Notations

$$\mathcal{X} = \text{Ensemble de variable}$$
$$\mathcal{S} = \mathcal{P}(\mathcal{T} \cup \mathcal{X}) \times \mathcal{T}$$
$$\{\Gamma, x, P \vdash Q\} = \{\Gamma, x \vdash P \to Q\} = \{\Gamma \vdash \forall x.\ P \to Q\}$$
$$\Big\{\{\Gamma \vdash a \wedge b\}\Big\} \simeq \{\Gamma \vdash a\} \cup \{\Gamma \vdash b\}$$

Opérateurs s^+, s^- et sc^{all}

$$s^+ : \left\{ \begin{array}{ccc} \mathcal{S} & \to & \mathcal{P}(\mathcal{S}) \\ \{\Gamma \vdash a\} & \mapsto & \{\Gamma \vdash e^+(a) \wedge c^{all}(a)\} \end{array} \right. \qquad sc^{all} : \left\{ \begin{array}{ccc} \mathcal{S} & \to & \mathcal{P}(\mathcal{S}) \\ \{\Gamma \vdash a\} & \mapsto & \{\Gamma \vdash c^{all}(a)\} \end{array} \right.$$

Comment calculer s^+ lors d'une négation ?

$$\begin{aligned} s^+(\{\Gamma \vdash \neg a\}) &= \{\Gamma \vdash e^+(\neg a) \wedge c^{all}(\neg a)\} \\ &= \{\Gamma \vdash \neg e^-(a) \wedge c^{all}(a)\} \\ &= ? \end{aligned}$$

On introduit alors s^-, qui correspond “pousser” la négation au sein de la formule :

$$s^- : \left\{ \begin{array}{ccc} \mathcal{S} & \to & \mathcal{P}(\mathcal{S}) \\ \{\Gamma \vdash a\} & \mapsto & s^+(\{\Gamma \vdash \neg a\}) \end{array} \right.$$

Remarque :

$$sc^{all}(\{\Gamma \vdash a\}) \subset s^+(\{\Gamma \vdash a\}) \qquad sc^{all}(\{\Gamma \vdash a\}) \subset s^-(\{\Gamma \vdash a\})$$

Formules

$$\begin{aligned}
s^+ &\ (\{\Gamma \vdash a\}) &=\ & \{\Gamma \vdash a\} \\
s^- &\ (\{\Gamma \vdash a\}) &=\ & \{\Gamma \vdash \neg a\} \\
sc^{all} &\ (\{\Gamma \vdash a\}) &=\ & \emptyset
\end{aligned}$$

$$\begin{aligned}
s^+ &\ (\{\Gamma \vdash \neg a\}) &=\ & s^-(\{\Gamma \vdash a\}) \\
s^- &\ (\{\Gamma \vdash \neg a\}) &=\ & s^+(\{\Gamma \vdash a\}) \\
sc^{all} &\ (\{\Gamma \vdash \neg a\}) &=\ & sc^{all}(\{\Gamma \vdash a\})
\end{aligned}$$

$$\begin{aligned}
s^+ &\ (\{\Gamma \vdash a \land b\}) &=\ & s^+(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma \vdash b\}) \\
s^- &\ (\{\Gamma \vdash a \land b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma, e^-(a) \vdash b\}) \\
sc^{all} &\ (\{\Gamma \vdash a \land b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma \vdash b\})
\end{aligned}$$

$$\begin{aligned}
s^+ &\ (\{\Gamma \vdash a \to b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, e^-(a) \vdash b\}) \\
s^- &\ (\{\Gamma \vdash a \to b\}) &=\ & s^+(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma \vdash b\}) \\
sc^{all} &\ (\{\Gamma \vdash a \to b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma, e^-(a) \vdash b\})
\end{aligned}$$

$$\begin{aligned}
s^+ &\ (\{\Gamma \vdash a \lor b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, \neg e^+(a) \vdash b\}) \\
s^- &\ (\{\Gamma \vdash a \lor b\}) &=\ & s^-(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma \vdash b\}) \\
sc^{all} &\ (\{\Gamma \vdash a \lor b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma, \neg e^+(a) \vdash b\})
\end{aligned}$$

$$\begin{aligned}
s^+ &\ (\{\Gamma \vdash a \leftrightarrow b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, e^-(a) \vdash b\}) \\
& & & \cup\, sc^{all}(\{\Gamma \vdash b\}) \cup s^+(\{\Gamma, e^-(b) \vdash a\}) \\
s^- &\ (\{\Gamma \vdash a \leftrightarrow b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma, e^-(a) \vdash b\}) \\
& & & \cup\, s^+(\{\Gamma, \neg e^+(a) \vdash b\}) \\
sc^{all} &\ (\{\Gamma \vdash a \leftrightarrow b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma, e^-(a) \vdash b\}) \\
& & & \cup\, sc^{all}(\{\Gamma \vdash b\}) \cup sc^{all}(\{\Gamma, e^-(b) \vdash a\})
\end{aligned}$$

$$\begin{aligned}
s^+ &\ (\{\Gamma \vdash \forall x.\ a\}) &=\ & s^+(\{\Gamma, c \vdash a[x \leftarrow c]\}) \\
s^- &\ (\{\Gamma \vdash \forall x.\ a\}) &=\ & \{\Gamma \vdash \exists x.\ \neg e^+(a)\} \cup sc^{all}(\{\Gamma, c \vdash a[x \leftarrow c]\}) \\
sc^{all} &\ (\{\Gamma \vdash \forall x.\ a\}) &=\ & sc^{all}(\{\Gamma, c \vdash a[x \leftarrow c]\})
\end{aligned}$$

$$\begin{aligned}
s^+ &\ (\{\Gamma \vdash \exists x.\ a\}) &=\ & \{\Gamma \vdash \exists x.\ e^+(a)\} \cup sc^{all}(\{\Gamma, c \vdash a[x \leftarrow c]\}) \\
s^- &\ (\{\Gamma \vdash \exists x.\ a\}) &=\ & s^-(\{\Gamma, c \vdash a[x \leftarrow c]\}) \\
sc^{all} &\ (\{\Gamma \vdash \exists x.\ a\}) &=\ & sc^{all}(\{\Gamma, c \vdash a[x \leftarrow c]\})
\end{aligned}$$

$$\begin{aligned}
s^+ &\ (\{\Gamma \vdash a \ \&\&\ b\}) &=\ & s^+(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, e^-(a) \vdash b\}) \\
s^- &\ (\{\Gamma \vdash a \ \&\&\ b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma, e^-(a) \vdash b\}) \\
sc^{all} &\ (\{\Gamma \vdash a \ \&\&\ b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma, e^-(a) \vdash b\})
\end{aligned}$$

$$\begin{aligned}
s^+ &\ (\{\Gamma \vdash a \ ||\ b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, \neg e^+(a) \vdash b\}) \\
s^- &\ (\{\Gamma \vdash a \ ||\ b\}) &=\ & s^-(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma, \neg e^+(a) \vdash b\}) \\
sc^{all} &\ (\{\Gamma \vdash a \ ||\ b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma, \neg e^+(a) \vdash b\})
\end{aligned}$$

$$\begin{aligned}
s^+ &\ (\{\Gamma \vdash a \ \texttt{by}\ b\}) &=\ & s^+(\{\Gamma \vdash b\}) \cup sc^{all}(\{\Gamma \vdash a\}) \\
& & & \cup\, \{\Gamma, e^-(b) \vdash e^+(a)\} \\
s^- &\ (\{\Gamma \vdash a \ \texttt{by}\ b\}) &=\ & s^-(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma \vdash b\}) \\
& & & \cup\, \{\Gamma, e^-(b) \vdash e^+(a)\} \\
sc^{all} &\ (\{\Gamma \vdash a \ \texttt{by}\ b\}) &=\ & sc^{all}(\{\Gamma \vdash b\}) \cup sc^{all}(\{\Gamma \vdash a\}) \\
& & & \cup\, \{\Gamma, e^-(b) \vdash e^+(a)\}
\end{aligned}$$

$$\begin{aligned}
s^+ &\ (\{\Gamma \vdash a \ \texttt{so}\ b\}) &=\ & s^+(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, e^-(a) \vdash b\}) \\
s^- &\ (\{\Gamma \vdash a \ \texttt{so}\ b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, e^-(a) \vdash b\}) \\
& & & \cup\, \{\Gamma, e^-(a) \vdash \neg e^-(b)\} \\
sc^{all} &\ (\{\Gamma \vdash a \ \texttt{so}\ b\}) &=\ & sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, e^-(a) \vdash b\})
\end{aligned}$$

Let

Le connecteur `let` correspond à l'évaluation d'une formule avec une nouvelle variable définie dans le contexte. D'où :

$$\{\Gamma \vdash \texttt{let}\ c \coloneqq b\ \texttt{in}\ a\} \simeq \{\Gamma, c \coloneqq b \vdash a\}$$

Soit les formules :

$$\begin{array}{rlcl}
e^{+} & (\texttt{let}\ c \coloneqq b\ \texttt{in}\ a) & = & \texttt{let}\ c \coloneqq b\ \texttt{in}\ e^{+}(a) \\
e^{-} & (\texttt{let}\ c \coloneqq b\ \texttt{in}\ a) & = & \texttt{let}\ c \coloneqq b\ \texttt{in}\ e^{-}(a) \\
s^{+} & (\{\Gamma \vdash \texttt{let}\ c \coloneqq b\ \texttt{in}\ a\}) & = & s^{+}(\{\Gamma, c \coloneqq b \vdash a\}) \\
s^{-} & (\{\Gamma \vdash \texttt{let}\ c \coloneqq b\ \texttt{in}\ a\}) & = & s^{-}(\{\Gamma, c \coloneqq b \vdash a\}) \\
sc^{all} & (\{\Gamma \vdash \texttt{let}\ c \coloneqq b\ \texttt{in}\ a\}) & = & sc^{all}(\{\Gamma, c \coloneqq b \vdash a\})
\end{array}$$

Mais si on "compilait" `let` en :

$$\{\Gamma \vdash \texttt{let}\ c \coloneqq b\ \texttt{in}\ a\} \simeq \{\Gamma \vdash \forall c.\ c = b \rightarrow a\} \qquad \{\Gamma, \texttt{let}\ c \coloneqq b\ \texttt{in}\ a \vdash X\} \simeq \{\Gamma, \exists c.\ c = b \wedge a \vdash X\}$$

Les définitions pour s^{+}, s^{-} et sc^{all} restent valables, mais pour e^{+} et e^{-} :

$$\begin{array}{rlcl}
e^{+} & (\texttt{let}\ c \coloneqq b\ \texttt{in}\ a) & = & \forall c.\ c = b \rightarrow e^{+}(a) \\
e^{-} & (\texttt{let}\ c \coloneqq b\ \texttt{in}\ a) & = & \exists c.\ c = b \wedge e^{-}(a)
\end{array}$$

Ce qui élimine la construction `let` de la logique utilisée en sortie.

If

Une "compilation" du `if` dans la logique :

$$\texttt{if}\ a\ \texttt{then}\ b\ \texttt{else}\ c \equiv (a \rightarrow b) \wedge (\neg a \rightarrow c) \equiv (a \wedge b) \vee (\neg a \wedge c)$$

Extension au `if` :

$$\begin{array}{rlcl}
e^{+} & (\texttt{if}\ a\ \texttt{then}\ b\ \texttt{else}\ c) & = & (e^{-}(a) \rightarrow e^{+}(b)) \wedge (\neg e^{+}(a) \rightarrow e^{+}(c)) \\
e^{-} & (\texttt{if}\ a\ \texttt{then}\ b\ \texttt{else}\ c) & = & (e^{-}(a) \wedge e^{-}(b)) \vee (\neg e^{+}(a) \wedge e^{-}(c)) \\
s^{+} & (\{\Gamma \vdash \texttt{if}\ a\ \texttt{then}\ b\ \texttt{else}\ c\}) & = & sc^{all}(\{\Gamma \vdash a\}) \cup s^{+}(\{\Gamma, e^{-}(a) \vdash b\}) \cup s^{+}(\{\Gamma, \neg e^{+}(a) \vdash c\}) \\
s^{-} & (\{\Gamma \vdash \texttt{if}\ a\ \texttt{then}\ b\ \texttt{else}\ c\}) & = & sc^{all}(\{\Gamma \vdash a\}) \cup s^{-}(\{\Gamma, e^{-}(a) \vdash b\}) \cup s^{-}(\{\Gamma, \neg e^{+}(a) \vdash c\}) \\
sc^{all} & (\{\Gamma \vdash \texttt{if}\ a\ \texttt{then}\ b\ \texttt{else}\ c\}) & = & sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma, e^{-}(a) \vdash b\}) \cup sc^{all}(\{\Gamma, \neg e^{+}(a) \vdash c\})
\end{array}$$

Match

Une “compilation” du `match` dans la logique :

$$
\begin{array}{rcl}
M & \coloneqq & \begin{array}{l}\texttt{match}\ e\ \texttt{with}\\ \begin{array}{llll} \mid & A & \mapsto & a\\ \mid & B(x) & \mapsto & b\\ \mid & \_ & \mapsto & c\end{array}\\ \texttt{end}\end{array}\\
& \equiv & (e = A \rightarrow a) \wedge (\forall x.\ e = B(x) \rightarrow b) \wedge (p(e) \rightarrow c)\\
& \equiv & (e = A \wedge a) \vee (\exists x.\ e = B(x) \wedge b) \vee (p(e) \wedge c)
\end{array}
$$

Avec

$$p(e) \coloneqq e \neq A \wedge (\forall x.\ e \neq B(x))$$

Extension au `match` :

$$
\begin{array}{rlcl}
e^{+} & (M) & = & (e = A \rightarrow e^{+}(a)) \wedge (\forall x.\ e = B(x) \rightarrow e^{+}(b)) \wedge (p(e) \rightarrow e^{+}(c))\\
e^{-} & (M) & = & (e = A \wedge e^{-}(a)) \vee (\exists x.\ e = B(x) \wedge e^{-}(b)) \vee (p(e) \wedge e^{-}(c))\\
s^{+} & (\{\Gamma \vdash M\}) & = & s^{+}(\{\Gamma, e = A \vdash a\}) \cup s^{+}(\{\Gamma, cst, e = B(cst) \vdash b[x \leftarrow cst]\})\\
& & & \cup\, s^{+}(\{\Gamma, e \neq A, \forall x.\ e \neq B(x) \vdash c\})\\
s^{-} & (\{\Gamma \vdash M\}) & = & s^{-}(\{\Gamma, e = A \vdash a\}) \cup s^{-}(\{\Gamma, cst, e = B(cst) \vdash b[x \leftarrow cst]\})\\
& & & \cup\, s^{-}(\{\Gamma, e \neq A, \forall x.\ e \neq B(x) \vdash c\})\\
sc^{all} & (\{\Gamma \vdash M\}) & = & s^{all}(\{\Gamma, e = A \vdash a\}) \cup sc^{all}(\{\Gamma, cst, e = B(cst) \vdash b[x \leftarrow cst]\})\\
& & & \cup\, s^{all}(\{\Gamma, e \neq A, \forall x.\ e \neq B(x) \vdash c\})
\end{array}
$$

Preuves

Atomes

$$s^+(\{\Gamma \vdash a\}) = \{\Gamma \vdash e^+(a) \wedge c^{all}(a)\} = \{\Gamma \vdash a \wedge \top\} = \{\Gamma \vdash a\}$$

$$\begin{aligned} s^-(\{\Gamma \vdash a\}) &= s^+(\{\Gamma \vdash \neg a\}) = \{\Gamma \vdash e^+(\neg a) \wedge c^{all}(\neg a)\} = \{\Gamma \vdash \neg e^-(a) \wedge c^{all}(a)\} \\ &= \{\Gamma \vdash \neg a \wedge \top\} = \{\Gamma \vdash \neg a\} \end{aligned}$$

$$sc^{all}(\{\Gamma \vdash a\}) = \{\Gamma \vdash c^{all}(a)\} = \{\Gamma \vdash \top\} = \{\} \in \mathcal{S}$$

Négation

$$s^+(\{\Gamma \vdash \neg a\}) = s^-(\{\Gamma \vdash a\})$$

$$s^-(\{\Gamma \vdash \neg a\}) = s^+(\{\Gamma \vdash \neg\neg a\}) = s^+(\{\Gamma \vdash a\})$$

$$sc^{all}(\{\Gamma \vdash \neg a\}) = \{\Gamma \vdash c^{all}(\neg a)\} = \{\Gamma \vdash c^{all}(a)\} = sc^{all}(\{\Gamma \vdash a\})$$

Conjonction

$$\begin{aligned} s^+(\{\Gamma \vdash a \wedge b\}) &= \{\Gamma \vdash e^+(a \wedge b) \wedge c^{all}(a \wedge b)\} \\ &= \{\Gamma \vdash e^+(a) \wedge e^+(b) \wedge c^{all}(a) \wedge c^{all}(b)\} \\ &= \{\Gamma \vdash (e^+(a) \wedge c^{all}(a)) \wedge (e^+(b) \wedge c^{all}(b))\} \\ &= \{\Gamma \vdash e^+(a) \wedge c^{all}(a)\} \cup \{\Gamma \vdash e^+(b) \wedge c^{all}(b)\} \\ &= s^+(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma \vdash b\}) \end{aligned}$$

$$\begin{aligned} sc^{all}(\{\Gamma \vdash a \wedge b\}) &= \{\Gamma \vdash c^{all}(a \wedge b)\} \\ &= \{\Gamma \vdash c^{all}(a) \wedge c^{all}(b)\} \\ &= sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma \vdash b\}) \end{aligned}$$

$$\begin{aligned} s^-(\{\Gamma \vdash a \wedge b\}) &= s^+(\{\Gamma \vdash \neg(a \wedge b)\}) \\ &= s^+(\{\Gamma \vdash \neg a \vee \neg b\}) \\ &= s^+(\{\Gamma \vdash a \rightarrow \neg b\}) \\ &= sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, e^-(a) \vdash \neg b\}) \\ &= sc^{all}(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma, e^-(a) \vdash b\}) \end{aligned}$$

Implication

$$
\begin{aligned}
s^+(\{\Gamma \vdash a \to b\}) &= \{\Gamma \vdash e^+(a \to b) \wedge c^{all}(a \to b)\} \\
&= \{\Gamma \vdash (e^-(a) \to e^+(b)) \wedge c^{all}(a) \wedge (e^-(a) \to c^{all}(b))\} \\
&= \{\Gamma \vdash c^{all}(a) \wedge (e^-(a) \to e^+(b) \wedge c^{all}(b))\} \\
&= \{\Gamma \vdash c^{all}(a)\} \cup \{\Gamma \vdash e^-(a) \to e^+(b) \wedge c^{all}(b)\} \\
&= \{\Gamma \vdash c^{all}(a)\} \cup \{\Gamma, e^-(a) \vdash e^+(b) \wedge c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, e^-(a) \vdash b\})
\end{aligned}
$$

$$
\begin{aligned}
sc^{all}(\{\Gamma \vdash a \to b\}) &= \{\Gamma \vdash c^{all}(a \to b)\} \\
&= \{\Gamma \vdash c^{all}(a) \wedge (e^-(a) \to c^{all}(b))\} \\
&= \{\Gamma \vdash c^{all}(a)\} \cup \{\Gamma \vdash e^-(a) \to c^{all}(b)\} \\
&= \{\Gamma \vdash c^{all}(a)\} \cup \{\Gamma, e^-(a) \vdash c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma, e^-(a) \vdash b\})
\end{aligned}
$$

$$
\begin{aligned}
s^-(\{\Gamma \vdash a \to b\}) &= s^+(\{\Gamma \vdash \neg(a \to b)\}) \\
&= s^+(\{\Gamma \vdash a \wedge \neg b\}) \\
&= s^+(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma \vdash \neg b\}) \\
&= s^+(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma \vdash b\})
\end{aligned}
$$

Disjonction

$$
\begin{aligned}
s^+(\{\Gamma \vdash a \vee b\}) &= s^+(\{\Gamma \vdash \neg\neg a \vee b\}) \\
&= s^+(\{\Gamma \vdash \neg a \to b\}) \\
&= sc^{all}(\{\Gamma \vdash \neg a\}) \cup s^+(\{\Gamma, e^-(\neg a) \vdash b\}) \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, \neg e^+(a) \vdash b\})
\end{aligned}
$$

$$
\begin{aligned}
s^-(\{\Gamma \vdash a \vee b\}) &= s^+(\{\Gamma \vdash \neg(a \vee b)\}) \\
&= s^+(\{\Gamma \vdash \neg a \wedge \neg b\}) \\
&= s^+(\{\Gamma \vdash \neg a\}) \cup s^+(\{\Gamma \vdash \neg b\}) \\
&= s^-(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma \vdash b\})
\end{aligned}
$$

$$
\begin{aligned}
sc^{all}(\{\Gamma \vdash a \vee b\}) &= sc^{all}(\{\Gamma \vdash \neg\neg a \vee b\}) \\
&= sc^{all}(\{\Gamma \vdash \neg a \to b\}) \\
&= sc^{all}(\{\Gamma \vdash \neg a\}) \cup sc^{all}(\{\Gamma, e^-(\neg a) \vdash b\}) \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma, \neg e^+(a) \vdash b\})
\end{aligned}
$$

Équivalence

$$\begin{aligned}
s^+(\{\Gamma \vdash a \leftrightarrow b\}) &= s^+(\{\Gamma \vdash (a \to b) \wedge (b \to a)\}) \\
&= s^+(\{\Gamma \vdash a \to b\}) \cup s^+(\{\Gamma \vdash b \to a\}) \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma \vdash b\}) \cup s^+(\{\Gamma, e^-(a) \vdash b\}) \cup s^+(\{\Gamma, e^-(b) \vdash a\})
\end{aligned}$$

$$\begin{aligned}
s^-(\{\Gamma \vdash a \leftrightarrow b\}) &= s^-(\{\Gamma \vdash (a \wedge b) \vee (\neg a \wedge \neg b)\}) \\
&= s^-(\{\Gamma \vdash a \wedge b\}) \cup s^-(\{\Gamma \vdash \neg a \wedge \neg b\}) \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma, e^-(a) \vdash b\}) \cup sc^{all}(\{\Gamma \vdash \neg a\}) \cup s^-(\{\Gamma, e^-(\neg a) \vdash \neg b\}) \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma, e^-(a) \vdash b\}) \cup s^+(\{\Gamma, \neg e^+(a) \vdash b\})
\end{aligned}$$

$$\begin{aligned}
sc^{all}(\{\Gamma \vdash a \leftrightarrow b\}) &= sc^{all}(\{\Gamma \vdash (a \to b) \wedge (b \to a)\}) \\
&= sc^{all}(\{\Gamma \vdash a \to b\}) \cup sc^{all}(\{\Gamma \vdash b \to a\}) \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma, e^-(a) \vdash b\}) \cup sc^{all}(\{\Gamma \vdash b\}) \cup sc^{all}(\{\Gamma, e^-(b) \vdash a\})
\end{aligned}$$

Quantificateur Universel

$$\begin{aligned}
s^+(\{\Gamma \vdash \forall x.\ a\}) &= \{\Gamma \vdash e^+(\forall x.\ a) \wedge c^{all}(\forall x.\ a)\} \\
&= \{\Gamma \vdash (\forall x.\ e^+(a)) \wedge (\forall x.\ c^{all}(a))\} \\
&= \{\Gamma \vdash \forall x.\ e^+(a) \wedge c^{all}(a)\} \\
&= \{\Gamma, c \vdash e^+(a[x \leftarrow c]) \wedge c^{all}(a[x \leftarrow c])\} \\
&= s^+(\{\Gamma, c \vdash a[x \leftarrow c]\})
\end{aligned}$$

$$\begin{aligned}
s^-(\{\Gamma \vdash \forall x.\ a\}) &= s^+(\{\Gamma \vdash \neg(\forall x.\ a)\}) \\
&= s^+(\{\Gamma \vdash \exists x.\ \neg a\}) \\
&= \{\Gamma \vdash e^+(\exists x.\ \neg a) \wedge c^{all}(\exists x.\ \neg a)\} \\
&= \{\Gamma \vdash \exists x.\ e^+(\neg a)\} \cup \{\Gamma \vdash \forall x.\ c^{all}(\neg a)\} \\
&= \{\Gamma \vdash \exists x.\ \neg e^+(a)\} \cup \{\Gamma \vdash \forall x.\ c^{all}(a)\} \\
&= \{\Gamma \vdash \exists x.\ \neg e^+(a)\} \cup \{\Gamma, c \vdash c^{all}(a[x \leftarrow c])\} \\
&= \{\Gamma \vdash \exists x.\ \neg e^+(a)\} \cup sc^{all}(\{\Gamma, c \vdash a[x \leftarrow c]\})
\end{aligned}$$

$$\begin{aligned}
sc^{all}(\{\Gamma \vdash \forall x.\ a\}) &= \{\Gamma \vdash c^{all}(\forall x.\ a)\} \\
&= \{\Gamma \vdash \forall x.\ c^{all}(a)\} \\
&= \{\Gamma, c \vdash c^{all}(a[x \leftarrow c])\} \\
&= sc^{all}(\{\Gamma, c \vdash a[x \leftarrow c]\})
\end{aligned}$$

Quantificateur Existentiel

$$\begin{aligned}
s^+(\{\Gamma \vdash \exists x.\ a\}) &= \{\Gamma \vdash e^+(\exists x.\ a) \land c^{all}(\exists x.\ a)\} \\
&= \{\Gamma \vdash (\exists x.\ e^+(a)) \land (\forall x.\ c^{all}(a))\} \\
&= \{\Gamma \vdash \exists x.\ e^+(a)\} \cup \{\Gamma \vdash \forall x.\ c^{all}(a)\} \\
&= \{\Gamma \vdash \exists x.\ e^+(a)\} \cup \{\Gamma, c \vdash c^{all}(a[x \leftarrow c])\} \\
&= \{\Gamma \vdash \exists x.\ e^+(a)\} \cup sc^{all}(\{\Gamma, c \vdash a[x \leftarrow c]\})
\end{aligned}$$

$$\begin{aligned}
s^-(\{\Gamma \vdash \exists x.\ a\}) &= s^+(\{\Gamma \vdash \neg(\exists x.\ a)\}) \\
&= s^+(\{\Gamma \vdash \forall x.\ \neg a\}) \\
&= s^+(\{\Gamma, c \vdash \neg a[x \leftarrow c]\}) \\
&= s^-(\{\Gamma, c \vdash a[x \leftarrow c]\})
\end{aligned}$$

$$\begin{aligned}
sc^{all}(\{\Gamma \vdash \exists x.\ a\}) &= \{\Gamma \vdash c^{all}(\exists x.\ a)\} \\
&= \{\Gamma \vdash \forall x.\ c^{all}(a)\} \\
&= \{\Gamma, c \vdash c^{all}(a[x \leftarrow c])\} \\
&= sc^{all}(\{\Gamma, c \vdash a[x \leftarrow c]\})
\end{aligned}$$

Conjonction Asymétrique

$$\begin{aligned}
s^+(\{\Gamma \vdash a \mathbin{\&\&} b\}) &= \{\Gamma \vdash e^+(a \mathbin{\&\&} b) \land c^{all}(a \mathbin{\&\&} b)\} \\
&= \{\Gamma \vdash e^+(a) \land (e^-(a) \to e^+(b)) \land c^{all}(a) \land (e^-(a) \to c^{all}(b))\} \\
&= \{\Gamma \vdash e^+(a) \land c^{all}(a) \land (e^-(a) \to e^+(b) \land c^{all}(b))\} \\
&= \{\Gamma \vdash e^+(a) \land c^{all}(a)\} \cup \{\Gamma \vdash e^-(a) \to e^+(b) \land c^{all}(b)\} \\
&= s^+(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, e^-(a) \vdash b\})
\end{aligned}$$

$$\begin{aligned}
s^-(\{\Gamma \vdash a \mathbin{\&\&} b\}) &= s^-(\{\Gamma \vdash a \land b\}) \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma, e^-(a) \vdash b\})
\end{aligned}$$

$$\begin{aligned}
sc^{all}(\{\Gamma \vdash a \mathbin{\&\&} b\}) &= \{\Gamma \vdash c^{all}(a \mathbin{\&\&} b)\} \\
&= \{\Gamma \vdash c^{all}(a) \land (e^-(a) \to c^{all}(b))\} \\
&= \{\Gamma \vdash c^{all}(a)\} \cup \{\Gamma \vdash e^-(a) \to c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup \{\Gamma, e^-(a) \vdash c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma, e^-(a) \vdash b\})
\end{aligned}$$

Disjonction Asymétrique

$$
\begin{aligned}
s^+(\{\Gamma \vdash a \mid\mid b\}) &= \{\Gamma \vdash e^+(a \mid\mid b) \wedge c^{all}(a \mid\mid b)\} \\
&= \{\Gamma \vdash (e^+(a) \vee e^+(b)) \wedge c^{all}(a) \wedge (e^+(a) \vee c^{all}(b))\} \\
&= \{\Gamma \vdash c^{all}(a)\} \cup \{\Gamma \vdash (e^+(a) \vee e^+(b)) \wedge (e^+(a) \vee c^{all}(b))\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup \{\Gamma \vdash e^+(a) \vee e^+(b) \wedge c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup \{\Gamma \vdash \neg e^+(a) \rightarrow e^+(b) \wedge c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup \{\Gamma, \neg e^+(a) \vdash e^+(b) \wedge c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, \neg e^+(a) \vdash b\}) \qquad\qquad (= s^+(\{\Gamma \vdash a \vee b\})\ !)
\end{aligned}
$$

$$
\begin{aligned}
s^-(\{\Gamma \vdash a \mid\mid b\}) &= s^+(\{\Gamma \vdash \neg(a \vee (\neg a \wedge b))\}) \\
&= s^+(\{\Gamma \vdash \neg a \wedge \neg(\neg a \wedge b)\}) \\
&= s^+(\{\Gamma \vdash \neg a\}) \cup s^+(\{\Gamma \vdash \neg(\neg a \wedge b)\}) \\
&= s^-(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma \vdash a \vee \neg b\}) \\
&= s^-(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, \neg e^+(a) \vdash \neg b\}) \\
&= s^-(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma, \neg e^+(a) \vdash b\})
\end{aligned}
$$

$$
\begin{aligned}
sc^{all}(\{\Gamma \vdash a \mid\mid b\}) &= \{\Gamma \vdash c^{all}(a \mid\mid b)\} \\
&= \{\Gamma \vdash c^{all}(a) \wedge (e^+(a) \vee c^{all}(b))\} \\
&= \{\Gamma \vdash c^{all}(a)\} \cup \{\Gamma \vdash e^+(a) \vee c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup \{\Gamma \vdash \neg e^+(a) \rightarrow c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup \{\Gamma, \neg e^+(a) \vdash c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma, \neg e^+(a) \vdash b\})
\end{aligned}
$$

By

$$
\begin{aligned}
s^{+}(\{\Gamma \vdash a \texttt{ by } b\}) &= \{\Gamma \vdash e^{+}(a \texttt{ by } b) \wedge c^{all}(a \texttt{ by } b)\} \\
&= \{\Gamma \vdash e^{+}(b) \wedge c^{all}(a) \wedge c^{all}(b) \wedge (e^{-}(b) \rightarrow e^{+}(a))\} \\
&= \{\Gamma \vdash e^{+}(b) \wedge c^{all}(b)\} \cup \{\Gamma \vdash c^{all}(a)\} \cup \{\Gamma \vdash e^{-}(b) \rightarrow e^{+}(a)\} \\
&= s^{+}(\{\Gamma \vdash b\}) \cup sc^{all}(\{\Gamma \vdash a\}) \cup \{\Gamma, e^{-}(b) \vdash e^{+}(a)\}
\end{aligned}
$$

$$
\begin{aligned}
s^{-}(\{\Gamma \vdash a \texttt{ by } b\}) &= s^{+}(\{\Gamma \vdash \neg(a \texttt{ by } b)\}) \\
&= \{\Gamma \vdash e^{+}(\neg(a \texttt{ by } b)) \wedge c^{all}(\neg(a \texttt{ by } b))\} \\
&= \{\Gamma \vdash \neg e^{-}(a \texttt{ by } b) \wedge c^{all}(a \texttt{ by } b)\} \\
&= \{\Gamma \vdash \neg e^{-}(a) \wedge c^{all}(a) \wedge c^{all}(b) \wedge (e^{-}(b) \rightarrow e^{+}(a))\} \\
&= \{\Gamma \vdash e^{+}(\neg a) \wedge c^{all}(\neg a)\} \cup sc^{all}(\{\Gamma \vdash b\}) \cup \{\Gamma, e^{-}(b) \vdash e^{+}(a)\} \\
&= s^{+}(\{\Gamma \vdash \neg a\}) \cup sc^{all}(\{\Gamma \vdash b\}) \cup \{\Gamma, e^{-}(b) \vdash e^{+}(a)\} \\
&= s^{-}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma \vdash b\}) \cup \{\Gamma, e^{-}(b) \vdash e^{+}(a)\}
\end{aligned}
$$

$$
\begin{aligned}
sc^{all}(\{\Gamma \vdash a \texttt{ by } b\}) &= \{\Gamma \vdash c^{all}(a \texttt{ by } b)\} \\
&= \{\Gamma \vdash c^{all}(a) \wedge c^{all}(b) \wedge (e^{-}(b) \rightarrow e^{+}(a))\} \\
&= \{\Gamma \vdash c^{all}(a)\} \cup \{\Gamma \vdash c^{all}(b)\} \cup \{\Gamma \vdash e^{-}(b) \rightarrow e^{+}(a)\} \\
&= sc^{all}(\{\Gamma \vdash b\}) \cup sc^{all}(\{\Gamma \vdash a\}) \cup \{\Gamma, e^{-}(b) \vdash e^{+}(a)\}
\end{aligned}
$$

So

$$\begin{aligned}
s^+(\{\Gamma \vdash a \ \mathtt{so}\ b\}) &= \{\Gamma \vdash e^+(a \ \mathtt{so}\ b) \wedge c^{all}(a \ \mathtt{so}\ b)\} \\
&= \{\Gamma \vdash e^+(a) \wedge c^{all}(a) \wedge (e^-(a) \rightarrow e^+(b) \wedge c^{all}(b))\} \\
&= \{\Gamma \vdash e^+(a) \wedge c^{all}(a)\} \cup \{\Gamma \vdash e^-(a) \rightarrow e^+(b) \wedge c^{all}(b)\} \\
&= s^+(\{\Gamma \vdash a\}) \cup \{\Gamma, e^-(a) \vdash e^+(b) \wedge c^{all}(b)\} \\
&= s^+(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, e^-(a) \vdash b\})
\end{aligned}$$

$$\begin{aligned}
s^-(\{\Gamma \vdash a \ \mathtt{so}\ b\}) &= s^+(\{\Gamma \vdash \neg(a \ \mathtt{so}\ b)\}) \\
&= \{\Gamma \vdash e^+(\neg(a \ \mathtt{so}\ b)) \wedge c^{all}(\neg(a \ \mathtt{so}\ b))\} \\
&= \{\Gamma \vdash \neg e^-(a \ \mathtt{so}\ b) \wedge c^{all}(a \ \mathtt{so}\ b)\} \\
&= \{\Gamma \vdash \neg(e^-(a) \wedge e^-(b)) \wedge c^{all}(a) \wedge (e^-(a) \rightarrow e^+(b) \wedge c^{all}(b))\} \\
&= \{\Gamma \vdash (\neg e^-(a) \vee \neg e^-(b)) \wedge c^{all}(a) \wedge (e^-(a) \rightarrow e^+(b) \wedge c^{all}(b))\} \\
&= \{\Gamma \vdash (e^-(a) \rightarrow \neg e^-(b)) \wedge c^{all}(a) \wedge (e^-(a) \rightarrow e^+(b) \wedge c^{all}(b))\} \\
&= \{\Gamma \vdash c^{all}(a)\} \cup \{\Gamma \vdash e^-(a) \rightarrow \neg e^-(b) \wedge e^+(b) \wedge c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup \{\Gamma, e^-(a) \vdash \neg e^-(b)\} \cup \{\Gamma, e^-(a) \vdash e^+(b) \wedge c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup \{\Gamma, e^-(a) \vdash \neg e^-(b)\} \cup s^+(\{\Gamma, e^-(a) \vdash b\})
\end{aligned}$$

$$\begin{aligned}
sc^{all}(\{\Gamma \vdash a \ \mathtt{so}\ b\}) &= \{\Gamma \vdash c^{all}(a \ \mathtt{so}\ b)\} \\
&= \{\Gamma \vdash c^{all}(a) \wedge (e^-(a) \rightarrow e^+(b) \wedge c^{all}(b))\} \\
&= \{\Gamma \vdash c^{all}(a)\} \cup \{\Gamma \vdash e^-(a) \rightarrow e^+(b) \wedge c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup \{\Gamma, e^-(a) \vdash e^+(b) \wedge c^{all}(b)\} \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, e^-(a) \vdash b\})
\end{aligned}$$

If

Rappel :

$$\texttt{if } a \texttt{ then } b \texttt{ else } c \equiv (a \to b) \land (\neg a \to c) \equiv (a \land b) \lor (\neg a \land c)$$

$$\begin{aligned}
s^+(\{\Gamma \vdash \texttt{if } a \texttt{ then } b \texttt{ else } c\}) &= s^+(\{\Gamma \vdash (a \to b) \land (\neg a \to c)\}) \\
&= s^+(\{\Gamma \vdash a \to b\}) \cup s^+(\{\Gamma \vdash \neg a \to c\}) \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, e^-(a) \vdash b\}) \\
&\quad \cup sc^{all}(\{\Gamma \vdash \neg a\}) \cup s^+(\{\Gamma, e^-(\neg a) \vdash c\}) \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup s^+(\{\Gamma, e^-(a) \vdash b\}) \cup s^+(\{\Gamma, \neg e^+(a) \vdash c\})
\end{aligned}$$

$$\begin{aligned}
s^-(\{\Gamma \vdash \texttt{if } a \texttt{ then } b \texttt{ else } c\}) &= s^-(\{\Gamma \vdash (a \land b) \lor (\neg a \land c)\}) \\
&= s^-(\{\Gamma \vdash a \land b\}) \cup s^-(\{\Gamma \vdash \neg a \land c\}) \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma, e^-(a) \vdash b\}) \\
&\quad \cup sc^{all}(\{\Gamma \vdash \neg a\}) \cup s^-(\{\Gamma, e^-(\neg a) \vdash c\}) \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup s^-(\{\Gamma, e^-(a) \vdash b\}) \cup s^-(\{\Gamma, \neg e^+(a) \vdash c\})
\end{aligned}$$

$$\begin{aligned}
sc^{all}(\{\Gamma \vdash \texttt{if } a \texttt{ then } b \texttt{ else } c\}) &= sc^{all}(\{\Gamma \vdash (a \to b) \land (\neg a \to c)\}) \\
&= sc^{all}(\{\Gamma \vdash a \to b\}) \cup sc^{all}(\{\Gamma \vdash \neg a \to c\}) \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma, e^-(a) \vdash b\}) \\
&\quad \cup sc^{all}(\{\Gamma \vdash \neg a\}) \cup sc^{all}(\{\Gamma, e^-(\neg a) \vdash c\}) \\
&= sc^{all}(\{\Gamma \vdash a\}) \cup sc^{all}(\{\Gamma, e^-(a) \vdash b\}) \cup sc^{all}(\{\Gamma, \neg e^+(a) \vdash c\})
\end{aligned}$$

Match

On pose :

$$\begin{aligned}
M \quad &\coloneqq \quad \begin{array}{l} \texttt{match } e \texttt{ with} \\ \begin{array}{llll} | & A & \mapsto & a \\ | & B(x) & \mapsto & b \\ | & _ & \mapsto & c \end{array} \\ \texttt{end} \end{array} \\
&\equiv \quad (e = A \to a) \land (\forall x.\ e = B(x) \to b) \land (p(e) \to c) \\
&\equiv \quad (e = A \land a) \lor (\exists x.\ e = B(x) \land b) \lor (p(e) \land c)
\end{aligned}$$

Avec

$$p(e) \coloneqq e \neq A \land (\forall x.\ e \neq B(x))$$

De plus, puisque $\{\Gamma, a \land b \vdash X\} = \{\Gamma, a, b \vdash X\}$, on a lorsque $p(e)$ est une hypothèse :

$$\{\Gamma, p(e) \vdash X\} = \{\Gamma, e \neq A, \forall x.\ e \neq B(x) \vdash X\}$$

$$
\begin{aligned}
&s^+(\{\Gamma \vdash M\}) \\
&\quad = s^+(\{\Gamma \vdash (e = A \to a) \land (\forall x.\ e = B(x) \to b) \land (p(e) \to c)\}) \\
&\quad = s^+(\{\Gamma \vdash e = A \to a\}) \cup s^+(\{\Gamma \vdash \forall x.\ e = B(x) \to b\}) \cup s^+(\{\Gamma \vdash p(e) \to c\}) \\
&\quad = sc^{all}(\{\Gamma \vdash e = A\}) \cup s^+(\{\Gamma, e^-(e = A) \vdash a\}) \cup s^+(\{\Gamma \vdash \forall x.\ e = B(x) \to b\}) \cup s^+(\{\Gamma \vdash p(e) \to c\}) \\
&\quad = \emptyset \cup s^+(\{\Gamma, e = A \vdash a\}) \cup s^+(\{\Gamma, cst \vdash (e = B(x) \to b)[x \leftarrow cst]\}) \cup s^+(\{\Gamma \vdash p(e) \to c\}) \\
&\quad = s^+(\{\Gamma, e = A \vdash a\}) \cup s^+(\{\Gamma, cst \vdash e = B(cst) \to b[x \leftarrow cst]\}) \cup s^+(\{\Gamma \vdash p(e) \to c\}) \\
&\quad = s^+(\{\Gamma, e = A \vdash a\}) \cup sc^{all}(\{\Gamma, cst \vdash e = B(cst)\}) \cup s^+(\{\Gamma, cst, e^-(e = B(cst)) \vdash b[x \leftarrow cst]\}) \\
&\qquad \cup s^+(\{\Gamma \vdash p(e) \to c\}) \\
&\quad = s^+(\{\Gamma, e = A \vdash a\}) \cup \emptyset \cup s^+(\{\Gamma, cst, e = B(cst) \vdash b[x \leftarrow cst]\}) \cup sc^{all}(\{\Gamma \vdash p(e)\}) \\
&\qquad \cup s^+(\{\Gamma, e^-(p(e)) \vdash c\}) \\
&\quad = s^+(\{\Gamma, e = A \vdash a\}) \cup s^+(\{\Gamma, cst, e = B(cst) \vdash b[x \leftarrow cst]\}) \cup \emptyset \cup s^+(\{\Gamma, p(e) \vdash c\}) \\
&\quad = s^+(\{\Gamma, e = A \vdash a\}) \cup s^+(\{\Gamma, cst, e = B(cst) \vdash b[x \leftarrow cst]\}) \cup s^+(\{\Gamma, p(e) \vdash c\}) \\
&\quad = s^+(\{\Gamma, e = A \vdash a\}) \cup s^+(\{\Gamma, cst, e = B(cst) \vdash b[x \leftarrow cst]\}) \cup s^+(\{\Gamma, e \neq A, \forall x.\ e \neq B(x) \vdash c\})
\end{aligned}
$$

$$
\begin{aligned}
&s^-(\{\Gamma \vdash M\}) \\
&\quad = s^-(\{\Gamma \vdash (e = A \land a) \lor (\exists x.\ e = B(x) \land b) \lor (p(e) \land c)\}) \\
&\quad = s^-(\{\Gamma \vdash e = A \land a\}) \cup s^-(\{\Gamma \vdash \exists x.\ e = B(x) \land b\}) \cup s^-(\{\Gamma \vdash p(e) \land c\}) \\
&\quad = s^-(\{\Gamma, e^-(e = A) \vdash a\}) \cup s^-(\{\Gamma, cst \vdash (e = B(x) \land b)[x \leftarrow cst]\}) \cup s^-(\{\Gamma, e^-(p(e)) \vdash c\}) \\
&\quad = s^-(\{\Gamma, e = A \vdash a\}) \cup s^-(\{\Gamma, cst \vdash e = B(cst) \land b[x \leftarrow cst]\}) \cup s^-(\{\Gamma, p(e) \vdash c\}) \\
&\quad = s^-(\{\Gamma, e = A \vdash a\}) \cup s^-(\{\Gamma, cst, e^-(e = B(cst)) \vdash b[x \leftarrow cst]\}) \cup s^-(\{\Gamma, p(e) \vdash c\}) \\
&\quad = s^-(\{\Gamma, e = A \vdash a\}) \cup s^-(\{\Gamma, cst, e = B(cst) \vdash b[x \leftarrow cst]\}) \cup s^-(\{\Gamma, p(e) \vdash c\}) \\
&\quad = s^-(\{\Gamma, e = A \vdash a\}) \cup s^-(\{\Gamma, cst, e = B(cst) \vdash b[x \leftarrow cst]\}) \cup s^-(\{\Gamma, e \neq A, \forall x.\ e \neq B(x) \vdash c\})
\end{aligned}
$$

$$
\begin{aligned}
&s^{all}(\{\Gamma \vdash M\}) \\
&\quad = s^{all}(\{\Gamma \vdash (e = A \to a) \land (\forall x.\ e = B(x) \to b) \land (p(e) \to c)\}) \\
&\quad = s^{all}(\{\Gamma \vdash e = A \to a\}) \cup s^{all}(\{\Gamma \vdash \forall x.\ e = B(x) \to b\}) \cup s^{all}(\{\Gamma \vdash p(e) \to c\}) \\
&\quad = s^{all}(\{\Gamma, e^-(e = A) \vdash a\}) \cup sc^{all}(\{\Gamma, cst \vdash (e = B(x) \to b)[x \leftarrow cst]\}) \cup s^{all}(\{\Gamma, e^-(p(e)) \vdash c\}) \\
&\quad = s^{all}(\{\Gamma, e = A \vdash a\}) \cup sc^{all}(\{\Gamma, cst \vdash e = B(cst) \to b[x \leftarrow cst]\}) \cup s^{all}(\{\Gamma, p(e) \vdash c\}) \\
&\quad = s^{all}(\{\Gamma, e = A \vdash a\}) \cup sc^{all}(\{\Gamma, cst, e^-(e = B(cst)) \vdash b[x \leftarrow cst]\}) \cup s^{all}(\{\Gamma, p(e) \vdash c\}) \\
&\quad = s^{all}(\{\Gamma, e = A \vdash a\}) \cup sc^{all}(\{\Gamma, cst, e = B(cst) \vdash b[x \leftarrow cst]\}) \cup s^{all}(\{\Gamma, p(e) \vdash c\}) \\
&\quad = s^{all}(\{\Gamma, e = A \vdash a\}) \cup sc^{all}(\{\Gamma, cst, e = B(cst) \vdash b[x \leftarrow cst]\}) \cup s^{all}(\{\Gamma, e \neq A, \forall x.\ e \neq B(x) \vdash c\})
\end{aligned}
$$