The lottery problem

Alewyn Petrus Burger
Stellenbosch University

The Lottery Problem

Alewyn P Burger[†], Werner R Gründlingh[*] & Jan H van Vuuren[*]

September 16, 2003

Abstract

Suppose a lottery scheme consists of randomly selecting a winning t–set from a universal m–set, while a player participates in the scheme by purchasing a playing set of any number of n–sets from the universal set prior to the draw, and is awarded a prize if k or more elements in the winning t–set match those of at least one of the player's n–sets in his playing set ($k \leq \{t, n\} \leq m$). Such a prize is called a k–prize. The player may wish to construct his playing set in such a way that it will guarantee him a k–prize, no matter which winning t–set is chosen from the universal set. In this paper the following lottery problem is considered: What is the smallest possible cardinality of a playing set that will guarantee the player a k–prize? This number, denoted $L(m, n, t; k)$, is called the lottery number, and such a smallest playing set is called an $L(m, n, t; k)$–set. We introduce the notion of a *lottery graph* and demonstrate how standard results from graph domination theory, not previously associated with lottery numbers lead to simple, closed–form bound formulations for the lottery number, which are usually better than the best analytic covering bounds available (yet weaker than bounds obtained from lottery design constructions). Another novel contribution of this paper is the development of a search procedure for characterising all possible overlapping structures of $L(m, n, t; k)$–sets. This technique is used to establish 19 new lottery numbers and to improve upon best known bounds for a further 20 lottery numbers.

Keywords: lottery problem, covering problem, combinatorial design, graph domination.

AMS Classification: 05B05, 05B07, 05B40, 51E10, 62K05, 62K10.

1 Introduction

Practical situations sometimes occur in which it is beneficial to construct a set $\mathcal{N}$ of (unordered) n–sets from some universal m–set $\mathcal{U}_m$ ($n \leq m$) in such a manner that, given a randomly selected t–set w from $\mathcal{U}_m$ ($t \leq m$), there exists at least one n–set in $\mathcal{N}$ in which at least k of the elements match those in w, for some specified $k \leq \min\{t, n\}$. Moreover, in such applications it often holds that the smaller the cardinality of the set $\mathcal{N}$, the more beneficial the particular construction.

One such instance is encountered when a participant of a lottery scheme wishes to play in a manner so as to ensure that he wins a prize. Suppose the lottery scheme $\langle m, n, t; k\rangle$ consists of randomly selecting a winning t–set, w, from the universal set $\mathcal{U}_m = \{1, 2, \ldots, m\}$, while a player participates in the scheme by purchasing a playing set $\mathcal{N}$ of any number of n–sets from $\mathcal{U}_m$ prior to the draw, and is awarded a prize, called a *k–prize*, if at least k elements of w match those of at least one of the player's n–sets in $\mathcal{N}$. In such a case the player may wish to construct his playing set so as to be guaranteed a k–prize, no matter which winning t–set w is chosen from $\mathcal{U}_m$. With this objective in

[*]Department of Applied Mathematics, Stellenbosch University, Private Bag X1, Matieland, 7602, Republic of South Africa, fax: +27 21 8083778, email: vuuren@sun.ac.za

. [†] Department of Mathematics and Statistics, University of Victoria, Box 3045, Victoria, BC, Canada, V8W 3P4, email: alewynburger@hotmail.com

Region, Country or US State	Lottery
Illinois, Lithuania	$\langle 30,5,5;k\rangle$
Iowa	$\langle 30,6,6;k\rangle$
Chile	$\langle 30,7,7;k\rangle$
Colorado	$\langle 32,5,5;k\rangle$
Michigan	$\langle 33,5,5;k\rangle$
Norway	$\langle 34,7,7;k\rangle$
Connecticut, Latvia, Slovak Republic, Slovenia	$\langle 35,5,5;k\rangle$
Kansas	$\langle 35,6,6;k\rangle$
Hungary, Sweden	$\langle 35,7,7;k\rangle$
Kazakhstan, Yugoslavia	$\langle 36,5,5;k\rangle$
Maine, New Hampshire, Vermont, Wisconsin	$\langle 36,6,6;k\rangle$
China, Denmark	$\langle 36,7,7;k\rangle$
Jamaica	$\langle 37,6,6;k\rangle$
Iceland, Nebraska	$\langle 38,5,5;k\rangle$
Australia, Delaware	$\langle 38,6,6;k\rangle$
Malta, New York, Pennsylvania	$\langle 39,5,5;k\rangle$
Croatia	$\langle 39,7,7;k\rangle$
Czech Republic	$\langle 40,5,5;k\rangle$
Ghana, Kazakhstan, Louisiana, New Zealand, Perú	$\langle 40,6,6;k\rangle$
Arizona	$\langle 41,6,6;k\rangle$
Minnesota	$\langle 42,5,5;k\rangle$
Belgium, Colorado, Ireland, Maine, Malaysia, Massachusetts, New Hampshire, Phillipines, Puerto Rico, Taiwan, Vermont	$\langle 42,6,6;k\rangle$
Japan	$\langle 43,6,6;k\rangle$
Portugal, Uruguay	$\langle 44,5,5;k\rangle$
Australia, Connecticut, Missouri	$\langle 44,6,6;k\rangle$
Argentina, Australia, Austria, Croatia, Hungary, Israel, Netherlands, Perú, Phillipines, Singapore, Switzerland, Ukraine, Yugoslavia	$\langle 45,6,6;k\rangle$
California	$\langle 47,5,5;k\rangle$
Hong Kong	$\langle 47,6,6;k\rangle$
British Columbia, Québec, Western Canada	$\langle 47,7,7;k\rangle$
Denmark, Finland, Oregon	$\langle 48,6,6;k\rangle$
Malaysia	$\langle 49,4,4;k\rangle$
British Columbia, Colorado, Connecticut, Delaware, France, Georgia, Germany, Greece, Idaho, Iowa, Kansas, Kentucky, Louisiana, Malaysia, Maryland, Massachusetts, Minnesota, Missouri, Montana, New Hampshire, New Jersey, New Mexico, Ohio, Phillipines, Poland, Québec, Rhode Island, Slovak Republic, South Africa, South Dakota, Spain, Turkey, United Kingdom, Virginia, Washington, Western Canada, Wisconsin	$\langle 49,6,6;k\rangle$
Georgia, Illinois, Maryland, Massachusetts, Michigan, New Jersey, Virginia	$\langle 50,6,6;k\rangle$
Michigan	$\langle 51,6,6;k\rangle$
Illinois	$\langle 52,6,6;k\rangle$
Florida	$\langle 53,6,6;k\rangle$
India, Texas	$\langle 54,6,6;k\rangle$
Delaware, Minnesota, Montana, Nebraska, New Hampshire, South Dakota, West Virginia	$\langle 55,5,5;k\rangle$

Table 1.1: Lotteries operating in 2002, gathered from an extensive Internet survey [16].

mind, the player may well wonder what the cardinality of such a smallest feasible set $\mathcal{N}$ might be and how to go about constructing such a set of smallest cardinality. To make the above description precise, let $\Phi(\mathcal{A}, s)$ denote the set of all (unordered) s–sets from a set $\mathcal{A}$, so that $|\Phi(\mathcal{A}, s)| = \binom{|\mathcal{A}|}{s}$. We consider the following problem.

Definition 1 (The lottery problem) *Define a* lottery set *for* $\langle m, n, t; k\rangle$ *as a subset* $\mathcal{L}(\mathcal{U}_m, n, t; k) \subseteq \Phi(\mathcal{U}_m, n)$ *with the property that, for any element* $\phi_t \in \Phi(\mathcal{U}_m, t)$*, there exists an element* $l \in \mathcal{L}(\mathcal{U}_m, n, t; k)$ *such that* $\Phi(\phi_t, k) \cap \Phi(l, k) \neq \emptyset$*. Then the* lottery problem *is: what is the smallest possible cardinality of a lottery set* $\mathcal{L}(\mathcal{U}_m, n, t; k)$*? Denote the answer to this problem by the* lottery number $L(m, n, t; k)$*. We refer to a lottery set of cardinality* $L(m, n, t; k)$ *as an* $L(m, n, t; k)$*–set for* $\langle m, n, t; k\rangle$*.* ■

In a world wide survey of 147 national and US state lotteries in operation during 2002 (summarised in Table 1.1), *all* of the schemes took $t = n$, if one ignores the practice of allowing so–called "bonus numbers." (A bonus number is an additional number drawn at random by the governing body of a lottery, but which induces a different price structure in terms of realised monetory winnings when k–overlaps occur between playing sets and the t numbers drawn by the governing body, involving the bonus number. Intuitively speaking, the bonus number is "weaker" than the t winning numbers chosen by the governing body from $\mathcal{U}_m$.) However, practical situations sometimes occur in which one is interested in the value of a lottery number, where $t \neq n$. For example, in the construction of a so called *wheel* or *reduced system*, the player predicts that the winning n–set for the lottery scheme $\langle m, n, n; k\rangle$ will come from some subset (consisting of m' numbers) of $\mathcal{U}_m$. The player might wish to construct a lottery set so that a k–intersection is guaranteed if t $(\leq n)$ of the winning numbers belong to this subset. This is equivalent to constructing a lottery set for a scheme of the form $\langle m', n, t; k\rangle$, where $m \geq m' \geq n \geq t \geq k$. There are several Internet websites that list such reduced system constructions (see, for example, [5, 30, 31, 33, 44]). In this paper we sometimes confine our attention to the sub–class of lottery problems where $(m \geq)$ $n \geq t$ $(\geq k)$, but we show later that this is, in fact, without loss of generality.

In the case where $t = k$, the lottery problem is known as the *covering problem*, where the objective is to determine a *covering set* $\mathcal{C}(\mathcal{U}_m, n; k)$ of minimal cardinality $C(m, n; k)$ consisting of n–sets from $\mathcal{U}_m$ so that *every* k–subset from $\mathcal{U}_m$ is contained in at least one element of $\mathcal{C}(\mathcal{U}_m, n; k)$. That is, for any $\phi_k \in \Phi(\mathcal{U}_m, k)$, there exists an element $c \in \mathcal{C}(\mathcal{U}_m, n; k)$ such that $\phi_k \cap \Phi(c, k) \neq \emptyset$. This problem is a well–studied topic in the combinatorial literature [13, 15, 19, 21, 23, 24, 25, 29, 34, 35, 36, 37, 38, 39, 48, 53]. Note that, although in both the lottery and covering problems minimum cardinality sets consisting of n–nets from $\mathcal{U}_m$ are sought, the difference is that in the covering problem we wish to have "covered" *all* k–subsets from $\mathcal{U}_m$, whilst in the lottery problem we require something weaker: we need only cover *at least one* k–subset in the winning n–set $w \in \Phi(\mathcal{U}_m, n)$, no matter what w might be. Therefore the covering number $C(m, n; k)$ is an upper bound on the lottery number $L(m, n, t; k)$, as will be shown later.

On the other hand, if $n = k$ then the lottery problem is known as the *Turán problem* [17, 22, 50], which has also received significant interest in the combinatorial literature since its inception in the 1940s. The Turán number, $T(m, n; t)$, is defined, for all $n \leq t$, as the minimal number of n–subsets from $\mathcal{U}_m$ with the property that each t–subset of $\mathcal{U}_m$ contains at least one of the n–sets as subset. The Turán number is related to the covering number by means of the identity $T(m, n; t) = C(m, m - n; m - t)$.

To the best knowledge of the authors the only documented complete classes of lottery numbers are the class $L(m, n, n; 1) = \lceil (m - n + 1)/n \rceil$ (proved by Bate in his PhD dissertation [3]) and the class of lottery numbers $L(m, 3, 3; 2)$ (derived first by Bate [3] and later independently by Brouwer & Voorhoeve [7], as summarised in Theorem 1).

Theorem 1 (The class of lottery numbers $L(m, 3, 3; 2)$) *For all $l \geq 1$,*
(a) $L(2l + 1, 3, 3; 2) = \underline{C}(l, 3; 2) + \underline{C}(l + 1, 3; 2)$,
(b) $L(4l, 3, 3; 2) = \underline{C}(2l - 1, 3; 2) + \underline{C}(2l + 1, 3; 2)$, *and*
(c) $L(4l + 2, 3, 3; 2) = 2\underline{C}(2l + 1, 3; 2)$. ■

Here the notation $\underline{C}(m, n; k)$ denotes the well–known *Schönheim covering bound* [46],

$$C(m, n; k) \geq \underline{C}(m, n; k) := \left\lceil \frac{m}{n} \left\lceil \frac{m-1}{n-1} \cdots \left\lceil \frac{m-k+1}{n-k+1} \right\rceil \cdots \right\rceil \right\rceil. \tag{1.1}$$

Small values of lottery numbers are also known within other classes, such as the numbers shown in Table 1.2, which are due to Bate & Van Rees [4]. Furthermore, listings of best known upper and lower bounds on yet undetermined values of $L(m, n, t; k)$ for all allowable combinations of

m	6–10	11–15	16–20	21–25	26–30	31–33	34	35–36	37	38–39		
$L(m,6,6;2)$	1	2	3	4	5	7	8	9	10	11		
m	40	41–42	43	44–45	46	47	48	49–50	51	52	53	54
$L(m,6,6;2)$	12	13	14	15	16	17	18	19	20	21	22	23

Table 1.2: The lottery numbers $L(m,6,6;2)$ for $6 \leq m \leq 54$, due to Bate & Van Rees [4].

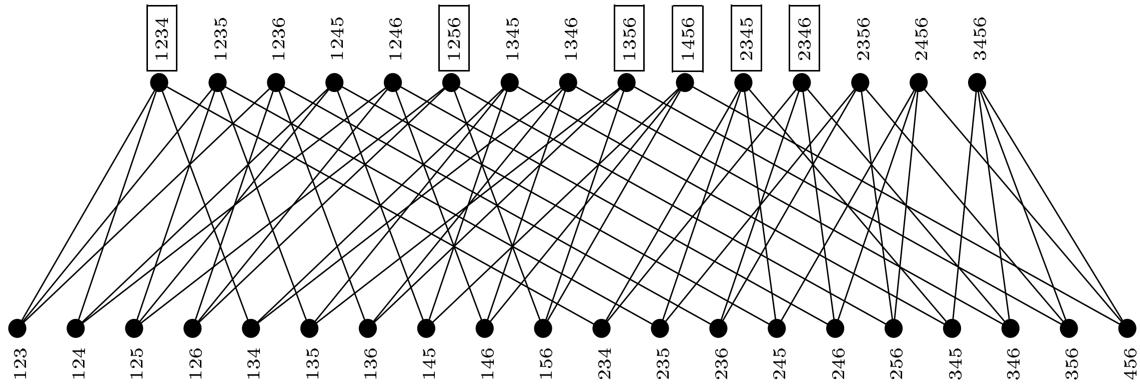


Figure 1.1: The lottery graph $G\langle 6,4,3;3\rangle$. A bipartite dominating set is indicated by blocked vertex labels, thereby establishing the upper bound $L(6,4,3;3) \leq 6$.

the parameters $m \leq 20$, $n \leq 12$, $t \leq 10$ and $k \leq 4$ also appear in [26, 28]. Finally, the lottery equivalence

$$\langle m,n,t;k\rangle \equiv \langle m,m-n,m-t;m+k-n-t\rangle \tag{1.2}$$

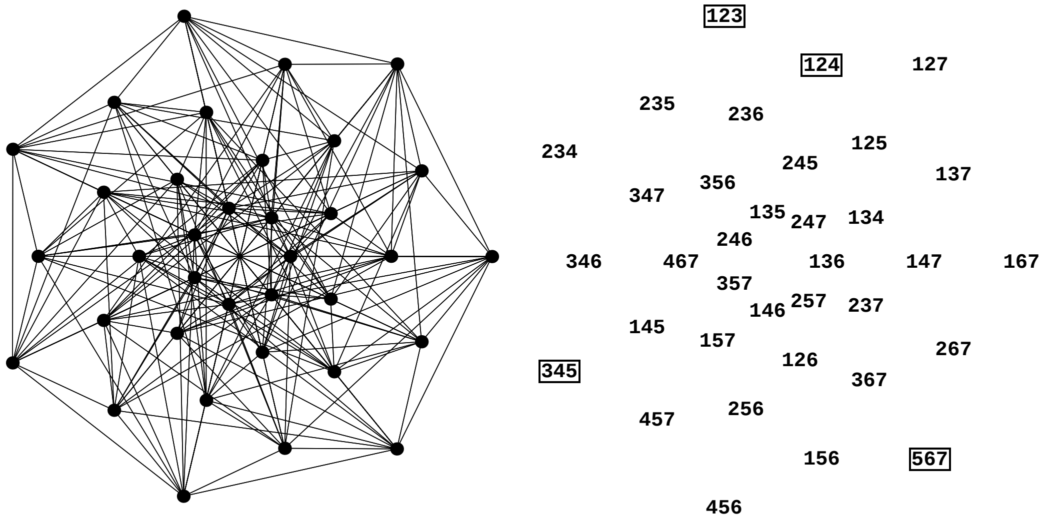
is also known for all $1 \leq k \leq \{n,t\} \leq m$ satisfying $m+k > n+t$ [3].

The lottery, covering and Turán problems may also be stated as graph theoretic problems. Let $V(G)$ denote the vertex set of a graph G. Then a set $\mathcal{D} \subseteq V(G)$ is called a dominating set of G if each vertex of G which is not in $\mathcal{D}$, is adjacent to a vertex in $\mathcal{D}$. A dominating set $\mathcal{D}$ is *minimal dominating* if no proper subset of $\mathcal{D}$ is a dominating set of G. The minimum cardinality of a minimal dominating set of G is denoted by $\gamma(G)$ (called the *lower domination number* of G).

Consider a bipartite graph $G\langle m,n,t;k\rangle$ whose vertex set $V(G\langle m,n,t;k\rangle)$ may be partitioned as $\{\Phi(\mathcal{U}_m,t),\Phi(\mathcal{U}_m,n)\}$, where a vertex $v_t \in \Phi(\mathcal{U}_m,t)$ is joined to a vertex $v_n \in \Phi(\mathcal{U}_m,n)$ if $\Phi(v_t,k)\cap\Phi(v_n,k) \neq \emptyset$. We call such a graph the *lottery graph*. Then the lottery problem translates to finding a special type of minimal dominating set $\mathcal{D}$ in $G\langle m,n,t;k\rangle$, with the properties that $\mathcal{D} \subseteq \Phi(\mathcal{U}_m,n)$ and that $\mathcal{D}$ dominates every vertex in $\Phi(\mathcal{U}_m,t)$. This kind of domination is known as *bipartite graph domination* [20]. We illustrate the graph theoretic nature of the lottery and covering problems by means of a small example.

Example 1 *Consider the lottery scheme $\langle 6,4,3;3\rangle$. The order of the corresponding lottery graph $G\langle 6,4,3;3\rangle$ is $|\Phi(\mathcal{U}_6,4)|+|\Phi(\mathcal{U}_6,3)| = \binom{6}{4}+\binom{6}{3} = 15+20 = 35$ and it is (4,3) bi–regular, as may be seen in Figure 1.1. From the figure it is clear that $\mathcal{L}(\mathcal{U}_6,4,3;3) = \{\{1,2,3,4\},\{1,2,5,6\},\{1,3,5,6\},\{1,4,5,6\},\{2,3,4,5\},\{2,3,4,6\}\} \subseteq \Phi(\mathcal{U}_6,4)$ dominates the set $\Phi(\mathcal{U}_6,3)$, so that $\mathcal{L}(\mathcal{U}_6,4,3;3)$ is both a lottery and a covering set for $\langle 6,4,3;3\rangle$. From this we conclude that $L(6,4,3;3) = C(6,4;3) \leq 6$. Although it is not easy to see, this minimal dominating set is in fact one of minimum cardinality and consequently $L(6,4,3;3) = C(6,4;3) = 6$.* ■

In the special case where $t = n$, the lottery graph may be simplified considerably by defining its vertex set as $V(G\langle m,n,n;k\rangle) = \Phi(\mathcal{U}_m,n)$, and joining two vertices u and v if and only if $\Phi(u,k)\cap\Phi(v,k) \neq \emptyset$. Then the lottery problem translates to finding the lower domination number $\gamma(G\langle m,n,n;k\rangle)$ of the lottery graph $G\langle m,n,n;k\rangle$ in the classical sense, as illustrated in the next example.



(a) Lottery graph $G\langle 7, 3, 3; 2\rangle$ (b) Vertex labels for $G\langle 7, 3, 3; 2\rangle$

Figure 1.2: A small lottery graph (special case where $t = n$). A dominating set is indicated by blocked vertex labels, thereby establishing the upper bound $L(7, 3, 3; 2) \leq 4$.

Example 2 *Consider the lottery scheme $\langle 7, 3, 3; 2\rangle$. The order of the corresponding lottery graph is $|\Phi(\mathcal{U}_7, 3)| = \binom{7}{3} = 35$ and it is 10–regular, as may be seen in Figure 1.2. From the figure it is clear that $\mathcal{L}(\mathcal{U}_7, 3, 3; 2) = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}\}$ is a dominating set for the graph. In fact, this set is a so–called Steiner system [52], because each of the $\binom{35}{2} = 21$ two–sets from $\mathcal{U}_7$ appear exactly once in some element of $\mathcal{L}(\mathcal{U}_7, 3, 3; 2)$. Hence it is known that $C(7, 3; 2) = 7$. However, the much smaller set $\mathcal{L}'(\mathcal{U}_7, 3, 3; 2) = \{\{1, 2, 3\}, \{5, 6, 7\}, \{3, 4, 5\}, \{1, 2, 4\}\}$ is also a (minimal) dominating set for the lottery graph $G\langle 7, 3, 3; 2\rangle$. We conclude that $L(7, 3, 3; 2) \leq 4$. It is not easy to see, but this minimal dominating set in fact has minimum cardinality and hence $L(7, 3, 3; 2) = 4 < 7 = C(7, 3; 2)$.* ■

Since the lottery problem is a combinatorial minimisation problem, it is natural to attempt an integer programming approach toward finding lottery numbers. However, such an approach is known to be very inefficient [16]. Other algorithmic approaches towards establishing bounds for lottery numbers include the use of greedy algorithms, simulated annealing, genetic algorithms and exhaustive backtracking searches [28].

Our approach in this paper will be to utilise results from graph domination theory by virtue of the introduction of a so–called *lottery graph* instead, in order to establish general, closed–form analytic bounds for lottery numbers of the form $L(m, n, n; k)$, where $1 \leq k \leq n \leq m$. In §3 we prove certain properties of the lottery graph, and in §4 we summarise both upper and lower bounds from the considerable body of combinatorial literature on the lottery problem, as well as bounds emanating from our graph theoretic approach towards the problem. An exhaustive search procedure for characterising all possible overlapping structures of $L(m, n, t; k)$–sets is developed in §5, and this technique is then used in §6 to establish 19 new lottery numbers and to improve upon best known bounds for a further 20 lottery numbers. However, we start, in §2, by establishing the boundedness (and hence existence) of solutions to the lottery problem, and then proceed to find values for certain basic lottery numbers.

2 Existence & basic properties of numbers

The existence of solutions to the lottery problem is settled by the following theorem for all feasible values of m, n, t and k. The theorem asserts that the task of finding bounds for the lottery number $L(m,n,t;k)$ may be embedded in that of finding bounds for the covering number $C(m,n;k)$.

Theorem 2 (Existence of lottery numbers)
The numbers $L(m,n,t;k)$ and $C(m,n;k)$ exist for all $1 \leq k \leq \{n,t\} \leq m$ and, in fact,

$$\left\lfloor \frac{m}{\max\{n,t\}} \right\rfloor \leq L(m,n,t;k) \leq C(m,n;k) \leq \binom{m}{k}. \tag{2.1}$$

Proof: Let $s = \max\{n,t\}$ and suppose $\mathcal{L}$ is any lottery set of cardinality $L(m,n,t;k) < \lfloor m/s \rfloor$ for $\langle m,n,t;k \rangle$. Then the elements of $\mathcal{L}$ collectively contain at most $n(\lfloor m/s \rfloor - 1) \leq s(\lfloor m/s \rfloor - 1) \leq s(m/s-1) = m-s$ distinct elements of $\mathcal{U}_m$. Hence there exists a subset $\mathcal{U}'_m \subset \mathcal{U}_m$ of cardinality at least s, whose elements do not appear in $\mathcal{L}$. Since $t \leq s$, it therefore follows that there exists a $w \in \Phi(\mathcal{U}'_m, t)$ such that $\Phi(\mathcal{L},k) \cap \Phi(w,k) = \emptyset$, contradicting the fact that $\mathcal{L}$ is a lottery set for $\langle m,n,t;k \rangle$ and thereby establishing the first inequality in (2.1).

Let $\mathcal{L}$ be any lottery set of cardinality $L(m,n,k;k)$ for $\langle m,n,k;k \rangle$. Then, for every element of $\Phi(\mathcal{U}_m,t)$, there exist, as subsets, $\binom{t}{k}$ elements of $\Phi(\mathcal{U}_m,k)$. But for each of these elements $\phi_k \in \Phi(\mathcal{U}_m,k)$ there exists an $\ell \in \mathcal{L}$ such that $\phi_k \cap \Phi(\ell,k) \neq \emptyset$. Consequently, for every $\phi_t \in \Phi(\mathcal{U}_m,t)$ there exists at least one (but probably more than one) $\ell \in \mathcal{L}$ such that $\Phi(\phi_t,k) \cap \Phi(\ell,k) \neq \emptyset$. We conclude that $L(m,n,t;k) \leq L(m,n,k;k) = C(m,n;k)$ thereby establishing the second inequality in (2.1).

The upper bound in (2.1) follows since there are $\binom{m}{k}$ different k–sets in the set $\mathcal{U}_m$. By adding $n-k$ elements of the remaining $m-k$ elements of $\mathcal{U}_m$ to each of these k–sets in an arbitrary fashion, a covering set is obtained for $\langle m,n,k;k \rangle$. Hence $C(m,n;k) \leq \binom{m}{k}$. ■

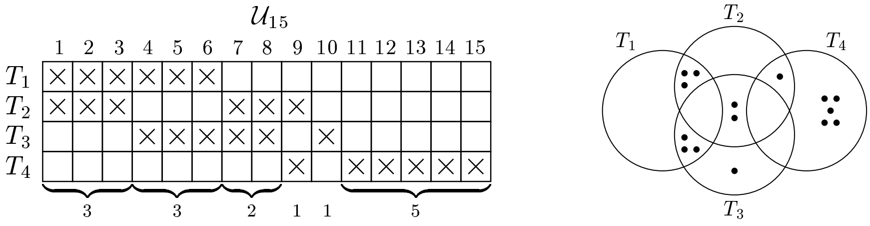
Note that while the upper bound in (2.1) is sharp for the special case $n = t = k$, this bound becomes dramatically weaker as the difference between n, t and k increases. We present improvements on the upper bound in §4.

It is possible to establish the following growth properties of the lottery number $L(m,n,t;k)$ as a function of its arguments. We obtained proofs of the results in the following theorem independently of, but after Li [27]; hence we omit the proofs.

Theorem 3 (Growth properties of lottery numbers)
(a) $L(m,n,t;k) \leq L(m',n,t;k)$ for all $1 \leq k \leq t \leq n \leq m \leq m'$.
(b) $L(m,n,t;k) \geq L(m,n',t;k)$ for all $1 \leq k \leq t \leq n \leq n' \leq m$.
(c) $L(m,n,t;k) \geq L(m,n,t';k)$ for all $1 \leq k \leq t \leq t' \leq n \leq m$.
(d) $L(m,n,t;k) \leq L(m,n,t;k')$ for all $1 \leq k \leq k' \leq t \leq n \leq m$. ■

The results of the above theorem may be summarised in a visually appealing way as $L(m\!\uparrow,n,t;k)\!\uparrow$, $L(m,n\!\uparrow,t;k)\!\downarrow$, $L(m,n,t\!\uparrow;k)\!\downarrow$ and $L(m,n,t;k\!\uparrow)\!\uparrow$. Of course the results $L(m\!\uparrow,n,t;k\!\uparrow)\!\uparrow$ and $L(m,n\!\uparrow,t\!\uparrow;k)\!\downarrow$ follow immediately from Theorem 3. Furthermore, in 1999 Li [27] was able to prove the following eight growth combinations: $L(m\!\uparrow,n\!\uparrow,t;k)\!\downarrow$, $L(m\!\uparrow,n,t\!\uparrow;k)\!\downarrow$, $L(m,n\!\uparrow,t;k\!\uparrow)\!\uparrow$, $L(m,n,t\!\uparrow;k\!\uparrow)\!\uparrow$, $L(m\!\uparrow,n\!\uparrow,t\!\uparrow;k)\!\downarrow$, $L(m\!\uparrow,n\!\uparrow,t;k\!\uparrow)\!\uparrow$, $L(m\!\uparrow,n,t\!\uparrow;k\!\uparrow)\!\uparrow$ and $L(m\!\uparrow,n\!\uparrow,t\!\uparrow;k\!\uparrow)\!\uparrow$. It is interesting to note that for some values of parameters m, n, t and k the trend $L(m,n\!\uparrow,t\!\uparrow;k\!\uparrow)\!\uparrow$ is observed, while for other values of these parameters the opposite is observed. The precise nature of the set(s) of separating growth parameters (m fixed and $n\!\uparrow$, $t\!\uparrow$ & $k\!\uparrow$) are not known [28].

It is possible to characterise small lottery numbers, and to determine explicitly certain basic lottery numbers, as we do in the following theorem. However, to do this we need a coding scheme whereby



(a) (b)

Figure 2.1: (a) Graphical representation of the set $\mathcal{L}$ in Example 3. (b) Overlapping 6–set structure of the set $\mathcal{L}$ in Example 3.

to capture the structure of an $L(m,n,t;k)$–set, $\mathcal{L} = \{T_1, T_2, \ldots, T_L(m,n,t;k)\}$. This may be achieved by defining the function

$$x^{(L)}_{(t_L t_{L-1} \cdots t_2 t_1)_2} = \left| \bigcap_{i=1}^{L} \left\{ \begin{array}{ll} T_i & \text{if } t_i = 1 \\ T_i' & \text{if } t_i = 0 \end{array} \right. \right|,$$

where $(t_L t_{L-1} \cdots t_2 t_1)_2$ denotes the binary representation of an integer in the range $\{0, \ldots, 2^L - 1\}$ and where T_i' denotes the complement $\mathcal{U}_m \backslash T_i$. This function induces the 2^L–integer vector

$$\vec{X}^{(L)} = \left(x^{(L)}_{(000\cdots 00)_2}, x^{(L)}_{(000\cdots 01)_2}, \ldots, x^{(L)}_{(111\cdots 11)_2} \right),$$

which represents all the information needed to describe the n–set overlapping structure of any lottery set of cardinality L for $\langle m,n,t;k \rangle$. The entries of the vector $\vec{X}^{(L)}$ add up to m and may be interpreted as follows:

- there are $x^{(L)}_{(000\cdots 00)_2}$ elements of $\mathcal{U}_m$ contained in no n–set of $\mathcal{L}$,
- there are $x^{(L)}_{(000\cdots 01)_2}$ elements of $\mathcal{U}_m$ contained in only the n–set T_1 of $\mathcal{L}$,
- there are $x^{(L)}_{(000\cdots 10)_2}$ elements of $\mathcal{U}_m$ contained in only the n–set T_2 of $\mathcal{L}$,
- there are $x^{(L)}_{(000\cdots 11)_2}$ elements of $\mathcal{U}_m$ contained in both the n–sets T_1 and T_2 of $\mathcal{L}$, *etc.*

Sometimes it is more convenient to write the subscripts of entries in the vector $\vec{X}^{(L)}$ in decimal form. We illustrate the above method of set structure encoding by means of an example.

Example 3 *The set* $\mathcal{L} = \{T_1, T_2, T_3, T_4\}$ *is an* $L(15,6,6;3)$*–set for* $\langle 15,6,6;3 \rangle$*, where* $T_1 = \{1,2,3,4,5,6\}$*,* $T_2 = \{1,2,3,7,8,9\}$*,* $T_3 = \{4,5,6,7,8,10\}$ *and* $T_4 = \{9,11,12,13,14,15\}$*. Figure 2.1(a) shows the set* $\mathcal{L}$ *in tabular form, while Figure 2.1(b) captures the overlapping* 6*–set structure of* $\mathcal{L}$ *graphically. It is easy to see, from Figure 2.1(b), that*

$$x^{(4)}_{(0011)_2} = x^{(4)}_3 = 3, \quad x^{(4)}_{(0101)_2} = x^{(4)}_5 = 3, \quad x^{(4)}_{(0110)_2} = x^{(4)}_6 = 2,$$

$$x^{(4)}_{(1010)_2} = x^{(4)}_{10} = 1, \quad x^{(4)}_{(0100)_2} = x^{(4)}_4 = 1 \text{ and } x^{(4)}_{(1000)_2} = x^{(4)}_8 = 5,$$

while $x^{(4)}_{(t_4 t_3 t_2 t_1)_2} = 0$ *for all other combinations of the bits* t_1*,* t_2*,* t_3 *and* t_4*. Therefore* $\vec{X}^{(4)} = (0,0,0,3,1,3,2,0,5,0,1,0,0,0,0,0)$ *captures the* 6*–set overlapping structure in Figure 2.1(b). Note that* $\vec{X}^{(4)}$ *is not unique for a given overlapping set structure: if we choose the names* T_1*,* T_2*,* T_3 *and* T_4 *in a different order, a different vector* $\vec{X}^{(4)}$ *results. In fact, there are* $15!/(3!3!2!5!) = 151\,351\,200$ *different ways to form a table such as shown in Figure 2.1(a) from the overlapping* 6*–set structure shown in Figure 2.1(b), if the order of the* 6*–set listing is not altered. To pinpoint a unique* $\vec{X}^{(4)}$ *representation for the given lottery set structure, we may consider all permutations of* T_1*,* T_2*,* T_3 *and* T_4*, as well as* $\{1, \ldots, 15\}$*, and choose, for example, the lexicographic first one.* ■

Theorem 4 (Basic lottery numbers) *For all* $1 \leq k \leq \{n,t\} \leq m$,
(a) $L(m,n,t;k) = 1$ *if and only if* $n + t \geq m + k$.
(b) $L(m,n,t;k) = 2$ *if and only if* $2k - 1 + \max\{m - 2n, 0\} \leq t \leq m + k - n - 1$.
(c) $L(m,n,t;k) = 3$ *if and only if*

$$t \leq \min\{2k - 2 + \max\{m - 2n, 0\}, m - n + k - 1\} \tag{2.2}$$

and

$$t \geq \left\{ \begin{array}{ll} 3k - 2 + \max\{m - 3n, 0\} & \text{if } m \geq 2n \\ \frac{3}{2}k - 1 + \max\{m - \frac{3}{2}n, 0\} & \text{if } m < 2n. \end{array} \right. \tag{2.3}$$

(d) $L(m,n,n;1) = \lfloor m/n \rfloor$.
(e) $L(m,n,1;1) = C(m,n;1) = \lceil m/n \rceil$.
(f) $L(m,n,n;n) = C(m,n;n) = \binom{m}{n}$.

The proof of Theorem 4(b) is similar to, but much simpler than that of Theorem 4(c). Hence we omit the proof of Theorem 4(b). Furthermore, since the result of Theorem 4(f) holds trivially, we only prove parts (a), (c), (d) and (e).

Proof: (a) If $n + t \geq m + k$, then $\Phi(u,k) \cap \Phi(v,k) \neq \emptyset$ for any $u \in \Phi(\mathcal{U}_m, n)$ and $v \in \Phi(\mathcal{U}_m, t)$. Therefore the lottery graph $G\langle m,n,t;k\rangle$ is a complete bipartite graph and hence $L(m,n,t;k) = 1$. The converse is proved by means of a contra–positive argument. Consider an arbitrary playing set $\mathcal{L}^{(1)} = \{T^{(1)}\}$ for $\langle m,n,t;k\rangle$ of cardinality 1, but suppose that $n+t < m+k$. Then $t \leq m-n+k-1$, and hence a winning n–set, w, for $\langle m,n,t;k\rangle$ may be chosen, consisting of $k-1$ elements from $\mathcal{L}$ and $m-n$ elements from $\mathcal{U}_m \backslash \mathcal{L}$, in which case $\Phi(T^{(1)},k) \cap \Phi(w,k) = \emptyset$, showing that $L(m,n,t;k) \neq 1$.

(c) It follows, by parts (a) and (b) of this theorem, that $L(m,n,t;k) \neq 1,2$ if and only if $t \leq m-n+k-1$ and $t \leq 2k-2+\max\{m-2n,0\}$. Therefore

$$L(m,n,t;k) > 2 \text{ if and only if } t \leq \min\{2k-2+\max\{m-2n,0\}, m-n+k-1\}. \tag{2.4}$$

We first prove the theorem for the case $m \geq 2n$. Suppose that (2.2) and the first inequality in (2.3) holds. Then it follows, by (2.4), that $L(m,n,t;k) > 2$. We now show that $L(m,n,t;k) \leq 3$. Construct a playing set $\mathcal{L}^{(2)} = \{T_1^{(2)}, T_2^{(2)}, T_3^{(2)}\}$ of cardinality 3 for $\langle m,n,t;k\rangle$, for which $x^{(3)}_{(000)_2}$ is a minimum. Then $x^{(3)}_{(000)_2} = \max\{m-3n,0\}$ and hence $\Phi(T_i^{(2)},k) \cap \Phi(w,k) \neq \emptyset$ for at least one $i \in \{1,2,3\}$, where w is an arbitrary winning set for $\langle m,n,t;k\rangle$, since $t > \max\{m-3n,0\}+3(k-1)$. Therefore $\mathcal{L}^{(2)}$ is a lottery set for $\langle m,n,t;k\rangle$, and we conclude that $L(m,n,t;k) = 3$.

Conversely, suppose $L(m,n,t;k) = 3$. Then (2.2) follows from (2.4). We show that $t \geq 3k-2+\max\{m-3n,0\}$ by proving that, for any $L(m,n,t;k)$–set,

$$\max\{m-3n,0\} \leq x^{(3)}_{(000)_2} \leq t-3k+2 \tag{2.5}$$

The first inequality in (2.5) is obvious. To prove the second inequality in (2.5), suppose, to the contrary, that $x^{(3)}_{(000)_2} > t-3k+2$ for some $L(m,n,t;k)$–set $\mathcal{L}^{(3)} = \{T_1^{(3)}, T_2^{(3)}, T_3^{(3)}\}$ for $\langle m,n,t;k\rangle$. It is easy to see that (as long as $x^{(3)}_{(100)_2}, x^{(3)}_{(010)_2}, x^{(3)}_{(001)_2} \geq k-1$) the larger $x^{(3)}_{(000)_2}$, the smaller $\max_{i\in\{1,2,3\}}\{T_i^{(3)} \cap w\}$, for any winning t–set, w. We therefore only have to consider the case $x^{(3)}_{(000)_2} = t-3k+3$. In this case, if the winning t–set, w, consists of $t-3k+3$ elements from $\mathcal{U}_m \backslash (T_1^{(3)} \cup T_2^{(3)} \cup T_3^{(3)})$, and $k-1$ elements from each of $T_1^{(3)} \backslash (T_2^{(3)} \cup T_3^{(3)})$, $T_2^{(3)} \backslash (T_1^{(3)} \cup T_3^{(3)})$ and $T_3^{(3)} \backslash (T_1^{(3)} \cup T_2^{(3)})$, it follows that $\Phi(T_i^{(3)},k) \cap \Phi(w,k) = \emptyset$ for all $i \in \{1,2,3\}$, contradicting the fact that $\mathcal{L}^{(3)}$ is a lottery set for $\langle m,n,t;k\rangle$. Note that this case is indeed possible, since $x^{(3)}_{(000)_2}, x^{(3)}_{(100)_2}, x^{(3)}_{(010)_2}, x^{(3)}_{(001)_2} \geq k-1$, due to the fact that $m \geq 2n+t-2k+2 = 2n+(k-1)+t-3k+3$, thereby establishing (2.5), from whence it follows that $t \geq 3k-2+\max\{m-3n, 0\}$. This completes the proof for the case $m \geq 2n$.

For the case $m < 2n$ we consider the complementary lottery problem $\langle m', n', t'; k'\rangle \equiv \langle m, m-n, m-t; m+k-n-t\rangle$ by virtue of (1.2). We then have $m' > 2n'$, which is the first case, proved above. Therefore $L(m', n', t'; k') = 3$ if and only if

$$t' \leq \min\{2k'-2+\max\{m'-2n', 0\}, m'-n'+k'-1\} \tag{2.6}$$

and

$$t' \geq 3k'-2+\max\{m'-3n', 0\}. \tag{2.7}$$

We only have to show that (2.7) is equivalent to the second inequality in (2.3). From (2.7) it follows that

$$\begin{aligned} m-t &\geq 3(m+k-n-t)-2+\max\{m-3(m-n), 0\} \\ \Leftrightarrow \quad 2t &\geq 3k-2+2m-3n+\max\{3n-2m, 0\} \\ \Leftrightarrow \quad 2t &\geq 3k-2+\max\{2m-3n, 0\}, \end{aligned}$$

which is equivalent to the second inequality in (2.3).

(d) Consider the set

$$\mathcal{L} = \left\{\{(k-1)n+1, (k-1)n+2, \ldots, kn\}_{k=1}^{\lfloor m/n\rfloor}\right\} \tag{2.8}$$

of n–sets from $\mathcal{U}_m$. Suppose that $m = in+j$ for some integers $i \geq 0$ and $0 \leq j < n$. If $j = 0$, then $\mathcal{L}$ is merely a partition of $\mathcal{U}_m$ into m/n n–sets and hence clearly a lottery set for $\langle m, n, n; 1\rangle$. On the other hand, if $j > 0$, then the elements of $\mathcal{L}$ collectively contain $n(\lfloor m/n\rfloor) > m-n$ elements of $\mathcal{U}_m$, in which case there exists, for any $\phi_n \in \Phi(\mathcal{U}_m, n)$, an $\ell \in \mathcal{L}$ such that $\Phi(\phi_n, 1) \cap \Phi(\ell, 1) \neq \emptyset$. In this case, therefore also, $\mathcal{L}$ is a lottery set for $\langle m, n, n; 1\rangle$. Hence we conclude that $\lfloor m/n\rfloor \leq L(m, n, n; 1) \leq \lfloor m/n\rfloor$ by Theorem 2.

(e) Suppose $\mathcal{L}$ is a lottery set of cardinality less than $\lceil m/n\rceil$ for $\langle m, n, 1; 1\rangle$. We consider two cases. First suppose that $n|m$, then the elements of $\mathcal{L}$ collectively contain at most $n(\lceil m/n\rceil - 1) = n(m/n-1) = m-n < m$ distinct elements of $\mathcal{U}_m$. Next suppose that $m = in+j$ (for some integers $i \geq 0$ and $0 < j < n$), then the elements of $\mathcal{L}$ collectively contain at most $n(\lceil m/n\rceil - 1) = n(\lceil i+j/n\rceil - 1) = n(i+1-1) = in < m$ distinct elements of $\mathcal{U}_m$. In both cases, therefore, there exists an element $\phi_1^* \in \Phi(\mathcal{U}_m, 1)$ such that $\phi_1^* \cap \Phi(\ell, 1) = \emptyset$ for all $\ell \in \mathcal{L}$, contradicting the fact that $\mathcal{L}$ is a lottery set for $\langle m, n, 1; 1\rangle$. Hence $L(m, n, 1; 1) \geq \lceil m/n\rceil$ for all $1 \leq n \leq m$. To prove that this lower bound is indeed sharp, consider the subset

$$\mathcal{L}^* = \left\{\{(k-1)n+1, (k-1)n+2, \ldots, kn\}_{k=1}^{\lceil m/n\rceil} \pmod{m}\right\}$$

of $\Phi(\mathcal{U}_m, n)$[1]. Since $n\lceil m/n\rceil \geq m$, there exists, for any $\phi_1 \in \Phi(\mathcal{U}_m, 1)$ with $k \geq 0$ and $0 \leq j < n$, an element $\ell \in \mathcal{L}^*$ such that $\phi_1 \cap \Phi(\ell, 1) \neq \emptyset$. Hence $\mathcal{L}^*$ is a lottery set of cardinality $\lceil m/n\rceil$ for $\langle m, n, 1; 1\rangle$, so that $L(m, n, 1; 1) \leq \lceil m/n\rceil$. ■

3 The lottery graph

In this section we explore certain properties of the bipartite lottery graph $G\langle m, n, t; k\rangle$. It is possible to prove the following property of the lottery graph $G\langle m, n, t; k\rangle$.

[1] Here modular arithmetic is assumed to occur over the set $\{1, 2, \ldots, m\}$ instead of over the usual set $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$.

Theorem 5 (Bi–regularity of the lotery graph $G\langle m,n,t;k\rangle$)
The lottery graph $G\langle m,n,t;k\rangle$ is bi–regular. Moreover, the degree of each vertex in $\Phi(\mathcal{U}_m,t) \subseteq V(G\langle m,n,t;k\rangle)$ is d_t, and the degree of each vertex in $\Phi(\mathcal{U}_m,n) \subseteq V(G\langle m,n,t;k\rangle)$ is d_n, where

$$d_t = \sum_{i=k}^{t}\binom{t}{i}\binom{m-t}{n-i} \quad \text{and} \quad d_n = \sum_{i=k}^{t}\binom{n}{i}\binom{m-n}{t-i} \tag{3.1}$$

for any $1 \leq k \leq t \leq n \leq m$.

Proof: Consider any vertex $v_t \in \Phi(\mathcal{U}_m,t)$ of the lottery graph $G\langle m,n,t;k\rangle$. The number of vertices $v_n \in \Phi(\mathcal{U}_m,n)$ whose labels have exactly i elements of $\mathcal{U}_m$ in common with the label of v_t, is $\binom{t}{i}\binom{m-t}{n-i}$. In $G\langle m,n,t;k\rangle$ two vertices v_t and v_n are joined if the labels of v_t and v_n have at least $k \leq i \leq t-1$ elements of $\mathcal{U}_m$ in common. Hence the degree d_t of every vertex $v_t \in \Phi(\mathcal{U}_m,t)$ is as defined in (3.1). It can be shown in a similar way that the degree d_n of every vertex $v_n \in \Phi(\mathcal{U}_m,n)$ is as defined in (3.1). ∎

It turns out that the equivalence relationship (1.2) is very easy to prove from a graph theoretic point of view. However, to prove this we first need the following definition. Let $u \in \Phi(\mathcal{U}_m,n)$ be a vertex of the lottery graph $G\langle m,n,t;k\rangle$. Then the *n–complement of the label of* u is defined as the $(m-n)$–set formed by taking all the elements of $\mathcal{U}_m$ that do not appear in the label of u. The notion of a t–complement may be defined in a similar fashion.

Theorem 6 (Lottery graph isomorphism)
$G\langle m,n,t;k\rangle \simeq G\langle m,m-n,m-t;m+k-n-t\rangle$ for all $1 \leq k \leq t \leq n < m$ satisfying $m+k > n+t$.

Proof: Let u' be the n–complement of the label of $u \in \Phi(\mathcal{U}_m,n)$ in $G\langle m,n,t;k\rangle$ and let v' be the t–complement of the label of $v \in \Phi(\mathcal{U}_m,t)$ in $G\langle m,n,t;k\rangle$. We have to show that the labels of u and v have a k–subset in common if and only if the vertices in $G\langle m,m-n,m-t,m+k-n-t\rangle$ with labels u' and v' are adjecent, *i.e.* if and only if u' and v' have an $(m+k-n-t)$–subset in common. Suppose the labels of u and v have k (or more) elements of $\mathcal{U}_m$ in common. The elements of $\mathcal{U}_m$ that occur in both u' and v' are those that occur in neither of the labels of u nor of v. Therefore the number of elements common to u' and v' is m less the number of (different) elements common to the labels of u and v, which is given by $m-[k+(n-k)+(t-k)] = m+k-n-t$. A similar argument may be used to prove the converse. Hence two vertices in $G\langle m,n,t;k\rangle$ are adjacent if and only if their complementary vertices are adjacent in $G\langle m,m-n,m-t;m+k-n-t\rangle$. ∎

The following corollary is a direct consequence of (1.2) and Theorem 6, and enables us to focus our attention on only half of the parameter values m, n, t and k, the other half being accounted for by the corollary.

Corollary 1 (Lottery number equivalence)
$L(m,n,t;k) = L(m,m-n,m-t;m+k-n-t)$ for all $1 \leq k \leq t \leq n < m$ satisfying $m+k > n+t$.

Note that $t \leq n$ if and only if $m-n \leq m-t$. Therefore it follows from Theorem 6 that our assumption that $t \leq n$ in some of the results of this paper is, in fact, without loss of generality.

Finally, the lottery graph exhibits a rich structure and strong symmetry, which remains unfathomed. Various lottery graphs (or their complements) are also members of well–studied general classes of graphs; for example, the complement of the lottery graph $G\langle 5,2,2;1\rangle \simeq G\langle 5,3,3;2\rangle$ is the intriguing Peterson graph. It is our opinion that the lottery graph and its properties deserves greater attention than is afforded here.

4 Lotteries of the form $\langle m, n, n; k\rangle$

In this section we consider lotteries for which $t = n$. Therefore note that in this section we do not take the lottery graph to be bipartite as in the previous section, but instead we assume the simpler graph with vertex set $V(G\langle m, n, n; k\rangle) = \Phi(\mathcal{U}_m, n)$, where two vertices are adjacent if their labels have at least k elements of $\mathcal{U}_m$ in common, as described in §1. The following result is, in part, a direct consequence of Theorem 5 in the special case where $t = n$ and will be used in the following subsections to establish analytic bounds for lottery numbers of the form $L(m, n, n; k)$.

Corollary 2 (Properties of the lottery graph $G\langle m, n, n; k\rangle$)
(a) The lottery graph $G\langle m, n, n; k\rangle$ is r–regular, where

$$r = \sum_{i=k}^{n-1} \binom{n}{i}\binom{m-n}{n-i} \tag{4.1}$$

for any $1 \leq k \leq n \leq m$.
(b) For all $1 \leq k < n < m$,

$$\begin{aligned} \text{Radius}(G\langle m,n,n;k\rangle) &= \text{diameter}(G\langle m,n,n;k\rangle) \\ &= \begin{cases} \lceil n/(n-k)\rceil & \text{if} \quad m \geq 2n, \\ \lceil (m-n)/(n-k)\rceil & \text{if} \quad m < 2n. \end{cases} \end{aligned}$$

(c) $\text{Radius}(G\langle m,k,k;k\rangle) = \text{diameter}(G\langle m,k,k;k\rangle) = \infty$ *for any $1 \leq k \leq m$.*

Proof: (a) By Theorem 5.

(b) By the vertex transitivity of the lottery graph $G\langle m, n, n; k\rangle$, the eccentricity[2] of all vertices are equal, so that radius$(G\langle m, n, n; k\rangle)$ = diameter$(G\langle m, n, n; k\rangle)$. Furthermore, two vertices v_0 and v, whose labels have as few elements as possible of $\mathcal{U}_m$ in common, are furthest apart in $G\langle m, n, n; k\rangle$. Let the labels of v_0 and v have x elements of $\mathcal{U}_m$ in common. Then $x = 0$ if $m \geq 2n$, and $x = 2n - m$ if $m < 2n$. Construct a shortest path $\mathcal{P} : v_0, v_1, v_2, \ldots, v_{\ell-1}, v$ in the following way. For any v_i select a neighbouring vertex v_{i+1} in $G\langle m, n, n; k\rangle$ so that the labels of v_{i+1} and v_i have as many elements of $\mathcal{U}_m$ in common as possible. Because the labels of any two neighbouring vertices in $G\langle m, n, n; k\rangle$ have at most $n - k$ elements of $\mathcal{U}_m$ that may differ, it follows that the labels of v_i and v have exactly $x + i(n - k)$ elements of $\mathcal{U}_m$ in common for all $0 \leq i \leq \ell - 1$. Now the label of $v_{\ell-1}$ has at least k elements in common with that of v, so that $x + (\ell - 1)(n - k) \geq k$. This yields $\ell \geq \lceil n/(n - k)\rceil$ if $m \geq 2n$, and $\ell \geq \lceil (m - n)/(n - k)\rceil$ if $m < 2n$. Since ℓ is maximal, the desired result follows.

(c) In this case the lottery graph $G\langle m, k, k; k\rangle$ consists of $\binom{m}{k}$ isolated vertices. ∎

4.1 Analytic lower bounds from graph theory and other fields

Corollary 2 enables us to establish a closed form general lower bound for the lottery number $L(m, n, n; k)$. In any order p graph G with maximal degree $\Delta(G)$, it holds that $\gamma(G) \geq \lceil p/(\Delta(G)+1)\rceil$ (see, for example, [20]). Therefore

$$L(m,n,n;k) \geq \left\lceil \frac{\binom{m}{n}}{r+1} \right\rceil \tag{4.2}$$

[2]The eccentricity of a vertex v in a graph G is the distance from v to a vertex that is furthest from v in G.

for all $1 \leq k \leq n \leq m$, where r is the degree of regularity of the lottery graph $G\langle m, n, n; k\rangle$, as defined in (4.1). It is interesting to note that in 1993 Nurmela & Östergård [40] were able to prove that

$$L(m, n, n; k) \geq \frac{\binom{m}{n}}{\sum_{i=k}^{n} \binom{n}{i}\binom{m-n}{n-i}}, \tag{4.3}$$

using non–graph theoretic techniques, but their result is equivalent to (4.2), by Corollary 2(a).

Apart from this lower bound, many other (non–graph theoretic) lower bounds exist in the literature, and many of them seem to do better than the basic graph theoretic domination bound for realistic values of m, n and k. For example, Schrijver [47] reports the bounds

$$L(m, n, n; k) \geq \max_{n \leq a \leq m} \left\{ \left\lceil \frac{(a-n+1)\binom{m}{a}}{\sum_{i=k}^{n} \binom{n}{i}\binom{m-n}{a-i}(i-k+1)} \right\rceil \right\} \tag{4.4}$$

and

$$L(m, n, n; k) \geq \max_{n \leq a \leq m} \left\{ \left\lceil \frac{\left\lceil \frac{a-n+1}{n-k+1} \right\rceil \binom{m}{a}}{\sum_{i=k}^{n} \binom{n}{i}\binom{m-n}{a-i}} \right\rceil \right\} \tag{4.5}$$

due to Sterboul [49], while the bound

$$L(m, n, n; k) \geq \left\lceil \binom{m}{k} \Big/ \binom{n}{k}^2 \right\rceil \tag{4.6}$$

follows by a theorem stating that

$$L(m, n, n; k) \geq \frac{T(m, n; k)}{\binom{n}{k}}, \tag{4.7}$$

where $T(m, n; k)$ denotes the Turán number, as mentioned in the introduction. The bound (4.6) may be improved, via (4.7), by utilisation of the Turán bound

$$T(m, n; k) \geq \frac{\binom{m}{k}}{\binom{n-1}{k-1}} \frac{m-n+1}{m-k+1},$$

which is due to De Caen [12], yielding the lottery bound

$$L(m, n, n; k) \geq \frac{\binom{m}{k}}{\binom{n-1}{k-1}\binom{n}{k}} \frac{m-n+1}{m-k+1}. \tag{4.8}$$

However, it is anticipated that the lower bound (4.2) may perhaps be improved to the extent that it supercedes the bounds (4.4)–(4.8). We hint here at a possible way of achieving this. Recall the following well–known inclusion–exclusion principle.

Proposition 1 (Inclusion–exclusion principle)
Suppose $\mathcal{D}$ is any subset of the vertices of a graph $\mathcal{G}$ and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{|\mathcal{D}|}$ denote the sets of vertices dominated by the vertices in $\mathcal{D}$ respectively. If we define

$$\begin{aligned}
\mathcal{D}_1 &= |\mathcal{A}_1| + |\mathcal{A}_2| + \cdots + |\mathcal{A}_{|\mathcal{D}|}| \\
\mathcal{D}_2 &= |\mathcal{A}_1 \cap \mathcal{A}_2| + |\mathcal{A}_1 \cap \mathcal{A}_3| + \cdots + |\mathcal{A}_{|\mathcal{D}|-1} \cap \mathcal{A}_{|\mathcal{D}|}| \\
\mathcal{D}_3 &= |\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3| + |\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_4| + \cdots + |\mathcal{A}_{|\mathcal{D}|-2} \cap \mathcal{A}_{|\mathcal{D}|-1} \cap \mathcal{A}_{|\mathcal{D}|}| \\
&\vdots \qquad \vdots \\
\mathcal{D}_{|\mathcal{D}|} &= |\mathcal{A}_1 \cap \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_{|\mathcal{D}|}|,
\end{aligned}$$

then

$$D = \sum_{i=1}^{|\mathcal{D}|} (-1)^{i+1} |\mathcal{D}_i| \tag{4.9}$$

vertices in $\mathcal{G}$ are dominated by $\mathcal{D}$. ■

Note that after each addition in (4.9) an upper bound on D is found, while a lower bound on D is established after each subtraction. Note also that the well–known domination bound (4.2) may be attained from (4.9) by truncating the series (4.9) after the first term. To see this, observe that, for the lottery graph $G\langle m, n, n; k\rangle$, $\mathcal{D}_1 = \sum_{i=1}^{|\mathcal{D}|} |\mathcal{A}_i| = |\mathcal{D}|(r+1)$, where r is defined as in (4.1). Hence, to dominate the entire vertex set of $G\langle m, n, n; k\rangle$, we require that

$$\binom{m}{n} = D \leq \mathcal{D}_1 = |\mathcal{D}|(r+1) \leq L(m, n, n; k)(r+1),$$

rendering the lower bound (4.2). This bound may be refined by rather truncating the series in (4.9) after the third term. In fact, a lower bound on $L(m, n, n; k)$ may be obtained by truncating the series in (4.9) after any odd–numbered term, and the lower bound of course improves the more this truncation is postponed. However, the calculation of $\mathcal{D}_i$ in (4.9) becomes a tedious and non–trivial task for $i \geq 2$. This calculation can only be accomplished once a counting method has been established to determine the terms $|\cap_{j=1}^{i} \mathcal{A}_{s_j}|$ in $\mathcal{D}_i$. These terms depend heavily on the overlapping properties of the labels of elements s_j in the dominating set $\mathcal{D}$ of Proposition 1.

The set structure encoding scheme described in §2 and illustrated in Example 3 paves the way to determine the values of $|\cap_{j=1}^{i} \mathcal{A}_{s_j}|$. In order to illustrate this, we need the notion of a **power matrix**, $\mathbf{A}^{(L)}$, which is defined as a $2^L \times L$ binary matrix whose entry in row i and column j is the j–th bit in the binary representation of the integer i (here we assume that binary numbers in the decimal range $\{0, \ldots, 2^L - 1\}$ are prefixed with zeros so that they all have length L).

Lemma 1 *Any L vertices of $G\langle m, n, n; k\rangle$, whose labels have an overlapping structure captured by the vector*

$$\vec{X}^{(L)} = (x_0^{(L)}, x_1^{(L)}, x_2^{(L)}, \ldots, x_{2^L-1}^{(L)}),$$

have

$$\alpha(m, \vec{X}^{(L)}) = \sum_{\substack{\vec{k} \leq \vec{y}^{(L)} \mathbf{A}^{(L)} < \vec{n} \\ \vec{y}^{(L)} \cdot \vec{1} = n \\ \vec{0} \leq \vec{y}^{(L)} \leq \vec{x}^{(L)}}} \prod_{i=0}^{2^L-1} \binom{x_i^{(L)}}{y_i^{(L)}} \tag{4.10}$$

mutual neighbours in $G\langle m, n, n; k\rangle$, where $\vec{0}$ and $\vec{1}$ represent 2^L–vectors containing respectively only zeros and ones, and where $\vec{k}$ and $\vec{n}$ represent L–vectors containing respectively only k's and n's.

Proof: We count all n–sets from $\mathcal{U}_m$ that have at least k elements of $\mathcal{U}_m$ in common with each of the L n–sets defined by the overlapping structure vector $\vec{X}^{(L)}$. Suppose we choose $y_i^{(L)}$ elements of $\mathcal{U}_m$ from the $x_i^{(L)}$ elements in the i–th n–set. The j–th entry in the vector $\vec{y}^{(L)}\mathbf{A}^{(L)}$ represents the sum of all the variables $(y_i^{(L)})$ that have a 1 in the j–th position of their indices, thus giving the total number of elements choosen from the j–th n–set. ■

We illustrate the above result by means of a simple example.

Example 4 *Consider the special case $\langle m, 6, 6; 3\rangle$. The overlapping structure of two 6–sets from $\mathcal{U}_m$, that have exactly one mutual element from $\mathcal{U}_m$, is captured by the vector*

$$\vec{X}^{(2)} = (x_{(00)_2}^{(2)}, x_{(01)_2}^{(2)}, x_{(10)_2}^{(2)}, x_{(11)_2}^{(2)}) = (m-11, 5, 5, 1).$$

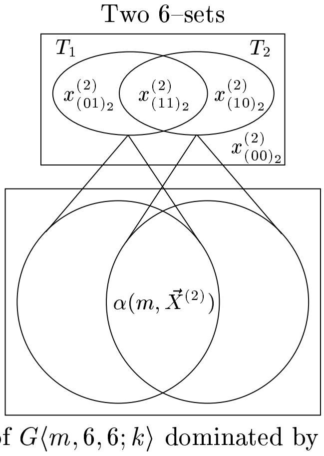


Figure 4.1: The number of vertices dominated by two vertices in $G\langle m, 6, 6; 3\rangle$ whose labels have $x^{(2)}_{(11)_2} = 1$ element of $\mathcal{U}_m$ in common (Example 4).

In order to count the number of mutual neighbours in $G\langle m, 6, 6; 3\rangle$ of these 6–sets (as depicted in Figure 4.1), we sum over all $\vec{y}^{(2)} \geq \vec{0}$ satisfying

$$(3,3) \leq (y^{(2)}_{(00)_2}, y^{(2)}_{(01)_2}, y^{(2)}_{(10)_2}, y^{(2)}_{(11)_2}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} < (6,6)$$

and $y^{(2)}_{(00)_2} + y^{(2)}_{(01)_2} + y^{(2)}_{(10)_2} + y^{(2)}_{(11)_2} = 6$ in (4.10). There are four such vectors, as shown in Table 4.1, yielding the result $\alpha(m, \vec{X}^{(2)}) = 100(m-11) + 300$. ■

$y^{(2)}_{(00)_2}$	$y^{(2)}_{(01)_2}$	$y^{(2)}_{(10)_2}$	$y^{(2)}_{(11)_2}$
1	2	2	1
0	2	3	1
0	3	2	1
0	3	3	0

Table 4.1: Valid choices for the vector $\vec{y}^{(2)}$ in Example 4.

Note that the formula for the degree of regularity of the lottery graph $G\langle m, n, n; k\rangle$ shown in (4.1) is a special case of (4.10), where $L = 1$.

4.2 Analytic upper bounds from graph theory

We now turn our attention to upper bounds for the lottery number. It is known that if G is a connected order p graph, then $\gamma(G) \leq \frac{2}{5}p$ [42] and that this bound may be further improved to $\gamma(G) \leq \frac{3}{8}p$ in the case of connected order p graphs with minimal degree at least 2 (with the exception of 7 small pathological graphs) [32]. Furthermore, $\gamma(G) \leq p - \Delta(G)$ for any order p graph [51], while $\gamma(G) \leq \delta(G)$ for any order p graph with diameter 2 [20] (here $\Delta(G)$ and $\delta(G)$ denote respectively the maximal and minimal vertex degrees of G). From these four bounds we deduce that

$$L(m, n, n; k) \leq \min\left\{\frac{3}{8}\binom{m}{n}, \binom{m}{n} - r\right\} \tag{4.11}$$

for all $1 \leq k \leq n \leq m$, where r is the degree of regularity of the lottery graph $G\langle m, n, n; k\rangle$, as defined in (4.1). Furthermore, if diameter$(G\langle m, n, n; k\rangle) = 2$, then

$$L(m, n, n; k) \leq r. \tag{4.12}$$

In 1985 Caro & Roditty [8, 9] proved that $\gamma(G) \leq p\{1 - \delta(G)[1/(\delta(G)+1)]^{1+1/\delta(G)}\}$ for any order p graph G with $\delta(G) \geq 7$. It therefore follows that

$$L(m, n, n; k) \leq \binom{m}{n}\left[1 - r\left(\frac{1}{r+1}\right)^{1+1/r}\right] \tag{4.13}$$

for realistic values of m, n and k, while the bound

$$L(m, n, n; k) \leq \binom{m}{n}\frac{1+\ln(r+1)}{r+1} \tag{4.14}$$

is a result of a well–known theorem [1, 2, 41] stating that, for any order p graph without isolated vertices, $\gamma(G) \leq p[1 + \ln(\delta(G)+1)]/[\delta(G)+1]$. It is also possible to derive the bound

$$L(m, n, n; k) \leq \frac{1}{r+1}\binom{m}{n}\sum_{j=1}^{r+1}\frac{1}{j} \tag{4.15}$$

from a theorem of Arnautov [2], stating that, for any order p graph without isolated vertices, $\gamma(G) \leq [p/(\delta(G)+1)]\sum_{j=1}^{\delta(G)+1} 1/j$. Sanchis [45] proved, in 1991, that if an order p graph G has $\gamma(G) \geq 2$ and $\Delta(G) \leq p - \gamma(G) - 1$, then $2q \leq [p - \gamma(G)][p - \gamma(G) + 1]$, where q is the size of G. From this we deduce that

$$L(m, n, n; k) \leq \left\lfloor \binom{m}{n} + \frac{1}{2} - \sqrt{r\binom{m}{n} + \frac{1}{4}} \right\rfloor \tag{4.16}$$

for all $1 \leq k < n \leq m$. Finally, in 1998 Clark, *et al.* [10] established the result that $\gamma(G) \leq p\left(1 - \prod_{i=1}^{\delta(G)+1}\frac{i}{i+1/\delta(G)}\right)$ for any graph G or order p and minimum degree $\delta(G)$. With the additional constraint that G is regular, this bound was further improved, yielding

$$L(m, n, n; k) \leq \binom{m}{n}\left(1 - \frac{r^3+r}{r^3+1}\prod_{j=1}^{r}\left(1+\frac{1}{jr}\right)^{-1}\right), \tag{4.17}$$

which is a slight improvement on the Arnautov bound (4.15) for realistic values of m, n and k.

4.3 Comparison of bounds

The relative merits of the bounds presented in §4.1 and §4.2 are compared in Table 4.2 in the realistic special case of $L(m, 6, 6; 3)$, for the parameter values $6 \leq m \leq 50$. Apart from the upper bounds presented in §4.2, the values of the corresponding covering numbers are of course also upper bounds, by (2.1). But these numbers are not known for realistic lottery values of m, n and k. Furthermore, analytic upper bounds available for these covering numbers seem to be worse than those obtained from the graph theoretic approach in §4.2. On the other hand, we expect even the graph theoretic upper bounds to be weak in comparison to the real lottery numbers, so that the only way to obtain good upper bounds seems to be via special lottery playing set constructions. In fact, far tighter upper bounds on the lottery numbers in Table 4.2 may be found via design theoretic constructions. For example, it is known that $L(49, 6, 6; 3) \leq 163$ via the construction in [5], which is considerably better than the best available analytic upper bound $L(49, 6, 6; 3) \leq 700$.

m	$\binom{m}{6}$	r	(4.6)	(4.2)	(4.5)	(4.4)	(4.8)	L	(4.17)	(4.15)	(4.13)	(4.14)
6	1	0	1	1	1	1	1	**1**	1	1	–	1
7	7	6	1	1	1	1	1	**1**	2	2	–	2
8	28	27	1	1	1	1	1	**1**	3	3	4	4
9	84	83	1	1	1	1	1	**1**	4	5	5	5
10	210	194	1	2	2	2	1	**2**	6	6	6	6
11	462	380	1	2	2	2	1	**2**	7	7	8	8
12	924	661	1	2	2	2	1	**2**	9	9	10	10
13	1 716	1 057	1	2	2	2	2	**2**	12	12	12	12
14	3 003	1 588	1	2	3	3	2	**4**	14	15	15	15
15	5 005	2 274	2	3	3	3	2	**4**	18	18	19	19
16	8 008	3 135	2	3	3	3	3	**5**	22	22	23	23
17	12 376	4 191	2	3	3	4	3	**6**	26	26	27	27
18	18 564	5 462	3	4	4	4	4	**7**	31	31	32	32
19	27 132	6 968	3	4	4	5	4	?	36	36	38	38
20	38 760	8 729	3	5	5	5	5	?	42	42	44	44
21	54 264	10 765	4	6	6	6	6	?	49	49	51	51
22	74 613	13 096	4	6	6	7	7	?	57	57	59	59
23	100 947	15 742	5	7	7	8	8	?	65	65	68	68
24	134 596	18 723	6	8	8	8	9	?	74	74	77	77
25	177 100	22 059	6	9	9	9	10	?	84	84	88	88
26	230 230	25 770	7	9	9	10	12	?	95	95	99	99
27	296 010	29 876	8	10	10	11	13	?	107	107	111	112
28	376 740	34 397	9	11	11	13	15	?	120	120	125	125
29	475 020	39 353	10	13	13	14	17	?	134	134	139	139
30	593 775	44 764	11	14	14	15	19	?	149	149	155	155
31	736 281	50 650	12	15	15	17	21	?	165	165	171	172
32	906 192	57 031	13	16	16	18	23	?	183	183	189	189
33	1 107 568	63 927	14	18	18	20	25	?	201	201	209	209
34	1 344 904	71 358	15	19	19	21	28	?	221	221	229	229
35	1 623 160	79 344	17	21	21	23	30	?	242	242	251	251
36	1 947 792	87 905	18	23	23	25	33	?	265	265	274	274
37	2 324 784	97 061	20	24	24	27	36	?	288	288	298	298
38	2 760 681	106 832	22	26	26	29	39	?	314	314	325	325
39	3 262 623	117 238	23	28	28	32	42	?	340	340	352	352
40	3 838 380	128 299	25	30	30	34	46	?	369	369	381	381
41	4 496 388	140 035	27	33	33	36	50	?	398	399	412	412
42	5 245 786	152 466	29	35	35	39	54	?	430	430	445	445
43	6 096 454	165 612	31	37	37	42	58	?	463	463	479	479
44	7 059 052	179 493	34	40	40	44	62	?	498	498	515	515
45	8 145 060	194 129	36	42	42	47	66	?	535	535	552	552
46	9 366 819	209 540	38	45	45	51	71	?	573	573	592	592
47	10 737 573	225 746	41	48	48	54	76	?	613	613	633	633
48	12 271 512	242 767	44	51	51	57	81	?	655	655	677	677
49	13 983 816	260 623	47	54	54	61	87	?	700	700	722	722
50	15 890 700	279 334	49	57	57	64	92	?	746	746	770	770

Table 4.2: Analytic Bounds on the lottery number $L(m, 6, 6; 3)$ for all $6 \le m \le 50$ in a comparative fashion. The table shows the order of the lottery graph, $\binom{m}{6}$; the degree of regularity of the lottery graph, r; known values of lottery numbers, L, and values of the lower and upper bounds presented in §§4.1–4.2.

4.4 Analytic bounds for special cases

The bounds cited in §4.1 and §4.2 are all general, bar the requirement that $n = t$. However, some good results are available for special classes of lottery numbers. For example, the following special case lower bound is due to Füredi, *et al.* [14], who used a result of Turán as modified by Erdös in [6], while the upper bound is due to Bate & Van Rees [4]:

$$\min_{a_1+\cdots+a_{n-1}=m}\left\lceil\sum_{i=1}^{n-1}\frac{a_i}{n}\left\lceil\frac{a_{i-1}}{n-1}\right\rceil\right\rceil \leq L(m,n,n;2) \leq \min_{a_1+\cdots+a_{n-1}=m}\left\lfloor\sum_{i=1}^{n-1}C(a_i,n;2)\right\rfloor. \tag{4.18}$$

Furthermore, in 1964 Hanani, *et al.* [18] proved that

$$L(m,n,n;2) \geq \frac{m(m-n+1)}{n(n-1)^2} \tag{4.19}$$

and this bound is asymptotically best possible, in the sense that

$$\lim_{m\to\infty} L(m,n,n;2)\frac{n(n-1)^2}{m(m-n+1)} = 1.$$

5 Characterisation of lottery set structures

Given an $L(m,n,t;k)$–set $\mathcal{L} = \{T_1, T_2, \ldots, T_{L(m,n,t;k)}\}$ for $\langle m,n,t;k\rangle$, it is possible to interchange the roles of elements in $\mathcal{U}_m$ in order to induce a different $L(m,n,t;k)$–set for $\langle m,n,t;k\rangle$. Although these $L(m,n,t;k)$–sets are different, they still have the same structure in terms of n–set overlappings. In this section we devise a search method to find (i) the number of different n–set overlapping structures of minimum cardinality lottery sets for $\langle m,n,t;k\rangle$, denoted by $\eta(m,n,t;k)$, and (ii) all $\eta(m,n,t;k)$ actual minimum cardinality lottery set structures for $\langle m,n,t;k\rangle$.

The search method consists of simply traversing a rooted tree of evolving overlapping structures as more and more n–sets T_i are added to form a potential lottery set $\mathcal{L}$ of cardinality $L(m,n,t;k)$. Level i of the tree consists of all possible structures of the vector $\vec{X}^{(i)}$ and is constructed from the nodes on level $i-1$ of the tree by appending 2^{i-1} integers to each of the existing vectors $\vec{X}^{(i-1)}$. These appendices are carried out in such a way that all possible different (new) n–set overlappings are considered when adding the i–th n–set T_i to an existing set of n–set overlappings $\{T_1, T_2, \ldots, T_{i-1}\}$ (*i.e.*, a node on level $i-1$ of the tree). This is done at all nodes on level $i-1$ of the tree. The tree has $L(m,n,t;k)+1$ levels in total. The first level consists of the node $\vec{X}^{(1)} = (m-n, n)$ only (the root), while the nodes $\vec{X}^{(L(m,n,t;k))}$ on level $L(m,n,t;k)$ of the tree represent potential (minimum cardinality) lottery set structures for $\langle m,n,t;k\rangle$. The $(L(m,n,t;k)+1)$–st level of nodes consists of adding t–sets to each node on level $L(m,n,t;k)$ of the tree in such a way that all possible different (new) overlappings are considered when adding the t–set $T_{L(m,n,t;k)+1}$ to an existing set of n–set overlappings $\{T_1, T_2, \ldots, T_{L(m,n,t;k)}\}$. This is done in order to carry out the graph *domination test* (*i.e.*, to test which nodes on level $L(m,n,t;k)$ actually represent *valid* lottery sets). This domination test is achieved by testing whether all nodes on level $L(m,n,t;k)+1$ of the tree whose corresponding final t–set $T_{L(m,n,t;k)+1}$ overlaps in at least k positions with at least one n–set of the existing $L(m,n,t;k)$ n–set overlapping structure represented by its parent node in the tree. If this is the case, then the n–set structure $\{T_1, T_2, \ldots, T_{L(m,n,t;k)}\}$ represented by the parent node constitutes a valid $L(m,n,t;k)$–set for $\langle m,n,t;k\rangle$.

The above tree is traversed in a depth–first fashion. So every time we move one level down in the tree an n–set is added (the lexicographic first one for that node, to be precise). The search procedure then calculates whether the concerned overlapping structure was encountered before. If this is the case, that structure is removed from the tree and the next n–set is added (to the same parent). If not, we proceed one level down in the tree, *etc.*, until a depth of $L(m,n,t;k)$ is reached, whereafter the domination test is performed. Non–repetition of the structures of the vectors $\vec{X}^{(i)}$ is accomplished by avoiding permutations. The following two additional rules were implemented to prune the tree at the domination test level, thereby speeding up the domination test:

(1) If $L(m-1,n,t-1;k) > L(m,n,t;k)$ and $x_0^{(L)} > 0$, then the structure corresponding to the vector $\vec{X}^{(L)}$ is *not* dominating, and may hence be omitted from the tree.

(2) If $\min\{x^{(L)}_{(100\cdots0)_2}, k-1\}+\cdots+\min\{x^{(L)}_{(000\cdots1)_2}, k-1\}+\min\{x_0^{(L)}, k-1\} \geq t$, then the structure corresponding to the vector $\vec{X}^{(L)}$ is *not* dominating, and may hence be omitted from the tree.

Rule (1) follows from the fact that if a specific element of $\mathcal{U}_m$ is not used in a lottery set of cardinality ℓ for $\langle m,n,t;k\rangle$, then $L(m-1,n,t-1;k) \leq \ell$. In rule (2) we add up the number of elements (not exceeding $k-1$ per set) that are in at most one n–set of the structure corresponding to $\vec{X}^{(L)}$. If there are t or more such elements, there exists a t–set having no k–intersection with any of the n–sets in the structure $\vec{X}^{(L)}$, and hence the structure does not represent a lottery set.

However, even with the above pruning rules in force, the number of nodes on level i of the tree typically grows very rapidly as i increases, as illustrated in the next example.

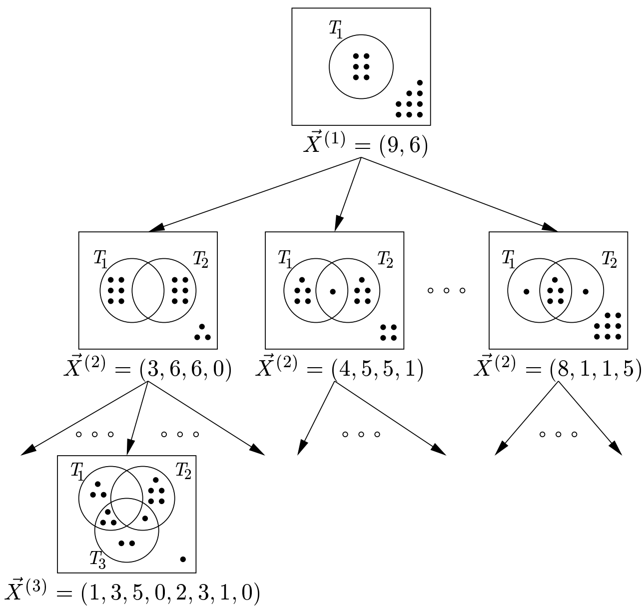


Figure 5.1: Part of the tree construction to determine all $\eta(15,6,6;3) = 4$ $L(15,6,6;3)$–set overlapping structures for the lottery $\langle 15,6,6;3\rangle$.

Example 5 *Table 5.1 shows the values of* $L(m_1,6,6;2)$ *and* $\eta(m_1,6,6;2)$ *for* $6 \leq m_1 \leq 30$ *and Table 5.2 shows the values of* $L(m_2,6,6;3)$ *and* $\eta(m_2,6,6;3)$ *for* $6 \leq m_2 \leq 17$*. Both tables were constructed using the tree characterisation method described above.*

To illustrate the growth and complexity of the tree characterisation method, consider the scheme $\langle 15,6,6;3\rangle$*, for which it is known that* $L(15,6,6;3) = 4$*. Part of the tree construction to determine all* $L(15,6,6;3)$*–set overlapping structures for the lottery* $\langle 15,6,6;3\rangle$ *is shown in Figure 5.1. The number of nodes on the different levels of the characterisation tree is given by the sequence* 1, 6, 68, 2 384, 252 065 (after *all node repetitions have been eliminated) and it takes less than* 1 *second*

m	6–10	11	12	13	14	15	16	17	18	19	20
$L(m,6,6;2)$	1	2	2	2	2	2	3	3	3	3	3
$\eta(m,6,6;2)$	1	4^a	4^b	3^c	2^d	1^e	14^f	8^g	5^h	2^i	1^j

m	21	22	23	24	25	26	27	28	29	30
$L(m,6,6;2)$	4	4	4	4	4	5	5	5	5	5
$\eta(m,6,6;2)$	29^k	13^l	5^m	2^n	1^o	34^p	11^q	4^r	1^s	1^t

Table 5.1: Known values of $L(m,6,6;2)$ and (new) values of $\eta(m,6,6;2)$ for $6 \le m \le 30$. Solution set structures are listed by superscript in Appendix A.

m	6–9	10	11	12	13	14	15	16	17
$L(m,6,6;3)$	1	2	2	2	2	4	4	5	6
$\eta(m,6,6;3)$	1	2^A	2^B	2^C	1^D	26^E	4^F	7^G	3^H

Table 5.2: Known values of $L(m,6,6;3)$ and (new) values of $\eta(m,6,6;3)$ for $6 \le m \le 17$. Solution set structures are listed by superscript in Appendix B.

to traverse the tree for $\langle 15,6,6;3\rangle$ *up to level* $i = 5$ *(including the domination test level) on an AMD Thunderbird 1.4GHz processor with 512Mb memory, yielding the result* $\eta(15,6,6;3) = 4$. *The* $\eta(15,6,6;3) = 4$ *possible* $L(15,6,6;3)$*–set overlapping structures are shown schematically in Figure 5.2(a)–(d). Similarly, the number of nodes on the different levels of the trees for* $\langle 16,6,6;3\rangle$ *and* $\langle 17,6,6;3\rangle$*, for which it is known that* $L(16,6,6;3) = 5$ *and* $L(17,6,6;3) = 6$*, are given by the sequences* 1, 6, 71, 2 643, 319 813 *and* 1, 6, 72, 2 795, 368 800, 129 820 402 *respectively (*after *all node repetitions have been eliminated). It takes approximately* 7 *minutes and* 60 *days,* 10 *hours,* 43 *minutes to traverse the trees for* $\langle 16,6,6;3\rangle$ *and* $\langle 17,6,6;3\rangle$ *up to levels* $i = 6$ *and* $i = 7$ *respectively (including the domination test levels) on the same processor as mentioned above. The* $\eta(16,6,6;3) = 7$ *possible* $L(16,6,6;3)$*–set overlapping structures are shown schematically in Figure 5.2(e)–(k), while the* $\eta(17,6,6;3) = 3$ *possible* $L(17,6,6;3)$*–set overlapping structures are shown schematically in Figure 5.2(l)–(n).* ∎

We anticipate that it would be possible to eliminate entire branches of the tree by means of intelligent rule–based pruning (for example, when it is clear that a certain branch would not be able to produce any *new* solutions further down the tree). Although such an approach may speed up the characterisation procedure significantly and allow larger trees to be traversed, we did not investigate this possibility. We are therefore limited in this paper to the characterisation of $L(m,n,t;k)$–set structures of lotteries for which $L(m,n,t;k) \le 5$, as demonstrated in the example above.

6 New lottery numbers and improved lottery bounds

According to [28] the lottery number for $\langle 18,6,6;3\rangle$ falls in the range $6 \le L(18,6,6;3) \le 7$. The tree for $\langle 18,6,6;3\rangle$ is too deep (large) to traverse in a realistic time–span, so that it is not feasible to determine the value of $L(18,6,6;3)$ using the tree characterisation procedure described in §5. However, it is possible to prove that $L(18,6,6;3) \ne 6$ via a construction method described below, from which the following result may then be deduced.

Theorem 7 $L(18,6,6;3) = 7$.

Consider the following method to construct lottery sets of the same cardinality for $\langle m-1,n,n;k\rangle$ from any lottery set for $\langle m,n,n;k\rangle$.

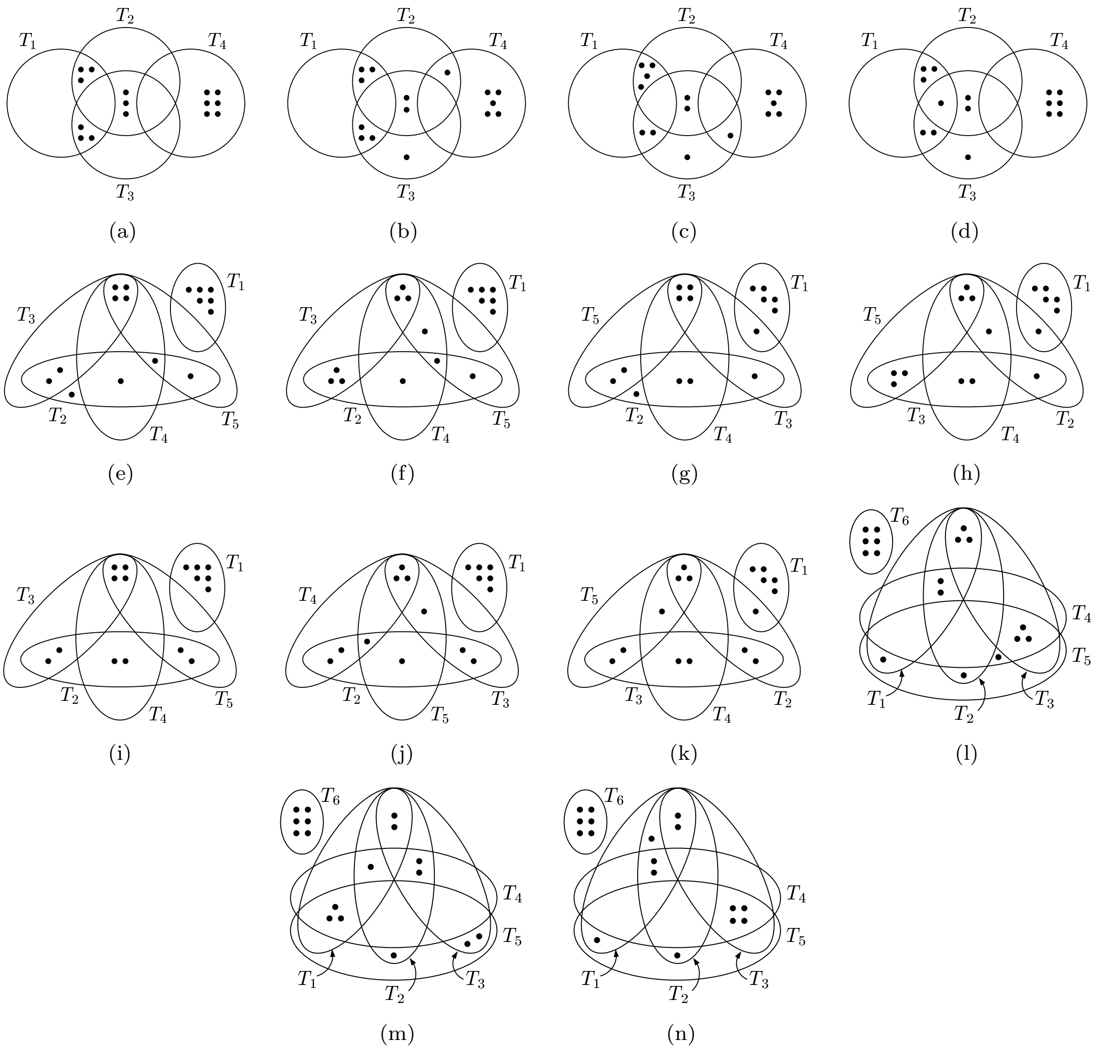


Figure 5.2: (a)–(d) The four 6–set overlapping $L(15, 6, 6; 3)$–set structures corresponding to the result $\eta(15, 6, 6; 3) = 4$. (e)–(k) The seven 6–set overlapping $L(16, 6, 6; 3)$–set structures corresponding to the result $\eta(16, 6, 6; 3) = 7$. (l)–(n) The three 6–set overlapping $L(17, 6, 6; 3)$–set structures corresponding to the result $\eta(17, 6, 6; 3) = 3$.

> **Construction.** Consider a tabular representation (such as in Figure 2.1(a)) of a lottery set for $\langle m, n, n; k\rangle$. Remove any column from this representation, and add a (valid) arbitrary element to the original n–sets that now have only $n - 1$ elements as a result of the deletion. The result is a tabular representation of a lottery set for $\langle m - 1, n, n; k\rangle$.

In order to prove Theorem 7, we first need the following intermediate result.

Lemma 2 *If there exist lottery sets for* $\langle 18, 6, 6; 3\rangle$ *of cardinality* 6*, all such sets must contain exactly one disjoint* 6*–set.*

Proof: Suppose there exist lottery sets of cardinality 6 for $\langle 18, 6, 6; 3\rangle$. Then such lottery sets may have at most one disjoint 6–set, otherwise some of their 6–sets will be forced to coincide exactly. Now suppose one such lottery set contains no disjoint 6–set. Then, keeping in mind that

$\langle m,n,t;k\rangle$	Previous $L(m,n,t;k)$	$L(m,n,t;k)$	$\eta(m,n,t;k)$	Time (sec)	$\vec{X}$
$\langle 17,7,5;3\rangle$	$5:7$	$6^b:7$	–	523	–
$\langle 18,7,5;3\rangle$	$5:8$	$6^e:8$	–	–	–
$\langle 19,8,5;3\rangle$	$5:6$	6^b	≥ 1	2 985	–
$\langle 20,8,5;3\rangle$	$5:7$	$6^e:7$	–	–	–
$\langle 18,6,6;3\rangle$	$6:7$	7^d	≥ 1	–	–
$\langle 19,7,6;3\rangle$	$4:5$	5^a	2	778	(07000000000000501020200020000000) (07000010000000400020200030000000)
$\langle 20,7,6;3\rangle$	$4:7$	$6^b:7$	–	825	–
$\langle 19,5,8;3\rangle$	$5:7$	$6^b:7$	–	18	–
$\langle 18,4,10;3\rangle$	$5:6$	6^b	≥ 1	2	–
$\langle 16,10,5;4\rangle$	$4:5$	5^a	1	136	(00000002000202000400000000000060)
$\langle 17,10,5;4\rangle$	$5:8$	$6^b:8$	–	631	–
$\langle 17,11,5;4\rangle$	$4:5$	5^a	11	171	Listed in Appendix C
$\langle 18,11,5;4\rangle$	$4:7$	$6^b:7$	–	905	–
$\langle 19,11,5;4\rangle$	$5:10$	$6^e:10$	–	–	–
$\langle 19,12,5;4\rangle$	$4:5$	5^a	2	1 191	(00000002000202010500000000000070) (00000003000202000400000001000070)
$\langle 20,12,5;4\rangle$	$5:7$	$6^b:7$	–	5 357	–
$\langle 14,6,7;4\rangle$	$5:8$	$6^b:8$	–	63	–
$\langle 15,7,7;4\rangle$	$4:5$	5^a	26	281	Listed in Appendix D
$\langle 16,7,7;4\rangle$	$4:7$	$6^b:7$	–	438	–
$\langle 17,7,7;4\rangle$	$5:11$	$6^e:11$	–	–	–
$\langle 17,8,7;4\rangle$	$4:5$	5^a	67	1 994	Listed in Appendix E
$\langle 18,8,7;4\rangle$	$4:6$	$5^{a,c}$	1	2 926	(00000060040000000400000000202000)
$\langle 19,8,7;4\rangle$	$4:9$	$6^b:9$	–	3 603	–
$\langle 20,8,7;4\rangle$	$4:10$	$6^e:10$	–	–	–
$\langle 19,9,7;4\rangle$	$4:6$	$5^{a,c}$	154	13 409	Listed in Appendix F
$\langle 20,9,7;4\rangle$	$4:6$	$5^{a,c}$	3	18 479	(00000060040000100500000000202000) (00000070040000000400000001202000) (00000160040000000400000000302000)
$\langle 16,6,8;4\rangle$	$4:7$	$6^b:7$	–	114	–
$\langle 18,7,8;4\rangle$	$4:6$	6^b	≥ 1	824	–
$\langle 19,7,8;4\rangle$	$4:9$	$6^e:9$	–	–	–
$\langle 15,5,9;4\rangle$	$5:8$	$6^b:8$	–	16	–
$\langle 17,6,9;4\rangle$	$4:6$	6^b	≥ 1	134	–
$\langle 18,6,9;4\rangle$	$5:6$	6^e	≥ 1	–	–
$\langle 19,6,9;4\rangle$	$5:10$	$6^e:10$	–	–	–
$\langle 19,7,9;4\rangle$	$4:5$	5^a	20	1 067	Listed in Appendix G
$\langle 20,7,9;4\rangle$	$4:6$	6^b	≥ 1	944	–
$\langle 16,5,10;4\rangle$	$5:6$	6^b	≥ 1	17	–
$\langle 19,6,10;4\rangle$	$5:6$	6^b	≥ 1	137	–
$\langle 20,6,10;4\rangle$	$5:8$	$6^e:8$	–	–	–
$\langle 18,5,11;4\rangle$	$5:7$	$6^b:7$	–	17	–

Table 6.1: New lottery numbers and improved bounds found via the characterisation technique described in §5. The second column contains previously best known bounds on lottery numbers, taken from [28]. The bound improvements or new lottery numbers obtained are listed in the third column, while column 5 shows the execution time required to implement the tree characterisation method on an AMD 1.8GHz processor with 256Mb of memory. The characterisation number $\eta(m,n,t;k)$ is listed, where applicable, in column 4. The corresponding n–set overlapping structures are listed in column 6 and in the appendices, using the $\vec{X}$–vector notation of §2. Bounds and new lottery numbers in column 3 are motivated as follows: ^{a}No lottery sets of cardinality 4 found. ^{b}No lottery sets of cardinality 5 found. ^{c}At least one lottery set of cardinality 5 found. ^{d}By Theorem 7. ^{e}Since $L(m+1,n,t;k) \geq L(m,n,t;k)$, by Theorem 3(a).

all $L(17,6,6;3)$–sets contain one disjoint 6–set (see Figure 5.2(l)–(n)), it is not difficult to see that it is only possible to construct lottery sets for $\langle 17,6,6;3\rangle$ via the above construction method if at least one element of $\mathcal{U}_{18}$ is not utilised in the $L(18,6,6;3)$–set (*i.e.*, if the tabular representation of the $L(18,6,6;3)$–set contains at least one empty column). However, if the tabular representation of the $L(18,6,6;3)$–set contains at least one empty column, then it must be true that $L(17,6,5;3) \leq 6$. But $7 \leq L(17,6,5;3) \leq 11$ [28], which is a contradiction, from which we deduce that the result of the lemma holds. ■

We are now in a position to prove Theorem 7.

Proof of Theorem 7: By contradiction. Suppose that $L(18,6,6;3) = 6$. Then it follows, by Lemma 2, that any $L(18,6,6;3)$–set must contain exactly one disjoint 6–set from $\mathcal{U}_{18}$. If, in the construction technique outlined above, a column involving the disjoint 6–set is deleted, the resulting lottery set for $\langle 17,6,6;3\rangle$ will have no disjoint 6–set, which contradicts the characterisation in Figure 5.2(l)–(n). We conclude that $6 < L(18,6,6;3) \leq 7$, which yields the desired result. ■

The above proof technique cannot be used to fix the lottery number for $\langle 19,6,6;3\rangle$, because, a complete characterisation of $L(18,6,6;3)$–set structures for $\langle 18,6,6;3\rangle$ is not currently known.

Apart from the new lottery number in Theorem 7, a further 18 new lottery numbers and improvements on best known bounds for a further 20 lottery numbers, as listed in [28], are given in Table 6.1. These results were all established using the tree characterisation method described in §5, and the time required to execute the search procedure in each case is also included in the table.

7 Conclusion

In this paper we considered an old problem in the combinatorial literature: the lottery problem, introduced as early as 1964. We introduced the novel notion of a *lottery graph* in §1, derived some of its properties in §3 and demonstrated in §4 how standard results from graph domination theory, not previously associated with the lottery problem, lead to simple, closed–form analytic bound formulations for the lottery number, which are usually better than the best analytic covering bounds available, though weaker than bounds obtained from specific, lottery set constructions. Another novel contribution of this paper was the development of a search procedure in §5 for characterising all possible overlapping structures of $L(m,n,t;k)$–sets. This technique was used in §6 to establish 19 new lottery numbers and to improve upon best known bounds for a further 20 lottery numbers.

Further work might include the following. (i) Investigations into the possibility of eliminating entire branches of the tree characterisation method described in §5 by means of intelligent rule–based pruning (for example, when it is clear that a certain branch would not be able to produce any *new* solutions downward in the tree). (ii) Parallel implementations of the tree characterisation method, enabling one to find slightly larger lottery numbers than was done in this paper. (iii) Improvement of the graph theoretic lower bound (4.2) via the approach outlined at the end of §4.1. (iv) Exploration of the rich structure and strong symmetry properties of the lottery graph.

Acknowledgements

This paper is based upon work which formed part of the second author's PhD study and he gratefully acknowledges financial assistance from the Harry Crossley Foundation, the Wilhelm Frank Stipendium Trust, the Department of Labour Scarce Skills Scholarship Programme and the Graduate Merit Bursary Fund at the University of Stellenbosch. Work towards this paper was also supported financially by the South African National Research Foundation under grant number GUN 2053755 and by Research Sub–Committee B at the University of Stellenbosch. Any opinions, findings and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the South African National Research Foundation or the Department of Labour.

References

[1] Alon N & Spencer JH: *The probabilistic method*, John Wiley & Sons, Inc., New York, 1992.

[2] Arnautov VI: *Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices* (Russian), Prikl. Mat. i Pragrammirovanie Vyp., **11** (1974), 3–8.

[3] Bate JA: *A Generalized Covering Problem*, PhD dissertation, University of Manitoba, Canada, 1978.

[4] Bate JA & Van Rees GHJ: *Lotto Designs*, J. Comb. Math. Comb. Comp., **28** (1998), 15–39.

[5] Belic R: *Lotto Systems and Toto Systems to win Wheel Game*, [Online], [cited 2002, April 23], Available from: `http://www.xs4all.nl/~rbelic/`

[6] Bollobas B: *Extended graph theory*, Academic Press, New York, 1978.

[7] Brouwer AI & Voorhoeve M: *Turán Theory and the Lotto Problem*, Mathematical Centrum Tracts, **106** (1995), 99–105.

[8] Caro Y & Roditty Y: *On the vertex–independence number and star decomposition of graphs*, Ars Combin., **29** (1985), 167–180.

[9] Caro Y & Roditty Y: *A note on the k–domination number of a graph*, Internat. J. Math. Sci., **13** (1990), 205–206.

[10] Clark WE, Fisher DC, Shekhtman B & Suen S: *Upper Bounds for the Domination Number of a Graph*, Congr. Numer., **132** (1998), 99–123, or [Online], [cited 2002, March 2], NEC Research Institute, *Research Index [NEC Research Institute, Citeseer, Computer Science]*, Available from: `http://www.citeseer.nj.nec.com/cs/`

[11] Colbourn CJ: *Winning the lottery*, pp. 578–584 in Colbourn CJ (Ed): *The CRC Handbook of Combinatorial Designs*, CRC Press, Boca Raton, 1996.

[12] De Caen D: *Extension of a theorem of Moon and Moser on complete subgraphs*, Ars Combin., **16** (1983), 5–10.

[13] Fort MK Jr & Hedlund GA: *Minimal coverings of pairs by triples*, Pacific J. Math., **8** (1958), 709–719.

[14] Füredi Z, Székely GJ & Zuber Z: *On the Lottery Problem*, J. Combin. Designs, **4** (1996), 5–10.

[15] Gordon DM, Patashnik O & Kuperberg G: New Constructions for Covering Designs, J. Combin. Designs, **3** (1995), 269–284.

[16] Gründlingh WR: *Two Combinatorial Problems involving Lottery Schemes*, PhD dissertation, University of Stellenbosch, South Africa, to be submitted.

[17] Guy RK: *Sequences associated with a problem of Turán and other problems*, pp 553–569 in *Combinatorial theory and its applications II* (Proc. Collog., Balatonfüred, 1969), North–Holland, Amsterdam, 1970.

[18] Hanani H, Ornstein D & Sós VT: *On the lottery problem*, Magyar Tud. Akad. Mat. Kutató Int. Közl., **9** (1964), 155–158.

[19] Hartman A, Mills WH & Mullin RC: *Covering triples by quadruples: An asymptotic solution*, J. Combin. Theory A, **41** (1986), 117–138.

[20] Haynes TW, Hedetniemi ST & Slater PJ: *Fundamentals of domination in graphs*, Marcel Dekker, New York, 1998.

[21] Horton JD, Mullin RC & Stanton RG: *Minimal coverings of pairs by quadruples*, Congr. Numer., **3** (1971), 495–516.

[22] Katona G, Nemetz T & Simonovits M: *On a Problem of Turán in the Theory of Graphs*, Mat. Fiz. Lapok, **15** (1964), 228–238.

[23] Kirkman TP: *On a problem in combinations*, Cambridge and Dublin Math. J., **2** (1847), 191–204.

[24] Center for Communication Research – La Jolla, *La Jolla Covering Repository Tables*, [Online], [cited 2003, September 7], Available from: `http://ww.ccrwest.org/cover/HIGH.html`

[25] Lamken ER, Mills WH, Mullin RC & Vanstone SA: *Coverings of pairs by quintuples*, J. Combin. Theory A, **44** (1987), 49–68.

[26] Li PC: *Ben Li's Lotto Tables Page*, [Online], [cited 2002, April 23], Available from: `http://www.cs.umanitoba.ca/~lipakc/lottotables.html`

[27] Li PC: *Some results on Lotto designs*, PhD dissertation, University of Manitoba, Canada, 1999.

[28] Li PC & van Rees GHJ: *Lotto design tables*, J. Combin. Designs, **10** (2002), 335–359.

[29] Lindner CC & Rosa A: *Steiner quadruple systems – a survey*, Discrete Math., **21** (1978), 147–181.

[30] The Lottery Institute: *The Lottery Institute*, [Online], [cited 2002, April 23], Available from: `http://www.members.tripod.com/~TheLotteryInstitute/`

[31] Lotto–Logix: *Lotto–Logix Wheeling Systems and Software*, [Online], [cited 2002, April 23], Available from: `http://www.lotto-logix.com/wheellinks.html`

[32] McCuaig W & Shepherd B: *Domination in graphs with minimum degree two*, J. Graph Theory, **13** (1989), 749–762.

[33] McLottery: *Lottery, lotto and powerball software for Mac computers*, [Online], [cited 2002, April 23], Available from: `http://www.chariot.net.au/~et/pages/`

[34] Mills WH: *On the covering of pairs by quadruples I*, J. Combin. Theory A, **13** (1972), 55–78.

[35] Mills WH: *On the covering of pairs by quadruples II*, J. Combin. Theory A, **15**, (1973), 138–166.

[36] Mills WH & Mullin RC: *Coverings and Packings* in Dinitz JH & Stinson DR (Eds): *Contemporary Design Theory: A Collection of Surveys*, John Wiley & Sons, Inc., New York, 1992.

[37] Mills WH & Mullin RC: *Covering pairs by quintuples: The case v congruent to* 3 (mod 4), J. Combin. Theory A, **49** (1988), 308–322.

[38] Mullin RC: *On covering pairs by quintuples: The cases $v \equiv 3$ or 11 modulo* 20, J. Comb. Math. Comb. Comp., **2** (1987), 133–146.

[39] Mullin RC: *On the determination of the covering numbers $C(2,5,v)$*, J. Comb. Math. Comb. Comp., **4** (1988), 123–132.

[40] Nurmela KJ & Östergård PRJ: *Upper Bounds for Covering Designs by Simulated Annealing*, Congr. Numer., **96** (1993), 93–111.

[41] Payan C: *Sur le nombre d'absorpsion d'un graphe simple*, Cahiers Centre d'Études Rech. Opér., **2.3.4** (1975), 171.

[42] Reed B: *Paths, stars, and the number three*, J. Combin. Prob. Comp., **5**(3) (1996), 277–295.

[43] Rödl V: *On a Packing and Covering Problem*, Eur. J. Math., **5** (1985), 69–78.

[44] Rosendahl P: *Peter's Lotto–Wheel's Page*, [Online], [cited 2002, April 23], Available from: `http://www.algonet.se/~peteros/`

[45] Sanchis LA: *Maximum number of edges in connected graphs with a given domination number*, Discrete Math., **87** (1991), 65–72.

[46] Schönheim J: *On Coverings*, Pacific J. Math., **14** (1963), 1 405–1 411.

[47] Schrijver A (Ed): *Packing and covering in combinatorics*, Mathematical Centre Tracts, **106**, Mathematisch Centrum, Amsterdam, 1979.

[48] Steiner J: *Combinatorische Aufgabe*, J. Reine Angew. Math., **45** (1853), 181–182.

[49] Sterboul F:, *Le problème du Loto*, Cahiers du Centre d'Études de Recherche Opérationnelle, **20** (1978), 443–449.

[50] Turán P: *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok, **48** (1941), 436–452.

[51] Walikar HB, Acharya BD & Sampathkumar E: *Recent developments in the theory of domination in graphs*, MRI Lecture Notes in Math., **1**, Mahta Research Instit., Allahabad, 1979.

[52] Wolfram Research, Inc., *Steiner Systsem*, [Online], [cited 2003, September 7], Available from: `http://mathworld.wolfram.com/SteinerSystem.html`

[53] Woolhouse WSB: *Prize question 1733*, Lady's and Gentleman's Diary, 1844.

Appendix A

Entry [a] in Table 5.1

1. (3224)	2. (0551)	3. (1442)	4. (2333)

Entry [b] in Table 5.1

1. (3333)	2. (0660)	3. (1551)	4. (2442)

Entry [c] in Table 5.1

1. (3442)	2. (1660)	3. (2551)

Entry [d] in Table 5.1

1. (3551)	2. (2660)

Entry [e] in Table 5.1

1. (3660)

Entry [f] in Table 5.1

1. (22246000)	2. (05415010)	3. (05505001)	4. (14414110)	5. (04426000)
6. (14325010)	7. (14415001)	8. (23324110)	9. (24224020)	10. (24314011)
11. (24404002)	12. (13336000)	13. (23235010)	14. (23325001)	

Entry [g] in Table 5.1

1. (23336000)	2. (05516000)	3. (15415010)	4. (15505001)	5. (24414110)
6. (14426000)	7. (24325010)	8. (24415001)		

Entry [h] in Table 5.1

1. (24426000)	2. (06606000)	3. (15516000)	4. (25415010)	5. (25505001)

Entry [i] in Table 5.1

1. (25516000)	2. (16606000)

Entry [j] in Table 5.1

1. (26606000)

Entry ^{k} in Table 5.1

1.	(1224600060000000)	2.	(1332500160000000)	3.	(1323501060000000)
4.	(1333500050001000)	5.	(0333600060000000)	6.	(1440400260000000)
7.	(1431401160000000)	8.	(1422402060000000)	9.	(1332411060000000)
10.	(1441500050000001)	11.	(1441400150001000)	12.	(1440500150010000)
13.	(1431500150100000)	14.	(0441500160000000)	15.	(1432500050000010)
16.	(1432401050001000)	17.	(1422501050100000)	18.	(1332501051000000)
19.	(0432501060000000)	20.	(1442400040002000)	21.	(0442500050001000)
22.	(1441410050000010)	23.	(1441311050001000)	24.	(0441411060000000)
25.	(0550500050000001)	26.	(0550400150001000)	27.	(1441401041001000)
28.	(0541401050001000)	29.	(0531501050100000)		

Entry ^{l} in Table 5.1

1.	(1333600060000000)	2.	(1441500160000000)	3.	(1432501060000000)
4.	(1442500050001000)	5.	(0442600060000000)	6.	(1441411060000000)
7.	(1550500050000001)	8.	(1550400150001000)	9.	(0550500160000000)
10.	(1541401050001000)	11.	(1531501050100000)	12.	(0541501060000000)
13.	(0551500050001000)				

Entry ^{m} in Table 5.1

1.	(1442600060000000)	2.	(1550500160000000)	3.	(1541501060000000)
4.	(1551500050001000)	5.	(0551600060000000)		

Entry ^{n} in Table 5.1

1.	(1551600060000000)	2.	(0660600060000000)

Entry ^{o} in Table 5.1

1. (1660600060000000)

Entry ^{p} in Table 5.1

1.	(02246000600000006000000000000000)	2.	(03325001600000006000000000000000)
3.	(03235010600000006000000000000000)	4.	(03335000500010006000000000000000)
5.	(04404002600000006000000000000000)	6.	(04314011600000006000000000000000)
7.	(04224020600000006000000000000000)	8.	(03324110600000006000000000000000)
9.	(04415000500000016000000000000000)	10.	(04414001500010006000000000000000)
11.	(04405001500100006000000000000000)	12.	(04315001501000006000000000000000)
13.	(04415001500000005000000010000000)	14.	(04325000500000106000000000000000)
15.	(04324010500010006000000000000000)	16.	(04225010501000006000000000000000)
17.	(03325010510000006000000000000000)	18.	(04325010500000005000000010000000)
19.	(04424000400020006000000000000000)	20.	(04425000500000005000000000001000)
21.	(04425000400010005000000010000000)	22.	(04414100500000106000000000000000)
23.	(04413110500010006000000000000000)	24.	(04414110500000005000000010000000)

25. (0550500050000000500000000000000001)
26. (0550500040000001500000001000000)
27. (0550400150000000500000000001000)
28. (0550400140001000500000001000000)
29. (0550300150001000500010000000000)
30. (0540400150001000501000000000000)
31. (0441401041001000600000000000000)
32. (0541401040001000500000001000000)
33. (0541301050001000500010000000000)
34. (0521501050100000501000000000000)

Entry ^{q} in Table 5.1

1. (0333600060000000600000000000000)
2. (0441500160000000600000000000000)
3. (0432501060000000600000000000000)
4. (0442500050001000600000000000000)
5. (0441411060000000600000000000000)
6. (0550500050000001600000000000000)
7. (0550400150001000600000000000000)
8. (0550500150000000500000001000000)
9. (0541401050001000600000000000000)
10. (0531501050100000600000000000000)
11. (0541501050000000500000001000000)

Entry ^{r} in Table 5.1

1. (0442600060000000600000000000000)
2. (0550500160000000600000000000000)
3. (0541501060000000600000000000000)
4. (0551500050001000600000000000000)

Entry ^{s} in Table 5.1

1. (0551600060000000600000000000000)

Entry ^{t} in Table 5.1

1. (0660600060000000600000000000000)

Appendix B

Entry ^{A} in Table 5.2

1. (1333)
2. (0442)

Entry ^{B} in Table 5.2

1. (1442)
2. (0551)

Entry ^{C} in Table 5.2

1. (1551)
2. (0660)

Entry ^{D} in Table 5.2

1. (1660)

Entry ^{E} in Table 5.2

1.	(0002111360000000)	2.	(0003111250001000)	3.	(0004111140002000)
4.	(0005111030003000)	5.	(0002022260000000)	6.	(0102012250001000)
7.	(0003122050000001)	8.	(0003121150000010)	9.	(0003022150001000)
10.	(0002122150010000)	11.	(0103012140002000)	12.	(0003112141001000)
13.	(0004121040001010)	14.	(0004022040002000)	15.	(0003122040011000)
16.	(0104012030003000)	17.	(0004112031002000)	18.	(0003032050000010)
19.	(0102022140101000)	20.	(0103022040001010)	21.	(0103013040001100)
22.	(0102023040011000)	23.	(0003023041001000)	24.	(0103013031002000)
25.	(0103022030102000)	26.	(0003122031101000)		

Entry ^{F} in Table 5.2

1.	(0003122160000000)	2.	(0004122050001000)	3.	(0003033060000000)
4.	(0103023050001000)				

Entry ^{G} in Table 5.2

1.	(0000000411112000600000000000000000)	2.	(0001000301113000600000000000000000)
3.	(0000000402202000600000000000000000)	4.	(0000000411202000510000000000000000)
5.	(0001000302102010600000000000000000)	6.	(0001022000032000500000000010000000)
7.	(0001000301203000510000000000000000)		

Entry ^{H} in Table 5.2

1. (00000003000200000110000010003000600000000000000000000000000000000)
2. (00010002000200000110000000004000600000000000000000000000000000000)
3. (00000001000002200003100020000000600000000000000000000000000000000)

Appendix C

1.	(000000010001010305000000000000060)	2.	(000000020001010204000000001000060)
3.	(000000030001010103000000002000060)	4.	(000000040001010002000000003000060)
5.	(000000020002020004000000000000061)	6.	(000000020002010104000000000000160)
7.	(000000020001020103010000001000060)	8.	(000000030002010003000000001000160)
9.	(000000020002020003000001010000060)	10.	(000000030001020002010000002000060)
11.	(000000020002020002010100010000060)		

Appendix D

1.	(0000003002000030050000000001010000)	2.	(0000004003000010040000000001010100)
3.	(0000004002000020040000000011010000)	4.	(0000003003000020040000010001010000)
5.	(0000005003000000003000000011010100)	6.	(0000005002000010030000000021010000)
7.	(0000004003000010030000100111010000)	8.	(0000006002000000002000000031010000)
9.	(0000013002000020040000000002010000)	10.	(0000004002000110040000000002010000)
11.	(0000014003000000003000000002010100)	12.	(0000014002000010030000000012010000)

13. (000001300300001003000010002010000)
14. (000000400300001003000100002010000)
15. (000000500300000003000000002011000)
16. (000000500200010003000000012010000)
17. (000001500200000002000000022010000)
18. (000002000200004003300000000010000)
19. (000002300200001003000000003010000)
20. (000001400200010003000000003010000)
21. (000002400200000002000000013010000)
22. (000000410300000003000000002020000)
23. (000101300200001003000000002020000)
24. (000100400200010003000000002020000)
25. (000101400200000002000000012020000)
26. (000102000200003002300000000020000)

Appendix E

1. (000000300200004006000000001010000)
2. (000000400300002005000000001010100)
3. (000000400200003005000000011010000)
4. (000000300300003005000010001010000)
5. (000000500400000004000000001010200)
6. (000000500300001004000000011010100)
7. (000000500200002004000000021010000)
8. (000000400400001004000010001010100)
9. (000000400300002004000010011010000)
10. (000000300400002004000020001010000)
11. (000000600300000003000000021010100)
12. (000000600200001003000000031010000)
13. (000000500300001003000010021010000)
14. (000000700200000002000000041010000)
15. (000001300200003005000000002010000)
16. (000000400200012005000000002010000)
17. (000001400300001004000000002010100)
18. (000001400200002004000000012010000)
19. (000001300300002004000010002010000)
20. (000000400300002004000100002010000)
21. (000000500300010004000000002010100)
22. (000000500300001004000000002011000)
23. (000000500200011004000000012010000)
24. (000000400300011004000010002010000)
25. (000001500300000003000000012010100)
26. (000001500200001003000000022010000)
27. (000001400300001003000010012010000)
28. (000000500300001003000100012010000)
29. (000000600300000003000000012011000)
30. (000000600200010003000000022010000)
31. (000001600200000002000000032010000)
32. (000002000200005004300000000010000)
33. (000002300200002004000000003010000)
34. (000001400200011004000000003010000)
35. (000000500200020004000000003010000)
36. (000002000300004003300010000010000)
37. (000002400300000003000000003010100)
38. (000002400200001003000000013010000)
39. (000002300300001003000010003010000)
40. (000001400300001003000100003010000)
41. (000001500300000003000000003011000)
42. (000001500200010003000000013010000)
43. (000000500300010003000100003010000)
44. (000002500200000002000000023010000)
45. (000002000200014003400000000010000)
46. (000002400200010003000000004010000)
47. (000000500300000104000000002020000)
48. (000000410300001004000000002020000)
49. (000000510300000003000000012020000)
50. (000001410300000003000000003020000)
51. (000101300200002004000000002020000)
52. (000100400200011004000000002020000)
53. (000000500201010004000000002020000)
54. (000101400300000003000000002020100)
55. (000101400200001003000000012020000)
56. (000101300300001003000010002020000)
57. (000100400300001003000100002020000)
58. (000100500300000003000000002021000)
59. (000100500200010003000000012020000)
60. (000000500300010003010000002020000)
61. (000101500200000002000000022020000)
62. (000002010240000003000030000020000)
63. (000102000200004003300000000020000)
64. (000102300200001003000000003020000)
65. (000101400200010003000000003020000)
66. (000102400200000002000000013020000)
67. (000202000200003002000030300000000)

Appendix F

1.	(000000300200005007000000000101000)	2.	(000000400300003006000000000101010)
3.	(000000400200004006000000001101000)	4.	(000000300300004006000001000101000)
5.	(000000500400001005000000000101020)	6.	(000000500300002005000000001101010)
7.	(000000500200003005000000002101000)	8.	(000000400400002005000001000101010)
9.	(000000400300003005000001001101000)	10.	(000000300400003005000002000101000)
11.	(000000600400000004000000001101020)	12.	(000000600300001004000000002101010)
13.	(000000600200002004000000003101000)	14.	(000000500400001004000001001101010)
15.	(000000500300002004000001002101000)	16.	(000000400400002004000002001101000)
17.	(000000700300000003000000003101010)	18.	(000000700200001003000000004101000)
19.	(000000600300001003000001003101000)	20.	(000000800200000002000000005101000)
21.	(000001300200004006000000000201000)	22.	(000000400200013006000000000201000)
23.	(000001400300002005000000000201010)	24.	(000001400200003005000000001201000)
25.	(000001300300003005000001000201000)	26.	(000000400300003005000100000201000)
27.	(000000500300011005000000000201010)	28.	(000000500300002005000000000201100)
29.	(000000500200012005000000001201000)	30.	(000000400300012005000001000201000)
31.	(000001500400000004000000000201020)	32.	(000001500300001004000000001201010)
33.	(000001500200002004000000002201000)	34.	(000001400400001004000001000201010)
35.	(000001400300002004000001001201000)	36.	(000001300400002004000002000201000)
37.	(000000500400001004000100000201010)	38.	(000000500300002004000100001201000)
39.	(000000400400002004000110000201000)	40.	(000000600400000004000000000201110)
41.	(000000600300010004000000001201010)	42.	(000000600300001004000000001201100)
43.	(000000600200011004000000002201000)	44.	(000000500400001004000001000201100)
45.	(000000500300011004000001001201000)	46.	(000001600300000003000000002201010)
47.	(000001600200001003000000003201000)	48.	(000001500300001003000001002201000)
49.	(000000600300001003000100002201000)	50.	(000000700300000003000000002201100)
51.	(000000700200010003000000003201000)	52.	(000001700200000002000000004201000)
53.	(000002000200006005300000000001000)	54.	(000002300200003005000000000301000)
55.	(000001400200012005000000000301000)	56.	(000000500200021005000000000301000)
57.	(000002000300005004300010000001000)	58.	(000002400300001004000000000301010)
59.	(000002400200002004000000001301000)	60.	(000002300300002004000001000301000)
61.	(000002000330002004000040000001000)	62.	(000001400300002004000100000301000)
63.	(000001500300010004000000000301010)	64.	(000001500300001004000000000301100)
65.	(000001500200011004000000001301000)	66.	(000001400300011004000001000301000)
67.	(000000500300011004000100000301000)	68.	(000000600300010004000000000301100)
69.	(000000600200020004000000001301000)	70.	(000000500300020004000001000301000)
71.	(000002500300000003000000001301010)	72.	(000002500200001003000000002301000)
73.	(000002400300001003000001001301000)	74.	(000001500300001003000100001301000)
75.	(000001600300000003000000001301100)	76.	(000001600200010003000000002301000)
77.	(000000600300010003000100001301000)	78.	(000002600200000002000000003301000)
79.	(000002000200015004400000000001000)	80.	(000002400200011004000000000401000)
81.	(000001500200020004000000000401000)	82.	(000003000300004003400000000001010)
83.	(000003000200005003400000001001000)	84.	(000002000300014003400010000001000)
85.	(000003000300004003300001000101000)	86.	(000003000240001003000040010010000)
87.	(000002500300000003000000000401100)	88.	(000002500200010003000000001401000)
89.	(000001500300010003000100000401000)	90.	(000003500200000002000000002401000)
91.	(000000500300001105000000000202000)	92.	(000000410300002005000000000202000)
93.	(000000600400000004000000000202001)	94.	(000000600300001040000000001202000)
95.	(000000500400001040000100000202000)	96.	(000000510400000004000000000202010)
97.	(000000510300001004000000001202000)	98.	(000000410400001004000001000202000)
99.	(000000610300000003000000002202000)	100.	(000001500300001040000000000302000)

101.	(00000141030000100400000000302000)	102.	(00000051030001000400000000302000)
103.	(00000151030000000300000001302000)	104.	(00010130020000300500000000202000)
105.	(00010040020001200500000000202000)	106.	(00000050020101100500000000202000)
107.	(00010140030000100400000000202010)	108.	(00010140020000200400000001202000)
109.	(00010130030000200400001000202000)	110.	(00010040030000200400010000202000)
111.	(00010050030001000400000000202010)	112.	(00010050030000100400000000202100)
113.	(00010050020001100400000001202000)	114.	(00010040030001100400001000202000)
115.	(00000050030001100401000000202000)	116.	(00000060030100000400000000202100)
117.	(00000060020101000400000001202000)	118.	(00000050030101000400001000202000)
119.	(00010150030000000300000001202010)	120.	(00010150020000100300000002202000)
121.	(00010140030000100300001001202000)	122.	(00010050030000100300010001202000)
123.	(00010060030000000300000001202100)	124.	(00010060020001000300000002202000)
125.	(00000060030001000301000001202000)	126.	(00010160020000000200000003202000)
127.	(00000201024000100400003000002000)	128.	(00000201030000400340000000002000)
129.	(00000241030000000300000000402000)	130.	(00010200020000500430000000002000)
131.	(00010230020000200400000000302000)	132.	(00010140020001100400000000302000)
133.	(00010050020002000400000000302000)	134.	(00000150020101000400000000302000)
135.	(00010200030000400330001000002000)	136.	(00010240030000000300000000302010)
137.	(00010240020000100300000001302000)	138.	(00010230030000100300001000302000)
139.	(00010140030000100300010000302000)	140.	(00000240030000100301000000302000)
141.	(00010150030000000300000000302100)	142.	(00010150020001000300000001302000)
143.	(00010050030001000300010000302000)	144.	(00000150030001000301000000302000)
145.	(00000250030000000300000000312000)	146.	(00000250020100000300000001302000)
147.	(00010250020000000200000002302000)	148.	(00010200020001400340000000002000)
149.	(00010240020001000300000000402000)	150.	(00020200020000400300003030000000)
151.	(00020300020000300200003030100000)	152.	(00010141030000000300000000303000)
153.	(00010201024000000300003000003000)	154.	(00020240020000000200000001303000)

Appendix G

1.	(00000002010101020500000000000070)	2.	(00000003010101010400000001000070)
3.	(00000004010101000300000002000070)	4.	(00000002000102020500000000100060)
5.	(00000003000102010400000001100060)	6.	(00000004000102000300000002100060)
7.	(00000002000202010500000000000070)	8.	(00010002000102010400000001000070)
9.	(00000002000202010400001001000060)	10.	(00000003010201000400000000000170)
11.	(00000003000202000400000001000070)	12.	(00000002010202000400000100000070)
13.	(00010003000102000300000002000070)	14.	(00000003010102000301000001000070)
15.	(00000003000202000300001002000060)	16.	(00000003000202000400000000100160)
17.	(00000003000103000400000000110060)	18.	(00000003000103000301000001100060)
19.	(00000003000202000300010001100060)	20.	(00010002000202000300010001000070)