Lecture 6

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2.3. Invertibility:

1. Invertible Matrices:

Defn: A matrix $A_{m\times n}$ is said have a "left inverse" if there is a matrix $B_{n\times m}$ such that $B.A = I_{n\times n}$, similarly said to have a "right inverse" if there is a matrix $C_{n\times m}$ such that

$A.C = I_{m\times m}$.

If there is a matrix D such that for some matrix A we have $A.D = D.A = I_{n\times n}$ then A & D must be square matrices of the same size & they are called "invertible".

RMK: If a square matrix $A_{n\times n}$ is invertible then the inverse is unique.

Proof: Let $D_{n\times n}$ be the inverse of $A_{n\times n}$ so $A.D = D.A = I_{n\times n}$.
Assume there is an other matrix $C_{n\times n}$ s.t. $A.C = C.A = I_{n\times n}$

$$A.C = I \Rightarrow D.(A.C) = D.I \Rightarrow \underbrace{(D.A)}_{I}.C = D \Rightarrow \underbrace{I.C}_{C} = D \Rightarrow C = D \quad \blacksquare$$

Hence inverse is unique.

Notation: If A is invertible then the inverse is denoted by "A^{-1}."

Thm 2.3.1: If $A_1, A_2, --, A_r$ are invertible matrices with inverses $A_1^{-1}, A_2^{-1}, --, A_r^{-1}$, then $A_1.A_2 \dots A_r$ is invertible.

Proof: By induction.

$r=1$ $\quad A_1$ is invertible by assumption.

$r=2$ $\quad A_1, A_2$ is invertible with inverses A_1^{-1}, A_2^{-1}

$$(A_1.A_2).(A_2^{-1}.A_1^{-1}) = A_1.\underbrace{(A_2.A_2^{-1})}_{I}.A_1^{-1} = A_1.I.A_1^{-1} = A_1.A_1^{-1} = I$$

$$(A_2^{-1}A_1^{-1})(A_1A_2) = - - - - - - - - = I$$

Assume $(A_1^{-1} \cdots A_{r-1}^{-1})$ is inverse of $(A_1 \cdots A_{r-1})$ then

$$(A_1 \cdots A_{r-1} \cdot \underbrace{A_r) (A_r^{-1}} \cdot A_{r-1}^{-1} \cdots A_1^{-1}) = (A_1 \cdots A_r) \cdot \underbrace{(A_r \cdot A_r^{-1})}_{I} \cdot (A_{r-1}^{-1} \cdots A_1^{-1})$$

$$= (A_1 \cdots A_{r-1}) \cdot (A_{r-1}^{-1} \cdots A_1^{-1}) = I$$

- - - - - - - exc.

Corollary 2.3.1: Elementary matrices are invertible.

Thm 2.3.2: Let $A_{n \times n}$ T.F.A.E

i) A is product of elementary matrices
ii) A is invertible
iii) A is not row equivalent to a matrix with zero row
iv) A is row equivalent to $I_{n \times n}$.

RMK: ① (ii) $\Rightarrow$ (iv) gives us a method to find the inverse.

$$[A \mid I] \longrightarrow \xrightarrow{\text{elem. row op.}} [I \mid P] \Rightarrow A \text{ is row eq. to } I$$

i.e. there a matrix P s.t. $I = PA \Rightarrow P = A^{-1}$.

② In general if $A \sim B$ then $B = P.A$ where P is invertible.

③ We can deduce that every matrix is row equivalent to a "unique" row reduced echelon matrix.

exc.

③' If $A \sim B$ then $A \sim R$ & $B \sim R$ where R is the unique row reduced echelon matrix.

2. Invertibility & Systems of Linear Eqns.

Thm 2.3.3: For a square matrix $A_{n \times n}$ T.F.A.E.
(i) A has a left inverse
(ii) The homogeneous system $AX = O$ has only trivial solution
(iii) A is invertible

(ii) The homogeneous system $AX=0$ has only trivial solution
(iii) A is invertible
(iv) A has a right inverse

Sketch of proof:

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (i)$$

(i) $\Rightarrow$ (ii): Assume B is the left inverse i.e. $B.A=I$

$$AX=0 \Rightarrow B(AX)=B.0 \Rightarrow \underbrace{(BA)}_{I}.X=0 \Rightarrow X=0$$

(ii) $\Rightarrow$ (iii): $X=0$ is the only soln. of $AX=0$ i.e. $[A\,|\,0] \to \cdots$ elementary row op. $\to [R\,|\,0]$ where R row reduced echelon, $RX=0$

$x=0$ is the only soln. of $RX=0$ then R can not have a zero $R_{n\times n}$ then $R=I$ so $A \sim I \Rightarrow A$ is invertible.

exc.

Corollary 2.3.3: $A_{n\times n}$ T.F.A.E

(i) A is invertible
(ii) Every system $AX=B$ has a unique soln.
(iii) Every system $AX=B$ has at most one soln.
(iv) There is a system $AX=B$ which has a unique soln.

Sketch of proof: (i) $\Rightarrow$ (ii) A is invertible means A^{-1} exists &

$$A^{-1}A=I \quad \text{so} \quad AX=B \Rightarrow A^{-1}(AX)=A^{-1}B \Rightarrow$$
$$\underbrace{(A^{-1}A)}_{I}X=A^{-1}B \Rightarrow X=A^{-1}B.$$

Exp: Let $A=\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix}$ a) Find A^{-1} if exists
b) Solve the system $AX=\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$

Soln:

a) $$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-\frac{3}{2}R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & -\frac{3}{2} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3}\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 1 \end{array}\right] \underset{-4R_3+R_2}{\xrightarrow{2R_3+R_1}} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & 1 & 5 & -4 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 1 \end{array}\right]$$

$$\underset{-2R_3}{\xrightarrow{\frac{1}{2}R_1}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 5 & -4 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right]$$

with $\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}$ and $\underbrace{\begin{matrix} 0 & -1 & 1 \\ 1 & 5 & -4 \\ 1 & 2 & -2 \end{matrix}}_{= A^{-1}}$

$\therefore$ A is invertible

b) $AX = \underbrace{\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}}_{B}$ since A is invertible $X = A^{-1}\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 5 & -4 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 27 \\ 13 \end{bmatrix}$

- CHAPTER 3 - VECTOR SPACES :

Defn: Let $\mathbb{R}$ ($\mathbb{C}$) be our scalar field, and V be a non-empty set.

If there is an operation $+ : V \times V \longrightarrow V$ called "addition" and a function $\cdot : \mathbb{R} \times V \xrightarrow{(v_1, v_2)} V$ called "scalar multiplication" such that the followings are satisfied

- V1 - $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$; $v_1, v_2, v_3 \in V$
- V2 - there is $0 \in V$ s.t. $v + 0 = 0 + v = v$; for all $v \in V$.
- V3 - for all $v \in V$ there is $v' \in V$ s.t. $v + v' = v' + v = 0$.
- V4 - $v_1 + v_2 = v_2 + v_1$ for all $v_1, v_2 \in V$.
- V5 - $c(v_1 + v_2) = cv_1 + cv_2$ for all $c \in \mathbb{R}$, $v_1, v_2 \in V$.
- V6 - $(c_1 + c_2)v = c_1 v + c_2 v$ for all $c_1, c_2 \in \mathbb{R}$, $v \in V$.
- V7 - $(c_1 . c_2) . v = c_1 \cdot (c_2 v)$ for $c_1, c_2 \in \mathbb{R}$, $v \in V$
- V8 - $1 . v = v$ for all $v \in V$.

is called a "vector space". $(V, +, .)$

Defn: Elements of a vector space are called "vectors".

exps: 1. $(\mathbb{R}, +, \cdot)$ is a vector space over $\mathbb{R}$

2. $(\mathbb{R}^2, \text{"}+\text{"}, \cdot)$ is a vector space over $\mathbb{R}$

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ (x_1, x_2) \mid x_1, x_2 \in \mathbb{R} \}$$

$$\text{"}+\text{"} : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\big((x_1, x_2), (y_1, y_2)\big) \longmapsto (x_1 + y_1, x_2 + y_2)$$

$$\cdot : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(c, (x_1, x_2)) \longmapsto (cx_1, cx_2)$$