The 2D Heat Equation

— Analysis and Numerical Approaches

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1 Formulation of Problem

The heat equation, as know as the diffusion equation, Let $\Omega \subset \mathbb{R}^d$ be an open set with boundry $\Gamma := \partial\Omega$, set $\Omega_T = \Omega \times]0,T[$, $\Gamma_T := \Gamma \times]0,T[$, Γ_T is called the *lateral* boundary of the cylinder Ω_T.

Consider the heat equation with τ-periodic boundary condition:

$$\begin{cases} \partial_t u - \Delta_x u &= f & \text{on } \Omega_T \\ u(x_i,t) &= u(x_i+\tau,t) & \text{on } \Gamma_T \\ u(x,0) &= u_0(x) & \text{on } \Omega \times \{t=0\} \end{cases} \tag{1.1}$$

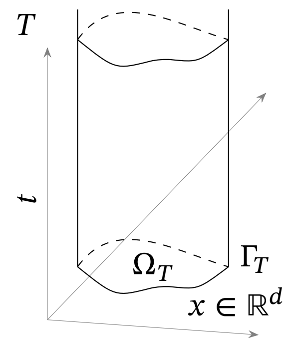

Figure 1: Region Ω_T

1.1 Fundamental Solution for Heat Equation

We note $|x| := \sqrt{\sum_1^d x_i^2}$.

✠ **Definition 1.1:** *The function*

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^d, t > 0 \\ 0 & x \in \mathbb{R}^d, t < 0 \end{cases} \tag{1.2}$$

is called the fundamental solution of the heat equation.

1.2 Spectrum Theory and Spectral Analysis

In this part, for analysis preliminaries, **[Folland1999Real]** offers a panorama on the theory of L^p spaces.

Given E, F two Banach spaces, an operator $T \in \mathcal{L}(E;F)$ is said to be copmact if the image of unit ball in E under T, i.e. $T(B_E)$ is relatively compact in F.

1.2.1 Riesz-Fredholm theory

▶ **Lemma 1.2** (Riesz): *Let E be a normed vector space (not necessary complete), $M \subsetneq E$ a proper closed linear subspace, then $\forall \varepsilon > 0$, $\exists u \in E$ s.t. $\|u\| = 1$ and $d(u,M) \geq 1 - \varepsilon$.*

▶ **Theorem 1.3:** *Let E be a normed vector space with compact unit ball B_E, then E is finite-dimensional.*

▶ **Theorem 1.4** (Fredholm alternative)**:** *Let $T \in \mathcal{L}(E)$ be a compact operator, then*

- $\operatorname{Ker}(I-T)$ *is finite-dimensional.*
- $\operatorname{Im}(I-T)$ *is closed. More precisely,* $\operatorname{Im}(I-T) = \operatorname{Ker}(I-T')^{\perp}$
- $\operatorname{Ker}(I-T) = \{0\} \iff \operatorname{Im}(I-T) = E$
- $\dim \operatorname{Ker}(I-T) = \dim \operatorname{Ker}(I-T')$

▷ *Proof:* Admitted. **[Brezis2011Functional]** □

✠ **Definition 1.5** (resolvent set, spectrum and eigenvalue)**:** *Let $T \in \mathcal{L}(E)$, the resolvent set, denoted by $\rho(T)$, is defined by*

$$\rho(T) := \{\lambda \in \mathbb{C}; (T-\lambda I) : E \to E \text{ is bijective } \}$$

The spectrum, denoted by $\sigma(T)$, is the complement of the resolvent set, i.e., $\sigma(T) = \mathbb{C} \backslash \rho(T)$. A complex number λ is said to be an eigenvalue of T if $\operatorname{Ker}(T-\lambda I) \neq \{0\}$*. The set of eigenvalues of T is denoted by $EV(T)$. The space* $\operatorname{Ker}(T-\lambda I)$ *is called the eigenspace of T, the elemet in it is called eigenvector.*

Remark 1: As E is a Banach space, the open mapping theorem tells us, the bijectivity of T equals that $T^{-1} \in \mathcal{L}^{-1}(E)$. Actually we have following consequnce :

▶ **Proposition 1.6:**

- *If $T \in \mathcal{L}(E)$ and $\|I\text{-}T\| < 1$ where I is the identity operator, then T is invertible, the series $\lim_{n\to\infty} \sum_{n=0}^{\infty} (I-T)^n = T^{-1}$ in $\mathcal{L}(E)$.*
- *The set of invertible operators in $\mathcal{L}(E)$, denoted as $GL(E)$, is an open set in $\mathcal{L}(E)$, and $GL(E) \to GL(E); T \mapsto T^{-1}$ is continuous. More precisely, if $S \in GL(E)$ and $\|T\text{-}S\| < \left\|T^{\text{-}1}\right\|^{-1}$, then $S \in GL(E)$.*

▶ **Theorem 1.7** (Gelfand)**:**

- *We have $\|T^n\|^{1/n} \xrightarrow{n\to\infty} \inf_n \|T^n\|^{1/n}$, we call this limite, denoted by $r(T)$, the spectral radius. Moreover, $r(T) \leq \|T\|$, and $\forall \lambda \in \sigma(T), |\lambda| \leq r(T)$. In particular, $\sigma(T)$ is a compact set in $\mathbb{C}$.*
- *For all $T \in \mathcal{L}(E)$, we have $\sigma(T) \neq \varnothing$. Moreover*

$$r(T) = \max_{\lambda \in \sigma(T)} \{|\lambda|\}$$

▷ *Proof:* **[Lax2002Functional]** □

✠ **Definition 1.8** (adjoint, self-adjoint)**:** *Let $A : \operatorname{Dom}(A) \subset E \to F$ be an unbounded linear operator that is densely defined. We shall introduce an unbounded operator $A' : \operatorname{Dom}(A') \subset F' \to E'$ as follows:*

$$\operatorname{Dom}(A') := \{v \in F' : \exists c \geq 0 \text{ s.t. } |\langle v, Au\rangle| \geq c\,\|u\|, \quad \forall u \in \operatorname{Dom}(A)\}$$

$${}_{F'}\langle v, Au\rangle_F = {}_{E'}\langle A'v, u\rangle_E, \quad \forall u \in \operatorname{Dom}(A), \forall v \in \operatorname{Dom}(A')$$

A bounded operator T is said to be self-adjoint if $T' = T$.

▶ **Theorem 1.9:** *Suppose that H is a separable Hilbert space, T is a compact self-adjoint operator. then there exists a Hilbert basis composed of eigenvectors of T.*

Our last statement is a fundamental result. It asserts that every compact self-adjoint operator may be diagonalized in some suitable basis.

1.2.2 Eigenfunctions and spectral decomposition

Now, we have sufficient tools to proceed the spectral analysis of heat equation. (More generally, the spectral analysis could be applied to other types of PDE [**Brezis2011Functional**; **Lax2002Functional**])

✠ **Definition 1.10** (Sobolev spaces, distribution)**:**

$$W^{m,p}(\Omega) := \left\{ u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega), \forall \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}_+^d \text{ and } 1 \leq \sum_{i=1}^{d} \alpha_i \leq d \right\}$$

We simply note $H^m(\Omega) := W^{m,2}(\Omega)$, since it's a Hilbert space.

At last, we define $H_0^m(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^m(\Omega)}$, that means the closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$. Where $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ called the set of test functions. Its dual space $\mathcal{D}'(\Omega)$ is called distribution.

The notion of distriburion generalized the notion of function. We could find that the Dirac mass at $x = a$: $\delta(x - a)$ is not a function, however, it's a distriburion. Important example: $L^1_{\text{loc}}(\Omega)$ is a distribution. For proof and more details, [**Gosle2020Distributions**].

▶ **Theorem 1.11** (the spectrum of Laplacian operator)**:** *Suppose $T : L^2(\Omega) \to L^2(\Omega); f \mapsto T_f$, where T_f is the weak solution (i.e. solution in $H_0^1(\Omega)$) of*

$$\begin{cases} -\Delta u_f &= f \quad \text{in } \Omega \\ \quad u_f &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.3}$$

Then T is compact and self-adjoint. Moreover T is positive defined.

▷ *Proof:* □

Assume that $\Omega \subset \mathbb{R}^d$ is a bounded open set, then there exist a Hilbert basis $\{e_n\}_n$ of $L^2(\Omega)$ s.t. $e_n \in H_0^1(\Omega) \cap C^\infty(\Omega), \forall n$ and a sequence $\{\lambda_n\}_n$ of real numbers with $\lambda_n > 0, \forall n$ and $\lambda_n \to +\infty$ s.t.

$$-\Delta e_n = \lambda_n e_n \quad \text{in } \Omega$$

We say that $\{\lambda_n\}_n$ are the eigenvalues of $-\Delta$ (with Dirichlet boundray condition) and the $\{e_n\}_n$ are the associated eigenfunctions.

▷ *Proof:* □

$$\begin{cases} -\Delta e_i &= \lambda_i e_i \quad \text{on } \Omega \\ \quad e_i &= 0 \quad \text{on } \Gamma \end{cases} \tag{1.4}$$

▶ **Corollary 1.12:** $\left\{\sqrt{\lambda_n} e_n\right\}_n$ *is a Hilbert basis for $H_0^1(\Omega)$ equipped with the inner product*

$$\langle v, w \rangle_{H_0^1(\Omega)} = \int_\Omega \nabla v \nabla w$$

.

▷ *Proof:*

$$\left\langle \nabla\sqrt{\lambda_n}e_n, \nabla\sqrt{\lambda_m}e_m \right\rangle_{L^2(\Omega)} = \frac{1}{\sqrt{\lambda_n}} \left\langle \sqrt{\lambda_n}\nabla Te_n, \nabla\sqrt{\lambda_m}e_m \right\rangle_{L^2(\Omega)}$$
$$= \frac{1}{\sqrt{\lambda_n}}\sqrt{\lambda_n\lambda_m}\int_\Omega e_n e_m$$

□

We seek for the solution of equation 1.1 in the form of series

$$u(x,t) = \sum_{i=1}^{\infty} a_i(t)e_i(x) \tag{1.5}$$

1.3 Maximum Principle

1.3.1 Weak maximum principle

1.3.2 Strong maximum principle

2 Numerical Approaches

2.1 Discrete and Fast Fourier Transform (DFT & FFT)

[**Stein2003Fourier**; **Schatzman2001Analyse**; **Mallat2008Wavelet**]

$$\widehat{f}(k) = \int_0^\tau f(x)e^{-2i\pi kx}\,\mathrm{d}x \xrightarrow{\text{discretization}} U_k = \frac{1}{N}\sum_{j=0}^{N-1} f(\frac{j}{N})e^{-2i\pi k\frac{j}{N}} \tag{2.1}$$

2.2 Finite Difference Method (FDM) for 1D Heat Equation

2.3 Finite Element Method (FEM) Approximation for 1D Heat Equation

2.4 FDM for 2D Heat Equation

2.5 FEM for 2D Heat Equation

3 Analysis of Algorithms

3.1 Consistency

3.2 Stability

3.3 Order of Convergence

3.4 Possibility of Improvement

4 Application on Economics and Finance

4.1 The Black-Scholes PDE for Option Pricing

4.2 Optimal Portfolio for Consumption and Investment