7.1 (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $m = 3$ $n = 2$

$j = 1$: $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $r_{11} = \|v_1\|_2 = \sqrt{2}$

$\therefore q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$j = 2$: $r_{12} = q_1^* a_2 = 0$

$v_2 = a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = q_2$ $r_{22} = 1$

$\therefore \hat{Q} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix}$ $\hat{R} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$ $\Big\}$ Reduced QR factorization

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $x + z = 0 \Rightarrow z = -x$

$y = 0$

$\sqrt{x^2 + y^2 + z^2} = 1$

$\Rightarrow \sqrt{2}\,x = 1 \Rightarrow x = 1/\sqrt{2}$

$\therefore Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$ $R = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\Big\}$ Full QR factorization

(b) $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$j = 1$:

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $r_{11} = \sqrt{2}$

$q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$j=2$:

$$r_{12} = q_1^* b_2 = \sqrt{2}$$

$$v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$r_{22} = \sqrt{3}$$

$$\left.\hat{Q} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \qquad \hat{R} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}\right\} \text{Reduced QR factorization}$$

$$x+z=0, \qquad x+y-z=0 \qquad \sqrt{x^2+4x^2+x^2} = 1$$

$$\Rightarrow 2x+y=0 \qquad x = 1/\sqrt{6}$$

$$\left.Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}\right\} \text{Full QR factorization}$$

7.2 $\quad A = \begin{bmatrix} a_1 & a_2 & a_3 \cdots & a_n \end{bmatrix}, \quad a_i \in \mathbb{C}^m$

$$A = \hat{Q}\hat{R}, \quad \text{where} \quad \hat{R} = [r_1, \dots, r_n]$$

$$a_i = \hat{Q} r_i$$

Since A has full rank, $r_{ii} \neq 0 \;\; \forall\, i$

From Gram-Schmidt decomposition :

$$q_n = \frac{a_n - \sum_{i=1}^{n-1} r_{in} q_i}{r_{nn}} \qquad \text{where} \quad r_{ij} = q_i^* a_j$$

$\hat{R}$ has elements such that $\forall\, (i,k)$ with $k \geq i$, $r_{ik} = 0$ if $k-i$ is odd. $\underbrace{\text{(checkerboard structure)}}$

$\boxed{\text{Base case}}$: $i=1$

↳ Proof is straightforward once this result is known.

$$r_{1k} = q_1^* a_k$$

Since q_1 is collinear with a_1, r_{1k} will be 0 when $a_1^* a_k = 0$
i.e., when k is even, i.e., when $k-1$ is odd.
So base case holds.

$\boxed{\text{Inductive case}}$: Assume property holds till $i-1$.

$$q_i = \frac{a_i - (r_{1i} q_1 + \dots + r_{i-1,i} q_{i-1})}{r_{ii}}$$

$$\text{So, } q_i^* a_k = \frac{1}{r_{ii}} \left[a_i^* a_k - \sum_{j=1}^{i-1} r_{ji} \underbrace{q_j^* a_k}_{= r_{jk}} \right]$$

$$= \frac{1}{r_{ii}} \left[a_i^* a_k - \sum_{j=1}^{i-1} \underbrace{r_{ji}}_{q_j^* a_i} \underbrace{r_{jk}}_{q_j^* a_k} \right]$$

$= 0$ if $k-i$ is odd

either of these must be 0 if $k-i$ is odd because then a_i and a_k are odd and even column (or even and odd column) and a_j must be either an even or an odd column.

7.3 $A = [a_1, \dots, a_m] \in \mathbb{C}^{m \times m}$

Hadamard's inequality : $|\det A| \le \prod_{j=1}^{m} \|a_j\|_2$

<u>Case 1</u>: A is not full rank $\Rightarrow$ A is not invertible
$\Rightarrow \det A = 0$

$\therefore$ Trivially holds.

Case 2: A has full rank.

(Now since this is a QR factorization exercise, we are tempted to use QR. But SVD also has results on norms. We need to realize here that $|\det Q| = 1$ which would make life easy if we use QR.)

$$|\det A| = |\det(QR)| = |\det Q \cdot \det R|$$

$$= |\det R| = \left| \prod_{j=1}^{m} r_{jj} \right|$$

Since R is upper Δr, its determinant is just the product of diagonal entries.

This is why we use QR and not SVD.

Now we need to compute r_{jj}. From Gram-Schmidt orthogonalization,

$$|r_{jj}| = \| a_j - \sum_{i=1}^{j-1} r_{ij} q_i \|_2 \quad , \text{ where } \quad r_{ij} = q_i^* a_j \quad (i \neq j)$$

We have:

$$|r_{jj}|^2 = \| a_j - \sum_{i=1}^{j-1} r_{ij} q_i \|_2^2 = \left(a_j^* - \sum_{i=1}^{j-1} r_{ij} q_i^*\right)\left(a_j - \sum_{i=1}^{j-1} r_{ij} q_i\right)$$

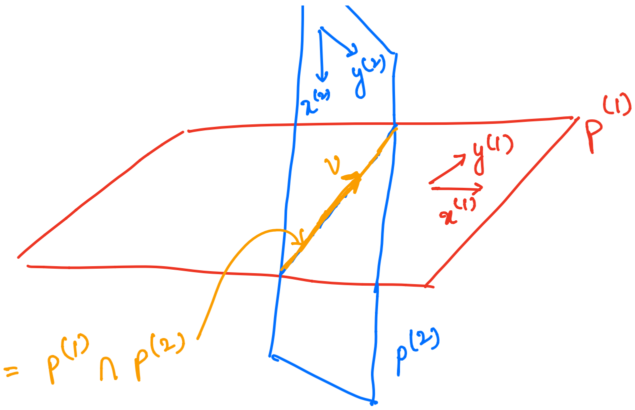
$$= \|a_j\|_2^2 - 2\sum_{i=1}^{j-1} r_{ij}^2 + \sum_{i=1}^{j-1} r_{ij}^2 \leq \|a_j\|_2^2$$

$$\Rightarrow \sum_{j=1}^{m} r_{jj} \leq \sum_{j=1}^{m} \|a_j\|_2 .$$

7.4 $\quad x^{(1)}, y^{(1)}, x^{(2)}, y^{(2)} \in \mathbb{R}^3$

$p^{(1)} = \langle x^{(1)}, y^{(1)} \rangle \qquad p^{(2)} = \langle x^{(2)}, y^{(2)} \rangle$

Let's draw a figure to understand this.



$P = P^{(1)} \cap P^{(2)}$

is a straight line

v lies on both $P^{(1)}$ and $P^{(2)}$, so it is orthogonal to both $x^{(1)} \times y^{(1)}$ and $x^{(2)} \times y^{(2)}$.

Consider the following QR factorizations (full):

$$[x^{(1)}, y^{(1)}] = [q_1^{(1)}, q_2^{(1)}, q_3^{(1)}] R^{(1)}$$

$\to$ scalar multiple of $x^{(1)} \times y^{(1)}$

$$[x^{(2)}, y^{(2)}] = [q_1^{(2)}, q_2^{(2)}, q_3^{(2)}] R^{(2)}$$

$\to$ scalar multiple of $x^{(2)} \times y^{(2)}$

Then to get v, we can compute another full QR

$$[q_3^{(1)}, q_3^{(2)}] = [q_{31}, q_{32}, q_{33}]$$

$\downarrow$

Scalar multiple of $q_3^{(1)} \times q_3^{(2)}$, i.e orthogonal to both, ie, lies on both $P^{(1)}$ and $P^{(2)}$.

7.5 (a) $A \in \mathbb{C}^{m \times n}$ $\quad A = \hat{Q}\hat{R}$ $\quad \hat{Q} \in \mathbb{C}^{m \times n}$ $\quad \hat{R} \in \mathbb{C}^{n \times n}$

$(\Rightarrow)$ All diagonal entries of $\hat{R}$ are non-zero.

We need to show that A has full rank, i.e, $Ax = 0 \Rightarrow x = 0$.

Let $x \in \mathbb{C}^n$ s.t $Ax = 0$

$$\Rightarrow \hat{Q}\hat{R}x = 0$$

Multiplying both sides by $\hat{Q}^*$, we get:

$$\hat{Q}^*\hat{Q}\hat{R}x = 0$$

$$\Rightarrow \hat{R}x = 0 \quad , \quad \text{since} \quad \hat{Q}^*\hat{Q} = I$$

Since $\hat{R}$ has all non-zero diagonal entries, this means that $x = 0$.

$\therefore$ A has full rank.

($\Leftarrow$) Given A has full rank.

Suppose, for sake of contradiction, that $\hat{R}$ has at least one zero diagonal entry. Let the smallest index for which the diagonal is 0 be k.

Case 1: $k = 1 \Rightarrow a_1 = 0$, so A does not have full rank.

Case 2: $k > 1 \Rightarrow a_k$ is linear combination of $q_1, \dots, q_{k-1}$.

$a_1, \dots, a_{k-1}$ are linearly independent, so

$$\langle a_1, \dots, a_{k-1} \rangle = \langle q_1, \dots, q_{k-1} \rangle$$

So, a_k is a linear combination of $a_1, \dots, a_{k-1}$.

$\therefore$ A does not have full rank, which is a contradiction.

$\Rightarrow$ All diagonal entries of R must be non-zero.

(b) $$a_k = \sum_{j=1}^{k-1} r_{jk} q_j + r_{kk} q_k \quad , \quad k = 1, \dots, n$$

$$\Rightarrow a_k \in \langle q_1, \dots, q_k \rangle$$

But $a_1, \dots, a_{k-1} \in \langle q_1, \dots, q_{k-1} \rangle$

$\therefore a_k \notin \langle a_1, \dots, a_{k-1} \rangle$ i.e., $\text{rank}(A_k) \geq \text{rank}(A_{k-1}) + 1$

If $\hat{R}$ has k non-zero diagonal elements

$$\boxed{\text{rank}(A) \geq k}$$