ex: let $A = \begin{bmatrix} 1 & 2a+2 & 3b-14 \\ 0 & a+1 & 2b-4 \\ 2 & a+1 & b-18 \end{bmatrix}$

For which values of a and b is the matrix A invertible?

as execise show that A is r.e. to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a+1 & 0 \\ 0 & 0 & b-2 \end{bmatrix}$$

If $a=-1$ OR $b=2$ then A is r.e to a matrix with a zero row, hence A is not invertible.

If $a \neq 1$ and $b \neq 2$ then $\begin{bmatrix} 1 & 0 & 0 \\ 0 & a+1 & 0 \\ 0 & 0 & b-2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Hence A is r.e to I, thus is invertible.

Recall: If $A \in \mathbb{R}^{m \times n}$ its transpose is $A^T \in \mathbb{R}^{n \times m}$ with $(A^T)_{ij} = A_{ji}$

Prop: let $A \in \mathbb{R}^{n \times n}$ be invertible. Then A^T is invertible with $(A^T)^{-1} = (A^{-1})^T$

Proof: (Recall $(AB)^T = B^T \cdot A^T$)

$$\left.\begin{aligned} A^T \cdot (A^{-1})^T &= (A^{-1} \cdot A)^T = I^T = I \\ (A^{-1})^T \cdot A^T &= (A \cdot A^{-1})^T = I^T = I \end{aligned}\right\} \Rightarrow (A^T)^{-1} = (A^{-1})^T$$

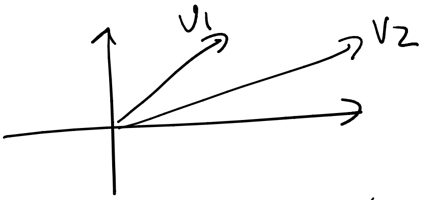
end of proof.

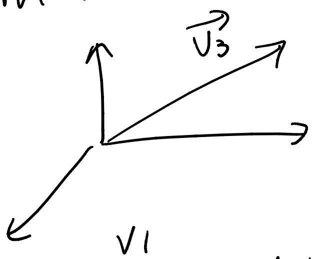
Vector Spaces

Motivation: Recall vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$:

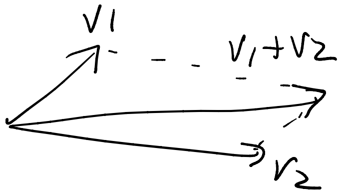
$\vec{v_3}$

<u>Motivation</u>: Recall vectors in $\mathbb{R}$ on $\mathbb{R}$:





We cand add vectors:



We can multiply vectors by real numbers (i.e., scalors)

Also, we have a special vector $\vec{0}$ (the zero vector)
s.t $\vec{0}+\vec{v}=\vec{v}$ for any other vector $\vec{v}$.

These operations have certain properties.
A vector space is a mathematical generalization of spaces in which such operations are possible.

<u>Definition</u>: A "vector space" (over $\mathbb{R}$) is a set V (whose elements are called "vectors") together with two operations:

1) $V \times V \longrightarrow V$ called "vector addition" (or simply "addition")
$(v_1, v_2) \longmapsto v_1 + v_2$

2) $\mathbb{R} \times V \longrightarrow V$ called "scalar multiplication"
$(c, v) \longmapsto c \cdot v$

satisfying the following

(V1) $v_1+(v_2+v_3) = (v_1+v_2)+v_3$ for any $v_1, v_2, v_3 \in V$

( ) $v_1 + v_2 = v_2 + v_1$ for any $v_1, v_2 \in V$

(V1) $v_1+(v_2+v_3) = (v_1+v_2)+v_3$ for any $v_1, v_2, v_3 \in$ ...

(V2) $v_1+v_2 = v_2+v_1$ for any $v_1, v_2 \in V$

(V3) There is an element 0_V (the "zero vector")
s.t. $0_V + w = w + 0_V = w$ for any $w \in V$.

(V4) For any $v \in V$ there is a vector $-v \in V$
s.t. $v + (-v) = 0_V = (-v) + v$
($-v$ is the "additive inverse" of v)(or simply "inverse" of v)

(V5) For any $c \in \mathbb{R}$ and $v_1, v_2 \in V$
$$c(v_1+v_2) = cv_1 + cv_2$$

(V6) For any $c_1, c_2 \in \mathbb{R}$, $v \in V$
$$(c_1 \underset{\substack{\uparrow \\ \text{addition in } \mathbb{R}}}{+} c_2)\, v = c_1 v \underset{\substack{\uparrow \\ \text{vector addition}}}{+} c_2 v$$

(V7) for any $c_1, c_2 \in \mathbb{R}$ and $v \in V$
$$(c_1 \underset{\substack{\uparrow \\ \text{mult. in } \mathbb{R}}}{\cdot} c_2)\, v = c_1 \overbrace{(c_2 v)}^{\text{is a vector}}$$
scalar mult.

(V8) For any $v \in V$, $\underset{\substack{\uparrow \\ \text{(number one in } \mathbb{R})}}{1} v = v$

<u>Prop</u>: let V be a vector space

1) There is a unique zero vector 0_V
2) For any $v \in V$, there is a unique $-v \in V$ s.t. $v + (-v) = 0_V$
3) For any $c \in \mathbb{R}$, $c \cdot 0_V = 0_V$
4) For any $v \in V$ $0 \cdot v = 0_V$
5) For any $v \in V$ $(-1) \cdot v = -v$

<u>Pf:</u> 1) Suppose $0_V \in V$ and $\bar{0}_V \in V$ satisfy (V3).
Then $0_V = 0_V + \bar{0}_V = \bar{0}_V$. Thus $0_V = \bar{0}_V$

Then $0_V \overset{\uparrow}{=} 0_V + \bar{0}_V \overset{\uparrow}{=} \bar{0}_V$ Thus $0_V = \bar{0}_V$

since $\bar{0}_V$ is a zero vector   since 0_V is a zero vector.

2) exercise

3) $c \cdot 0_V \overset{V3}{=} c \cdot (0_V + 0_V) \overset{V5}{=} c \cdot 0_V + c \cdot 0_V$

So we have $c \cdot 0_V = (c \cdot 0_V) + (c \cdot 0_V)$

$\Rightarrow \underbrace{c \cdot 0_V + (-(c \cdot 0_V))}_{0_V} = (c \cdot 0_V) + \underbrace{(c \cdot 0_V) + (-(c \cdot 0_V))}$

$= c \cdot 0_V + 0_V = \underline{c \cdot 0_V}$

4) exercise

5) $v + (-1)v \overset{V8}{=} 1v + (-1)v \overset{V6}{=} (1+(-1)) \cdot v = 0 \cdot v \overset{\text{part 3}}{=} 0_V$

end of proof.

Examples of vector spaces:

1) let $V = \mathbb{R}^{m \times n}$ ($= m \times n$ matrices.)

with $+$ matrix addition

with $c \cdot \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} c \cdot a_{11} & \cdots & c\,a_{1n} \\ \vdots & & \\ c\,a_{m1} & \cdots & c\,a_{mn} \end{bmatrix}$ as scalar multiplication

and $0_V = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$ = zero matrix

As exercise show this satisfies $V1, V2, \ldots, V8$!!

2) let $V = \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$

with $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$

$c \cdot (x_1, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n)$

$0_V = (0, 0, \ldots, 0)$

as exercise show this satisfies $V1, V2, \ldots, V8$.

For $n=2$ or $n=3$ this is the familiar space of

For $n=2$ or $n=3$ this is the familiar space of 2 dim (3 dim) vectors.

3) let $V = F([0,1]) = \{ f : [0,1] \to \mathbb{R} \mid f \text{ is a function} \}$

is a vector space with + defined as:

$$\boxed{(f+g)(x) = f(x) + g(x)} \quad \text{for any } x \in [0,1]$$

and scalar mult. defined by:

$$\boxed{(c \cdot f)(x) = c \cdot f(x)} \quad \text{for any } x \in [0,1] \text{ and } c \in \mathbb{R}.$$

also the zero vector = the zero function
i.e. the function which sends $x \mapsto 0$ for any $x \in [0,1]$.

One can check these satisfy V1, ..., V8,

For example, V6: let $c_1, c_2 \in \mathbb{R}$ and $f \in F([0,1])$

we need to show: $(c_1+c_2) \cdot f = c_1 f + c_2 f$

$$\big((c_1+c_2)f\big)(x) = (c_1+c_2) \cdot f(x) = c_1 f(x) + c_2 f(x)$$
$$= (c_1 f)(x) + (c_2 f)(x)$$
$$= (c_1 f + c_2 f)(x)$$

for any $x \in [0,1]$

thus, $(c_1+c_2)f = c_1 f + c_2 f$

4) Recall: A polynomial (with real coefficients) is an expression $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where $n \in \mathbb{N}$ and $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$.

If $a_n \neq 0$, the degree of this polynomial is n.

let $\mathbb{R}[x] = \{ a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{R}, n \in \mathbb{N} \}$ = the set of polynomials let $\mathbb{R}[x] = \{a_n x^n + \dots + a_1 x + a_0 \mid a_i \in \mathbb{R}, n \in \mathbb{N}\}$ of polynomials

$\mathbb{R}[x]$ is a vectorspace with: let $n \geq m$

$$(a_n x^n + \dots + a_m x^m + \dots + a_1 x + a_0) + (b_m x^m + \dots + b_1 x + b_0)$$
$$= a_n x^n + \dots + (a_m + b_m) x^m + \dots + (a_1 + b_1) x + (a_0 + b_0)$$

and

$$c \cdot (a_n x^n + \dots + a_1 x + a_0) = (c a_n) x^n + \dots + (c a_1) x + (c a_0)$$

with the zero polynomial (ie the poly. with all coefficients zero) is the vector.

As exercise verify that those satisfy $V1, \dots, V8$.

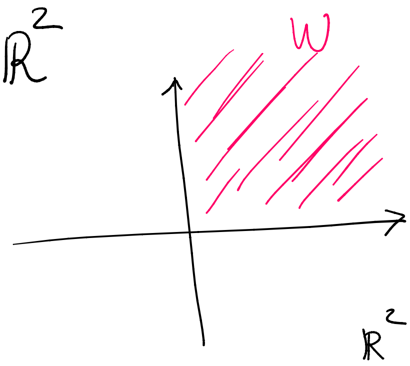
Subspaces:

Definition: let V be a vectorspace and $W \subseteq V$ be a subset.

1) If for any $w_1, w_2 \in W$ we have $w_1 + w_2 \in W$ then W is "closed under addition"

2) If for any $c \in \mathbb{R}$ and $w \in W$, we have $cw \in W$, then W is "closed under scalar multiplication"

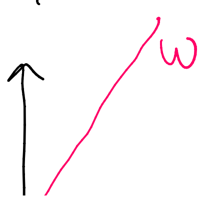
ex: 1) let $W = \{(x,y) \mid x > 0, y > 0\} \subseteq \mathbb{R}^2$



Clearly W is closed under $+$ (since if $(x_1, y_1), (x_2, y_2) \in W$ then $(x_1 + x_2, y_1 + y_2) \in W$).

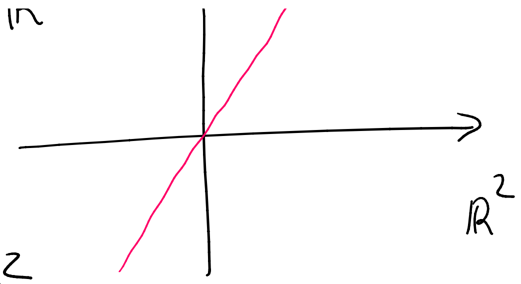
But W is not closed under $\cdot$ since $(1,1) \in W$, $-1 \in \mathbb{R}$, but $(-1) \cdot (1,1) = (-1,-1) \notin W$.

2) let $W = \{(x,y) \mid y = 2x\} \subseteq \mathbb{R}^2$

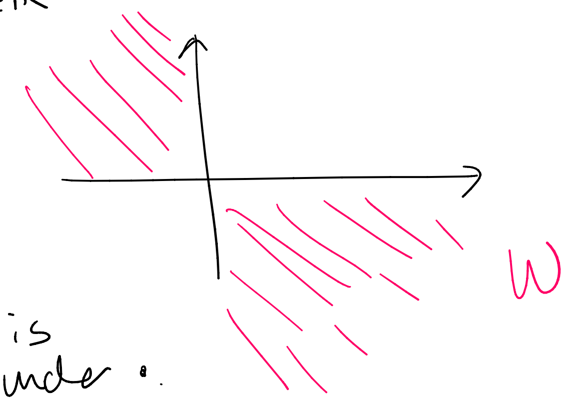


2) let $W = \{(x,y) \mid y = 2x\} \subseteq \mathbb{R}^2$

Clearly W is closed under both $+$ and $\cdot$



3) let $W = \{(x,y) \mid x \cdot y < 0\} \subseteq \mathbb{R}^2$



$(2,-1), (-1,2) \in W$

but $(2,-1) + (-1,2) = (1,1) \notin W$.

so W is NOT closed under $+$

also $0 \cdot \underbrace{(2,-1)}_{\in W} = (0,0) \notin W$. So W is not closed under $\cdot$.

4) let $W =$ invertible matrices in $\mathbb{R}^{n \times n} \subseteq \mathbb{R}^{n \times n}$

$I, -I \in W$ but $I + (-I) = 0 \notin W$.

so W is not closed under $+$.

So also not closed under $\cdot$

Also $0 \cdot \underbrace{I}_{\in W} = 0 \notin W$.

5) let $W = \{ f \in F([0,1]) \mid f \text{ is continuous} \} \subseteq F([0,1])$

W is closed under both $+$ and $\cdot$ (A result from Calculus)

6) let $W = \{ p \in \mathbb{R}[x] \mid p(0) = 0 \} \subseteq \mathbb{R}[x]$.

If $p, q \in W$ then $(p+q)(0) = p(0) + q(0) = 0 + 0 = 0$

so $p + q \in W$

If $c \in \mathbb{R}$, $p \in W$ then $(c \cdot p)(0) = c \cdot p(0) = c \cdot 0 = 0$

So $c \cdot p \in W$.

<u>Prop.</u>: let V be a vectorspace and $W \subseteq V$ be non-empty subset closed under $+$ and $\cdot$.

Then i) W contains the zero vector 0_V

non-empty subset ...

Then 1) W contains the zero vector 0_V

2) If $w \in W$ then $-w \in W$.

3) W itself is a vectorspace with the same operations

Pf: 2) If $w \in W$ then $-w = (-1)w \in W$ (since W is closed under $\cdot$)

1) Since $W \neq \emptyset$, let $w \in W$. Then $0 \cdot w = 0_V \in W$.

3) Since axioms $V1, \ldots, V8$ hold for V, using 1) and 2) they also hold for W.

end of proof

Definition: let V be a vectorspace and $W \subseteq V$. W is called a "subspace" of V if

1) $W \neq \emptyset$ 2) W is closed under $+$

3) W is closed under $\cdot$

Note: 1) A subspace must contain the zero vector, by previous prop.

2) If V is a vectorspace then $\{0_V\}$ and V are always subspaces of V. (These are the "trivial" subspaces")

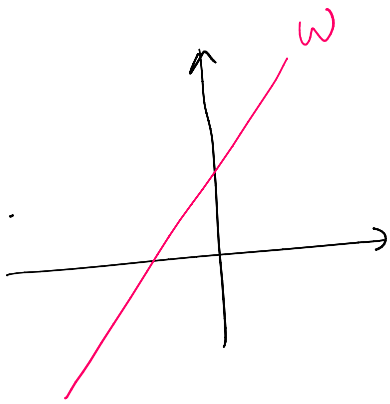
ex: 1) let $W = \{(x,y) \mid y = ax\} \subseteq \mathbb{R}^2$ → (W is a line passing through the origin)

$(0,0) \in W$, and W is closed under both $+$ and $\cdot$.

So, W is a subspace of $\mathbb{R}^2$

2) let $W = \{(x,y) \mid y = 2x+1\} \subseteq \mathbb{R}^2$

since $(0,0) \notin W$, W is not a subspace.



3) let $W = \{(x,y,z) \mid z = ax+by\} \subseteq \mathbb{R}^3$

$(0,0,0) \in W$ so $W \neq \emptyset$

(... $z = ax+by$ ...

$(0,0,0) \in W$ so $W \neq \emptyset$

If $(x_1, y_1, z_1), (x_2, y_2, z_2) \in W$ $\left(\text{ie } \begin{array}{l} z_1 = ax_1 + by_1 \\ z_2 = ax_2 + by_2 \end{array} \right)$

then $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in W$

similarly W is closed under $\cdot$ since $z_1 + z_2 = a(x_1 + x_2) + b(y_1 + y_2)$

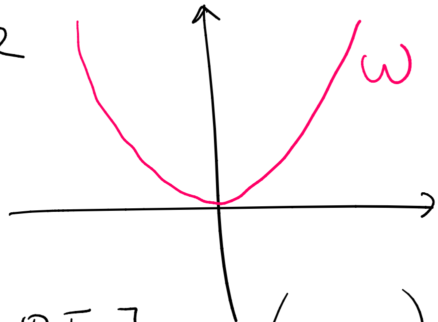
So, W is a subspace of $\mathbb{R}^3$.

(W is a plane in $\mathbb{R}^3$ passing through the origin).

4) let $W = \{(x,y) \mid y = x^2\} \subseteq \mathbb{R}^2$

is not a subspace

since $(1,1) \in W$ but $2(1,1) \notin W$.



5) $W = \{p(x) \in \mathbb{R}[x] \mid \deg p(x) \leq n\} \subseteq \mathbb{R}[x]$ $(n \geq 1)$

= polynomials of degree at most n

If $p(x) = a_k x^k + \cdots + a_1 x + a_0 \in W$ i.e $k \leq n$

$q(x) = b_\ell x^\ell + \cdots + b_1 x + b_0 \in W$ $\ell \leq n$

then $\deg(p(x) + q(x)) \leq n$ and $\deg(c \cdot p(x)) \leq n$

Clearly $W \neq \emptyset$, thus is a subspace of $\mathbb{R}[x]$.

6) let $W = \{p(x) \in \mathbb{R}[x] \mid \deg p(x) = n\} \subseteq \mathbb{R}[x]$

$x^n \in W$, $-x^n \in W$ $(n \geq 1)$

but $x^n + (-x^n) = 0 \notin W$

So W is not a subspace.

7) let $W = \{A \in \mathbb{R}^{n \times n} \mid A_{11} + A_{22} + \cdots + A_{nn} = 0\} \subseteq \mathbb{R}^{n \times n}$

Is W a subspace? Clearly, $0 \in W$.

zero matrix

Is W a subspace? Clearly, $0 \in W$.
zero matrix

If $A, B \in W$ then

$(A+B)_{11} + (A+B)_{22} + \cdots + (A+B)_{nn}$

$= A_{11} + B_{11} + A_{22} + B_{22} + \cdots + A_{nn} + B_{nn} = 0 \Rightarrow A+B \in W$

Also $(cA)_{11} + \cdots + (cA)_{nn} = c A_{11} + \cdots + c A_{nn}$

$= c(A_{11} + \cdots + A_{nn}) = 0$

so $cA \in W$.

Thus W is a subspace of $\mathbb{R}^{n \times n}$.

<u>Prop</u>: If W_1 and W_2 are subspaces of V then $W_1 \cap W_2$ is a subspace of V.

<u>Pf</u>: exercise!

<u>Definition</u>: Let V be a vector space and $v_1, v_2, \ldots, v_n \in V$. An element $v \in V$ is a "linear combination of $v_1, \ldots, v_n$" if one can write $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ for some scalars $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

<u>ex</u>: 1) $V = \mathbb{R}^{2\times 2}$ $\quad \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

so $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ is a linear combination of

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

2) $(1,-2,3) \in \mathbb{R}^3$ $\quad (1,-2,3) = 1\cdot(1,0,0) + (-2)(0,1,0) + 3\cdot(0,0,1)$

So $(1,-2,3)$ is a linear comb. of $(1,0,0), (0,1,0), (0,0,1)$.

(in fact this is true for any vector in $\mathbb{R}^3$!)

Definition: Let V be a vector space and

<u>Definition</u>: let V be a vectorspace and $S = \{v_1, v_2, \dots, v_n\} \subseteq V$

let $\langle S \rangle$ be the set of all linear combinations of $v_1, v_2, \dots, v_n$.

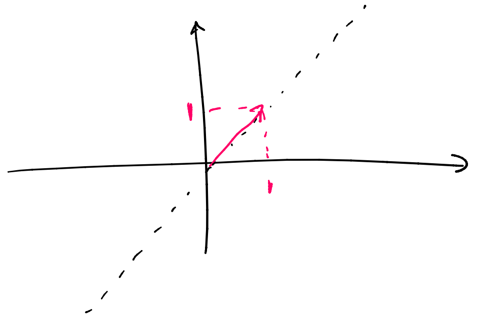
i.e $\langle S \rangle = \{c_1 v_1 + c_2 v_2 + \dots + c_n v_n \mid c_1, c_2, \dots, c_n \in \mathbb{R}\}$

$\langle S \rangle$ is the "span of S" (or "linear span of S")

<u>ex</u>: 1) $S = \{(1,1)\} \subseteq \mathbb{R}^2$

then $\langle S \rangle = \{c(1,1) \mid c \in \mathbb{R}\} = \{(c,c) \mid c \in \mathbb{R}\}$

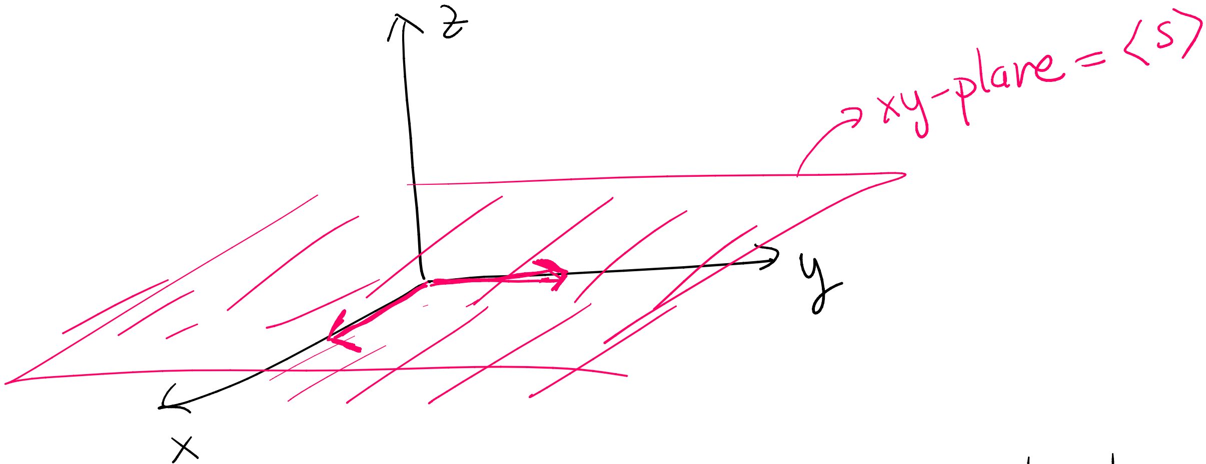
$= \text{the line } y = x \text{ in } \mathbb{R}^2$



2) let $S = \{(1,0,0), (0,1,0)\} \subseteq \mathbb{R}^3$

$\langle S \rangle = \{c_1(1,0,0) + c_2(0,1,0) \mid c_1, c_2 \in \mathbb{R}\}$

$= \{(c_1, c_2, 0) \mid c_1, c_2 \in \mathbb{R}\} = xy\text{-plane}$



3) $\langle 0_V \rangle = \{0_V\}$

<u>Remark</u>: Some books write $\text{Span}(S)$ instead

3) $\langle 0_V \rangle = \{0_V\}$

Remark: write Span(S) instead of $\langle S \rangle$.

<u>Definition</u>: let V be a vector space and $S = \{v_1, \dots, v_n\} \subseteq V$. If $\langle S \rangle = V$ then S is called a "spanning set of V"

OR we say "S spans V".

<u>ex</u>: 1) $S = \{(1,0),(0,1)\} \subseteq \mathbb{R}^2$ then $\langle S \rangle = \mathbb{R}^2$

So S is a spanning set of $\mathbb{R}^2$

2) $S = \{ \overset{v_1}{(1,2,-1)}, \overset{v_2}{(-3,1,0)}, \overset{v_3}{(2,-3,2)} \} \subseteq \mathbb{R}^3$

show that $\langle S \rangle = \mathbb{R}^3$

This means, for any $(a,b,c) \in \mathbb{R}^3$ we must show that there are scalars $c_1, c_2, c_3 \in \mathbb{R}$ s.t

$$(a,b,c) = c_1(1,2,-1) + c_2(-3,1,0) + c_3(2,-3,2)$$

$$\iff$$

$$(a,b,c) = (c_1 - 3c_2 + 2c_3,\ 2c_1 + c_2 - 3c_3,\ -c_1 + 2c_3)$$

$\iff$
$$\begin{aligned} c_1 - 3c_2 + 2c_3 &= a \\ 2c_1 + c_2 - 3c_3 &= b \\ -c_1 + \quad\ \ 2c_3 &= c \end{aligned}$$
has a solution for any $(a,b,c) \in \mathbb{R}^3$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & a \\ 2 & 1 & -3 & b \\ -1 & 0 & 2 & c \end{array}\right] \xrightarrow[-2R_1+R_2]{R_1+R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & a \\ 0 & 7 & -7 & b-2a \\ 0 & -3 & 4 & a+c \end{array}\right]$$

$\downarrow \frac{3}{7}R_2 + R_3$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & a \\ 0 & 7 & -7 & b-2a \\ 0 & 0 & 1 & \frac{3}{7}(b-2a)+(a+c) \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & \frac{3}{7}(b-2a)+(a+c) \end{array}\right]$$

So the system has a unique solution independent of a, b and c.

Hence, $(a,b,c) \in \langle S \rangle$ for any $(a,b,c) \in \mathbb{R}^3$

thus $\langle S \rangle = \mathbb{R}^3$.

<u>ex</u>: $S = \{(1,0,1), (2,1,-1)\} \subseteq \mathbb{R}^3$

Describe the elements of $\langle S \rangle$.

$$\langle S \rangle = \{ c_1(1,0,1) + c_2(2,1,-1) \mid c_1, c_2 \in \mathbb{R} \}$$
$$= \{ (c_1 + 2c_2, c_2, c_1 - c_2) \mid c_1, c_2 \in \mathbb{R} \}$$

So, $(a,b,c) \in \langle S \rangle \iff$ there are $c_1, c_2 \in \mathbb{R}$

s.t $c_1 + 2c_2 = a$

$c_2 = b$

$c_1 - c_2 = c$

$$\iff \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ has a solution.}$$

$$\left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & b \\ 1 & -1 & c \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & b \\ 0 & -3 & c-a \end{array}\right] \xrightarrow{3R_2+R_3} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & b \\ 0 & 0 & c-a+3b \end{array}\right]$$

System has a solution $\iff c - a + 3b = 0$

i.e. $(a,b,c) \in \langle S \rangle \iff c - a + 3b = 0$

(for example $(3, 1, 0) \in \langle S \rangle$ but $(1,1,1) \notin \langle S \rangle$)

$\langle S \rangle = \langle (1,0,1), (2,-1,1) \rangle = \{(a,b,c) \in \mathbb{R}^3 \mid c - a + 3b = 0\}$

i.e $\langle S\rangle = \langle (1,0,1),(2,-1,1)\rangle = \{(a,b,c)\in\mathbb{R}^3 \mid c-a+3b=0\}$

<u>Theorem</u>: let V be a vecterspace and $S \subseteq V$. $S = \{v_1,\dots,v_n\}$

Then $\langle S\rangle$ is a subspace of V.

<u>Pf</u>: Recall $\langle S\rangle = \{c_1v_1+c_2v_2+\cdots+c_nv_n \mid c_1,\dots,c_n\in\mathbb{R}\}$

For $c_1=c_2=\cdots=c_n=0$ then $0_V\in\langle S\rangle$. ie $\langle S\rangle\neq\emptyset$

If $c_1v_1+\cdots+c_nv_n,\ c_1'v_1+\cdots+c_n'v_n\in\langle S\rangle$

then $(c_1v_1+\cdots+c_nv_n)+(c_1'v_1+\cdots+c_n'v_n)$

$=(c_1+c_1')v_1+(c_2+c_2')v_2+\cdots+(c_n+c_n')v_n\in\langle S\rangle$

If $d\in\mathbb{R}$, $d(c_1v_1+\cdots+c_nv_n)=(dc_1)v_1+\cdots+(dc_n)v_n\in\langle S\rangle$

So, $\langle S\rangle$ is closed under $+$ and $\cdot$

Thus $\langle S\rangle$ is a subspace of V

end of proof.