1.6. Elementary Matrices:

Defn: If a matrix E is obtained from the identity matrix $I_{n\times n}$ by a single elementary row operation is called an elementary matrix

Type 1 - $cR_i + R_j \rightsquigarrow \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \ddots & \ddots & \\ & & c & & \ddots & \\ & & & & & 1 \end{bmatrix}_{n\times n}$ $\begin{array}{l} \to i\text{-th row} \\ \\ \to j\text{-th row} \end{array}$ OR $\begin{bmatrix} 1 & & & \\ & \ddots & c & \\ & & \ddots & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}_{n\times n}$ $\begin{array}{l} \to j\text{-th row} \\ \\ \to i\text{-th row} \end{array}$

$I_{n\times n} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n\times n}$

exp: $n = 3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2R_2+R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2R_2+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

are elementary matrices of Type I.

Type II - $R_i \longleftrightarrow R_j \rightsquigarrow \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 0 \cdots 1 & \\ & & 1 \cdots 0 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}$

exp: $n = 3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

elementary matrix of Type II.

Type III - $cR_i \rightsquigarrow \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & c & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}_{n\times n}$ $\rightsquigarrow i\text{-th row}$

exp: $n = 3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \end{bmatrix} \xrightarrow{3R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \end{bmatrix}$$

elementary matrix of

exp: $n=3$ $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{3R_3} \begin{bmatrix}1&0&0\\0&1&0\\0&0&3\end{bmatrix}$ } elementary matrix of Type III

Lemma: For any matrix $A_{m\times n}$ and elementary row operation $\mathcal{E}$
$\mathcal{E}(A) = \mathcal{E}(I) . A$ where $I_{m\times m}$

Proof: consider $\mathcal{E}$ as type I elementary row operation

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \xrightarrow[(i<j)]{\mathcal{E}:\ cR_i+R_j} \begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ cR_i+R_j \\ \vdots \\ R_m \end{bmatrix} = \mathcal{E}(A)$$

$$\mathcal{E}(I).A = \begin{bmatrix} e_1 \\ \vdots \\ e_i \\ \vdots \\ ce_i+e_j \\ \vdots \\ e_m \end{bmatrix} . \begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \\ R_m \end{bmatrix} = \begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ cR_i+R_j \\ \vdots \\ R_m \end{bmatrix}$$

where $e_i = (0,\dots,0,\underset{i\text{-th}}{1},0,\dots 0)$

$(0,\dots 0, c, 0, \dots, 1, \dots, 0)$

exp: $A = \begin{bmatrix}1&2&0&1\\0&1&3&2\\2&0&1&0\end{bmatrix}_{3\times 4} \xrightarrow{\mathcal{E}:\ 2R_3+R_1} \begin{bmatrix}5&2&2&1\\0&1&3&2\\2&0&1&0\end{bmatrix}$

$I = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{2R_3+R_1} \begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix} = \mathcal{E}(I)$

$\mathcal{E}(I).A =$

Consider $\mathcal{E}^{-1}$: $-2R_3+R_1$

$$\mathcal{E}^{-1}(\mathcal{E}(A)) = A$$ by Lemma we have

$$A = \mathcal{E}^{-1}(\mathcal{E}(A)) = \underbrace{\mathcal{E}^{-1}(I)\mathcal{E}(I)}_{I} A$$

<u>Notation:</u> Elementary matrices are represented by E_i.

<u>e.g.</u> $A \xrightarrow{\mathcal{E}_1} \_ \xrightarrow{\mathcal{E}_2} \_ \xrightarrow{\mathcal{E}_3} B$ $\quad B = \mathcal{E}_3(\mathcal{E}_2(\mathcal{E}_1(A)))$

<u>OR</u> $B = \mathcal{E}_3(I).\mathcal{E}_2(I).\mathcal{E}_1(I).A = E_3.E_2.E_1.A$

<u>exp:</u> $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ not elementary matrix

for example

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 1 & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{E_2} \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 2 \end{bmatrix} = B$$

$$A \xrightarrow{2R_2 + R_1} \_ \xrightarrow{R_2 \leftrightarrow R_3} \_$$

$$B = \mathcal{E}_2(\mathcal{E}_1(A)) = \mathcal{E}_2(I).\mathcal{E}_1(I).A = \underbrace{E_2.E_1}.A$$

<u>Thm:</u> If $A_{m\times n}$ is row equivalent to B then there is a matrix $P_{m\times m}$ such that $B = P.A$.

$$A \sim B \iff B = P.A$$

$$B \sim A \iff A = Q.B$$

Proof: $A \sim B$ then $B = \mathcal{E}_k(\dots(\mathcal{E}_1(A))) =$

$$\underbrace{\mathcal{E}_k(I).\,\mathcal{E}_{k-1}(I)\dots\dots\mathcal{E}_1(I)}.\,A = \underbrace{E_k.\,E_{k-1}\dots E_1}_{\substack{\parallel \\ P}}.\,A$$

$B \sim A \Rightarrow \quad A = \mathcal{E}_1^{-1}\dots(\mathcal{E}_{k-1}^{-1}(\mathcal{E}_k^{-1}(B))) =$

$$\mathcal{E}_1^{-1}(I).\,\mathcal{E}_2^{-1}(I)\dots\dots\mathcal{E}_{k-1}^{-1}(I).\,\mathcal{E}_k^{-1}(I).\,B$$

$$= \underbrace{E_1^{-1}.\,E_2^{-1}\dots\dots E_k^{-1}}_{Q}.\,B$$

RMK: How to find P if $B = P.A$?

$$[A \mid I] \longrightarrow \dots \longrightarrow [B \mid P]$$

$$\mathcal{E}_2(I)\mathcal{E}_1(I) = \mathcal{E}_2(\mathcal{E}_1(I))$$

exp: Find row reduced echelon matrix R & P such that $R = P.A$.

$$\left[\begin{array}{cccc|ccc} -1 & 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 3 & 0 & 1 & 0 \\ 1 & -1 & 4 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3]{R_1+R_2} \left[\begin{array}{cccc|ccc} -1 & 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 1 & 0 \\ 0 & 0 & 6 & 4 & 1 & 0 & 1 \end{array}\right]$$

(the left block is labelled A, the right block I)

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{cccc|ccc} -1 & 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right] \xrightarrow[-R_1]{\frac{1}{3}R_2} \left[\begin{array}{cccc|ccc} 1 & -1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{2R_2+R_1} \left[\begin{array}{cccc|ccc} 1 & -1 & 0 & 7/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 2/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right]$$

$R =$ row reduced echelon; P

$$\underbrace{\begin{bmatrix} 1/3 & 2/3 & 0 \\ 1/3 & 1/3 & 0 \\ -1 & -2 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} -1 & 1 & 2 & -1 \\ 1 & -1 & 1 & 3 \\ 1 & -1 & 4 & 5 \end{bmatrix}}_{A} \overset{?}{=} R$$

CHAPTER 2 - Systems of Linear Eqns.

2.1. Defn: A system of m linear equations in n unknowns is called a system of linear eqns.

We can represent any system of linear eqns. by matrices as follows:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} (*)$$

$x_1, \dots, x_n$ are unknowns

a_{ij} are coefficients.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{B} \Rightarrow \underline{A.X = B}$$

We will represent this by a matrix so called "augmented matrix"

$$\Rightarrow \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

with A under the coefficient part and B under the last column.

"Augmented matrix" of the system $(*)$

matrix

$$\underset{\substack{\nearrow \\ \text{coefficient matrix}}}{A} \qquad \qquad B$$

By a solution of a system we mean a set S such that

$A.S = B$ where $S_{n\times 1}$

e.g. • $x+y=1$ $\rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -1 & 2 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 3 \end{array}\right]$ ↯

$-x-y=2$

$+$ ———

$0+0=3$

Defn: When the system has at least one soln. then it is called "consistent" otherwise it is called "inconsistent."

$$\text{system} \begin{cases} \text{consitent} \begin{cases} \text{unique soln.} \\ \text{infinitely may solns.} \end{cases} \\ \text{inconsistent} \end{cases}$$

e.g.

• $x+y=1$ $\longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$

$2x+2y=2$

———

$x+y=1$

e.g.

• $x+y=1$ $\longrightarrow \underbrace{\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 0 \end{array}\right]} \xrightarrow{-R_1+R_2} \underbrace{\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & -1 \end{array}\right]}$

$x-y=0$

———

$x=\frac{1}{2}$ $y=\frac{1}{2}$

$$\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \boxed{\begin{array}{l} x+y=1 \\ -2y=-1 \end{array}} \begin{array}{l} \leftarrow x=\frac{1}{2} \\ \rightarrow y=\frac{1}{2} \end{array}$$

Defn: Two systems $A_1X=B_1$ & $A_2X=B_2$ are said to be equivalent systems is $[A_1 | B_1]$ & $[A_2 | B_2]$ are row equivalent.

Thm: Equivalent systems have the same solutions.

Rmk: The converse of above thm is not true!
Just consider two systems with no solution.

Proof: If two systems are equivalent the $[A_1|B_1] \sim [A_2|B_2]$

$\Rightarrow$ there are P, Q product of elementary matrices such that

$$[A_2|B_2] = P.[A_1|B_1] \ \& \ [A_1|B_1] = Q.[A_2|B_2] \Rightarrow$$

$\Rightarrow A_2 = PA_1 \ \& \ B_2 = PB_1$ $\qquad$ $A_1 = QA_2 \ \& \ B_1 = QB_2$

If $A_1.S = B_1 \Rightarrow \underbrace{PA_1}S = \underbrace{PB_1}$
$A_2.S = B_2$

if $A_2.S = B_2 \Rightarrow \underbrace{QA_2}S = \underbrace{QB_2}$
$A_1 S = B_1$

Hence equivalent systems has same solutions. ∎