① separable equations = $y' = \frac{dy}{dx}$ $\int (x \text{ terms})\, dx = \int (y \text{ terms})\, dy$ (non-homo)

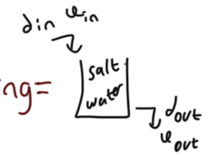
② homogeneous equations = $v = \frac{y}{x}$ $y = vx$ $y' = v'x + v$ → becomes separable

③ integrating factor on linear equations = $y' + p(t)y = q(t)$ → multiply both sides

$\mu = e^{\int p(t)dt}$ $\int (y \cdot \mu)' = \int q(t) \cdot \mu$

if $y' + ay = b$ (a, b const., $a > 0$) $\lim_{t \to \infty} y = \frac{b}{a}$

④ modelling =



$Q(t) = $ salt

$Q'(t) = Q_{in} - Q_{out} = d_{in} v_{in} - d_{out} v_{out}$

⑤ linear $= y' + p(t)y = q(t)$ $f(t,y) = q(t) - p(t)y$ → superposition

nonlinear $= y^2, yy', e^y, \sin y$ $f(t,y) = y'$ → finite time blow-up → existence and uniquenes for initial values

undefined point $\frac{1}{0}$

⑤ existence and uniqueness for initial values →

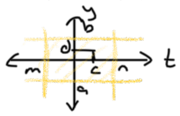
linear = leave y' alone → $\frac{q(t)}{p(t)}$ $\frac{1}{0}$ points → find like $(-\infty, a)$ (a,b) (b, ∞)

$y_0(m) = n$, choose the interval which m in it

nonlinear = leave y' alone, ∂y derivative respect to y right side, find for both $\neq \frac{1}{0}$ intervals, where blow up point inside (with solving eq)

if $y_0(t_0)$ at undefined point → there are more than one solution

↳ ex: $y \neq a, b$
$t \neq m, n$
$y(c) = d$



→ $\underset{t}{(m,n)} \times \underset{y}{(a,}$

• if $y = \frac{1}{\sqrt{x-a}\,(y-b)}$ → has unique solution $x \neq a$, $y \neq b$

→ no solution $x \neq a$ $y(x) = $ anything

→ more than one solution $y(t) = b$ anything ($t \neq a$)

⑥ exact equations = for $M dx + N dy = 0$, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact sol: $\int M dx + c(y) = \int N dy + c(x)$

⑥ integrating factor = if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ → $\frac{M_y - N_x}{N}$ = only x → $e^{\int a \cdot dx}$ (a)

or

$\frac{N_x - M_y}{M}$ = only y → $e^{\int b dy}$ (b) → multiply eq. becomes exact

☆ any separable equation is also exact

⑦ systems of first order equations = n first order ODE, n variables

$x_1' = a_1(t)\, x_1 + a_2(t)\, x_2 + b_1(t)$
$x_2' = a_3(t)\, x_1 + a_4(t)\, x_2 + b_2(t)$ } linear system

if $b_1(t) = b_2(t) = 0$
↳ homogenous system

⑨ eigen values & vectors = $(A - \lambda I)\, v = 0$ $\det(A - \lambda I) = 0$

* $Ax = 0$ has non-trivial solution if A is non-invertible ⇔ $\det(A) = 0$

⑩ linear independence of funcs = f_1, f_2, f_3 → ① give t values, 3x3 matrix, no 000 row
→ ② $\begin{vmatrix} f_1' & \cdots \end{vmatrix} \neq 0$

11) constant coefficient systems = → real, two roots ⇒ $x = c_1 e^{\lambda_1 t}[v_1] + c_2 e^{\lambda_2 t}[v_2]$

12) → complex ⇒ $e^{a+bi}[v] \to e^{a}(\cos bt + i \sin t)[v] = c_1 e^{at}[\ ] + c_2 e^{at}[$ real terms img

* periodic solutions → complex eigenvalues

13) fundamental matrices = $\frac{d\Psi}{dt} = A\Psi$ and Ψ is invertible. → if A is constant $\Psi(t) = \Phi(t) = e^{At}$, $\Phi(0) = I$

$e^{At} = P e^{Dt} P^{-1} = [v_1\ v_2] \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} P^{-1}$ (P^{-1} → constant no need)

13) Jordan form = $\begin{bmatrix} 1&0&1 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$ → 2 free - 2 block $\begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1&t&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}$ 2 block

$\begin{bmatrix} 1&0&1 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$ → 1 free - 1 block

$e^{At} = P . e^{Jt}$ $e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$ ex

13) repeated eigen values = $\lambda \times 2$ v_1 → put instead $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ end → v_2 $x = c_1 e^{\lambda t}[v_1] + c_2 e^{\lambda t}\left([v_1].t + [v_2]\right)$

14) non-homogeneous system = $x' = At + b$ $x = \Psi(t).\boxed{v(t)}$ → $\int \Psi^{-1}(t).b(t).dt$

- variation of parameters - * when $\lambda_1 \neq \lambda_2$, real and negative → $\begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix} = -A^{-1}.b$ → $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

15) converting higher order ode's into a first order system = $y^{(n)} + a_1(t)\,y^{(n-1)} + \dots + a_n(t).y = b(t)$

$x_1 = y$
$x_2 = y'$
$\vdots$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & 0 & 0 & 1 \\ -a_n & -a_{n-1} & \dots & & -a_1(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b(t) \end{bmatrix}$$

16) homogenous equations with constant coefficients (higher order) = $y^{(n)} + a_1 y^{(n-1)} + \dots + a_n.y = 0$

char eq. → $P(\lambda) = 0$ → $y^{(n)} = \lambda^n$, $y' = \lambda$, $y = 1$ (put λ instead of y

* distinct eigenvalues → $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \dots$

* repeated → $y = e^{\lambda t}[c_1 + c_2 t + c_3 t^2 + \dots + c_n t^{n-1}]$

17) method of undetermined coefficients = non-homo / constant coef.

find $y_{Homo}(t)$,

$M(D).L(D).y = 0$ → solve → eliminate

$y'' - y$ → $(D^2 - 1)\,y.(\text{eliminated}) = b(t)$ y_P

$b(t)$	annihilator $M(D)$
$e^{\lambda t}$	$D - \lambda$
1	D
$te^{\lambda t}$	$(D-\lambda)^2$
t	D^2
$\cos(at)$	$D^2 + a^2$
$e^{(a+bi)t}$	$D - (a+bi)t$

18) method of variation of parameters = non-homo / constant coef → if no annihilator

$y_1 . v_1' + y_2 v_2' = 0$

$y_1' v_1' + y_2' v_2' = 0$

$y = y_1 \int \frac{-y_2 . b(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 . b(t)}{W(y_1, y_2)} dt$

solution of y_H v_1 v_2

$n \geq 3$ $v_1' = \frac{D_x}{D}$ → their det, $b(t)$ in x's column, D: Ψ, put as → $\int v_1'$

19) mechanical systems = $F(t) = m.u''(t) + \gamma u'(t) + k u(t)$ $m, \gamma, k > 0$

$F(t) = \gamma = 0$ $u(t) = c_1 \cos(w_0 t) + c_2(\sin w_0 t)$ $\boxed{R = \sqrt{c_1^2 + c_2^2}}$ = amplitude

$f(t) = 0$ $\Delta > 0$ overdamped; $\Delta = 0$ critical damping; $\lim_{t \to \infty} u(t) = 0$ stops, $\Delta < 0$ underdamped

$F(t)$ → $\sin w_1 t + \cos w_2$ w_0 = natural frequency = $\sqrt{\frac{k}{m}}$

resonance $w = w_1$ or $w = w_2$

(21) Series solutions near an ordinary point = $y = \sum_{n=0}^{\infty} a_n x^n$ if x_0 is ordinary point,
$\rightarrow$ around $x = x_0$

$\longrightarrow a_0 y_1 + a_2 y_2$

$y' = \cdots$

$y'' = \cdots$

- first make equal the exponents of x
- then the starting indicies

(22) regular singular point = $x = x_0$ singular point

① $\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} = \alpha$ ✓ $P(x)y'' + Q(x)y' + R(x$

② $\lim_{x \to x_0} (x - x_0)^2 \cdot \frac{R(x)}{P(x)} = \beta$ ✓ $\Rightarrow$ otherwise irregular

(23) euler equations = a, b constants $\rightarrow x^2 y'' + a x y' + b y = 0$ $\quad y(x) = x^r \quad y'' + (\alpha - 1) y' + \beta y = 0$

$x = x_0$ regular sing $(x=0)$ no solution at $x = 0$ $\quad$ $\hookrightarrow$ indicial equation

$\Delta > 0 \quad y_H = c_1 x^{r_1} + c_2 x^{r_2}$ $\quad$ if $x_0 \neq 0 \rightarrow (x - x_0)^{r_1} \ldots$, if $x > x_0 \rightarrow |x - x_0|$

$\Delta = 0 \quad y_H = c_1 x^{r_1} + c_2 x^{r_1} \ln x$

$\Delta < 0 \quad y_H = c_1 x^a \cos(b \ln x) + c_2 x^a \sin(b \ln x)$

(24) series solution near a regular singular point = $x = x_0 \rightarrow$ ordinary point : 2 sol.
singular = no sol (series)
regular singular p. = if $r_1 - r_2$ integer $\rightarrow$ ①
not integer $\rightarrow$ 2 sol

① find α and β

② solve $r^2 + (\alpha - 1) r + \beta = 0$

③ $y_1 = \sum_{n=0}^{\infty} a_n x^{n + r_1} \quad y_2 = \sum_{n=0}^{\infty} b_n x^{n + r_2}$ $\quad$ (if a_0, a_1 are free, give random) $\quad y = c_1 y_1 + c_2 y_2$

$\rightarrow$ continuous func but not analytic (no derivative)

$y = \sum_{n=0}^{\infty} a_n (x-4)^n$ of $17y'' - 6xy = 0$, $y(4) = 0$, $y'(4) = 136$

$a_n = \frac{y^{(n)}(4)}{n!}$! 0

$a_0 = \frac{y(4)}{0!} = 0$

$a_1 = \frac{y'(4)}{1!} = 136$

$a_2 = \frac{y''(4)}{2} = 0$

$a_3 = \frac{y'''(4)}{3!} = \frac{192}{6} = 32$

$\vdots$

$17 y''(x) = 6x\, y(x)$

$x = 0 \quad 17 y''(0) = 0$

$17 y'''(x) = 6 y(x) + 6x\, y'(x)$

$17 y'''(4) = 6 y(4) + 6 \cdot 4 \cdot y'(4)$

$y'''(4) = 192$

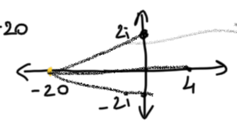
$\vdots$

$(x-4)(x^2+4) y' - y = 0$ $\quad y = \sum_{n=0}^{\infty} a_n (x+20)^n$ $\quad$ centered at $x_0 = -20$

radius of converge?

$y'' - \frac{y}{(x-4)(x^2+4)} = 0$ $\quad x_0 = -20$

$x = 4, \pm 2i$



choose smallest

$\sqrt{(20)^2 + 2^2} = \sqrt{404}$

$X(t) = \Psi(t) \int \left(\Psi(t)^{-1} \cdot b(t) \right) dt$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$(k \cdot A)^{-1} = \frac{1}{k} \cdot A^{-1}$

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annihilator of = $t e^{3t} + \cos(7t)$

$(D-3)^2 \cdot (D^2+49)$ → product

$\Phi(t) = e^{At}$ → $t \to 0$ → $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Psi(t) = [y_1 \quad y_2]$

linear combination of fundamental matrix

$\Phi(t) = [c_1 y_1 + c_2 y_2 \quad c_3 y_1 + c_4 y_2]$

$t=0 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad t=0 \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

in euler equation:

$\Delta < 0 \to y(x) = c_1 e^{a+bi} + c_2 e^{a-bi}$

$y(x) = c_3 \cdot x^a \cdot \cos(b \cdot \ln x) + c_4 x^a \cdot \sin(b \cdot \ln x)$

to use Variation of parameters y'' coefficient must be 1.

in spring-mass system if you recalculate with $F(t)$ old c_1 and c_2 change too, recalculate them too

$mg = kl$ $\quad \downarrow +$ way

max displacement → $R = \sqrt{c_1^2 + c_2^2}$

the radius of converge of a series solution is the distance to the nearest singular point → at least !

$r \geq k$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \to y'(x) = \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right) a_n x^{n-\frac{1}{2}}$$