5.1 $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $A^* = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$

$$A^*A = \begin{bmatrix} 5 & 4 \\ 4 & 4 \end{bmatrix}$$

$$A^*A - \lambda I = 0 \Rightarrow \begin{bmatrix} 5-\lambda & 4 \\ 4 & 4-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (\lambda - 5)(\lambda - 4) - 16 = 0$$

$$\Rightarrow \lambda^2 - 9\lambda + 4 = 0$$

$$\lambda = \frac{9 \pm \sqrt{81-16}}{2} = \frac{9 \pm \sqrt{65}}{2}$$

$$\therefore \sigma_{max} = \left(\frac{9+\sqrt{65}}{2}\right)^{1/2} \qquad \sigma_{min} = \left(\frac{9-\sqrt{65}}{2}\right)^{1/2}$$

5.2 T.P: Set of full rank matrices is a dense subset of $\mathbb{C}^{m \times n}$.

Use $\|\cdot\|_2$ for proof

$\Downarrow$

we need to show that if we perturb A by ε to get A_ε

$\|A - A_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0$

Let $A = U \Sigma V^*$

$$A_\varepsilon \triangleq U(\Sigma + \varepsilon I_{m \times n}) V^*$$

we know that if Q is unitary, $\|QA\|_2 = \|A\|_2$

so $\|A\|_2 = \|\Sigma\|_2$ for the SVD.

$$\|A - A_\varepsilon\|_2 = \| U (\varepsilon I_{m \times n}) V^* \|_2 = \varepsilon$$

$$\therefore \lim_{\varepsilon \to 0} \|A - A_\varepsilon\|_2 = 0$$

5.3 $A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$

(a) $A^* A = \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} = \begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix}$

$$\det(\lambda I - A^* A) = (\lambda - 104)(\lambda - 146) - 72 * 72 = 0$$

$$\Rightarrow \lambda^2 - 250\lambda + (146 * 104 - 72 * 72) = 0$$

$$\Rightarrow \lambda^2 - 250\lambda + 10000 = 0$$

$$\lambda = 50, 200$$

$$\sigma_1 = \sqrt{200} = 10\sqrt{2}$$

$$\sigma_2 = \sqrt{50} = 5\sqrt{2}$$

Also, $AA^* = \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix}$

** Columns of U and V are eigenvectors of $A^* A$ and AA^*, respectively (recall Ex 4.1)

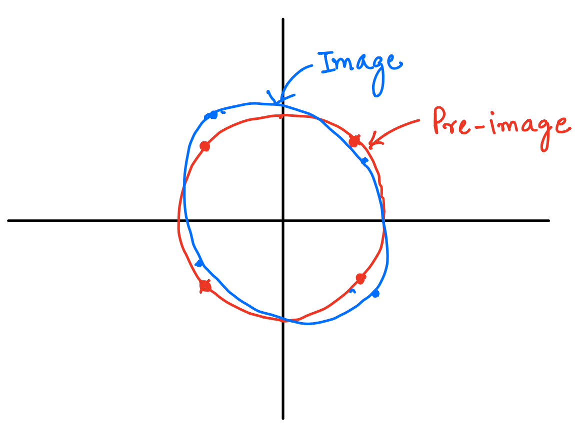
We can compute them to get:

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad V = \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$$

(b) $\sigma_1 = 10\sqrt{2}$, $\sigma_2 = 5\sqrt{2}$

Left singular vectors : $\pm \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\pm \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

Right singular vectors : $\pm \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$, $\pm \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$



(c) $\|A\|_2 = \sigma_{max} = 10\sqrt{2}$

$\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2} = 5\sqrt{10}$

$\|A\|_1 = \max^m$ column sum of $|A| = \max(12, 16) = 16$

$\|A\|_\infty = \max^m$ row sum of $|A| = \max(13, 15) = 15$

(d) $A^{-1} = (U \Sigma V^*)^{-1} = V \Sigma^{-1} U^*$

$$= \frac{1}{5\sqrt{2}} \begin{bmatrix} -3 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{10\sqrt{2}} & 0 \\ 0 & \frac{1}{5\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{100} \begin{bmatrix} -3 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{100} \begin{bmatrix} 5 & -11 \\ 10 & -2 \end{bmatrix}$$

(e) $\det(\lambda I - A) = 0$

$\Rightarrow (\lambda + 2)(\lambda - 5) + 110 = 0$

$\Rightarrow \lambda^2 - 3\lambda + 100 = 0$

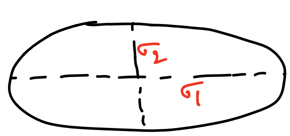
$\therefore \lambda = \dfrac{3 \pm \sqrt{9 - 400}}{2} = \dfrac{3 \pm i\sqrt{391}}{2}$

(f) can be easily verified

$\boxed{\det A = \lambda_1 \lambda_2}$ $\boxed{|\det A| = \sigma_1 \sigma_2}$

(g) The axes of the ellipsoid are σ_1 and σ_2.

Area $= \pi \cdot \sigma_1 \cdot \sigma_2 = 100\pi$



5.4 $\quad A \in \mathbb{C}^{m\times m} \qquad A = U\Sigma V^*$

$$B = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \qquad B^* = B$$

$$B = X \Lambda X^{-1}$$

X: contain linearly independent eigenvectors of B

Λ: diagonal matrix whose entries are the eigenvalues of B.

We cannot compute these matrices directly since we don't know A. So we must use properties.

$$B = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} = \begin{bmatrix} 0 & V\Sigma^* U^* \\ U\Sigma V^* & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I_{m\times m} \\ I_{m\times m} & 0 \end{bmatrix}\begin{bmatrix} U\Sigma V^* & 0 \\ 0 & V\Sigma^* U^* \end{bmatrix}$$

(simple row transformation to get a diagonal looking matrix)

$$= \underbrace{\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}}\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma^* \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}$$

suppose this is X, then if this is X^{-1}, we are done. Check!

$$X^{-1} = \begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix}\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} = \begin{bmatrix} 0 & U^* \\ V^* & 0 \end{bmatrix}$$

not the same! BUT

$$\begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix} = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} X^{-1}$$

Plug this in

$$B = X \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma^* \end{bmatrix}\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} X^{-1}$$

$$\underbrace{B = X \begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} X^{-1}}_{} \qquad \{ \Sigma = \Sigma^* \}$$

this is still not eigenvalue decomposition because Λ needs to be a diagonal matrix

need to do some matrix multiplication magic and update X and X^{-1}.

$$\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} = \frac{1}{2} \underbrace{\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}} \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} \underbrace{\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}}$$

need to have these equal so that X and X^{-1} are not broken

$$\therefore \quad B = X \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} X^{-1}$$

$$\text{for} \quad X = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}$$