18 - method of variation of parameters

for non-homogenous equations with constant coefficients $\rightarrow y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = b(t)$

☆ if $b(t)$ does not have a polynomial annihilator $\rightarrow$ convert the ode into a first order $n\times n$ system.

$$y_1 \cdot v_1' + y_2 v_2' = 0$$
$$y_1' \cdot v_1' + y_2' v_2' = 0$$

$\boxed{2\times 2 \text{ case}}$

$y'' + a_1 y' + a_2 y = 0$ $\qquad$ y_1, y_2 are solutions

$x_1 = y$
$x_2 = y'$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b(t) \end{bmatrix} \rightarrow x' = Ax + b \rightarrow \Psi = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

$$y = x_1 = y_1 \underbrace{\boxed{\int \frac{-y_2\, b(t)}{W(y_1, y_2)} \cdot dt}}_{v_1} + y_2 \underbrace{\boxed{\int \frac{y_1 \cdot b(t) \cdot dt}{W(y_1, y_2)}}}_{v_2}$$

ex:

$$4y'' + y = 2(\sec(t/2)) \rightarrow y'' + \frac{y}{4} = \frac{\sec(t/2)}{2}$$

$y_H = 4\lambda^2 + 1 = 0$
$\lambda = \pm \frac{1}{2} i \rightarrow \{\underbrace{\cos\frac{t}{2}}_{y_1}, \underbrace{\sin\frac{t}{2}}_{y_2}\}$

$$W(y_1, y_2) = \begin{vmatrix} \cos(\frac{t}{2}) & \sin(t/2) \\ \frac{-\sin(\frac{t}{2})}{2} & \frac{\cos(t/2)}{2} \end{vmatrix} = \frac{1}{2}$$

$$y = \cos(t/2) \int \frac{-\sin(t/2) \cdot \frac{\sec(t/2)}{2}}{\frac{1}{2}} dt + \sin(t/2) \int \frac{\cos(t/2) \cdot \frac{\sec(t/2)}{2}}{1/2} dt$$

$$= \cos(t/2)\,(2\ln|\cos(t/2)| + c_1) + \sin(t/2)\,(t + c_2)$$
$$= c_1 \cdot \cos(t/2) + c_2 \cdot \sin(t/2) + 2\ln|\cos(t/2)| + t\sin(t/2)$$

$\boxed{\text{general } n\times n \text{ case}}$

find $v'_{1,2,3\dots}$ from cramers rule. $\frac{D_x}{D}$!

$$y = y_1 \int v_1' + y_2 \int v_2' + \dots + y_n \int v_n'$$

cramer's rule:

Cramer's rule: (solving linear systems with the help of determinant)

$2x - 3y = 9$

$4x + 3y = 9$

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 3 \end{bmatrix} \Rightarrow \det(A) = 18$$

$$\left(\begin{matrix}y'\text{nin} \\ \text{katsayı}\end{matrix}\right) A_1 = \begin{bmatrix} 9 & -3 \\ 9 & 3 \end{bmatrix} \Rightarrow \det(A_1) = 54 \qquad \left(\begin{matrix}x'\text{in} \\ \text{katsayı}\end{matrix}\right) A_2 = \begin{bmatrix} 2 & 9 \\ 4 & 9 \end{bmatrix} \Rightarrow \det(A_2) = -18$$

$$x = \frac{A_1}{A} = \frac{54}{18} = 3 \qquad y = \frac{A_2}{A} = \frac{-18}{18} = -1$$

$$y_1 \cdot v_1' + y_2 v_2' = 0$$
$$y_1' \cdot v_1' + y_2' v_2' = 0$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ b(t) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & b(t) \end{vmatrix}$$

$$v_1' = \frac{W_1}{W} \qquad v_2' = \frac{W_2}{W}$$

→ integrate to find v_1 and v_2

or just solve this equations

3x3 case:

$4x - y + z = 12$

$2x + 2y + 3z = 1$

$5x - 2y + 6z = 22$

$$D = \begin{vmatrix} 4 & -1 & 1 \\ 2 & 2 & 3 \\ 5 & -2 & 6 \end{vmatrix}$$

$$D_x = \begin{vmatrix} 12 & -1 & 1 \\ 1 & 2 & 3 \\ 22 & -2 & 6 \end{vmatrix} \qquad x = \frac{D_x}{D}$$

$$D_y = \begin{vmatrix} 4 & 12 & 1 \\ 2 & 1 & 3 \\ 5 & 22 & 6 \end{vmatrix} \qquad y = \frac{D_y}{D}$$

$$D_z = \begin{vmatrix} 4 & -1 & 12 \\ 2 & 2 & 1 \\ 5 & -2 & 22 \end{vmatrix} \qquad z = \frac{D_z}{D}$$

example: $y''' - y' = t$ (using variation of parameters)

$\lambda^3 - \lambda = 0 \quad \lambda = 0, 1, -1 \quad \{1, e^t, e^{-t}\}$

$$D = \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = 2 \qquad v_1' = \frac{-2t}{2} = -t$$

$$D_x = \begin{vmatrix} 0 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ t & e^t & e^{-t} \end{vmatrix} = -2t \qquad v_2 = \frac{te^{-t}}{2}$$

$$D_y = \begin{vmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & t & e^{-t} \end{vmatrix} = te^{-t} \qquad v_3' = \frac{te^t}{2}$$

$$D_z = \begin{vmatrix} 1 & e^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & t \end{vmatrix} = te^t$$

$$y = 1 \cdot \int -t\, dt + e^t \int \frac{te^{-t}}{2} dt + e^{-t} \int \frac{t}{}$$

$$y = c_1 + c_2 e^t + c_3 e^{-t} - \frac{t^2}{2}$$