2 - probability

sample space = the set of all possible outcomes of a random experiment
Ω
↳ tossing a coin's sample space $= \{H, T\}$

event = any collection of possible outcomes of an experiment (any subset of the sample space)
↳ sample space of tossing two coins $= \{HH, HT, TH, TT\}$ → if sample space is n
↳ an event $= \{HT, TH\} \subset \{HH, HT, TH, TT\}$ → 2^n possible events

$\varnothing$ = empty event

$P\{E\}$ = probability of event E

cardinality = set's size

the set of all events' cardinality = the power set of sample space (2^n)

two dice experiment → event example $= \{1, 1\}$ ✗ $\{(1,1)\}$ ✓

event algebra

- complementation: A^c or $\bar{A}$ $\{x: x \in \Omega \text{ and } x \notin A\}$
- disjoint = $A \cap B = \varnothing$ (mutually exclusive events) 

- exhausive = if the union of the events equals the sample space ($A \cup B = \Omega$) 

probability

$P(A)$ = the relative frequency of occurrence of an event in a large number of experiments

$P(\varnothing) = 0$ $P(\Omega) = 1$ $1 \geq P(A) \geq 0$

$$\boxed{\begin{aligned} A \cup B &= 1 - A' \times B' \\ &= A + B - A \times B \end{aligned}}$$

★ mutually exclusive = empty intersections

example

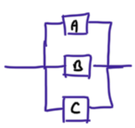
probability of first event: 0.7
second event: 0.5

→ probability of occurance of any event: $1 - [0.3 \times 0.5]$
or
$0.7 + 0.5 - (0.7) \times (0.5)$
$= 0.85$

systems' reliability:

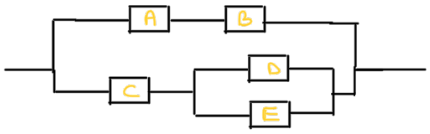
A: 0.01
B: 0.02
C: 0.02
} the probability of crush

→ $(0.99) \times (0.98) \times (0.98) = 0.95$



→ $1 - [(0.01) \times (0.02) \times (0.02)] = 0.99$

example



calculate the reliability of the system, if each component is operable with probability 0.92 independently.

$A \cap B = 0.92 \times 0.92 = 0,8464$

↳ $(A \cap B) \cup [C \cap (D \cup E)]$

$A \cap B = X$
$C \cap (D \cup E) = Y$
$D \cup E = Z$

$X \cup Y = 1 - X'Y'$

$X' = 1 - A.B = 1 - (0.92)^2 = 1 - 0,8464$
$Y' = 1 - C.Z$
$Z = 1 - D'.E' = 1 - (0.08)^2 = 0.9936$
$Y' = 1 - (0.92).(0.9936) = 1 - 0,914112$

$\rightarrow 1 - (1 - 0.8464).(1 - 0.914112) = 1 - (0,1536) \times (0,085888)$
$= 1 - 0,0131924 = 0,9868076$

conditional probability

if A occurs when B is known to occur $\rightarrow P(A|B)$

the occurence of A

without condition: $P(A) = \frac{A}{\Omega}$

with condition: $P(A|B) = \frac{P(A \cap B)}{P(B)}$

example: tossing a dice $P\{A|B\} = ?$

A = numbers are even $2, 4, 6 \rightarrow \frac{3}{6}$ $A \cap B = 2, 4 \rightarrow \frac{2}{6}$

B = numbers < 5 $1, 2, 3, 4 \rightarrow \frac{4}{6}$

↳ with formula $\rightarrow P\{A|B\} = \frac{\frac{2}{6}}{\frac{4}{6}} = \frac{1}{2}$

↳ without formula $\rightarrow$ B is known to occur 1, 2, 3, 4 evens are 2, 4

$P\{A|B\} = \frac{2}{4} = \frac{1}{2}$

intersection = $P\{A \cap B\} = P\{B\}.P\{A|B\}$ $\rightarrow$ independent: $P\{A|B\} = P\{A\}$
general case | case

↳ $P\{A \cap B\} = P\{A\}.P\{B\}$

! mutually independent does not mean that they are disjoint.

↳ if they are disjoint then $P\{B|A\} = 0$
$A|B$

example:

80% of the flights arrive on time $\rightarrow A$

90% of the flights depart on time → D
75% of the flights arrive and depart on time → A∩D

① if a flight depart on time, probability of arrive on time

$$P\{A|D\} = \frac{P(A\cap D)}{P(D)} = \frac{0.75}{0.9} = 0.83$$

② if a flight arrive on time, probability of depart on time

$$P\{D|A\} = \frac{P(D\cap A)}{P(A)} = \frac{0.75}{0.8} = 0.93$$

③ are the events independent? NO

$P\{A|D\} \neq P\{A\}$, $P\{D|A\} = P\{D\}$, $P(A\cap D) \neq P(A).P(D)$

example: tossing a dice

A: numbers are even → 2, 4, 6

B: numbers > 2 → 3, 4, 5, 6

$P(A\cap B) = \frac{2}{6}$ $P(A) = \frac{3}{6}$ $P(B) = \frac{4}{6}$

$P(A|B) = \frac{2/6}{4/6} = \frac{1}{2}$ $P(B|A) = \frac{2/6}{3/6} = \frac{2}{3}$

$\left.\begin{array}{l} P\{A|B\} = P\{A\} \\ P\{B|A\} = P\{B\} \end{array}\right\}$ they are independent.
but they are not disjoint.

bayes' rule

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

when $P(A\cap B)$ is unknown:

$P(A|B).P(B) = P(B|A).P(A)$ $P(A|B) = \frac{P(B|A).P(A)}{P(B)}$

example: X, Y, Z students forgot to sign their papers, their probabilities of getting good grades: X → 0.8, Y → 0.7, Z → 0.5. There are two good, one bad paper the probability that the bad exam belongs to Z?

base condition → $\boxed{+}$ $\boxed{+}$ $\boxed{-}$ P(B) →

good good bad

$\left.\begin{array}{l} (0.8).(0.7).(0.5) \\ + \\ (0.2)(0.7).(0.5) \\ + \\ (0.8).(0.3).(0.5) \end{array}\right\} 0.47$

P(A): Z gets bad paper
= 0.5

$P(A) = 0.5$
$P(B) = 0.47$

$P(A|B) = ?$

$P(A|B) = \frac{P(B|A)}{P(B)}.P(A)$

$P(B|A) = (0.8).(0.7) = 0.56$

$P(B|A)$ → in base condition, if Z gets the bad grade the probability of X and Y gets the good ones.

$$P(A|B) = \frac{(0.56).(0.5)}{0.47} = \underline{0.5957}$$

without formula:

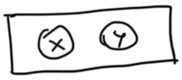
	X	Y	Z	
bad	0.2	0.3	0.5	grades
good	0.8	0.7	0.5	

2 good
1 bad

$$\frac{(0.8)(0.7)(0.5)}{(0.8)(0.7)(0.5)+(0.8)(0.3).(0.5)+(0.2).(0.7)(0.5)} = 0.595$$

notes

- $\{1\}$ is an event that outcome is 1.
- $\{1, 6\}$ → an event example of throwing one (1) die → subset of Ω
- A and $\bar{A}$ are not independent ($A.\bar{A} \neq A\cap\bar{A}$) because they are mutually exclusive.



$$P(X|Y) = \frac{\overset{0}{P(X\cap Y)}}{P(Y)} \neq P(X)$$

- if A is independent from B ($P(A|B)=P(A)$), then B is also independent from A ($P(B|A)=P(B)$) too.
- $P(A|B) = \frac{P(A\cap B)}{P(B)}$ if they are independent $P(A\cap B) = P(A).P(B)$

↳ $\frac{P(A).\cancel{P(B)}}{\cancel{P(B)}} = P(A) = P(A|B)$

law of total probability

if we do not know $P(A)$ directly, and B, C, D ... are exhaustive (without A) and mutually exclusive.

$$P(A) = P(A\cap B) + P(A\cap C) + P(A\cap D)\ldots$$
$$= P(A|B).P(B) + P(A|C).P(C) + P(A|D).P(D)\ldots$$

in two events

$$P(A) = P(A\cap B) + P(A\cap\bar{B})$$
$$= \underline{P(A|B)P(B) + P(A|\bar{B}).P(\bar{B})}$$

in bayes rule $$P(B|A) = \frac{P(A|B).P(B)}{P(A)} = \frac{P(A|B).P(B)}{P(A|B).P(B) + P(A|\bar{B}).P(\bar{B})}$$

if we do not know it

example: the test is positive when the disease exists = 95%
has positive false rate = 1%
and 0.5% of the population has the disease
probability of having disease when the test is positive?

given

$P(P|D) = 0.95$
$P(P|\bar{D}) = 0.01$
$P(D) = 0.005$
$P(D|P) = ?$

bayes rule

$$P(D|P) = \frac{P(P|D).P(D)}{P(P)}$$ with the law of total probability

$$= \frac{\overbrace{P(P|D)}^{0.95}.\overbrace{P(D)}^{0.005}}{\underbrace{P(P|D)}_{0.95}.\underbrace{P(D)}_{0.005} + \underbrace{P(P|\bar{D})}_{0.01}.\underbrace{P(\bar{D})}_{(1-0.005)}} = 0.323$$

↳ we can also refer that:
$P(P|D) = 0.95 \rightarrow P(\bar{P}|D) = 0.05$
$P(P|\bar{D}) = 0.01 \rightarrow P(\bar{P}|\bar{D}) = 0.99$

without formula

	+	−	
0.995	0.01	0.99	→ do not have disease
0.005	0.95	0.05	→ have disease

$$\frac{(0.005)(0.95)}{(0.005).(0.95) + (0.995)(0.01)} = 0.323$$