relations

let A and B sets. a binary relation from a to b is a subset of $A \times B$. (bağıntı)
↳ essentially, a binary relation is a set R of ordered pairs where the first element of each ordered pair comes from A, and the second element comes from B.

example

$A = \{1,2,3\}$ $B = \{a,b,c\}$ → $R = \{(1,a),(1,b),(2,b),(2,c),(3,a)\}$ is a relation from A to B.
↳ it does not have to contain all but it can

$1 R a$, $1 \not R c$

* a function is a relation whee each elemnt of B is mapped to by only one element of A.

binary relation on a set:
a subset of $A \times A$, or a relation from set A onto itself.
the number of relations from set A to itself: $2^{n^2} = 2^{|A|^2}$

properties

(dönüşlü)
reflexive relations: if $(a,a) \in R$ for every $a \in A$
↳ $R: \{(a,b) \mid a \leq b\}$ → (a,a) $a \leq a$ so it is reflexive ✓
$R: \{(a,b) \mid a+b \leq 3\}$ → (a,a) $2a \nleq 3$ so it is not reflexive ✗

symmetric relations: if $(b,a) \in R$ whenever $(a,b) \in R$
↳ $R: \{(a,b) \mid a \leq b\}$ → (a,b) (b,a) not always ✗
$R: \{(a,b) \mid a+b \leq 3\}$ ✓

anti-symmetric relation: if $\forall a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a=b$
↳ $R: \{(a,b) \mid a \leq b\}$ $(2,2)$ $(2,2)$ ✓
$R: \{(a,b) \mid a+b \leq 3\}$ $(1,2)$ $(2,1)$ ✗
$R: \{(a,b) \mid a > b\}$ (no counter example) ✓

(geçişli)
transitive relation: if wherever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ $\forall a,b,c \in A$
↳ $R: \{(a,b) \mid a \leq b\}$ $(1,1)(1,1)$ $a \leq b$ $b \leq c$ → $a \leq c$ ✓
$R: \{(a,b) \mid a+b \leq 3\}$ $(2,1)$ $(1,2)$ $(2,2)$ ✗
$R: \{(a,b) \mid a > b\}$ $a > b$ $b > c$ $a > c$ ✓

equivalence relations

a relation on a set A is called an equivalence relation if it is reflexive a relation on a set A is called an equivalence relation if it is reflexive, symmetric and transitive.

↳ two elements related by an equivalence relation are said to be equivalent, denoted $a \sim b$.

equivalence classes: the set of all elements eqivalent to an element a of A through the equivalence relation R is called an eqivalence class of a, $[a]_R$

composing relations

$(a,b) \in R_1$, $(b,c) \in R_2 \rightarrow (a,c) \in R_2 \circ R_1$

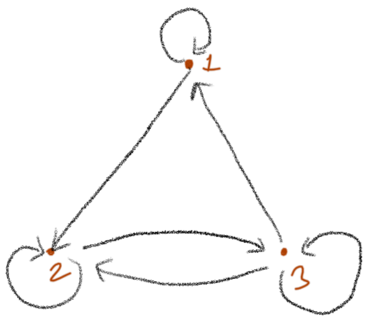
ex: $R_1 = \{(1,3)\}$ $\quad R_2 \circ R_1 = R_2(R_1) = \{(1,4),(1,5)\}$

$R_2 = \{(3,4),(3,5)\}$

* $R = \{(1,1),(2,1)(3,2),(4,3)\}$

$R \circ R = \{(1,1),(2,1),(3,1),(4,2)\}$

representing relations using digraphs



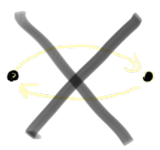
the relation R on $\{1,2,3\}$ where

$R = \{(1,1),(1,2),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

- reflexive ✓ (each node has a loop)
- symmetric ✗ (every node must be connected mutually)
- antisymmetric ✗ (olmamalı)
- transitive ✗ $(2 \rightarrow 3)(3 \rightarrow 1)$ but no $(2 \rightarrow 1)$

partial orders (POSET)

- antisymmetric
- reflexive
- transitive



example:

$A = \{1,2,3,4\}$

$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$ is (A,R) POSET?

↳ reflexive: (1,1), (2,2), (3,3), (4,4) ✓

↳ antisymmetric: (1,2) ✓ (2,1) ✗
(1,3) ✓ (3,1) ✗
(1,4) ✓ (4,1) ✗
(2,4) ✓ (4,2) ✗
(2,3) ✓ (3,2) ✗
(3,4) ✓ (4,3) ✗ ✓

(A,R) is POSET

↳ ...

↳ transitive: (1,2) (2,3) → (1,3) ✓
(1,2) (2,4) → (2,4) ✓
(1,3) (3,4) → (1,4) ✓
(2,3) (3,4) → (2,4) ✓

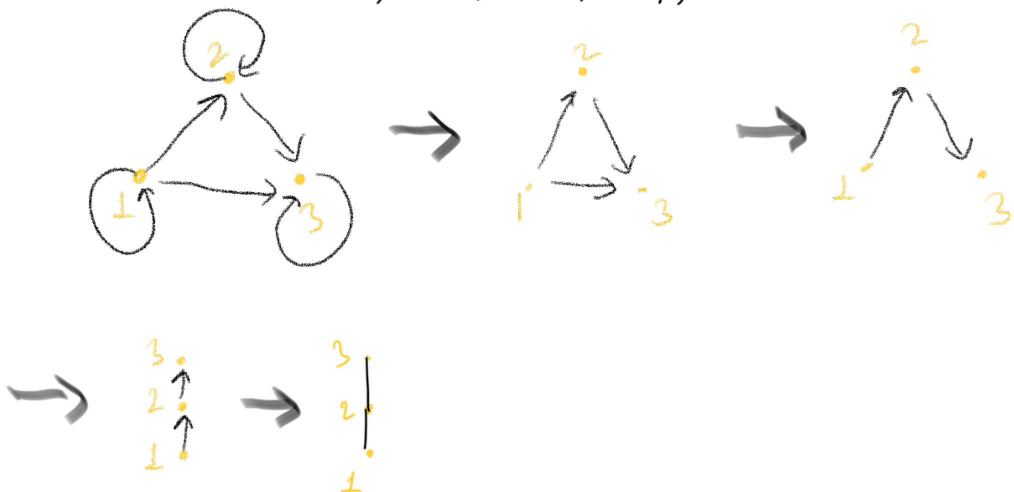
$A = \{1,2,3\}$
$P(A)\ \subseteq$

hasse diagram (ordering diagram)
↳ graphical representation of a poset

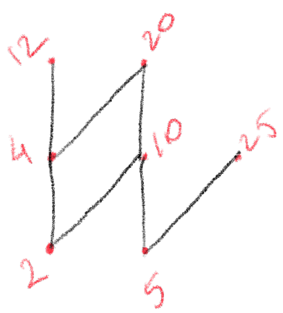
1. draw directed graph representation
2. remove loops (reflexivity)
3. remove the transitive nodes
4. arrange each edge so that all arrows point upwards
5. remove all arrowheads

example hasse diagram for $(\{1,2,3\}, \subseteq)$

$R = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$



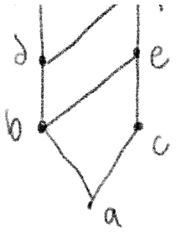
example $(\{2,4,5,10,12,20,25\}, |)$



maximal = 12, 20, 25
minimal = 2, 5
greatest = no unique element, so no greatest elem
least = no unique element, so no least elem

example

maximal = h, j
minimal = g



greatest = no unique element, no greatest element
least = a

$A = \{a, b, c, d, e, f, g, h, j\}$

$B = \{a, b, c\}$ (a subset)

upper bound (B→A) GLB (greatest lower bound)

	a	b	c	d	e	f	g	h	j
a	✓	✓	✓	✓	✓	✓	✓	✓	✓
b	x	✓	x	✓	✓	✓	✓	✓	✓
c	x	x	✓	x	✓	✓	x	✓	✓

aRj ✓
bRj ✓

$= \{e, f, h, j\}$

least bound (A→B) LUB (least upper bound)

$\{a, b, c\} = \{e\}$ (en aşağıda hepsini bağlayan node)

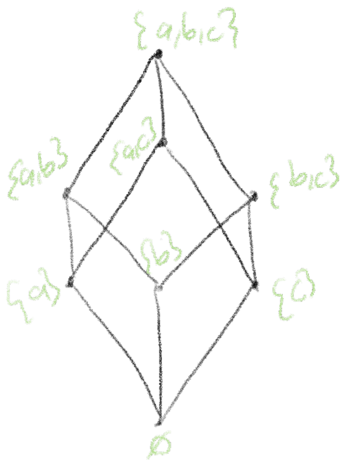
lower bound (A→B)

	a	b	c
a	✓	✓	✓
b	x	✓	x
c	x	x	✓
d	x	x	x
e	x	x	x
f	x	x	x
g	x	x	x
h	x	x	x

aRa aRb aRc

$LB = \{a\}$
$GLB = \{a\}$

example: the hasse diagram for $(P(\{a,b,c\}), \subseteq)$ → power set



Lattice

every pair of elements has both a least upper bound and a greatest lower bound (partially ordered set)

closure of relations

when relation does not have a certain property, by adding some elements to the relation, making relation to have that property

reflexive closure:

$R = \{(1,1),(1,2),(2,1),(3,2)\} \xrightarrow{\text{add } (2,2),(3,3)}$ reflexive (R on $A = \{1,2,3\}$)

↳ add all pairs in form $(a,a) \in A$

symmetric closure:

$R = \{(1,1),(1,2),(2,2),(2,3),(3,1),(3,2)\} \xrightarrow{\text{add } (2,1)\ (1,3)}$ symmetric

↳ add the inverse of the R

transitive closure:

↳ we cannot produce transitive closure by adding pairs.