MATH 219

Spring 2020-21

Lecture 4

Lecture notes by Özgür Kişisel

Content: Differences between linear and nonlinear equations (section 2.4)).

Suggested Problems: (Boyce, Di Prima, 10th edition)

§2.4: 1, 3, 5, 6, 8, 9, 12, 14, 16, 23, 27, 30

Recall that a first order ODE is called linear if it can be written in the form

$$\frac{dy}{dt} + p(t)y = q(t)$$

for some functions $p(t)$ and $q(t)$. This definition does not give us much insight about the adjective "linear"; it doesn't tell us much about distinguishing features of linear equations. In this lecture we will study linear equations and non-linear equations from a more theoretical point of view. The characteristic feature of linear equations is the principle of superposition, which we discuss first. Afterwards, we investigate the existence and uniqueness of solutions for initial value problems. There are some subtle differences between linear and non-linear equations in this respect. The possibility of finite time blow-up for non-linear equations is the most notable of these differences. We conclude the lecture by stating some other differences between linear and nonlinear equations.

1 Principle of Superposition

The characteristic feature of linear equations is that one can combine some given solutions in order to produce new solutions.

Definition 1.1 *Say $f_1, \cdots, f_n$ are functions of t. A function of the form $f = c_1f_1 + c_2f_2 + \cdots + c_nf_n$, where $c_1, \cdots, c_n$ are constants, is called a **linear combination** of $f_1, \cdots, f_n$. If furthermore $c_1 + \cdots + c_n = 1$, then f is called an **affine linear combination** of $f_1, \cdots, f_n$.*

Proposition 1.1 *Suppose that $f_1, f_2, \cdots, f_n$ are solutions of the first order linear equation $y' + p(t)y = q(t)$. Then any affine linear combination of $f_1, \cdots, f_n$ is also a solution. If furthermore $q(t)$ is the zero function, then any linear combination is a solution.*

Proof: Since $f_1, \cdots, f_n$ are solutions, we have

$$\begin{aligned} f_1' + p(t)f_1 &= q(t) \\ f_2' + p(t)f_2 &= q(t) \\ &\cdots \\ f_n' + p(t)f_n &= q(t) \end{aligned}$$

We multiply the first equation by c_1, second equation by c_2, ... , the nth equation by c_n and add them up. We get

$$(c_1f_1 + \cdots + c_nf_n)' + p(t)(c_1f_1 + \cdots + c_nf_n) = (c_1 + \cdots + c_n)q(t).$$

Therefore, if $c_1+\cdots+c_n = 1$ then the right hand side becomes $q(t)$ and consequently $f = c_1f_1 + \cdots + c_nf_n$ is a solution of the equation. If $q(t) = 0$, then the value of $c_1 + \cdots + c_n$ doesn't matter and any linear combination of $f_1, \cdots, f_n$ is a solution. $\square$

The theorem above describes us precisely in which sense combinations of solutions of linear equations are again solutions. This is called the **principle of superposition**.

Example 1.1 *The principle of superposition fails in general for non-linear equations. For instance, consider the nonlinear ODE $y' = y^2$. It is separable and we can solve it as follows:*

$$\begin{aligned} \int \frac{dy}{y^2} &= \int dt \\ -\frac{1}{y} &= t + c \\ y(t) &= -\frac{1}{t+c} \end{aligned}$$

But a linear combination of two solutions of this form is almost never in the same form, therefore it will almost never be a solution. For instance, $-\frac{1}{t}$ and $-\frac{1}{t-1}$ are solutions. But

$$-c_1\frac{1}{t} - c_2\frac{1}{t-1} = -\frac{(c_1+c_2)t - c_1}{t(t-1)}$$

is not a solution (unless $c_1 = 0, c_2 = 1$ or $c_1 = 1, c_2 = 0$, but these are very special cases).

2 Existence and Uniqueness Theorems

A first order ODE typically has infinitely many solutions because of the constant c appearing in the solution. Our experience so far is that if an additional initial condition $y(t_0) = y_0$ is given then the constant and therefore the solution is uniquely determined. One can ask whether this is always the case. The answer to this question is given by the existence-uniqueness theorem. Let us begin investigating this problem starting from the case of linear equations.

Theorem 2.1 *Suppose that we have an initial value problem $y' + p(t)y = q(t)$ and $y(t_0) = y_0$ where the functions $p(t), q(t)$ are continuous on an open interval (a, b) and $t_0 \in (a, b)$. Then this initial value problem has a unique solution $y(t)$ which is valid over (at least) the whole open interval (a, b).*

Remark 2.1 *It is possible that $a = -\infty$, $b = +\infty$ or both.*

Proof: We already derived a formula for the general solution of a first order linear equation in section 2:

$$y = \frac{\int \mu(t)q(t)dt}{\mu(t)}$$

where $\mu(t) = e^{\int p(t)dt}$. If $p(t)$ and $q(t)$ are both continuous on (a, b) then they are integrable. Same holds for $\mu(t)$. Also, $\mu(t) \neq 0$ since it is an exponential function; so division by $\mu(t)$ does not cause a problem. Therefore $y(t)$ is defined for all $t \in (a, b)$. What about the constant? Suppose that $F(t)$ is one of the antiderivatives of $\mu(t)q(t)$. Then we can rewrite the solution as

$$y = \frac{F(t) + c}{\mu(t)}$$

If we put $y(t_0) = y_0$ then we see that $c = \mu(t_0)y_0 - F(t_0)$, therefore it is uniquely determined by the initial condition. Hence the solution is unique. $\square$

Now, let us look at the case of non-linear equations. An arbitrary first order equation (linear or non-linear) can be written in the form

$$\frac{dy}{dt} = f(t, y)$$

provided that we can isolate the term $\frac{dy}{dt}$ in the equation by using some algebraic operations. We will assume that this can indeed be done and work with equations in the form above. Notice that if the equation is linear then $f(t, y) = q(t) - p(t)y$.

The first thing to think about regarding the existence-uniqueness theorem for the non-linear case, is its statement. In general, it will be impossible to decompose $f(t, y)$ in the form $q(t) - p(t)y$ or in any reasonable simple form. In the linear case, the continuity of $p(t)$ and $q(t)$ implies the continuity of the two variable function $f(t, y)$. However, it implies more than just that: Notice that $\frac{\partial f}{\partial y} = -p(t)$. Therefore the continuity of $p(t)$ in the linear case is equivalent to the continuity of $\frac{\partial f}{\partial y}$. This gives a good idea about a possible correct formulation in the non-linear case:

Theorem 2.2 *Suppose that we have an initial value problem* $y' = f(t, y)$ *and* $y(t_0) = y_0$ *where* $f(t, y)$ *and* $\frac{\partial f}{\partial y}$ *are both continuous on an open rectangle* $(a, b) \times (c, d)$ *containing* (t_0, y_0). *Then there exists a unique solution of this initial value problem, defined possibly for* t *belonging to a smaller subinterval of* (a, b).

Since f and $\frac{\partial f}{\partial y}$ are functions of two variables, it is necessary to check continuity on an open rectangle containing the initial point, rather than on an interval. An actual rectangle is not absolutely necessary, any open set containing the initial point would do the same job. The proof of this theorem requires a knowledge of some important facts from mathematical analysis, and it is beyond the scope of these lectures. The phrase "possibly smaller subinterval" at the end of the statement is related to the notion of "finite time blow-up". This will be explained in detail in the next section.

Example 2.1 *We want to see that the condition* $\frac{\partial f}{\partial y}$ *really plays an important role in the theorem and cannot be entirely discarded. For this purpose, consider the ODE*

$$y' = y^{1/3}$$

together with the initial condition $y(0) = 0$. *Then* $f(t, y) = y^{1/3}$ *is continuous everywhere, however* $\frac{\partial f}{\partial y} = \frac{1}{3}y^{-2/3}$ *is not continous on the line* $y = 0$. *This line* *contains the initial point* $(0,0)$*, so the conditions of the existence-uniqueness theorem are not satisfied here. Let us indeed see that there are multiple solutions to this initial value problem. The equation is separable:*

$$\begin{aligned} \int \frac{dy}{y^{1/3}} &= \int dt \\ \frac{3}{2}y^{2/3} &= t+c \\ y &= \pm\left(\frac{2}{3}(t+c)\right)^{3/2} \end{aligned}$$

The condition $y(0)=0$ *gives* $c=0$*. But there are still at least two solutions of the initial value problem, namely*

$$y_1(t) = \begin{cases} (\frac{2}{3}t)^{3/2}, & t \geq 0 \\ 0, & t < 0 \end{cases}, \qquad y_2(t) = \begin{cases} -(\frac{2}{3}t)^{3/2}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Also, in the derivation we assumed that $y \neq 0$*. The function* $y_3 = 0$ *is yet a third solution. (In fact, literally speaking, one can construct infinitely many solutions to this initial value problem. Can you see how?)*

3 Finite Time Blow-up

In the previous section it was mentioned that the domain of the solution of a non-linear initial value problem may not extend to the largest interval on which the relevant functions are continuous. This phenomenon is called **finite time blow-up**, and is especially disturbing in any theoretical or practical consideration of differential equations. Let us see that this can indeed happen, by looking at an example.

Example 3.1 *Consider the non-linear ODE* $y' = y^2$ *together with the initial condition* $y(0) = y_0 > 0$*. We obtained the general solution to this ODE to be*

$$y(t) = -\frac{1}{t+c}.$$

Since $y(0) = y_0$*, we have* $y_0 = -\frac{1}{c}$*, so* $c = -\frac{1}{y_0}$*. Then*

$$y(t) = -\frac{1}{t - \frac{1}{y_0}}.$$

Now, the conditions of the existence-uniqueness theorem are satisfied for this ODE. Indeed, $f(t, y) = y^2$ *and* $\frac{\partial f}{\partial y} = 2y$ *are continuous on all of* $\mathbb{R}^2$*. However, the solution is defined only on the interval* $(-\infty, \frac{1}{y_0})$ *and not on all of* $(-\infty, \infty)$ *(We say the solution* ***blows up*** *at* $\frac{1}{y_0}$*). Notice that the place of the blow-up depends on the initial condition, hence it can not be quickly predicted from the ODE only. One actually has to solve the ODE in order to understand what happens.*

4 Some Other Differences

There are some further differences between linear and nonlinear equations, two of which we remark below:

- The general solution of a linear first order ODE can be written as a formula in terms of the functions $p(t)$ and $q(t)$. This is not the case for nonlinear ODE's. Even if one can find all solutions, not all solutions need to correspond to special values of a constant in some general formula. It might be difficult to understand how the solution depends on the parameters and functions in the original ODE.

- When solving a nonlinear ODE, one sometimes needs to leave the final solution in an implicit form. This is never necessary for a first order linear ODE since the formula expresses y explicitly in terms of t.