21 - series solutions near an ordinary point

$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$

$\boxed{x_0}$ ordinary point = if functions $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ are analytic

$y'' + \frac{Q(x)}{P(x)}\,y' + \frac{R(x)}{P(x)}\,y = 0$

otherwise singular point. ($\frac{1}{x^2+1} \to$ complex roots!)

solve for: $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

$\boxed{\text{ex}}$ $y'' - xy = 0$ around $x_0 = 0$?

$y'' + 0y' - xy = 0 \quad x_0 = 0$ analytic in 0 and x ✓

$y = \sum_{n=0}^{\infty} a_n . x^n \qquad y' = \sum_{n=1}^{\infty} n . a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n.(n-1)\, a_n\, x^{n-2}$

$$\sum_{n=2}^{\infty} n.(n-1)\, a_n\, x^{n-2} - x . \sum_{n=0}^{\infty} a_n . x^n = 0$$

$$\sum_{n=2}^{\infty} n.(n-1)\, a_n\, x^{n-2} - \sum_{n=0}^{\infty} a_n . x^{n+1} = 0$$

make the powers of x same

$n \to n+2 \qquad n \to n-1$

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n - \sum_{n=1}^{\infty} a_{n-1} . x^n = 0$$

$$\sum_{n=1}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n - 2a_2 - \sum_{n=1}^{\infty} a_{n-1} . x^n = 0$$

then the starting indices

$$\underbrace{\sum_{n=1}^{\infty} [(n+2)(n+1)\, a_{n+2} - a_{n-1}]\, x^n}_{0} \underbrace{- 2a_2 x^0}_{0} = 0$$

$a_2 = 0 \qquad (n+2)(n+1)\, a_{(n+2)} = a_{n-1}$

$n=3 \quad a_5 = \frac{a_2}{5.4} = 0 \qquad n=4 \quad a_6 = \frac{\overbrace{a_3}^{0}}{6.5}$

$2, 5, 8, 11 \to a_n = 0$

$n=2 \quad a_4 = \frac{a_1}{4.3} \quad a_7 = \frac{a_1}{7.6.4.3} \qquad n=1 \quad a_3 = \frac{a_0}{3.2} \quad a_6 = \frac{a_0}{6.5.3.2} \quad a_9 = \frac{a_0}{9.8. \dots}$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 \left(1 + \frac{x^3}{3.2} + \frac{x^6}{6.5.3.2} + \frac{x^9}{9.8.6.5.3.2} + \dots\right) + a_1 \left(x + \frac{x^4}{4.3} + \frac{x^7}{7.6.5.4} + \frac{x^{11}}{10.9.7.6.4.3} + \dots\right)$$

- converges for all x
- it is linear combination of two solutions $c_1 y_1 + c_2 y_2 = 0$

$\boxed{\text{ex}}$ $(1-x)\,y'' + xy' - y = 0, \quad y(0) = -3 \quad y'(0) = 2$, first 5 nonzero terms? around $x_0 = 0$

$y'' + \frac{x}{1-x}\,y' - \frac{1}{1-x}\,y = 0 \qquad x = 1$ singular $\neq 0$

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n . a_n . x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n.(n-1) . a_n\, x^{n-2} \qquad y = a_0 + a_1 x + a_2 x^2 + \dots$

$y(0) = \boxed{a_0 = -3}$

$y'(0) = \boxed{a_1 = 2}$

$2a_2 + a_0 = 0$

$\boxed{a_2 = \frac{3}{2}}$

$$(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} (n+1)n a_{n+1} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n+1)n a_{n+1} + n a_n - a_n\right] x^n - (2a_2 + a_0)x^0 = 0$$

$$(n+2)(n+1)a_{n+2} - (n+1)n a_{n+1} + (n-1)a_n = 0$$

$n=1$ $\quad 6a_3 - 2a_2 = 0 \quad \boxed{a_3} = \frac{2}{6}\cdot\frac{3}{2} = \boxed{\frac{1}{2}}$

$n=2$ $\quad 12a_4 - 6a_3 + a_2 = 0 \quad \boxed{a_4} = \dfrac{\frac{-3}{2} + 6\cdot\frac{1}{2}}{12} = \dfrac{\cancel{3}}{2\cdot\cancel{12}\,4} = \boxed{\frac{1}{8}}$