Week2

Tuesday, 11 February, 2020 9:13 AM

Defn: 1) A square matrix A is "symmetric" if $A = A^T$

i.e., $A_{ij} = A_{ji}$ for any i and j.

ex: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 0 \end{bmatrix}$ is a symmetric matrix.

2) A square matrix A is "skew symmetric" if $A = -A^T$

i.e. $A_{ij} = -A_{ji}$ for any i and j.

Defn: 1) let $A \in \mathbb{R}^{m \times n}$. The first nonzero entry of each row is the "leading entry" of that row. (if it exists)

ex: $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 5 & 7 \\ 0 & 0 & 0 \\ 0 & 2 & 5 \end{bmatrix}$ 1, 1, 2 are the leading entries of A.

2) If the leading entries of A are of the form

$A_{1j_1}, A_{2j_2}, A_{3j_3}, \ldots, A_{rj_r}$ where

$j_1 < j_2 < j_3 < \cdots < j_{r-1} < j_r$ and the zero rows of A are at the bottom

then A is called an "Echelon Matrix".

Such a matrix is of this form:

$$\begin{bmatrix} 0 \cdots 0 & A_{1j_1} & \text{———} & & & \\ 0 \cdots 0 & 0 \cdots & A_{2j_2} & \text{———} & & \\ 0 & \cdots & \cdots & 0 & A_{3j_3} & \text{———} \\ \vdots & & & & & A_{rj_r} \\ 0 & \cdots & & & & 0 \\ 0 & \cdots & & & & 0 \\ 0 & \cdots & & & & 0 \end{bmatrix}$$

ex: $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -4 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ is NOT an echelon matrix.

$B = \begin{bmatrix} 0 & 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 7 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is an echelon matrix.

3) let A be an echelon matrix.

If i) The leading entries of A are all equal to 1.

ii) Each leading entry is the only nonzero entry in its column

then A is called a "Row reduced echelon matrix".

ii) Each leading entry ... column

then A is called a "Row reduced echelon matrix".

ex: $A = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a row-reduced echelon matrix.

$B = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$ is an echelon matrix but not a row-reduced echelon matrix

Elementary Row Operations:

let $A \in \mathbb{R}^{m\times n}$ with rows $R_1, R_2, \dots, R_m$ considered as $1\times n$ matrix.

The following operations on the rows of A are "elementary row operations"

Type I: Add cR_i to R_j where $i \neq j$ and $c \in \mathbb{R}$. (Notation: $cR_i + R_j$)

Type II: Interchange R_i and R_j Type equation here. (Notation: $R_i \leftrightarrow R_j$)

Type III: Multiply R_i by a non-zero scalar $c \in \mathbb{R}$ (Notation: cR_i)

ex:

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 4 & 5 & 7 & 1 \\ 6 & 8 & 3 & 11 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 3 & 9 & -5 \\ 6 & 8 & 3 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 1 & 7 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 3 \\ 1 & 7 \\ 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 \\ 1 & 4/3 \end{bmatrix}$$

ex: one can perform several elem. row ops. on a matrix.

$$\begin{bmatrix} 2 & 4 & 1 \\ 1 & 0 & -1 \\ 2 & 7 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 2 & 7 & 4 \\ 1 & 0 & -1 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 2 & 7 & 4 \\ 1 & 0 & -1 \\ 0 & 4 & 3 \end{bmatrix}$$

$$\downarrow \tfrac{1}{2}R_1$$

$$\begin{bmatrix} 1 & 7/2 & 2 \\ 1 & 0 & -1 \\ 0 & 4 & 3 \end{bmatrix}$$

Remark: We can think of elem. row operations functions $f: \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$

<u>Remark</u>: We can think of elem. row operations as a function $\mathcal{E}: \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{m\times n}$.

<u>Proposition</u>: Each elementary row operation has an inverse of the same type.

<u>Pf</u>: The inverse of $cR_i + R_j$ is $-cR_i + R_j$

$$A = \begin{bmatrix} \bar{R}_1 \\ \vdots \\ \bar{R}_i \\ \vdots \\ \bar{R}_j \\ \vdots \\ \bar{R}_m \end{bmatrix} \xrightarrow{cR_i+R_j} \begin{bmatrix} \bar{R}_1 \\ \vdots \\ \bar{R}_i \\ \vdots \\ c\bar{R}_i+\bar{R}_j \\ \vdots \\ \bar{R}_m \end{bmatrix} \xrightarrow{-cR_i+R_j} \begin{bmatrix} \bar{R}_1 \\ \vdots \\ \bar{R}_i \\ \vdots \\ \bar{R}_j \\ \vdots \\ \bar{R}_m \end{bmatrix}$$

Similarly, the inverse of $R_i \leftrightarrow R_j$ is $R_i \leftrightarrow R_j$
the inverse of cR_i $(c\neq 0)$ is $\frac{1}{c}R_i$ end of proof.

<u>Definition</u>: let $A, B \in \mathbb{R}^{m\times n}$.
We say "A is row equivalent to B" if B can be obtained from A by a sequence of elementary row operations.
[i.e., there exists elem. row operations $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_r$ s.t.
$B = \mathcal{E}_r \mathcal{E}_{r-1} \cdots \mathcal{E}_3 \mathcal{E}_2 \mathcal{E}_1(A)$]

<u>ex</u>: $A = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 4 & -1 & 6 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix}$

are Row equivalent since:

$$A = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 4 & -1 & 6 & 2 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 2 & 4 & 3 & 1 \\ 0 & -9 & 0 & 0 \end{bmatrix}$$

$$\downarrow -\tfrac{1}{9}R_2$$

$$\begin{bmatrix} 2 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xleftarrow{-4R_2+R_1} \begin{bmatrix} 2 & 4 & 3 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\downarrow R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix} = B$$

<u>Theorem</u>: let $A, B, C \in \mathbb{R}^{m\times n}$. (r.e. = row equiv.)

1) A is row equivalent to itself
2) If A is r.e. to B then B is r.e. to A.
3) If A is r.e. to B and B is r.e. to C then

2) If A is r.e. to B then ...

3) If A is r.e. to B and B is r.e. to C then A is r.e. to C.

Pf: 1) Clear.

2) Suppose $B = \mathcal{E}_r \mathcal{E}_{r-1} \cdots \mathcal{E}_2 \mathcal{E}_1 (A)$

Then $A = \mathcal{E}_1^{-1} \mathcal{E}_2^{-1} \cdots \mathcal{E}_{r-1}^{-1} \mathcal{E}_r^{-1} (B)$

so B is r.e. to A.

3) Suppose $B = \mathcal{E}_r \mathcal{E}_{r-1} \cdots \mathcal{E}_2 \mathcal{E}_1 (A)$ $\quad$ $\mathcal{E}_1, \dots, \mathcal{E}_r$, are

$C = \mathcal{E}_k' \mathcal{E}_{k-1}' \cdots \mathcal{E}_2' \mathcal{E}_1' (B)$ $\quad$ $\mathcal{E}_1', \dots, \mathcal{E}_k'$ elem. row ops.

Then $C = \underbrace{\mathcal{E}_k' \mathcal{E}_{k-1}' \cdots \mathcal{E}_2' \mathcal{E}_1' \mathcal{E}_r \mathcal{E}_{r-1} \cdots \mathcal{E}_2 \mathcal{E}_1}\, (A)$

which means A is r.e. to C.

end of proof.

Theorem: Every matrix is row equivalent to a row reduced echelon matrix.

Pf: sketch by an example:

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 4 \\ 0 & 2 & 6 & 8 & -1 \\ 0 & 1 & 5 & 6 & 1 \\ 0 & 10 & 5 & 10 & 5 \end{bmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 2 & 6 & 8 & -1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 5 & 6 & 1 \\ 0 & 10 & 5 & 10 & 5 \end{bmatrix} \xrightarrow[-5R_1+R_4]{-\frac{1}{2}R_1+R_3} \begin{bmatrix} 0 & 2 & 6 & 8 & -1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 2 & 3/2 \\ 0 & 0 & -25 & -30 & 10 \end{bmatrix}$$

$$\xrightarrow[25R_2+R_4]{-2R_2+R_3} \begin{bmatrix} 0 & 2 & 6 & 8 & -1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -2 & -13/2 \\ 0 & 0 & 0 & 20 & 110 \end{bmatrix} \xrightarrow{10R_3+R_4} \overbrace{\begin{bmatrix} 0 & 2 & 6 & 8 & -1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -2 & -13/2 \\ 0 & 0 & 0 & 0 & 45 \end{bmatrix}}^{\text{an echelon matrix}}$$

$$\xrightarrow[R_3+R_2]{-6R_2+R_1} \begin{bmatrix} 0 & 2 & 0 & -4 & -25 \\ 0 & 0 & 1 & 0 & -5/2 \\ 0 & 0 & 0 & -2 & -13/2 \\ 0 & 0 & 0 & 0 & 45 \end{bmatrix} \xrightarrow[-2R_3+R_1]{\frac{1}{45}R_4} \begin{bmatrix} 0 & 2 & 0 & 0 & -12 \\ 0 & 0 & 1 & 0 & -5/2 \\ 0 & 0 & 0 & -2 & -13/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\frac{5}{2}R_4+R_2,\ \frac{13}{2}R_4+R_3]{12R_4+R_1} \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_3 \;\downarrow\; \frac{1}{2}R_1}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$$ a matrix in row reduced echelon form.

Defn: A matrix which is obtained from the identity matrix by applying one elementary row operation is called an "elementary matrix".

ex: identity matrix

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=I} \xrightarrow{-5R_3+R_1} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$$ an elementary matrix.

Nototation: If $\mathcal{E}$ is an elementary row op. then the corresponding elem. matrix is denoted by $\mathcal{E}(I)$.

Thm: let $\mathcal{E}$ be an elem. row op. let $A \in \mathbb{R}^{m\times n}$ matrix.

Then, $\mathcal{E}(A) = \mathcal{E}(I)\cdot A$

i.e., $\mathcal{E}(A)$ is obtained from A by multiplying $\mathcal{E}(I)$ by A.

ex: let $\mathcal{E} = -5R_2 + R_1$ $\quad A = \begin{bmatrix} 2 & 7 \\ 1 & -2 \end{bmatrix}$

$$\mathcal{E}(A) = \begin{bmatrix} -3 & 17 \\ 1 & -2 \end{bmatrix}$$

$$\mathcal{E}(I) = \mathcal{E}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$$

$=$

$$\mathcal{E}(I)\cdot A = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -3 & 17 \\ 1 & -2 \end{bmatrix}$$

Proof: Observe: If $A = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}_{m\times n}$ $R_1, \ldots, R_m$ are the rows of A considered as $1\times n$ matrices

and B is another matrix

then $AB = \begin{bmatrix} R_1B \\ \vdots \\ R_mB \end{bmatrix}$ wher $R_1B, \ldots, R_mB$ are rows of AB

let $\mathcal{E} = cR_i + R_j$ let $A = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}$ then $\mathcal{E}(A) = \begin{bmatrix} R_1 \\ \vdots \\ cR_i+R_j \\ \vdots \\ R_m \end{bmatrix}$

let $I = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$ where $e_i = [0 \cdots 0\, 1\, 0 \cdots 0]$
i-th position

Then

$$\mathcal{E}(I) = \begin{bmatrix} e_1 \\ \vdots \\ ce_i+e_j \\ \vdots \\ e_m \end{bmatrix}$$

$$\mathcal{E}(I)\cdot A = \begin{bmatrix} e_1 \\ \vdots \\ ce_i+e_j \\ \vdots \\ e_m \end{bmatrix} A = \begin{bmatrix} e_1A \\ e_2A \\ \vdots \\ (ce_i+e_j)A \\ \vdots \\ e_m \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ cR_i+R_j \\ \vdots \\ R_m \end{bmatrix} = \mathcal{E}(A)$$

(Note: $e_1A = [1\ 0\ 0 \cdots 0]A = [1\ 0 \cdots 0]\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} = R_1$
in general $e_iA = R_i$)

Similarly this is true for other type of row operations
end of proof.

ex: $A = \begin{bmatrix} x & y & z \\ a & b & c \\ k & l & m \end{bmatrix}$ we want $\mathcal{E} = R_2 \leftrightarrow R_3$

$$\mathcal{E}(I) = \mathcal{E}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathcal{E}(I)\cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix} = \begin{bmatrix} x & y & z \\ k & l & m \end{bmatrix} = \mathcal{E}(A)$$

$$\mathcal{E}(I)\cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x & y & z \\ a & b & c \\ k & \ell & m \end{bmatrix} = \begin{bmatrix} x & y & z \\ k & \ell & m \\ a & b & c \end{bmatrix} = \mathcal{E}(A)$$

Prop: Suppose B is obtained from A by a sequence of elementary row operations $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_r$

Then $B = P\cdot A$ where $P = \underbrace{\mathcal{E}_r(I)\cdots\mathcal{E}_2(I)\,\mathcal{E}_1(I)}_{\text{a product of elementary matrices.}}$

Pf: Clear!

ex: $A = \begin{bmatrix} 2 & 1 \\ -1 & 5 \end{bmatrix} \xrightarrow[\mathcal{E}_1]{-R_1+R_2} \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \xrightarrow[\mathcal{E}_2]{R_1 \leftrightarrow R_2} \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}$

$\downarrow 4R_2 \;)\, \mathcal{E}_3$

$\begin{bmatrix} -3 & 4 \\ 8 & 4 \end{bmatrix} = B$

$\mathcal{E}_1(I) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$\mathcal{E}_2(I) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{E}_3(I) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$B \overset{?}{=} \underbrace{\mathcal{E}_3(I)\mathcal{E}_2(I)\,\mathcal{E}_1(I)}_{P}\,A$

$P = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 4 & 0 \end{bmatrix}$

$B \overset{?}{=} \begin{bmatrix} -1 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 5 \end{bmatrix}$

$= \begin{bmatrix} -3 & 4 \\ 8 & 4 \end{bmatrix}$ ✓

Systems of Linear Equations:

An $m\times n$ system of linear equations is:

An $m \times n$ system of (linear) equations is:

$$\text{System } S \begin{cases} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = b_m \end{cases}$$

(n unknowns, m equations)

If system has solution(s) we will call it "consistent" and "inconsistent" otherwise.

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \rightarrow \text{The "Coefficient matrix"}$$

$m \times n$ → # of unknowns, # equations

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

A solution to System S is a $n \times 1$ matrix $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$AX = B \rightarrow$ "matrix equation"

System S is completely determined by the matrices A and B.

We write this an "augmented matrix"

$$[A \mid B]_{m \times (n+1)} \rightarrow \text{augmented matrix corresponding to system } S$$

<u>Defn</u>: Two $m \times n$ systems S_1 and S_2 are "equivalent" if their corresponding augmented matrices are row equivalent.

<u>Theorem</u>: Equivalent systems have the same solutions.

Theorem: Equivalent systems

Pf: Suppose S_1 is the system $[A_1 \mid B_1]$

—— S_2 —— $[A_2 \mid B_2]$

Suppose they are equivalent,

i.e, $[A_1 \mid B_1]$ and $[A_2 \mid B_2]$ are row equivalent.

From the above observations, there is a matrix P s.t.

$$[A_2 \mid B_2] = P[A_1 \mid B_1]$$

$$\Rightarrow A_2 = PA_1 \quad \text{and} \quad B_2 = PB_1 \quad (\text{Why?})$$

let X be a solution to the system $[A_1 \mid B_1]$.

This means $A_1 X = B_1$.

$$\Rightarrow P(A_1 X) = PB_1 \Rightarrow (PA_1)\cdot X = PB_1$$

$$\Rightarrow A_2 \cdot X = B_2$$

So, X is also a solution to the system $[A_2 \mid B_2]$.

Similarly, there is a matrix Q s.t $[A_1 \mid B_1] = Q[A_2 \mid B_2]$

$$\Rightarrow A_1 = QA_2 \quad \text{and} \quad B_1 = QB_2$$

If Y is a solution of $[A_2 \mid B_2]$ then $A_2 Y = B_2$

$$\Rightarrow QA_2 Y = QB_2 \Rightarrow A_1 \cdot Y = B_1$$

$\Rightarrow$ Y is a solution of $[A_1 \mid B_1]$

end of proof.

So, Given a system, one can reduce its augmented matrix to echelon form (OR row reduced echelon form).

i.e. the system will be equivalent to a system of the form:

$$\begin{aligned}
A_{1j_1} x_{j_1} + \cdots\cdots &= b_1 \\
A_{2j_2} x_{j_2} + \cdots &= b_2 \\
\ddots \quad & \vdots \\
A_{rj_r} x_{j_r} + \cdots &= b_r \\
0 &= b_{r+1} \\
0 &= 0 \\
& \vdots \\
0 &= 0
\end{aligned}$$

(A system in echelon form)

This process is called "Gaussian Elimination" (OR "Gaus-Jordan Elimination")

There are 3 possibilities:

Possibility 1: $b_{r+1} \neq 0$ the system has no solutions! (i.e. it is inconsistent)

ex:

$$\begin{aligned}
x + y - z &= 2 \\
2x + 3y - z &= 0 \\
3x + 4y - 2z &= 1
\end{aligned}
\qquad
\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & 3 & -1 & 0 \\ 3 & 4 & -2 & 1 \end{array}\right]$$

$\updownarrow$ equivalent systems.

$\downarrow$ Gaussian Elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

(echelon matrix)

$$\begin{aligned}
x + y - z &= 2 \\
y + z &= -4 \\
0 &= -1 = b_{r+1}
\end{aligned}$$

has no solutions!

Possibility 2 $\quad b_{r+1} = 0$

Possibility 2.1 $\{x_{j_1}, x_{j_2}, \dots, x_{j_r}\} =$ set of all unknowns.

In this case there is a unique solution and it can be found by "back substitution"

<u>ex</u>:

$$\begin{aligned} x - 3y + z &= 4 \\ 2x - 8y + 8z &= -2 \\ -6x + 3y - 15z &= -9 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & -9 \end{array}\right]$$

$\downarrow$ (G.E)

$$\begin{aligned} x - 3y + z &= 4 \\ -y + 3z &= -5 \\ -18z &= 36 \end{aligned} \leftarrow \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & -18 & 36 \end{array}\right] \text{(echelon form)}$$

$$\overset{eq3}{\Longrightarrow} z = -2 \overset{eq2}{\Longrightarrow} y = -1 \overset{eq1}{\Longrightarrow} x = 3 \quad (\text{"Back subst."})$$

So the unique solution of the system is $\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$.

Possibility 2.2 There are more unknowns than $\{x_{j_1}, \dots, x_{j_r}\}$.

let us call $x_{j_1}, x_{j_2}, \dots, x_{j_r}$ "basic variables"
and the remaining ones "free variables"

In this case the basic variables can be written in terms of the free variables, and there will be infinitely many solutions:

<u>ex</u>:

$$\begin{aligned} x + y - z + t + u &= 1 \\ -x + 2y + 3z - t + 2u &= -1 \\ 2x + y - z + 2t - u &= 2 \end{aligned}$$

$$2x + y - z + 2t - u = 2$$
$$x + 6y + 4z + t + 4u = 1$$
$$8y + 7z + 6u = 0$$
$$3x + 7y + 3z + 3t + 3u = 3$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 2 & 3 & -1 & 2 & -1 \\ 2 & 1 & -1 & 2 & -1 & 2 \\ 1 & 6 & 4 & 1 & 4 & 1 \\ 0 & 8 & 7 & 0 & 6 & 0 \\ 3 & 7 & 3 & 3 & 3 & 3 \end{array}\right] \xrightarrow{G.E.} \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 5 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x + y - z + t + u = 1$$
$$-y + z - 3u = 0$$
$$5z - 6u = 0$$

x, y, z are basic var.

t, u are free var.

$$z \overset{eq3}{=} \frac{6}{5}u \qquad y \overset{eq2}{=} -\frac{9}{5}u \qquad x \overset{eq1}{=} 1 - t + 2u$$

So, a general solution of the system is of the form:

$$\begin{bmatrix} 1 - t + 2u \\ -9/5\, u \\ 6/5\, u \\ t \\ u \end{bmatrix}, \quad t, u \in \mathbb{R}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 2 \\ -9/5 \\ 6/5 \\ 0 \\ 1 \end{bmatrix} \qquad u, t \in \mathbb{R}.$$