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Middle East Technical University Department of Computer Engineering

Discrete Computational Structures
Take Home Exam 1

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Question 1 (7 pts)

a) Construct a truth table for the following compound proposition.

$$(q \to \neg p) \leftrightarrow (p \leftrightarrow \neg q)$$

(3.5/7 pts)

p	q	$\neg p$	$\neg q$	$q \to \neg p$	$p \leftrightarrow \neg q$	$(q \to \neg p) \leftrightarrow (p \leftrightarrow \neg q)$
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	F	F

b) Show that whether the following conditional statement is a tautology by using a truth table.

$$[(p \vee q) \wedge (r \rightarrow p) \wedge (r \rightarrow q)] \rightarrow r$$

(3.5/7 pts)

Truth Table for $a \rightarrow r$ where $a = (p \vee q) \wedge (r \rightarrow p) \wedge (r \rightarrow q)$

p	q	r	$\neg p$	$\neg q$	$\neg r$	$p \vee q$	$r \rightarrow p$	$r \rightarrow q$	$(p \vee q) \wedge (r \rightarrow p)$	a	$a \rightarrow r$
T	T	T	F	F	F	T	T	T	T	T	T
T	T	F	F	F	T	T	T	T	T	T	F
T	F	T	F	T	F	T	T	F	T	F	T
T	F	F	F	T	T	T	T	T	T	T	F
F	T	T	T	F	F	T	F	T	F	F	T
F	T	F	T	F	T	T	T	T	T	T	F
F	F	T	T	T	F	F	F	F	F	F	T
F	F	F	T	T	T	F	T	T	F	F	T

Therefore, the conditional statement is not a tautology.

Question 2 (8 pts)

Show that $(p \to q) \land (p \to r)$ and $(\neg q \lor \neg r) \to \neg p$ are logically equivalent. Use tables 6,7 and 8 given under the section "*Propositional Equivalences*" in the course textbook and give the reference to the table and the law in each step.

$(p \to q) \land (p \to r)$	$(\neg q \lor \neg r) \to \neg p$
$(\neg p \lor q) \land (\neg p \lor r)$ (Table 7) $\neg p \lor (q \land r)$ (Table 6 - Distributive laws) $(q \land r) \lor \neg p$ (Table 6 -Commutative laws)	$\neg(\neg q \lor \neg r) \lor \neg p$ (Table 7) $(q \land r) \lor \neg p$ (Table 6 - De Morgan's laws)

$(q \land r) \lor \neg p \equiv (q \land r) \lor \neg p$ Therefore, $(p \to q) \land (p \to r) \equiv (\neg q \lor \neg r) \to \neg p$.

Question 3 (30 pts, 2.5 pts each)

Let F(x, y) mean that x is the father of y; M(x, y) denotes x is the mother of y. Similarly, H(x, y), S(x, y), and B(x, y) say that x is the husband/sister/brother of y, respectively. You may also use constants to denote individuals, like Sam and Alex. You can use $\lor, \land, \to, \neg, \forall, \exists$ rules and quantifiers. However, you are not allowed to use any predicate symbols other than the above to translate the following sentences into predicate logic. $\exists!$ and exclusive-or (XOR) quantifiers are forbidden:

[label=**0**)]Everybody has a mother. Everybody has a father and a mother. Whoever has a mother has a father. Sam is a grandfather. All fathers are parents. All husbands are spouses. No uncle is an aunt. All brothers are siblings. Nobody's grandmother is anybody's father. Alex is Ali's brother-in-law. Alex has at least two children. Everybody has at most one mother.

1) $\forall y \exists x M(x, y)$
2) $\forall y[\exists x M(x, y) \land \exists z F(z, y)]$
3) $\forall y[\exists x M(x, y) \to \exists z F(z, y)]$
4) $\exists y[F(Sam, y) \land ((\exists x M(y, x) \lor \exists z F(y, z))]$
5) $\forall x[\exists y F(x, y) \to \exists z(F(x, z) \lor \exists t M(t, z))]$
6) $\forall x \exists y[H(x, y) \to (H(x, y) \lor H(y, x))]$
7) $\forall x \exists y \exists z \exists t \exists m[(B(x, y) \land (F(y, z) \lor M(y, z))) \to \neg(S(x, t) \land (F(t, m) \lor M(t, m)))]$
8) $\forall x \forall y \exists t \exists z[B(x, y) \to ((M(t, x) \land M(t, y)) \lor (F(z, x) \land F(z, y)))]$
9) $\forall x \exists y \exists t \exists z \exists m[(M(x, y) \land (M(y, t) \lor F(y, z))) \to \neg F(x, m)]$
10) $\exists x[H(Alex, x) \land B(Ali, x)]$
11) $\exists x \exists y[F(Alex, x) \land F(Alex, y)]$
12) $\forall z \exists x \exists y[M(x, z) \land \neg M(y, z)]$

Question 4 (25 pts)

Prove the following claim by natural deduction. Use only the natural deduction rules $\vee$, $\wedge$, $\rightarrow$, $\neg$ introduction and elimination. If you attempt to make use of a lemma or equivalence, you need to prove it by natural deduction too.

a) $p \rightarrow q, r \rightarrow s \vdash (p \vee r) \rightarrow (q \vee s)$

(12.5/25 pts)

1[illegible].

1 :	$p \rightarrow q$	Premise
2 :	$r \rightarrow s$	Premise
	3 : p	Assumption
	4 : r	Assumption
	5 : q	1,3 $\rightarrow$ e
	6 : s	2,4 $\rightarrow$ e
	7 : $p \vee r$	3,4 $\vee i$
	8 : $q \vee s$	5,6 $\vee i$
9:	$(p \vee r) \rightarrow (q \vee s)$	38 $\rightarrow i$

b) $(p \rightarrow (r \rightarrow \neg q)) \rightarrow ((p \wedge q) \rightarrow \neg r)$

(12.5/25 pts)

1 :	$(p \rightarrow (r \rightarrow \neg q)) \rightarrow ((p \wedge q) \rightarrow \neg r)$	Assumption
2 :	$p \rightarrow (r \rightarrow \neg q)$	Assumption
3 :	$(p \wedge q) \rightarrow \neg r$	1,2$\rightarrow$ e
	4 : p	Assumption
	5 : q	Assumption
	6 : r	Assumption
	7 : $p \wedge q$	4,5 $\wedge i$
	8 : $\neg r$	3,7 $\rightarrow$ e
	9 : $\perp$	6,8 $\neg$ e
	10 : $\neg r$	6,8,9 $\neg$ e
	11 : $(p \wedge q) \rightarrow \neg r$	7,10 $\rightarrow$ i
12 :	$(p \rightarrow (r \rightarrow \neg q)) \rightarrow ((p \wedge q) \rightarrow \neg r)$	2,4-11 $\rightarrow i$

Question 5 (30 pts)

Prove the following claim by natural deduction. Use only the natural deduction rules $\vee$, $\wedge$, $\rightarrow$, $\neg$ introduction and elimination. If you attempt to make use of a lemma or equivalence, you need to prove it by natural deduction too.

a) $\forall x P(x) \vee \forall x Q(x) \vdash \forall x (P(x) \vee Q(x))$

(12.5/25 pts)

$1:$	$\forall x P(x) \vee \forall x Q(x)$	Premise
	$2:\ a \mid P(a)$	$1\ \forall x e$
	$3:\ a \mid Q(a)$	$1\ \forall x e$
	$4:\ a \mid P(a) \vee Q(a)$	$2{,}3\ \vee e$
5:	$\forall x (P(x) \vee Q(x))$	2-4 $\forall x i$

b) $\forall x P(x) \rightarrow S \vdash \exists x (P(x) \rightarrow S)$

(17.5/25 pts)

$1:$	$\forall x P(x) \rightarrow S$	Premise
	$2:\ a \mid P(a)$	$1\ \forall x e$
	$3:\ a \mid P(a) \rightarrow S$	$1, \rightarrow e$
4:	$\exists x (P(x) \rightarrow S)$	2,3 $\exists x i$