9 - parameter estimation

method of moments

k^{th} population moment $= \mu_k = E(X^k)$

k^{th} sample moment $= m_k = \frac{1}{n}\sum_{i=1}^{n} x_i^k$ (estimates from the sample $X_1, X_2 \dots X_n$)

central moments

k^{th} population central moment $= \mu_k' = E(X-\mu_1)^k$ $(k \geq 2)$ $(k=1 \to 0)$

k^{th} sample central moment $= m_k' = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^k$

↳ first sample moment = the sample mean $\to \bar{x}$

↳ second population central moment = variance, $V(x)$ ($\frac{1}{n-1}$ instead $\frac{1}{n}$) → unbiased

- simply mathing the first k paratemets of populition and sample.

ex: in poisson → one parameter $\mu_1 = E(x) = \lambda$

↳ thus, one equation: $\mu_1 = \lambda = m_1 = \bar{x}$

ex: in gamma → α and λ, $m_1 = \bar{x} = 48.233$ $m_2' = S^2 = 679.2122$

2 equations $\mu_1 = E(x) = \frac{\alpha}{\lambda} = m_1$ $\hat{\alpha} = m_1^2/m_2' = 3.4227$

$\mu_2 = Var(x) = \frac{\alpha}{\lambda^2} = m_2'$ $\hat{\lambda} = m_1/m_2 = 0.0710$

ex: cdf $F(x) = 1 - \left(\frac{x}{\sigma}\right)^{-\theta}$ two paramters

↳ pdf $f(x) = F'(x) = \theta . \sigma^{\theta} . x^{-\theta-1}$

$\mu_1 = E(x) = \int_{\sigma}^{\infty} x f(x)\,dx = \frac{\theta\sigma}{\theta-1} = m_1$

$\mu_2 = E(x^2) = \int_{\sigma}^{\infty} x^2 f(x)\,dx = \frac{\theta\sigma^2}{\theta-2} = m_2$

$\hat{\theta}_{mom} = \sqrt{\frac{m_2}{m_2 - m_1^2}} + 1$

$\hat{\sigma}_{mom} = \frac{m_1 \hat{\theta} - 1}{\hat{\theta}}$

method of maximum likehood

maximum likelihood estimator = the parameter value that maximize the likelihood of the observed sample

discrete distribution = maximize the joint pmf of data $P(x_1, \dots, x_n)$

continuous distribution = maximize the joint density $f(x_1, \dots, x_n)$

sample: 3 4 8 6 5 → $P(x=3) . P(x=4) . P(x=8) . P(x=6) . P(x=5)$ (independent) (joint distributio)

λ = unknown

$P = \frac{e^{-\lambda}.\lambda^3}{3!} . \frac{e^{-\lambda}.\lambda^4}{4!} . \frac{e^{-\lambda}.\lambda^8}{8!} . \frac{e^{-\lambda}.\lambda^6}{6!} . \frac{e^{-\lambda}.\lambda^5}{5!}$ ↳ to take derivative take ln of both side

$(xy)'$ → hard as product

$k = xy$ $\ln k = \ln x + \ln y$

$(\ln k)' = (\ln x)' + (\ln y)'$

sum is easier

$P\{X = (X_1, X_2, \ldots X_n)\} = P(X) = \prod_{i=1}^{n} P(X_i) \xrightarrow[\text{likehood}]{\text{to maximize}}$ derivative $\rightarrow \dfrac{\partial P(x)}{\partial \sigma} = 0$

$\ln \prod_{i=1}^{n} P(X_i) = \sum_{i=1}^{n} \ln P(X_i) \xrightarrow{\text{take derivative}}$ make $= 0$ find unknown paparmeter

estimation of standard errors

$$\sigma(\hat{\lambda}) = \sqrt{Var(\hat{\lambda})} = \sqrt{E(\hat{\lambda}^2) - E^2(\hat{\lambda})}$$

$k=2$ $\quad$ $k=1$

$var(x) = \underbrace{E(x^2)}_{\text{second moment}} - \underbrace{E(x)}_{\text{first moment}}^2$