The 2D Heat Equation
— Analysis and Numerical Approaches

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Outline

An Evolutional Problem

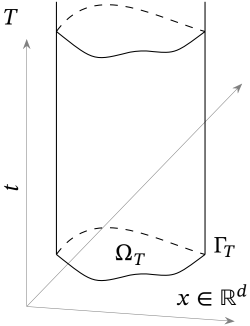

Figure: Region Ω_T

- Let $\Omega \subset \mathbb{R}^d$ be an open set with boundry $\Gamma := \partial\Omega$
- Set $\Omega_T = \Omega \times]0, T[$, $\Gamma_T := \Gamma \times]0, T[$, Γ_T is called the *lateral* boundary of the cylinder Ω_T.
- Consider the heat equation with τ-periodic boundary condition:

$$\begin{cases} \partial_t u - \Delta_x u = f & \text{on } \Omega_T \\ u(x_i, t) = u(x_i + \tau, t) & \text{on } \Gamma_T \\ u(x, 0) = u_0(x) & \text{on } \Omega \times \{t = 0\} \end{cases}$$

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Schemes based on FDM

[cf. Luc16, ch.08]

- The Explicit Euler Three Point Finite Difference Scheme
- The Implicit Euler and Leapfrog Schemes
- The Crank-Nicolson Scheme

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Meshes in 2D

[cf. Luc16, ch.06; Zil21]

- Rectangular Q_1 Finite Elements
- Triangular P_1 and P_2 Lagrange Elements

(a) Rectangular Q_1 Finite Elements (b) Triangular P_1 Lagrange Elements (c) Triangular P_2 Lagrange Elements

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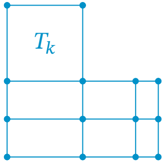
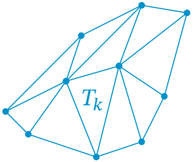
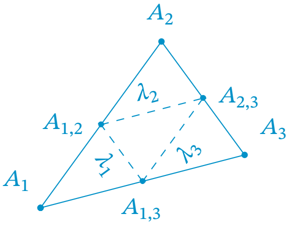


(a) Rectangular Q_1 Finite Elements



(b) Triangular P_1 Lagrange Elements



(c) Triangular P_2 Lagrange Elements

The Riesz-Fredholm Theory

[cf. Bre11, ch.06; Sch01, ch.08]

Outline

[cf. Luc16; Zil21]

All algorithms are implemented by Python 3.9 on 64-bit machine.

- Assemble with sparse matrix: `import scipy.sparse`
- JIT (Just-In-Time) compile: `import numba`
- Parallel computation for large volume data processing

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The Black-Scholes PDE for the Pricing of Financial Derivatives

[cf. Nef00, ch.12-13]

$$
\begin{aligned}
-rF + \partial_t F + \partial_t F \partial_t S + \frac{\sigma_t^2}{2} \Delta_s F \partial_t^2 S = 0 \\
S_t \in [0, +\infty[, \quad t \in [0, T] \\
F(T) = (S_T - K)_+
\end{aligned}
$$

References:

[Bre11] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer New York, 2011.

[Eva10] Lawrence Evans. *Partial Differential Equations*. American Mathematical Society, Mar. 2010.

[Luc16] Hervé Le Dret ; Brigitte Lucquin. *Partial Differential Equations: Modeling, Analysis and Numerical Approximation*. Springer International Publishing, 2016.

[Nef00] Salih N. Neftci. *An Introduction to the Mathematics of Financial Derivatives*. Elsevier Science & Techn., June 2000. 527 pp.

[Sch01] Michelle Schatzman. *Analyse numérique : une approche mathématique : cours et exercices*. Paris: Dunod, 2001.

[Zil21] Matthieu Bonnivard; Adina Ciomaga; Alessandro Zilio. "Méthodes numériques pour les EDO et les EDP". Notes de cours M1 Mathématiques. 2021.

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Thank you for listening!

Questions?

All resources including this diapositive, the final report, code and user's manual are submitted to:

Project Github: https://github.com/derrring/2d_heat_equation

Welcome to give us your valuable feedbacks :)

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