CENG 384 - Signals and Systems for Computer Engineers
Spring 2022
Homework 2

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1. (a) $x(t) - 2\frac{dx(t)}{dt} + 3y(t) - 2\int y(t)dt = \frac{dy(t)}{dt}$

   If we take derivative of both sides,

   $\frac{dx(t)}{dt} - 2\frac{d^2x(t)}{dt} + 3\frac{dy(t)}{dt} - 2y(t) = \frac{d^2y(t)}{dt}$

   It can be written also as,

   $2y(t) - 3y'(t) + y''(t) = x'(t) - 2x''(t)$.

   (b) The given system is a causal LTI system. Assume that $y(t) = y_h(t) + y_p(t)$ such that our homogenous part is $y_h(t) = Ce^{st}$
   Therefore,
   $2Ce^{st} - 3Cse^{st} + Cs^2e^{st} = 0$
   $C(s^2 - 3s + 2) = 0 \xrightarrow[C \neq 0]{} s = 2, 1$. So, $y_h(t) = C_1e^{2t} + C_2e^t$.
   Let's write the particular solution, $y_p(t) = Kx(t)$ where $x(t) = (e^{-t} + e{-}2t)u(t)$
   $y_p(t) = Ae^{-t}u(t) + Be^{-2t}u(t)$

   $[2Ae^{-t} + 2Be^{-2t} - 3(-Ae^{-t} - 2Be^{-2t}) + (Ae^{-t} + 4Be^{-2t})]u(t) = (-e^{-t} - 2e^{-2t} - 2(e^{-t} + 4e^{-2t}))u(t)$

   $6Ae^{-t} + 12Be^{-2t} = -3^{-t} - 10e^{-2t}$

   $(6A + 3)e^{-t} + (10 + 12B)e^{-2t} = 0 \rightarrow A = \frac{-1}{2}, B = \frac{-5}{6}$

   Since $y(t) = y_h(t) + y_p(t)$, $y(t) = [C_1e^{2t} + C_2e^t + \frac{1}{2}e^{-t} - \frac{5}{6}e^{-2t}]u(t)$

   By assuming the system is initially at rest, $y(0) = 0$ and $y'(0) = 0$
   Using $y(0) = 0$, $C_1 + C_2 + \frac{-1}{2} - \frac{5}{6} = 0$ and

   Using $y'(0) = 0$, $2C_1 + C_2 + \frac{1}{2} + \frac{10}{6} = 0$

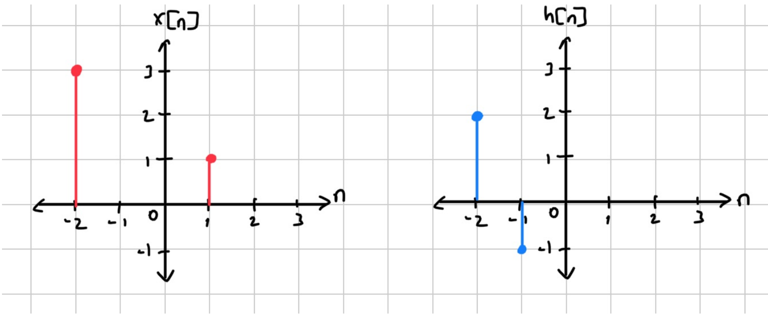
   Solving these two equations give us $C_1 = \frac{-7}{2}, C_2 = \frac{29}{6}$

   Finally, the output $y(t)$ is $y(t) = [\frac{-7}{2}e^{2t} + \frac{29}{6}e^t - \frac{1}{2}e^{-t} - \frac{5}{6}e^{-2t}]u(t)$.

2. (a) We know that

$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

The signals $x[n]$ and $h[n]$ are:

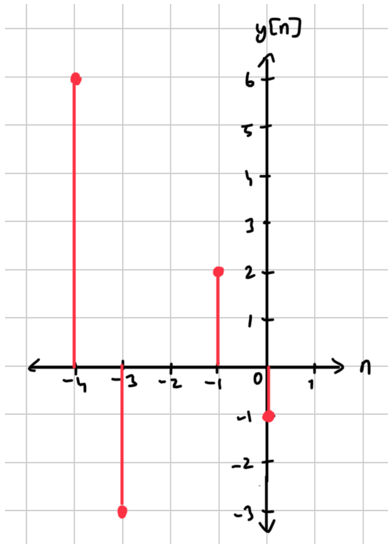


$y[n] = x[-2]h[n+2] + x[1]h[n-1]$ since x is 0 for other values

$= 3h[n+2] + h[n-1]$

$= 3(2\delta[n+4] - \delta[n+3]) + (2\delta[n+1] - \delta[n])$

$= -\delta[n] + 2\delta[n+1] - 3\delta[n+3] + 6\delta[n+4]$



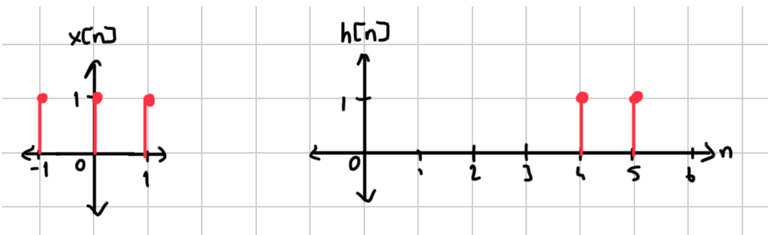
(b) $x[n] = u[n+1] - u[n-2] = \delta[n+1] + \delta[n] + \delta[n-1]$

$h[n] = u[n-4] - u[n-6] = \delta[n-4] + \delta[n-5]$

We know that

$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

The signals $x[n]$ and $h[n]$ are:

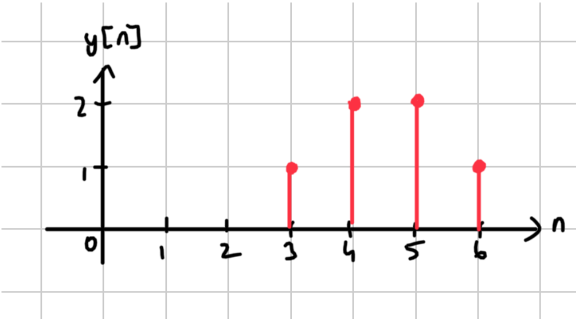


$y[n] = x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1]$ since x is 0 for other values

$= h[n+1] + h[n] + h[n-1]$

$= \delta[n-3] + \delta[n-4] + \delta[n-4] + \delta[n-5] + \delta[n-5] + \delta[n-6]$

$= \delta[n-3] + 2\delta[n-4] + 2\delta[n-5] + \delta[n-6]$



3. (a) $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-\frac{1}{2}(t-\tau)}u(t-\tau)\,d\tau$

If last part is non-zero when $\tau > 0$ and $t - \tau > 0$, then $0 < \tau < t$. So,

$y(t) = \int_0^t e^{-\tau} e^{\frac{-t}{2}} e^{\frac{\tau}{2}}\,d\tau$

$= e^{\frac{-t}{2}} \int_0^t e^{\frac{-\tau}{2}}\,d\tau$

$= e^{\frac{-t}{2}}(-2e^{\frac{-t}{2}} + 2)u(t)$

$= (-2e^{-t} + 2e^{\frac{-t}{2}})u(t)$.

(b) $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

$= \int_{-\infty}^{\infty} (u(\tau) - u(\tau-4))(e^{-3(t-\tau)}u(t-\tau))\,d\tau$

For $t < 0$, $y(t) = 0$

For $0 < t < 4$, $y(t) = \int_0^t e^{-3(t-\tau)}\,d\tau = e^{-3t}\int_0^t e^{3\tau}\,d\tau = e^{-3t}(\frac{e^{3t}-1}{3}) = \frac{1-e^{-3t}}{3}$

For $4 < t$, $y(t) = \int_0^4 e^{-3(t-\tau)}\,d\tau = e^{-3t}\int_0^4 e^{3\tau}\,d\tau = \frac{e^{-3t}}{3}(e^{12}-1) = \frac{e^{12-3t}-e^{-3t}}{3}$

4. (a) $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau-3)d\tau = \int_{-\infty}^{t-3} e^{-(t-3-\tau')}x(\tau')d\tau'$
Since impulse response is $h(t)$ when the convolution integral is $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$
$h(t) = e^{-(t-3)}u(t-3)$

(b) We have
$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
$= \int_3^{\infty}(u(\tau+2) - u(\tau-1))e^{-(t-\tau-3)})d\tau$
If $t < 1$ no intersection area so $y(t) = 0$
If $1 < t < 4$ the value of $x(t)$ is 1 in t-3. So, $= \int_{-2}^{t-3} e^{-(t-\tau-3)})d\tau = (e^{\tau+3-t})|_{-2}^{t-3} = 1 - e^{1-t}$
If $4 < t$ since the value of $x(t)$ is 0 other than $[-2, 1]$, $y(t) = \int_{-2}^{1} e^{-(t-\tau-3)})d\tau = (e^{\tau+3-t})|_{-2}^{1} = e^{4-t} - e^{1-t}$
Therefore,

$$y(x) = \begin{cases} 0 & t < 1 \\ 1 - e^{1-t} & 1 < t < 4 \\ e^{4-t} - e^{1-t} & 4 < t \end{cases}$$

5. (a) $h_1^{-1}[n] * h_1[n] = \delta[n]$

$h_1^{-1}[n] - Ah_1^{-1}[n-1] = \delta[n]$

$(\frac{1}{2})^n u[n] - A(\frac{1}{2})^{n-1} u[n-1] = \delta[n]$

$(\frac{1}{2})^n (u[n] - 2Au[n-1]) = \delta[n]$

$2A = 1, A = \frac{1}{2}$

Since $(u[n] - u[n-1]) = \delta[n]$, $(\frac{1}{2})^n(u[n] - u[n-1]) = (\frac{1}{2})^n \delta[n]$, which is 1 for n=0 and 0 otherwise. Therefore, the equation equals $\delta[n]$

$h_1^{-1}[n] - \frac{1}{2}h_1^{-1}[n-1] = \delta[n]$

$h_1^{-1}[n](\delta[n] - \frac{1}{2}\delta[n-1]) = \delta[n]$ where, $(\delta[n] - \frac{1}{2}\delta[n-1]) = h_1[n]$

Finally, $h_1[n] * h_1[n] = \sum_{k=-\infty}^{\infty} h[k]h[n-k] = h[0]h[n] + h[1]h[n-1]$
$= \delta[n] - \frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2] = \delta[n] - \delta[n-1] + \frac{\delta[n-2]}{4}$

(b) $h_0[n] * (h_1[n] * h_1[n]) = h[n]$

$h_0[n] * (\delta[n] - \delta[n-1] + \frac{\delta[n-2]}{4}) = h[n]$

$h_0[n] - h_0[n-1] + \frac{h_0[n-2]}{4} = h[n]$

When $n < 0, h_0[n] \to 0$ since $h[n] < 0$ when $n < 0$.

For n = 0,

$h_0[0] - h_0[-1] + \frac{h_0[-2]}{4} = h[0] = 4$. So, $h_0[0] = 4$

For n = 1,
$h_0[1] - h_0[0] + \frac{h_0[-1]}{4} = h[1] = 0$. So $h_0[1] = 4$

For n = 2,
$h_0[2] - h_0[1] + \frac{h_0[0]}{4} = h[2] = 1$. So $h_0[2] = 4$

For n = 3,
$h_0[3] - h_0[2] + \frac{h_0[1]}{4} = h[3] = -3$. So $h_0[3] = 0$

For n = 4,
$h_0[4] - h_0[3] + \frac{h_0[2]}{4} = h[4] = 1$. So $h_0[4] = 0$

For n = 5,
$h_0[5] - h_0[4] + \frac{h_0[3]}{4} = h[5] = 0$. So $h_0[5] = 0$

For $n \geq 3, h_0[n] \to 0$.

So, $h_0[n] = 4(\delta[n] + \delta[n-1] + \delta[n-2])$.

(c) $y[n] = x[n] * h_0[n]$

$= \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} (\delta[k] + \delta[k-2])4(\delta[n-k] + \delta[n-k-1] + \delta[n-k-2])$

$y[n] = x[0]h_0[n] + x[2]h_0[n-2]$ since x is 0 for other k values.

$= 4(\delta[n] + \delta[n-1] + \delta[n-2]) + 4(\delta[n-2] + \delta[n-3] + \delta[n-4])$

$= 4\delta[n] + 4\delta[n-1] + 8\delta[n-2] + 4\delta[n-3] + 4\delta[n-4]$