

MATH 219

Fall 2020

Lecture 5

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Content: Exact equations and integrating factors (section 2.6).

Suggested Problems: (Boyce, Di Prima, 9th edition)

§2.6: 3, 5, 9, 10, 13, 18, 21, 24, 30, 32

1 Exact equations

Let us consider the case of an arbitrary first order ODE once again. Say that our independent variable is x rather than t . Suppose for a moment that we found the solutions of the equation and that they can be written in the implicit form

$$F(x, y) = c.$$

by leaving the constant c alone. Of course, when c changes, the solution curve will change. We can easily write dy/dx in terms of x : Take the derivative of both sides with respect to x . By the chain rule,

$$\begin{aligned}\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{\partial F/\partial x}{\partial F/\partial y}\end{aligned}$$

The question is whether we can reverse this process. Namely, given the ODE, can we recover such a function $F(x, y)$? One important remark at this point is that, even if we could, given the ODE we just know the *ratio* of $\partial F/\partial x$ to $\partial F/\partial y$, but F and the values of the partial derivatives themselves are not uniquely determined at all. In some favourable cases, the functions appearing in the particular way we write the ODE will a priori be equal to the derivatives of a certain function F . To better understand this, let us write the ODE in a more symmetric form:

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0.$$

An equivalent way of writing this equation is

$$M(x, y)dx + N(x, y)dy = 0.$$

Derivatives with respect to x are suppressed in this notation. Instead of $\frac{d}{dx}$ we write d etc.¹ With this notation, we have

$$d(F(x, y)) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy.$$

Definition 1.1 *The equation $M(x, y)dx + N(x, y)dy = 0$ is called **exact** on a domain D if there exists a differentiable function $F(x, y)$ on D such that the left hand side of the equation can be written as $d(F(x, y))$ on D . A function $F(x, y)$ satisfying this condition is called a **potential** for this equation.*

By the chain rule, the condition on $F(x, y)$ is equivalent to the pair of equations

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N$$

We emphasize that the potential $F(x, y)$ must be defined as a single valued function on the whole domain D .

Example 1.1 *The equation $ydx + (x + 2y)dy = 0$ is exact on \mathbb{R}^2 . Indeed, if $F(x, y) = xy + y^2$ then*

$$\begin{aligned} d(F(x, y)) &= d(xy + y^2) \\ &= ydx + (x + 2y)dy \end{aligned}$$

Remark 1.1 Recall from multivariable calculus that a vector field $\langle M(x, y), N(x, y) \rangle$ is called conservative if it can be written in the form ∇F for some function $F(x, y)$. It is clear that the equation $Mdx + Ndy = 0$ is exact if and only if the vector field $\langle M, N \rangle$ is conservative.

¹From a more advanced perspective, this is an equality of differential 1-forms. We will not pursue this viewpoint here

If the equation $Mdx + Ndy = 0$ is exact with potential F , then it can be rewritten as $dF = 0$. Consequently, the equations $F(x, y) = c$ for arbitrary values of c give us all solutions of the ODE in an implicit form.

Example 1.2 Solve the initial value problem $ydx + (x + 2y)dy = 0$, $y(1) = 5$.

Solution: We saw above that the equation is exact with $F(x, y) = xy + y^2$ a potential. Therefore the solutions of the equation are $xy + y^2 = c$. Using the initial condition, we find that $c = 1 \times 5 + 5^2 = 30$. Hence the solution is $xy + y^2 = 30$ (in implicit form).

Suppose that $M(x, y)$ and $N(x, y)$ are themselves continuously differentiable on a common domain D . As in the case of conservative vector fields, a necessary condition for the existence of a potential function is

$$\frac{\partial M}{\partial y} = \frac{\partial F}{\partial x \partial y} = \frac{\partial F}{\partial y \partial x} = \frac{\partial N}{\partial x}$$

This condition is not always sufficient for the existence of a potential F . However, if the domain is *simply connected*, then it is. A simply connected domain, intuitively, is a domain with no interior holes. An example of a simply connected domain is a rectangle. We will formulate and use the result in this particular case.

Theorem 1.1 (Test for exactness) Suppose that $M, N, \partial M/\partial y$ and $\partial N/\partial x$ are continuous on a rectangle R . Then $Mdx + Ndy = 0$ is exact if and only if $\partial M/\partial y = \partial N/\partial x$ at each point of R .

Proof: Fix x_0 . The functions $F(x, y)$ satisfying the equation $\partial F/\partial x = M$ can be found by integrating M along a line segment from (x_0, y) to (x, y) , since each such line segment remains in the rectangle R :

$$F(x, y) = \int_{(x_0, y)}^{(x, y)} M(s, y) ds$$

The result is any antiderivative of M with respect to x plus a function of y to be determined. Namely, it is of the form $F(x, y) = R(x, y) + h(y)$ where $\partial R/\partial x = M$.

The question is whether or not we can always choose $h(y)$ so that the equation $\partial F/\partial y = N$ is also satisfied. We need

$$\begin{aligned}\frac{\partial F}{\partial y} &= \frac{\partial R}{\partial y} + h'(y) = N(x, y) \\ h'(y) &= N(x, y) - \frac{\partial R}{\partial y}\end{aligned}$$

The last equation has a solution for $h'(y)$ (and consequently for $h(y)$) if and only if its right hand side is independent of x . In order to test whether this is true or not, let us look at its partial derivative with respect to x :

$$\begin{aligned}\frac{\partial}{\partial x} \left(N(x, y) - \frac{\partial R}{\partial y} \right) &= \frac{\partial N}{\partial x} - \frac{\partial^2 R}{\partial x \partial y} \\ &= \frac{\partial N}{\partial x} - \frac{\partial^2 R}{\partial y \partial x} \\ &= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \\ &= 0.\end{aligned}$$

Therefore we can solve for $h(y)$, and the equation is exact. This completes the proof. \square

Example 1.3 Find the value of the constant a for which the ODE

$$3e^y dx + (2y + axe^y) dy = 0$$

is exact. Solve the equation for this value of a .

Solution: We have $M(x, y) = 3e^y$ and $N(x, y) = 2y + axe^y$. Since

$$\frac{\partial M}{\partial y} = 3e^y \quad \frac{\partial N}{\partial x} = ae^y$$

the equality holds if and only if $a = 3$. Since M, N and their partial derivatives are all continuous on \mathbb{R}^2 , we can apply the theorem and conclude that the ODE is exact for $a = 3$. Now,

$$\begin{aligned}\frac{\partial F}{\partial x} &= 3e^y \\ F(x, y) &= 3xe^y + h(y) \\ \frac{\partial F}{\partial y} &= 3xe^y + h'(y) = 2y + 3xe^y \\ h'(y) &= 2y.\end{aligned}$$

Therefore $h(y) = y^2$ is a solution and $F(x, y) = 3xe^y + y^2$ is a potential. The solutions of the ODE are

$$3xe^y + y^2 = c$$

where $c \in \mathbb{R}$ is a constant. \square

Example 1.4 Solve the initial value problem

$$\left(\frac{2xy}{x^2 + 1} - 2x \right) dx - (2 - \ln(x^2 + 1))dy = 0, \quad y(5) = 0.$$

Determine the largest interval on which the solution is valid.

Solution: Here, $M(x, y) = \frac{2xy}{x^2 + 1} - 2x$ and $N(x, y) = -2 + \ln(x^2 + 1)$. We compute

$$\frac{\partial M}{\partial y} = \frac{2x}{x^2 + 1} = \frac{\partial N}{\partial x}$$

Both M, N and their partial derivatives are continuous on \mathbb{R}^2 (which can be viewed as an infinite rectangle). Therefore, by the test for exactness, the equation is exact. Let us find a potential.

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{2xy}{x^2 + 1} - 2x \\ F(x, y) &= y \ln(x^2 + 1) - x^2 + h(y) \\ \frac{\partial F}{\partial y} &= \ln(x^2 + 1) + h'(y) = -2 + \ln(x^2 + 1) \\ h'(y) &= -2 \\ h(y) &= -2y. \end{aligned}$$

We deduce that $F(x, y) = y \ln(x^2 + 1) - x^2 - 2y$ is a potential. All solutions of the ODE are $y \ln(x^2 + 1) - x^2 - 2y = c$. Using the initial condition $y(5) = 0$, we find that $0 \ln(5^2 + 1) - 5^2 - 2 \times 0 = c$, therefore $c = -25$. So,

$$\begin{aligned} y \ln(x^2 + 1) - x^2 - 2y &= -25 \\ y(\ln(x^2 + 1) - 2) &= x^2 - 25 \\ y &= \frac{x^2 - 25}{\ln(x^2 + 1) - 2}. \end{aligned}$$

This function is defined if and only if $\ln(x^2 + 1) - 2 \neq 0$, namely for $x^2 + 1 \neq e^2$. The interval of definition, which must be a connected interval, could then be either of $(-\infty, -\sqrt{e^2 - 1})$, $(-\sqrt{e^2 - 1}, \sqrt{e^2 - 1})$ or $(\sqrt{e^2 - 1}, \infty)$ but since the initial point $x = 5$ belongs to the last one, the answer is $(\sqrt{e^2 - 1}, \infty)$.

2 Integrating Factors

Recall from the lecture on first order linear equations that an ODE of the form $y' + p(t)y = q(t)$ can be solved by multiplying the equation by an integrating factor $\mu(t)$. In this case, the equation for $\mu(t)$ turned out to be easy to solve and we even got a formula $\mu(t) = \exp(\int p(t)dt)$.

Let us now suppose that we have an ODE of the form $M(x, y)dx + N(x, y)dy = 0$. If the equation is exact, then we know what to do. If it is not exact, we may try to find an integrating factor $\mu(x, y)$ such that, after multiplication with μ , the new ODE

$$\mu M dx + \mu N dy = 0$$

is exact. Let us assume that all of these functions and their partial derivatives are continuous on a rectangle R , so that we can use the test for exactness. Then the new equation is exact if and only if

$$\begin{aligned}\frac{\partial(\mu M)}{\partial y} &= \frac{\partial(\mu N)}{\partial x} \\ \frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} &= \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}.\end{aligned}$$

The problem that we encounter here is that this new differential equation for μ is terribly difficult to solve. It is not even an ODE, it is a PDE. Therefore, finding an integrating factor in this very general setting is a hopelessly difficult task. Only when there is some additional information that tells us something about the form of the integrating factor, this method could be useful.

Example 2.1 Show that $\mu(x, y) = (x^2 + y^2)^{-1}$ is an integrating factor for the ODE

$$(3x^2 + x + 3y^2)dx + (7x^2 + y + 7y^2)dy = 0$$

and use it to find all solutions of this ODE.

Solution: The original equation is not exact (please check this). If we multiply the ODE throughout by $\mu(x, y)$, we get

$$\left(3 + \frac{x}{x^2 + y^2}\right) dx + \left(7 + \frac{y}{x^2 + y^2}\right) dy = 0.$$

The functions $3 + \frac{x}{x^2 + y^2}$ and $7 + \frac{y}{x^2 + y^2}$ are defined on $\mathbb{R}^2 - \{(0,0)\}$. Since this set is not a rectangle (in fact, it is not a simply connected domain), we cannot use the test for exactness here. We should directly show that a potential function exists:

$$\begin{aligned}\frac{\partial F}{\partial x} &= 3 + \frac{x}{x^2 + y^2} \\ F(x, y) &= 3x + \frac{1}{2} \ln(x^2 + y^2) + h(y) \\ \frac{\partial F}{\partial y} &= \frac{y}{x^2 + y^2} + h'(y) = 7 + \frac{y}{x^2 + y^2} \\ h'(y) &= 7 \\ h(y) &= 7y.\end{aligned}$$

Therefore $F(x, y) = 3x + 7y + \frac{1}{2} \ln(x^2 + y^2)$ is a potential. Existence of a potential implies that $\mu(x, y)$ is indeed an integrating factor. The solutions of the ODE are

$$3x + 7y + \frac{1}{2} \ln(x^2 + y^2) = c$$

where $c \in \mathbb{R}$ is a constant.