differential equations

first order differential equations

$y' + p(t)y = g(t)$

1- separable equations (y'liler).dy = (x'liler) dx
2- homogenous equations ($y' + p(t)y = 0$)
3- linear equations ($y^2, y.y', e^y, \sin y$ içermeyen)
4- method of integrating factors ($\mu = e^{\int p(t)dt}$)
5- exact equations ($Mdx + Ndy = 0$)

systems of first order linear equations

$$x_1' = ax_1 + bx_2 + b_1(t)$$
$$x_2' = cx_1 + dx_2 + b_2(t)$$

$$\frac{dx}{dt} = x' = Ax + b(t)$$

1) homogenous linear systems with constant coefficients ($b(t) = 0$)

distinct and real

$x_1' = 2x_1 - 4x_2$, $x_2' = -x_1 - x_2$ $\quad x' = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} x \quad \lambda_1 = 3 \rightarrow v = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \quad \lambda_2 = -2 \rightarrow v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x = c_1 e^{3t}\begin{bmatrix} -4 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

complex

$x_1' = -4x_1 + 10x_2$, $x_2' = -5x_1 + 6x_2$ $\quad x' = \begin{bmatrix} -4 & 10 \\ -5 & 6 \end{bmatrix} x \quad \lambda = 1 \mp 5i \rightarrow v = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \quad x' = e^{(1+5i)t}\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

$$(\cos 5t + i\sin 5t)\begin{bmatrix} 1-i \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 5t + i\sin 5t - i\cos 5t + \sin 5t \\ \cos 5t + i\sin 5t \end{bmatrix} = \underbrace{\begin{bmatrix} \cos 5t + \sin 5t \\ \cos 5t \end{bmatrix}}_{} + i\underbrace{\begin{bmatrix} \sin 5t - \cos 5t \\ \sin t \end{bmatrix}}_{}$$

$x = c_1 e^{t}[\;] + c_2 e^{t}[\;]$

repeated

$e^{At} = P.\Delta.P^{-1}$
↳ normalde $\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$ $\quad e^{Jt} = e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, \; e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

$x_1' = 7x_1 + x_2$, $x_2' = -4x_1 + 3x_2$ $\quad x' = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} x \quad (\lambda = 5) \times 2$

$v_1 \Rightarrow \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad v_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \quad v_2 \Rightarrow \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix}$

$$x = c_1 e^{5t}\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{5t}\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix}\right) = e^{5t}\begin{bmatrix} -\frac{c_1}{2} + -\frac{tc_2}{2} - \frac{c_2}{4} \\ c_1 + tc_2 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{4} \\ 1 & 0 \end{bmatrix} e^{5t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} = e^{5t}\begin{bmatrix} 2tc_3 + c_3 + c_4 t \\ -4tc_3 + c_4 - 2tc_4 \end{bmatrix}$$

same

$e^{At} = PDP^{-1}$ ile her denk. sistemi çözülebilir ama uzun.

2) nonhomogenous linear systems ($x' = A(t)x + b$)

fundemental matrix (not unique)

the method of undetermined coefficients

$$x' = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} e^{2t} \\ t \end{bmatrix}$$

$\lambda_1 = 2 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 1 \to \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$$\Psi(t) = \begin{bmatrix} e^{2t} & -3e^{t} \\ 0 & e^{t} \end{bmatrix}$$

$$\Psi^{-1}(t) = \begin{bmatrix} e^{-2t} & 3e^{-2t} \\ 0 & e^{-t} \end{bmatrix}$$

$v = \Psi^{-1}(t) . b . dt$

$$v = \int \begin{bmatrix} e^{-2t} & 3e^{-2t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} e^{2t} \\ t \end{bmatrix} . dt$$

$x = \Psi v$

$$x = \begin{bmatrix} e^{2t} & -3e^{t} \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} t - \frac{3}{2} t e^{-2t} - \frac{1}{4} e^{-2t} \\ -te^{-t} - e^{-t} \end{bmatrix} + \underbrace{\begin{bmatrix} e^{2t} & -3e^{t} \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\text{homogenous part}}$$

higher order differential equations

$$y^{(n)} + a_1(t).y^{(n-1)} + a_2(t)\,y^{(n-2)} + \ldots + a_n(t).y = b(t)$$

Converting to first order system

$x_1 = y$
$x_2 = y' = x_1'$
$\vdots$
$x_n = y^{(n-1)} = x_{n-1}'$

$x_1' = x_2$
$x_2' = x_3$
$\vdots$
$x_n' = -a_n(t)x_1 - \ldots - a_1(t)x_n + b(t)$

$$y''' + 3y'' - ty' + 2y = \cos t \to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & t & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix}$$

$$t^3 y''' + t^2 y'' - 2ty' + 2y = 0 \to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{2}{t^3} & \frac{2}{t^2} & -\frac{1}{t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

homogenous system but not linear so we do not know the solution

1 homogenous equations with constant coefficients

$$y'' + 2y' - 3y = 0 \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{bmatrix} 0-\lambda & 1 \\ 3 & -2-\lambda \end{bmatrix} \Rightarrow \lambda^2 + 2\lambda - 3 = 0$$

$y = 1$

same

you don't have to do this step

distinct and real

$y'' + 2y' - 3y = 0 \to \lambda^2 + 2\lambda - 3 = 0$ $\quad \lambda_1 = -3$, $\lambda_2 = 1$ $\quad y = c_1 e^{-3t} + c_2 e^{t}$

complex

$y'' - 6y' + 13y = 0 \to \lambda^2 - 6\lambda + 13 = 0 \quad \lambda = 3 \pm 2i \to e^{3t}\cos 2t,\ e^{3t}\sin 2t$

$y = c_1 e^{3t}\cos 2t + c_2 e^{3t}\sin 2t$

repeated

for repeated root $\left(c_1 + c_2 t + c_3 \frac{t^2}{2} \ldots\right) e^{\lambda t}$

$y'' - 2y' + y = 0 \to \lambda^2 - 2\lambda + 1 = 0 \quad (\lambda = 1) \times 2$ $\quad y = c_1 e^{t} + c_2 t e^{t}$

2 non-homogenous with constant coefficients

the method of undetermined coefficients

$y'' - y = e^{2t}$ $\quad (D^2 - 1)\, y = e^{2t}$

① homogenous solution $\Rightarrow \{ e^{-t}, e^{t} \}$

② annihilator of $b(t)$ $\Rightarrow (D-2)$

③ $(D-2)(D^2-1)y = e^{2t} \cdot (D-2) = 0$ cancels $\Rightarrow \{ e^{2t}, e^{-t}, e^{t} \}$

④ $(D^2-1)(ce^{2t}) = e^{2t}$

2 kere türev al demek

$= 4ce^{2t} - ce^{2t} = e^{2t}$

$c = \frac{1}{3}$ $\quad \Rightarrow \frac{e^{2t}}{3} = y_p$

$y_{general} = y_h + y_p = c_1 e^{-t} + c_2 e^{t} + \frac{e^{2t}}{3}$

annihilator

$e^{3t} \to D - 3$

$1 \, (e^{0t}) \to D$

$te^{3t} \to (D-3)$

$\cos 3t \to (D^2+9)$

$e^{(a+bi)t} \to D - (a+bi$

$\left.\begin{matrix} e^{at}\sin bt / \\ e^{at}\cos bt \end{matrix}\right\} D^2 - 2 \dots + (a^2 \dots$

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the method of variation of parameters

second order, linear ode

$y'' + a_1(x)y' + a_2(x)\, y = 0$

solution $\Rightarrow y(x) = c_1 y_1(x) + c_2 y_2(x)$

* in general it is not possible to find $y_1(x)$ and $y_2(x)$ (exception: constant coefficients)

* but, if we know the homogenous solution, we can always find particular solution with variation of parameters