# CENG 384 - Signals and Systems for Computer Engineers
# Spring 2022
# Homework 4

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1. (a) $\int_{-\infty}^{t} x(t)\,dt + \frac{\partial x(t)}{\partial t} - \int_{-\infty}^{t} x(t)\,dt - \int_{-\infty}^{t} y(t)\,dt + -2y(t) = y(t)$

   For easiness we can write it as,

   $x'(t) + x''(t) - x(t) - y(t) - 2y'(t) = y''(t)$

   (b) To find the frequency response $H(jw)$ of the system, we will take the Fourier transform of the system, using the differentiation in time property of the Fourier transform.

   $jwX(jw) + (jw)^2 X(jw) - X(jw) - 2jwY(jw) - Y(jw) = (jw)^2 Y(jw)$

   $X(jw)(jw + (jw)^2 - 1) = Y(jw)(2jw + 1 + (jw)^2)$

   We know that the frequency response of a system satisfies the following equation. $Y(jw) = X(jw)H(jw)$
   So,

   $H(jw) = \frac{Y(jw)}{X(jw)} = \frac{(jw)^2 + jw - 1}{(jw+1)^2} = 1 - \frac{jw+2}{(jw+1)^2}$

   $\frac{jw+2}{(jw+1)^2}$ can also be written as,

   $\frac{jw+2}{(jw+1)^2} = \frac{A}{jw+1} + \frac{B(jw)+C}{(jw+1)^2}$

   So, $Ajw + A + Bjw + C = jw + 2$ and $A = 1, B = 0$ and $C = 1$ values satisfy the equation.

   Therefore, $H(jw) = 1 - \frac{1}{jw+1} - \frac{1}{(jw+1)^2}$.

   (c) $H(jw) \leftrightarrow h(t)$
   Using the table of basic continuous time Fourier transform pairs (Table 4.2 in our textbook):

   $h(t) = \delta(t) + (-e^{-t} - te^{-t})u(t)$

   (d) $X(jw) \leftrightarrow x(t)$

   Using the table of basic continuous time Fourier transform pairs (Table 4.2 in our textbook):

   $x(t) = \frac{1}{jw+1}$

   We know that the frequency response of a system satisfies the following equation. $Y(jw) = X(jw)H(jw)$
   So,

   $Y(jw) = \frac{1}{jw+1}(1 - \frac{1}{jw+1} - \frac{1}{(jw+1)^2}) = \frac{1}{jw+1} - \frac{1}{(jw+1)^2} - \frac{1}{(jw+1)^3}$

   Since $Y(jw) \leftrightarrow y(t)$, using Table 4.2 we can find that,
   $y(t) = (e^{-t} - te^{-t} - \frac{t^2}{2}e^{-t})u(t)$

2. (a) The impulse response of a system is the inverse Fourier transform of the frequency response of the system. From the b part of the question the frequency response of the system is $H(jw) = \frac{2sinw}{w}$
From the table of basic Fourier transform pairs, $\frac{2sinwT_1}{w}$ is

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$$

In our frequency response $T_1 = 1$, then the impulse response is,

$$h(t) = \begin{cases} 1 & |t| < 1 \\ 0 & |t| > 1 \end{cases}$$

We can also write this equation in the form of difference of unit step functions as:
$h(t) = u(t+1) - u(t-1)$

(b) To find the frequency response $H(jw)$ of the system, we will take the Fourier transform of the system, using the differentiation in time property and time shifting property of the Fourier transform.
$jwY(jw) = e^{jw}X(jw) - e^{-jw}X(jw)$
We know that the frequency response of a system satisfies the following equation. $Y(jw) = X(jw)H(jw)$
If we substitute this in the previous equation.
$jwX(jw)H(jw) = e^{jw}X(jw) - e^{-jw}X(jw)$
$jwH(jw) = e^{jw} - e^{-jw}$
$H(jw) = \frac{e^{jw} - e^{-jw}}{jw}$
We can further simplify this equation bu substituting $e^{jw} - e^{-jw} = 2jsinw$
$H(jw) = \frac{2jsinw}{jw} = \frac{2sinw}{w}$

3. (a) The overall frequency response of the system is,

$h[n] = h_1[n] * h_2[n]$

Using the basic discrete-time Fourier transform pairs table,

$H_1(e^{jw}) = H_2(e^{jw}) = \frac{1}{1 - \frac{1}{2}e^{-jw}}$

Since $H(e^{jw}) \leftrightarrow h(e^{jw})$, using convolution property of discrete-time Fourier transform we can write that,
$H_1(e^{jw})H_2(e^{jw}) = H(e^{jw})$

So,

$H(e^{jw}) = \frac{1}{(1 - \frac{1}{2}e^{-jw})^2}$

(b) A periodic signal with series representation

$x[n] = \sum_{<N>} a_k e^{jk(2\pi/N)n}$

has a Fourier transform

$X(e^{jw}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N})$

The fundamental period of the signal $x[n]$ is $N = 6$. The signal can be written as

$x[n] = (1/2j)e^{j(\frac{\pi}{3}n + \frac{\pi}{4})} - (1/2j)e^{-j(\frac{\pi}{3}n + \frac{\pi}{4})} = (1/2j)e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{6}n} - (1/2j)e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}n}$

From this we obtain the non-zero Fourier series coefficents $a_k$ of $x[n]$ are

$a_1 = a_{1+N} = a_{1+2N}.. = (1/2j)e^{j\frac{\pi}{4}}$,
$a_{-1} = a_{-1+N} = a_{-1+2N}.. = -(1/2j)e^{-j\frac{\pi}{4}}$

Therefore, we obtain
$X(e^{jw}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{6})$

(c) We know that the frequency response of a system satisfies the following equation. $Y(jw) = X(jw)H(jw)$

So, $Y(e^{jw}) = (2\pi a_1\delta(w-\frac{\pi}{3}) - 2\pi a_{-1}\delta(w+\frac{\pi}{3}))\frac{1}{(1-\frac{1}{2}e^{-jw})^2}$
$Y(e^{jw}) = ((\pi/j)e^{j\frac{\pi}{4}}\delta(w-\frac{\pi}{3}) - (\pi/j)e^{-j\frac{\pi}{4}}\delta(w+\frac{\pi}{3}))\frac{1}{(1-\frac{1}{2}e^{-jw})^2}$

4. (a) Using the table of basic discrete time Fourier transform pairs:
$H(e^{jw}) = 2 + \frac{1}{1-\frac{e^{-jw}}{2}} = \frac{6-2e^{-jw}}{2-e^{-jw}}$

(b) Since $H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$, we can substitute as:
$\frac{Y(e^{jw})}{X(e^{jw})} = \frac{6-2e^{-jw}}{2-e^{-jw}}$
$2Y(e^{jw}) - e^{-jw}Y(e^{jw}) = 6X(e^{jw}) - 2e^{-jw}X(e^{jw})$
By using the time shifting property of the discrete-time Fourier transform and using the basic discrete time Fourier transform pairs table, we can take the inverse Fourier transform of this equation to find the difference equation of this system as:
$2y[n] - y[n-1] = 6x[n] - 2x[n-1]$

(c) We can represent $x[n] = (-1)^n$ as $x[n] = e^{j\pi n}$.
The Fourier transform of $x[n]$ from the table:
$X(e^{jw}) = 2\pi\sum_{l=-\infty}^{\infty}\delta(\omega-\pi-2\pi l)$
Since $Y(e^{jw}) = X(e^{jw})H(e^{jw})$
$Y(e^{jw}) = 2\pi\delta(\omega-\pi)\frac{6-2e^{-jw}}{2-e^{-jw}}$