17 - method of undetermined coefficients

for non-homogeneous equations with constant coefficients

↳ these kind of equations can be solved with variation of parameter method, but it is easier with undetermined coeffients (only special forms)

$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_n y = b(t) \rightarrow y(t) = \underbrace{y_h(t)}_{b(t)=0} + \underbrace{y_p(t)}_{\text{guess}}$

ex: $y'' - y = 6e^{3t} \rightarrow \boxed{y_h(t)} = \lambda^2 - 1 = 0 \quad = \boxed{c_1 e^{t} + c_2 e^{-t}}$
$\lambda = 1, -1$

guess: Ae^{3t} $\quad (Ae^{3t})'' - Ae^{3t} = 6e^{3t} \rightarrow (9A - A)e^{3t} = 6e^{3t} \quad A = \frac{3}{4}$

$\boxed{y_p(t)} = \boxed{\frac{3}{4}e^{3t}}$ $\quad y(t) = y_h(t) + y_p(t)$
$= c_1 e^{t} + c_2 e^{-t} + \frac{3}{4}e^{3t}$

ex: $y'' + 3y' + 2y = \cos t \rightarrow y_h(t) = \lambda^2 + 3\lambda + 2 = 0 \quad = c_1 e^{-2t} + c_2 e^{-t}$
$\lambda = -2, -1$

guess = $A\cos t + B\sin t \rightarrow (A+3B)\cos t + (B-3A)\sin t = \cos t$

$\left.\begin{matrix} A+3B=1 \\ B-3A=0 \end{matrix}\right\rbrace \quad 10B = 3 \quad B = \frac{3}{10} \quad A = \frac{1}{10}$

$y(t) = c_1 e^{-2t} + c_2 e^{-t} + \frac{\cos t}{10} + \frac{3\sin t}{10}$

ex: $y'' - y = e^{t} \rightarrow y_h(t) = \lambda^2 - 1 = 0 \quad = c_1 e^{t} + c_2 e^{-t}$
$\lambda = 1, -1$

guess = can't be $\boxed{Ae^{t}} \rightarrow Ate^{t}$

$(Ate^{t})'' - Ate^{t} = e^{t}$
$2Ae^{t} = e^{t} \quad A = \frac{1}{2}$
$y(t) = c_1 e^{t} + c_2 e^{-t} + \frac{t}{2}e^{t}$

$\boxed{\text{annihilators (how to guess } y_p)}$

$D = \frac{d}{dt} \rightarrow$ differentiation operator $\quad Dy \rightarrow y'$, $D^2 y \rightarrow y''$ $\quad (D^3 + 4D - 7)y = y''' + 4y' - 7y$

$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_n y = b(t) \rightarrow (D^{(n)} + a_1 D^{(n-1)} + a_2 D^{(n-2)} + \dots + a_n)y = b(t)$

if $\lambda \rightarrow D \rightarrow$ char. equation $\rightarrow (D-\lambda_1)(D-\lambda_2)\dots(D-\lambda_n).y = b$

$L(D).b(t) = 0$

function $(b(t))$		$\rightarrow$ annihilator	
$e^{\lambda t}$	e^{3t}	$D-\lambda$	$D-3$
1	e^{0t}	D	$D-0$
$te^{\lambda t}$	te^{3t}	$(D-\lambda)^2$	$(D-3)^2$
t	te^{0t}	D^2	$(D-0)^2$

Steps: $L(D).y = b(t)$

① annihilator of $b(t) = M(D)$

② $M(D).L(D).y = \underbrace{b(t).M(D)}_{0} = 0$

③ solve $M(D)\,L(D)y=0$ for y_H

④ eliminate dublicates (main y_H and ③ y_H)

$\cos(at)$ — $\cos(3t)$ — (D^2+a^2) — (D^2+3)

$e^{(a+bi)t}$ — $D-(a+bi)$

$e^{at}\sin bt \,/\, e^{at}\cos bt$ (same for both) — $D^2-2aD+(a^2+b^2)$

why?

$e^{\lambda t}(D-\lambda) \to (e^{\lambda t})' - \lambda e^{\lambda t} = 0$ ✓

$(\cos(at)).(D^2+a^2) = (\cos(at))'' + a^2\cos(at)$
$= -a^2\cos(at) + a^2(\cos(at)) = 0$

annihilators of combinations of them:

$$f(t) = e^{-2t} + t^2 e^{-2t} + \cos(\sqrt{3}t) + t$$

$$(D+2) . (D+2)^2 . (3+D^2) . D^2 \to M(D) = (D+2)^3 (D^2+3) . D^2$$

ex: $y'' - y = e^{2t}$

$y_H = \lambda^2 - 1 = 0 \quad \lambda = 1, -1 \quad \to \{e^{t}, e^{-t}\}$

$e^{2t} \to (D-2)$

$\boxed{(D-2)\,\boxed{(D^2-1)\,y}} = e^{2t}(D-2) \boxed{=0} \qquad y = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t}$

$D = 2, 1, -1$ (put it to original eq)

$\boxed{(D^2-1)(\cancel{c_1 e^{t}} + \cancel{c_2 e^{-t}}} + c_3 e^{2t}) = e^{2t}$

$4c_3 e^{2t} - c_3 e^{2t} = e^{2t} \qquad 3c_3 e^{2t} = e^{2t} \qquad c_3 = \frac{1}{3}$

$\hookrightarrow y = c_1 e^{t} + c_2 e^{-t} + \frac{e^{2t}}{3}$

ex: $y''' + 4y' = t$

$y_H \to \lambda^3 + 4\lambda = 0 \quad \lambda = 0, -2i, +2i \; \{e^{-2it}, e^{+2it}, 1\}$

$D^2 . (D^3 + 4D) y = t \quad D^2 = 0 \quad \to D^3 (D^2+4) . y = 0$

$D = 0^{\times 3}, -2i, +2i$

$\hookrightarrow \cancel{\cos(2t)}, \cancel{\sin(2t)}, \cancel{e^{0t}}, te^{0t}, t^2 e^{0t}$

$\to t, t^2$

$e^{(a+bi)t} \to e^{a}(\cos(bt) + i\sin(bt))$

$e^{2i} \to e^{0}(\cos(2t) + i\sin(2t))$
$= \cos 2t + i\sin 2t$
$= c_1\cos 2t, c_2 \sin 2t$

$y_H = \{\cos 2t, \sin 2t, 1\}$

$(D^3 + 4D)(c_4 t + c_5 t^2) = t$!

$4c_4 + 4.2c_5 t = t$

$c_4 = 0 \quad c_5 = \frac{1}{8}$

$y = c_1(\cos 2t) + c_2(\sin 2t) + c_3 + \frac{t^2}{8}$

$\hookrightarrow$ with guess: $\lambda^3 + 4\lambda = 0 \qquad y_H = \sin 2t, \cos 2t, 1$

$b(t) = At$

$y''' + 4y' = t \to 0 + 4A = t$ ✗

$b(t) = At^2$

$0 + 8At = t \to A = \frac{1}{8}$

not safe
does not
give all the
solutions