4 - modelling with first order equations

- determining the dependent and independent variables
- relating the rate of change in the problem to the variables
- finding the initial problems

example: a partical moving along the x-axis

$v(t) = x'(t) \qquad a(t) = v'(t) = x''(t) = \frac{F(t)}{m}$

$F = m.a \rightarrow a = \frac{F}{m}$

when $F(t)$ is constant=

$v'(t) = a'(t)$

$v(t) = v(t_0) + \int_{t_0}^{t} a . d\tau = v(t_0) + a(t - t_0)$

only depends on t

$x'(t) = v(t)$

$x(t) = x(t_0) + \int_{t_0}^{t} v(\tau) . d\tau = x(t_0) + v(t_0).(t - t_0) + \frac{a(t-t_0)^2}{2}$

$F(t)$ is constant + additional restraining force propotional to its velocity =

(kısıtlayıcı)

k: constant

$a(t) = v'(t) = \frac{F(t) - k.v(t)}{m}$

$\int_{v_0}^{v} \frac{dv}{F_0 - kv} = \int_{t_0}^{t} \frac{d\tau}{m}$

depends on v and t (separable)

$v = \frac{F_0}{k} + \left(v_0 - \frac{F_0}{k}\right) e^{-\frac{k}{m}(t - t_0)}$

$t \to \infty \qquad v = \frac{F_0}{k}$ (limiting velocity)

example: initially $= 1000$ tl

rate $\rightarrow$ proportional to money at that time

at the end of the second year $= 2000$ tl $\rightarrow$ end of fifth year?

$M(0) = 1000 \qquad M(2) = 2000 \qquad M(5) = ?$

$M'(t) = k.M \qquad \frac{dm}{dt} = km \qquad \int \frac{1}{m} dm = \int k . dt \rightarrow \ln|m| = kt + c \rightarrow m = ce^{kt}$

$M(0) = c = 1000$

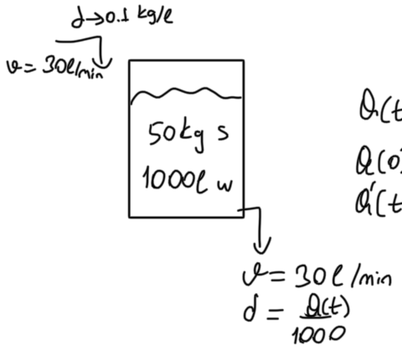
$M(2) = 1000e^{2k} = 2000 \qquad e^{2k} = 2 \qquad e^{k} = \sqrt{2}$

$M(5) = 1000e^{5k} = 1000(\sqrt{2})^5 = 4000\sqrt{2}$

Example: $t=0 \to$ 50kg salt in 1000l of water

water containing 100 g of salt per liter entering and leaving at a rate of 30l/min

limiting amount of salt $(t \to \infty)$?



$Q(t) \to$ amount of salt

$Q(0) = 50$

$Q'(t) = Q(in) - Q(out) = (0.1)(30) - 30 . \frac{Q(t)}{1000}$

$Q'(t) = 3 - \frac{3Q(t)}{100}$

$\frac{dQ(t)}{dt} = \frac{300 - 3Q(t)}{100} \to \int \frac{1}{300-3Q(t)} \, dQ(t) = \int \frac{1}{100} \, dt$

$= \frac{\ln|300 - 3Q(t)|}{-3} = \frac{t}{100} + c \to 300 - 3Q(t) = e^{-\frac{3t}{100}} . c$

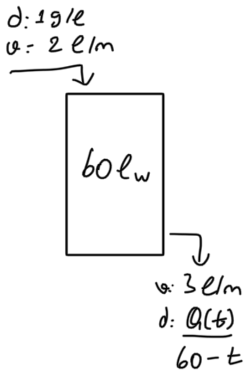
$Q(t) = 100 - \frac{c e^{-\frac{3t}{100}}}{3}$ $\qquad Q(0) = 100 - \frac{c}{3} = 50$ $\qquad c \to 150$

$t \to \infty = 100 - \frac{50}{e^{\frac{3t}{100}}} \to 100$ limiting salt

example: 60 l pure water

in → 1 g per liter salt with rate 2 l/min

out → with rate 3 l/min → empty after 1 hour



$Q(t) \to$ salt $\qquad Q(0) = 0$

$Q'(t) = {}^{v}Q_{in} - {}^{v}Q_{out} = 2 - \frac{3Q(t)}{60-t}$

$60Q'(t) - t.Q'(t) + 3Q(t) = 2$

$Q'(t) + \frac{3}{60-t} Q(t) = \frac{2}{60-t}$

$\mu = e^{\int \frac{3}{60-t} dt}$

$= e^{-3\ln|60-3t|}$

$= (60-t)^{-3}$

$\frac{Q(t)}{(60-t)^3} = \int \frac{2}{(60-t)^3} dt \to \frac{Q(t)}{(60-t)^3} = \frac{2}{-2}(60-t)^{-2} + c$

$Q(t) = 60 - t + c(60-t)^3$

$Q(0) = 60 + 60^3 . c = 0$ $\qquad c = \frac{-1}{3600}$

$Q(t) = 60 - t - \frac{(60-t)^3}{3600}$

max salt → $Q'(t) = -1 + \frac{3}{3600}(60-t)^2 = 0$ $\qquad \left(\frac{60-t}{1200}\right)^2 = 1$ when $t = 25.36$

salt → 25.0J