induction and recursion

induction and recursion

→ tümevarım

$$\left[P(1) \land \forall k\,(P(k) \to P(k+1))\right] \to \forall n\, P(n)$$

- if we assume that $P(k)$ is true, then $P(k+1)$ must als be true. (we do not assume that $P(k)$ is true for all positive integers) for $\forall n > n_0$ → initial conditio

$P(1) \Rightarrow$ is basis step. it does not have to be 1. it can be any integer. (including negative integers and zer

- Can be used only to prove results obtained by some other way
- ! it is not a tool for discovering formulae or theorems.

- domain is all positive integers

example-1

show that $1 + q + q^2 + q^3 + \dots + q^{n-1} = \dfrac{q^n - 1}{q - 1}$ $\quad q \neq$

step 1

basis $n=1$ $\quad 1 = \dfrac{q^1 - 1}{q - 1} = 1$ ✓

step 2

induction step:
assume that $p(k) = 1+q+q^2+\dots+q^{k-1} = \frac{q^k-1}{q-1}$

$p(k+1) = \frac{q^{k+1}-1}{q-1}$?

step 3

$$\underbrace{1+q+q^2+\dots+q^{k-1}}_{\frac{q^k-1}{q-1}} + q^k = \frac{q^k-1+q^{k+1}-q^k}{q-1} = \frac{q^{k+1}-1}{q-1}$$

due to induction hypotesis

example-2

let S be a set with n element, prove by induction $|P(S)| = 2^n$ $P(n)$

1) basis

$n=1 \longrightarrow S=\{a\} \quad P(S)=\{\emptyset, \{a\}\} \quad |P(S)|=2=2^1$

2) induction step

induction hypotesis: assume than $P(k) \rightarrow |P(k)| = 2^k$

$P(k+1) \qquad S' = S \cup \{x\} \qquad |S| = k$

$|S'| = k+1$

i) any subset A of S' which $x \in A \rightarrow 2^k$ subsets

ii) any subset A of S' which $x \notin A \rightarrow 2^k$ subsets

+

total number of subsets of S' $\quad 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

NOTATIO

$3^{k+1} | x \rightarrow$ means x is divisible by 3^k (without curry, or floating point.)

$2 | 8$

strong induction

$(\forall n > n_0 \quad P(n))$

1. basis (n_0)
2. induction step

   ↳ induction hypotesis: assume that $P(i) \quad \forall i \quad \underbrace{n_0 \leq i \leq k}_{\text{strong assumption}}$

   ↳ prove it for $P(k+1)$

other names: • second principle of mathematical inducti

• complete induction

* we show that the conditional statement $[P(1) \wedge P(2) \wedge \ldots P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

example

any number ≥ 2 can be written as a product of primes.

proof (strong induction)

basis $n=2 \rightarrow 2$ is a prime number

induction step

assume that $\forall i \quad 2 \leq i \leq k.\ P(i)$ is true

$k+1$

i) $k+1$ prime $\rightarrow P(k+1)$ ✓

ii) $k+1$ composite $\rightarrow k+1 = p \cdot q$

$$2 \leq p, q < k+1$$

by using induction hypotesis each of p & q can be written as a product of primes (like recursion)

Thus, $k+1$ is also product of primes.

recursive

structual induction = to prove the results about recursively defined sets

* recursively defined functions are well defined that is, the value of the function at any integer is determined in an unambiguous way → kesin / açık / tam

means that: from two parts of the definition, we get the same result whether we use the fist def. or second definition.

Why Mathematical Induction is Valid

Why is mathematical induction a valid proof technique? The reason comes from the well-ordering property, listed in Appendix 1, as an axiom for the set of positive integers, which states that every nonempty subset of the set of positive integers has a least element. So, suppose we know that $P(1)$ is true and that the proposition $P(k) \to P(k+1)$ is true for all positive integers k. To show that $P(n)$ must be true for all positive integers n, assume that there is at least one positive integer for which $P(n)$ is false. Then the set S of positive integers for which $P(n)$ is false is nonempty. Thus, by the well-ordering property, S has a least element, which will be denoted by m. We know that m cannot be 1, because $P(1)$ is true. Because m is positive and greater than 1, $m - 1$ is a positive integer. Furthermore, because $m - 1$ is less than m, it is not in S, so $P(m-1)$ must be true. Because the conditional statement $P(m-1) \to P(m)$ is also true, it must be the case that $P(m)$ is true. This contradicts the choice of m. Hence, $P(n)$ must be true for every positive integer n.

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.