5 - markov chain

6.1. A small computer lab has 2 terminals. The number of students working in this lab is recorded at the end of every hour. A computer assistant notices the following pattern:

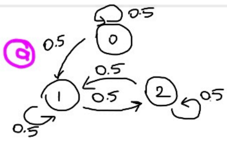
- If there are 0 or 1 students in a lab, then the number of students in 1 hour has a 50-50% chance to increase by 1 or remain unchanged.
- If there are 2 students in a lab, then the number of students in 1 hour has a 50-50% chance to decrease by 1 or remain unchanged.

(a) Write the transition probability matrix for this Markov chain.

(b) Is this a regular Markov chain? Justify your answer.

(c) Suppose there is nobody in the lab at 7 am. What is the probability of nobody working in the lab at 10 am?

a



steady states

$0.5x = x$ $x=0$

$0.5x + 0.5y + 0.5z = y \to y = z$

$0.5y + 0.5z = z \to y = z$

$x+y+z=1$

$x=0$

$y=0.5$

$z=0.5$

$[0, 0.5, 0.5]$ → zero!

b no $(0, \pi_1, \pi_2)$ between $[1, 2]$

c

	0.5	0.5	0	0
	0	0.5	0.5	1
	0	0.5	0.5	2
to	0	1	2	

from

$P^3 =$

$\boxed{0.125}$	0.5	0.375	0
0	0.5	0.5	1
0	0.5	0.5	2
0	1	2	

$\boxed{0.125}$

6.2. A computer system can operate in two different modes. Every hour, it remains in the same mode or switches to a different mode according to the transition probability matrix

$$\begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}.$$

(a) Compute the 2-step transition probability matrix. P^2

(b) If the system is in Mode I at 5:30 pm, what is the probability that it will be in Mode I at 8:30 pm on the same day?

a $P^2 \to \begin{pmatrix} 0.52 & 0.48 \\ 0.48 & 0.52 \end{pmatrix}$

b $P^3 \to \begin{pmatrix} \boxed{0.496} & 0.504 \\ 0.504 & 0.496 \end{pmatrix}$ 1 1

3 hours

$\boxed{0.496}$

6.3. Markov chains find direct applications in genetics. Here is an example.

An offspring of a black dog is black with probability 0.6 and brown with probability 0.4. An offspring of a brown dog is black with probability 0.2 and brown with probability 0.8.

(a) Write the transition probability matrix of this Markov chain.

(b) Rex is a brown dog. Compute the probability that his grandchild is black.

a

0.6	0.4	Black
0.2	0.8	Brown
Black	Brown	

from, to

b $P^2 =$

0.44	0.56	black
$\boxed{0.28}$	0.72	brown
black	brown	

$\boxed{0.28}$

6.4. Every day, Eric takes the same street from his home to the university. There are 4 street lights along his way, and Eric has noticed the following Markov dependence. If he sees a green light at an intersection, then 60% of time the next light is also green, and 40% of time the next light is red. However, if he sees a red light, then 70% of time the next light is also red, and 30% of time the next light is green.

(a) Construct the transition probability matrix for the street lights.

(b) If the first light is green, what is the probability that the third light is red?

(c) Eric's classmate Jacob has *many* street lights between his home and the university. If the *first* street light is green, what is the probability that the *last* street light is red? (Use the steady-state distribution.)

a

0.6	0.4	G
0.3	0.7	R
G	R	

to, from

$P^4 =$

0.4332	0.5668	G
0.4251	0.5749	R
G	R	

to, from

b $P^3 =$

0.48	$\boxed{0.52}$	G
0.39	0.61	R
G	R	

to → , from

$\boxed{0.52}$

c $(x,y) = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}(x,y)$

$0.6x + 0.3y = x$, $0.4x + 0.7y = y$ → $0.3y = 0.4x$ $y = \frac{4x}{3}$

$x+y=1$ $\frac{7x}{3}=1$ $x=\frac{3}{7}$ $y=\frac{4}{7}$

$G \to \frac{3}{7}$ $\boxed{R \to \frac{4}{7}}$ steady states does not depend on initial state!!

6.5. The pattern of sunny and rainy days on planet Rainbow is a homogeneous Markov chain with two states. Every sunny day is followed by another sunny day with probability 0.8. Every rainy day is followed by another rainy day with probability 0.6. Compute the probability that April 1 next year is rainy on Rainbow.

0.8	0.2	S
0.4	0.6	R
S	R	

to, from

$0.8x + 0.4y = x$, $0.2x + 0.6y = y$ → $0.4y = 0.2x$

$x = 2y$

$x+y=1$ $3y=1$ $y=\frac{1}{3}$ $x=\frac{2}{3}$

S R $[\frac{2}{3}, \frac{1}{3}]$

$\boxed{\frac{1}{3}}$

6.6. A computer device can be either in a busy mode (state 1) processing a task, or in an idle mode (state 2), when there are no tasks to process. Being in a busy mode, it can finish a task and enter an idle mode any minute with the probability 0.2. Thus, with the probability 0.8 it stays another minute in a busy mode. Being in an idle mode, it receives a new task any minute with the probability 0.1 and enters a busy mode. Thus, it stays another minute in an idle mode with the probability 0.9. The initial state X_0 is idle. Let X_n be the state of the device after n minutes.

(a) Find the distribution of X_2.

(b) Find the steady-state distribution of X_n.

a x

0.8	0.2	1 (busy)
0.1	0.9	2 (idle)
1	2	

from, to

x_2 $\begin{pmatrix} 0.66 & 0.34 \\ 0.17 & 0.83 \end{pmatrix}$

b $(a,b) = (a,b)\begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}$

$0.8a + 0.1b = a$

$0.2a + 0.9b = b$

$b = 2a$ $3a=1$

$a+b=1$ → $[\frac{1}{3}, \frac{2}{3}]$

6.7. A Markov chain has the transition probability matrix

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$0.3x + y = x$, $0.7x = y$, $y = z$

$x+y+z=1$

$x + 0.7x + 0.7x = 1$

$2.4x = 1$

$x = 0.416$

$y = 0.292$

$z = 0.292$

(a) Fill in the blanks. +

(b) Show that this is a regular Markov chain. + ↓ non of them is zero

(c) Compute the steady-state probabilities. + $[0.416, 0.292, 0.292]$

6.8. A Markov chain has 3 possible states: A, B, and C. Every hour, it makes a transition to a *different* state. From state A, transitions to states B and C are equally likely. From state B, transitions to states A and C are equally likely. From state C, it always makes a transition to state A. Find the steady-state distribution of states.

0	0.5	0.5	A
0.5	0	0.5	B
1	0	0	C
A	B	C	

to, from

$(a,b,c) = (a,b,c)\begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}$

$= \begin{cases} 0.5b + c = a \\ 0.5a = b \\ 0.5a + 0.5b = c \end{cases}$

$0.25a + c = a$

$c = 0.75a$

$a + b + c = 1$

$a + 0.5a + 0.75a = 1$

$2.25a = 1$

$a = 0.444$

$b = 0.222$

$c = 0.333$

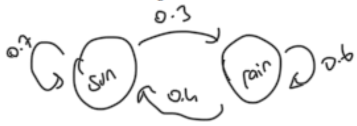
$[0.444, 0.222, 0.333]$

Example 6.7 (Weather forecasts). In some town, each day is either sunny or rainy. A sunny day is followed by another sunny day with probability 0.7, whereas a rainy day is followed by a sunny day with probability 0.4.

It rains on Monday. Make forecasts for Tuesday, Wednesday, and Thursday.

$\boxed{\text{rainy}} \xrightarrow{0.6} \boxed{\text{rainy}}$, $\boxed{\text{rainy}} \xrightarrow{0.4} \boxed{\text{sunny}}$

$\boxed{\text{sunny}} \xrightarrow{0.3} \boxed{\text{rainy}}$, $\boxed{\text{sunny}} \xrightarrow{0.7} \boxed{\text{sunny}}$



Monday: rainy → Tuesday: sunny (0.4), rainy (0.6) → Wednesday:

sunny $\overbrace{(0.4)(0.7) + (0.6)(0.4)}^{0.52}$

rainy $\underbrace{(0.4)(0.3) + (0.6)(0.6)}_{0.48}$

let sunny = 1, rainy = 2

notation = $p_{11} = 0.7$, $p_{12} = 0.3$, $p_{21} = 0.4$, $p_{22} = 0.6$

Example 6.8 (Weather, continued). Suppose now that it does not rain yet, but meteorologists predict an 80% chance of rain on Monday. How does this affect our forecasts?

Monday: rainy (0.8), sunny (0.2) → Tuesday:

rainy $(0.8)\cdot(0.6) + (0.2)(0.3)$ → 0.54

sunny $(0.8)(0.4) + (0.2)(0.7)$ → $0.32 + 0.14$ → 0.46

...

$\boxed{\text{matrix approach}}$

$$P = \begin{array}{c|cc} & R & S \\ \hline \text{from } R & 0.6 & 0.4 \\ S & 0.3 & 0.7 \end{array} \quad \text{to } R \ S$$

$$P^2 = \begin{array}{cc|c} 0.48 & 0.52 & R \\ 0.39 & 0.61 & S \\ \hline R & S & \end{array}$$

- if monday is R → tuesday R → 0.6, S → 0.4, wednesday R → 0.48, S → 0.52

$$P_h = P_0 \cdot P^h$$

moday R → 0.8, S → 0.2

$$\overbrace{[0.8, 0.2]}^{P_0} \times \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$$

$$\text{tuesday} = \Big[\underbrace{\overset{R}{0.48 + 0.06}}_{0.54},\ \underbrace{\overset{S}{0.32 + 0.14}}_{0.46}\Big]$$

probabilities

$$
P = \begin{array}{c|c}
\begin{pmatrix} p_{11} + p_{12} + \cdots + p_{1n} = & 1 \\ p_{21} + p_{22} + \cdots + p_{2n} = & 1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots & \\ p_{n1} \quad p_{n2} \quad \cdots \quad p_{nn} & \end{pmatrix} & \begin{array}{c} \text{From state:} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \\ \hline
\text{To state:} \quad 1 \quad 2 \quad \cdots \quad n &
\end{array}
$$

Example 6.9 (Shared device). A computer is shared by 2 users who send tasks to a computer remotely and work independently. At any minute, any connected user may disconnect with probability 0.5, and any disconnected user may connect with a new task with probability 0.2. Let $X(t)$ be the number of concurrent users at time t (minutes). This is a Markov chain with 3 states: 0, 1, and 2.

			from
0.64	0.32	0.4	0
0.4	0.5	0.1	1
0.25	0.5	0.25	2
0	1	2	
to			

P

$P_{00} = (0.8)(0.8) = 0.64$ $\quad P(_{01}) = (0.2)(0.8) + (0.8)(0.2)$ $\quad P_{02} = (0.2)(0.2)$

$P_{10} = (0.5)(0.8) = 0.4$ $\quad P_{11} = (0.5)(0.8) + (0.5)(0.2) = 0.5$ $\quad P_{12} = (0.5)(0.2) = 0.1$

$P_{20} = (0.5).(0.5) = 0.25$ $\quad P_{21} = (0.5).(0.8) + (0.5)(0.2) = 0.5$ $\quad P_{22} = (0.5)(0.5) = 0.25$

↳ if 2 useres are connected at $p_0 \to (0,0,1)$

↳ probability of 1 connected user at $t=2 \to p_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \cdot p^2 \to (a, b, c)$ user #0 1 2 answer

steady state

$\pi_x = \lim_{h \to \infty} P_h(x)$ computed $\pi P = \pi$

ex:

$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$

$(\pi_1, \pi_2) = (\pi_1, \pi_2) \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$

same equations

$0.4\pi_2 = 0.3\pi_1$

$0.7\pi_1 + 0.4\pi_2 = \pi_1$
$0.3\pi_1 + 0.6\pi_2 = \pi_2$

(1) $0.3\pi_1 = 0.4\pi_1$

* does not depend on initial state

$\pi_1 + \frac{3}{4}\pi_1 = 1 \quad \pi_1 = \frac{4}{7} \simeq 0.57$ ← (2) $\pi_1 + \pi_2 = 1$

$\pi_2 = \frac{3}{7} \simeq 0.43$

$(0.57, 0.43)$

↳ in long term 0.57 → days are sunny

0.43 → are rainy

steady state

Given the transition probability matrix P for a three-state Markov chain, in which the set of states is {1, 2, 3} and

$$P = \begin{bmatrix} 0.15 & 0.60 & 0.25 \\ 0.40 & 0.40 & 0.20 \\ 0.80 & 0.10 & 0.10 \end{bmatrix}$$

compute the steady-state probabilities.

$\pi_1 =$ 0.334 ✖

$\pi_2 =$ 0.334 ✖

$\pi_3 =$ 0.334 ✖

Write your answers by providing up to 4 digits after the decimal point.

$0.15x + 0.4y + 0.8z = x$
$0.6x + 0.4y + 0.1z = y$
$0.25x + 0.2y + 0.1z = z$

$y = 1.08654x$
$z = 0.519231x$

$x + y + z = 1$

$\frac{+ \quad x}{1}$

$2.605777x = 1$

$x = 0.383763$ ✓
$y = 0.416974$ ✓
$z = 0.199261$ ✓

DEFINITION 6.9

A Markov chain is regular if

$$p_{ij}^{(h)} > 0$$

find some p^n that all are non-zero

for some h and all i, j. That is, for some h, matrix $P^{(h)}$ has only non-zero entries, and h-step transitions from any state to any state are possible.

Any regular Markov chain has a steady-state distribution.

Example 6.15. A Markov chain with transition probability matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.9 & 0 & 0 & 0.1 \end{pmatrix}.$$

is also regular. Matrix P contains zeros, and so do P^2, P^3, P^4, and P^5. The 6-step transition probability matrix

$$P^{(6)} = \begin{pmatrix} .009 & .090 & .900 & .001 \\ .001 & .009 & .090 & .900 \\ .810 & .001 & .009 & .180 \\ .162 & .810 & .001 & .027 \end{pmatrix}$$

contains no zeros and proves regularity of this Markov chain.

also be seen from the transition diagram in Figure 6.4. Based on this diagram, any state j can be reached in 6 steps from any state i. Indeed, moving counterclockwise through this

Example 6.16 (Irregular Markov chain, absorbing states). When there is a state i with $p_{ii} = 1$, a Markov chain cannot be regular. There is no exit from state i; therefore,

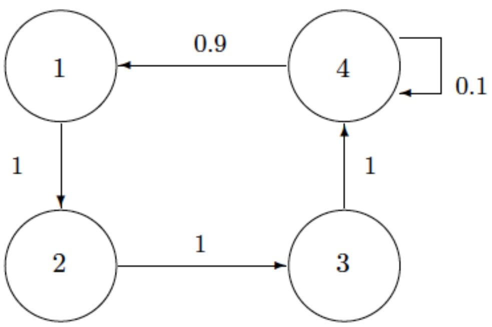


FIGURE 6.4: *Transition diagram for a regular Markov chain in Example 6.15.*

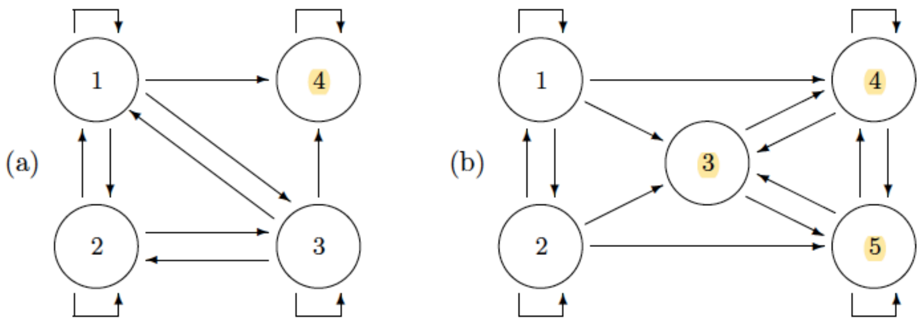


FIGURE 6.5: *Absorbing states and absorbing zones (Example 6.16).*

Notice that both Markov chains do have steady-state distributions. The first process will eventually reach state 4 and will stay there for good. Therefore, the limiting distribution of $X(h)$ is $\pi = \lim P_h = (0, 0, 0, 1)$. The second Markov chain will eventually leave states 1 and 2 for good, thus its limiting (steady-state) distribution has the form $\pi = (0, 0, \pi_3, \pi_4, \pi_5)$. $\Diamond$