24 - series solutions near a regular singular point

$P(x) \cdot y'' + Q(x) y' + R(x) y = 0$

$\longrightarrow$ ordinary point : two solutions can be found

$x = x_0$ $\longrightarrow$ singular point : no series solution

$\longrightarrow$ regular singular point : at least one solution can be represented

when $x = x_0$ is regular singular point =

$(x-x_0)^2 y'' + (x-x_0) A(x) y' + B(x) \cdot y = 0$

$(x-x_0)^2 y'' + (x-x_0)\left[a_0 + a_1(x-x_0) + \dots\right] y' + \left[b_0 + b_1(x-x_0) + \dots\right] y = 0$

$(x-x_0)^2 y'' + (x-x_0)\, a\, y' + b y = 0$ $\longrightarrow$ as euler equation

if $r_1 - r_2$ is not an integer $\Rightarrow$ 2 solutions

else only y_1

$\boxed{\text{example}}$ $2x^2 \cdot y'' + 3x y' + (2x^2 - 1) y = 0$ when $x_0 = 0$

$y'' + \frac{3\cancel{x}}{2x^{\cancel{2}}} y' + \frac{2x^2-1}{2x^2} y = 0$

$\lim_{x \to 0} x \cdot \frac{3}{2x} = \boxed{\frac{3}{2} = \alpha}$

$\lim_{x \to 0} \cancel{x^2} \frac{(2x^2-1)}{2\cancel{x^2}} = \boxed{\frac{-1}{2} = \beta}$

indicial equation

$r^2 + (\underbrace{\tfrac{3}{2}}_{\alpha} - 1) r \underbrace{- \tfrac{1}{2}}_{\beta} = 0$ $\qquad r^2 - \frac{1}{2} r - \frac{1}{2} = 0$

$2r^2 - r - 1 = 0$ $\qquad r = -1, \frac{1}{2}$ $\qquad r_1 - r_2 \neq$ integer $\rightarrow$ 2 solutions

$y_1 = \sum_{n=0}^{\infty} a_n x^{n-1}$ $\qquad y_2 = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$ $\qquad y = \sum_{n=0}^{\infty} a_n x^{n+r}$

y_1: $y = \sum_{n=0}^{\infty} a_n x^{n-1}$

$y' = \sum_{n=1}^{\infty} a_n (n-1) \cdot x^{n-2}$

$y'' = \sum_{n=2}^{\infty} a_n (n-1)(n-2) x^{n-3}$

$2x^2 \sum_{n=2}^{\infty} a_n (n-1)(n-2) x^{n-3} + 3x \sum_{n=1}^{\infty} a_n (n-1) x^{n-2} + (2x^2-1) \sum_{n=0}^{\infty} a_n x^{n-1} = 0$

$2 \sum_{n=2}^{\infty} a_n (n-1)(n-2) x^{n-1} + 3 \sum_{n=1}^{\infty} a_n (n-1) x^{n-1} + \underbrace{2 \sum_{n=0}^{\infty} a_n x^{n+1}}_{2 \sum_{n=2}^{\infty} a_{n-2} x^{n-1}} - \sum_{n=0}^{\infty} a_n x^{n-1} = 0$

$-(a_0 \cancel{x^{-1}} +$

$x = 0$

$a_1 = 0$

$\sum_{n=2}^{\infty} \left[2a_n (n-1)(n-2) + 3a_n (n-1) + 2a_{n-2} - a_n\right] x^{n-1} - (a_0 x^{-1} + a_1) = 0$

$a_n [2\cancel{n^2} - \cancel{6n} + 4 + 3\cancel{n} - 3 - 1] + a_{n-2} \cdot 2$

$a_n (2n^2 - 3n) + a_{n-2} 2 = 0$

let $a_0 = 1$

$a_n = \frac{2 a_{n-2}}{(3n - 2n^2)}$

$a_1 = 0$ $\qquad a_3, a_5, a_7 \dots = 0$

$a_2 = \frac{2a_0}{-2} = -a_0 = -1$ $\qquad a_4 = \frac{1}{10}$ $\qquad a_6 = -\frac{1}{270}$

$y_1 = \sum_{n=0}^{\infty} a_n x^{n-1} = x^{-1}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\right)$

$= x^{-1}\left(1 + 0 - 1 \cdot x^2 + 0 + \dots\right)$

a_0 and b_0 are free, can be

$y_2: \sum_{n=0} b_n x^{n+\frac{1}{2}}$ $\quad b_0 = 1 \quad b_1 = 0 \quad b_2 = -\frac{7}{14} \quad a_3 = 0 \quad a_4 = \frac{1}{154}$ taken as 1.

$$y_2 = x^{\frac{1}{2}}\left(1 + 0 - \frac{7}{14}x^2 + 0 + \frac{x^4}{154} \cdots\right)$$

$$y = c_1 y_1 + c_2 y_2$$