the laplace transform

$f(t) \longrightarrow F(s)$: integral transform variable $t \to s$

$t \geq 0$

$$L\{f(t)\} = F(s) = \int_0^\infty f(t).\underbrace{e^{-st}}_{\text{kernel}}.dt$$

• func must be piecewise continuous

$L\{t^n\} = \frac{n!}{s^{n+1}}$

$L\{e^{at}\} = \frac{1}{s-a} \to s > a$

$L\{\sin(at)\} = \frac{a}{a^2+s^2}$

$L\{\cos(at)\} = \frac{s}{a^2+s^2}$

$L\{e^{at}.t^n\} = \frac{n!}{(s-a)^{n+1}}$

$s > 0$

$L\{e^{at}.\sin bt\} = \frac{b}{(s-a)^2+b^2}$

$L\{e^{at}.\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$

$L\{y\} = Y(s)$

$L\{y'\} = s.Y(s) - y(0)$

$L\{y''\} = s^2.Y(s) - sy(0) - y'(0)$

$L\{y'''\} = s^3.Y(s) - s^2 y(0) - sy'(0) - y''(0)$

ex: $y' - 2y = e^{2t}$, $y(0) = 1$ $\to$ $L\{y'\} - 2.L\{y\} = L\{e^{2t}\}$

$s.Y(s) - y(0) - 2.Y(s) = \frac{1}{s-2}$

$(s-2).Y(s) = \frac{1}{s-2} + 1$

$Y(s) = \frac{s-1}{(s-2)^2}$

$\frac{A}{s-2} + \frac{B}{(s-2)^2} = \frac{s-1}{(s-2)^2}$ $(s-2)$

$A = B = 1$

$Y(s) = \frac{1}{s-2} - \frac{1}{(s-2)^2}$ $\quad y(t) = e^{2t} - e^{2t}.t$

$\boxed{\text{Step functions}}$

$$u_a(t) = u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

• $u_0(t) = u(t)$
• $u_1(t)$

$L\{u_a(t)\} = \frac{e^{-as}}{s}$

$L\{u_a(t).f(t-a)\} = e^{-as}.L\{f(t)\}$

$\to L\{u_2(t).(t-2)^3\} = e^{-2s}.L\{t^3\} = e^{-2s}.\frac{6}{s^4}$

$t \to t+2$

$\to L\{u_3(t).t\} = e^{-3s}.L\{t+3\} = e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$

inverse step func

$L^{-1}\{e^{-4s}.\underbrace{\frac{s}{s^2+25}}_{\cos 5t}\} = u_4(t).\cos 5(t$

$= u_4(t).\cos(5t-2$

ex: $y'' + 3y' + 2y = u_2(t) - u_3(t)$ $\quad y(0) = y'(0) = 0$

$$s^2Y(s) - s\cancel{y(0)} - \cancel{y'(0)} + 3\left[sY(s) - \cancel{y(0)}\right] + 2y(s) = \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}$$

$$Y(s)\left[s^2+3s+2\right] = \frac{e^{-2s}-e^{-3s}}{s} \qquad Y(s) = \frac{e^{-2s}-e^{-3s}}{s}\cdot\frac{1}{(s+2)}\cdot\frac{1}{(s+1)}$$

$$Y(s) = \left(e^{-2s}-e^{-3s}\right)\underbrace{\left(\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right)}$$

$$\mathcal{L}^{-1} \rightarrow \frac{1}{2} - e^{-t} + \frac{1}{2}$$

$$\hookrightarrow u_2(t)\left(\frac{1}{2} - e^{-(t-2)} + \frac{e^{-2(t-2)}}{2}\right) - u_3(t)\left(\frac{1}{2} - e^{-(t-3)} + \frac{e^{-2(t-2)}}{2}\right)$$

$$\boxed{\text{discontinuous forcing functions}}$$

ex: $m=1$, $k=4$ $\quad y''+4y=F(t)$ $\quad F(t) = \begin{cases} 0 & t<3 \\ 3t-9 & 3\le t<6 \\ 15-t & 6\le t<15 \\ 0 & 15\le t \end{cases}$

subtract only previous one

$$F(t) = (3t-9-0)u_3(t) + (\underbrace{15-t-(3t-9)}_{24-4t})u_6(t) + (\underbrace{0-(15-t)}_{t-15})u_{15}(t)$$

$$y''+4y = (3t-9)u_3(t) + (24-4t)u_6(t) + (t-15)u_{15}(t) \qquad y(0)=y'(0)=0$$

$$Y(s).s^2 - Y(0).s - y'(0) + 4y(s) = e^{-3s}.\frac{3}{s^2} - e^{-6s}.\frac{4}{s^2} + e^{-15s}.\frac{1}{s^2}$$

$\mathcal{L}\{\delta(t)\} = 1$

ex: $y''+9y = 15\delta(t-3\pi) + 12\delta(t-6\pi)$ $\quad y(0)=y'(0)=0$

$$s^2y(s) - sy(0) - y'(0) + 9y(s) = 15e^{-3\pi s} + 12e^{-6\pi s}$$

! $\mathcal{L}\{\delta(t-5)\} = e^{-5s}$

$\mathcal{L}\{u_{10}(t)\} = \frac{e^{-10s}}{s}$

$$\mathcal{L}\{f\,.\,g\} = \mathcal{L}\{f\}.\mathcal{L}\{g\} = \left(\int_0^\infty e^{-s\eta}f(\eta).d\eta\right).\left(\int_0^\infty e^{-s\tau}g(\tau)d\tau\right) \qquad \eta = t-\tau$$

ex: $y''+5y'+4y = g(t)$ $\quad y(0)=y'(0)=0$ $\quad y(t)$ in terms of $g(t)=?$

$$\Rightarrow Y(s).\underline{(s^2+5s+4)} = \mathcal{L}\{g(t)\}$$

$$\hookrightarrow \frac{1}{3}\left(\frac{1}{s+1} - \frac{1}{s+4}\right) \xrightarrow{\mathcal{L}^{-1}} \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} = f(t) \qquad y(t) = \int_0^t\left(\frac{1}{3}e^{-(t-\tau)} - \frac{1}{3}e^{-4(t-\tau)}\right)g(\tau)$$

ex: $\mathcal{L}\left\{\int_0^t (t-\tau)^2.\cos 2\tau.d\tau\right\} \rightarrow \frac{2}{s^3}.\frac{s}{(s^2+4)}$

$t^2 \qquad \cos 2t$

ex: $f(t) = \int_0^t e^{-(t-\tau)}.\sin\tau.d\tau$ $\qquad \mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}\}.\mathcal{L}\{\sin t\} = \frac{1}{1+s}.\frac{1}{1+s^2}$

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periodic functions $\rightarrow$ $f(x+T) = f(x)$

periodic functions → $f(x+T) = f(x)$

even functions

symmetric to y axis

$f(x) = f(-x)$

→ $\cos x$

odd functions

symmetric to orijing

$f(x) = -f(-x)$

→ $\sin x$, $\tan x$, $\cot x$

integrals: $\int_{-a}^{a} f(x).dx = 2.\int_{0}^{a} f(x).dx$ $\qquad$ $\int_{-a}^{a} f(x)\,dx = 0$

$\boxed{\text{fourier series}}$

period: 2L $\qquad f(x) = a_0 + \sum_{n=1}^{\infty} a_n.\cos\left(\frac{n x \pi}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n x \pi}{L}\right)$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x).dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x).\cos\left(\frac{n x \pi}{L}\right)dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n x \pi}{L}\right)dx$

ex: $f(x) = x \quad , \quad -3 \leq x \leq 3$

↓

$f(x) = -f(-x)$

odd

$L = 3$

$a_0 = \frac{1}{6}\int_{-3}^{3} x.dx = 0$

$a_n = \frac{1}{3}\int_{-3}^{3} .x.\cos\left(\frac{n x \pi}{3}\right)dx = 0$ odd

$b_n = \frac{1}{3}\int_{-3}^{3} x.\sin\left(\frac{n x \pi}{3}\right)dx = \frac{2}{3}\int_{0}^{3} x\sin\left(\frac{x n \pi}{3}\right)dx$

T	I
x	$\sin(kx)$
1	$-\frac{\cos(kx)}{k}$
0	$-\frac{\sin(kx)}{k^2}$

$-\frac{x\cos(kx)}{k} + \frac{\sin(kx)}{k^2}$

$\rightarrow \frac{2}{\cancel{3}}\left[\frac{-x.\cos\left(\frac{x n \pi}{3}\right)\cancel{3}}{n\pi} + \frac{\cancel{3}\sin\left(\frac{x\pi n}{3}\right)3}{n^2\pi^2}\right]$

$2\left[\frac{-3\cos(n\pi)}{n\pi} + \frac{3\cancel{\sin(n\pi)}}{n^2\pi^2}\right] \Rightarrow b_n$

0

$f(x) = \frac{1}{3}\sum_{n=1}^{\infty} .b_n.\sin\left(\frac{n x \pi}{L}\right) \qquad n \rightarrow 1 \; 2 \; 3 \; 4$

$f(x) = \frac{6}{\pi}\sin\left(\frac{x\pi}{3}\right) - \frac{3}{\pi}\sin\left(\frac{2\pi x}{3}\right) \cdots$

ex: $f(x) = \begin{cases} 1 & 0 < x < 2 \\ -1 & -2 < x < 0 \end{cases}$ $\qquad$ Period=4

$L = 2 \qquad f(x) = a_0 + \frac{1}{L}\sum_{n=1}^{\infty} a_n.\cos\left(\frac{n\pi x}{L}\right) + \frac{1}{L}\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x).dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x).\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x).\sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{4}\int_{-2}^{0} 1.dx + \frac{1}{4}\int_{0}^{2}(-1)\,dx \to \frac{2}{4} + \frac{1}{4}.(-2) = 1$$

$$x\Big|_{-2}^{0} \qquad (-x)\Big|_{0}^{2}$$

$$a_1 = \frac{1}{2}\int_{-2}^{0}\cos\left(\frac{\pi x n}{2}\right)dx + \frac{1}{2}\int_{0}^{2}-\cos\left(\frac{\pi x n}{2}\right)dx \to \frac{1}{2}\left(\frac{\sin(-n\pi)}{-\frac{n\pi}{2}}\right) + \frac{1}{2}\left(\frac{\sin(n\pi)}{\frac{n.\pi}{2}}\right)$$

$$\frac{\sin\left(\frac{\pi x n}{2}\right)}{\frac{\pi n}{2}}\Bigg|_{-2}^{0} \qquad \frac{-\sin\left(\frac{\pi x n}{2}\right)}{\frac{n\pi}{2}}\Bigg|_{0}^{2} = 0$$

$n = 1, 2, 3$

$$b_0 = \frac{1}{2}\int_{-2}^{0}\sin\left(\frac{\pi x n}{2}\right)dx + \frac{1}{2}\int_{0}^{2}-\sin\left(\frac{\pi x n}{2}\right)dx \to \frac{1}{2}\,\frac{(2\cos(-\pi n)-2)}{n\pi} + \frac{1}{2}\left(\frac{2\cos(\pi n)}{n\pi} - \frac{2}{n\pi}\right.$$

$$\frac{-\cos\left(\frac{\pi x n}{2}\right).2}{n\pi}\Bigg|_{-2}^{0} \qquad \frac{2.\cos\left(\frac{\pi x n}{2}\right)}{n\pi}\Bigg|_{0}^{2} = \frac{\cos -\pi n}{n\pi} - \frac{2}{n\pi}$$

$$\frac{-2}{n\pi} - \left(\frac{2\cos(-\pi n)}{n\pi}\right) \qquad \frac{2\cos(\pi n)}{n\pi} - \frac{2}{n\pi}$$