4.1 (a) $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$A^TA - \lambda I = 0 \iff \begin{bmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} = 0$$

$$\lambda = 9, 4$$

singular values $= 3, 2$

$S = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \det(S) = 6$

$$S^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

For $\lambda = 9$:

$$(A^TA - \lambda I)\vec{x}_1 = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 \\ -4x_2 \end{bmatrix} = 0 \quad \therefore x_2 = 0$$

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For $\lambda = 4$:

$$(A^TA - \lambda I)\vec{x}_2 = 0$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 5x_1 \\ x_2 \end{bmatrix} = 0 \quad \therefore x_1 = 0$$

$$\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U = [x_1 \ x_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V = AUS^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Note: other computations can be done similarly.

Some theory :-

$$A = U \Sigma V^*$$

$$\Rightarrow AA^* = (U\Sigma V^*)(U\Sigma V^*)^*$$

$$= U \Sigma V^* V \Sigma^* U^*$$

$$= U (\Sigma \Sigma^*) U^*$$

$$\Rightarrow (AA^*)U = U(\Sigma\Sigma^*)$$

Similarly,

$$(A^*A)V = V(\Sigma\Sigma^*)$$

So U and V are eigenvectors of AA^* and A^*A, respectively.

4.2 Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}_{m \times n}$

Then, $B = \begin{bmatrix} a_{m1} & \cdots & a_{21} & a_{11} \\ \vdots & & & a_{12} \\ \vdots & & & \vdots \\ a_{mn} & \cdots & \cdots & a_{1n} \end{bmatrix}_{n \times m}$

Note, B is obtained from A by first taking a transpose and then doing some column swaps (essentially a mirror image).

$$B = A^T U$$

↳ U is a unitary matrix.

→ A and A^T have the same singular values since

$$\det(A - \lambda I) = \det(A^T - \lambda I)$$

since diagonal remains same.

→ Let $B = AU$

$$BB^* = AUU^*A^* = AA^*$$

$\therefore$ Singular values remain the same.

4.4 $A, B \in \mathbb{C}^{m \times m}$

unitarily equivalent if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{m \times m}$.

T.P : $\Leftrightarrow$ A and B have the same singular values.

($\Rightarrow$) Let $A = U_1 \Sigma_1 V_1^*$ and $B = U_2 \Sigma_2 V_2^*$

$$A = QBQ^*$$

$$\Rightarrow U_1 \Sigma_1 V_1^* = Q U_2 \Sigma_2 V_2^* Q^*$$

$$\Rightarrow U_1 \Sigma_1 V_1^* = \underbrace{(QU_2)} \Sigma_2 \underbrace{(QV_2)}^* = A$$

unitary matrices

But SVD of a matrix is unique . $\therefore \Sigma_1 = \Sigma_2$

($\Leftarrow$) Not true. Example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$$

$\rightarrow$ same singular values but not unitarily equivalent

$\Downarrow$

stronger condition

4.5 $A = \mathbb{R}^{m \times n}$

$A^* A = A^T A = \mathbb{R}^{n \times n}$, i.e., real and symmetric.

($A^* = A^T$ since A is real)

$A^* A = V D V^T$

$\hookrightarrow$ real diagonal matrix

$\hookrightarrow$ real orthogonal matrix

If $m > n$, we can add $m-n$ zero rows to V to get another real matrix, U.