CENG 384 - Signals and Systems for Computer Engineers
Spring 2022
Homework 3

ALAN, Şeyma
e2380053@ceng.metu.edu.tr

TINMAZ, Derya
e2380947@ceng.metu.edu.tr

May 15, 2022

---

1. (a) The Fourier series representation of continuous-time periodic signal with the fundamental frequency w_0 is $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$, and its spectral coefficients are a_k.
   To find fundamental period $\frac{\pi}{5}T_0 = 2\pi k_1$ and $\frac{\pi}{4}T_0 = 2\pi k_2$
   $T_0 = 10k_1 = 8k_2$ for $k_1 = 4$ and $k_2 = 5$, $T_0 = 40$
   Fundamental frequency is $w_0 = \frac{2\pi}{T_0} = \frac{2\pi}{40} = \frac{\pi}{20}$

   Since $sin(wt) = \frac{e^{jwt} - e^{-jwt}}{2j}$ and $cos(wt) = \frac{e^{jwt} + e^{jwt}}{2}$, x(t) can be written as,

   $x(t) = \frac{1}{2j}e^{j\frac{\pi}{5}t} - \frac{1}{2j}e^{-j\frac{\pi}{5}t} + \frac{1}{2}e^{j\frac{\pi}{4}t} + \frac{1}{2}e^{-j\frac{\pi}{4}t} = -\frac{1}{2}je^{j4w_0 t} + \frac{1}{2}je^{-j4w_0 t} + \frac{1}{2}e^{j5w_0 t} + \frac{1}{2}e^{-j5w_0 t}$

   So the nonzero Fourier series coefficients for $x(t)$ are $a_4 = \frac{-1}{2}j$, $a_{-4} = \frac{1}{2}j$, $a_5 = \frac{1}{2}$, $a_{-5} = \frac{1}{2}$

   (b) The Fourier series representation of discrete-time periodic signal with the fundamental frequency w_0 is $x[n] = \sum_{k=<N>} a_k e^{jkw_0 n}$, and its spectral coefficients are a_k.
   To find fundamental period $4\pi N_1 = 2\pi k_1$ and $2\pi N_1 = 2\pi k_2$
   $N_1 = \frac{k_1}{2} = k_2$ for $k_1 = 2$ and $k_2 = 1$, $N_1 = 1$
   The fundamental period of $e^{j\pi n}$ is $N_2\pi = 2\pi k$, $N_2 = 2$
   Overall fundamental period N_0 is least common multiple $(N_1, N_2) = 2$
   Fundamental frequency is $w_0 = \frac{2\pi}{N_0} = \frac{2\pi}{2} = \pi$
   Since $sin(wn) = \frac{e^{jwn} - e^{-jwn}}{2j}$ and $cos(wn) = \frac{e^{jwn} + e^{jwn}}{2}$, x[n] can be written as,

   $x[n] = \frac{1}{2} + e^{j\pi n} - \frac{1}{2}je^{j4\pi n} + \frac{1}{2}je^{-j4\pi n} + \frac{1}{2}e^{j2\pi n} + \frac{1}{2}e^{-j2\pi n}$
   $= \frac{1}{2} + e^{jw_0 n} - \frac{1}{2}je^{j4w_0 n} + \frac{1}{2}je^{-j4w_0 n} + \frac{1}{2}e^{j2w_0 n} + \frac{1}{2}e^{-j2w_0 n}$
   So the nonzero Fourier series coefficients for $x[n]$ are $a_0 = \frac{1}{2}$, $a_1 = 1$, $a_4 = \frac{-j}{2}$, $a_4 = \frac{j}{2}$, $a_2 = \frac{1}{2}$, $a_{-2} = \frac{1}{2}$.

2. Using the Fourier series synthesis equation
   $x[n] = a_1 e^{jw_0 n} + a_{-1}e^{-jw_0 n} + a_2 e^{2jw_0 n} + a_{-2}e^{-2jw_0 n} + a_3 e^{3jw_0 n} + a_{-3}e^{-3jw_0 n}$
   (since other coefficients are zero no need to write them)
   $x[n] = 2je^{jw_0 n} - 2je^{-jw_0 n} + 2e^{2jw_0 n} + 2e^{-2jw_0 n} + 2je^{3jw_0 n} - 2je^{-3jw_0 n}$
   Since $2jsin(wn) = e^{jwn} - e^{-jwn}$ and $2cos(wn) = e^{jwn} + e^{jwn}$, $x[n]$ can be written as
   $x[n] = 2j(2jsin(w_0 n)) + 2(2cos(2w_0 n)) + 2j(2jsin(3w_0 n))$
   $= -4sin(w_0 n) + 4cos(2w_0 n) - 4sin(3w_0 n)$
   The fundamental frequency is $w_0 = \frac{2\pi}{N} = \frac{2\pi}{7}$
   $= -4sin(\frac{2\pi}{7}n) + 4cos(\frac{4\pi}{7}n) - 4sin(\frac{6\pi}{7}n)$
   Since $cos(\theta) = sin(\theta + \frac{\pi}{2})$
   $= -4sin(\frac{2\pi}{7}n) + 4sin(\frac{4\pi}{7}n + \frac{\pi}{2}) - 4sin(\frac{6\pi}{7}n)$

3. (a) x(t) $= \frac{-1}{2} j e^{j\frac{\pi}{8}t} + \frac{1}{2} j e^{-j\frac{\pi}{8}t}$
   Since $w_0 = \frac{\pi}{8}$ the nonzero Fourier series coefficients are, $a_1 = -\frac{1}{2}j$ and $a_{-1} = \frac{1}{2}j$

   (b) $y(t) = \frac{1}{2} e^{j\frac{\pi}{8}t} + \frac{1}{2} e^{-j\frac{\pi}{8}t}$
   Since $w_0 = \frac{\pi}{8}$ the nonzero Fourier series coefficients are, $b_1 = \frac{1}{2}$ and $b_{-1} = \frac{1}{2}$

   (c) Using the multiplication property we know that,
   $z(t) = x(t)y(t) \longleftrightarrow F.S.c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$.

   $a_k \longrightarrow \frac{-j\delta[k-1]}{2} + \frac{j\delta[k+1]}{2}$

   $b_k \longrightarrow \frac{\delta[k-1]}{2} + \frac{\delta[k+1]}{2}$
   Therefore, $c_k = a_k * b_k = \frac{-j}{4}\delta[k-2] + \frac{j}{4}\delta[k+2]$

   This implies that, the nonzero Fourier series coefficients of $z(t)$ are $c_2 = c_{-2}^* = \frac{-j}{4}$

4. Since $x(t)$ is real and odd, its Fourier series coefficients a_k are purely imaginary and odd. Therefore, $a_k = -a_{-k}$ and $a_0 = 0$. $a_2 = 3j$ is given, then $a_{-2} = -3j$. Also, since it is given that $a_k = 0$ for $|k| > 2$, the only unknown Fourier series coefficients are a_1 and a_{-1}. Using Parseval's relation,
   $\frac{1}{T}\int_{<T>} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$
   for the given signal we have
   $\frac{1}{4}\int_0^4 |x(t)|^2 dt = \sum_{k=-2}^{2} |a_k|^2$
   Using the information given in iv. along with the above equation
   $|a_0|^2 + |a_1|^2 + |a_{-1}|^2 + |a_2|^2 + |a_{-2}|^2 = 18$
   $0 + |a_1|^2 + |a_{-1}|^2 + 9 + 9 = 18$
   $|a_1|^2 + |a_{-1}|^2 = 0$
   Therefore, $a_1 = a_{-1} = 0$
   The signal which satisfy the given information is
   $x(t) = 3je^{2w_0jt} - 3je^{-2w_0jt}$
   Since $w_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$
   $x(t) = 3je^{\pi jt} - 3je^{-\pi jt} = 3j(2jsin(\pi t)) = -6sin(\pi t)$

5. (a) We know from table 5.2 and table 3.2 that:

Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1+\frac{1}{2})]}{N\sin[2\pi k/2N]}, \; k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1+1}{N}, \; k = 0, \pm N, \pm 2N, \ldots$

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$

We can conclude the following shifted signal formula:

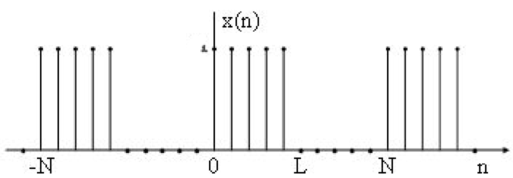

Figure 1

$$c_k = \begin{cases} \frac{L}{N}, & k = 0, \pm N, \pm 2N \ldots \\ \frac{1}{N} e^{-j\pi k\frac{(L-1)}{N}} \frac{\sin(\pi kL/N)}{\sin(\pi k/N)}, & \text{otherwise} \end{cases}$$

Then the spectral coefficients of $x[n]$:

$$a_k = \begin{cases} \frac{5}{9} & k = 0, N, -N, +2N, -2N.. \\ \frac{1}{9} e^{-j\pi k\frac{4}{9}} \frac{sin(\pi k5/9)}{sin(\pi k/9)} & otherwise \end{cases}$$

(b) The spectral coefficients of $y[n]$:

$$b_k = \begin{cases} \frac{4}{9} & k = 0, N, -N, +2N, -2N.. \\ \frac{1}{9} e^{-j\pi k\frac{3}{9}} \frac{sin(\pi k4/9)}{sin(\pi k/9)} & otherwise \end{cases}$$

(c) We know that $b_k = H(jkw_0)a_k$ where $H(jkw_0)$ is the frequency response.

So, frequency response is $\frac{b_k}{a_k}$,

$$H(jkw_0) = \begin{cases} \frac{4}{5} & k = 0, N, -N, +2N, -2N.. \\ e^{j\pi k\frac{1}{9}} \frac{sin(\pi k4/9)}{sin(\pi k5/9)} & otherwise \end{cases}$$