Middle East Technical University Department of Computer Engineering

CENG 223

Discrete Computational Structures

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Homework 3

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Question 1

$2^{22} + 4^{44} + 6^{66} + 8^{80} + 10^{110}$
$= 2^{2.10+2} + 4^{4.10+4} + 6^{6.10+6} + 8^{8.10} + 10^{11.10}$
$= (2^{10})^2.2^2 + (4^{10})^4.4^4 + (6^{10})^6.6^6 + (8^{10})^8 + (10^{10})^{11} \pmod{11}$
(since 11 is prime and 2, 4, 6, 8 and 10 are integers not divisible by 11, from Fermat's little theorem)
$\equiv (1)^2.2^2 + (1)^4.4^4 + (1)^6.6^6 + (1)^8 + (1)^{11} \pmod{11}$
$\equiv 4 + 256 + 46656 + 1 + 1 \pmod{11}$
$\equiv 4 + 3 + 5 + 1 + 1 \pmod{11}$
$\equiv 14 \pmod{11}$
$\equiv 3 \pmod{11}$

Question 2

Successive uses of the division algorithm give:
$7n + 4 = (5n + 3).1 + (2n + 1)$
$5n + 3 = (2n + 1).2 + (n + 1)$
$2n + 1 = (n + 1).1 + (n)$
$n + 1 = (n).1 + 1$
$n = (1).n$
Hence, $gcd(5n + 3, 7n + 4) = 1$, because 1 is the last nonzero remainder.

Question 3

$m^2 = n^2 + kx$ is given.
$m^2 - n^2 = kx$
$(m - n)(m + n) = kx$
Proof by contradiction:
Let's assume that x does not divide $(m-n)$ and x does not divide $(m+n)$, but x divides $(m-n)(m+n)$ (since $x.k = (m - n)(m + n)$).
So, $gcd(x, (m - n)) = 1$ and $gcd(x, (m + n)) = 1$

If gcd(a, b) = 1, by Bézout's theorem(Theorem is provided in our book (Kenneth H. Rosen, Discrete Mathematics and Its Applications)) there are integers s and t such that $sa + tb = 1$.
Given that we can construct linear combinations(where a,b,c,and d are integers):
$a.x + b.(m - n) = 1$
$c.x + d.(m + n) = 1$
Multiplying the left and the right sides of the two equations above we get:
$a.x.c.x + a.x.d.(m + n) + b.(m - n).c.x + b.(m - n).d.(m + n) = 1$
can be written as:
$x[a.c.x + a.d.(m + n) + b.(m - n).c] + (m - n)(m + n)[(b.d)] = 1$ means x does not divide $(m - n)(m + n)$
This is a contradiction. Therefore, x divides $(m - n)$ or x divides $(m + n)$.

Question 4

Let $P(n)$ be the proposition that the sum of the following positive integers, $1 + 4 + 7 + ... + (3n - 2)$ for all n such that $n \geq 1$ is $\frac{n.(3n-1)}{2}$. We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, ..$ Namely, we must show that $P(1)$ is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3, ...$

Basic Step: $P(1)$ is true, because $1 = \frac{1.(3.1-1)}{2}$. (The left-hand side of this equation is 1, because 1 is the sum of the first integer of the series. The right-hand side is found by substituting 1 for n in $\frac{n.(3n-1)}{2}$.

Inductive Step: For the inductive hypothesis we assume that P (k) holds for an arbitrary positive integer k. That is, we assume that
$1 + 4 + 7 + ... + (3k - 2)$ is $\frac{k.(3k-1)}{2}$
Under this assumption, it must be shown that $P(k + 1)$ is true, namely, that
$1 + 4 + 7 + ... + (3k - 2) + (3(k + 1) - 2) = \frac{(k+1).(3(k+1)-1)}{2}$
$1 + 4 + 7 + ... + (3k - 2) + (3k + 1) = \frac{(k+1).(3k+2)}{2}$
is also true. When we add $(k + 1)$th term to both sides of the equation in $P(k)$, we obtain
$1 + 4 + 7 + ... + (3k - 2) + (3(k + 1) - 2) = \frac{k.(3k-1)}{2} + (3(k + 1) - 2)$
$1 + 4 + 7 + ... + (3k - 2) + (3k + 1) = \frac{k.(3k-1)}{2} + (3k + 1)$
$= \frac{k.(3k-1)}{2} + \frac{2.(3k+1)}{2}$
$= \frac{3k^2-k}{2} + \frac{6k+2}{2}$
$= \frac{3k^2-k+6k+2}{2}$
$= \frac{3k^2+3k+2k+2}{2}$
$= \frac{3k(k+1)+2(k+1)}{2}$
$= \frac{(3k+2)(k+1)}{2}$
This last equation shows that $P(k + 1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.
We have completed the basis step and the inductive step, so by mathematical induction we know that $P(n)$ is true for all n such that $n \geq 1$. That is, we have proven that $1 + 4 + 7 + ... + (3n - 2)$ is $\frac{n.(3n-1)}{2}$ for all n such that $n \geq 1$.