Lotto Designs

Article *in* Journal of Combinatorial Mathematics and Combinatorial Computing · January 1998

2 authors:

J. A. Bate
University of Manitoba

G. H. J. van Rees
University of Manitoba

Lotto Designs

J. A. Bate and G. H. J. van Rees

Department of Computer Science
University Of Manitoba
Winnipeg, Manitoba
Canada R3T 2N2

bate/vanrees@cs.umanitoba.ca

Dedicated to Anne Penfold Street

Abstract

Let $L(n,k,p,t)$ be the minimum number of subsets of size k (*k-subsets*) of a set of size n (*n-set*) such that any p-subset intersects at least one of these k-subsets in at least t elements. The value of L(n,6,6,2) is determined for $n \leq 54$.

1. Introduction

There are many lotteries in the world. Colbourn catalogues some of them in his survey paper [4]. Most of these lotteries are fairly new and people are interested in devising efficient lottery betting schemes as evidenced by several recent papers [5],[6] and [9], by the marketing of many betting schemes and by several newsgroups on the internet.

The lotteries we refer to work in the following way. The government picks p distinct random numbers from a set of n numbers. Before this happens, the people are allowed to buy tickets and choose k numbers per ticket. If the p numbers chosen by the government intersect the k numbers from a ticket in exactly t numbers, then the buyer gets a prize. The larger t is, the greater the prize. For very small values of t no prize is given. The question to be examined in this paper is "how many tickets are required to be bought in order to guarantee a buyer an intersection of t numbers and the prize that goes with it?"

More formally, we can define $LD(n,k,p,t;j)$ to be a set of j k-subsets (*blocks* or simply *k-sets*) of a n-set such that any p-subset intersects at least one k-subset in at least t elements. Let *L(n,k,p,t)* be the minimum number of blocks in any LD(n,k,p,t;j). If the LD(n,k,p,t;j) has j = L(n,k,p,t), then this will be called a minimal LD(n,k,p,t:j) and indicated by *LD*(n,k,p,t;j)*. The fundamental question in this paper is what is L(n,k,p,t)?

2. History

The first results on this subject were given by Bate in 1978 [1] but they were little noticed. He proved and we state the following obvious lemma.

Lemma 2.1 $L(n,k,p,1) = \lceil (n-p+1)/k \rceil$.

Since the problem has been solved for t = 1, we will always assume that t > 1 in this paper. This will eliminate some annoying trivialities. It has also been observed, by many, that a LD(n,k,t,t) is a (n,k,t)-cover and that a LD(n,k,p,k) is a (n,p,k)-Turan Design. An *(n,k,t)-cover* is a set of k-subsets (blocks) of a n-set such that every t-subset occurs in some k-subset. An *(n,k,p)-Turan Design* is a set of k-subsets (blocks) of an n-set such that every p-subset contains at least one of those k-subsets. The two configurations are related as follows.

Theorem 2.2 *If the blocks of an (n,k,t)-cover are complemented, then an (n,n-t,n-k)-Turan Design is produced.*

The minimum number of blocks in an (n,k,t)-cover is denoted by *C(n,k,t)*. A very useful lower bound is the Schonheim bound [11].

Lemma 2.3 $C(n,k,t) \geq \lceil n/k \lceil (n-1)/(k-1) ... \lceil (n-t+1)/(k-t+1) \rceil ... \rceil \rceil$

Proof: Every element of the n-set must be contained in at least C(n-1,k-1,t-1) blocks. Hence C(n,k,t) ≥ nC(n-1,k-1,t-1)/k. By Lemma 2.1 C(n,k,1) ≥ n/k and the result follows by iteration. ❑

See Mullin & Mills [10] for a survey on covers and Turan designs and Gordon et al. [7] for recent results. The problem for L(n,3,3,2) was first solved by Bate in [1] and later, independently, by Brouwer in [3]. We use Brouwer’s formulation of the result.

Lemma 2.4 $L(2m+1,3,3,2) = C(m,3,2)+C(m+1,3,2)$.

$L(4m,3,3,2) = C(2m-1,3,2)+C(2m+1,3,2)$.

$L(4m+2,3,3,2) = 2C(2m+1,3,2)$.

The next result from Furedi et al. [6] uses Turan's Theorem as modified by Erdos in Bollobas[2].

Theorem 2.5

$L(n,k,p,2) \geq min[a_1+...+a_{p-1}=n] \quad (a_1\lceil(a_1-1)/(k-1)\rceil + ... + a_{p-1}\lceil(a_{p-1}-1)/(k-1)\rceil)/k$

In order to get an upper bound, we can divide the n-set into p-1 parts of sizes $a_1,...,a_{p-1}$. For each part, constuct a an $(a_i,k,2)$-cover. Now any p-subset meets one of these (p-1) parts in at least 2 elements and since there is a cover on each part, any p-subset must intersect one of the blocks in 2 elements. Thus we can state the following theorem.

Theorem 2.6

$L(n,k,p,2) \leq min[a_1+...+a_{p-1}=n] \qquad (C(a_1,k,2) + ... + C(a_{p-1},k,2))$

We will call a design constructed in this way a "sum of disjoint covers". In many cases the two bounds meet and therefore determine the value of L(n,k,p,2). Some of these cases are listed in Furedi et al.[6]. When this happens, the minimum set $\{a_1,...,a_{p-1}\}$ is usually the same for both Theorem 2.5 and Theorem 2.6 and each of the disjoint covers meets the Schonheim bound. However, most minimal covers with $k \geq 6$ do not meet the Schonheim bound and in this case ascertaining the value of L(n,k,p,2) is much more difficult. In the next section, we give some general results and then in Section 4 we find L(n,6,6,2) for $n \leq 54$.

3. General Results

Before trying to find specific values of L(n,k,p,t), we first prove some useful general results.

Lemma 3.1 *L(n,k,p,t) ≤ L(n+1,k,p,t).*

Proof: Assume L(n,k,p,t) > L(n+1,k,p,t). Consider an LD*(n+1,k,p,t;w). Pick an element, x, and delete every occurrence of that element from the design. To the blocks which have been shortened add any element, y, not in the block and not equal to x. The element y exists because if it did not exist then k = n+1 and k > n in a LD(n,k,p,t) which is a contradiction. Clearly, the new design has fewer than L(n,k,p,t) blocks. This is a contradiction. ❑

The following lemma appears in [1]. It allows us to handle elements of the n-set that occur in only one block in an efficient manner.

Lemma 3.2 *If an LD(n,k,p,t;b) exists with b ≥ n/k, then there exists an LD(n,k,p,t;b) in which every element of the design occurs in some block.*

Proof: Since b ≥ n/k, either every element occurs in the design and the lemma is true or there is some element, x, that occurs in more than one block and an element, say y, that occurs in no block. Let A be one of the blocks containing x, and replace the x in A with y. The modified design must also be a LD(n,k,p,t;b). To prove this consider a p-set, P, which intersects no k-set in a t-set in the modified design. However, in the original design P did intersect at least one k-set in at least t elements. That k-set had to be A and that must have been the only k-set intersected by P in t elements in the original design. Then that p-set must contain x but not y and must contain exactly t-1 other elements from block A. But in that case the p-set $(P-\{x\})\cup\{y\}$ does not intersect any k-set in at least t elements in the original design, which is a contradiction. So the modified design is also an LD(n,k,p,t;x). If there are any more elements that occur in no blocks, this modification can be repeated until all elements occur in at least one block of an LD(n,k,p,t;x). ❑

Next we will examine Lotto Designs with t = 2 very closely. We will borrow some suggestive graph theory terminology. An *independent set* in a lotto design is a set of elements, no pair of which occurs together in any block of the design. It is *maximal* if the set can not be enlarged. The blocks containing elements of a particular maximal independent set are called the *independent blocks*. A *maximum* independent set has the largest cardinality of any independent set in a particular lotto design. The next Lemma is very useful.

Lemma 3.3 *In any LD*(n,k,p,2;w) with an independent set of size p-1, every element of the design must occur in the independent blocks of a maximal independent set.*

Proof: If an element does not occur in any block with some element of a maximal independent set, then that set could be enlarged, giving an independent set of size p, which is a contradiction. ❑

The maximum size of an independent set when t = 2 is p-1. Let f_i be the number of elements in a lotto design that occur in i blocks. If I is an independent set, let A_I be a vector, in non-decreasing order, of the frequencies of the elements in I.

Lemma 3.4 *An LD*(n,k,p,2;w) with n > k(p-2) implies the existence of an LD*(n,k,p,2;w) which has an independent set of size p-1.*

Proof: If p = 2 the theorem is obviously true. For n > k(p-2) and p > 2, assume that the size of the largest independent set in any LD*(n,k,p,2;w) is b and that b < p-1. Consider the LD*(n,k,p,2;w) which has the lexicographically least vector A_I where the cardinality of I is b. Since every element of the design must occur in the independent blocks, if every independent element had frequency 1 there could be at most k(p-2) elements in the design. This contradicts n > (p-2)k, so at least one element, x, of the independent set must have frequency at least 2.

Consider two blocks containing x, B_1 and B_2. If the blocks are identical, the design is not minimal so let y be in B_1 and not in B_2. Replace x in B_2 with y. If the modified design is still a LD*(n,k,p,2;w), then the set I is still an independent set, which must still be maximal (by assumption), but which has a lexicographically lower A_I (since the frequency of x has decreased) which is a contradiction.

Now we must show that the modified design is still a LD*(n,k,p,2;w). The new design has the same n, k, and w as the old one. Now consider any p-set that does not contain both x and y. Then it contains an (p-1)-set not containing either x or y. Since the (p-1)-set cannot be an independent set, as its size is too large, this (p-1)-set meets some block in the original design in 2 elements. It still meets that same block in 2 elements in the new design. A p-set containing both x and y contains an (p-1)-set $S = \{x,z_1,...z_{p-2}\}$, containing x but not y. Now S must also intersect some block in the original design in 2 elements. It will intersect that same block in 2 elements in the new design unless perhaps that block was the modified block and x was one of the intersecting elements. Let the two intersecting elements be x,z_j. But in that case the original p-set, $S\cup\{y\}$, still intersects B_2 in y,z_j. So the modified design is still a LD*(n,k,p,2;w), which completes the proof. ❑

The next lemma allows us to handle nicely the elements that occur only once in the lotto design. It turns out that there is a lotto design on the same number of blocks that has these elements of frequency one occurring in blocks that only contain such elements. So we define a block of a lotto design to be *isolated* if it contains only elements of frequency one. The elements in an isolated block are called *isolated* elements.

Theorem 3.5 *If $n \geq k(p-1)$ then an LD(n,k,p,2;x) implies the existence of an LD(n,k,p,2;x) in which there are $f_1 = rk$ elements of frequency one occurring in r isolated blocks.*

Proof: Let there be r isolated blocks in the LD(n,k,p,2;x). In a lotto design with t = 2, an independent set may contain at most p–1 elements and since a isolated block can always contribute one to the size of any independent set, $r \leq p-1$. If r = p-1, then the original design has exactly (p-1)k elements with frequency one and the lemma holds. If r = p-2, and we examine the non-isolated blocks, we see that they form a subdesign $S = LD(n_1,k,2,2;x-r)$, where $n_1 = n-k(p-2) \geq k$ since $n \geq k(p-1)$. Since p = t = 2, S is a cover in which every pair of its n_1 elements must occur. If $n_1 = k$, we have the previous case. If $n_1 > k$, then any element of frequency one in S cannot occur with all the remaining n_1-1 elements as required. So there are no more elements of frequency one in the original lotto design and the design fits the lemma. Therefore let $r \leq p-3$ which implies that there are at least 2k non-isolated elements.

Let us consider this lotto design. Suppose that an element, a, has frequency 1 and that it appears in a block, B, with an element, b, which has frequency greater than 1. Let $X = \{x_1,...x_{k-2}\}$ be the set of remaining elements in that block. Let us delete all other occurrences of the element b in the whole design. Any p-set containing ab, ax_i, or bx_i intersects B in a pair. Any p-set that does not contain b will still intersect the same block in a pair as it did before. We must still account for p-sets containing b but not any x_i or a, ie $\{b,y_1,...,y_{p-1}\}$ where y_i is not an element of X and $y_i \neq a$. But the p-set $\{a,y_1,...,y_{p-1}\}$ must intersect some block in a pair y_iy_j, so the p-set $\{b,y_1,...,y_{p-1}\}$ will also intersect this block in that pair.

Now we must show that it is possible to replace the deleted b's. Consider block B and one other block C which contained an occurrence of b (now deleted). Since there are at least 2k non-isolated elements, there must be some non-isolated element that does not occur in B or C. Use that element to replace b in block C. In this way all the deleted b's may be replaced. Thus we have increased the number of elements of frequency one in block B, but the design is still an LD(n,k,p,2;x). This procedure can be repeated until a lotto design obeying the restrictions of the lemma is produced. ❑

Lemma 3.6 *Any maximal independent set from a LD(n,k,p,t) must contain one element from each isolated block.*

Proof: If no element from an isolated block appears in a maximal independent set, then any of the elements in that block could be added to make a larger independent set, which is a contradiction. ❑

4. L(n,6,6,2) for n $\leq$ 54

Examining Colbourn’s list, we find that the most popular value for k and p in existing lotteries is 6. So we will take the general graph-theoretical ideas of Furedi et al.[6] and apply them in a design-theoretical way to the specific case of k = p = 6. By examining these cases closely we can improve on the results given for those cases (L(45,6,6,2) = 14 or 15 for the Hungarian and Austrian Lotteries and L(49,6,6,2) = 16, 17, 18 or 19 for the German, French, British and Canadian Lotteries).

Consider L(n,6,6,2). Clearly, we have the following lemma.

Lemma 4.1 *For $6 \leq n \leq 30$, the values of L(n,6,6,2) are as given below.*

n	L(n,6,6,2)
6-10	1
11-15	2
16-20	3
21-25	4
26-30	5

Let us plot the Furedi lower bound from Lemma 2.5 (F) against the sum of disjoint covers upper bound from Lemma 2.6 (C) for $31 \leq n \leq 54$:

n	F	C	n	F	C	n	F	C
31	7	7	39	10	11	47	15	17
32	7	7	40	11	12	48	15	18
33	7	7	41	12	13	49	16	19
34	8	8	42	12	13	50	16	19
35	8	9	43	13	14	51	16	20
36	9	9	44	13	15	52	18	21
37	10	10	45	13	15	53	18	22
38	10	11	46	15	16	54	18	23

In the rest of the paper, we want to find the minimum number of blocks in lotto designs with certain parameters. These parameters will obey the restrictions of Lemmas 3.2, 3.4 and Theorem 3.5 and so we can always restrict our research to "nice" optimal lotto designs. For this reason we define the following.

A *nice* LD(n,k,p,t;x) is a lotto design wherein each element occurs at least once, the elements of frequency one occur only in isolated blocks and there is an independent set of size p-1.

Theorem 4.2 *There exists a nice LD*(n,k,p,t;x) for $x \geq n/k$ and $n \geq k(p-1)$.*

Proof: Apply Lemmas 3.2, 3.4 and Theorem 3.5. ❑

Theorem 4.3 *For a nice LD(n,k,p,2;x) with an independent set, I, and b corresponding independent blocks, $n \leq (k-1)b + (p-1)$.*

Proof: Every element of a lotto design with t = 2 must occur in some independent block of any maximal independent set, by Lemma 3.3. There are at most (p-1) elements in the independent set, and at most (k-1)b other elements in the independent blocks, which gives the required result. ❑

Before we go on, we must define a few terms. *Singles* are elements that occur just once in the independent blocks of a particular independent set I, and are not themselves in I. An *m-clique* is the set of m blocks that contain all the occurrences of a particular independent element, x, of a particular independent set. The following lemma is needed.

Lemma 4.4 *If two singles occur in different blocks of the same i-clique which is part of a maximum independent set, then that pair of singles must occur together in some block which is not an independent block.*

Proof: Suppose that the two elements a and b occur in different blocks of some i-clique whose independent element is x. Assume that a and b do not occur together in any block. Then if we delete x and add a and b to the independent set, we get a larger independent set which is a contradiction. ❑

The following lemma which appears in Hartman's thesis [8] is an adaptation of a result of Shannon[12].

Lemma 4.5 *If a multigraph has e edges and maximum degree d, then there is a set of* $\lceil e / \lfloor 3d/2 \rfloor \rceil$ *disjoint edges.*

We use this to prove the following lemma:

Lemma 4.6 *In a nice lotto design with k = 6, there exists a maximal independent set I with at least* $\lceil f_2/9 \rceil$ *2's in* A_I.

Proof: Consider the blocks of the lotto design as vertices and let two vertices be joined by one edge for every element of frequency two that occurs in both corresponding blocks. Then $e = f_2$ and $d = k = 6$ in the graph. Applying Lemma 4.5 shows that there are at least $\lceil f_2/9 \rceil$ elements of frequency 2, no two of which appear together in any block, and which therefore may be included in an independent set I. If the set is not maximal, other elements may be added until it is maximal. ❑

Since we will be considering designs and portions of designs in which most elements have frequencies of 1 or 2, it will be convenient to measure the extent to which there are elements with higher frequencies. The *excess frequency count* of a set of elements E, efc(E), is the number $\sum_{i>2} (i-2)g_i$ where g_i is the number of elements of frequency i in the set E. If no set of elements is specified, then all of the elements in the design are assumed. For example, if efc = 3 then the lotto design has either three elements of frequency 3, one each with frequencies 3 and 4, or one element of frequency 5.

We are now ready to study some of the lotto designs in general. We will first look at the number of singles that appear in 2-cliques and 3-cliques. Let S refer to the set of such singles under consideration. Using Lemma 4.4, we will obtain a lower bound for efc(S).

Consider a 2-clique which contains 5 or fewer singles (S). It is possible that all of these singles appear in the same block in the clique, and therefore no additional blocks are needed to hold pairs of these singles, nor does any element require a frequency greater than 2, giving a trivial lower bound of 0 for efc(S).

If there are 6 singles in S, then they cannot all appear in the same block of a 2-clique, and there must be pairs of singles which must appear together elsewhere, by Lemma 4.4. All 6 singles could appear in the same “extra” block (outside of the independent blocks). Again, no element will require a frequency greater than 2, giving a lower bound of 0 for efc(S).

If there are 7 singles in a 2-clique then the minimum efc(S) is 2. Suppose efc(S) is 1 or less. There are 2 cases to consider. In the first case let there be 5 singles in one block {x,a,b,c,d,e} and 2 in the other {x,f,g,-,-,-}. One of f or g (say, f) must occur just once more (since efc(S) < 2) and it must appear with abcde, forcing the extra block {f,a,b,c,d,e}. But a, b, c, d and e must occur again with g giving efc(S) ≥ 5, which is a contradiction. In the second case let a, b, c, and d appear in one block {x,a,b,c,d,–} and e, f and g in another {x,e,f,g,–,–}. At least two of e, f or g (say, e and f) must each occur just once more and hence must occur with a, b, c and d in the extra block {e,f,a,b,c,d}. But a, b, c and d must each occur at least once more with g. Hence a, b, c and d have frequency at least 3 giving efc(S) ≥ 4, which is also a contradiction, proving that efc(S) ≥ 2. The configuration {x,1,2,3,4,5}, {x,6,7,...}, {1,2,3,4,6,7} and {5,6,7,...} shows that this minimum may be obtained.

If there are 8 singles in a 2-clique then the minimum efc(S) is 3, and in addition there is a unique configuration of the 8 singles in the extra blocks which will achieve this minimum value. Suppose that efc(S) is 2 or less. There are two cases to consider. In the first case, let the independent blocks be {x,1,2,3,4,-} and {x,5,6,7,8,-}. If efc(S) is 3 or less, one of 5, 6, 7 or 8 (say, 5) must occur only once more and hence occurs with 1, 2, 3 and 4 in the block {5,1,2,3,4,-}. There is not room in that block for all of 6, 7 and 8. Say 6 does not occur in that block. Then 1, 2, 3 and 4 must occur in another block with 6. This implies efc(S) ≥ 4 which is a contradiction. In the second case, let the independent blocks be {x,1,2,3,4,5} and {x,6,7,8,-,-}. Suppose efc(S) ≤ 3. If one of 6, 7 or 8, say 8, has frequency 2, then it occurs in one non-independent block with 1, 2, 3, 4 and 5. But 1, 2, 3, 4 and 5 must still appear with 6 and 7, giving efc(S) ≥ 5, which is also a contradiction. Therefore all of 6, 7, and 8 must have frequency 3, giving efc(S) ≥ 3. To achieve the minimum value of 3, all of 1..5 must have frequency 2, and each must appear in an extra block with all of 6..8. The configuration {x,1,2,3,4,5}, {x,6,7,8,...}, {6,7,8,1,2,3}, {6,7,8,4,5,–} is the unique way in which the minimum can be obtained, up to isomorphism.

If there are 9 singles in a 2-clique then the minimum efc(S) is 7. The 2-clique must look like {x,1,2,3,4,5} and {x,6,7,8,9,-}. Suppose efc(S) is 6 or less. Then at least 9 - 6 = 3 singles must appear only once more. If one single from each block (say, 1 and 6) each appear only once more, then the extra block {6,1,2,3,4,5} is forced but then 1 cannot appear with 7, 8, or 9 as required. If three singles from the second block (say, 6, 7, and 8) each appear only once more, this forces the extra blocks {6,1,2,3,4,5}, {7,1,2,3,4,5}, and {8,1,2,3,4,5} but this gives efc(S) ≥ 10 since each of 1..5 appear 4 times. which is a contradiction. If three singles from the first block (say, 1, 2 and 3) each appear only once more, this forces two blocks such as {1,6,7,8,9,-} and {2,6,7,8,9,-}. But now all of 6...9 must appear a 4th time in order to form pairs with 3, 4 and 5, giving efc(S) ≥ 8, which is again a contradiction. This gives efc(S) ≥ 7. The configuration {x,1,2,3,4,5}, {x,6,7,8,9,...}, {1,2,6,7,8,9}, {3,4,5,6,7,8}, {3,4,5,9,...} shows that this minimum can be obtained.

If there are 10 singles in a 2-clique then the minimum efc(S) is 10. The 2-

clique must look like {x,1,2,3,4,5} and {x,6,7,8,9,10}. Suppose efc(S) is 9 or less. Without loss of generality let 6 be a single of frequency 2. This forces the extra block {6,1,2,3,4,5}. There cannot be a second single of frequency 2 for the following reasons. If there were a second single of frequency 2 from the other block (say, 1), then 1 could not appear again with 7-10, as required. If there were a second single of frequency 2 from the same block (say, 7), this would force the extra block {7,1,2,3,4,5} but then all of 1-5 must appear a fourth time, giving efc(S) $\geq$ 10, which is a contradiction. Since no element other than 6 may have frequency 2, and efc(S) $\leq$ 9, all of 1-5 and 7-10 must have frequency 3. Each of 1-5 must appear exactly once more, forcing two blocks such as {1,7,8,9,10,–} and {2,7,8,9,10,–} but now 7-10 may not appear again and therefore cannot appear with 3-5, as required. This shows that efc(S) $\geq$ 10. The configuration {x,1,2,3,4,5}, {x,6,7,8,9,10}, {1,2,3,6,7,8}, {1,2,3,4,9,10}, {4,5,6,7,8,9} and {5,10,...} shows that this minimum can be met.

Now we will consider 3-cliques. What is the minimum efc(S) for a given set of singles S in a 3-clique?

For 10 or fewer singles in S the answer is the same as in the 2-clique case. If the singles are not split into the 3 blocks as (2,4,4), (3,3,3) or (3,3,4) then the singles that occur in the block with the fewest singles can be put with the singles in the block with the second fewest singles. So such a configuration with a particular efc(S) would give a configuration in a 2-clique with the same efc(S). Therefore it cannot give a smaller minimum efc(S) value than the corresponding 2-clique case. The specific configurations identified above are quickly ruled out. If efc(S) were less than 7 in the (3,3,3) case or less than 10 in the (2,4,4) or (3,3,4) cases, then at least one element would have to appear only once more, and it would not have enough room to occur with all the singles. So the previous results on 2-cliques also hold for 3-cliques, and Table 1 may serve for both.

We can summarize this information in Table 1.

Num. of singles	5	6	7	8	9	10
Minimum efc(S)	0	0	2	3	7	10

Table 1: Minimum efc(S) in 2-cliques and 3-cliques with a set S of up to 10 singles

For a set S of $r \geq 11$ singles in a 3-clique, we must have efc(S) $\geq$ r. Assume efc(S) < r. Then at least one single, a, must have frequency 2, so it may occur in only one more non-independent block. This means that it occurs with at least $r-5 \geq 6$ other singles in that block. But blocks have only 6 elements in them so this configuration can not be realized.

Now a 3-clique with a set S of 11 singles and efc(S) = 11 can be realized as shown by the following blocks: {x,1,2,3,4,5}, {x,6,7,8,9,10}, {x,11,...}, {1,2,3,6,7,8}, {1,2,3,9,10,11}, {4,5,6,7,8,11}, {4,5,9,10,...}. A 3-clique with a set S of 12 singles and efc(S) = 12 can be realized as shown by the following blocks: {x,1,2,3,4,–}, {x,5,6,7,8,–}, {x,9,10,11,12,–}, {1,2,5,6,9,10}, {1,2,7,8,11,12}, {3,4,5,6,11,12}, {3,4,7,8,9,10}.

If there is a set S of 13 singles in a 3-clique then efc(S) ≥ 16. Suppose efc(S) = 15. Consider the first case: {x,1,2,3,4,5}, {x,6,7,8,9,10} and {x,11,12,13,...}. There can be no singles of frequency 1 or 2 as such singles could not appear with all of the others, as required. So all but at most two elements have frequency 3. So one of 11, 12, or 13, say 11, occurs in precisely two more blocks with the elements from 1 to 10. These two blocks each contain at most 5 + 6 = 11 pairs which have not already appeared in the independent blocks (5 involving 11 and at most 6 involving only 1..10). Then in the remaining blocks we must have at least 33 pairs of singles and 16 occurrences of those singles for an average of 2.06. But no block or part block can attain that ratio. The best one can do is {1,2,6,7,12,13} which has 12 pairs of singles and 6 occurrences for an average of 2. Now consider the other case: {x,1,2,3,4,5}, {x,6,7,8,9,..} and {x,10,11,12,13,...}. One of 10, 11, 12 or 13, say 10, must have frequency 3 so there must be a block containing 10 and none of 11, 12 or 13. As in the previous case, if we calculate the ratio of the pairs of singles which must yet appear to the occurrences of those singles in the remaining blocks, we get 45/22 = 2.05. This is still unattainable. So 13 singles require efc(S) ≥ 16.

Suppose we have a set, S, of 15 singles in a 3-clique. The 3-clique and the other forced blocks would almost be a (16,6,2)-covering design. A minimal covering design requires C(16,6,2) = 10 (see [7]) complete blocks whereas our 3-clique and forced blocks need not be complete. By using the minimal (16,6,2)-covering design we obtain a configuration with efc(S) = 7x6 - 15 = 27. Is it possible to have efc(S) = 26? If so then there must be at least 4 singles of frequency 3, and at least two of them must appear in the same independent block. Let the independent blocks be {x,1,2,3,4,5}, {x,6,7,8,9,10} and {x,11,12,13,14,15}, and let 1 and 2 have frequency 3, without loss of generality. This forces each of them to appear in a pair of bocks whose other elements are 6..15 in some order. But now 6..15 have all appeared 3 times already, and none have yet appeared with 3..5 and so none of these may have frequency 3. Therefore the other two singles of frequency 3 must be two of the elements 3..5. But this forces two more pairs of blocks, each containing 6..15, and so all of 6..15 must have frequency ≥ 5, giving efc(S) ≥ 30 which is a contradiction.

A 3-clique with a set S of 14 singles and efc(S) = 25 can be obtained by deleting an element of frequency 4 from the minimal (16,6,2)-cover [13]. Note that efc(S) is 25 and not 24 since we are not considering the independent element which generates the 3-clique itself.

Suppose that a 3-clique with a set S of 14 singles exists with efc(S) = 24. Let the 3-clique consist of the following 3 blocks: {x,1,2,3,4,5}, {x,6,7,8,9,10}, {x,11,12,12,14,–}. At least 4 of the singles must have frequency 3. (None may have frequency less than 3, and if all had frequency 4 it would give efc(S) = 28.) Consider 11, 12 13 and 14. If any of these has frequency 3, it must occur exactly twice in the non-independent blocks, and must appear exactly once with 1..10 and with no other elements. If 2 or more of {11,12,13,14}, say 11 and 12, have frequency = 3, then there are 4 blocks in which 11, 12, and 1..10 each appear twice. But then none of 1..10 may have frequency 3 (since they have already appeared 3 times, and have not yet appeared with 13 or 14), and so 13 and 14 must be the other two elements of frequency 3. But this forces 1..10 to each appear 2 more times, giving efc(S) ≥ 30 which is a contradiction. Suppose just one of {11,12,13,14}, say 11, has frequency 3. Now some other element, say 1, must have frequency 3. This forces, without loss of generality, the following non-independent blocks: {11,1,6,7,8,9}, {11,10,2,3,4,5}, {1,10,12,13,14,–}. Since a frequency 3 single occurs with no element more than once except for one, the only other element that could have frequency 3 is 10. This means $f_3 < 4$, which is a contradiction. Hence let 11, 12 ,13 and 14 have frequency 4. Let 2 singles from distinct independent blocks have frequency 3, say 1 and 6. This forces either {{1,6,11,12,13,14}, {1,7,8,9,10,–}, {6,2,3,4,5,–}} or {{1,6,11,12,13,–}, {1,7,8,9,10,14}, {6,2,3,4,5,14}}. In either case there is at most one more element that could possibly have frequency 3. This means $f_3 < 4$, which is a contradiction. Finally let us assume that 1, 2, 3 and 4 all have frequency 3. So each occurs in exactly two non-independent blocks of the form {X,A,B,C,D,-} and {X,E,F,G,H,I} where A-I is 6-14 and X is 1, 2, 3 or 4. The number of elements, and hence efc(S), in the non-independent blocks will be minimized when the dash is another of the elements 1, 2, 3 or 4, giving two groups of three blocks of the form {X,A,B,C,D,Y}, {X,E,F,G,H,I} and {Y,E,F,G,H,I}. Now 5 occurs at least twice more with new occurrences of 6..14. This is a total of 2x18 + 2 + 9 = 47 occurrences of 1..14 in the non-independent blocks, which corresponds to efc(S) = 47 – 14 = 33, which is a contradiction. Every possibility has now been considered, and therefore if there is a 3-clique with a set S of 14 singles we must have efc(S) ≥ 25.

We summarize this information in the following table.

Num. of singles	11	12	13	14	15
Minimum efc(S)	11	12	16	25	27

Table 2: Minimum efc(S) in 3-cliques with a set S of 11 to 15 singles

In order to use the information in Tables 1 and 2 we need one more lemma.

Lemma 4.7 *In a nice lotto design LD(n,k,p,2) with a maximal independent set I, and s isolated blocks, the minimum number of non-isolated singles in the b independent blocks of that independent set is 2n-2p+2-(k-1)(b-s).*

Proof: Let b_i be the number of non-independent elements that occur i times in the independent blocks. We wish to minimize b_1. By Lemma 3.3 we have

$$\sum b_i = n-p+1 \qquad (1)$$

and by counting the non-independent elements in the independent blocks, we have

$$\sum ib_i=(k-1)b. \qquad (2)$$

Taking 2(1) - (2) and noting that $b_i \geq 0$ gives

$$b_1 \geq 2n-2p+2-(k-1)b.$$

Since there are at most (k-1)s isolated singles, the result follows. ❑

We are now ready to study the individual lotto designs where the Furedi bound and the "sum of disjoint covers" bound are not equal. In general, our proofs will be indirect. By examining the frequencies of elements in the design, we attempt to prove that an independent set with a certain vector A_I exists, which leads to a contradiction. The lexicographically least vectors A_I are the easiest to rule out. We can restrict our attention to nice lotto designs since the restrictions in Theorem 4.2 hold in this section. If there is no nice lotto design of a certain size then there is no lotto design of that size.

Lemma 4.8 *L(35,6,6,2) = 9.*

Proof: Assume that a nice LD(35,6,6,2;8) exists. Recall that f_i denotes the number of elements with frequency i in the design. There are 6x8 = 48 positions in the design to be occupied by 35 elements, each of which appears at least once in a nice lotto design. Therefore $f_1 \geq 22$. By Theorem 3.5 f_1 must be a multiple of 6. By simple counting, the following cases arise.

case	f_1	f_2	efc
1	24	≥9	2
2	30	≥0	8

When f_1=30, Lemma 3.6 shows that there must be an independent set I with A_I = (1,1,1,1,1), but by Theorem 4.3 we see that n ≤ 30 in this case. In this and all of the lemmas that follow, we have n > 30 and so case 2 in which f_1=30 may be eliminated.

Case 1 By Lemma 3.6, there are 4 isolated elements in any maximal independent set. By Lemma 4.6 there is at least one element of frequency 2 in some maximal independent set. So there must be some maximal independent set with A_I = (1,1,1,1,2). By Lemma 4.7, there are at least 10 singles in the 2-clique. Checking Table 1, we see that there can be at most 7 singles in the 2-clique since efc = 2 and a set S of more than 7 singles would give efc(S) > 2. This is a contradiction. ❑

Lemma 4.9 *L(38,6,6,2) = 11.*

Proof: Assume there exists a nice LD(38,6,6,2:10). By the same methods that were used in Lemma 4.8, the following cases are generated:

case	f_1	f_2	efc
1	18	≥18	2
2	24	≥6	8

Case 1 By Lemma 3.6, there are 3 isolated elements in any maximal independent set. By Lemma 4.6 there are at least two elements of frequency 2 in some maximal independent set. So there must be some independent set with $A_I = (1,1,1,2,2)$. By Lemma 4.7 there are at least 16 singles in the two 2-cliques. Checking Table 1 and dividing the efc between the two sets of singles in the two cliques in all possible ways, we see that there are at most 6+7 = 13 singles in the 2-cliques when the efc is 2. This is a contradiction.

Case 2 Lemmas 3.6 and 4.6 require $A_I = (1,1,1,1,2)$, but this is ruled out by Theorem 4.3. ❑

Corollary *L(39,6,6,2) = 11.*

Proof: Lemma 3.1

Lemma 4.10 *L(40,6,6,2) = 12.*

Proof: Assume there exists a nice LD(40,6,6,2:11). By the same methods that were used in Lemma 4.8, the following cases are generated:

case	f_1	f_2	efc
1	18	≥18	4
2	24	≥6	10

Case 1 As in Lemma 4.9, there is a maximal independent set with $A_I = (1,1,1,2,2)$ and there must be at least 20 singles in the two 2-cliques. Checking Table 1, we see that there can be at most 7+7 = 14 or 8+6 = 14 singles in the two 2-cliques when the efc is 4. This is a contradiction.

Case 2 Lemmas 3.6 and 4.6 require $A_I = (1,1,1,1,2)$, but this is ruled out by Theorem 4.3. ❑

Lemma 4.11 *L(41,6,6,2) = 13.*

Proof: Assume there exists a nice LD(41,6,6,2;12). By the same methods that were used in Lemma 4.8, the following cases are generated:

case	f_1	f_2	efc
1	12	≥27	2
2	18	≥15	8
3	24	≥3	14

Case 1 Calculating as before, there must exist a maximal independent set with $A_I = (1,1,2,2,2)$ and by Lemma 4.7 there must be at least 22 non-isolated singles in the independent blocks. However, consulting Table 1, we see that with efc = 2, the most non-isolated singles we could have in the three 2-cliques is 7+6+6=19, which is a contradiction.

Cases 2 and 3 Lemmas 3.6 and 4.6 require $A_I = (1,1,1,2,2)$ or $A_I = (1,1,1,1,2)$, but these are ruled out by Theorem 4.3. ❑

Corollary *L(42,6,6,2) = 13.*

Proof: Lemma 3.1.

Lemma 4.12 *L(43,6,6,2) = 14.*

Proof: Assume there exists a nice LD(43,6,6,2;13). By the same methods that were used in Lemma 4.8, the following cases are generated:

case	f_1	f_2	efc
1	12	≥27	4
2	18	≥15	10
3	24	≥3	16

Case 1 As in Lemma 4.11, there must be a maximal independent set with $A_I = (1,1,2,2,2)$ and by Lemma 4.7 there must be at least 26 non-isolated singles in the independent blocks. However, consulting Table 1, with efc = 4, the most non-isolated singles we could have in the three 2-cliques is 8+6+6=20, which is a contradiction.

Cases 2 and 3 Lemmas 3.6 and 4.6 require $A_I = (1,1,1,2,2)$ or $A_I = (1,1,1,1,2)$, but these are ruled out by Theorem 4.3. ❑

In the rest of the paper, elements of frequency 3 become more important to our arguments, and so the next two lemmas help to handle them.

Lemma 4.13 *In a nice LD(n,k,p,2;x),* $f_2+f_3 \geq (4n - kx - 3f_1)/2$.

Proof: Counting appearances of elements in the design, we see that f_1 elements appear once, f_2+f_3 elements appear at least twice each, and $n–f_1–f_2–f_3$ elements appear at least four times each. Since there are a total of kx appearances of elements in the design, we get

$$f_1 + 2(f_2+f_3) + 4(n-f_1-f_2-f_3) \leq kx$$

and simplifying gives the result. ❑

Lemma 4.14 *In a nice LD(n,k,p,2;x) with an independent set I containing T elements of frequency 2, if*

$$f_2 + f_3 > (1+2(k-1))T$$

then there exists an independent set I^+ containing T elements of frequency 2 and one additional element of frequency 2 or 3.

Proof: The T 2-cliques in the independent blocks contain at most the T elements of frequency 2, plus at most 2(k-1)T other elements of frequency 2 or 3. If $f_2+f_3 > (1+2(k-1))T$, there must exist an element of frequency 2 or 3 which does not appear in any of the 2-cliques, and may therefore be added to the independent set I giving a larger independent set I^+. ❑

We will now resume our discussion of the values of L(n,6,6,2).

Lemma 4.15 *L(44,6,6,2) = 15.*

Proof: Assume there exists a nice LD(44,6,6,2;14). By the same methods that were used in Lemma 4.8, the following cases are generated:

case	f_1	f_2	efc
1	6	≥36	2
2	12	≥24	8
3	18	≥12	14
4	24	≥0	20

Case 1 By Lemmas 3.6 and 4.6, there must exist a maximal independent set with A_I = (1,2,2,2,2) and by Lemma 4.7, there must be 28 non-isolated singles in the independent blocks. However, consulting Table 1, with efc = 2, the most non-isolated singles we could have in the four 2-cliques is 7+6+6+6 = 25, which is a contradiction.

Case 2 By Lemmas 3.6 and 4.6, there must exist a maximal independent set with A_I = (1,1,2,2,2) and by Lemma 4.7, there must be at least 28 non-isolated singles in the independent blocks. However, consulting Table 1, with efc = 8, the most singles we could have in the three 2-cliques is 8+8+7 = 23, which is a contradiction.

Case 3 Lemmas 3.6 and 4.6 require A_I = (1,1,1,2,2), but this is ruled out by Theorem 4.3.

Case 4 By Lemmas 4.13, 4.14 (with T=0) and 3.6, there must be an independent set I with A_I = (1,1,1,1,2) or A_I = (1,1,1,1,3), but both are ruled out by Theorem 4.3. ❑

Corollary *L(45,6,6,2) = 15.*

Proof: Lemma 3.1.

Lemma 4.16 *L(46,6,6,2) = 16.*

Proof: Assume there exists a nice LD(46,6,6,2;15). By the same methods that were used in Lemma 4.8, the following cases are generated:

case	f_1	f_2	efc
1	6	≥36	4
2	12	≥24	10
3	18	≥12	16
4	24	≥0	22

Case 1 By Lemmas 3.6 and 4.6, there must exist a maximal independent set with A_I = (1,2,2,2,2) and by Lemma 4.7, there must be at least 32 non-isolated singles. However, using Table 1, with a total efc = 4, the most singles we could have in the four 2-cliques is 7+7+6+6 = 26, which is a contradiction.

Cases 2, 3 Lemmas 3.6 and 4.6 require A_I = (1,1,2,2,2) or A_I = (1,1,1,2,2), but both are ruled out by Theorem 4.3.

Case 4 By Lemmas 4.13, 4.14 (with T=0) and 3.6, there must be an independent set I with A_I = (1,1,1,1,2) or A_I = (1,1,1,1,3), but both are ruled out by Theorem 4.3. ❑

Lemma 4.17 *L(47,6,6,2) = 17.*

Proof: Assume there exists a nice LD(47,6,6,2;16). By the same methods that were used in Lemma 4.8, the following cases are generated:

case	f_1	f_2	efc
1	0	≥45	2
2	6	≥33	8
3	12	≥21	14
4	18	≥9	20
5	24	≥0	26

Case 1 By Lemma 4.6, there must exist a maximal independent set with A_I = (2,2,2,2,2) and by Lemma 4.7, there must be at least 34 non-isolated singles. However, using Table 1 with a total efc = 2, the most singles we could have in the five 2-cliques is 7+6+6+6+6 = 31, which is a contradiction.

Case 2 By Lemmas 3.6 and 4.6, there must exist a maximal independent set with A_I = (1,2,2,2,2). By Lemma 4.7, there must be at least 34 non-isolated singles. However, using Table 1 with a total efc = 8, the most singles that we can get in the four 2-cliques is 8+8+7+6 = 29, which is a contradiction.

Case 3 Lemmas 3.6 and 4.6 require A_I = (1,1,2,2,2), but this is ruled out by Theorem 4.3.

Case 4 By Lemmas 3.6, 4.6, 4.13, and 4.14 (with T=1), there must be an independent set I with A_I = (1,1,1,2,2) or A_I = (1,1,1,2,3), but both are ruled out by Theorem 4.3.

Case 5 By Lemmas 3.6, 4.6, 4.13, and 4.14 (with T=0), there must be an independent set I with A_I = (1,1,1,1,2) or A_I = (1,1,1,1,3), but both are ruled out by Theorem 4.3. ❑

In all of the following lemmas, the highest cases may all be ruled out by applying Lemmas 3.6, 4.6, 4.13, 4.14, and Theorem 4.3 in the same way as in cases 3, 4, and 5 above. We will no longer list these simple cases.

Lemma 4.18 *L(48,6,6,2) = 18.*

Proof: Assume there exists a nice LD(48,6,6,2;17). By the same methods that were used in Lemma 4.8 the following cases are generated (the simple ones have been omitted):

case	f_1	f_2	efc
1	0	≥42	6
2	6	≥30	12
3	12	≥18	18

Case 1 By Lemma 4.6, there must exist a maximal independent set with A_I = (2,2,2,2,2). By Lemma 4.7, there must be at least 36 non-isolated singles. With an efc of 6, the largest number of non-isolated singles in the five 2-cliques is 8+8+6+6+6 = 34, which is a contradiction.

Case 2 By Lemmas 3.6 and 4.6, there must exist a maximal independent set with A_I = (1,2,2,2,2). By Lemma 4.7, there must be at least 36 non-isolated singles. With efc = 12, the maximum number of non-isolated singles in the four 2-cliques is 8+8+8+8 = 32, which is a contradiction.

Case 3 By Lemmas 3.6, 4.6, 4.13, and 4.14 (with T=2), there must be an independent set I with A_I = (1,1,2,2,2) or A_I = (1,1,2,2,3). The former is ruled out by Theorem 4.3. So there must be an independent set with A_I = (1,1,2,2,3). By Lemma 4.7 there must be at least 31 non-isolated singles in the design. With an efc of 18, Tables 1 and 2 show, by considering all of the possible combinations of two 2-cliques and one 3-clique, that there are at most 12+8+8 = 28 non-isolated singles which is a contradiction. ❑

Lemma 4.19 *L(49,6,6,2) = 19.*

Proof: Assume there exists a nice LD(49,6,6,2;18). By the same methods that were used in Lemma 4.8 the following cases are generated (the simple ones have been omitted):

case	f_1	f_2	efc
1	0	≥39	10
2	6	≥27	16
3	12	≥15	22

Case 1 By Lemma 4.6, there must be a maximal independent set with A_I = (2,2,2,2,2). By Lemma 4.7, the number of non-isolated singles in the independent blocks must be at least 38. Consulting Table 1, and considering all possibilities, we see that with efc = 10, the largest number of non-isolated

singles in the five 2-cliques is 8+8+7+7+6=36 which is a contradiction.

Case 2 By Lemmas 3.6, 4.6, 4.13, and 4.14 (with T=3), there must be an independent set I with A_I = (1,2,2,2,2) or A_I = (1,2,2,2,3).

If there exists an independent set I with A_I = (1,2,2,2,2), then by Lemma 4.7 we know that there are at least 38 non-isolated singles in the independent blocks. With efc = 16, Table 1 allows us to determine that the maximum number of non-isolated singles in four 2-cliques is 9+8+8+8 = 33 which is a contradiction.

If there exists an independent set I with A_I = (1,2,2,2,3), then by Lemma 4.7 we know that there are at least 33 non-isolated singles in the independent blocks. The efc of the entire design is 16, but one element of the independent set has frequency 3, leaving a maximum efc of 15 for the non-isolated singles in the independent blocks. With efc = 15, Tables 1 and 2 allow us to determine that the maximum number of non-isolated singles in the three 2-cliques and the 3-clique is 12+8+6+6 = 32 which is a contradiction.

Case 3 By Lemmas 3.6, 4.6, 4.13, and 4.14 (with T=2), there must be an independent set I with A_I = (1,1,2,2,2) or A_I = (1,1,2,2,3). Theorem 4.3 eliminates the first case. In the second case, we know by Lemma 4.7 that there are at least 33 non-isolated singles in the independent blocks. The efc of the design is 22, but one of the independent elements has frequency 3, leaving a maximum efc of 21 for the non-isolated singles in the independent blocks. Checking Tables 1 and 2 we see with a total efc of 21 there can be at most 11+9+8=28 non-isolated singles in one 3-clique and two 2-cliques which is a contradiction

Corollary *L(50,6,6,2)=19.*

Proof: Lemma 3.1.

Lemma 4.20 *L(51,6,6,2)=20.*

Proof: Assume there exists a nice LD(51,6,6,2;19). By the same methods that were used in Lemma 4.8 the following cases are generated (the simple ones have been omitted):

case	f_1	f_2	efc
1	0	≥39	12
2	6	≥27	18

Case 1 Using Lemma 4.6, we find that there must be a maximal independent set with A_I = (2,2,2,2,2). By Lemma 4.7, the number of singles in the independent blocks must be at least 42. With an efc of 12, consulting Table 1 shows that the largest number of singles in five 2-cliques is 8+8+8+8+6 = 38 which is a contradiction.

Case 2 By Lemmas 3.6, 4.6, 4.13, and 4.14 (with T=3), there must be an independent set I with $A_I = (1,2,2,2,2)$ or $A_I = (1,2,2,2,3)$. The former is ruled out by Theorem 4.3.

If there exists an independent set with $A_I = (1,2,2,2,3)$, then by Lemma 4.7, we know that there are at least 37 non-isolated singles in the independent blocks. With efc = 18, Tables 1 and 2 allow us to determine that the maximum number of non-isolated singles in three 2-cliques and one 3-clique is 6+8+8+12 = 34 which is a contradiction. ❑

Lemma 4.21 *L(52,6,6,2)=21.*

Proof: Assume there exists a nice LD(52,6,6,2;20). By the same methods that were used in Lemma 4.8 the following cases are generated (the simple ones have been omitted – note that although the cases with f_1=12 or 24 are simple, the case with f_1=18 is not):

case	f_1	f_2	efc
1	0	≥36	16
2	6	≥24	22
3	18	≥0	34

Case 1 If f_2>36 then by Lemma 4.6, there must be a maximal independent set with $A_I = (2,2,2,2,2)$. If f_2=36 then a simple count will reveal that f_3=16 and so f_2+f_3=52 and by Lemma 4.14 there must be an independent set I with either $A_I = (2,2,2,2,2)$ or $A_I = (2,2,2,2,3)$.

If there exists an independent set I with $A_I = (2,2,2,2,2)$, then by Lemma 4.7, the number of singles in the independent blocks must be at least 44. With a total efc of 16, using Table 1 we can determine that the largest number of singles in five 2-cliques is 8+8+8+8+8=40, which is a contradiction.

If an independent set I with $A_I = (2,2,2,2,2)$ does not exist, then an independent set I with $A_I = (2,2,2,2,3)$ must exist, and also all of the remaining 32 non-independent elements of frequency 2 must appear in the 2-cliques. This means that all of the singles in the 3-clique must have frequency 3 or more, and so if there are x singles in the 3-clique, the minimum efc for those singles is x, which is an improvement on the minimum efc values given in Table 1. By Lemma 4.7, the total number of singles in the independent blocks must be at least 39. The total efc of the design is 16, but there is an element of frequency 3 in the independent set itself, leaving a total efc of 15 for the singles in the independent blocks. Using Tables 1 and 2, and keeping in mind that the minimum efc of x singles in the 3-clique is x, we can determine that the largest possible number of singles in the 3-clique and the four 2-cliques is 12+8+6+6+6 = 38, which is a contradiction.

Case 2 By Lemmas 3.6, 4.6, 4.13, and 4.14 (with T=3), there must be an independent set I with $A_I = (1,2,2,2,2)$ or with $A_I = (1,2,2,2,3)$. Theorem 4.3 rules out the former. If there is an independent set with $A_I = (1,2,2,2,3)$, then by

Lemma 4.7, the total number of singles in the independent blocks must be at least 39. The total efc of the design is 22, but there is an element of frequency 3 in the independent set itself, leaving a total efc of 21 for the singles in the independent blocks. Using Tables 1 and 2 we can determine that the largest possible number of singles in the 3-clique and the three 2-cliques is 12+8+8+8=36, which is a contradiction.

Case 3 If $f_2 > 0$ then by Lemmas 3.6, 4.6, 4.13, and 4.14 (with T=1), there must be an independent set I with $A_I = (1,1,1,2,2)$ or $A_I = (1,1,1,2,3)$, but both are ruled out by Theorem 4.3. If $f_2 = 0$ then a simple count will reveal that $f_3 = 34$ and therefore there must be a pair of elements of frequency 3 which do not appear together, giving an independent set I with $A_I = (1,1,1,3,3)$, which is also ruled out by Theorem 4.3. ❑

Lemma 4.22 *L(53,6,6,2)=22.*

Proof: Assume there exists a nice LD(53,6,6,2;21). By the same methods that were used in Lemma 4.8 the following cases are generated:

case	f1	f2	efc
1	0	≥33	20
2	6	≥21	26
3	12	≥9	32
4	18	≥0	38

Case 1 If $f_2 > 36$ then by Lemma 4.6, there must be an independent set I with $A_I = (2,2,2,2,2)$. If there is no independent set I with $A_I = (2,2,2,2,2)$, then $33 \le f_2 \le 36$ and consequently $f_3 \ge 14$. By Lemma 4.6, there must be an independent set I with $A_I = (2,2,2,2,x)$. Also, all of the elements of frequency 2 must appear in the 2-cliques, leaving room for at most 40-(33-4) = 11 elements of frequency 3 in the four 2-cliques. There must be at least 14-11=3 other elements of frequency 3, one of which may be added to the independent set I giving $A_I = (2,2,2,2,3)$.

If there is an independent set I with with $A_I = (2,2,2,2,2)$, then using Lemma 4.7, we know that there are at least 46 non-isolated singles in the independent blocks. Using Table 1, with a total efc = 20, we can see that there can be at most 9+8+8+8+8 = 41 non-isolated singles in five 2-cliques, which is a contradiction.

If there is an independent set I with with $A_I = (2,2,2,2,3)$, then using Lemma 4.7, we know that there are at least 41 non-isolated singles in the independent blocks. The total efc is 20, but one of the independent elements has frequency 3, leaving a total efc of 19 for the non-independent elements in the independent blocks. As in Case 1 of Lemma 4.21, all of the singles in the 3-clique must have frequency 3 (else an independent set I with $A_I = (2,2,2,2,2)$ would exist), and so the minimum efc of a set of x singles in the 3-clique is x, not the values given in Table 1. By checking Tables 1 and 2, with this modification, and with a total efc of 19, we can see that there can be at most 40 non-isolated singles (11+8+8+7+6 and several other ways) in one 3-clique and four 2-cliques, which is a contradiction.

Case 2 By Lemmas 3.6, 4.6, 4.13, and 4.14 (with T=3), there must be an independent set I with A_I = (1,2,2,2,2) or (1,2,2,2,3). The former is eliminated by Theorem 4.3. In the latter case, by Lemma 4.7, there are at least 41 non-isolated singles in the independent blocks. The total efc is 26, but one of the independent elements has frequency 3, leaving a total efc of 25 for the non-independent elements in the independent blocks. From Tables 1 and 2, with a total efc of 25, there can be at most 37 non-isolated singles (13+8+8+8 or 12+9+8+8) in one 3-clique and three 2-cliques, which is a contradiction.

Case 3 If $f_2 > 9$, then by Lemmas 3.6, 4.6, 4.13, and 4.14 (with T=2), there must be an independent set I with A_I = (1,1,2,2,2) or (1,1,2,2,3), both of which are ruled out by Theorem 4.3. Therefore $f_2 = 9$ and counting will reveal that all of the remaining elements must have frequency 3, giving $f_3 = 32$. By Lemmas 3.6, 4.6, and 3.4, there must be an independent set I with A_I = (1,1,2,3,3). By Lemma 4.7, we know that there are at least 36 non-isolated singles in the independent blocks. The total efc is 32, but there are two independent elements of frequency 3, leaving a total efc of 30 for the non-independent elements in the independent blocks. By checking Tables 1 and 2, with a total efc of 30, we can see that there can be at most 32 non-isolated singles (13+12+7 or 13+11+8 or 12+11+9) in two 3-cliques and one 2-clique, which is a contradiction.

Case 4 If $f_2 > 0$, then by Lemmas 3.6, 4.6, 4.13, and 4.14 (with T=1), there must be an independent set I with A_I = (1,1,1,2,2) or (1,1,1,2,3), both of which are ruled out by Theorem 4.3. Therefore $f_2 = 0$ and counting will reveal that $f_3 \geq 32$. Since an element of frequency 3 can appear with at most 15 other elements of frequency 3, there must be two elements of frequency 3 which do not appear together, giving an independent set I with A_I = (1,1,1,3,3) which is also ruled out by Theorem 4.3. ❑

Lemma 4.23 *L(54,6,6,2)=23.*

Proof: Assume that there exists a nice LD(54,6,6,2;22). By the same methods that were used in Lemma 4.8 the following cases are generated:

case	f1	f2	efc
1	0	≥30	24
2	6	≥18	30
3	12	≥6	36
4	18	≥0	42

Case 1 If $f_2 > 36$ then by Lemma 4.6 there must be an independent set I

with $A_I = (2,2,2,2,2)$. If $f_2 \leq 36$ then counting will reveal that $f_2 + f_3 \geq 48$ and then by Lemmas 4.6 and 4.14 there must be an independent set I with $A_I = (2,2,2,2,2)$ or $A_I = (2,2,2,2,3)$.

If there is an independent set I with $A_I = (2,2,2,2,2)$, then we know by using Lemma 4.7 that there are at least 48 non-isolated singles in the independent blocks. Consulting Table 1 with a total efc of 24 shows that there can be a maximum of 42 non-isolated singles (10+8+8+8+8 or 9+9+8+8+8) in five 2-cliques, which is a contradiction.

If there is an independent set I with $A_I = (2,2,2,2,3)$, then we know by using Lemma 4.7 that there are at least 43 non-isolated singles in the independent blocks. The total efc is 24, but there is one independent element of frequency 3, leaving a total efc of 23 for the non-independent elements in the independent blocks. By consulting Tables 1 and 2, with a total efc of 23, it can been seen that it is possible to get 43 non-isolated singles in just two ways. The distribution of singles is either (12,8,8,8,7) or (11,8,8,8,8) in the 3-clique and the four 2-cliques. Since there are a total of 22 blocks, and 11 independent blocks, there must be exactly 11 extra blocks, containing 66 elements. But there is a set S of 43 singles, and efc(S)=23, giving a minimum of 43+23=66 occurrences of these singles in the extra blocks, and so no other element may appear in the extra blocks, and the minimum efc value must be met exactly for the set of singles that appears in each clique. The unique configuration which covers all of the pairs of a set S of 8 singles with efc(S)=3, as shown in the derivation of Table 1, is {1,2,3,6,7,8}, {4,5,6,7,8,–}. The one remaining unspecified element must be one of the singles from another clique. But this will increase efc(T) for the set of singles, T, of that clique, since an isolated element will not help in covering the pairs from T, and the minimum efc(T) value will still be required elsewhere. Since both of the possible distributions contain several sets of 8 singles, the required total efc value of 23 is not possible.

Case 2 If $f_2 > 18$ then by Lemmas 3.6 and 4.6 an independent set I with $A_I = (1,2,2,2,x)$ exists. Counting will show that $f_2 + f_3 \geq 33$ but only 30 elements can appear in the three 2-cliques, and so an independent set with x=2 or 3 exists. If $f_2 = 18$ then counting will show that $f_3 = 30$ (all of the remaining elements have frequency 3), and Lemmas 3.6, 4.6, and 3.4 show that an independent set I exists with $A_I = (1,2,2,2,2)$ or (1,2,2,2,3) or (1,2,2,3,3). Theorem 4.3 rules out the first of these. The other two are considered below.

If there is an independent set I with $A_I = (1,2,2,2,3)$, then by Lemma 4.7 we know that there are at least 43 non-isolated singles in the independent blocks. The total efc is 30, but there is one independent element of frequency 3, leaving a total efc of 29 for the non-independent elements in the independent blocks. By consulting Tables 1 and 2, with a total efc of 29, it can been seen that there is a maximum of 38 singles (12+10+8+8 or 12+9+9+8) in one 3-clique and three 2-cliques, which is a contradiction.

If there is an independent set I with $A_I = (1,2,2,3,3)$, then by Lemma 4.7 we know that there are at least 38 non-isolated singles in the independent blocks. The total efc is 30, but there are two independent elements of frequency 3, leaving a total efc of 28 for the non-independent elements in the independent blocks. By consulting Tables 1 and 2, with a total efc of 28, it can been seen that there five ways in which 38 non-isolated singles may be placed in two 2-cliques and two 3-cliques with a total efc of 28: 12+12+8+6, 12+12+7+7, 12+11+8+7, 12+10+8+8, and 11+11+8+8. We also know that $f_2 = 18$ (since the cases that arise when $f_2 > 18$ were ruled out above) and counting will show that $f_3 = 30$ and there are no elements with frequency greater than 3. Also, no independent set I with $A_I = (1,2,2,2,x)$ may exist (it has previously been ruled out), and so there can be no sets of 3 independent elements of frequency 2. This means that all of the other 16 elements of frequency 2 must appear with the two independent elements of frequency 2 in the two 2-cliques, else a third independent element of frequency 2 could be chosen. Let the two 2-cliques be

```
x f _ _ _ _ (block 1)
x g _ _ _ _ (block 2)
y _ _ _ _ _ (block 3)
y _ _ _ _ _ (block 4)
```

where x and y are the two independent elements of frequency 2. Consider the elements of frequency 2 that appear only once in these four blocks. Suppose that 7 such elements appear in blocks 1 and 2 with x. Then two of them (f and g) must be in different blocks. Also, f and g must appear together in another block elsewhere, else three independent elements of frequency 2 (y, f, and g) could be chosen. But this means that all 7 such elements that appear with x must appear together in some other block, since every such element in block 1 must appear with g, and every such element in block 2 must appear with f. Since there are only 6 elements in a block, this is impossible. Therefore there can be at most 6 such elements that appear in blocks 1 and 2, and similarly at most 6 that appear in blocks 3 and 4. Therefore at most 12 elements of frequency 2 appear only once in these 4 blocks. But all 16 elements of frequency 2 must appear, and therefore at least four of them must appear twice. This uses up all of the available space, and so every element in these four blocks must have frequency 2 (12 appear once and 4 appear twice). Therefore the maximum efc in each of these two cliques is 0, and the maximum number of singles in each is 6. But the five possible configurations listed previously contain only values greater than 6, and so this case can be ruled out.

Case 3 If $f_2 > 9$ then by Lemmas 3.6, 4.6, 4.13 and 4.14 there must exist an independent set I with $A_I = (1,1,2,2,2)$ or $(1,1,2,2,3)$, which are ruled out by Theorem 4.3. If $6 \le f_2 \le 9$ then $f_3 \ge 30$. By Lemmas 3.6, 4.6, 4.13 and 4.14 there must exist an independent set I with $A_I = (1,1,2,3,x)$, and since there is room for at most 25 other elements to appear in the 2-clique and the 3-clique, but $f_3 \geq 30$, an additional independent element of frequency 3 may be found, giving an independent set I with $A_I = (1,1,2,3,3)$. By Lemma 4.7 there are at least 38 non-isolated singles in the independent blocks. The total efc is 36, but there are two independent elements of frequency 3, leaving a total efc of 34 for the non-independent elements in the independent blocks. By consulting Tables 1 and 2, with a total efc of 34, it can been seen that there is a maximum of 34 singles (12+12+10) in two 3-cliques and one 2-clique, which is a contradiction.

Case 4 If $f_2 > 0$ then by Lemmas 3.6, 4.6, 4.13 and 4.14 there must exist an independent set I with $A_I = (1,1,1,2,2)$ or $(1,1,1,2,3)$, which are ruled out by Theorem 4.3. If $f_2 = 0$ then $f_3 \geq 30$ and so two independent elements of frequency 3 must exist, giving an independent set I with $A_I = (1,1,1,3,3)$, which is also ruled out by Theorem 4.3. ❑

References

1. J. A. Bate, *A Generalized Covering Problem*, Ph.D. thesis, University of Manitoba (1978)
2. B. Bollobas, *Extended Graph Theory*, Academic Press, New York (1978)
3. A. I. Brouwer & M. Vorhoeven, *Turan Theory and the Lotto Problem*, Math Centrum Tracts **106** (1995) 99-105
4. C. J. Colbourn, *Winning the Lottery*, in CRC Handbook of Combinatorial Designs, edited by C.J. Colbourn & J.H. Dinitz, CRC Press, New York (1996) 578-584
5. T. Etzion, V. Wei & Z. Zhang, *Bounds on the Sizes of Constant Weight Covering Codes*, Designs, Codes and Cryptography **5** (1995) 217-239
6. Z. Furedi, G.J. Szekely & Z. Zuber, *On the Lottery Problem*, Journal of Combinatorial Theory **4** (1996) 5-10
7. D.M. Gordon, O. Patashnik & S. Kuperberg, *New Constructions for Coverings*, Journal of Combinatorial Designs **3** (1995) 269-284
8. C. Hartman, *Extremal Problems in Graph Theory*, Ph.D. thesis, Indiana State University (1977)
9. N. Henz, *The distribution of spaces on Lottery tickets*, Fibonacci Quarterly **33** (1995) 426-431
10. R. C. Mullin & W. H. Mills, *Coverings and Packings*, in Contempory Design Theory A collection of Surveys, Ed. J. H. Dinitz & D. R. Stinson, Wily Press Toronto (1992) 371-395
11. J. Schonheim, *On Coverings*, Pacific Journal of Mathematics **14** (1964) 1405-1411
12. S. Shannon, *A Theorem on Coloring the Lines of a Network*, Journal of Mathematical Physics **28** (1949) 148-151
13. D.M. Gordon, *La Jolla Covering Repository*, (Internet site) URL is http://sdcc12.ucsd.edu/~xm3dg/cover.html (May 19, 1998)