23 - euler equations

when $A(x)$, $B(x)$ are constant functions:

$x = x_0 \to$ regular singular point

$(x-x_0)^2 y'' + (x-x_0)\, a y' + by = 0$

when $x_0 = 0 \to x^2 \cdot y'' + axy' + by = 0 \longrightarrow$ since $x=0$, singular point

↳ solution is in $(-\infty, 0)$ or $(0, \infty)$

$y(x) = x^r$ $\quad x^r \left[\underbrace{a'r^2 + b'r + c'}_{\text{indicial equation}}\right] = 0$ $\quad y'' + (\alpha - 1)y' + \beta y = 0$

$\Delta = (\alpha - 1)^2 - 4\beta$

$\Delta > 0 \quad y_H = c_1 x^{r_1} + c_2 x^{r_2} = c_1 (x - x_0)^{r_1} + c_2 (x - x_0)^{r_2}$

$\Delta = 0 \quad y_H = c_1 x^{r_1} + c_2 \cdot x^{r_1} \ln x$

$\Delta < 0 \quad y_H = c_1 x^a \cos(b \ln x) + c_2 x^a \sin(b \cdot \ln x)$

when $(x_0 > 0)$ $x_0 > x$

↳ if $x_0 < x \to |x - x_0|$

$\boxed{ex}$ $x^2 \cdot y'' - 2xy' + 2y = 0$

$y(x) = x^r$

$y'(x) = r \cdot x^{r-1}$

$y''(x) = r(r-1)\, x^{r-2}$

$x^2 \cdot (r(r-1)x^{r-2}) - 2x\, r\, x^{x-1} + 2x^r = 0$

$x^r \left[r^2 - r - 2r + 2 \right] = 0$

$r^2 - 3r + 2 = 0$

$r = 2, 1$

$y_H(x) = c_1 x^2 + c_2 x$

$\boxed{ex}$ $x^2 y'' - xy' + y = 0$

$r^2 - 2r + 1 = 0$

$r = 1 \ \cancel{2}$

$y_H(x) = c_1 x + c_2 x \ln x$

$\boxed{ex}$ $x^2 y'' - 5xy' + 13y = 0$

$x^2 \left(r(r-1)\, x^{r-2} \right) - 5x \cdot r\, x^{r-1} + 13 \cdot x^r = 0$

$x^r \left[r^2 - r - 5r + 13 \right] = 0$

$r^2 - 6r + 13$

$r = 3 \pm 2i$

$y_H = c_1 x^3 \cdot \cos(2 \ln x) + c_2 x^3 \sin(2 \ln x)$