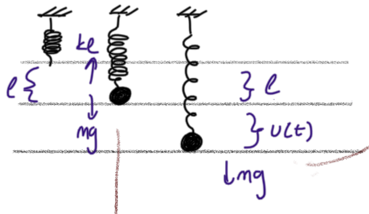
19 - mechanical systems

$\boxed{\text{undamped spring-mass systems}}$

undamped = sönümlenmemiş, gücü azaltılmamış



no movement: $k\ell = mg$
constant

moving

$F = mg - k(\ell + u(t)) = mg - k\ell - ku(t) = -ku(t)$ (with $k\ell = mg$)

$F = m.a = -k.u(t)$

$F = m.u''(t) = -k.u(t)$

velocity $= (u(t))'$
acceleration $= (u(t))''$

$m.u''(t) + k.u(t) = 0$ → second order, homogeneous linear, constant coefficient equation

$my'' + ky = 0$

characteristic equation: $m\lambda^2 + k = 0$ $\quad \lambda = \pm i\sqrt{\frac{k}{m}}$ $\quad w_0 = \sqrt{\frac{k}{m}}$ (natural frequency)

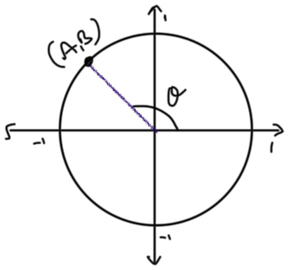
general solution: $y = u(t) = c_1 \cos(w_0 t) + c_2 \sin(w_0 t)$

another formula

let $R = \sqrt{c_1^2 + c_2^2}$

$$u(t) = \frac{R\,(c_1 \cos(w_0 t) + c_2 \sin(w_0 t))}{R} = \sqrt{c_1^2+c_2^2}\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}} \cos w_0 t \right.$$

$$= \sqrt{c_1^2+c_2^2}\left(\underbrace{\frac{c_1}{\sqrt{c_1^2+c_2^2}}}_{A} \cos(w_0 t) + \underbrace{\frac{c_2}{\sqrt{c_1^2+c_2^2}}}_{B} \sin(w_0 t)\right)$$

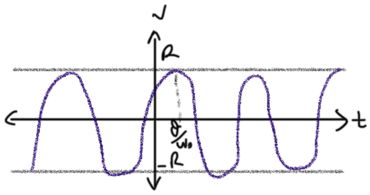


$A = \cos\theta$
$B = \sin\theta$

$u(t) = R\,(\cos\theta . \cos(w_0 t) + \sin\theta . \sin(w_0 t))$

$u(t) = R \cos(w_0 t - \theta)$

R: amplitude
θ: phase
w_0: frequency



$\boxed{\text{effect of dampling}}$

- damping force = like friction force, proportional to its velocity (assume)

$F = m.u''(t) = -k.u(t) - \gamma . u'(t)$ $\qquad m\,u''(t) + \gamma . u'(t) + k\,u(t) = 0$
constants

characteristic equation ⇒ $m\lambda^2 + \gamma . \lambda + k = 0$

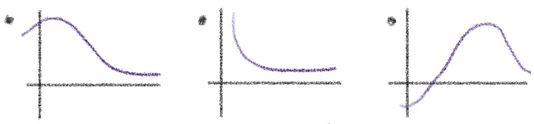
$m, \lambda, k > 0$!

① overdamped case : $\Delta > 0$

$\lambda_1, \lambda_2 < 0$

$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

$\lim_{t \to \infty} u(t) = 0$
(the object stops)



has at most one local max or min

② critical damping : $\Delta = 0$

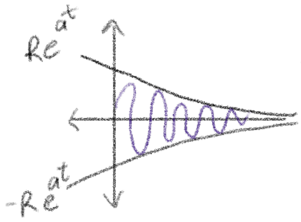
$$u(t) = c_1 . e^{\frac{-\gamma}{2m}t} + c_2 . t . e^{\frac{-\gamma}{2m}t}$$

③ underdamped: $\Delta < 0$ (smal γ)

$\lambda = a \pm bi$

$$u(t) = c_1 e^{at} . \cos(bt) + c_2 . e^{at} . \sin(bt)$$

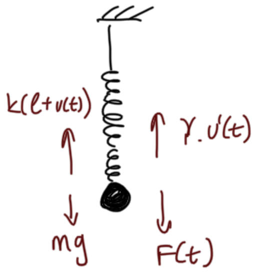
$\hookrightarrow R . e^{at} . \cos(wt - \theta)$



* has infinitely many local max and min
* amplitude goes to 0

$\boxed{\text{mechanical systems with a forcing term - forced vibration}}$

$F(t)$ = positive external force (depends on only time)



$$F(t) = m . u''(t) + \gamma . u'(t) + k . u(t)$$

second order, linear, constant coefficient, non-homogenous equation

$\hookrightarrow$ solve this equation, using variation of parameters

- $x' = A . t + b \qquad x = \Psi(t) . \vartheta(t) \qquad \vartheta(t) = \int \Psi^{-1}(t) . b(t) . dt$
- $y = y_1 \int \frac{-y_2 . b(t)}{W(y_1, y_2)} . dt + y_2 \int \frac{y_1 . b(t)}{W(y_1, y_2)} dt$

$$u(t) = u_1(t) \int \frac{-u_2(t) . F(t)/m}{W} . dt + u_2 \int \frac{u_1(t) . F(t)/m}{W} . dt$$

$\boxed{\text{special case}}$

$F(t) = \cos(wt) \qquad \gamma = 0$

w = external frequency

w_0 = natural frequency $= \sqrt{\frac{k}{m}}$

$$m . u''(t) + k . u(t) = \cos(wt) \quad (w > 0)$$

$$y_{\text{homogenous}} = \lambda = \pm i \underbrace{\sqrt{\frac{k}{m}}}_{w_0} \qquad u(t) = c_1 . \cos(w_0 t) + c_2 \sin(w_0 t) \quad (w_0 > 0)$$

using undetermined coeffients method:

case 1 $\quad w \neq w_0$

$A = \frac{1}{m(w_0^2 - w^2)} \qquad u_p(t) = \frac{1}{m(w_0^2 - w^2)} \cos(wt)$ — particular

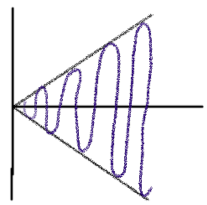
$B = 0$

$$u(t) = c_1 . \cos(w_0 t) + c_2 . \sin(w_0 t) + \frac{1}{m(w_0^2 - w^2)} \cos(wt)$$

* when $w \to w_0$, amplitude goes to ∞.

case 2 $w = w_0 \rightarrow$ resonance

$u_p(t) = A.t.\cos(w_0 t) + Bt \sin(w_0 t)$ — particular



* not periodic
* amplitude $\rightarrow \infty$

$\boxed{\text{example}}$ $u'' + \gamma u' + 4u = 0$ $u(0) = 1$ $u'(0) = 0$

* underdamped, $\gamma = ?$ $\gamma^2 - 16 < 0$ $\underline{0 < \gamma < 4}$
  $\gamma > 0$

* $\gamma = 4$, $u(t) = ?$ $u'' + 4u' + 4u = 0$ $u(t) = c_1 . e^{-2t} + c_2 t e^{-2t}$
  $\lambda^2 + 4\lambda + 4 = 0$ $u'(0) \Rightarrow c_1 = 1, \; c_2 = 2$
  $\lambda = -2$ $\underline{u(t) = e^{-2t} + 2te^{-2t}}$

* $\gamma = 0$, $F(t) = 4\sin(wt) + 4\sin(3wt)$, resonance occurs $w = ?$

$ku'' + \gamma u' + mu = F(t)$

$u'' + 4u = 4\sin(wt) + 4\sin(3wt)$

$W_0 = \text{natural frequency} = \sqrt{\frac{k}{m}} = \sqrt{4} = 2$

* resonance occurs when one of the external frequencies equals to the natural frequency

$w = 2$ $\underline{w = 2 \text{ or } \frac{2}{3}}$

$3w = 2$