recurrance relations

- equation that expresses a_n in terms of one or more previous terms
  ↳ initial conditions are required to define terms

example

the number of bacteria doubles every hour. if the colony begins with 5 bacteria, how many will be present in n hours?

$$\left.\begin{array}{l} a_0=5 \\ a_1=10=2(a_0) \\ a_2=20=2(a_1) \end{array}\right\} \boxed{\begin{array}{l} a_0=5 \\ a_n=2a_{n-1} \end{array}}$$

example

let $a_0=5$ and $a_n=2a_{n-1}$ find the explicit function

$$\left.\begin{array}{l} a_0=5 \\ a_1=2(5) \\ a_2=2(2.5) \end{array}\right\} \boxed{a_n=2^n.5}$$

example

let $H_0=1$ and $H_n=2H_{n-1}+1$ $\quad H_n$ function?

$H_n=2(2H_{n-2}+1)+1=2^2H_{n-2}+2+1$

$2(2(2H_{n-3}+1)+1)+1=2^3H_{n-3}+2^2+2+1$

↳ $2^{n-1}\underbrace{H_1}_{1}+2^{n-2}+2^{n-3}+\ldots+2+1 \rightarrow \boxed{2^n-1}$

classification of recurance relations

① order

$a_n=a_{n-1}+5^n$ → does not matter (1st order)

$a_n=a_{n-2}+a_{n-1}$ (2nd)

$a_n=a_{n-3}$ (3rd)

② homogeneous – non-homogeneous

$a_n=3a_{n-1}+a_{n-2}$ (homogeneous)

$a_n=3a_{n-1}+a_{n-3}+3^n$ (non homogeneous)

$3^n/n^2/2$

③ linear – nonlinear

a_n'li terimlein ussü alındığında veya birbirleriyle çarpıldığında lineerlik bozulur.

$a_n=5a_{n-1}+\cancel{n\,a_{n-2}}$ → linear

$a_n=(a_{n-1})(a_{n-2})$ → nonlinear

④ constant coefficients

a_n'li terimler n'li terimlerle çarpım halinde olmadığında

$a_n=3a_{n-1}+n$ (constant)

$a_n=3^na_{n-2}+1$ (not constant)

homogenous recurence relations

example $(r_1 \neq r_2)$

$a_n - a_{n-1} - 6a_{n-2} = 0$ where $a_0 = 1$, $a_1 = 8$.

$\hookrightarrow \dfrac{r^n - r^{n-1} - 6r^{n-2}}{r^{n-2}} = \dfrac{0}{r^{n-2}} \rightarrow \boxed{r^2 - r - 6 = 0} \rightarrow$ ① characteristic polynomial

$\hookrightarrow (r-3)(r+2) = 0$ $\boxed{r = -2, 3} \rightarrow$ ② factors

$\hookrightarrow \boxed{a_n = \alpha(3)^n + \beta(-2)^n} \rightarrow$ ③ determine form of a_n

$\hookrightarrow a_0 = 1$ $a_1 = 8$ $\quad a_0 = \alpha + \beta = 1$ $\quad a_1 = 3\alpha - 2\beta = 8$ $\quad \boxed{\alpha = 2 \quad \beta = 1} \rightarrow$ ④ find coefficients

$a_n = 2.3^n - (-2)^n$

example $(r_1 = r_2)$

$a_n - 4a_{n-1} + 4a_{n-2} = 0$ where $a_0 = 1$, $a_1 = 3$.

$\hookrightarrow \dfrac{r^n - 4r^{n-1} + 4r^{n-2}}{r^{n-2}} = \dfrac{0}{r^{n-2}} \rightarrow r^2 - 4r + 4 = 0$ $\quad (r=2) \times 2$

for third root

$\hookrightarrow \boxed{a_n = \alpha(2)^n + \beta n(2)^n} + \gamma n^2(2^n)$

$a_0 = \alpha = 1$ $\quad \alpha = 1$

$a_1 = 2\alpha + 2\beta = 3$ $\quad \beta = \frac{1}{2}$

$a_n = 2^n + \dfrac{n2^n}{2} \rightarrow 2^n + n2^{n-1}$

non-homogenous recurence relations

$a_n = \underbrace{a_n^{(h)}}_{\text{homogeneus}} + \underbrace{a_n^{(p)}}_{\text{particular}}$

we guess the particular solutions

$f(n)$	a_n^p
c	A
n	$A_1n + A_0$
n^2	$A_2n^2 + A_1n + A_0$
r^n	A_r^n

example

$a_{n+1} - 2a_n = 2^n$ $\quad n \geq 0$ $\quad a_0 = 1$

$r^2 - 2r = 0$ $\quad r = 2, 0$

$a_n^{(h)} = \alpha 2^n + \beta 0^n$

$a_n^h = \alpha 2^n$

a_n^p: $\quad a_{n+1} - 2a_n = 2^n$ $\quad r^n \rightarrow \boxed{A2^n} . n$

2^n in the homogenous solution, multiply by

$(n+1)A\,2^{n+1} - n2A\,2^n = 2^n$

$2A(n+1) - 2An = 1$

$2A = 1$ $\quad A = \frac{1}{2}$ $\quad a_n^p = \dfrac{2^n . n}{2} = 2^{n-1} . n$

$a_n = a_n^h + a_n^p$

$a_n = \alpha 2^n + n2^{n-1}$ $\quad a_0 = \alpha = 1$

$a_n = 2^n + n2^{n-1}$