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July 31, 2025

Mean Field Limit and Mean Field Games

— From Finite Crowds to Infinite Wisdom —

International Workshop
on Applied Mathematics and Interdisciplinary Research
BISTU, Beijing

The N-Player Games

- Each player $i = 1, \ldots, N$ has a state X_t^i and a control α_t^i.
- The state evolves according to the dynamics:

$$dX_t^i = \alpha_t^i dt + \sigma d{W_t}^i$$

  where ${W_t}^i$ are independent Brownian motions.
- Each player wants to minimize a cost function $J^i(\alpha^1, \ldots, \alpha^N)$ that depends on their own state and the states of all other players.
- The **Nash equilibrium** is a set of strategies $(\alpha^1, \ldots, \alpha^N)$ such that no player can unilaterally improve their cost by changing their strategy.

Key Idea: Each player's action depends on the actions of all other N-1 players.

The Problem: $N \uparrow \implies$ Complexity $\uparrow\uparrow\uparrow$

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The Stoßzahlansatz and the early kinetic theory of gas

The hard-sphere gas: A simple mechanical model?

- Evolution of velocity distribution described by the Boltzmann equation.
- System converges to the Maxwellian equilibrium distribution.
- Entropy increases over time (H-theorem).

The Stoßzahlansatz (collision number assumption): The velocities of any two colliding particles are uncorrelated before they collide.

Crucial simplification, but cannot be true:

1. Collisions inherently create correlations between particles.
2. The microscopic laws are time-reversible, but the Boltzmann equation is not.

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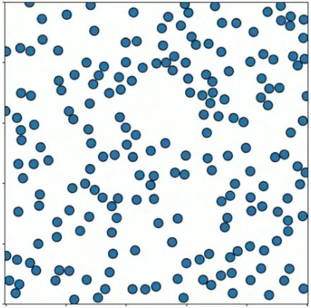

Figure: The Boltzmann equation describes the evolution of the distribution of velocities in a gas, assuming uncorrelated collisions.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left[\frac{\partial f}{\partial t}\right]_{\text{diffusion}} + \left[\frac{\partial f}{\partial t}\right]_{\text{force}} + \left[\frac{\partial f}{\partial t}\right]_{\text{collision}}$$

$$\Longrightarrow \partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = Q(f, f)$$

The collision term $Q(f, f)$ is quadratic, capturing the idea that the collision rate depends on the probability of two particles meeting.

The "Aha!" Moment: The Mean Field Paradigm

The core idea is to replace the complex, direct interactions between individual players with a simplified, averaged interaction.

As $N \to \infty$, the Law of Large Numbers suggests the random empirical measure of player states converges to a deterministic distribution:

$$m_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i} \quad \xrightarrow{N\to\infty} \quad m_t$$

where m_t is the distribution of a "representative" agent at time t.

The Paradigm Shift: Instead of tracking every player, we model a single representative player interacting with the continuous, deterministic distribution m_t of the entire population. The problem becomes tractable.

The Hilbert's 6th Problem

> *Was die Axiome der Wahrscheinlichkeitsrechnung angeht, so scheint es mir wünschenswert, daß mit der logischen Untersuchung derselben zugleich eine strenge und befriedigende Entwickelung der Methode der mittleren Werte in der mathematischen Physik, speciell in der kinetischen Gastheorie Hand in Hand gehe.*
>
> — *David Hilbert, Mathematische Probleme, 1900*

> *So regt uns beispielsweise das Boltzmannsche Buch über die Principe der Mechanik an, die dort angedeuteten Grenzprocesse, die von der atomistischen Auffassung zu den Gesetzen über die Bewegung der Continua führen, streng mathematisch zu begründen und durchzuführen.*

This involves deriving macroscopic equations (like fluid dynamics or kinetic equations) from the fundamental laws governing individual particles.

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Toward a rigorous mathematical kinetic theory

Microscopic scale:

- N identical particles in a space E.
- N^d-dimensional dynamical system.

Mesoscopic scale when $N \to +\infty$:

- $m_t \in \mathcal{P}(E)$ distribution of a typical particle.
- Compute the evolution of statistical quantities.

Goal: extend this framework to other types of particle systems...

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Formalizing the Particle Systems I

Definition (N-Particle System)

Given a state space E, an N-particle system is a E^N-valued Markov process $\mathcal{X}_t^N = (X_t^1, \ldots, X_t^N)$. Its law at time t is denoted by $m_t^N \in \mathcal{P}(E^N)$ and is characterized by the (weak-forward) Kolmogorov equation:

$$\forall \varphi_N \in C_b(E^N), \quad \frac{d}{dt}\mathbb{E}[\varphi_N(\mathcal{X}_t^N)] = \langle m_t^N, L_N \varphi_N \rangle$$

where $L_N : C_b(E^N) \to C_b(E^N)$ is the infinitesimal generator of Markov process $\mathcal{X}_t^N$.

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Two Main Types of Interaction Kernels

1. Collisional (Kac Type)

- Particles interact through random, pairwise "collisions".
- Generator acts on pairs:

$$\overline{L}_N \varphi_N = \frac{1}{N} \sum_{i<j} L^{(2)} \diamond_{ij} \varphi_N$$

- Leads to Boltzmann-type equations in the limit.

2. Mean-Field (McKean-Vlasov Type)

- Particles interact continuously via an averaged force.
- Generator is a sum of one-body operators depending on the empirical measure μ_{x^N}:

$$\overline{L}_N \varphi_N = \sum_{i=1}^{N} L_{\mu_{x^N}} \diamond_i \varphi_N$$

- Leads to Fokker-Planck-type equations in the limit.

McKean-Vlasov diffusion model

Assumption: Indistinguishability

The process is symmetric: $\forall \pi \in \mathfrak{S}_N, (X_t^{\pi(1)}, ..., X_t^{\pi(N)}) \sim (X_t^1, ..., X_t^N)$. The N-particle system can thus be represented by its **empirical measure**:

$$m_{\mathcal{X}_t^N} := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i} \in \mathcal{P}(E)$$

Each particle i feels a small force of size $1/N$ from each of the other particles:

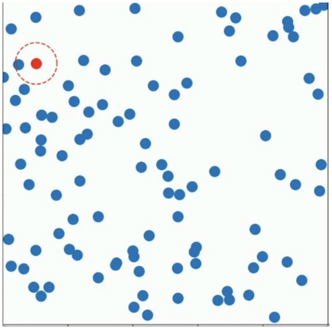
$$\mathrm{d}X_t^i = F * m_{\mathcal{X}_t^N}(X_t^i)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^i \tag{2.1}$$

where $F * m(x) = \int_E F(y-x)m(\mathrm{d}y)$ is the mean field interaction term, representing the average effect of all other particles on particle i.

Self-propulsion and short-range repulsion:

$$\frac{\mathrm{d}X_t^i}{\mathrm{d}t} = V_t^i$$

$$\frac{\mathrm{d}V_t^i}{\mathrm{d}t} = (1 - |V_t^i|^2) \cdot V_t^i - \frac{1}{N} \sum_{j=1}^{N} \nabla_{x^i} e^{\frac{-|X_t^i - X_t^j|}{R}}$$



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The Limit: McKean-Vlasov SDE

In a McKean-Vlasov model, each particle's drift depends on the distribution of all other particles.

The N-particle system:

$$\mathrm{d}X_t^i = F * m_{\mathcal{X}_t^N}(X_t^i)\mathrm{d}t + \sigma \mathrm{d}W_t{}^i$$

where $F * m(x) = \int_E F(y-x)m(\mathrm{d}y)$ is the mean-field interaction term.

As $N \to \infty$, we expect the system to be described by a **nonlinear** SDE, where the interaction is with the law m_t of the process itself:

$$\mathrm{d}X_t = F * m_t(X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad m_t = \mathrm{Law}(X_t)$$

The evolution of the density m_t is given by the **nonlinear Fokker-Planck equation**:

$$\partial_t m_t + \nabla_x \cdot ((F * m_t)m_t) - \frac{\sigma^2}{2}\Delta_x m_t = 0$$

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The Key Concept: Propagation of Chaos

How do we justify replacing the N-particle system with the limiting nonlinear process?

Definition (Propagation of Chaos)

A sequence of N-particle systems is said to be "chaotic" if, starting from independent and identically distributed (i.i.d.) initial conditions, any finite collection of particles becomes asymptotically independent as $N \to \infty$.

Formally, if the initial law is $m_0^N = m_0^{\otimes N}$, then for any fixed $k \geq 1$ and time $t > 0$, the k-particle marginal distribution $m_t^{k,N}$ converges weakly to the product measure:

$$m_t^{k,N} \xrightarrow{N\to\infty} m_t^{\otimes k}$$

where m_t is the solution to the limiting nonlinear Fokker-Planck equation.

In essence, chaos means that in a large crowd, any two players become "strangers" to each other, their fates linked only anonymously through the mass.

Proving Chaos: The Coupling Method

1. **Construct two systems driven by the same noise:**
   - The "true" N-particle system: $dX_t^i = F * m_t^N(X_t^i)dt + \sigma dW_t^i$
   - An i.i.d. system of "ideal" particles following the limit dynamics: $\mathrm{d}\overline{X}_t^i = F * m_t(\overline{X}_t^i)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^i$
2. **Show they stay close:** The goal is to prove that the distance between a true particle and its ideal counterpart vanishes as $N \to \infty$.

Theorem (McKean, 1966)

If the interaction kernel F is Lipschitz continuous, then for any finite time horizon $T > 0$:

$$\lim_{N\to\infty} \sup_{i\in\{1,\dots,N\}} \mathbb{E}\left[\sup_{t\in[0,T]} |X_{t,i} - \overline{X}_t|^2\right] = 0, \quad \forall T > 0.$$

This guarantees that the N-particle system is well-approximated by the McKean-Vlasov limit.

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From Limit to Game: Introducing Rationality

So far, the "particles" have been passive. What if they are rational agents making optimal decisions?

- The mean field limit describes the aggregate behavior of a large population.
- Optimal control theory describes how a single agent should act to minimize a cost.

Mean Field Games (MFG) combine these two ideas.

The Core MFG Idea

A single agent optimizes their strategy assuming the population's distribution evolves along a given path $\{m_t\}_{t\in[0,T]}$. At equilibrium, this assumed path must coincide with the actual distribution generated by all agents following their optimal strategies. It's a fixed-point problem.

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The Optimal Control Problem

Assume the population distribution's evolution, $\{m_t\}_{t\in[0,T]}$, is a known, given flow. The problem for our representative agent becomes much simpler. They want to choose a control α_t to minimize:

$$J(\alpha; m_t) = \mathbb{E}\left[\int_0^T \mathcal{L}(X_t, \alpha_t, m_t)\,\mathrm{d}t + G(X_T, m_T)\right]$$

Subject to their personal dynamics: $\mathrm{d}X_t = \alpha_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$. This is a standard stochastic optimal control problem!

Core Setup

- Agents: Large population of beachgoers
- State Space: Positions on a one-dimensional beach $x \in [0, 1]$
- Objective: Choose location to balance two competing goals:
  1. Proximity to an ice cream stall at position x_{stall}
  2. Avoidance of overcrowded areas

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Figure: A beach with an ice cream stall at x_{stall}. Agents choose their positions to balance proximity to the stall and avoidance of crowded areas.

A Static Reward Function

Each agent choosing location x with population density $m(x)$ is considered to receive reward:

$$r(x, m) = -|x - x_{stall}| - \ln(m(x))$$

- Proximity Term: $-|x - x_{stall}|$ penalizes distance from stall
- Congestion Term: $-\ln(m(x))$ strongly penalizes crowded locations

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State and Control:

- State: $x_t \in [0,1]$ (position on the beach)
- Control: α_t
- Dynamics: $\mathrm{d}x_t = \alpha_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$

Running Cost:

$$\mathcal{L}(x,\alpha,m) = \underbrace{|x - x_{stall}|}_{\text{Proximity Cost}} + \underbrace{\lambda \ln(m(x,t))}_{\text{Weighted Congestion Cost}} + \underbrace{\frac{1}{2}\alpha^2}_{\text{Movement Cost}}$$

The parameter $\lambda > 0$ is the crucial crowd aversion parameter. It scales the importance of the congestion penalty relative to the proximity cost, directly governing an agent's tolerance for being in a crowd.

Coupled PDE System:

1. Hamilton-Jacobi-Bellman Equation (backward):

$$-\frac{\partial u}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} \underbrace{+ \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = |x - x_{stall}| + \lambda \ln(m)}_{\text{Hamiltonian: Fenchel-Legendre transformation of Lagrangian w.r.t } \alpha}$$

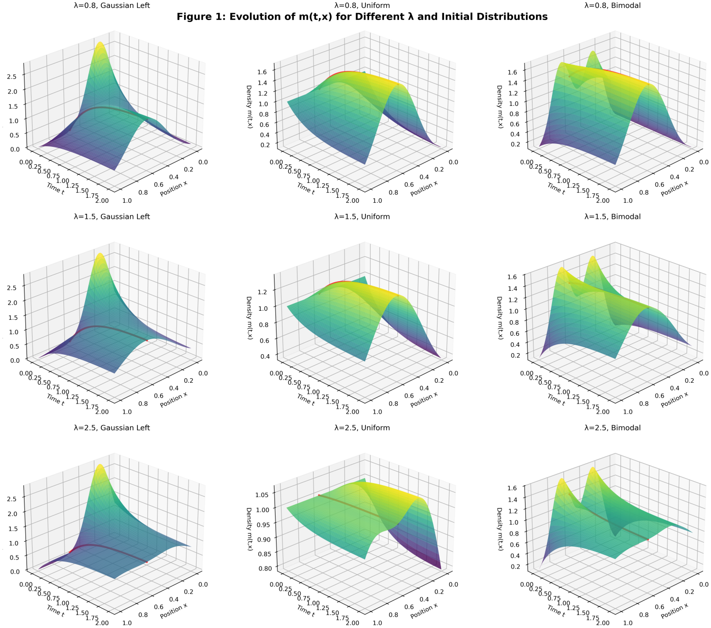
2. Fokker-Planck-Kolmogorov Equation (forward):

$$\frac{\partial m}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 m}{\partial x^2} - \mathrm{div}\left[m\frac{\partial u}{\partial x}\right] = 0$$

3. Initial Condition/ Terminal Condition:

$$m(x,0) = m_0(x), \quad u(x,T) = G(x, m_T)$$

Figure 1: Evolution of m(t,x) for Different λ and Initial Distributions



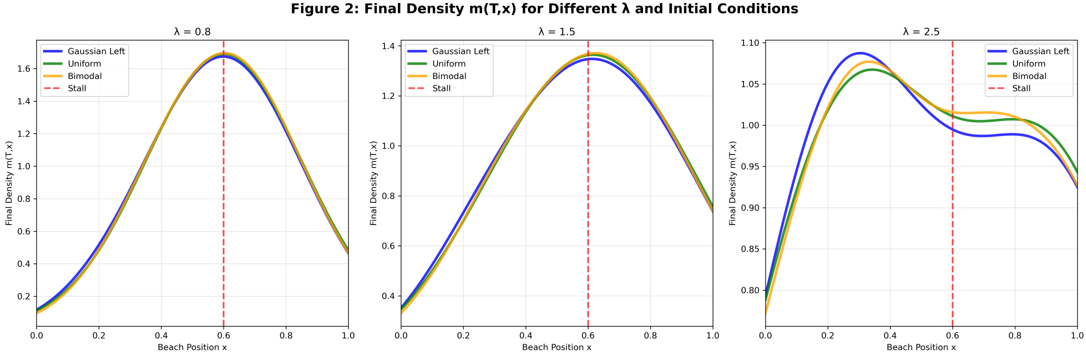


Figure: Final population density $m(x,T)$ at time T. The density has adapted to the presence of the ice cream stall, illustrating the balance between proximity and crowd avoidance.

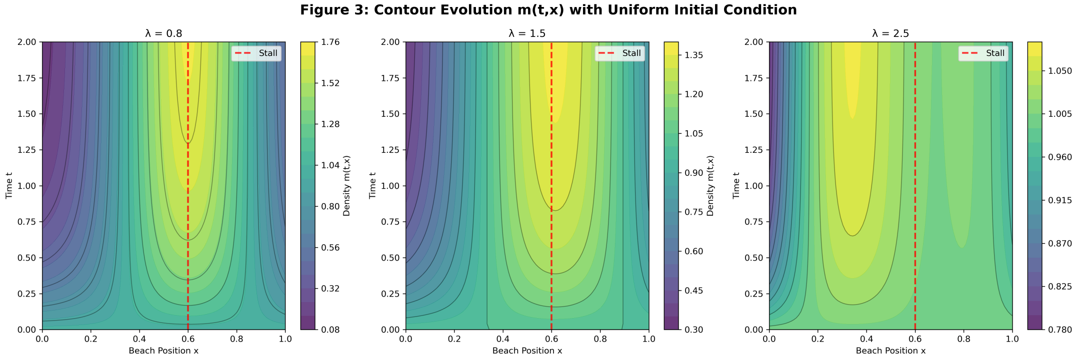
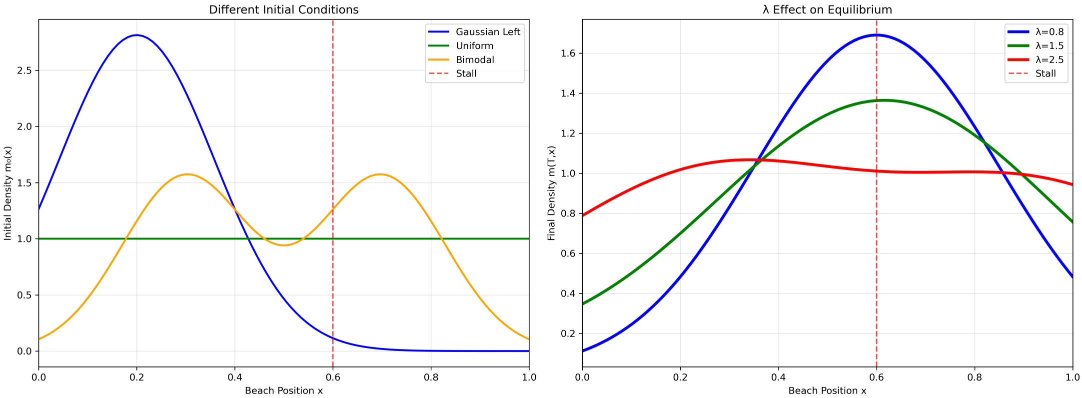


Figure: Evolution of the control $u(x,t)$ over time. The contours illustrate how agents' optimal control strategies change as they adapt to the evolving population density and their own positions relative to the ice cream stall.



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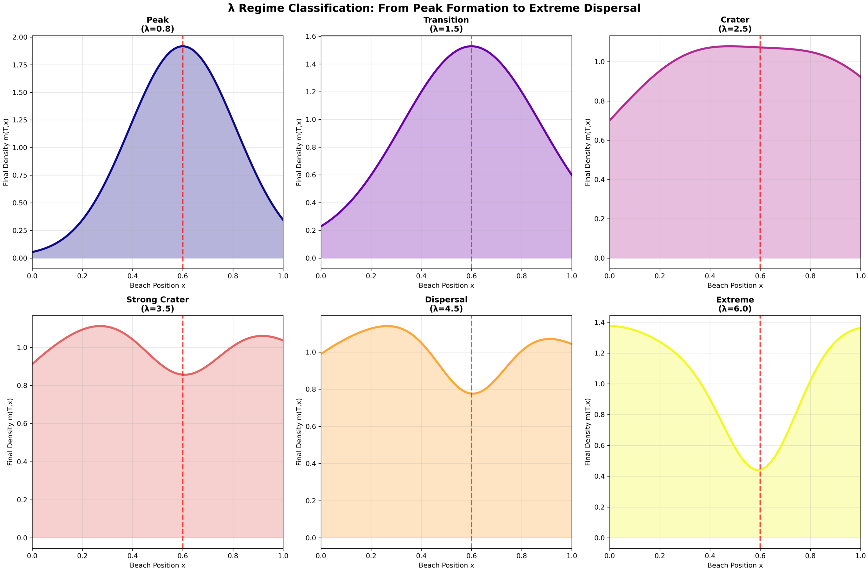

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- A Single Peak Equilibrium would look like a single mountain, with its summit located directly above the ice cream stall at x_{stall}.
- A Crater Equilibrium would look like a mountain range with two peaks, with a deep valley or ”crater” in between. The lowest point of this crater would be centered over the ice cream stall.

Unexpected Consequences: the neighborhood of the ice cream stall x_{stall} becomes a **no-go zone** for agents, as they avoid the crowding effect.

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Retrospect of Hilbert's 6th Problem: A Partial Success

1. Kolmogorov: Axiomatization of probability theory.
2. Classical Mechanics: The symplectic geometry and variational calculus.
3. Statistical and Hydrodynamics [OPEN]: The Navier-Stokes equations; The singular limits; The arrow of time in macroscopic systems; The turbulence.

McKean-Vlasov and Kac: The connection between kinetic theory and mean-field theory, remains to be unified.

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Thanks for your attention!

Outline

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