Chicken Mcnugget Theorem

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1 McDonald's Chicken McNuggets

In the earliest time, chicken nuggets are sold in packages of 9 and 20. People had wondered what was the largest amount of chicken nuggets they could never buy assuming they bought only using the packages of 9 and 20 given and did not remove any.

We would wonder if there even such a amount in the first place? But before anything else, let us have a definition.

Definition 1.1 *A number is "expressible in* $\{m, n\}$*" if it can be expressed in the form* $am + bn$ *where* a, b *are* ***positive*** *integers*

Well, let us try this for a few values that start with small pack sizes as a special case to check for the existence of the largest amount inexpressible in $\{m, n\}$

Example 1.2 *For packages of* 3 *and* 5*, this was the results I got:*
1*:not expressible in* $\{3, 5\}$
2*:not expressible in* $\{3, 5\}$
3*:expressible as* 3 *in* $\{3, 5\}$
4*:not expressible in* $\{3, 5\}$

5:*expressible as* 5 *in* $\{3,5\}$
6:*expressible as* 2×3 *in* $\{3,5\}$
7:*not expressible in* $\{3,5\}$
8:*expressible as* $3+5$ *in* $\{3,5\}$
9:*expressible as* 3×3 *in* $\{3,5\}$
10:*expressible as* 2×5 *in* $\{3,5\}$
11:*expressible as* $5+2(3)$ *in* $\{3,5\}$
12:*expressible as* $4(3)$ *in* $\{3,5\}$
$\vdots$

It appears that from 8 onwards, all numbers are expressible in $\{3,5\}$.
Another definition:

Definition 1.3 *Let L;*$\{m,n\}$ *the largest number which is inexpressible in* $\{m,n\}$

Since there is a L;$\{3,5\}$, we might guess there is a L;$\{9,20\}$ exists.We note that $13 = 10+3, 14 = 11+3, 15 = 12+3$. Since $10, 11, 12$ is expressible in $\{3,5\}$, then $13, 14, 15$ are expressible in $\{3,5\}$(by adding 1 package of 3 to the set $10, 11, 12$). And if we continue this process, we realise that actually all numbers greater than or equal to 8 are expressible in $\{3,5\}$.

This method is certainly not very useful, since for large values of m, n, we have to find m consecutive numbers which are expressible in $\{m,n\}$ to determine L;$\{m,n\}$

2 Chicken McNugget Theorem

So, people had came up with the Chicken McNugget Theorem to be able to find out L;$\{m,n\}$ quickly.

The Chicken McNugget Theorem states that:

Theorem 2.1 *For relatively prime integers m and n, the L;*$\{m,n\}$*, is $mn - m - n$.*

For example, for $m = 7$ and $n = 11$, we get the L;$\{7,11\}$ as $(7)(11) - 7 - 11 = 59$.
The numbers have to be relatively prime because if they have a common factor k, then only multiples of k are expressible.

Proof(Self-created):
First, we will prove that $mn - m - n$ is not expressible in $\{m, n\}$. Then, we will prove that all numbers above it are expressible in $\{m, n\}$. These two statements combined give the Chicken McNugget Theorem

Now, if $mn+m+n = am+bn$, where a, b are positive integers, then rearranging gives

$$(n - a - 1)m = (b + 1)n.$$

Since $a \geq 0$, we have $n - a - 1 < n$ and $\gcd(m, n) = 1$, which gives $(n - a - 1)m$ is not a multiple of n. But this is impossible since $\frac{(n-a-1)(m)}{n} = b + 1$ is an integer.

Next, we have to show that all numbers above $mn - m - n$ are expressible in $\{m, n\}$. Bezout's Lemma (see[1]) states that we can find integers x, y such that $xm + yn = 1$ for any positive integers m, n.

Lemma 2.2 *All positive integers can be expressed as $am+bn$, where a, b are integers and m, n are positive integers.*

This comes directly from $xm + yn = 1$ multiplied by k which gives $kxm+kyn = k$, where k is the number we want to express as $am+bn$.

Now, we note that a, b cannot both be negative since it would make $am + bn = k$ negative. If any one of them is negative, we change

$$am + bn$$

into

$$(a + n)m + (b - m)n.$$

Now, by doing this repeatedly, we can end up with $a + n, b - m$ are positive, which means k is expressible in $\{m, n\}$.
We can also end up with the case, where for certain a, b, c, d, a is negative, b is positive; $a + n$ is positive, $b - m$ is negative. Let use find the largest number in this case.

The greatest number in this case is achieved by maximizing a where

$a < 0$ and $b = m-1$ to get $(-1)m+(m-1)n$, which gives $mn-m-n$. Hence, all numbers above $mn - m - n$is expressible in $\{m, n\}$

3 The Chicken McNugget Theorem for 3 variables?

Is there any formula to find the L;$\{m, n, p\}$? Well, no one had found one yet. But there is indeed a formula for a certain special case of this.

We can find the L;$\{m, n, p\}$ if any two of the three numbers have greatest common factor k

If the two numbers out of the three have a common factor, then let the 3 numbers be km, kn and p, where $\gcd(m, n) = 1$ Then $akm + bkn + cp = k(am + bn) + cp$. Note that for $am + bn$, all numbers above $mn - m - n$ are expressible.

Now, we let $k(am + bn) + cp = k[(mn - m - n + 1) + d] + cp = k(m - 1)(n - 1) + dk + cp$($d$ is non-negative).

For $dk + cp$, all numbers above $kp - k - p$ are expressible. Hence, all numbers above or equal to $k(m - 1)(n - 1) + kp - k - p + 1$ are expressible.

From here, we can say that the $Largest; km, kn, p$ is $k(m - 1)(n - 1) + kp - k - p$.

Example 3.1 *If I have packages of* 6 *nuggets,* 9 *nuggets and* 20 *nuggets, then the we apply the formula to get Largest;*$6, 9, 20$*(*6 *and* 20 *has common factor 2) is* $2(3 - 1)(10 - 1) + 2(9) - 2 - 9 = 43$.

Note that we can do this in another way too (6 *and* 9 *has a common factor* 3*). The value here is* $3(2 - 1)(3 - 1) + 3(20) - 3 - 20 = 43$*, which is the same as the answer we got.*

Note we can extend this to four variables or more depending on the cases but I will not include this here.

4 Determining whether a number is expressible

If McDonald's only offer nuggets in packages of 9 and 20 and I have 7 friends who wants 15 nuggets each, can I actually buy $7 \times 13 = 105$ nuggets?
We know that $mn - m - n$ is actually Largest;(m, n). So is there a way to determine whether a number below $mn-m-n$ is expressible in $\{m, n\}$?

A possible method is to use the following Lemma:

Lemma 4.1 *Exactly one element in* $(k, mn-m-n-k)$ *is expressible in* $\{m, n\}$.

Proof(Self-created):

Case 1: If k is expressible in $\{m, n\}$, then $k = am + bn$, $mn - m - n - k = (n - a - 1)m - (b + 1)n$

$mn - m - n - k$ is always not expressible since $n - a - 1 < n$ and $b + 1$ is positive.

Case 2: If k is not expressible in $\{m, n\}$, then we can express it as $xm + yn$, where one and only one of x, y is negative (Bezout's Lemma). Here, if $y > -m$, then $x < n$. If not, the number would be expressible in $\{m, n\}$ by adding m to y and n to x. Hence, $mn-m-n-k = mn-m-n-(xm+yn) = (n-x)m+(m-1-y)n$. Now here, $m - 1 - y > 0$ and $n - x > 0$. Hence, $mn - m - n - k$ is expressible in $\{m, n\}$

Example 4.2 *For* 9*-piece package and* 20*-piece package and we want to determine whether* 91 *is expressible in* $\{m, n\}$. *We can just look at* $151 - 105 = 46$ *Since* 46 *is expressible in* $\{m, n\}$, *we have that* 111 *is inexpressible in* $\{m, n\}$.

This section actually helps to cut down the work of guess and check to **see whether you can buy a certain amount of nuggets**.

4.1 A consequence of the above section

Lemma 4.3 *A consequence is that exactly* $\frac{(m-1)(n-1)}{2}$ *numbers are expressible in* $\{m, n\}$ *from* 1 *to* $mn - m - n$

This can be easily seen by reflecting across the value $\frac{(m-1)(n-1)}{2}$ for the number line from 0 to $mn - m - n$

5 Determining how to express an expressible number

Now, we know that we can buy 105 nuggets given packages of 9 and 20 nuggets.

So how do we actually determine the number of 9-piece and 20-piece packages each to buy?(Instead of going through guess and check.) For example, $105 = 5(9) + 3(20)$, which means 5 packages of 9-piece +5 packages of 20-piece.

We notice that the last digit of each number helps us in determining the multiple of 9 because multiples of 20 end in 0. So a clue may be to take (mod 20). The method to find the a, b in $am + bn = k$, where k is the number of nuggets we want to buy, are as follows (Basically solving a Diophantine equation):

Take mod n.
Then we will have the equation $k_1 \equiv am \pmod{n}$, where k_1 is remainder on dividing k by n. We now solving the mod equation to get the value of a. Now, we can also get the value of b

Example 5.1 *For* 9 *and* 20 *, to make* 105*. Take* 105 *mod* 9*, we have* $9a + 20b = 105 \implies 2b \equiv 6 \pmod{9} \implies b \equiv 3 \pmod{9}$*. So we can take* $b = 3$ *and consequently* $a = 5$*.*

This section actually helps to determine **what package-types to select when buying nuggets**.

6 Classical Water pouring question

If we have an infinite source of water and a 5-liter jug and 3-liter jug, we know that by Chicken McNugget Theorem, we would be able to get any amount of water for above 7 liters.

So how do we get, for example, 4 litres? Note that we can pour away water too, in this case, so this is an "extension" of the Chicken McNugget Theorem. But, contrary to the Chicken McNugget Theorem, all amounts of water can be obtained here .
This statement comes from Lemma 2.2.

6.1 General Solution

We have $3(2) - 5 = 1$ and $3(8) - 5(4) = 4$. This give rise to the following steps(I have omitted the pouring from one jug to another because it does not change the total amount of water we have.):
Add 3 liters
Add 3 liters
Subtract 5 liters
Add 3 liters
Add 3 liters
Subtract 5 liters
Add 3 liters
Add 3 liters
Subtract 5 liters
Add 3 liters
Subtract 5 liters
Add 3 liters

But instead, we can proceed more easily by using the equations $5(2) - 3(2) = 4$ and $3(3) - 5 = 4$. So for these two, we have the moves:

Add 5 liters.
Subtract 3 liters
Add 5 liters
Subtract 3 liters

Add 3 liters
Add 3 liters
Subtract 5 liters
Add 3 liters

If we use this equation:$3(8) - 5(4) = 4$, then we would need 12 steps instead of the previous 4 steps.This will take longer than before (We usually want to finish the pouring as fast as possible). So we do a little bit of editing by changing our aim to finding the optimal solution.

The optimal method is to find the values of x, y such $|x| + |y|$ is minimised. This will result in the least amount of pouring. Indeed, 4 pours is the optimal pouring method, since 1 pour will give 5 or 3 litres. 2 pours will give 2 or 6 or 8 or 10 litres. 3 pours will give 1 or 3 or 5 or 7 or 9 or 11 or 13 or 15 litres.

6.2 Optimal General Solution

To find optimal solution, we can solve the equation $ax + by = k$ for x and y to get $x = my + r$, $y = mx + s$ for every integer m. Then by finding the absolute value of the two values, we choose m such that $|x| + |y|$ is minimised, then we can find the optimal solution!

Example 6.1 *So to find the optimal solution for $5x + 3y = 4$, we have $x = 3k + 2$ and $y = -5k - 2$, by solving the Diophantine equation, for integers k. Note that if k changes from 0 to 1, both $|x|$ and y increases. If k changes from -1 to -2, then both $|x|$ and $|y|$ increases. So $|x| + |y|$ is minimised for either the case $k = 0$ or $k = -1$ Both cases require 4 steps to reach the final amount of water. (Two optimal pouring)*

Another example:

Example 6.2 *We have 7-liter and 11-liter jugs. And we want to get 5 liters. So we have $7x + 11y = 5$. This gives $x = 11k + 7$, $y = -7k - 4$, for an integer k.Note that if k changes from 0 to 1, both $|x|$ and y increases. If k changes from -1 to -2, then both $|x|$ and $|y|$ increases. So $|x| + |y|$ is minimised for either the case $k = 0$ or $k = -1$*

$k = 0$ *gives* $7(7) - 4(11) = 5$ *and*
$k = -1$ *gives* $3(11) - 4(7) = 5$. *Hence, only* $k = -1$ *is the only optimal solution.*

This section helps in determining **what is the fastest way to pour the water**.

7 References

[1] Bezout'sLemma:http://www.artofproblemsolving.com/Wiki/index.php/Bezout%27s_Lemma
[2] ChickenMcNuggetTheorem:http://www.artofproblemsolving.com/Wiki/index.php/Chicken_McNugget_Theorem