Lecture 5

Tuesday, 18 February 2020 04:42

$AX = B$ ↓ column matrix

Rmk:

Assume that
$$\begin{aligned} --- &= b_1 \\ --- &= b_2 \\ &\vdots \\ 0 &= b_n \end{aligned}$$
is the reduced system

a) If $b_n \neq 0$ then the eqn. $0 = b_n$ has no solution then the system is inconsistent

b) If we do not have such an eqn. then the system is consistent.

c) For a consistent system if the number of the equations and the number of unknowns are equal and echelon form of the augmented matrix has no zero rows then the system has a unique solution.

Exp: Find all values of a & b for which the system has

$$\begin{aligned} 2x - y + 2az + t &= b \\ -2x + ay - 3z &= 4 \\ 2x - y + (2a+1)z + (a+1)t &= 0 \\ -2x + y + (1-2a)z - 2t &= -2b-2 \end{aligned}$$

a) a unique soln,
b) infinitely many solns.
c) no solution.

Soln:
$$\left[\begin{array}{cccc|c} 2 & -1 & 2a & 1 & b \\ -2 & a & -3 & 0 & 4 \\ 2 & -1 & 2a+1 & a+1 & 0 \\ -2 & 1 & (1-2a) & -2 & -2b-2 \end{array}\right] \xrightarrow[\text{----}]{\text{exc.}} \left[\begin{array}{cccc|c} 2 & -1 & 2a & 1 & b \\ 0 & \underbrace{a-1}_{a=1} & 2a-3 & 1 & b+4 \\ 0 & 0 & 1 & a & -b \\ 0 & 0 & 0 & \underbrace{-1-a}_{a=-1} & -2 \end{array}\right]$$

Consider last row $(-1-a)t = -2$ if $-1-a = 0$ then $0.t = -2$ hence no soln.

$\therefore$ If $a = -1$ then the system is inconsistent i.e. no soln.

put $a \neq -1$ but $a = 1$

$$\left[\begin{array}{cccc|c} 2 & -1 & 2 & 1 & b \\ 0 & 0 & -1 & 1 & b+4 \end{array}\right] \xrightarrow[\text{---}]{\text{2 row op.}} \left[\begin{array}{cccc|c} 2 & -1 & 2 & 1 & b \\ 0 & 0 & -1 & 1 & b+4 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 2 & -1 & 2 & 1 & b \\ 0 & 0 & -1 & 1 & b+4 \\ 0 & 0 & 1 & 1 & -b \\ 0 & 0 & 0 & -2 & -2 \end{array}\right] \xrightarrow[\text{---}]{\text{2 row op.}} \left[\begin{array}{cccc|c} 2 & -1 & 2 & 1 & b \\ 0 & 0 & -1 & 1 & b+4 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

again $0=2$ & the system is inconsistent i.e (no soln.)

Hence; for $a=1,-1$ the system has no solution (c)

for $a \neq \pm 1$ then the system has a unique soln. and we do not have infinitely many solns.

$$\left[\begin{array}{cccc|c} 2 & -1 & 2a & 1 & b \\ 0 & a-1 & 2a-3 & 1 & b+4 \\ 0 & 0 & 1 & a & -b \\ 0 & 0 & 0 & -1-a & -2 \end{array}\right] \Rightarrow \begin{array}{l} 2x-y+2az+t=b \\ (a-1)y+(2a-3)z+t=b+4 \\ z+at=-b \\ (-1-a)t=-2 \Rightarrow t=\frac{2}{a+1} \end{array}$$

$x = - - - -$

$y = - - - -$

$z = -b - \frac{2a}{a+1}$

2.2. Systems of Homogeneous Equations:

Defn: If the system is of the form $AX = 0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ then it is called "homogeneous system".

Obviously $A.0 = 0 \Rightarrow A. \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}$

this soln. i.e. $x_1 = x_2 = \cdots = x_m = 0$ is calle "trivial solution".

Rmk: A homogeneous system is alwasy consistent & the question is are there any non-trivial solns.

exp: $(*) \begin{cases} x+3y+3z+2t = b_1 \\ 2x+6y+9z+7t = b_2 \\ -x-3y+3z+4t = b_3 \end{cases} \longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 2 & 6 & 9 & 7 & b_2 \\ -1 & -3 & 3 & 4 & b_3 \end{array}\right] \rightarrow$ exc. $- - - -$

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 3 & -2b_1+b_2 \\ 0 & 0 & 0 & 0 & \underbrace{5b_1-2b_2+b_3} \end{array}\right]$$

(*) is inconsistent if $5b_1 - 2b_2 + b_3 \neq 0$. Otherwise it is consistent.

When $5b_1 - 2b_2 + b_3 = 0$ then we have infinitely many solns, and any two of the unknowns will be independent variables & the other two will be dependent variables.

e.g. $b_1 = b_2 = b_3 = 0$

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{array}{l} x+3y+3z+2t=0 \Rightarrow x+3y-t=0 \Rightarrow \underline{x=t-3y} \\ 3z+3t=0 \Rightarrow \underline{z=-t} \end{array}$$

let $t = \alpha$ $y = \beta$ $\Rightarrow z = -\alpha$ $x = \alpha - 3\beta$

solutions are
$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \alpha - 3\beta \\ \beta \\ -\alpha \\ \alpha \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}} + \beta \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \; ; \text{ s.t. } \alpha, \beta \in \mathbb{R}.$$

are called fundamental solutions of (*) when $b_1 = b_2 = b_3 = 0$

Let $b_1 = 2$, $b_2 = 5$, $b_3 = 0$, the corresponding augmented matrix's echelon form is

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 2 \\ 0 & 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x+3y+3z+2t=2 \\ 3z+3t=1 \Rightarrow z = \dfrac{1-3t}{3} \end{array}$$

$x + 3y + 1 - 3t + 2t = 2 \Rightarrow x = -3y + t + 1$

solutions are
$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -3y+t+1 \\ y \\ \frac{1-3t}{3} \\ t \end{bmatrix} = \underbrace{\underbrace{t \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{fund. soln.}} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1/3 \\ 0 \end{bmatrix}}_{\text{particular soln.}}}_{\text{general solution of the system } AX = B}$$

$t = \alpha$ $y = \beta$

solution to the corresponding homogeneous system.

general solution of the system $AX = B$.