6 - families of continuous distributions

uniform distribution

generating random number from a given interval → has constant density

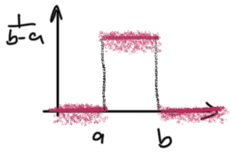
☆ probability is only determined by the length of the interval, not by its location. (independent of t)

interval: $[a,b]$

density: $\frac{1}{b-a}$

expectation: $\frac{a+b}{2}$

variance: $\frac{(b-a)^2}{12}$



standard uniform distribution = when $a=0$, $b=1$ > $[0,1]$ density $=1$ expectation $=\frac{1}{2}$

var $=\frac{1}{12}$

uniform (x) → standard uniform (y)

$X=[a,b]$ $Y=\frac{X-a}{b-a}$

exponential distribution

the waiting time for the next event

to model time → like continuous version of the geometric distribution which counts the number of trials before success

in a sequence of rare events: number of events → poisson

time between events → exponential

$F(x)=1-\frac{1}{e^{\lambda x}}$ (cdf)

density $= f(x)=\frac{\lambda}{e^{\lambda x}}$ (pdf) $\lambda=$ average number of events in a time unit

expectation $=\frac{1}{\lambda}$

variance $=\frac{1}{\lambda^2}$

ex: if it occurs every half a minute

$E(x)=0.5=\frac{1}{\lambda}$ $\lambda=2$

memoryless property = having waited for t minutes does not affect the future waiting time

in continuous distribution → only in exponential

discrete distribution → only in geometric

gamma distribution

the total time of observing α rare and independent events each with exponential waiting times

→ consists of α independent steps

→ each step takes exponential(λ) time

- widely used to model non-integer variables
- amount of time, money

(cdf) $F(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \int_0^t x^{\alpha-1} \cdot e^{-\lambda x} \cdot dx$

density $= f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-\lambda}$, $x>0$

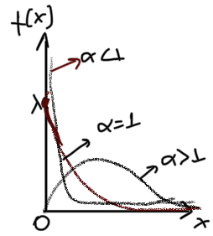
$(\Gamma(\alpha) = (\alpha-1)!)$

expectation $= \frac{\alpha}{\lambda}$

variance $= \frac{\alpha}{\lambda^2}$

* α does not need to be an integer

when $\alpha = 1$ it is exponential distribution.



gamma$(1, \lambda)$ = exponential (λ)

gamma$(\alpha, \frac{1}{2})$ = chi-square(2α)

gamma-poisson formula for easy calculation

!

$\alpha = 3$ events

in 5 minutes ⇒ $\lambda = 0.2$ per

probability of $t < 12$

$P(t<12) = P(\alpha \geq 3) = 1 - F(2) = 0.430$ (0.570)

poisson $\lambda = \lambda \cdot t = 2.4$
$(0.2)\cdot(12)$

λ = average number of events

ex:

$\mu = 12$ $\quad P(8 < T < 10)$ $\quad \frac{\alpha}{\lambda} = 12$ $\quad \lambda = \frac{3}{4}$

$\sigma = 4$ $\quad P(10) - P(8)$ $\quad \frac{\alpha}{\lambda^2} = 4^2$ $\quad \alpha = 9$

$P(t \leq 10) = P(x \geq 9) = 1 - P(8)$, $\lambda_1 = \frac{3}{4} \cdot 10 = 7.5$ → 0.338

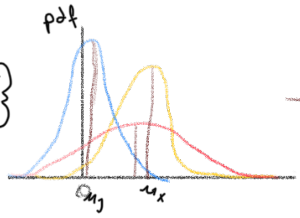
$P(t \leq 8) = P(x \geq 9) = 1 - P(8)$, $\lambda_2 = \frac{3}{4} \cdot 8 = 6$ → 0.153

$0.338 - 0.153 = \underline{0.185}$

normal (gaussian) distribution

to model sums, averages, and errors. also physical variables like weight, height, temp..

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}$



→ know as bell-shaped curve

symmetric

centered at μ → location parameter

its spread being controlled by σ → scale parameter

standard normal distribution(Z) (table is given for this)

normal distribution with standard parameters $\mu = 0$, $\sigma = 1$

$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ → pdf

$Z = \frac{X - \mu}{\sigma}$ ⇒ $X = \mu + \sigma Z$

$\Phi(x) = \int_{-\infty} \frac{1}{\sqrt{2\pi}} e^{\cdot} dz \rightarrow$ cdf

$z < -3.9 \rightarrow \Phi(z) \simeq 0$

$z > 3.9 \rightarrow \Phi(z) \simeq 1$

* $P(z < -k) = P(z > k)$ because of symmetry

$\boxed{P(z < 1.25) = P(z \leq 1.25)}$

how to use table:

$\Phi(a.bc)$ row $\rightarrow$ a.b column $\rightarrow$ 0.0c

example

row column

$\mu = 900$ $\quad P(600 < x < 1200)$ $\Big\}$ $P(1200 > x) = \Phi\left(\frac{1200-900}{200}\right) = P(1.5) \rightarrow$ 1.5 0.00

$\sigma = 200$ $\quad P(600 > x) = \Phi\left(\frac{600-900}{200}\right) = P(-1.5)$ -1.5 -0.00

$= 0.9332 - 0.0668 = 0.8664$

↓

inverse example

$P(X < \text{number}) = 0.03$

$\Phi\left(\frac{\text{number} - 900}{200}\right) = 0.03$ from table $\Phi(-1.88) = 0.33$

$\frac{\text{number} - 900}{200} = -1.88 \rightarrow \text{number} = 900 - 376 = \underline{524}$

$\Phi^{-1}(\alpha) = z_{1-\alpha}$

$\boxed{\text{central limit theorem}}$

sums of random independent variables with same expectation(μ) and standard deviation(σ) from any distribution.

(random variables from same distributio

$$S_n = \sum_{i=1}^{n} X_i = X_1 + X_2 + \dots + X_n$$

$S_n \rightarrow \infty$

$\frac{S_n}{n} \rightarrow \frac{\sigma^2}{n} \rightarrow 0$

as $n \rightarrow \infty$, the standardized sum converges in distribution to a standard normal random variable.

$$Z_n = \frac{S_n - E(S_n)}{Std(S_n)} = \frac{S_n - n\mu}{\sigma\sqrt{n}} \longrightarrow F_{Z_n}(z) = P\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq z\right\} \rightarrow \Phi(z)$$

* it can be applied to any distribution to compute probabilities about S_n, as long as n is large $(n > 30)$

normal approximation to binomial distribution

$n \to$ large
$0.05 \leq p \leq 0.95$ } binomial$(n,p) \approx$ normal$\left(\mu = np \;,\; \sigma = \sqrt{np(1-p)}\right)$

$p < 0.05 \Rightarrow$ poisson

↳ Continuity correction ⇒ is needed bc we approximate a discreate (binomial) distribution by a continuous (normal) distribution (not other way around)

expending the intervals by 0.5: $P\{x-0.5 < \underset{\text{binomial } x}{X} < x+0.5\}$

example

$n=200$
$p=0.2$

$P(x<50)$ } $\mu = 200 \cdot 0.2 = 40$
$\sigma = \sqrt{200 \cdot (0.2) \cdot (0.8)} = 5.6569$

$\to P(x \leq 49) \Rightarrow P(x < 49.5)$

$$P\{X<49.5\} = P\left\{\frac{X-\mu}{\sigma} < \frac{49.5-40}{5.6569}\right\} = P\{Z<1{,}6793\} \Rightarrow \Phi(1{,}6793) = 0.9535$$

9.5

Table A4. Standard Normal distribution

$$\Phi(z) = \boldsymbol{P}\{Z \leq z\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx$$

z	-0.09	-0.08	-0.07	-0.06	-0.05	-0.04	-0.03	-0.02	-0.01	-0.00
-(3.9+)	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
-3.8	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
-3.7	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
-3.6	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0002	.0002
-3.5	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
-3.4	.0002	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003
-3.3	.0003	.0004	.0004	.0004	.0004	.0004	.0004	.0005	.0005	.0005
-3.2	.0005	.0005	.0005	.0006	.0006	.0006	.0006	.0006	.0007	.0007
-3.1	.0007	.0007	.0008	.0008	.0008	.0008	.0009	.0009	.0009	.0010
-3.0	.0010	.0010	.0011	.0011	.0011	.0012	.0012	.0013	.0013	.0013
-2.9	.0014	.0014	.0015	.0015	.0016	.0016	.0017	.0018	.0018	.0019
-2.8	.0019	.0020	.0021	.0021	.0022	.0023	.0023	.0024	.0025	.0026
-2.7	.0026	.0027	.0028	.0029	.0030	.0031	.0032	.0033	.0034	.0035
-2.6	.0036	.0037	.0038	.0039	.0040	.0041	.0043	.0044	.0045	.0047
-2.5	.0048	.0049	.0051	.0052	.0054	.0055	.0057	.0059	.0060	.0062
-2.4	.0064	.0066	.0068	.0069	.0071	.0073	.0075	.0078	.0080	.0082
-2.3	.0084	.0087	.0089	.0091	.0094	.0096	.0099	.0102	.0104	.0107
-2.2	.0110	.0113	.0116	.0119	.0122	.0125	.0129	.0132	.0136	.0139
-2.1	.0143	.0146	.0150	.0154	.0158	.0162	.0166	.0170	.0174	.0179
-2.0	.0183	.0188	.0192	.0197	.0202	.0207	.0212	.0217	.0222	.0228
-1.9	.0233	.0239	.0244	.0250	.0256	.0262	.0268	.0274	.0281	.0287
-1.8	.0294	.0301	.0307	.0314	.0322	.0329	.0336	.0344	.0351	.0359
-1.7	.0367	.0375	.0384	.0392	.0401	.0409	.0418	.0427	.0436	.0446
-1.6	.0455	.0465	.0475	.0485	.0495	.0505	.0516	.0526	.0537	.0548
-1.5	.0559	.0571	.0582	.0594	.0606	.0618	.0630	.0643	.0655	.0668

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.9+	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

-1.4	.0681	.0694	.0708	.0721	.0735	.0749	.0764	.0778	.0793	.0808
-1.3	.0823	.0838	.0853	.0869	.0885	.0901	.0918	.0934	.0951	.0968
-1.2	.0985	.1003	.1020	.1038	.1056	.1075	.1093	.1112	.1131	.1151
-1.1	.1170	.1190	.1210	.1230	.1251	.1271	.1292	.1314	.1335	.1357
-1.0	.1379	.1401	.1423	.1446	.1469	.1492	.1515	.1539	.1562	.1587
-0.9	.1611	.1635	.1660	.1685	.1711	.1736	.1762	.1788	.1814	.1841
-0.8	.1867	.1894	.1922	.1949	.1977	.2005	.2033	.2061	.2090	.2119
-0.7	.2148	.2177	.2206	.2236	.2266	.2296	.2327	.2358	.2389	.2420
-0.6	.2451	.2483	.2514	.2546	.2578	.2611	.2643	.2676	.2709	.2743
-0.5	.2776	.2810	.2843	.2877	.2912	.2946	.2981	.3015	.3050	.3085
-0.4	.3121	.3156	.3192	.3228	.3264	.3300	.3336	.3372	.3409	.3446
-0.3	.3483	.3520	.3557	.3594	.3632	.3669	.3707	.3745	.3783	.3821
-0.2	.3859	.3897	.3936	.3974	.4013	.4052	.4090	.4129	.4168	.4207
-0.1	.4247	.4286	.4325	.4364	.4404	.4443	.4483	.4522	.4562	.4602
-0.0	.4641	.4681	.4721	.4761	.4801	.4840	.4880	.4920	.4960	.5000