Lecture 8

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Examples of Subspaces:

1 - $V = \mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$

$$\text{"}+\text{"} : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$((x_1,y_1),(x_2,y_2)) \longmapsto (x_1+x_2,\ y_1+y_2)$$
$$\text{"}\cdot\text{"} : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(c,(x,y)) \longmapsto (cx, cy)$$

a) $W = \{(x,0) \mid x \in \mathbb{R}\} \underset{\text{subset}}{\subseteq} V$

i) $(0,0) \in W$ so $W \neq \emptyset$

ii) Let $(x_1,0)$ & $(x_2,0) \in W$ where $x_1, x_2 \in \mathbb{R}$

$$(x_1,0) \text{"}+\text{"} (x_2,0) = (x_1+x_2, 0+0) = (\underbrace{x_1+x_2}_{\in \mathbb{R}}, 0) \in W$$

So, W is closed under addition

iii) Let $c \in \mathbb{R}$, $(x,0) \in W$ $\quad c\cdot(x,0) = (\underbrace{cx}_{\in\mathbb{R}}, 0) \in W$

$\therefore$ W is a subspace of V.

b) $U = \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\} \subseteq V$

i) $(1,1) \in U$ so $U \neq \emptyset$.

ii) Given any two elements $(x_1,y_1), (x_2,y_2) \in U$ so $x_1, x_2, y_1, y_2 > 0$

$$(x_1,y_1) + (x_2,y_2) = (\underbrace{x_1+x_2}_{>0}, \underbrace{y_1+y_2}_{>0}) \in U$$

So, U is closed under addition

iii) Let $c \in \mathbb{R}$, $(x,y) \in U$ then

$$c(x,y) = (cx, cy)$$

take $c = -1$ $\quad (-1)(x,y) = (-x,-y)$ but $x,y > 0 \Rightarrow -x,-y < 0$
so $(-x,-y) \notin U$ hence U is **NOT** closed under scalar multiplication.

$\therefore$ U is <u>not</u> a subspace.

c) $\mathcal{L} = \{ (x,y) \in \mathbb{R}^2 \mid xy \geq 0 \}$ is not a subspace because

$\mathcal{L}$ is not closed under addition : let $(-3,-4)$ & $(1,5) \in \mathcal{L}$
$(-3,-4) + (1,5) = (-2,1) \notin \mathcal{L}$.

2- $V = \mathbb{R}^{n\times n}$ $\quad W = \{\text{upper triangular matrices}\} \underset{\text{subspace}}{\subseteq} V$

$U = \{\text{diagonal matrices}\} \underset{\text{subspace}}{\subseteq} V$

$\mathcal{L} = \{\text{invertible matrices}\}$ is NOT a subspace of V since

Exc: {symmetric matrices} & {skew-symmetric matrices} ??

Exc: $\mathcal{F} = \{ f \mid f : [a,b] \longrightarrow \mathbb{R} \}$

Show that the subset of i) continuous functions on $[a,b]$ is a subspace of $\mathcal{F}$
ii) diff'ble functions on $[a,b]$ is a subspace of $\mathcal{F}$.

Thm 2.2.1 : Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$)

If $\emptyset \neq W \subset V$ which is closed under addition & scalar multiplication then
i) $\vec{0}_V \in W$.
ii) if $\vec{v} \in W$ then $-v \in W$.

Proof: i) Since W is closed under scalar multiplication then $c\vec{w} \in W$ for any $c \in \mathbb{R}$ in particular $c = 0$
$0.\vec{w} = \vec{0}_V \in W$.

ii) exc.

2.3. LINEAR SPAN

Defn: Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$)
i) If $\vec{v}_1, \dots, \vec{v}_r$ are elements of V then every vector of the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_r\vec{v}_r$ with $c_i \in \mathbb{R}$ ($\mathbb{C}$) is a "linear

i) If $\vec{v}_1, \dots, \vec{v}_r$ are elements of V then every vector of the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_r\vec{v}_r$ with $c_i \in \mathbb{R}$ ($\mathbb{C}$) is a "linear combination" of $\vec{v}_1, \dots, \vec{v}_r$.

ii) For a subset S of V the set of all linear combinations of elements of S is called 'linear span" of S and it is denoted by $\langle S \rangle$ or $\text{Span}(S)$.
When $S = \emptyset$ then $\text{Span}\,\emptyset = \vec{0}$.

iii) If $V = \langle S \rangle$ that is to say if every vector of V is a linear combination of elements of S, then we say V is "spanned" or "generated" by S. S is called "generating set" or "spanning set" of V.

Exp: $V = P_2[x] = \{ ax^2 + bx + c \mid a, b, c \in \mathbb{R} \}$

$$S = \{1, \; 1-x, \; 2x^2\} \underset{\text{subset}}{\subseteq} V$$

$$\langle S \rangle = \{ c_1 . 1 + c_2(1-x) + c_3(2x^2) \mid c_1, c_2, c_3 \in \mathbb{R} \}$$

$$= \{ c_1 + c_2 - c_2 x + 2c_3 x^2 \mid c_1, c_2, c_3 \in \mathbb{R} \}$$

$$= \{ (c_1 + c_2) + (-c_2)x + (2c_3)x^2 \mid c_1, c_2, c_3 \in \mathbb{R} \}$$

if we fix a, b, c then $a = 2c_3$ $\quad b = -c_2$ $\quad c = c_1 + c_2$ a linear system

So we try to solve this system $c_2 = -b$ $\quad c_3 = \frac{a}{2}$ $\quad c_1 = c + b$

$\therefore V = \langle S \rangle$

2 - Let $V = \mathbb{C}^3 = \{ (z_1, z_2, z_3) \mid z_k \in \mathbb{C} \}$ $\quad z_k = a_k + i b_k$ $\quad i = \sqrt{-1}$

over $\mathbb{C}$ with componentwise addition & scaler multiplication

Consider $S = \{ (-i, 2, 3), (0, -2i, 5) \}$ Does $\langle S \rangle = V$?

$$\langle S \rangle = \{ c_1(-i, 2, 3) + c_2(0, -2i, 5) \mid c_1, c_2 \in \mathbb{C} \}$$

$\langle S\rangle = \{c_1(-i,2,3) + c_2(0,-2i,5) \mid c_1,c_2 \in \mathbb{C}\}$

$= \{(-ic_1,\ 2c_1 - 2c_2 i,\ 3c_1 + 5c_2) \mid c_1,c_2 \in \mathbb{C}\}$

find (z_1,z_2,z_3) s.t.

$$\left.\begin{aligned} -c_1 i &= z_1 \\ -2c_1 - 2c_2 i &= z_2 \\ 3c_1 + 5c_2 &= z_3 \end{aligned}\right\} \text{linear system}$$

$(0,3,5)$

$$\begin{aligned} -c_1 i &= 0 &&\Rightarrow c_1 = 0 \\ -2c_1 - 2c_2 i &= 3 &&\Rightarrow c_2 = \tfrac{3}{2} i \\ 3c_1 + 5c_2 &= 5 &&\Rightarrow 3.0 + 5.\tfrac{3}{2} i \neq 5 \ \lightning \end{aligned}$$

but no solution!

$\therefore\ V \neq \langle S\rangle$

$\supsetneq$

Thm 2.3.1: If S is any subset of a vector space V over $\mathbb{R}$ (or $\mathbb{C}$) then $\langle S\rangle$ is subspace of V.

Proof: If $S=\emptyset$ then $\langle S\rangle = \langle\emptyset\rangle = \{\vec{0}\} \underset{\text{subspace}}{\subseteq} V$

Assume $S\neq\emptyset$ then it has at least one vector $\vec{v}\in S$.

i) $\langle S\rangle \neq \emptyset$ since $1.\vec{v} = \vec{v} \in \langle S\rangle$

ii) Let $\vec{v}_1,\dots,\vec{v}_n \in S$; $a_1\vec{v}_1 + \dots + a_n\vec{v}_n$, $b_1\vec{v}_1 + \dots + b_n\vec{v}_n \in \langle S\rangle$

$$(a_1\vec{v}_1 + \dots + a_n\vec{v}_n) + (b_1\vec{v}_1 + \dots + b_n\vec{v}_n) = (a_1+b_1)\vec{v}_1 + \dots + (a_n+b_n)\vec{v}_n \in \langle S\rangle$$

So $\langle S\rangle$ is closed under addition

iii) Let $a_1\vec{v}_1 + \dots + a_n\vec{v}_n \in \langle S\rangle$ & let $c\in\mathbb{R}$ (or $\mathbb{C}$)

$$c.(a_1\vec{v}_1 + \dots + a_n\vec{v}_n) = (ca_1)\vec{v}_1 + (ca_2)\vec{v}_2 + \dots + (ca_n)\vec{v}_n \in \langle S\rangle$$

$\therefore\ \langle S\rangle \underset{\text{subspace}}{\subseteq} V$.

Exp: $V=\mathbb{R}^4$ $\quad S = \{\overbrace{(2,-1,2,1)}^{\vec{v}_1}, \overbrace{(3,4,-1,1)}^{\vec{v}_2}\}$

$\langle S\rangle \subseteq \mathbb{R}^4$

$$\langle S \rangle \underset{\text{subspace}}{\subseteq} \mathbb{R}^4$$

i) Is $(0,-11,8,1)$ in $\langle S \rangle$?

$$\langle S \rangle = \{ c_1\vec{v}_1 + c_2\vec{v}_2 \mid c_1, c_2 \in \mathbb{R} \} = \{ c_1(2,-1,2,1) + c_2(3,4,-1,1) \mid c_1, c_2 \in \mathbb{R} \}$$

If $\vec{v} = (0,-11,8,1)$ in $\langle S \rangle$ then $c_1(2,1,2,1) + c_2(3,4,-1,1) = (0,-11,8,1)$

$$(2c_1+3c_2, -c_1+4c_2, 2c_1-c_2, c_1+c_2) = (0,-11,8,1)$$

$$\begin{aligned} 2c_1 + 3c_2 &= 0 \\ -c_1 + 4c_2 &= -11 \\ 2c_1 - c_2 &= 8 \\ c_1 + c_2 &= 1 \end{aligned} \Longrightarrow \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -11 \\ 8 \\ 1 \end{bmatrix} \Rightarrow$$

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ -1 & 4 & -11 \\ 2 & -1 & 8 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{\text{exc.}} \cdots \longrightarrow \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & -1/2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{aligned} c_2 &= -2 \\ c_1 &= 3 \end{aligned}$$

$$\vec{v} = 3\vec{v}_1 - 2\vec{v}_2$$

$\therefore \quad \vec{v} \in \langle S \rangle$.

Consider $\vec{w} = (2,3,1,2) \overset{?}{\in} \langle S \rangle$?

$$\left.\begin{aligned} 2c_1 + 3c_2 &= 2 \\ -c_1 + 4c_2 &= 3 \\ 2c_1 - c_2 &= 1 \\ c_1 + c_2 &= 2 \end{aligned}\right\} \Rightarrow \cdots \left[\begin{array}{cc|c} 2 & 3 & 2 \\ -1 & 4 & 3 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{array}\right] \longrightarrow \xrightarrow{\text{exc.}} \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & -1/2 & 2 \\ 0 & 0 & -15 \\ 0 & 0 & 25 \end{array}\right]$$

no soln.

$\therefore \quad \vec{w} \notin \langle S \rangle$

Hence $\langle S \rangle \neq \mathbb{R}^4$