The 2D Heat Equation

— Analysis and Numerical Approaches

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Last Updated on: March 9, 2022

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1 Formulation of Problem

The heat equation, as know as the diffusion equation, Let $\Omega \subset \mathbb{R}^d$ be an open set with boundry $\Gamma := \partial\Omega$, set $\Omega_T = \Omega\times]0,T[$, $\Gamma_T := \Gamma\times]0,T[$, Γ_T is called the *lateral* boundary of the cylinder Ω_T.

Consider the heat equation with τ-periodic boundary condition:

$$\begin{cases} \partial_t u - \Delta_x u &= f & \text{on } \Omega_T \\ u(x_i,t) &= u(x_i+\tau,t) & \text{on } \Gamma_T \\ u(x,0) &= u_0(x) & \text{on } \Omega\times\{t=0\} \end{cases} \tag{1.1}$$

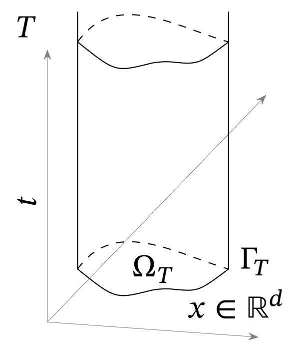

Figure 1: Region Ω_T

1.1 Fundamental Solution for Heat Equation

We note $|x| = \sqrt{\sum_1^d x_i^2}$.

✠ **Definition 1.1:** *The function*

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^d, t>0 \\ 0 & x\in\mathbb{R}^d, t<0 \end{cases} \tag{1.2}$$

is called the fundamental solution of the heat equation.

1.2 Spectrum Theory and Analysis

Given E, F two Banach spaces, an operator $T \in \mathcal{L}(E;F)$ is said to be copmact if the image of unit ball in E under T, i.e. $T(B_E)$ is relatively compact in F.

Let $T \in \mathcal{L}(E)$, the resolvent set, denoted by $\rho(T)$ The spectrum, denoted by $\sigma(T)$ A real number λ is said to be an eigenvalue of T if $\operatorname{Ker}(T-\lambda I) \neq \{0\}$

1.3 Maximum Principle

1.3.1 The strong maximum principle

2 Numerical Approaches - Case 1D

2.1 Discrete and Fast Fourier Transform (DFT & FFT)

[cf. Sha03; Sch01, ch.08]

$$\widehat{f}(k) = \int_0^\tau f(x) e^{-2i\pi kx} \, \mathrm{d}x \xrightarrow{\text{discretization}} U_k = \frac{1}{N} \sum_{j=0}^{N-1} f(\frac{j}{N}) e^{-2i\pi k \frac{j}{N}} \tag{2.1}$$

2.2 Finite Difference Method (FDM)

2.3 Finite Element Method (FEM) Approximation

3 Numerical Approaches - Case 2D

3.1 Discrete and Fast Fourier Transform (DFT & FFT)

3.2 Finite Difference Method (FDM)

3.3 Finite Element Method (FEM) Approximation

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