Lecture 7

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Elementary Properties: Given arbitrary scalar c & vector $\vec{v}$ in a vector space V. We have

i) $c.\vec{0} = 0.\vec{v} = \vec{0}$ for all $c \neq 0$ & $\vec{v} \in V$

ii) $(-1)\vec{v} = -\vec{v}$

iii) $(-c)\vec{v} = c(-\vec{v}) = -(c\vec{v})$

Proof: i) $\vec{0} = \vec{0} + \vec{0}$ $\quad c.\vec{0} = c.(\vec{0}+\vec{0}) = c.\vec{0} + c.\vec{0}$

$$\vec{0} = c\underline{\vec{0}} + (-c)\vec{0} = c(\vec{0}+\vec{0}) + (-c\vec{0}) = c\vec{0} + \underbrace{(c\vec{0} + (-c\vec{0}))}_{\vec{0}} \Rightarrow c\vec{0} = \vec{0} \quad //$$

$$0.\vec{v} = (0+0).\vec{v} = 0.\vec{v} + 0.\vec{v} \qquad \vec{0} = 0.\vec{v} + (-0.\vec{v}) = 0.\vec{v} + 0\vec{v} + (-0\vec{v})$$
$$= 0.\vec{v} + \vec{0} \Rightarrow 0.\vec{v} = \vec{0} \quad // \blacksquare$$

Examples of Vector Spaces:

1- $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$

"+" : $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\quad ((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \longrightarrow (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$

"." : $\mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\quad (c, (x_1, \ldots, x_n)) \longmapsto (cx_1, cx_2, \ldots, cx_n)$

Claim: $(\mathbb{R}^n, +, .)$ is a vector space.

exc.

2- Let $\mathcal{M}$ be the set of all $m\times n$ - matrices over $\mathbb{R}$

Claim: $(\mathcal{M}, "+", ".")$ is a vector space.

$A, B \in \mathcal{M}$ $\qquad (A "+" B)_{ij} := (a_{ij} + b_{ij})$

$\qquad c.(A)_{ij} := c.a_{ij}$

3- Let $\mathcal{F} = \{ f \mid f: [a,b] \longrightarrow \mathbb{R}\}$ / over $\mathbb{R}$

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$$(f \text{"+"} g)(x) := f(x) + g(x)$$
$$(cf)(x) := c.f(x)$$

<u>Claim:</u> $(\mathcal{F}, \text{"+"}, \text{"}\cdot\text{"})$ is a vector space.

4- The space of polynomials with real coefficients: $P_\infty[x]$

Given $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$
$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

$$p(x) \text{"+"} q(x) := (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$$
$$c \cdot p(x) := (ca_0) + (ca_1)x + (ca_2)x^2 + \cdots$$

$(P_\infty[x], \text{"+"}, \text{"}\cdot\text{"})$ is a vector space.

$P_3[x] = \{ \text{polynomials of degree} \leq 3 \}$

$P_n[x] = \{ \text{polynomials of degree} \leq n \}$ $\quad n \in \mathbb{Z}_{\geq 0}$

2.2. SUBSPACES:

<u>Defn:</u> Let V be a vector space over $\mathbb{R}$ ($\mathbb{C}$) & W be a non-empty subset of V

i) W is called "closed under addition" if the sum of two vectors in W is also in W

ii) W is called "closed under scalar multiplication" if for any scalar $c \in \mathbb{R}(\mathbb{C})$ $c.\vec{w} \in W$ for any $\vec{w} \in W$.

<u>Defn:</u> A subset W of V is called a "<u>subspace</u>" of V over $\mathbb{R}$ ($\mathbb{C}$) if

1- $W \neq \emptyset$
2- W is closed under addition
3- W is closed under scalar multiplication.

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