Jiongyi Wang
jiongyiwang@keio.jp

Peng Lab
Keio University

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A Structure-Preserving Approach to Mean Field Games

What is a Mean Field Game?

- MFGs model scenarios with a very large number of rational, interacting agents.
- Each agent has a negligible impact on the whole system, but the collective behavior (the “mean field”) influences each individual.
- Lies at the intersection of:
  - Game Theory
  - Stochastic Analysis & Control Theory
- Applications: Crowd dynamics, financial markets, traffic flow, etc.

jiongyiwang@keio.jp

The core task is to find a **Nash Equilibrium**: a state where no single agent can improve their outcome by unilaterally changing their strategy $\implies$ a steady distribution (of strategies) among agents.

jiongyiwang@keio.jp

I: Optimal Control for Agent's Behavior

The cost of a single player (agent) is given by the functional:

$$J(t,x,\alpha) = \underbrace{\mathbb{E}\left[\int_t^T \mathcal{L}(X_s,\alpha_s,m(s))ds + G(X_T,m(T))\right]}_{\text{cost function } J(\alpha_t)\text{= running cost } \mathcal{L}\text{+ terminal cost } G}$$

$$u(t,x) = \inf_{\alpha\in\mathcal{A}} J(t,x,\alpha)$$

Then u satisfies Hamilton-Jacobi-Bellman equation (Consequence of Dynamic Programming Principle):

$$-\partial_t u + H(x,m,Du) = 0$$

The Hamiltonian H is defined as:

$$H(x,m,p) = \sup_{\alpha\in\mathcal{A}}\{-\langle b(x,\alpha,m),p\rangle - \mathcal{L}(x,\alpha,m)\}$$

II: Dynamics of Interacting Systems

Each agent controls her own dynamics. A single agent's behavior satisfies SDE:

$$\begin{aligned} \mathrm{d}X_s &= b(X_s, \alpha_s, m(s))\,\mathrm{d}s + \sigma(X_s, \alpha_s, m(s))\,\mathrm{d}B_s \\ \Longleftrightarrow X_t &= X_0 + \int_0^t b(X_s, \alpha_s, m(s))\,\mathrm{d}s + \int_0^t \sigma(X_s, \alpha_s, m(s))\,\mathrm{d}B_s \end{aligned}$$

- X lives in $\mathcal{X}$, α is the control (taking values in admissible set $\mathcal{A}$) and B_s is a given d - dimensional Brownian motion
- b and σ are assumed to be smooth enough for the solution $\{X_t\}_{t\in[0,T]}$ to exist

III: Evolution of Population Distribution

$$\mathrm{d}X_s = b(X_s, \alpha_s, m(s))\,\mathrm{d}s + \sigma(X_s, \alpha_s, m(s))\,\mathrm{d}B_s$$

The distribution of the population of agents is described by the probability density function $m(t,x)$, which evolves according to the Fokker-Planck equation:

$$\frac{\partial m}{\partial t}(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) - \operatorname{div}\Big(m(t,\cdot)H_p(\cdot, m(t,\cdot), \nabla u(t,\cdot))\Big)(x) = 0$$

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The MFG System

$$
\begin{cases}
-\partial_t u - \frac{\sigma^2}{2}\Delta u + H(x, m, Du, D^2 u) = 0 & \text{in } [0,T) \times \Omega \quad (1.1a) \\
-\partial_t m - \frac{\sigma^2}{2}\Delta m - \operatorname{div}\left(m \cdot H_p(x, m, Du, D^2 u)\right) = 0 & \text{in } (0,T] \times \Omega \quad (1.1b) \\
u(T,x) = G(x, m(T,x)), \quad m(0,x) = m_0(x), & \text{in } \{T\}/\{0\} \times \Omega \quad (1.1c)
\end{cases}
$$

1. (1.1a) characterized an optimization problem approximating a large number of agents (players), where the behavior of each player is governed by the rest of the players via the mean-field.
2. (1.1b) characterized the evolution of population distributions of agents, which is governed by the optimal control of each agent.
3. The MFG system possesses a forward-backward structure, coupling the HJB and FP equations.

The Finite Difference Method (FDM)

Common approach: discretize the PDE system on a space-time grid.

HJB Discretization

- Use finite differences for derivatives.
- Requires a stable scheme (e.g., upwinding, semi-Lagrangian) to handle the non-linear Hamiltonian.

FP Discretization

- Also use finite differences.
- The divergence term requires careful treatment, often using a weak formulation.

However, this forward-backward structure is notoriously difficult to solve and can lead to numerical issues.

jiongyiwang@keio.jp

A Critical Flaw: Lack of Mass Conservation

Lens through no-flux Neumann boundary condition.

- Preserve the total mass of the system. (Since reflexive near $\partial\Omega$)
- Numerical errors can accumulate, leading to non-physical results. (Reveal the problems that hidden under periodic boundary conditions)
- The total mass should remain constant (e.g., 1.0 for normalized densities).

A Numerical Example

A simple 1D MFG system with no-flux Neumann BC:

$$
\begin{aligned}
\Omega_T &= (0,1) \times (0,1), \\
\mathcal{L}(t,x) &= 50\left(\frac{1}{10}\cos(2\pi x) + \cos(4\pi x) + \frac{1}{10}\sin(2\pi(x - \frac{\pi}{8}))\right), \\
u(T,x) &= g(x) = 0, \\
m(0,x) &= m_0(x) \propto \frac{1}{(20\pi)^{1/2}} \exp\left(-\frac{1}{20}\|x\|^2\right), \\
H(x,p,m) &= \frac{1}{4}\|p\|^2 - \mathcal{L}(t,x) - \|m\|_\infty^2.
\end{aligned}
$$

Note that the initial distribution m_0 should be truncated and normalized to ensure the total mass is 1.

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Result: The numerical scheme fails to conserve mass.

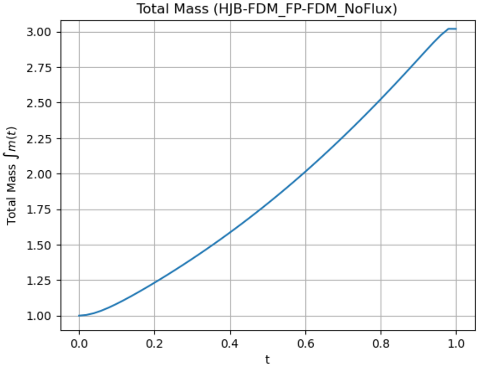

Figure: Simulation shows total mass erroneously increasing from 1.0 to over 3.0, due to the reflexion at boundaries.

$\implies$ This motivates searching for structure-preserving numerical methods.

jiongyiwang@keio.jp

Boltzmann, Vlasov, and McKean: A Historical Perspective

The particle method is inspired by the historical development of kinetic theory and the evolution of statistical mechanics.

1. **Boltzmann's Kinetic Theory:** Describes how gas particles interact and evolve over time.
2. **Vlasov's Plasma Theory:** Extends Boltzmann's ideas to systems with long-range interactions.
3. **McKean-Vlasov Equations:** Formalizes the idea of a large number of interacting agents (propagation of chaos), leading to the concept of MFG.

These theories [1–3] provide a solid foundation for understanding how individual agents' dynamics can be modeled as a collective behavior.

jiongyiwang@keio.jp

From Eulerian PDE to Lagrangian Particles

Instead of tracking the density $m(t,x)$ on a fixed grid (Eulerian view), we track the trajectories of individual agents (Lagrangian view).

- The Fokker-Planck equation for the density $m(t,x)$ is equivalent to a system SDEs for N particles.
- Each particle's motion is governed by Langevin dynamics:

$$dX_t = \underbrace{\alpha^*(t, X_t, m(t,\cdot))}_{\text{Optimal Drift}} dt + \underbrace{\sigma dW_t}_{\text{Random Noise}}$$

- The density $m(t,x)$ is simply the empirical distribution (i.e., histogram) of the particles.

$$\widehat{m}_N(t,x) = \frac{1}{N}\sum_{j=1}^{N} \delta(x - X_{t,j})$$

The Hybrid Solver Algorithm

We can create a **hybrid solver**: use a grid for the HJB equation but particles for the FP equation.

1. **Solve HJB (Grid):** Given a density $m(t, x)$, solve the HJB equation backward on a grid to find the optimal velocity field $\alpha^*(t, x) \propto -\nabla u(t, x)$.
2. **Evolve Particles:** Move N particles forward in time according to the SDE, using the velocity field computed in Step 1.
3. **Reconstruct Density:** At each time step, reconstruct the density $m(t, x)$ from the new particle positions (e.g., using kernel density estimation without renormalization of density).
4. **Iterate:** Repeat steps 1-3 until the solution (u, m) converges.

Result: Mass is Naturally Conserved!

By evolving particles, mass conservation is inherent, since the number of particles is constant.

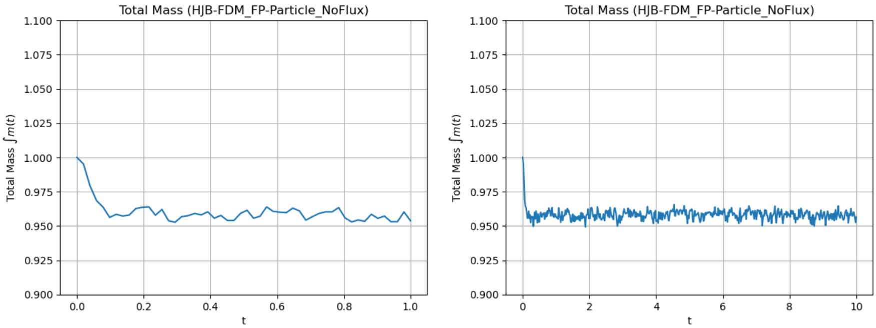

Figure: Simulation shows total mass is conserved around 1.0.

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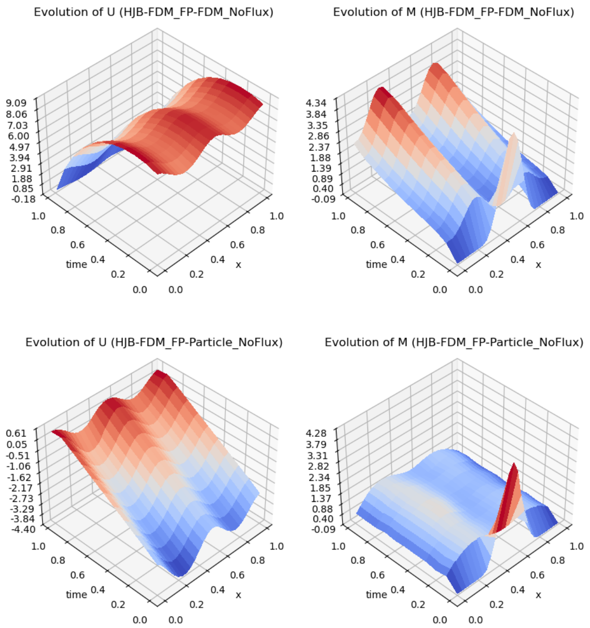

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Advantages of the Particle Method for FP:

- **Inherently preserves mass and positivity.**
- More intuitive, as it directly simulates agent dynamics.
- Avoids the stability issues of a dedicated FP grid-based solver.

Motivation: Can We Get Rid of the Grid Entirely?

The hybrid method was successful, but it still relied on a grid to solve the HJB equation.

Question: Can we develop a fully mesh-free method where the HJB equation is also solved directly on the particles?

Answer: Yes, using a particle-collocation approach.

jiongyiwang@keio.jp

Key Idea: Lagrangian View of the HJB Equation

The HJB equation can be interpreted from the perspective of a moving particle.

$$\alpha^*(t,X,\nabla_X u) = \arg\max_{\alpha\in\mathcal{A}}\{\underbrace{-\langle b(x,\alpha,m),p\rangle}_{L^\alpha[u](t,X(t))} - \mathcal{L}(x,\alpha,m)\}$$

$$\underbrace{\frac{\partial u}{\partial t} + L^{\alpha^*}[u](t,X^*(t))}_{\text{Expected rate of change of } u \text{ along optimal path}} = \underbrace{-\mathcal{L}(t,X^*(t),\alpha^*)}_{\text{Negative of instantaneous cost}}$$

- This re-frames the PDE as a statement about the evolution of the value function u along the optimal particle trajectories.
- This naturally suggests solving for the value function u at the particle locations.
- **The Challenge:** To evaluate the HJB equation, we need derivatives of u (e.g., ∇u), but we only have values of u at scattered particle locations.

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The Full Particle-Collocation Algorithm

A fully Lagrangian, mesh-free solver for Mean Field Games.

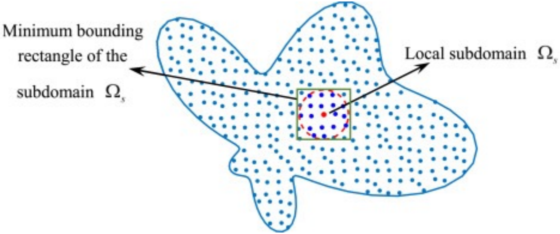


Figure: Illustration of the particle-collocation concept.

Approximating Derivatives without Grid

We can approximate derivatives at a particle X_{j_0} by using the values of u at its neighboring particles.

1. For a particle X_{j_0}, identify its neighbors $\{X_{j_l}\}$ within a radius δ.
2. Assume the value at a neighbor is given by a Taylor expansion around X_{j_0}:

$$u(X_{j_l}) \approx u(X_{j_0}) + \nabla u(X_{j_0}) \cdot (X_{j_l} - X_{j_0}) + \frac{1}{2}\langle D^2 u(X_{j_0})(X_{j_l} - X_{j_0}), X_{j_l} - X_{j_0}\rangle + \dots$$

3. Create a system of linear equations for each particle's neighborhood. Solving this system (e.g., via weighted least-squares) to find the unknown derivatives $(\nabla u, \nabla^2 u)$ at X_{j_0}.

Result: A method to solve the HJB equation that is completely mesh-free.

jiongyiwang@keio.jp

Iterative Fixed-Point Algorithm:

1. **HJB Solve (Collocation):** For a given particle distribution, solve for the value function u and its gradients ∇u at each particle location using the collocation method.

2. **Particle Evolution (SDE):** Use the computed gradients ∇u to define the optimal control, and evolve all particles forward one time step via the SDE.

3. **Repeat** until convergence.

jiongyiwang@keio.jp

Stability: Maximum Principle and Monotonicity

Maximum Principle: The solution $u(t, x)$ should satisfy the maximum principle, meaning it should not create new extrema during evolution.

- This is crucial for stability and physical realism.

Monotonicity: The numerical scheme should be monotone, meaning it does not introduce new extrema in the solution.

- This is essential for ensuring that the numerical solution converges to the correct physical behavior.

Monotonicity $\implies$ Discrete Maximum Principle.

- FDM (Kushner-Dupuis, Semi-Lagrangian) with upwinded scheme is monotone
- Hybrid method with particles is monotone
- Particle-collocation method is not naturally monotone.

jiongyiwang@keio.jp

Monotonicity Regained: A Fix

Solution: Use a monotone collocation scheme to approximate the derivatives.

- Use a weighted least-squares fit that preserves the maximum principle.
- This ensures that the computed gradients ∇u are consistent with the maximum principle.

Theorem (A Monotonicity Theorem, contribution of this thesis)

The constraint-optimized particle-collocation method is monotone, preserving the structure of the MFG system.

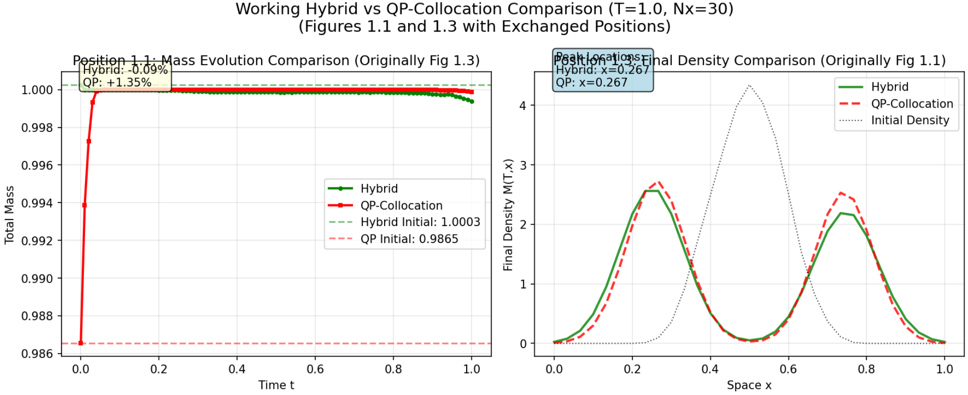
jiongyiwang@keio.jp

A Theorem on Numerical Order

Theorem (Godunov's order barrier theorem)

Linear numerical schemes for solving PDEs, having the property of not generating new extrema (monotonicity), can be at most first-order accurate.

jiongyiwang@keio.jp



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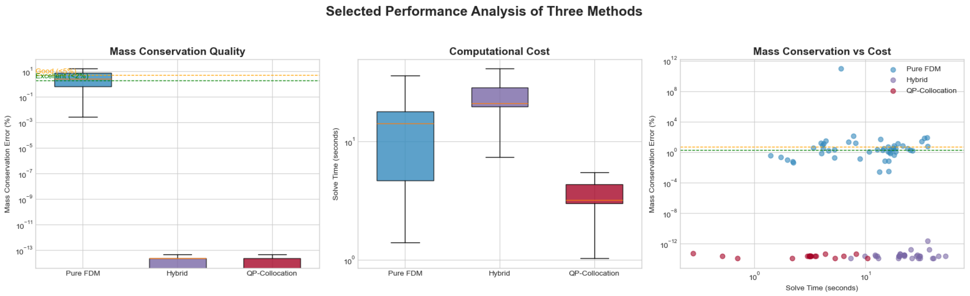

Figure: Pressure test: Quality of mass conservation and computational cost of three methods under various extreme conditions.

jiongyiwang@keio.jp

Summary of Contributions

- We highlighted a key structural flaw (lack of mass conservation) in standard finite difference methods for certain MFG problems.
- We demonstrated that a hybrid particle method successfully preserves these crucial physical properties.
- We proposed a novel and general **particle-collocation framework**, which provides a fully mesh-free and structure-preserving approach to solving MFG systems.
- This method is naturally suited for parallel computation, as the collocation step can be performed independently for each particle.

jiongyiwang@keio.jp

Known Issues

The convergence criterion for all particle-based methods need to be carefully chosen:

- Oscillations in the density due to the discrete nature of particles and noise.
- High sensitivity of the HJB equation to the distribution of particles.

Commnon issue for all stochastic particle methods.

jiongyiwang@keio.jp

Future Work

- Complete the full convergence analysis for the coupled particle-collocation framework.
- Implement and test the framework on higher-dimensional MFG problems.
- Explore adaptive techniques, such as adding or removing particles in regions of low/high density to improve accuracy and efficiency.
- Further investigate the connection to other numerical solvers (e.g., based on Finite Element Methods) to enhance theoretical guarantees.

jiongyiwang@keio.jp

Thank you for your attention!

Outline

jiongyiwang@keio.jp

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jiongyiwang@keio.jp