Lecture 2

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2 - Multiplication by Scalars:

Given any scalar number c and any m×n matrix A, the scalar multiplication is defined as follows:

$$(cA)(i,j) := ca_{ij} \quad \text{for all} \quad i = 1, \dots, m \ \& \ j = 1, \dots, n$$

Thm 1.2: Let, c, c_1, c_2 be scalar & A, A_1, A_2 be m×n - matrices. Then we have

1- $c.(A_1 + A_2) = cA_1 + cA_2$
2- $(c_1 + c_2)A = c_1 A + c_2 A$
3- $(c_1 c_2)A = c_1(c_2 A)$
4- $1.A = A$

3- Matrix Multiplication:

Consider

$$\left.\begin{aligned} 2u + v + w &= 5 \\ u + 2v + 3w &= 6 \\ -u + v - 2w &= 1 \end{aligned}\right\} \qquad \underset{3\times \boxed{3}}{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & -2 \end{bmatrix}} . \underset{\boxed{3}\times 1}{\begin{bmatrix} u \\ v \\ w \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}}$$

$$2u + v + w = 5$$
$$1u + 2v + 3w = 6$$
$$-u + v - 2w = 1$$

Defn: Let A be m×n - matrix & B be a n×r - matrix then define A.B as follows:

$$(AB)(i,j) := \sum_{k=1}^{n} a_{ik} b_{kj}$$

e.g. $A = \underset{2\times\boxed{2}}{\begin{bmatrix} 2 & 7 \\ -3 & 4 \end{bmatrix}}$ $\qquad B = \underset{\boxed{2}\times 3}{\begin{bmatrix} -1 & 7 & 0 \\ 0 & 3 & -1 \end{bmatrix}}$

$$\begin{bmatrix} 2 & 7 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 7 & 0 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 35 & -7 \\ 3 & -9 & -4 \end{bmatrix}$$

$$(AB)(1,1) = \sum^{2} a_{1k} b_{k1} = a_{11} b_{11} + a_{12} b_{21} = 2(-1) + 7.0 = -2$$

$$(AB)(1,1) = \sum_{k=1}^{2} a_{1k} b_{k1} = a_{11}b_{11} + a_{12}b_{21} = 2(-1) + 7.0 = -2$$

Thm 1.3: Let $A_{m\times n}$, $B_{n\times r}$ & $C_{r\times s}$ be matrices. $A'_{m\times n}$, $B'_{n\times r}$

1- $(AB)C = A(BC)$,
2- $A(B+B') = AB + AB'$
3- $(A+A')B = AB + A'B$
4- $I_{m\times m}.A = A$ & $A.I_{n\times n} = A$
5- $c(A.B) = (cA)B = A(cB)$ for any scalar c.

Sketch of Proof: 1- $$((AB).C)(i,j) = \sum_{k=1}^{r} (AB)_{ik} C_{kj} = \sum_{k=1}^{r}\left(\sum_{l=1}^{n} A_{il}B_{lk}\right) C_{kj}$$

$$= \sum_{l=1}^{n} A_{il} \underbrace{\left(\sum_{k=1}^{r} B_{lk} C_{kj}\right)} = \sum_{l=1}^{n} A_{il} (B.C)_{lj} = (A.(BC))(i,j)$$

for all $i=1,\dots,m$ $j=1,\dots,s$.

Rmk: If A is an $n\times n$- square matrix then $A.A := A^2$

$$\underbrace{A.A\dots A}_{k\text{-times}} = A^k \qquad A = A^1$$

Defn: Let A be $m\times n$- matrix. The "transpose" A^T of A is defined as $(A^T)(i,j) = a_{ji}$

e.g. $$A = \begin{bmatrix} -3 & 1 & 2 \\ -4 & 1 & 3 \end{bmatrix}_{2\times 3} \qquad A^T = \begin{bmatrix} -3 & -4 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}_{3\times 2}$$

Thm 1.4. If $A_{m\times n}$, $C_{m\times n}$, $B_{n\times r}$ are matrices then

1- $(A+C)^T = A^T + C^T$ proof: $$(A+C)^T(i,j) = (A+C)(j,i)$$
$$= A(j,i) + C(j+i)$$
$$= A^T(i,j) + C^T(i,j)$$
$$= (A^T+C^T)(i,j)$$

$$= (A^T + C^T)(i,j)$$

2- $(cA)^T = c.A^T$

3- $(A^T)^T = A$

4- $(A.B)^T = B^T.A^T$ <u>proof:</u> $(AB)^T(i,j) = (A.B)(j,i)$

$$= \sum_{k=1}^{n} a_{jk} b_{ki} = \sum_{k=1}^{n} b_{ki} a_{jk}$$

$$= \sum_{k=1}^{n} (B^T)(i,k)\,(A^T)(k,j)$$

$$= (B^T.A^T)(i,j) \blacksquare$$

<u>RMK:</u> 1) Given $A_{m\times n}$, $B_{n\times r}$ $A.B$ is defined but $B.A$ is defined only if $r=m$.

2) For square matrices of the same size then is $AB=BA$?

No, let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

$$A.B = \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix} \neq B.A = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$$

<u>1.3 SPECIAL MATRICES:</u>

A. Diagonal Matrices:

Let A be $n\times n$- square matrix. A is called "diagonal" matrix if

$$A(i,j) = \begin{cases} a_{ii} & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

$$A = \begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \end{bmatrix} = \text{diag}(a_{11}, a_{22}, ..., a_{nn})$$

$$A = \begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{nn} \end{bmatrix}_{n\times n} = \mathrm{diag}(a_{11}, a_{22}, \dots, a_{nn})$$

B. Triangular Matrices:

Defn: A square matrix $A_{n\times n}$ is said to be "upper triangular" if all the entries below the diagonal are zero.

i.e. $A(i,j) = 0$ for $j < i$. $A_{n\times n}$ is said to be "lower triangular" if all the entries above the diagonal are zero.

i.e. $A(i,j) = 0$ for $i < j$.

RMK: A diagonal matrix is both upper & lower triangular.

Thm: 1.5: The set of upper triangular (resp. lower triangular) matrices are closed under addition, scalar multiplication & matrix multiplication.

Proof: exc.

C. Symmetric & Skew-Symmetric Matrices:

Defn: A square matrix $A_{n\times n}$ which satisfies $A^T = A$ is called "symmetric", if $A^T = -A$ then it is called "skew-symmetric".

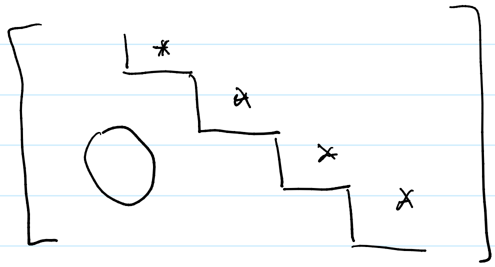
$$A^T = A \iff a_{ij} = a_{ji} \quad \forall i,j.$$

$A^T = -A \iff a_{ij} = -a_{ji} \quad \forall i,j. \Rightarrow$ if $i=j$ then $a_{ii} = -a_{ii} \iff a_{ii} = 0$

$\therefore$ for a skew-symmetric matrix has 0 on its diagonal.

D. Echelon Matrices:

Defn: Let $A_{m\times n}$. The first non-zero entry of the i-th row is called "leading entry" of the i-th row. If the leading entries of the matrix A are of the form $a_{1j_1}, a_{2j_2}, \dots, a_{rj_r}$ with $j_1 < j_2 < j_3 < \dots < j_r$



with $j_1 < j_2 < j_3 < \dots < j_r$

then A is called an "echelon matrix"

e.g. $\begin{bmatrix} \overset{a_{11}}{\underline{1}} & 0 & 2 & 1 & 5 \\ 0 & \overset{a_{22}}{\underline{2}} & 0 & 0 & 0 \\ 0 & 0 & \overset{a_{33}}{\underline{1}} & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 5}$ $\quad a_{11}, a_{22}, a_{33}$

$1 < 2 < 3$

"echelon matrix"

e.g. $\begin{bmatrix} \underline{1} & 0 & 2 & 1 & 5 \\ 0 & 0 & 0 & \underline{1} & 6 \\ 0 & 0 & 0 & 0 & \underline{2} \end{bmatrix}_{3\times 5}$ $\quad a_{11}, a_{24}, a_{35}$

$1 < 4 < 5$

"echelon matrix"

e.g. $\begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}_{3\times 5}$ "not echelon matrix"

Defn: An echelon matrix is called "row-reduced echelon" matrix if

i) each leading entry is 1

ii) each leading entry is the unique non-zero entry in its column.

e.g. $\begin{bmatrix} \underline{1} & 0 & 0 & 0 \\ 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ "row-reduced echelon"

e.g. $\begin{bmatrix} \underline{1} & 0 & 0 & 2 & 1 \\ 0 & \underline{1} & 0 & 1 & 5 \\ 0 & 0 & \underline{1} & 2 & 3 \end{bmatrix}$ "row-reduced echelon"

$$\begin{bmatrix} 0 & 0 & \underline{\underline{1}} & 2 & 3 \end{bmatrix}$$

e.g. $\begin{bmatrix} \underline{\underline{1}} & \cancel{2} & 1 & 0 & 0 \\ 0 & \underline{\underline{1}} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \underline{\underline{1}} \end{bmatrix}$ "echelon matrix"

<u>not</u> "row-reduced"