Gökhan Benli
Office: M227 - Math. Dept.
E-mail: benli@metu.edu.tr
Office Hours: Tuesday 14:40 – 15:30
Thursday 14:30 – 15:20
OR: Arrange office hours by email.

MATH 260 – BASIC LINEAR ALGEBRA

Motivation: Consider the system of linear equations

$$S1 \left\{ \begin{array}{ll} 2x - 2y + z = 6 & (E1) \\ x + y - z = 1 & (E2) \\ 4x + 2y + z = 2 & (E3) \end{array} \right. \xrightarrow[-2E_1+E_3 \to E_3]{-\frac{1}{2}E_1+E_2 \to E_2} \left. \begin{array}{r} 2x - 2y + z = 6 \\ 2y - \frac{3}{2}z = -2 \\ 6y - z = -10 \end{array} \right\} S2$$

Note S1 and S2 have the same solutions!

$$\downarrow -3E_2 + E_3 \to E_3$$

$$z = -8/7, \quad y = -13/7, \quad x = 12/7 \quad \longleftarrow \quad \left. \begin{array}{r} 2x - 2y + z = 6 \\ 2y - 3/2\, z = -2 \\ 7/2\, z = -4 \end{array} \right\} S_3$$

This process of elimination is called "Gaussian Elimination".

Note: What matters in this process are the coefficients of the system. This motivates the use of "matrices".

We could write:

$$S1: \left[\begin{array}{ccc:c} 2 & -2 & 1 & 6 \\ 1 & 1 & -1 & 1 \\ 4 & 2 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc:c} 2 & -2 & 1 & 6 \\ 0 & 2 & -3/2 & -2 \\ 0 & 6 & -1 & -10 \end{array}\right] : S2$$

$$\downarrow$$

This motivates use of matrices:

$$S_3: \left[\begin{array}{ccc:c} 2 & -2 & 1 & 6 \\ 0 & 2 & -3/2 & -2 \\ 0 & 0 & 7/2 & -4 \end{array}\right]$$

MATRICES:

Defn: 1) (Informal) Let m, n be two positive natural numbers. "An $m \times n$ matrix (with real entries)" is an array of real numbers

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}_{m \times n}$$

where $A_{ij} \in \mathbb{R}$ for any $1 \le i \le m$ and $1 \le j \le n$

$\boxed{m \times n \text{ is read as "} m \text{ by } n \text{"}}$

m and n are the "dimensions of A"
m is the number of "rows"
n ______ "columns"

A_{ij} is the (i,j)-th entry of A.

2) (Formal) An $m \times n$ matrix is a function

$$A: \{1, \ldots, m\} \times \{1, \ldots, n\} \longrightarrow \mathbb{R}.$$
$$\left(\{(i,j) \mid 1 \le i \le m,\ 1 \le j \le n\} \right)$$

$$A(i,j) = A_{ij}$$

ex: 1) $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 0 \end{bmatrix}_{2 \times 3}$ $\quad A_{22} = 2$, $A_{23} = 0$ etc...

2) Let B be the 3×2 matrix with $B_{ij} = (-1)^{i+j}(i+2j)$

explicitly: $B = \begin{bmatrix} 3 & -5 \\ -4 & 6 \\ 5 & -7 \end{bmatrix}_{3 \times 2}$ $\quad i = 2,\ j = 2 \quad B_{22} = 6$

3) Let $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$ (The "Kronecker Delta")

Let C be the 2×2 matrix with $C_{ij} = \delta_{ij}$

explicitly: $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Defn: let $A = \begin{bmatrix} A_{11} & \cdots & A_{1j} & \cdots & A_{1n} \\ \vdots & & & & \\ A_{i1} & \cdots & A_{ij} & \cdots & A_{in} \\ \vdots & & & & \vdots \\ A_{m1} & \cdots & A_{mj} & \cdots & A_{mn} \end{bmatrix}_{m \times n}$ be an $m \times n$ matrix

1). For $1 \leq i \leq m$, the $1 \times n$ matrix $[A_{i1}\ A_{i2} \cdots A_{in}]_{1 \times n}$ is the "i-th row" of A

- For $1 \leq j \leq n$, the $m \times 1$ matrix $\begin{bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{mj} \end{bmatrix}$ is the "j-th "column of A

2) The enties $A_{11}, A_{22}, A_{33}, \ldots, A_{kk}, \ldots$ are the "diagonal entries of A".

3) If $m = n$ then A is a "square matrix".

4) The set of all $m \times n$ matrices (with real entries) is denoted by $\mathbb{R}^{m \times n}$.

Note: If A is an $m \times n$ matrix
B ——— $r \times s$ matrix then

$A = B$ means $m = r$, $n = s$ and $A_{ij} = B_{ij}$ for any $1 \leq i \leq m$, $1 \leq j \leq n$.

We can add matrices of the same dimension:

If A is an $m \times n$ matrix
B ——— then

the $m \times n$ matrix $A + B$ is defined as

$$(A+B)_{ij} = A_{ij} + B_{ij} \quad \text{for any} \quad 1 \leq i \leq m,\ 1 \leq j \leq n$$

ex: $\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 12 \end{bmatrix}$

Definition: let O be the matrix whose enties are all zero. This is the "zero matrix".

Prop: let A, B, C be $m \times n$ matrices. then

1) $(A+B) + C = A + (B+C)$
2) $A + B = B + A$
3) $O + A = A$
zero matrix.

Proof: Exercise!

Terminology: let us call real number "scalars" from now on.

We can multiply a matrix by a scalar:

let $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}$ be a scalar.
We define $cA \in \mathbb{R}^{m \times n}$ by $(cA)_{ij} = c A_{ij}$
(usually: scalar $\to$ lower case
matrices $\to$ upper case)

ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$ $\quad 2A = \begin{bmatrix} 2 & 4 & 6 \\ 6 & 8 & 10 \end{bmatrix}$

Prop: let $c, d, e \in \mathbb{R}$ and $A, B \in \mathbb{R}^{m \times n}$, then

1) $c(A+B) = cA + cB$
2) $(c+d)A = cA + dA$ $\quad$ 3) $(cd)A = c(dA)$

Pf: Clear!

Notation: If $A \in \mathbb{R}^{m \times n}$ let us denote $(-1) \cdot A$ by $-A$.
So, $A + (-A) = O \to$ zero matrix.

We can also multiply matrices: same

i.e
# of colms of A
= # of rows of B

Defn: let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times r}$
$\quad$ matrix $\quad AB \in \mathbb{R}^{m \times r}$ as follows:

We can also multiply matrices.

Defn: let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times r}$ (same; n = # of rows of B)

Define an $m \times r$ matrix $AB \in \mathbb{R}^{m \times r}$ as follows:

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj}$$

$$= \sum_{k=1}^{n} A_{ik}B_{kj} \quad \text{where } i \in \{1, \ldots, m\},\ j \in \{1, \ldots, r\}$$

$$\underbrace{\begin{array}{c} \\ i\text{-th row} \rightarrow \\ \\ \end{array}\begin{bmatrix} & & & \\ A_{i1} & A_{i2} & A_{i3} & \cdots & A_{in} \\ & & & \end{bmatrix}_{m \times n}}_{A} \underbrace{\begin{array}{c} j\text{-th column} \\ \begin{bmatrix} & B_{1j} & \\ & B_{2j} & \\ & \vdots & \\ & B_{nj} & \end{bmatrix}_{n \times r}\end{array}}_{B} = \underbrace{\begin{bmatrix} & \vdots & \\ \cdots & \sum_{k=1}^{n} A_{ik}B_{kj} & \\ & & \end{bmatrix}_{m \times r}}_{AB}$$

ex

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 5 & 6 & 0 \end{bmatrix}_{2 \times 3} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}_{3 \times 2}$$

Same

So, we can mult. A by B

$$A \cdot B = \begin{bmatrix} 2\cdot 1 + 4 \cdot 0 + (-1)\cdot(1) & 2\cdot(1) + 4\cdot 1 + (-1)\cdot 0 \\ 5 \cdot 1 + 6 \cdot 0 + 0 \cdot 1 & 5\cdot 1 + 6 \cdot 1 + 0 \cdot 0 \end{bmatrix}_{2 \times 2}$$

$$= \begin{bmatrix} 1 & 6 \\ 5 & 11 \end{bmatrix}$$

ex:

$$A = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}_{2 \times 2} \qquad B = \begin{bmatrix} 3 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}_{3 \times 2}$$

$A \cdot B$ has no meaning since # of cols of $A \neq$ # of rows of B

$$B \cdot A = \begin{bmatrix} 7 & 5 \\ -1 & -3 \\ 1 & -1 \end{bmatrix}_{3 \times 2}$$

ex:

$$A = \begin{bmatrix} 2 & 3 & -2 \\ 1 & 1 & -1 \\ 3 & 5 & 4 \end{bmatrix}_{3 \times 3} \qquad B = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3 \times 1}$$

$$A \cdot B = \begin{bmatrix} 2x + 3y - 2z \\ x + y - z \\ 3x + 5y + 4z \end{bmatrix}_{3 \times 1}$$

BA has no meaning!

Note: For $A \cdot B$ to make sense, we must have Number of columns of A = Number of rows of B.

BIG QUESTION: Why do we define matrix multiplication in this way?

There are several reasons for this.

One reason is coming from representing linear systems:

Consider the linear system:

$$S \begin{cases} 3x + y + 2z - t = 5 \\ 2x + 6y - z + t = -1 \\ x + y + t = 4 \end{cases}$$

4 unknowns and 3 equations

let $A = \begin{bmatrix} 3 & 1 & 2 & -1 \\ 2 & 6 & -1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}_{3 \times 4}$ → the "Coefficient Matrix"

$$\begin{bmatrix} 2 & 6 & -1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}_{3\times 4}$$

$$B = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}_{3\times 1} \qquad X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

Solving the system S is equivalent to solving:

$$\underbrace{\begin{bmatrix} 3 & 1 & 2 & -1 \\ 2 & 6 & -1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}}_{A}{}_{3\times 4} \underbrace{\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}}_{X}{}_{4\times 1} = \underbrace{\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}}_{B}$$

$$\begin{bmatrix} 3x + y + 2z - t \\ 2x + 6y - z + t \\ x + y \quad + t \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$$ has a solution $\iff$ S has a solutions

So, one can represent S by the "matrix equation"

$$A \cdot X = B$$

Properties of Matrix Multiplication:

Proposition: let $A, A_1, A_2 \in \mathbb{R}^{m\times n}$
$B, B_1, B_2 \in \mathbb{R}^{n\times r}$
$C \in \mathbb{R}^{r\times s}$, $d \in \mathbb{R}$

1) $(A\cdot B)\cdot C = A\cdot(B\cdot C)$
2) $A\cdot(B_1+B_2) = A\cdot B_1 + A\cdot B_2$
$(A_1+A_2)\cdot B = A_1\cdot B + A_2\cdot B$
3) $d\,(A\cdot B) = (dA)\cdot(B) = A\cdot(dB)$

Proof: 1) dim. of $(AB)\cdot C$ is $m\times s$
$A\cdot(BC)$ is $m\times s$.

We will show $(AB)\cdot C$ and $A\cdot(BC)$ have the same entries:

$$(\underbrace{(A\cdot B)}_{\text{first}}\cdot \underbrace{C}_{\text{second}})_{ij} = \sum_{k=1}^{r} (A\cdot B)_{ik}\cdot C_{kj}$$

number of columns of the first matrix i.e AB

$$= \sum_{k=1}^{r}\left(\sum_{\ell=1}^{n} A_{i\ell}\cdot B_{\ell k}\right)\cdot C_{kj}$$

# of colms. of the first matrix i.e. A

$$= \sum_{k=1}^{r}\sum_{\ell=1}^{n} \underbrace{(A_{i\ell}\cdot B_{\ell k})\cdot C_{kj}}$$

since $A_{i\ell}, B_{\ell k}, C_{kj} \in \mathbb{R}$

$$= \sum_{k=1}^{r}\sum_{\ell=1}^{n} A_{i\ell}\,(B_{\ell k}\cdot C_{kj})$$

Think!

$$= \sum_{\ell=1}^{n}\sum_{k=1}^{r} A_{i\ell}\,(B_{\ell k}\cdot C_{kj})$$

$$= \sum_{\ell=1}^{n} A_{i\ell}\underbrace{\left(\sum_{k=1}^{r} B_{\ell k}\cdot C_{kj}\right)}$$

$$= \underbrace{\sum_{\ell=1}^{n} A_{i\ell}\,(B\cdot C)_{\ell j}}$$

Thus

$$A\cdot(B\cdot C) = (A\cdot B)\cdot C$$

$$= \underbrace{\sum_{l=1} A_{il} \, (B\cdot C)_{lj}}$$

$$= (A\cdot(B\cdot C))_{ij}$$

2, 3 are exercise!

Remark: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$AB = \begin{bmatrix} 19 & * \\ * & * \end{bmatrix} \neq BA = \begin{bmatrix} 23 & * \\ * & * \end{bmatrix}$$

So even if both AB and BA are defined, we could have $AB \neq BA$.

Definition: Recall $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ (Kronecker Delta)

The $n \times n$ I_n with $(I_n)_{ij} = \delta_{ij}$ is the "$n\times n$ identity matrix".

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ etc...}$$

Prop: let $A \in \mathbb{R}^{m\times n}$ then

1) $A \cdot I_n = A$
2) $I_m \cdot A = A$

ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 0 \end{bmatrix}$

Pf: Clear!

Note: In $\mathbb{R}^{n\times n}$ we can multiply and add any two elements.

In particular, if $A \in \mathbb{R}^{n\times n}$ we can define

$A^0 = I$, $\quad A^1 = A$, $\quad A^2 = A\cdot A$, $A^3 = A\cdot A\cdot A$, etc...

In particular, if A ∈ ...

$$A^0 = I_n, \quad A^1 = A, \quad A^2 = A \cdot A, \quad A^3 = A \cdot A \cdot A, \text{ etc...}$$

ex: $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$ $\quad \left(B = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \right)$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 5 \\ 3 & -3 \end{bmatrix} + \begin{bmatrix} 5 & 7 \\ 7 & 26 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 11 & 23 \end{bmatrix}$$

$$(A+B)^2 = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}^2 = \begin{bmatrix} 10 & 12 \\ 18 & 22 \end{bmatrix}$$

ex: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Rk! We can have $A \neq 0$ $\; B \neq 0$ but $AB = 0$ for matrices!

Definition: let $A \in \mathbb{R}^{m \times n}$. The "transpose of A" is the matrix $A^T \in \mathbb{R}^{n \times m}$ defined as

$$(A^T)_{ij} = A_{ji} \qquad \begin{matrix} i \in \{1, \dots, n\} \\ j \in \{1, \dots, m\} \end{matrix}$$

ex: $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 7 \end{bmatrix}_{3 \times 2}$ $\quad A^T = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & 7 \end{bmatrix}_{2 \times 3}$

Prop: let $A, B \in \mathbb{R}^{m \times n}$ $\quad C \in \mathbb{R}^{n \times r}$ $\quad d \in \mathbb{R}$

1) $(A^T)^T = A$ $\quad$ 2) $(dA)^T = d(A^T)$ $\quad$ 3) $(A+B)^T = A^T + B^T$

4) $(A \cdot C)^T = C^T \cdot A^T$

$$\underbrace{\quad}_{r \times m} \quad \underbrace{\overset{r\times n}{\quad}\ \overset{n \times m}{\quad}}_{r \times m}$$

Proof 1), 2), 3) easy to observe!

4) $$\left((A \cdot C)^T\right)_{ij} = (A \cdot C)_{ji} = \sum_{k=1}^{n} A_{jk} \cdot C_{ki} = \sum_{k=1}^{n} (A^T)_{kj} (C^T)_{ik}$$

$$= \sum_{k=1}^{n} (C^T)_{ik} \cdot (A^T)_{kj}$$

$$= \left(C^T \cdot A^T\right)_{ij}$$

end of proof.

Special types of Matrices:

Definition: A square matrix A is a "diagonal matrix" if $A_{ij} = 0$ for any $i \neq j$

i.e., $$A = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ 0 & A_{22} & 0 & \cdots & 0 \\ 0 & 0 & A_{33} & 0 \cdots & 0 \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & & A_{nn} \end{bmatrix}$$ → The "diagonal" of A.

ex: $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \pi \end{bmatrix}$ are diagonal matrices.

Notation: we write $A = \mathrm{diag}(A_{11}, A_{22}, \ldots, A_{nn})$

$= \mathrm{diag}(2, -1)$ $\qquad = \mathrm{diag}(1, \sqrt{2}, \pi)$

Remark: It is easy to multiply diagonal matrices:

$$\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 35 \end{bmatrix}$$ in general

$$\mathrm{diag}(A_{11}, A_{22}, \ldots, A_{nn}) \cdot \mathrm{diag}(B_{11}, \ldots, B_{nn})$$
$$= \mathrm{diag}(A_{11} \cdot B_{11}, \ldots, A_{nn} \cdot B_{nn})$$

$\mathrm{diag}(A_{11}, A_{22}, \ldots, A_{nn}) \cdot \mathrm{diag}(B_{11}, \ldots, B_{nn})$

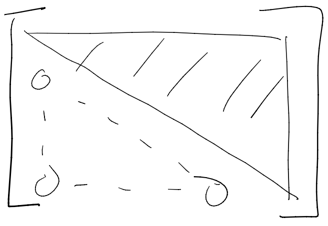
$$= \mathrm{diag}(A_{11} \cdot B_{11}, \ldots, A_{nn} \cdot B_{nn})$$

Also, sum and scalar products of diagonal matrices are diagonal.

<u>Definition</u>: Let A be a diagonal matrix. If all diagonal entries of A are the same then A is a "scalar matrix".

$$A = \begin{bmatrix} d & 0 & \cdots & 0 \\ 0 & d & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & d \end{bmatrix} = \mathrm{diag}(d, d, d, \ldots, d) = dI$$

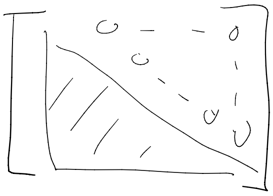
<u>Definition</u>: A square matrix A is "upper triangular" if $A_{ij} = 0$ for $i > j$.

$A =$  $_{n \times n}$

ex $= \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \end{bmatrix}$ are upper triangular.

A square matrix A is "lower triangular" if $A_{ij} = 0$ for $i < j$.

$A =$  $_{n \times n}$

Any diagonal matrix is both upper triangular and lower triangular.