
GAME
Theory

Game Theory

Game theory is a mathematical and cognitive science that analyzes the strategic decisions of a player, taking into account the reactions of other players. Essentially, it is a discipline that examines the process of players optimizing their decisions based on the decisions of other players. This can be applied in various social, economic, and political contexts, modeling various situations and interactions.

Game theory is often studied using mathematical models and graphs. The fundamental elements of a game typically include players, strategies, and rewards (or gains). Essentially, for any given strategy chosen by each player, there is a matrix representing the gains or losses of the players.

- An action is a move you can make at a stage in a game.
- A strategy is a plan conditional on any possible contingency.
- An equilibrium is a set of strategies such that neither player wishes to deviate.
- A static game is one that is played just once at the same time.
- A dynamic game is one in which players move sequentially or repeatedly.

Game theory can be broadly categorized into two main types:

1. **Zero-Sum Games**
2. **Non-Zero-Sum Games**

Zero-Sum Games

In game theory, a zero-sum game refers to a type of game where the total amount of utility or payoff remains constant, and any gain by one player is offset by an equivalent loss by another player. The term "zero-sum" implies that the overall amount of wealth, resources, or utility in the system remains unchanged, akin to a constant sum.

Let's explore different types of zero-sum games:

1. **Constant-Sum Games:**

•Constant-sum games are a specific type of zero-sum game where the total payoff remains constant, but the payoff for each player is not necessarily zero. The critical aspect is that the sum of payoffs across all players remains fixed.

2. **Symmetric Zero-Sum Games:**

•In symmetric zero-sum games, the payoff matrix is symmetric. This means that if player A receives a certain payoff for a particular strategy against player B, then player B receives the same payoff for the reverse situation. Chess and checkers are examples of symmetric zero-sum games.

3. **Non-Zero-Sum Games:**

•While zero-sum games imply that the total payoff is constant, non-zero-sum games allow for variations in the total payoff. In these games, cooperation between players can lead to outcomes where the sum of payoffs is not fixed.

4. **Matrix Games:**

•Zero-sum games are often represented as matrix games, where the players' strategies and corresponding payoffs are presented in a matrix format. This representation simplifies the analysis of different strategies and outcomes.

5. **Strictly Determined Games:**

•Strictly determined zero-sum games are those where one player has a dominant strategy, forcing the other player to adopt a particular strategy. This dominance makes the outcome of the game predictable.

6. **Mixed Strategies:**

•In mixed strategies, players randomize their choices based on probabilities. While individual plays may not be zero-sum, the expectation of the payoffs over repeated plays is zero-sum.

7. **Extensive Form Games:**

•Zero-sum games can also be represented in extensive form, where the players make sequential moves, and the game unfolds over a series of actions and reactions. Poker is an example of a zero-sum game often analyzed in extensive form.

8. **Pareto Optimal Outcomes:**

•Zero-sum games may have Pareto optimal outcomes where it is not possible to improve the payoff of one player without reducing the payoff of the other. Finding Pareto optimal solutions is essential in cooperative scenarios.

9. **Repeated Games:**

•In repeated zero-sum games, players interact with each other over multiple rounds, influencing each other's decisions over time. Strategies may evolve based on past interactions.

A key question in game theory is to reason about the behavior we should expect to see when players take part in a given game.

Key Characteristics:

1.Constant Total Payoff:

The fundamental characteristic of a zero-sum game is that the sum of the players' payoffs or utilities is fixed. Any gain by one player must be balanced by an equal loss suffered by another player.

2.Competitive Nature:

Zero-sum games are inherently competitive. The interests of the players are directly opposed, and one player's success is directly tied to another player's failure.

Nash Equilibrium

Nash Equilibrium is a central concept in game theory, introduced by mathematician and Nobel laureate John Nash. In the context of a strategic interaction among multiple decision-makers (players), Nash Equilibrium represents a state where no player has an incentive to unilaterally deviate from their chosen strategy given the strategies chosen by the other players.

Key Elements:

1. **Strategic Interaction:**
   Nash Equilibrium applies to situations where each player's action affects the outcomes for all players. It is a strategic setting where individuals or entities make decisions considering the actions of others.
2. **Simultaneous Decision-Making:**
   In many scenarios, players make decisions simultaneously, not knowing the choices of others. Each player aims to maximize their own payoff based on the assumption that others' strategies remain unchanged.
3. **No Unilateral Deviation:**
   In Nash Equilibrium, no player can improve their payoff by changing their strategy independently. Given the strategies chosen by other players, each player's decision is optimal, and any deviation would result in a suboptimal outcome.
4. **Mutual Consistency:**
   Nash Equilibrium is a mutual consistency of strategies. It's a situation where, once everyone knows the strategies chosen by others, no player wishes to change their strategy unilaterally.

Types of Nash Equilibrium:

1. **Pure Nash Equilibrium:** In a pure Nash Equilibrium, each player chooses a specific, non-randomized strategy.
2. **Mixed Nash Equilibrium:** Players may use randomization or mixed strategies to determine their actions. The

The Prisoner's Dilemma

- Scenario: the police have arrested two suspects for a crime.
  - They tell each prisoner they'll reduce his/her prison sentence if he/she betrays the other prisoner.
  - Each prisoner must choose between two actions:
  - cooperate with the other prisoner, i.e., don't betray him/her
  - defect (betray the other prisoner).
- Payoff = – (years in prison):

Prisoner's Dilemma

Agent 1 \ Agent 2	C	D
C	–2, –2	–5, 0
D	0, –5	–4, –4

- Each player has only two strategies, each of which is a single action
- Non-zero-sum
- Imperfect information: neither player knows the other's move until after *both* players have moved

Strategies

- Suppose the agents agent 1, agent 2, ..., agent n
- For each i, let S_i = {all possible strategies for agent i}
  - s_i will always refer to a strategy in S_i
- A **strategy profile** is an n-tuple $S = (s_1, \ldots, s_n)$, one strategy for each agent
- **Utility** $U_i(S)$ = payoff for agent i if the strategy profile is S
- s_i **strongly dominates** s_i' if agent i always does better with s_i than s_i'

$$\forall\ s_1, \ldots, s_{i-1},\ s_{i+1}, \ldots, s_n,$$
$$U_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) > U_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$$

- s_i **weakly dominates** s_i' if agent i never does worse with s_i than s_i', and there is at least one case where agent i does better with s_i than s_i',

$$\forall\ s_1, \ldots, s_{i-1},\ s_{i+1}, \ldots, s_n,\ U_i(\ldots, s_i, \ldots) \geq U_i(\ldots, s_i', \ldots)$$

and

$$\exists\ s_1, \ldots, s_{i-1},\ s_{i+1}, \ldots, s_n\quad U_i(\ldots, s_i, \ldots) > U_i(\ldots, s_i', \ldots)$$

(Strict) Best Response

"Best Response" is a strategy employed by a player with the aim of maximizing profit or utility based on the strategies chosen by other players. The best response of a player is determined in reaction to the strategies of other players and allows the player to optimize their own decision.

The key features of Best Response are as follows:

Maximum Benefit: The best response of a player is the strategy that provides the player with the highest possible profit or utility, taking into account the strategies of other players.

Reactive Decision: The best response of a player is a reactive decision based on the current strategies of other players. It involves adaptation to the choices of other players.

Optimal Strategy: Best response represents the optimal strategy that enables the player to do the best in the current situation. This strategy represents the most effective response to the actions of other players.

Relationship with Nash Equilibrium: The best response of a player can contribute to the existence of a Nash Equilibrium in the game. Nash Equilibrium refers to a situation where each player's best

Bimatrix games (Notation)

- P_1: payoff matrix for player 1
- P_2: payoff matrix for player 2
- When player 1 plays S and player 2 plays T:
  - player 1 gets payoff $P_1(S,T)$
  - player 2 gets payoff $P_2(S,T)$

(Strict) Best Response (Definition)

- Strategy S is a **best response** for player 1 to strategy T of player 2 if $P_1(S,T) \geq P_1(S',T)$ for **all** strategies S' of player 1.
- Strategy S is a **strict best response** for player 1 to strategy T of player 2 if $P_1(S,T) > P_1(S',T)$ for **all** strategies $S' \neq S$ of player 1.

Dominant Strategy

Dominant Strategy is a strategy for a player that, regardless of the strategies chosen by other players, maximizes their own payoff. This strategy ensures that, no matter what strategy the player decides to follow, their payoff will always be the highest compared to other available strategies.

Key features of Dominant Strategy include:

1. **Maximizes Payoff:** The dominant strategy for a player is the one that, without considering the strategies chosen by other players, provides the highest payoff in every situation.
2. **Independent Decision-Making:** Dominant strategy implies that a player makes decisions independently of the current strategies of other players. It is expected that the player consistently chooses the same strategy in every situation.
3. **Renders Other Strategies Irrelevant:** A dominant strategy indicates that the player is better off than with any other strategy, making other available strategies less relevant or suboptimal.
4. **Beneficial Outcome for Everyone:** Since the player employing the dominant strategy achieves the best possible outcome for themselves, the situation where this strategy is chosen becomes the most favorable for all players involved.

Dominant strategy is a crucial concept in the analysis of game theory, representing a powerful strategic position for a player to maximize their gains.

Bimatrix games (Notation)

- P_1: payoff matrix for player 1
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- When player 1 plays S and player 2 plays T:
  - player 1 gets payoff $P_1(S,T)$
  - player 2 gets payoff $P_2(S,T)$

(Strictly) Dominant Strategy (Definition)

- Strategy S is a **dominant strategy** for player 1 if $P_1(S,T) \geq P_1(S',T)$ for **all** strategies S' of player 1, and **all** strategies T of player 2.
- Strategy S is a **strictly dominant strategy** for player 1 if $P_1(S,T) > P_1(S',T)$ for **all** strategies $S' \neq S$ of player 1, and **all** strategies T of player 2.

- A dominant strategy is a best response to all of the opponent's strategies.

Battle of the Sexes

- Suppose both agents randomize, and the husband's mixed strategy s_h is

  $s_h(\text{Opera}) = p; \quad s_h(\text{Football}) = 1 - p$

- Expected utilities of the wife's actions:

  $$U_w(\text{Football}, s_h) = 0p + 1(1 - p)$$
  $$U_w(\text{Opera}, s_h) = 2p$$

Wife \ Husband	Opera	Football
Opera	2, 1	0, 0
Football	0, 0	1, 2

- If the wife mixes between her two actions, they must have the same expected utility
  - ➢ If one of the actions had a better expected utility, she'd do better with a pure strategy that *always* used that action
  - ➢ Thus $0p + 1(1 - p) = 2p$, so $p = 1/3$
- So the husband's mixed strategy is $s_h(\text{Opera}) = 1/3; \quad s_h(\text{Football}) = 2/3$

Battle of the Sexes

- A similar calculation shows that the wife's mixed strategy s_w is

  $$s_w(\text{Opera}) = 2/3, \quad s_w(\text{Football}) = 1/3$$

Wife \ Husband	Opera	Football
Opera	2, 1	0, 0
Football	0, 0	1, 2

- In this equilibrium,
  - P(wife gets 2, husband gets 1) $= (2/3)\ (1/3) = 2/9$
  - P(wife gets 1, husband gets 2) $= (1/3)\ (2/3) = 2/9$
  - P(both get 0) $= (1/3)(1/3) + (2/3)(2/3) = 5/9$
- Thus the expected utility for each agent is 2/3
- Pareto-dominated by both of the pure-strategy equilibria
  - In each of them, one agent gets 1 and the other gets 2

Pareto-Optimality: The first definition is Pareto-optimality, named after the Italian economist Vilfredo Pareto who worked in the late 1800's and early 1900's. A choice of strategies — one by each player — is Pareto-optimal if there is no other choice of strategies in which all players receive payoffs at least as high, and at least one player receives a strictly higher payoff.

Finding Nash Equilibria

Matching Pennies

- Each agent has a penny
- Each agent independently chooses to display his/her penny heads up or tails up
- Easy to see that in this game, no pure strategy could be part of a Nash equilibrium
  - For each combination of pure strategies, one of the agents can do better by changing his/her strategy
    - for (Heads,Heads), agent 2 can do better by switching to Tails
    - for (Heads,Tails), agent 1 can do better by switching to Tails
    - for (Tails,Tails), agent 2 can do better by switching to Heads
    - for (Tails,Heads), agent 1 can do better by switching to Heads
- But there's a mixed-strategy equilibrium:
  - (s,s), where $s(\text{Heads}) = s(\text{Tails}) = ½$

Agent 2 / Agent 1	Heads	Tails
Heads	1, –1	–1, 1
Tails	–1, 1	1, –1

FACILITY LOCATION

Figure 6.19: In the Facility Location Game, each player has strictly dominated strategies but no dominant strategy.

- You can build your facility at location A, C or D.
- Your competitor can build their facility at B, D, or F.
- Each location contains customers, who will buy from nearest facility.
- Goal: Maximize number of customers at your facility.
- Where would you not build your facility?

A strategy is **strictly dominated** if there is some other strategy available to the same player that produces a strictly higher payoff in response to every choice of strategies by the other players.

		Firm 2		
		B	D	F
Firm 1	A	1, 5	2, 4	3, 3
	C	4, 2	3, 3	4, 2
	E	3, 3	2, 4	5, 1

Figure 6.20: Facility Location Game

- A is strictly dominated (by C).
- F is strictly dominated (by D).
- Strictly dominated strategies are never part of a Nash Equilibrium.
- When searching for Nash equilibria, we can remove them.
- Among remaining strategies, C and D are dominant.
- (C,C) is a pure Nash equilibrium.

		Firm 2	
		B	D
Firm 1	C	4,2	3,3
	E	3,3	2,4

Figure 6.21: Smaller Facility Location Game

EXPECTED VALUE

Expected value is a statistical measure that calculates the average outcome of a particular event, taking into account all possible outcomes and their probabilities. In game theory, it is calculated by multiplying the potential outcomes by their respective probabilities and adding them together.

Here's a general formula for calculating the expected value:

$$E(U) = \sum_i P_i \cdot U_i$$

Where:

- $E(U)$ is the expected value of the utility.
- P_i is the probability of outcome i.
- U_i is the utility associated with outcome i.

Consider a game where a player can choose between two strategies, A and B. The possible outcomes and associated utilities are as follows:

- **Strategy A:**
  - Outcome X with probability 0.3 and utility 4
  - Outcome Y with probability 0.7 and utility 2

- **Strategy B:**
  - Outcome X with probability 0.6 and utility 3
  - Outcome Y with probability 0.4 and utility 1

The expected value for Strategy A would be:

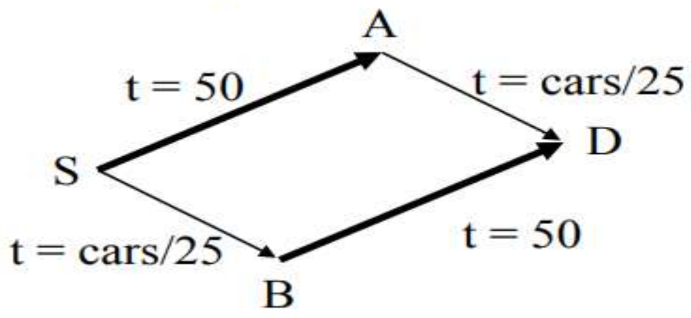
$E(UA)=(0.3\cdot 4)+(0.7\cdot 2)=1.2+1.4=2.6$$E(UA)=(0.3\cdot 4)+(0.7\cdot 2)=1.2+1.4=2.6$

Similarly, calculate the expected value for Strategy B.

NETWORK ROUTING GAMES

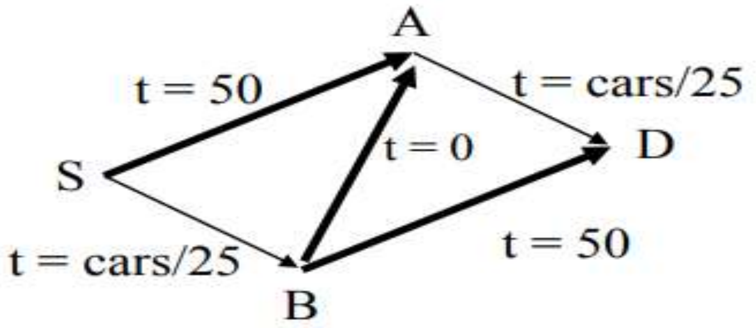
Theorem 1

Every network routing game contains at least one pure Nash equilibrium.



- **Road Networks**
- Suppose that 1,000 drivers wish to travel from *S* (start) to *D* (destination)
  - Two possible paths:
    - $S \rightarrow A \rightarrow D$ and $S \rightarrow B \rightarrow D$
  - The roads $S \rightarrow A$ and $B \rightarrow D$ are very long and very wide
    - $t = 50$ minutes for each, no matter how many drivers
  - The roads $S \rightarrow B$ and $A \rightarrow D$ are very short and very narrow
    - Time for each = (number of cars)/25
  - Nash equilibrium:
    - 500 cars go through A, 500 cars through B
    - Everyone's time is 50 + 500/25 = 70 minutes
    - If a single driver changes to the other route
      - There now are 501 cars on that route, so his/her time goes up

Braess's Paradox



- Suppose we add a new road from B to A
- It's so wide and so short that it takes 0 minutes
- New Nash equilibrium:
  - All 1000 cars go S→B→A→D
  - Time is 1000/25 + 1000/25 = 80 minutes
- To see that this is an equilibrium:
  - If driver goes *S→A→D*, his/her cost is 50 + 40 = 90 minutes
  - If driver goes *S→B→D*, his/her cost is 40 + 50 = 90 minutes
  - Both are dominated by S→B→A→D
- To see that it's the ***only*** Nash equilibrium:
  - For every traffic pattern, compute the times a driver would get on all three routes
  - In every case, S→B→A→D dominates *S→A→D* and *S→B→D*
- Carelessly adding capacity can actually be hurtful!

Social Cost

- Social cost refers to the overall cost or disutility associated with a particular outcome in a game, taking into account the preferences or utilities of all players.
- It is a measure of the collective welfare or well-being of all participants in the game.
- The goal is often to minimize the social cost to achieve a more socially desirable or efficient outcome.

Price of Anarchy

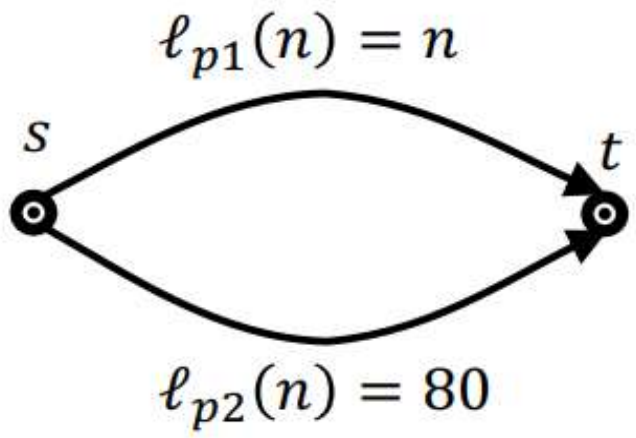
- The price of anarchy quantifies the degradation in the social cost or efficiency of a system when individuals pursue their own self-interest rather than a socially optimal outcome.
- It is the ratio of the social cost of the worst-case Nash equilibrium (where each player independently optimizes their own utility) to the social cost of a socially optimal outcome (where a central authority optimizes the collective welfare).
- A higher price of anarchy indicates a larger gap between the individualistic behavior of players and the socially optimal outcome.

(Pure) Nash Equilibrium

A strategy profile is a **Nash equilibrium** if no player has incentive to unilaterally deviate.

Example 3

100 players



Denote by x be number of players on p1 and $100 - x$ on p2.
In **Nash equilibrium**, we have:

- $\ell_{p1}(x) \leq \ell_{p2}(100 - x + 1)$
- $\ell_{p2}(100 - x) \leq \ell_{p1}(x + 1)$

Hence, $x \leq 80$ and $80 \leq x + 1$.
In Nash equilibrium 79 or 80 players on p1.

Optimal solution minimizes

$$f(x) = x \cdot \ell_{p1}(x) + (100 - x) \cdot \ell_{p2}(100 - x)$$
$$= x^2 + (100 - x) \cdot 80 = x^2 - 80x + 100$$

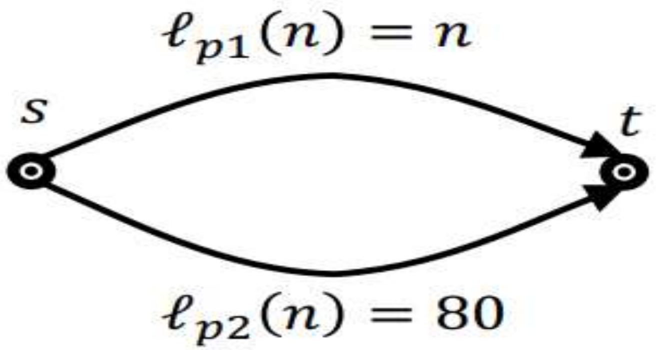
Which x minimizes $f(x)$?
$2x - 80 = 0$. In optimal solution 40 players on p1.

Price of Anarchy (Definition)

Price of Anarchy (PoA) is the social cost of a **worst** NE compared to the optimal social cost.

Example 3

100 players



Cost of NE with $x = 80$:

$$80 \cdot \ell_{p1}(80) + 20 \cdot \ell_{p2}(20) = 6400 + 1600 = 8000$$

Cost of NE with $x = 79$:

$$79^2 + 21 \cdot 80 = 7921$$

Cost of optimal solution with $x = 40$:

$$40^2 + 60 \cdot 80 = 6400$$

Price of Anarchy:

$$\frac{8000}{6400} = 1.25$$

Theorem 2

For any network routing game with linear latency functions, $\mathrm{PoA} \leq 2.5$

Price of Stability

The "Price of Stability" (PoS) is a concept in game theory that measures how much worse the total cost in a Nash Equilibrium of a game is compared to the globally best-known cost.

•**Price of Stability (PoS) = 1:** In this case, the total cost in the Nash Equilibrium of the game is the same as the total cost in the globally best-known situation.

•**Price of Stability (PoS) > 1:** If PoS is greater than 1, it indicates that the total cost in the Nash Equilibrium of the game is worse than the total cost in the globally best-known situation.

The Price of Stability is used to assess how socially stable equilibrium points are in a system. This is particularly important in resource allocation problems, network

Theorem 3

For any network routing game with linear latency functions, $\mathrm{PoS} \leq 2$