MIDDLE EAST TECHNICAL UNIVERSITY
DEPARTMENT OF COMPUTER ENGINEERING

CENG 384

Spring 2022
Homework 2

Regulations:

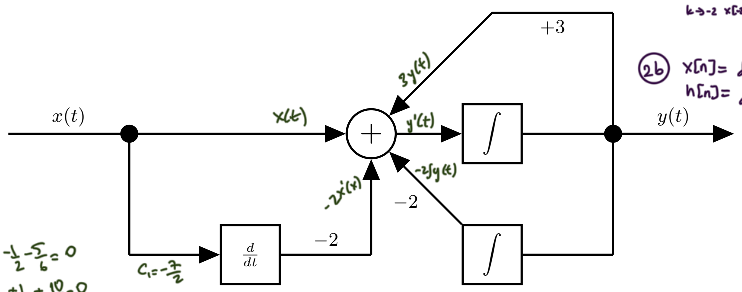
- **Grouping:** You are strongly encouraged to work in pairs.
- **Drawing Plots:** Clearly label the coordinate axes and make sure that your plots are not open to different interpretations.
- **Submission:** You need to submit a pdf file named 'hw2.pdf' to the odtuclass page of the course. You need to use the given template 'hw2.tex' to generate your pdf files. Otherwise you will receive zero.
- **Deadline:** 23:55, 17 April, 2022 (Sunday).
- **Late Submission:** Not allowed.

1. (30 pts) Consider an LTI system given by the following block diagram:

1-a
$y'(t) = x(t) - 2x'(t) + 3y(t) - 2\int y(t)$
$y''(t) = x'(t) - 2x''(t) + 3y'(t) - 2y(t)$
$y''(t) - 3y'(t) + 2y(t) = x'(t) - 2x''(t)$

1-b
$y(t) = y_H(t) + y_p(t)$
$y_H(t) = Ce^{\alpha t}$ $\qquad y_H = C_1 e^{2t} + C_2 e^{t}$
$C\alpha^2 e^{\alpha t} - 3C\alpha e^{\alpha t} + 2Ce^{\alpha t} = 0$
$\alpha^2 - 3\alpha + 2 = 0 \quad \alpha = 2,1$
$4e^{-2t}$
$y_p(t) = (Ae^{-t} + Be^{-2t})u(t)$
$Ae^{-t} + 4Be^{-2t} + 3Ae^{-t} + 6Be^{-2t} + 2Ae^{-t} + 2Be^{-2t} = -e^{-t} - 2e^{-2t} - 2e^{-t} - 8e^{-2t}$
$6A = -3 \quad A = -\frac{1}{2} \quad 12B = -10 \quad B = -\frac{5}{6}$
$y(t) = C_1 e^{2t} + C_2 e^{t} - \frac{1}{2}e^{-t} - \frac{5}{6}e^{-2t}$
$y(t) = \frac{-7}{2}e^{-2t} + \frac{29}{6}e^{t} - \frac{1}{2}e^{-t} - \frac{5}{6}e^{-2t}$

$C_1 + C_2 - \frac{1}{2} - \frac{5}{6} = 0$
$2C_1 + C_2 + \frac{1}{2} + \frac{10}{6} = 0$
$C_1 = -\frac{7}{2}$
$C_2 = \frac{29}{6}$



2a
$\sum_{k=-\infty}^{\infty} x[k]h[n-k]$
$k \to 1 \quad x[1]h[n-1] = 2\delta[n+1] - \delta[n]$
$k \to -2 \quad x[-2]h[n+2] = 3(2\delta[n+4] - \delta[n+3]) = 6\delta[n+4] - 3\delta[n+3]$
$\hookrightarrow 6\delta[n+4] - 3\delta[n+3] + 2\delta[n+1] - \delta[n]$

2b
$x[n] = \delta[n-1] + \delta[n] + \delta[n+1]$ $\qquad h[k]x[n-k]$
$h[n] = \delta[n-4] + \delta[n-5]$
$k \to 4 \quad h[4]x[n-4] = \delta[n-5] + \delta[n-4] + \delta[n-3]$
$k \to 5 \quad h[5]x[n-5] = \delta[n-6] + \delta[n-5] + \delta[n-4]$
$\hookrightarrow \delta[n-3] + 2\delta[n-4] + 2\delta[n-5] + \delta[n-6]$

   (a) (15 pts) Find the differential equation which represents this system.

   (b) (15 pts) Find the output $y(t)$, when the input $x(t) = (e^{-t} + e^{-2t})u(t)$. Assume that the system is initially at rest.

2. (10 pts) Evaluate the following convolutions.

   (a) (5 pts) Given $x[n] = \delta[n-1] + 3\delta[n+2]$ and $h[n] = 2\delta[n+2] - \delta[n+1]$, compute and draw $y[n] = x[n] * h[n]$.

   (b) (5 pts) Given $x[n] = u[n+1] - u[n-2]$ and $h[n] = u[n-4] - u[n-6]$, compute and draw $y[n] = x[n] * h[n]$.

3a
$\int_0^t e^{-\frac{\tau}{2}} e^{-(t-\tau)} d\tau = e^{-t}\int_0^t e^{\frac{\tau}{2}} d\tau = 2e^{-t} e^{\frac{\tau}{2}}\Big|_0^t = 2e^{-t}(e^{\frac{t}{2}} - 1) = (2e^{-\frac{t}{2}} - 2e^{-t})u(t)$
$0 \quad t < 0$

3. (10 pts) Evaluate the following convolutions.

   (a) (5 pts) Given $h(t) = e^{-\frac{1}{2}t}u(t)$ and $x(t) = e^{-t}u(t)$, find $y(t) = x(t) * h(t)$.

   (b) (5 pts) Given $h(t) = e^{-3t}u(t)$ and $x(t) = u(t) - u(t-4)$, find $y(t) = x(t) * h(t)$.

3b
$\int_0^t e^{-3(t-\tau)} d\tau = e^{-3t}\int_0^t e^{3\tau} d\tau = \frac{e^{-3t}}{3} \cdot e^{3\tau}\Big|_0^t = \frac{e^{-3t}}{3}(e^{3t} - 1) = \frac{1 - e^{-3t}}{3} \quad 0 < t < 4$
$= \frac{e^{-3t}}{3} e^{3\tau}\Big|_0^4 = \frac{e^{-3t}}{3}(e^{12} - 1) = \frac{e^{12-3t} - e^{-3t}}{3} \quad 4 < t$

4. (20 pts) Consider a continuous LTI system, initially at rest, given by the following input and output equation:

4a

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau - 3)d\tau = \int_{-\infty}^{t-3} e^{-(t-3-\tau')} x(\tau')d\tau' \qquad (1)$$

$h(t) = e^{-(t-3)}u(t-3)$

   (a) (10 pts) Find and plot the impulse response $h(t)$.

   (b) (10 pts) Find the response of the system when the input $x(t)$ is shown as:

4b
$t < 1 \to 0$
$1 < t < 4 \to \int_{-2}^{t-3} e^{-(t-\tau-3)} d\tau = e^{3-t} \cdot e^{\tau}\Big|_{-2}^{t-3} = e^{3-t}(e^{t-3} - e^{-2}) = 1 - e^{1-t}$
$4 < t \to e^{3-t} \cdot e^{\tau}\Big|_{-2}^{1} = e^{3-t}(e - e^{-2}) = e^{4-t} - e^{1-t}$

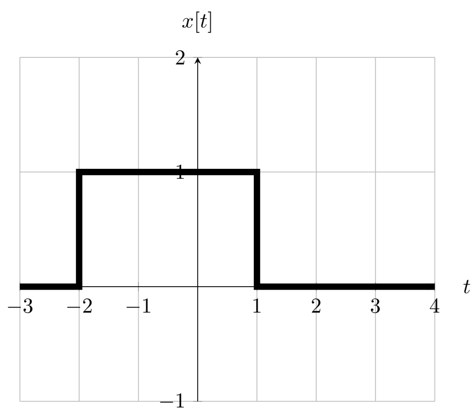
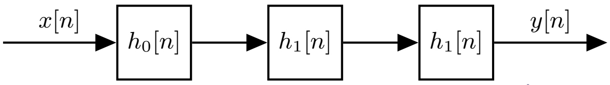


Figure 1: t vs. $x(t)$.

5. (30 pts) Consider a discrete LTI system represented by the following block diagram which is initially at rest:



where

$$h_1^{-1}[n] = (\frac{1}{2})^n u[n] \tag{2}$$

(5a) $h_1^{-1}[n] * h_1[n] = \delta[n]$

$h_1^{-1}[n] - A\, h_1^{-1}[n-1] = \delta[n]$

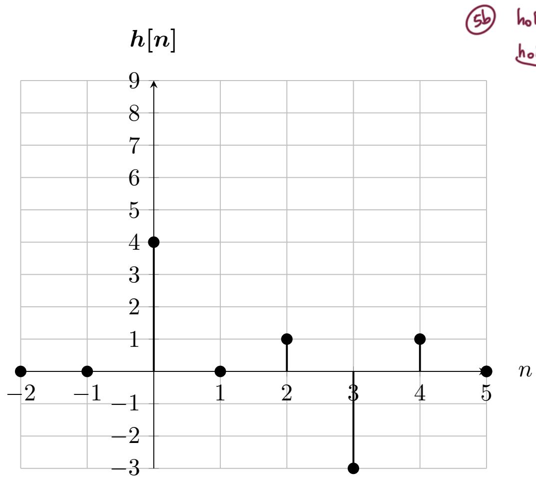
$(\frac{1}{2})^n u[n] - A(\frac{1}{2})^{n-1} u[n-1] = \delta[n]$

$A = \frac{1}{2}$

$h_1^{-1}[n] \underbrace{(\delta[n] - \frac{1}{2}\delta[n-1])}_{h_1[n]} = \delta[n]$

$h_1[n] * h_1[n] = \delta[n] - \delta[n-1] + \frac{1}{4}\delta[n-2]$

and the overall impulse response of the system is:



(5b) $h_0[n] * h_1[n] * h_1[n] = h[n]$

$\underbrace{h_0[n] (\delta[n] - \delta[n-1] + \frac{1}{4}\delta[n-2])}= h[n]$

$h_0[n] - h_0[n-1] + \frac{1}{4} h_0[n-2] = h[n]$

$h_0[0] = h[0] = 4$

$h_0[1] - \underbrace{h_0[0]}_{4} = h[1] = 0 \quad h_0[1] = 4$

$h_0[2] - \underbrace{h_0[1]}_{4} + \frac{1}{4}\underbrace{h_0[0]}_{1} = h[2] = 1 \quad h_0[2] = 4$

$h_0[3] - \underbrace{h_0[2]}_{4} + \frac{1}{4}\underbrace{h_0[1]}_{1} = h[3] = -3 \quad h_0[3] = 0$

$h_0[4] - \underbrace{h_0[3]}_{0} + \frac{1}{4}\underbrace{h_0[2]}_{1} = h[4] = 1 \quad h_0[4] = 0$

$h_0[5] - \underbrace{h_0[4]}_{0} + \frac{1}{4}\underbrace{h_0[3]}_{0} = h[5] = 0 \quad h_0[5] = 0 \ \vdots$

$\Big\} \; h_0[n] = 4(\delta[n] + \delta[n-1] + \delta[n-2])$

Figure 2: n vs. $h[n]$.

(5c) $y[n] = x[n] * h_0[n]$

$\sum_{k=-\infty}^{\infty} x[k]\, h_0[n-k]$

$k=0 \quad x[0]\, h_0[n] \rightarrow 4\delta[n] + 4\delta[n-1] + 4\delta[n-2]$

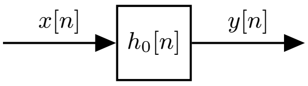
$k=2 \quad x[2]\, h_0[n-2] \rightarrow 4\delta[n-2] + 4\delta[n-3] + 4\delta[n-4]$

$\Big\} \; 4\delta[n] + 4\delta[n-1] + 8\delta[n-2] + 4\delta[n-3] + 4\delta[n-4]$

(a) (10 pts) Find $h_1[n] * h_1[n]$.

(b) (10 pts) Find the impulse response $h_0[n]$.

(c) (10 pts) Consider a discrete LTI system represented by the following block diagram which is initially at rest:



What is the response of this system when $x[n] = \delta[n] + \delta[n-2]$?