Peer-graded Assignment #1

Last Updated on: September 12, 2020

Contents

1 Problem 1, sets in the complex plane

$\boxed{1.}$ $\Gamma_1 = \{z \in \mathbb{C} : |z - 3 - 2i| \leq 1\}$

Solution: $\Gamma_1 = \overline{B(x, 1)}$ is the closed ball with radius 1 centered at $x = 3 + 2i$ on the complex plane $\mathbb{C}$. Just noticed the boundray $\partial\Gamma_1 \subset \Gamma_1$. ■

$\boxed{2.}$ $\Gamma_2 = \{z \in \mathbb{C} : \operatorname{Im} z = 2\}$

Solution: There is no constraint on the real part of z, however the only value allowed in the imaginary part is 2, Γ_2 correspond to a line in complex plane. ■

$\boxed{3.}$ $\Gamma_3 = \{z \in \mathbb{C} \setminus \{0\} : 0 < \arg z < \pi/6\}$

Solution: Use the fact that $\{z : \operatorname{Arg} z = \theta\}$ is a line passing $(0, 0)$ and with an angle θ between (with respect to the positive direction of) real axis. Noticed that $z \in \mathbb{C} \setminus \{0\}$ indiactes that $(0, 0)$ is excluded on the graph. Γ_3 is open in $\mathbb{C}$ hence the boundray is not attained. ■

$\boxed{4.}$ $\Gamma_4 = \{z \in \mathbb{C} : |z - 1| < |z|\}$

Solution: Let $z = x + iy$ s.t $x, y \in \mathbb{R}$. By squaring two sides,

$$\begin{aligned} |z-1|^2 \leq |z|^2 &\implies (x-1)^2 + y^2 \leq x^2 + y^2 \\ &\implies x - \frac{1}{2} \geq 0 \end{aligned} \tag{1.1}$$

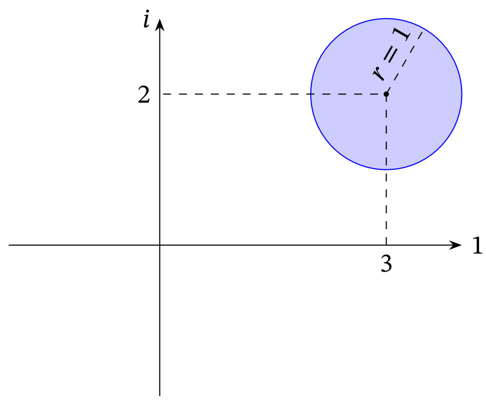
that means Γ_4 indiactes the hypograph of function $x = \frac{1}{2}$. It's also open under the usual topology of $\mathbb{C}$. ■

We could hereby draw all 4 graphs together in the Figure 1.

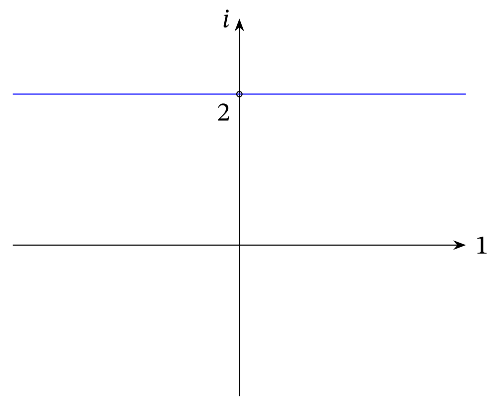
2 Problem 2, The nth roots of unity

By definithon, the nth roots of a number z is w that satisfy the equation:

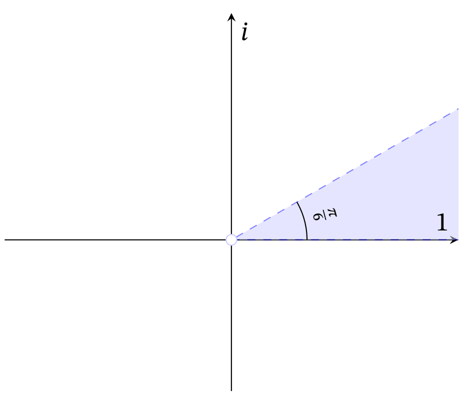
$$w^n = re^{i\theta}, \quad \theta = \operatorname{Arg} z \in [-\pi, \pi) \tag{2.1}$$



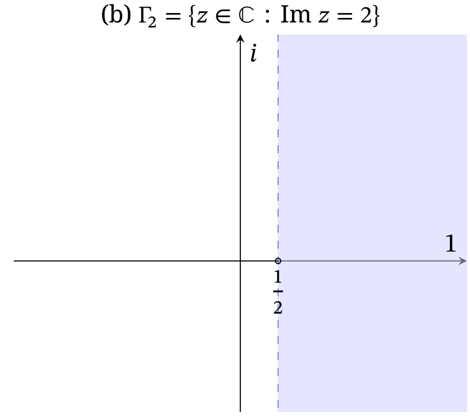
(a) $\Gamma_1 = \{z \in \mathbb{C} : |z - 3 - 2i| \leq 1\}$



(b) $\Gamma_2 = \{z \in \mathbb{C} : \operatorname{Im} z = 2\}$

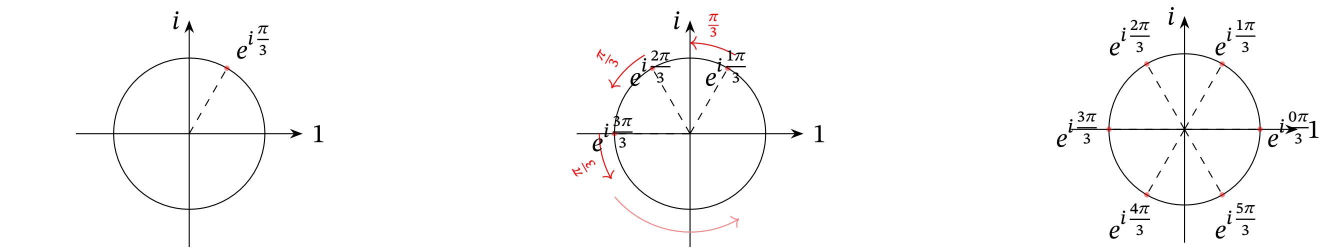


(c) $\Gamma_3 = \{z \in \mathbb{C} \setminus \{0\} : 0 < \arg z < \pi/6\}$



(d) $\Gamma_4 = \{z \in \mathbb{C} : |z - 1| < |z|\}$

Figure 1: The four graphs in Problem 1



(a) pick and draw the generator $e^{\frac{i\pi}{3}}$

(b) rotate counterclockwise with angle $\frac{\pi}{3}$ and repeat

(c) all 6th roots of unity

Figure 2: A feasible process to draw 6th unit roots

simply noticed that we also require that $\mathrm{Arg} w \in [-\pi, \pi)$:

$$w = \left\{ \sqrt[n]{r} e^{\frac{i\theta}{n} + \frac{2k\pi}{n}} \ : \ k \in \{0, 1, \dots, n-1\} \right\} \tag{2.2}$$

in our case to find nth unit roots, it equals to solve equation 2.1 with $r = 1, \theta = 0$, and our desired results can be directly inferred from formula 2.2:

$$U_n = \left\{ e^{\frac{2k\pi}{n}} \ : \ k \in \{0, 1, \dots, n-1\} \right\} \tag{2.3}$$

it's quite obvious that $\# U_n = n$, thus all 6th roots of unity are enumerated in U_6, which is

$$U_6 = \left\{ e^0, e^{\frac{i\pi}{3}}, e^{\frac{2i\pi}{3}}, e^{i\pi}, e^{\frac{4i\pi}{3}}, e^{\frac{5i\pi}{3}} \right\} \tag{2.4}$$

2.1 How to draw roots of unit

Although the polar coordinate of complex number $z = x + iy$ can be calculate with the help of $\cos\theta + i \sin\theta = e^{i\theta}$, it's a little bit cumbersome to decide explicitly the value of θ:

$$\theta = \begin{cases} \arccos \frac{x}{r} & y \geq 0, r \neq 0 \\ -\arccos \frac{x}{r} & y < 0 \\ NaN & r = 0 \end{cases} \quad \Bigg| \quad r = \sqrt{x^2 + y^2} \tag{2.5}$$

Espacially when the value of $\frac{x}{r}$ is not such special as $1, \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}$, etc. Intutively U_6 is cyclotomic, so its elements are uniformly distributed over the unit circle, that's sufficient for us to locate their position. To get a little bit more algebraic (and rigorous), U_n is essentially a cyclic group of order n, one of its generators is $e^{\frac{i\pi}{n}}$ (there are many generators in the same cyclic group, in our case any $e^{\frac{ik\pi}{3}}$ with $\gcd(k, 6) = 1$ is a generator, our construction can be started from any generator). Thus, when we draw $e^{\frac{ik\pi}{3}}$, it means that we rotate counterclockwise the generator $e^{\frac{1\pi}{3}}$ by k times. We show this process in Figure 2.