Homework 2

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3.1-1) Prove $\max(f(n), g(n)) = \Theta(f(n)+g(n))$

$\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2, and n_0 s.t.

$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0\}$

$0 \leq \max(f(n), g(n))$

$0 \leq \min(f(n), g(n)) + \max(f(n), g(n))$

$0 \leq 2\max(f(n), g(n))$

Since $\min(f(n), g(n)) + \max(f(n), g(n)) = f(n)+g(n)$

Then $0 \leq \max(f(n), g(n)) \leq f(n)+g(n) \leq 2\max(f(n), g(n))$

Therefore $f(n)+g(n) = \Theta(\max(f(n), g(n)))$

3.1-2) Prove $(n+a)^b = \Theta(n^b)$

$\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2, and n_0 s.t.

$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0\}$

$0 \leq c_1(n+a)^b \leq n^b \leq c_2(n+a)^b$

$0 \leq c_1(n+a) \leq n \leq c_2(n+a)$

if $n_0 \geq |a|$ then

$c_1 = \frac{1}{2}$ $c_2 = 2$

s.t

$0 \leq \frac{1}{2}(n+a) \leq n \leq 2(n+a)$

$0 \leq \frac{1}{2}(n+a)^b \leq n^b \leq 2(n+a)^b$

Therefore $(n+a)^b = \Theta(n^b)$

Prove:

3.1-4) $2^{n+1} = O(2^n)$ and $2^{2n} = O(2^n)$

$2 \cdot 2^n = O(2^n)$

constant $\cdot 2^n = O(2^n)$

There fore, $\boxed{\text{True}}$

$(2^2)^n = O(2^n)$

$4^n > 2^n$

Therefore $4^n = \omega(2^n)$

so $\boxed{\text{False}}$

3-2:

	A	B	O	o	Ω	ω	Θ
a	$\lg^k n$	n^ϵ	yes	yes	no	no	no
b	n^k	c^n	~~yes~~ yes	~~no~~ yes	~~yes~~ no	~~yes~~ no	no
c	$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
d	2^n	$2^{n/2}$	no	no	yes	yes	no
e	$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
f	$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

a) $\lim_{n\to\infty} \frac{\lg^k n}{n^\epsilon}$ if l'Hospital's rule is applied k times we have

$\lim_{n\to\infty} \frac{\lg^k n}{n^\epsilon} = 0$

therefore $\lg^k n = o(n^\epsilon)$

b) $\lim_{n\to\infty} \frac{n^k}{c^n}$ if l'Hospital's rule is applied k times we have

$\lim_{n\to\infty} \frac{n^k}{c^n} = 0$ therefore $n^k = o(c^n)$

c) $n^{\sin n} \not> \sqrt{n}$ since $\sin(n)$ flucuates from -1 to 1 the asymtotic relation is never strictly more or less for some $n \geq n_0$.

$n^{\sin n} \not< \sqrt{n}$

d) $\lim_{n\to\infty} \frac{2^n}{2^{n/2}} = \infty$ therefore $2^n = \omega(2^{n/2})$

$\leftarrow$ smaller compared to n

e) idenity $x^{\log_b y} = y^{\log_b x}$ so therefore $n^{\lg c} = c^{\lg n}$

f) Stirling's approximation: $\lg(n!) = \Theta(n \lg n)$ and $\lg(n^n) = n \lg(n)$ therefore $\lg(n!) = \Theta(n \lg n)$

3-4) a) $f(n) = O(g(n))$ implies $g(n) = O(f(n))$

$O(g(n)) = \{ f(n) \mid \exists\, c \,\&\, n_0 > 0$ s.t $0 \leq f(n) \leq c g(n)\ \forall n \geq n_0 \}$

$O(f(n)) = \{ g(n) \mid \exists\, c \,\&\, n_0 > 0$ s.t $0 \leq g(n) \leq c f(n)$

Let $f(n) = n$ & $g(n) = n^2$ ⟵ counter example

$n \leq c n^2\ \forall\, 2 \geq n_0$

however

$cn^2 \not\leq n$ $\boxed{\text{false}}$, $g(n) \neq O(f(n))$.

b) $f(n) + g(n) = \Theta(\min(f(n), g(n)))$

counter-example $f(n) = n$ $g(n) = 1$

$n + 1 \neq \Theta(1)$ $\boxed{\text{false}}$

d) $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$

counter-example $f(n) = 2n$ $g(n) = n$

$2n = O(n)$ ✓

$2^{2n} = O(2^n) \Rightarrow (4)^n \neq O(2^n)$

$4^n = \omega(2^n)$ $\boxed{\text{false}}$

4.3-1)

$T(n) = T(n-1) + n = O(n^2)$

Prove $T(n) \leq cn^2$ for some $c > 0$

I.H. $T(n-1) \leq c(n-1)^2$

$T(n) \leq c(n-1)^2 + n$

$\leq c(n^2 - 2n + 1) + n$

$\leq n^2 c - 2nc + c + n$

$\leq cn^2$ if $c \geq 1$

Q.E.D. inductive step.

4.3-2) $T(n) = T(\lceil n/2 \rceil) + 1 = O(\lg n)$

Prove $T(n) \le c\lg(n)$ for some $c > 0$

$T(\lceil n/2 \rceil) \le c \lg \lceil \frac{n}{2} \rceil$ I.H

$T(n) \le c \lg \lceil n/2 \rceil + 1$

$\le c \lg n - c \lg 2 + c$

$\le c \lg n$ for $c \geq 1$

Q.E.D. Inductive step

4.1.5-1) a) $T(n) = 2T(n/4) + 1$

$a = 2$ $b = 4$ $f(n) = 1$

$n^{\log_4 2} = n^{1/2}$

$n^{1/2} \geq 1$

$1 = O(n^{1/2-\epsilon})$ if, $\epsilon = 1/4$

case 1 $T(n) = \Theta(n^{1/2})$

b) $T(n) = 2T(n/4) + \sqrt{n}$

$a = 2$ $b = 4$ $f(n) = \sqrt{n}$

$n^{\log_b a} = n^{1/2}$

$n^{1/2} = \sqrt{n}$

case 2 $T(n) = \Theta(\sqrt{n} \lg n)$

c) $T(n) = 2T(n/4) + n$

$a = 2$ $b = 4$ $f(n) = n$

$n^{\log_b a} = n^{1/2}$

$n^{1/2} \le n$

case 1

$n = \Omega(n^{1-\epsilon})$ if $\epsilon = 1/4$

$T(n) = \Theta(n)$

d) $T(n) = 2T(n/4) + n^2$

$a = 2$ $b = 4$ $f(n) = n^2$

$n^{\log_b a} = n^{1/2}$

$n^2 \geq n^{1/2}$

$n^{1/2} = \Omega(n^{2-\epsilon})$ if $\epsilon = 1$

$T(n) = \Theta(n^2)$

4-5) a) If more than half of the chips are bad Professor Diogene will have some chips say that good ones are bad and bad ones that say theirgood. Since the chips are opposite but symmetric the professor has no way to differentiate between good and bad chips.

b) If chips A and B are being compared among n chips. Compare chips A and B and if both say they're good then put all these chips in a new pile. Since there are n chips and comparing 2 chips there will be $n/2$ comparisons. Since the chance they are both good is about $1/4$ then we know on average half the size will be reduced.

half of the size will be removed

c) Let $T(n) = T(n/2) + (n/2)$ ← at least $n/2$ goodchips

Prove $T(n) \leq c_1 n$

let $T(n/2) \leq c_1 n/2$ I.H

$T(n) \leq c_1 \frac{n}{2} + \frac{n}{2}$

$T(n) \leq c_1 \frac{\cancel{2}n}{\cancel{2}}$

$T(n) \leq c_1 n$ if $c \geq 1$

∴ QED Inductive Step

Exercise 5.2-1)

Probability (you hire exactly one time) = $P(1)$

Case happens when the person at the front is better than the rest in the list. And each person has a $\frac{1}{n}$ chance of being selected

$$P(1) = \frac{1}{n}$$

Probability (you hire n times) = $P(n)$

Case happens when everyone in list gets hired.

$$P(n) = \frac{1}{n!}$$

5.2-3) $E[X] = \sum_{x=1}^{6} x \cdot Pr\{X = x\}.$

$$E[X_i] = Pr\{\text{sum of } n \text{ dice}\}$$

$$X = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$$

$$E[X] = 1E[X_1] + 2E[X_2] \ldots + 6E[X_6]$$

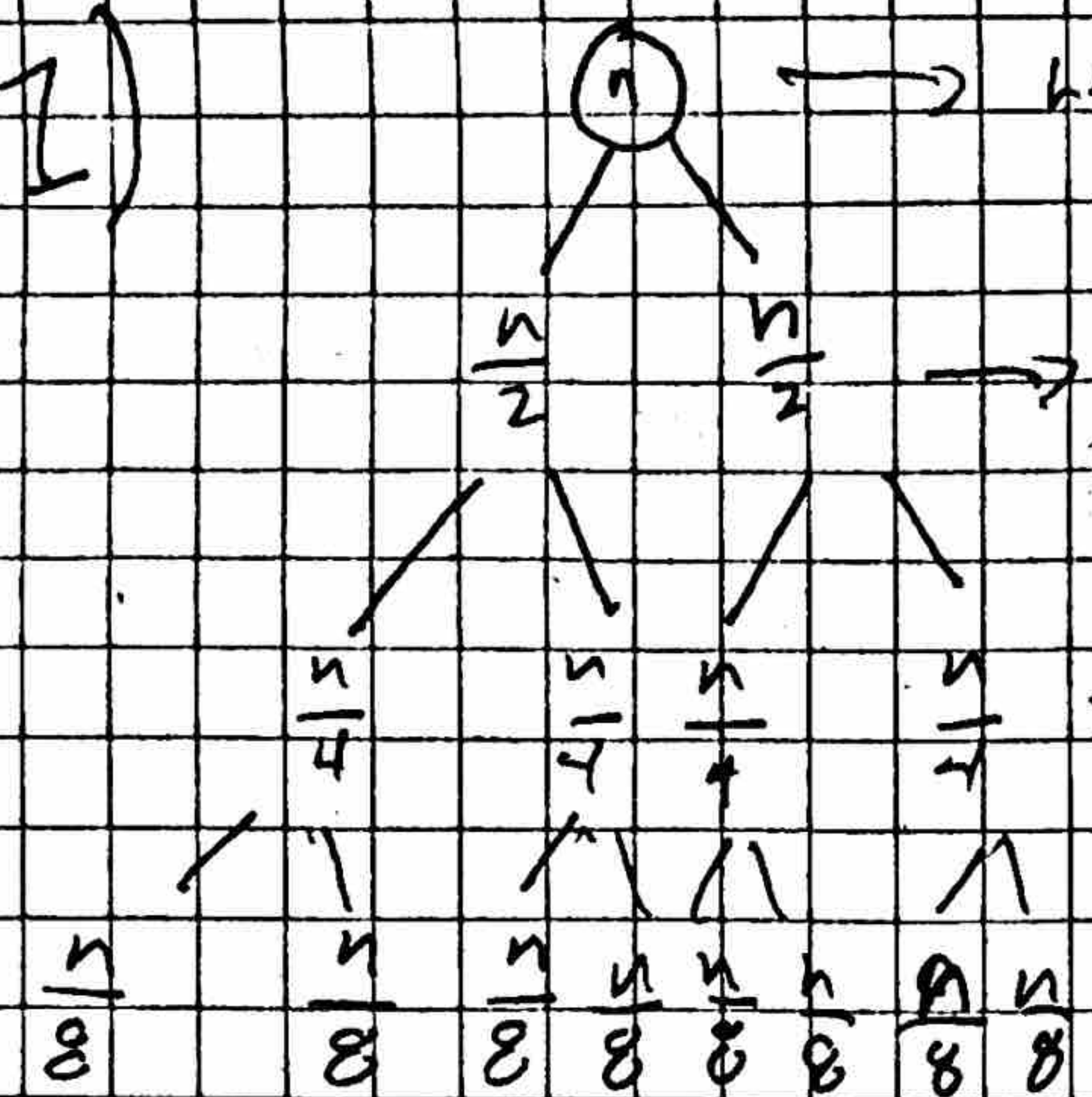
$$\therefore E[X_i] = n/6$$

$$E[X] = \sum_{x=1}^{6} x\,(n/6)$$

$$E[X] = (n/6)\,21$$

$$E[X] = 3.5n$$

6.1-1)



→ height is 0

→ height is 1 — ~~Max~~ for n min: 2 max: 3

→ height is 2 — min: 4 max: 7

→ height is 3 — min: 8 max: 15

$$\boxed{2^h \leq n \leq 2^{h+1} - 1}$$

H N H

6.1-2)

$$2^h \leq n \leq 2^{h+1} - 1$$

log by base 2

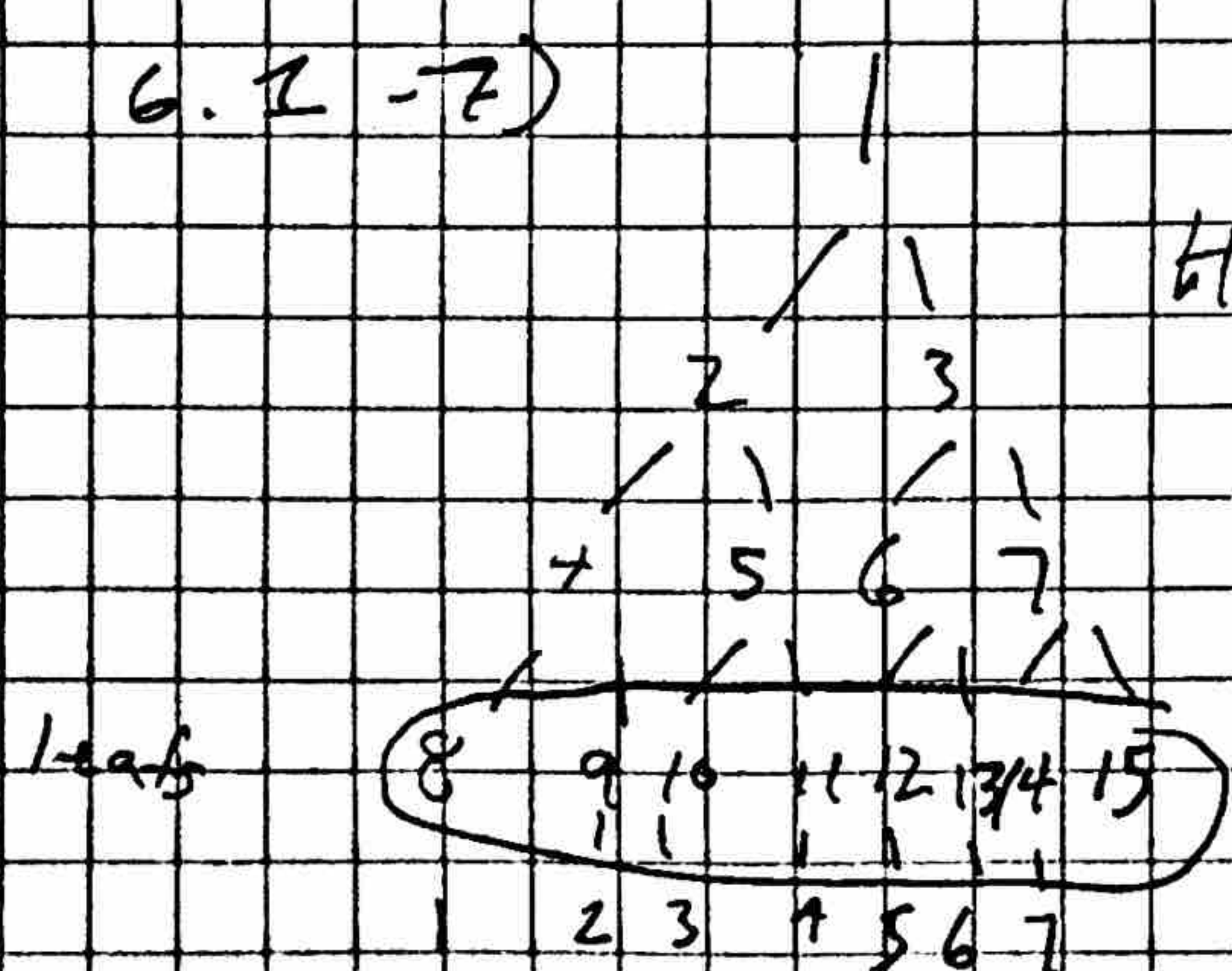
~~[scribbled out]~~

$$h \leq \lg n \leq h+1$$

$$\boxed{\therefore h = \lfloor \lg n \rfloor}$$

6.1-4) The smallest element in the max-heap will be a leaf node considering each parent node can have 1-2 leaf nodes that are smaller then the ~~[scribbled]~~ parent node.

6.1-7)



Heapsize = 15 = n-element heap

Show $8 = \lfloor n/2 \rfloor + 1$

$8 = \lfloor 15/2 \rfloor + 1 \Rightarrow 7 + 1 = 8$ ✓

$9 = \lfloor n/2 \rfloor + 2$

$9 = \lfloor 15/2 \rfloor + 2 \Rightarrow 7 + 2 = 9$ ✓

$13 = \lfloor 15/2 \rfloor + 6 \Rightarrow 7 + 6 = 13$ ✓

$14 = \lfloor 15/2 \rfloor + 7 \Rightarrow 7 + 7 = 14$ ✓