Lecture 3

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1.5. Elementory Row Operations & Row Equivalence:

<u>Defn:</u> Any of the following row operations are called "elementory row operation"

<u>Type 1</u> — Add $0 \neq c \in \mathbb{R}$, times i-th row to the j-th row ; $i \neq j$

denoted by $cR_i + R_j$

R_i — i-th row
R_j — j-th row

$$\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \\ R_n \end{bmatrix} \xrightarrow{cR_i + R_j} \begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ cR_i + R_j \\ \vdots \\ R_n \end{bmatrix}$$

<u>Type 2</u> — Interchange i-th row by j-th row, denoted by $R_i \longleftrightarrow R_j$

$$\begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \\ R_n \end{bmatrix} \xrightarrow{R_i \longleftrightarrow R_j} \begin{bmatrix} R_1 \\ \vdots \\ R_j \\ \vdots \\ R_i \\ \vdots \\ R_n \end{bmatrix}$$

<u>Type 3</u> - Multiply i-th row by a $0 \neq c \in \mathbb{R}$. (non-zero constant) denoted by cR_i

$$\begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_n \end{bmatrix} \xrightarrow{cR_i} \begin{bmatrix} R_1 \\ \vdots \\ cR_i \\ \vdots \\ R_n \end{bmatrix}.$$

Exp:

$$A = \begin{bmatrix} -1 & 0 & 3 & 5 \\ 0 & 5 & -7 & 8 \\ 1 & -11 & 0 & 0 \end{bmatrix} \underset{\frac{1}{2}R_2}{\overset{2R_2}{\rightleftarrows}} \begin{bmatrix} -1 & 0 & 3 & 5 \\ 0 & 10 & -14 & 16 \\ 1 & -11 & 0 & 0 \end{bmatrix} \underset{-3R_3+R_1}{\overset{3R_3+R_1}{\rightleftarrows}} \begin{bmatrix} 2 & -33 & 3 & 5 \\ 0 & 10 & -14 & 16 \\ 1 & -11 & 0 & 0 \end{bmatrix}$$

$$\underset{R_3 \leftrightarrow R_1}{\overset{R_1 \leftrightarrow R_3}{\rightleftarrows}} \begin{bmatrix} 1 & -11 & 0 & 0 \\ 0 & 10 & -14 & 16 \\ 2 & -33 & 3 & 5 \end{bmatrix} = B \qquad \text{" } A \sim B \text{ "}$$

Thm : Every elementary row operations has an inverse operation of the same type.

Proof:
- inverse of $cR_i + R_j$ is $-cR_i + R_j$
- inverse of cR_i is $\frac{1}{c}R_i$; $c \neq 0$
- inverse of $R_i \leftrightarrow R_j$ is $R_j \leftrightarrow R_i$.

Defn: Let A & B be $m \times n$ matrices. A & B are called row equivalent (or A is row equivalent to B) & denoted by $A \sim B$ if B is obtained from A by finitely many elementary row operations.

Thm: Being row equivalent is an "equivalence relation".
i.e. "$\sim$" satisfies
- reflexivity $A \sim A$
- symmetry $A \sim B \Rightarrow B \sim A$
- transitivity $A \sim B$ & $B \sim C \Rightarrow A \sim C$.

exp:

$$\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 + R_2} \begin{bmatrix} 2 & 3 \\ 0 & \frac{7}{2} \end{bmatrix} \xrightarrow{\frac{2}{7}R_2} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{-3R_2 + R_1} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Notation: • We denote elementary row operations by $\mathcal{E}_i$.

If A is row equivalent to B i.e. $A \sim B$ we write

$$B = \mathcal{E}_k \cdots \mathcal{E}_3 \mathcal{E}_2 \mathcal{E}_1(A)$$

- Also the inverse elementory row operation of $\mathcal{E}_i$ is denoted by $\mathcal{E}_i^{-1}$

<u>Thm:</u> Every matrix is row equivalent to a row reduced echelon matrix.

<u>Exp:</u>

$$A \begin{bmatrix} 1 & -1 & 1 & 3 & 1 \\ 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & -2 & -2 & -1 \\ 2 & 0 & 1 & 7 & 2 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & -1 & 1 & 3 & 1 \\ 0 & 3 & -2 & -2 & -1 \\ 0 & 3 & -2 & -2 & -1 \\ 2 & 0 & 1 & 7 & 2 \end{bmatrix} \xrightarrow{-2R_1+R_4}$$

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 1 \\ 0 & 3 & -2 & -2 & -1 \\ 0 & 3 & -2 & -2 & -1 \\ 0 & 2 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & -1 & 1 & 3 & 1 \\ 0 & 3 & -2 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{-\frac{3}{2}R_4+R_2}$$

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 1 \\ 0 & 0 & -\frac{1}{2} & -\frac{7}{2} & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & -1 & 1 & 3 & 1 \\ 0 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{7}{2} & -1 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 1 \\ 0 & 2 & -1 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{7}{2} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & -1 & 1 & 3 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{7}{2} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2+R_1}$$

$$\begin{bmatrix} 1 & 0 & 1/2 & 7/2 & 1 \\ 0 & 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & -1/2 & -7/2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & 0 & 1/2 & 7/2 & 1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & -1/2 & -7/2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3+R_1}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & -1/2 & -\frac{7}{2} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 7 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

"row reduced echelon"