12 - constant coefficient systems

$\frac{dx}{dt} = A.t$ when A is a constant matrix

ex: $x_1' = 2x_1 + x_2$
$x_2' = x_1 + 2x_2$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow x' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x$$

$$A = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda_1 = 3$

$\lambda_2 = 1$

$$\lambda_1 = 3 \rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$x_1 = x_2$

$$\lambda_2 = 1 \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$x_1 = -x_2$

solution $\Rightarrow$ $x^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$

$x^{(2)}(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{t}$

independent?

$$\begin{vmatrix} e^{3t} & -e^{t} \\ e^{3t} & e^{t} \end{vmatrix} = 2e^{4t} \neq 0$$

independent

$$x = c_1 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{t} \\ e^{t} \end{bmatrix}$$

$\boxed{\text{complex eigenvalues}}$

$$A = \begin{bmatrix} -4 & 10 \\ -5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} -4-\lambda & 10 \\ -5 & 6-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 26 = (\lambda-1)^2 + 25 = 0 \quad \lambda = 1 \pm 5i$$

euler's identity $= e^{ibt} = \cos(bt) + i\sin(bt)$

$$v \rightarrow \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \longrightarrow x' = e^{(1+5i)t}\begin{bmatrix} 1-i \\ 1 \end{bmatrix} \longrightarrow (\cos 5t + i\sin 5t)\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 5t + i\sin t - i\cos 5t + \sin 5t \\ \cos 5t + i\sin 5t \end{bmatrix} = \begin{bmatrix} \cos 5t + \sin 5t \\ \cos 5t \end{bmatrix} + i\begin{bmatrix} \sin 5t - \cos 5t \\ \sin t \end{bmatrix}$$

$$x = c_1 e^{t}\begin{bmatrix} \; \end{bmatrix} + c_2 e^{t}\begin{bmatrix} \; \end{bmatrix}$$

example $=$ $x' = \begin{pmatrix} -4 & 10 \\ -5 & 6 \end{pmatrix} x$

$\lambda = 1 + 5i \rightarrow \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

$\lambda = 1 - 5i \rightarrow \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -6-5i & 10 & 0 \\ -5 & 5-5i & 0 \end{array}\right] \rightarrow \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(1+5i)t}$$

$$e^{t}\left(\cos 5t + i\sin 5t\right)\begin{bmatrix} 1-i \\ 1 \end{bmatrix} = e^{t}\begin{bmatrix} \cos 5t - i\cos 5t + i\sin 5t + \sin 5t \\ \cos 5t + i\sin 5t \end{bmatrix}$$

$$= c_1 e^{t}\begin{bmatrix} \cos 5t + \sin 5t \\ \cos 5t \end{bmatrix} + c_2 e^{t}\begin{bmatrix} \sin 5t - \cos 5t \\ \sin 5t \end{bmatrix}$$