proposition 5.0.1 (Book Exercise 2.10)**.** *Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then, A is closed iff it contains all of its limit points*

Proof. ($\Rightarrow$) Suppose A is closed. Let p be a limit point of A. Suppose for contradiction that $p \notin A$, so $p \in A^c$. Since A^c is open, p must have some neighborhood in A^c, call it V. Since p is a limit point, we must have some point $a_0 \in A$ such that $a_0 \in V$ and $a_0 \neq p$. But then, $V \subseteq A^c$, so $a_0 \in A$ and $a_0 \in A^c$, which is absurd.

($\Leftarrow$) Suppose A contains all of its limit points. Let L denote the set of all limits point of A. Suppose for contradiction that we have $x_0 \in A^c$ with no neighborhood contained in A^c. Let V_{x_0} be an arbitrary neighborhood of x_0. So, V_{x_0} contains a point in A, say y_0, and clearly $y_0 \neq x_0$. But now this means that x_0 is a limit point of A (since every neighborhood of x_0 contains a point of A that is not x_0) that is not contained in A. This is obviously absurd.

□

6 Exercise 2.11

> **Definition 6.1.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then, A is dense in X if $\overline{A} = X$

Proposition 6.0.1 (Book Exercise 2.11)**.** *Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then, A is dense iff every nonempty open subset of X contains a point of A.*

Proof. ($\Rightarrow$) Suppose A is dense in X. Then we have that $\overline{A} = X$. Now, let $O \subseteq X$ be an arbitrary nonempty open subset of X. Let $k \in O$ be some point, and so O is a neighborhood for k. By **??**, O must contain a point of A, so we are done.

($\Leftarrow$) Suppose every nonempty open subset of X contains a point of A. Let $x \in X$ be arbitrary and let O_x be an arbitrary neighborhood of x. By **??**, $x \in A$ so $X \subseteq A$. Obviously, $A \subseteq X$ so $A = X$ and thus A is dense in X. □

7 Exercise 2.12

> **Recall** (Convergence in metric spaces). Let (X, d) be a metric space. Let $(x_n) \subseteq X$. Then, $\lim_{n\to\infty}(x_n) = x$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that when $n \geq N$, $d(x_n, x) < \varepsilon$

Proposition 7.0.1. *Let (X, d) be a metric space. Let $\mathcal{T}$ be the topological space on X. Let $(x_n) \subseteq X$ be a convergent sequence with limit x of the metric space. Then, for every neighborhood U of x, we have an $N \in \mathbb{N}$ such that when $n \geq N$, $x_n \in U$*

TL;DR: metric space convergence and topological convergence are equivalent

Proof. Let U be an arbitrary neighborhood of x. Now, let B be the largest open ball contained in U. Suppose that B has radius r. Then, we have $N \in \mathbb{N}$ such that when $n \geq N$, $d(x_n, x) < r$. This means that when $n \geq N$, $x_n \in B \subseteq U$.

Suppose that (x_n) converges topologically. Let $\varepsilon > 0$ be arbitrary and consider the open ball B_ε around x with radius ε. Since B_ε is a neighborhood of x, we have some $N \in \mathbb{N}$ such that whenever $n \geq N$, $x_n \in B_\varepsilon$. The rest is obvious. The obviousness completes the proof. □

8 Exercise 2.14

Proposition 8.0.1 (Textbook Exercise 2.14)**.** *Let $(X, \mathcal{T}_X)$ be a topological space. Let $A \subseteq X$. Let $(x_i) \subseteq A$ such that $(x_i) \to x \in X$. Then, $x \in A$*

Proof. Notice that since (x_i) is convergent each neighborhood U of x will have some $x_i \in A$ such that $x_i \in U$. Now apply **??** to complete the proof

Alternative proof: trivial, obvious, exercise □

9 Exercise 2.16

Proposition 9.0.1 (Book Proposition 2.15)**.** *A map between topological spaces is continuous if and only if the preimage of every closed subset is closed.*

Proof. ($\Rightarrow$) Let X and Y be topological spaces and suppose $f : X \to Y$ is a continuous map. Let $K \subseteq Y$ be closed. Then, $f^{-1}(Y \setminus K)$ is open. So, $X \setminus f^{-1}(Y \setminus K)$ is closed. We claim that $X \setminus f^{-1}(Y \setminus K) = f^{-1}(K)$. Obviously, $f^{-1}(K) \subseteq X \setminus f^{-1}(Y \setminus K)$. To show $f^{-1}(K) \supseteq X \setminus f^{-1}(Y \setminus K)$, suppose not. Then, there must be some $x_0 \in f^{-1}(K)$ such that $x_0 \notin X \setminus f^{-1}(Y \setminus K)$. This means $x_0 \in f^{-1}(Y \setminus K)$. So, $f(x_0) \in Y \setminus K$, so $f(x_0) \notin K$. But however, we said $x_0 \in f^{-1}(K)$. This is absurd.

($\Leftarrow$) This is trivial and tedious and so skipped. The argument is similar as to ($\Rightarrow$)

□

10 Exercise 2.18

Proposition 10.0.1 (Book Proposition 2.17)**.** *Let X, Y, Z be topological spaces. Then,*

(a) Every constant map $f : X \to Y$ is continuous.

(b) The identity map $\epsilon : X \mapsto X$, $\epsilon(x) = x$ is continuous

(c) If $f : X \to Y$ is continuous, the restriction of f to any open subset of X is also continuous.

Proof. For a), observe that the preimage of any open subset of Y is either the empty set or X (if it contains the point y_0 which everything in X is mapped to).

For b), notice that the preimage of any open set is itself.

c) is obviously true. □

11 Exercise 2.20

Proposition 11.0.1. *Let $X \approx Y$ denote that X is homeomorphic to Y. Then, $\approx$ is an equivalence relation on the class of all topological spaces.*

> **Recall.** An equivalence relation must be reflexive, transitive and antisymmetric.

> **Trivial Proof.** The proof is trivial □

Proper proof. For reflexivity, observe that any topological space is homeomorphic to itself.

For transitivity, suppose X, Y, Z are topological spaces and $\varphi : X \to Y$ and $\psi : Y \to Z$ are homeomorphisms. Define $\xi = \psi \circ \varphi$. Then ξ is bijective, and of course since the composition of continuous functions is continuous, ξ and ξ^{-1} are both continuous so ξ is a homeomorphism from X to Z.

For antisymmetry, notice that homeomorphism is a symmetric relation. □

12 Exercise 2.21

Proposition 12.0.1. *Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and let $f : X_1 \to X_2$ be a bijective map. Then, f is a homeomorphism iff $f(\mathcal{T}_1) = \mathcal{T}_2$ (aka, if $U \in \mathcal{T}_1$, $f(Y) \in \mathcal{T}_2$*

Proof. This proof is left as an exercise to the viewer. That's right, kiameimon $\square$