Week 3

ex:
$$\begin{aligned} x+y-z+t+u&=1\\ -x+2y+3z-t+2u&=-1\\ 2x+y-z+2t-u&=2\\ x+6y+4z+t+4u&=1\\ 8y+7z+6u&=0\\ 3x+7y+3z+3t+3u&=3 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1&1&-1&1&1&1\\ -1&2&3&-1&2&-1\\ 2&1&-1&2&-1&2\\ 1&6&4&1&4&1\\ 0&8&7&0&6&0\\ 3&7&3&3&3&3 \end{array}\right] \xrightarrow{G.E.} \left[\begin{array}{ccccc|c} 1&1&-1&1&1&1\\ 0&-1&1&0&-3&0\\ 0&0&5&0&-6&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{array}\right]$$ → a matrix in echelon form

x, y, z basic vars.
u, t free vars.

$$\begin{aligned} x+y-z+t+u&=1 && \text{eq 1}\\ -y+z-3u&=0 && \text{eq 2}\\ 5z-6u&=0 && \text{eq 3} \end{aligned}$$

We can find basic variables in terms of the free variables:

$$z \overset{eq3}{=} \frac{6}{5}u, \quad y \overset{eq2}{=} -\frac{9}{5}u, \quad x \overset{eq1}{=} 1-t+2u$$

So, the general solution is:

$$\begin{bmatrix} 1-t+2u\\ -9/5\,u\\ 6/5\,u\\ t\\ u \end{bmatrix} = \begin{bmatrix} 1\\0\\0\\0\\0 \end{bmatrix} + t\begin{bmatrix} -1\\0\\0\\1\\0 \end{bmatrix} + u\begin{bmatrix} 2\\-9/5\\6/5\\0\\1 \end{bmatrix} \quad t, u \in \mathbb{R}$$

We could also choose: x, y, u basic vars.
t, z free vars.

$$u = 5/6\, z, \quad y = -3/2\, z, \quad x = 1-\frac{5}{3}z-t$$

So the gen. soln. is:

$$\begin{bmatrix} 1-5/3\,z-t\\ -3/2\,z\\ z\\ t\\ 5/6\,z \end{bmatrix} = \begin{bmatrix} 1\\0\\0\\0\\0 \end{bmatrix} + t\begin{bmatrix} -1\\0\\0\\1\\0 \end{bmatrix} + z\begin{bmatrix} -5/3\\-3/2\\1\\0\\5/6 \end{bmatrix} \quad t, z \in \mathbb{R}$$

Find the values of a, b for which

$t, z \in \mathbb{R}$

ex: Find the values of a, b for which the system

$$\begin{aligned} 2x - y + 2az + t &= b \\ 2x + (a-1)y + 2az + t &= 1 \\ 2x - y + (2a+1)z + (a+1)t &= 0 \\ -2x + y + (1-2a)z - 2t &= -2b-2 \end{aligned}$$

a unique soln or infinitely many solns or no solution.

$$\left[\begin{array}{cccc|c} 2 & -1 & 2a & 1 & b \\ 2 & (a-1) & 2a & 1 & 1 \\ 2 & -1 & (2a+1) & (a+1) & 0 \\ -2 & 1 & 1-2a & -2 & -2b-2 \end{array}\right] \xrightarrow[R_1+R_4]{\substack{-R_1+R_2 \\ -R_1+R_3}} \left[\begin{array}{cccc|c} 2 & -1 & 2a & 1 & b \\ 0 & a & 0 & 0 & 1-b \\ 0 & 0 & 1 & a & -b \\ 0 & 0 & 1 & -1 & -b-2 \end{array}\right]$$

$$\downarrow -R_3 + R_4$$

$$\left[\begin{array}{cccc|c} 2 & -1 & 2a & 1 & b \\ 0 & a & 0 & 0 & 1-b \\ 0 & 0 & 1 & a & -b \\ 0 & 0 & 0 & -1-a & -2 \end{array}\right]$$

If $a \neq 0$ this is in echelon form.

In this case if $a = -1$ the system has no solutions for any $b \in \mathbb{R}$.

———— if $a \neq -1$ ———— a unique solution for any $b \in \mathbb{R}$

If $a = 0$ then the echelon form is

put $a=0$ and move R_2 to bottom.

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 1 & b \\ 0 & 0 & 1 & 0 & -b \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 1-b \end{array}\right]$$

In this case if $b \neq 1$ then system has no solutions

If $b = 1$ the system has ∞ ly many solutions.

Homogeneous Systems

A system of the form $AX = 0$ is a "homogeneous system"

zero matrix $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Such a system has always $X = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ as a solution.

This is the "trivial solution".

If this is not the unique solution, this system has other solutions called "nontrivial solutions"

If this is not the unique solution, this system has infinitely many other solutions called "non-trivial solutions".

ex:
$$\begin{aligned} x+y-z+t+u&=0\\ -x+2y+3z-t+2u&=0\\ 2x+y-z+2t-u&=0\\ x+6y+4z+t+4u&=0\\ 8y+7z+6u&=0\\ 3x+7y+3z+3t+3u&=0 \end{aligned} \xrightarrow{G.E.} \begin{aligned} x+y-z+t+u&=0\\ -y+z-3u&=0\\ 5z-6u&=0 \end{aligned}$$

$$x=-t+2u$$
$$y=-\tfrac{9}{5}u$$
$$z=\tfrac{6}{5}u$$

The general solution is:

$$\begin{bmatrix} -t+2u\\ -\frac{9}{5}u\\ \frac{6}{5}u\\ t\\ u \end{bmatrix} = t\begin{bmatrix} -1\\ 0\\ 0\\ 0\\ 0 \end{bmatrix} + u\begin{bmatrix} 2\\ -\frac{9}{5}\\ \frac{6}{5}\\ 0\\ 1 \end{bmatrix} \qquad u,t\in\mathbb{R}$$

if we denote $x_1=\begin{bmatrix} -1\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}$ $x_2=\begin{bmatrix} 2\\ -\frac{9}{5}\\ \frac{6}{5}\\ 0\\ 1 \end{bmatrix}$

$t=1$, $u=0$ $\qquad$ $t=0$, $u=1$

Any solution of the system is a "combination" of x_1 and x_2.
x_1 and x_2 are called the "fundamental solutions".

Invertible Matrices :

Definition: A matrix A is "invertible" if there is a matrix B such that $AB=I$ and $BA=I$.
(the identity matrix)

Note: Since both AB and BA are defined, an invertible matrix must be a square matrix!

Prop: let A be an invertible matrix.
There is a unique matrix B s.t $AB=BA=I$.

Pf: Suppose B_1 and B_2 are matrices s.t
$AB_1=B_1A=I$ and $AB_2=B_2A=I$.

Then $B_1=B_1\cdot I=B_1\cdot(AB_2)=(B_1\cdot A)\cdot B_2=I\cdot B_2=B_2$

Thus $B_1=B_2$ end of proof

Defn: let A be an invertible matrix. The unique matrix denoted by A^{-1} satisfying $A\cdot A^{-1}=A^{-1}\cdot A=I$

Defn: Let A be an $n \times n$ square matrix, the denoted by A^{-1} satisfying $A \cdot A^{-1} = A^{-1} \cdot A = I$ is the "inverse of A".

ex: $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

since $A \cdot A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^{-1} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

ex: $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ Claim: B has no inverse!

Suppose $B^{-1} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$

$$BB^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x + z &= 1 \\ y + t &= 0 \\ x + z &= 0 \\ y + t &= 1 \end{aligned} \quad \Rightarrow 1 = 0 \text{ not possible!}$$

Thus B is NOT invertible!

Note: Some matrices are invertible and some are not invertible!

0) Why do we care about invertible matrices?

Question: 1) How can we determine if a matrix is invertible?

2) How can we compute the inverse of A. (efficiently!)

Prop: If A_1 and A_2 are invertible $n \times n$ matrices then $A_1 \cdot A_2$ is invertible with $(A_1 A_2)^{-1} = A_2^{-1} \cdot A_1^{-1}$.

Proof: $(A_1 A_2) \cdot (A_2^{-1} A_1^{-1}) = A_1 \underbrace{(A_2 \cdot A_2^{-1})}_{=I} A_1^{-1} = A_1 \cdot A_1^{-1} = I$

$(A_2^{-1} A_1^{-1}) \cdot (A_1 A_2) = A_2^{-1} \underbrace{(A_1^{-1} \cdot A_1)}_{=I} \cdot A_2 = A_2^{-1} \cdot A_2 = I$

So, $(A_1 \cdot A_2)^{-1} = A_2^{-1} \cdot A_1^{-1}$ end of proof.

Corollary: If $A_1, A_2, \ldots, A_r$ are invertible then $A_1 \cdot A_2 \cdots A_r$ is invertible with

$(A_1 A_2 \cdots A_r)^{-1} = A_r^{-1} A_{r-1}^{-1} \cdots A_2^{-1} \cdot A_1^{-1}$

$A_1 \cdot A_2 \cdots A_r$ is invertible with

$$(A_1 \cdot A_2 \cdots A_r)^{-1} = A_r^{-1} \cdot A_{r-1}^{-1} \cdots A_2^{-1} \cdot A_1^{-1}$$

Pf: Previous prop + induction.

(as exercise: Find two invertible matrices A_1, A_2 s.t. $(A_1 + A_2)$ is not invertible)

Prop: Every elementary matrix is invertible. Therefore by above corollary a product of elementary matrices is invertible.

Proof: let $\mathcal{E}$ be an elementary row operation and let $\mathcal{E}(I)$ be the corresponding elementary matrix.
let $\mathcal{E}^{-1}$ be the inverse of the elem. row op. $\mathcal{E}$.

Then, $\underbrace{\mathcal{E}(I)}_{\text{elem. matrix}} \cdot \underbrace{\mathcal{E}^{-1}(I)} \overset{\text{a previous thm}}{=} \mathcal{E}(\mathcal{E}^{-1}(I)) = I$

similarly $\underbrace{\mathcal{E}^{-1}(I)}_{\text{elem. matrix}} \cdot \mathcal{E}(I) \overset{\text{a previous thm}}{=} \mathcal{E}^{-1}(\mathcal{E}(I)) = I$

$\Rightarrow (\mathcal{E}(I))^{-1} = \mathcal{E}^{-1}(I)$

i.e. $\mathcal{E}(I)$ is invertible.

end of proof.

Theorem: let $A \in \mathbb{R}^{n \times n}$. The following statements are equivalent:

1) A is a product of elementary matrices
2) A is invertible
3) A is NOT row equivalent to a matrix with a zero row.
4) A is row equivalent to the identity matrix.

Proof: $1) \Rightarrow 2)$ This is the previous proposition.

$2) \Rightarrow 3)$ Suppose A is invertible and suppose A is row equivalent to a matrix $B = \begin{bmatrix} R_1 \to \\ R_2 \to \\ \vdots \\ R_n \to \end{bmatrix}$ with $R_i = [0\ 0 \cdots 0]$.

Since A is r.e. to $B = P \cdot A$ for some matrix P, which is a product of elementary matrices. ($\Rightarrow P$ is invertible)

Since, P and A are invertible, we have that B is invertible.

But then we have $\underbrace{B \cdot B^{-1}}_{} = I \longrightarrow$ the ith row of this matrix is

But then we have $\underbrace{B \cdot B}_{} = I$ $\longrightarrow$ the ith row of this matrix is $[0\,0 \cdots 0\,1\,0 \cdots 0]$ ↑ ith col.

the i-th row of this matrix is $[0\,0 \cdots 0]$ since the i-th row of B is $[0 \cdots 0]$

So this is a contradiction.

3) ⇒ 4) A is equivalent to a matrix R in reduced row echelon form.

Since A is not r.e. to a matrix with a zero row, R has no zero rows.

So, R is an $n \times n$ matrix in reduced row echelon form and does not have a zero row.

Thus, $R = I$. (Why?)

4) ⇒ 1) Suppose A is r.e. to I.

So, $A = PI$ where P is a product of elem. matrices.

Hence $A = P$

end of proof.

This theorem provides an algorithm to find A^{-1} or to tell a matrix A is not invertible.

<u>Note:</u> Suppose A is r.e to B by elem. row operations say $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_r$.

<u>i.e</u> $B = \mathcal{E}_r \cdots \mathcal{E}_3 \mathcal{E}_2 \mathcal{E}_1 (A)$

in other words if $P = (\mathcal{E}_r \mathcal{E}_{r-1} \cdots \mathcal{E}_3 \mathcal{E}_2 \mathcal{E}_1)(I)$

then $B = PA$. So, to find P we should apply the row operations which produce B out of A to I.

<u>i.e:</u> $\left([A \mid I] \xrightarrow{\text{row.op.}} [B \mid P] \right) \Rightarrow B = PA$

Hence to find A^{-1}:

start with $[A \mid I] \longrightarrow [R \mid P]$

↳ row reduced echelon matrix

start with

↳ row reduced echelon matrix

Case 1: R has a zero row: In this case (by Thm) A is not invertible.

Case 2: R has no zero rows. In this $R = I$.

So, $[A \mid I] \xrightarrow{\text{row ops.}} [I \mid P] \implies PA = I$

Also $[P \mid I] \xrightarrow{\text{row ops in reverse order}} [I \mid A] \implies AP = I$

in other words $A^{-1} = P$.

Summary: start with $[A \mid I]$ and apply elem. row ops

to get $[R \mid P]$

if R has a zero row then A is not invertible

if R has no zero rows then $A^{-1} = P$

Shortly: $[A \mid I] \longrightarrow [I \mid A^{-1}]$

ex: $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ is A invertible? If yes, find A^{-1}.

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 + R_2} \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right]$$

$$\Big\downarrow -2R_2 + R_1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right] \xleftarrow[2R_2]{\frac{1}{2}R_1} \left[\begin{array}{cc|cc} 2 & 0 & 0 & -2 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \qquad \underbrace{\begin{matrix} 0 & -1 \\ 1 & 2 \end{matrix}}_{A^{-1}}$$

(lets check: $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

ex: same question: $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1 + R_3]{-2R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow R_2 + R_3$$

So A is R.E to

So A is R.E to a matrix with a zero row, thus by thm A is NOT invertible!

$$\downarrow R_2 + R_3$$

$$\underbrace{\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{array}\right]}_{\text{a matrix with a zero row}}$$

ex: $A = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$ same question.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 \\ 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 4 & -15 & -3 & 0 & 1 \end{array}\right]$$

$$\downarrow -4R_2 + R_3$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \underbrace{\begin{matrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{matrix}}_{A^{-1}} \xleftarrow[-5R_3+R_1]{4R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -4 & 1 \end{array}\right]$$

(Check!)

Corollary: Let $A, B \in \mathbb{R}^{n \times m}$.

A and B are row equivalent $\Longleftrightarrow$ $B = PA$ for some invertible matrix P.

Pf: ($\Rightarrow$) A and B are r.e

$\Longrightarrow B = P \cdot A$ where P is a product of elem. matrices.

We observed above that such a matrix P is invertible.

($\Leftarrow$) Suppose $B = P \cdot A$ where P is invertible.

But we observed that an invertible is a product of elementary matrices. So, A and B are r.e.

end of proof.

Theorem: A matrix is row equivalent to a unique matrix in row reduced echelon form.

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Theorem: A matrix is r.e. to a unique matrix in row reduced echelon form.

Pf: Skip! (Idea: Suppose A is r.e to R_1 in r.r.e.f
A —— R_2 ——

so R_1 and R_2 are row equivalent,
This means there is an invertible matrix P
s.t $R_1 = PR_2$. One can show $R_1 = R_2$)

Cor: Two $m \times n$ matrices are row equivalent
$\iff$ They are row equiv. to the same matrix in row reduced echelon form.

Pf: ($\Rightarrow$) Suppose A is r.e to B.
A is r.e to R_1 in r.r.e.f
B is —— R_2 ——
So R_1 is r.e. R_2 thus $R_1 = R_2$ by Thm.

($\Leftarrow$) Clear!
end of proof

ex: $A = \begin{bmatrix} 1 & -2 & 0 & 5 \\ -1 & 2 & 1 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 2 & 0 & -5 \\ -1 & 2 & 2 & 1 \\ 1 & -2 & -1 & 2 \end{bmatrix}$

Are A and B row equivalent?

Idea: Reduce A to a matrix R_1 in r.r.e.f.
—— B —— R_2 ——
and check if $R_1 = R_2$ → Yes, A and B are r.e
→ No, A and B are NOT r.e.

in our example both A and B are row equivalent to $\begin{bmatrix} 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ⇓ in r.r.e.f.
(Check this!)

So A and B are r.e.

Invertibility and systems of linear equations

Definition: Let $A \in \mathbb{R}^{n \times n}$.
" " " " if there is $B \in \mathbb{R}^{n \times n}$ s.t $BA = I$

Definition: Let $A \in \mathbb{R}^{n \times n}$.

A has a "left inverse" if there is $B \in \mathbb{R}^{n \times n}$ s.t $BA = I$

" " "right " " " $C \in \mathbb{R}^{n \times n}$ s.t $A \cdot C = I$

Theorem: Let $A \in \mathbb{R}^{n \times n}$. The following are equivalent:

1) A has a left inverse
2) The homogeneous system $AX = 0$ has only the trivial solution $X = 0$.
3) A is invertible
4) A has a right inverse.

Pf: $1) \Rightarrow 2)$: Suppose $B \cdot A = I$ for some B.

Then, $AX = 0 \Rightarrow B(AX) = B \cdot 0$

$\Rightarrow (BA) \cdot X = 0$

$\Rightarrow IX = 0 \Rightarrow X = 0$

so the only solution is $X = 0$.

$2) \Rightarrow 3)$ Suppose $AX = 0$ has only trivial solution $X = 0$

Let R be a matrix in r.r.e.f equivalent to A.

Then $RX = 0$ is equiv. to $AX = 0$,

hence $RX = 0$ has only trivial solution.

If R has a zero row then the system $RX = 0$ will have a free variable, thus a nonzero solution.

Hence the system $Ax = 0$ will have a nonzero solution

So, R cannot have a zero row and hence $R = I$.

So, A is invertible by a previous result.

$3) \Rightarrow 4)$ Clear!

$4) \Rightarrow 1)$ Suppose $A \cdot B = I$ for some $B \in \mathbb{R}^{n \times n}$.

So, B has a left inverse A.

By $1) \Rightarrow 2) \Rightarrow 3)$ B is invertible. So

$$A = A \cdot I = A \cdot (B \cdot B^{-1}) = (A \cdot B) \cdot B^{-1} = I \cdot B^{-1} = B^{-1}$$

So A is invertible and has a left inverse.

So A is invertible and has a left inverse.

end of proof

[Note: This tells us the following: $A, B \in \mathbb{R}^{n\times n}$
If $A \cdot B = I$ then $BA = I$ hence $B = A^{-1}$.]

Corollary: Let $A \in \mathbb{R}^{n\times n}$. Then the following are equivalent

1) A is invertible
2) $Ax = B$ has a unique solution for any B.
3) $Ax = B$ has ———— for some B.

Pf: 1) $\Rightarrow$ 2) Suppose A is invertible.
Then $Ax = B \iff A^{-1}(Ax) = A^{-1}B \iff x = A^{-1}B$

2) $\Rightarrow$ 3) Clear!

3) $\Rightarrow$ 1) Suppose there is B s.t $Ax = B$ has a unique solution say X_1.

Claim: $Ax = 0$ has only the trivial solution solution $x = 0$

(if not, there is $X_2 \neq 0$ s.t $AX_2 = 0$
But then $A(X_1 + X_2) = AX_1 + AX_2 = B + 0 = B$
hence $X_1 + X_2 \neq X_1$ is another solution to $Ax = B$)

This shows A is invertible

end of proof.