15 - higher order linear ode's / converting them into a first order system

nth order derivative $= y^{(n)} = \dfrac{d^n y}{dt^n}$

solutions intervals

singular points= the points which coeffients are undefined (when $y^{(n)}$ coefficient is 1)

↳ ex: $ty'' + 3y' = 1 \rightarrow y'' + \frac{3}{t}y' = \frac{1}{t} \rightarrow (-\infty, 0)$ or $(0, \infty)$ $t=0$ singular point

converting a higher order ode into a first order system

$$y^{(n)} + a_1(t)\,y^{(n-1)} + a_2(t)\,y^{(n-2)} + \dots + a_{n-1}(t).\,y^{(1)} + a_n(t).y = b(t)$$ → by $(n\times n)$ system

$$\begin{aligned}
x_1 &= y \\
x_2 &= y' = x_1' = x_2 \\
x_3 &= y^{(2)} = x_2' = x_3 \\
&\vdots \\
x_n &= y^{(n-1)} = x_{n-1}' = x_n \\
x_{n+1} &= y^{(n)} = x_n' = -a_n(t)\underbrace{y}_{x_1} - a_{n-1}(t)\underbrace{y'}_{x_2} - \dots - a_2(t)\underbrace{y^{(n-2)}}_{x_{n-1}} - a_1(t)\underbrace{y^{(n-1)}}_{x_n} + b(t)
\end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_n(t) & -a_{n-1}(t) & \cdots & -a_2(t) & -a_1(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{bmatrix}$$

ex:

$$y''' + 3y'' - ty' + 2y = \cos(t)$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & t & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos(t) \end{bmatrix}$$

Structure of the solution set

- solution of the system is n (nth order) linearly independent vectors $y''' \rightarrow y_1, y_2, y_3$
- the combination of these vectors with some constants are solutions $y(t) = c_1.y_1 + c_2 y_2 + c_3 y_3$

ex: check if $\left.\begin{matrix} y_1 = t \\ y_2 = t^2 \end{matrix}\right\}$ are the solution of the system when $t > 0$.

$y_2 = $ [illegible]
$y_3 = \frac{1}{t}$)

$t^3 y''' + t^2 y'' - 2t y' + 2y = 0$

$\rightarrow$ $t \rightarrow 0 + 0 - 2t \cdot 1 + 2t = 0$ ✓

$t^2 \rightarrow 0 + t^2 \cdot 2 - 2t \cdot 2t + 2t^2 = 0$ ✓

$\frac{1}{t} \rightarrow t^3 \cdot (-6) \cdot t^{-4} + t^2 \cdot 2 \cdot t^{-3} - 2t \cdot (-1) \cdot t^{-2} + 2t^{-1} = \frac{-6}{t} + \frac{2}{t} + \frac{2}{t} + \frac{2}{t} = 0$ ✓

$\hookrightarrow y(t) = c_1 t + c_2 t^2 + \frac{c_3}{t}$

check for independence $\longrightarrow$

$$W(y_1, y_2, y_3) = \begin{vmatrix} t & t^2 & \frac{1}{t} \\ 1 & 2t & \frac{-1}{t^2} \\ 0 & 2 & \frac{2}{t^3} \end{vmatrix}$$

$$= \frac{6}{t} \neq 0$$