13 - fundamental matrices / jordan form / repeated eigen values

fundamental set / basis = linearly independent set of solutions
↳ every solution is expressible as a linear combination of elements of the set

$\frac{dx}{dt} = A(t)x$ → homogenous linear system → if A is constant matrix: $\Phi(t) = e^{At}$ is fundamental matrix for the system with $\Phi(0) = I$

fundamental matrix Ψ ⇒ ① $\frac{d\Psi}{dt} = A.\Psi$

② Ψ is invertible

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

how to find e^{At} ⇒

$A^k = PD^kP^{-1} \rightarrow \boxed{e^{At} = Pe^{Dt}P^{-1}}$

$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

example: $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \rightarrow \begin{matrix} \lambda_1 = 3 \rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \lambda_2 = 5 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{matrix}$

$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ $P^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$e^{At} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -e^{3t} & e^{5t} \\ e^{3t} & e^{5t} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\frac{1}{2}$

Solution $x = c_1\begin{bmatrix} e^{3t} + e^{5t} \\ -e^{3t} + e^{5t} \end{bmatrix} + c_2\begin{bmatrix} -e^{3t} + e^{5t} \\ e^{3t} + e^{5t} \end{bmatrix} \longleftarrow = \frac{1}{2}\begin{bmatrix} e^{3t} + e^{5t} & -e^{3t} + e^{5t} \\ -e^{3t} + e^{5t} & e^{3t} + e^{5t} \end{bmatrix}$

1 independent → 1 Jordan block

$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Jordan form for repeated eigenvalues

$A = P.J.P^{-1} \rightarrow \boxed{e^{At} = Pe^{Jt}P^{-1}}$ → will be constant no need

example: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} \rightarrow (\lambda = 2) \times 3 \rightarrow \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \rightarrow$

$1-\lambda$ $1-\lambda$ $4-\lambda$

① $\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 0 \\ -3 & 2 & 2 & 0 \end{array}\right] \rightarrow \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

② $\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 2 & -1 & -1 & -1 \\ -3 & 2 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$= \begin{bmatrix} -1 \\ -1-k \\ k \end{bmatrix} \rightarrow k\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 & -2 \end{bmatrix}$

$$P = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 0 & 0 \end{bmatrix} \qquad J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

③ $$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 2 & -1 & -1 & -1 \\ -3 & 2 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \begin{bmatrix} -2 \\ -3-k \\ k \end{bmatrix} \rightarrow k\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix}$$

$$= P.e^{Jt} = \begin{bmatrix} 0 & -1 & -2 \\ -1 & -1 & -3 \\ 1 & 0 & 0 \end{bmatrix} e^{2t} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} c$$

$$= e^{2t} \begin{bmatrix} 0 & -1 & -t-2 \\ -1 & -t-1 & -\frac{t^2}{2}-t-3 \\ 1 & t & \frac{t^2}{2} \end{bmatrix} c = c_1 \begin{bmatrix} 0 \\ -e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{2t} \\ -te^{2t}-e^{2t} \\ te^{2t} \end{bmatrix} + c_3 \begin{bmatrix} (-t-2)e^{2t} \\ (-\frac{t^2}{2}-t-3)e^{2t} \\ \frac{t^2 e^{2t}}{2} \end{bmatrix}$$

Jordan forms $\rightarrow$ $\rightarrow \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{\begin{matrix} 1 & t \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{1} \end{bmatrix}_{(3\times3)} \quad , \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

example: $\begin{bmatrix} 7 & 0 & 1 \\ 0 & 6 & 0 \\ -1 & 0 & 5 \end{bmatrix} \rightarrow (\lambda = 6) \times 3$

① $$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \rightarrow \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$x_1 + x_3 = 0 \quad \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2 free/independent
2 Jordan block!

② $$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \end{array}\right] \rightarrow \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_3 = -1 \quad \begin{bmatrix} -k-1 \\ 0 \\ k \end{bmatrix} \Rightarrow k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$$J = \begin{bmatrix} 6 & 0 & 0 \\ 0 & \boxed{\begin{matrix} 6 & 1 \\ 0 & 6 \end{matrix}} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} e^{6t}$$

from first

$$P = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad x = Pe^{Jt}.c$$

when $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ x_2 is free

$$x = c_1 \begin{bmatrix} 0 \\ e^{6t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} e^{6t} \\ 0 \\ -e^{6t} \end{bmatrix} + c_3 \begin{bmatrix} te^{6t} \\ 0 \\ (-t+1)e^{6t} \end{bmatrix}$$

example:

$x' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} x \qquad (\lambda = 4) \times 2 \rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -3 & -3 & -1 \\ 3 & 3 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 3 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ 1 ind
1 Jordan

$3x_1 + 3x_2 = 1 \qquad \begin{bmatrix} \frac{1-3k}{3} \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} \leftarrow$

$$P = \begin{bmatrix} -1 & \frac{1}{3} \\ 1 & 0 \end{bmatrix} \qquad J = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e^{4t} \rightarrow \begin{bmatrix} e^{4t} & te^{4t} \\ 0 & e^{4t} \end{bmatrix}$$

$$= c_1 \begin{bmatrix} -e^{4t} \\ e^{4t} \end{bmatrix} + c_2 \begin{bmatrix} -te^{4t} + \frac{e^{4t}}{3} \\ te^{4t} \end{bmatrix} \rightarrow c_1 e^{4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} \frac{1}{3} - t \\ t \end{bmatrix}$$