5 - differences between linear and nonlinear equations

linear equations

$\frac{dy}{dt} + p(t) \cdot y = q(t)$

$f(t,y) = q(t) - p(t)y$

- principle of superposition

non-linear equations

includes: y^2, yy', e^y, $\sin(y)$

$f(t,y) = y'$

- possibility of finite time blow-up

- existence and uniqueness for initial value problems

$\boxed{\text{principle of superposition}}$

- linear equations → fails in non-linear equations
- combining given solutions to produce new solutions

$\boxed{\text{existence and uniqueness theorem}}$

- linear equations → $y(t_0) = y_0$ is given
  → $q(t)$, $p(t)$ are continuous on (a,b)
  → $t_0 \in (a,b)$

  } has unique solution valid for (a,b) → at least

- non-linear equations → $y' = f(t,y)$, and $y(t_0) = y_0$ are given
  → $f(t,y)$, $\frac{\partial f}{\partial y}$ are continuous on $(a,b) \times (c,d)$
  → y'e göre kısmi türev
  → $f(t_0, y_0) \in (a,b) \times (c,d)$

  } has a unique solution, t is in smaller subinterval of (a,b) (bc of finite time blow up)

$\boxed{\text{finite time blow-up}}$

- nonlinear equations
- the domain of the solution may not extend to the largest interval which the relavant functions are continuous

example:

$y' = y^2 \quad y(0) = y_0 > 0 \quad \rightarrow \quad y = \frac{-1}{t+c} \quad \overset{t=0}{y_0 = \frac{-1}{c}} \quad c = \frac{-1}{y_0} \quad \rightarrow \quad y(t) = -\frac{1}{t - \frac{1}{y_0}}$

- $f(t,y) = y^2$ } continuous on $\mathbb{R}^2$ → but solution is defined

- $\frac{\partial f}{\partial y} = 2y$ ✓ on ℝ → only on $\left(-\infty, \frac{1}{y_0}\right)$ (not all $(-\infty, \infty)$)

solution blows up at $\frac{1}{y_0}$ → depends on initial condition

☆ the general solution of a linear first order ode can be written as a formula in terms of the functions $p(t)$ and $q(t)$.

examples

① $(t-3)y' + (\ln t)y = 2t, \quad y(1) = 2$ interval of solution for given initial value, certain to exist?

$y' + \frac{\ln t}{t-3} y = \frac{2t}{t-3}$

$t > 0$ $(0,3)$ or $(3,\infty)$

$t-3 \neq 0$ ↳ given $y_0(1)$ so → $\underline{(0,3)}$

$t \neq 3$

② $t(t-4)y' + y = 0, \quad y(2) = 1$

$y' + \frac{y}{t(t-4)} = 0$ $t \neq 0, 4$ answer is

$(-\infty,0)$ or $(0,4)$ or $(4,\infty)$ → $\underline{(0,4)}$

③ $y' + (\tan t)y = \sin t, \quad y(\pi) = 0$

$\tan t = \frac{\sin t}{\cos t}$ $\cos t \neq 0$

$t \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$ → $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

④ $(\ln t)y' + y = \cot t, \quad y(2) = 3$

$y' + \frac{y}{\ln t} = \frac{\cot t}{\ln t}$ $\pi = 3.14$

$\underline{\underline{(1,\pi)}}$

$\ln t \neq 0$ $\cot t = \frac{\cos t}{\sin t}$ $\sin t \neq 0$

$t \neq 1$ $t > 0$ ↓

$(0,1)$ $(1,\infty)$ $0, \pi, 2\pi$

① $y' = 2ty^2 \quad y(0) = y_0$ interval, solution?

$\int \frac{dy}{y^2} = \int 2t\,dt$ $\quad -\frac{1}{y} = t^2 + c$ $\quad t = 0 \rightarrow c = -\frac{1}{y_0}$ $\quad -\frac{1}{y} = t^2 - \frac{1}{y_0} \rightarrow y = \frac{-1}{(t^2 \ldots)}$

$) y^2)$ $\quad y \quad y=y_0 \quad y_0 \quad y \quad y_0 \quad - \quad t \; \frac{-1}{y_0}$

blows up at $t=\sqrt{\frac{1}{y_0}}$

unique solution theorem:

$y' = f(t,y) = 2ty^2$ → contnious everywhere

$\frac{\partial f}{\partial y} = (2ty^2)\, dy = 4ty$ → continious everywhere

if $y_0 < 0$ never zero → unique solution exists $-\infty < t < \infty$

if $y_0 > 0$ unique solution exists $-\sqrt{\frac{1}{y_0}} < t < \sqrt{\frac{1}{y_0}}$

① superposition → $y' - 2y = 0$ show $\phi(t) = e^{2t}$ is a solution:

$2e^{2t} - 2e^{2t} = 0$ also show $c\,\phi(t)$ is a solution:

$2.ce^{2t} - 2ce^{2t} = 0$ → so it is linear ODE

→ $y' + y^2 = 0$ show $\phi(t) = \frac{1}{t}$ is a solution:

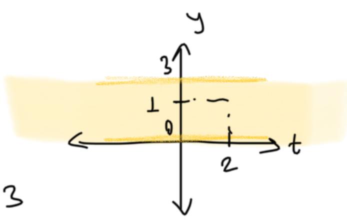
$-\frac{1}{t^2} + \frac{1}{t^2} = 0$ also show $c.\,\phi(t)$ is not a solution (when $c \neq 1, 0$)

$\frac{-c}{t^2} + \frac{c^2}{t^2} \neq 0$ → so it is nonlinear ODE.

☆ $y' = \frac{8+t^3}{3y-y^2}$ $\quad y(2)=1$ $\quad$ existence and uniqueness theorem?

$y' = \frac{8+t^3}{y(3-y)}$ $\quad t \in \mathbb{R}$

$\frac{\partial f}{\partial y} = \left((8+t^3)(3y-y^2)^{-1}\right)' = \frac{-1(8+t^3)(3-2y)}{(3y-y^2)^2}$ $\quad y \neq 0, 3$



$\underbrace{(-\infty, \infty)}_{t} \times \underbrace{(0,3)}_{y}$