6- exact equations and integrating factors

$\boxed{\text{Exact equations}}$

$M\,dx + N\,dy = 0$ → when we write the function in this form:

if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then it is exact equation

then $\int M\,dx = \int N\,dy$ → $\int M\,dx + c(y) = \int N\,dy + c(x)$

solution is $F(x,y) = c$

example

$(2y^2 e^{xy^2})\,dx + (2xy\,e^{xy^2})\,dy = 0$ is exact?

$M = (2y^2 e^{xy^2})\frac{1}{\partial y} = 4y e^{xy^2} + 2y^2 2xy\,e^{xy^2}$

$N = (2xye^{xy^2})\frac{1}{\partial x} = 2ye^{xy^2} + 2xy \cdot y^2 e^{xy^2}$

they are not equal so it is not exact

example

$(2xy - 9x^2)\,dx + (2y + x^2 + 1)\,dy = 0$

→ $M = (2xy - 9x^2)\frac{1}{\partial y} = 2x$
$N = (2y + x^2 + 1)\frac{1}{\partial x} = 2x$ same

→ $\int (2xy - 9x^2)\,dx = x^2y - 3x^3 + c(y)$
$\int (2y + x^2 + 1)\,dy = y^2 + x^2y + y + c$

→ $\cancel{x^2y} - 3x^2 + c(y) = y^2 + \cancel{x^2y} + y + c(x)$ → $c(x) = -3x^2$, $c(y) = y^2 + y$ → $x^2y - 3x^2 + y^2 + y = c$

example

$y\,dx + (x + 2y)\,dy = 0$, $y(1) = 5$?

$M = (y)\frac{1}{\partial y} = 1$
$N = (x + 2y)\frac{1}{\partial x} = 1$

$\int y\,dx = \int (x + 2y)\,dy$

$xy + c(y) = xy + y^2 + c(x)$
$c(y) = y^2$
$c(x) = 0$

$xy + y^2 = c$ $\quad x = 1,\ y = 5$ $\quad 5 + 25 = c$, $c = 30$ $\quad \underline{y^2 + xy - 30 = 0}$

example

example

$3e^y dx + (2y + axe^y)dy = 0$ for which a the equation is exact?

$M = (3e^y)\frac{1}{\partial y} = 3e^y$

$N = (2y + axe^y)\frac{1}{\partial x} = ae^y$

$3e^y = ae^y \quad \underline{a=3}$

↳ solution $\int 3e^y dx = 3xe^y + c(y)$

$\int (2y + 3xe^y)dy = y^2 + 3xe^y + c(x)$

$c(y) = y^2 \quad c(x) = 0$

$y^2 + 3xe^y = c$

example

$\left(\frac{2xy}{x^2+1} - 2x\right)dx - \left(2 - \ln(x^2+1)\right)dx = 0 \quad y(5) = 0 \quad ?$

$M = \left(\frac{2xy}{x^2+1} - 2x\right)\frac{1}{\partial y} = \frac{2x}{x^2+1}$

$N = (\ln(x^2+1) - 2)\frac{1}{\partial x} = \frac{2x}{x^2+1}$ ✓

$\int \left(\frac{2xy}{x^2+1} - 2x\right)dx = y\ln(x^2+1) - x^2 + c(y)$

$\int (\ln(x^2+1) - 2)dy = y\ln(x^2+1) - 2y + c(x)$

$c(x) = -x^2$

$c(y) = -2y$

$y\ln(x^2+1) - x^2 - 2y = c$

$x = 5 \quad y = 0$

$-25 = c$

$y(\ln(x^2+1) - 2) = x^2 - 25$

$y = \frac{x^2-25}{\ln(x^2+1) - 2}$

$\boxed{\text{integrating factors}}$

when $M dx + N dy = 0 \rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ is not exact. we find μ.

$M_y \qquad N_x$

$\frac{M_y - N_x}{N} \Rightarrow$ depends on only x or $\frac{N_x - M_y}{M} \Rightarrow$ depends on only y

↳ $\mu = e^{\int \left(\frac{M_y - N_x}{N}\right)dx}$

↳ $\mu = e^{\int \left(\frac{N_x - M_y}{M}\right)dy}$

then multiply equation with μ, it becomes exact $\Rightarrow \mu(Mdx + Ndy) = 0$

example

show that $\mu(x,y) = \frac{1}{x^2+y^2}$ is int. fac of $(3x^2 + x + 3y^2)dx + (7x^2 + y + 7y^2)dy = 0$

$$\left.\begin{array}{l} M = (3x^2 + x + 3y^2)\frac{1}{\partial y} = 6y \\ N = (7x^2 + y + 7y^2)\frac{1}{\partial x} = 14x \end{array}\right\} \neq$$

$$u.M = \frac{(3x^2 + x + 3y^2)}{x^2+y^2}\frac{1}{\partial y} = \left(3 + \frac{x}{x^2+y^2}\right)\frac{1}{\partial y} = x(x^2+y^2)^{-1} \rightarrow \frac{-x.2y}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$u.N = \frac{(7x^2+y+7y^2)}{x^2+y^2}\frac{1}{\partial x} = \left(7 + \frac{y}{x^2+y^2}\right)\frac{1}{\partial x} = y(x^2+y^2)^{-1} \rightarrow \frac{-y\,2x}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\int u.M = \int\left(3 + \frac{x}{x^2+y^2}\right)dx = 3x + \frac{\ln(x^2+y^2)}{2} + c(y)$$

$$c(x) = 3x$$
$$c(y) = 7y$$

$$\int u N = \int\left(7 + \frac{y}{x^2+y^2}\right)dy = 7y + \frac{\ln(x^2+y^2)}{2} + c(x)$$

$$\Rightarrow \frac{\ln(x^2+y^2)}{2} + 3x + 7y = c$$