6.1 P is an orthogonal projector.

↳ any projector that is also Hermitian, i.e., $P^* = P$

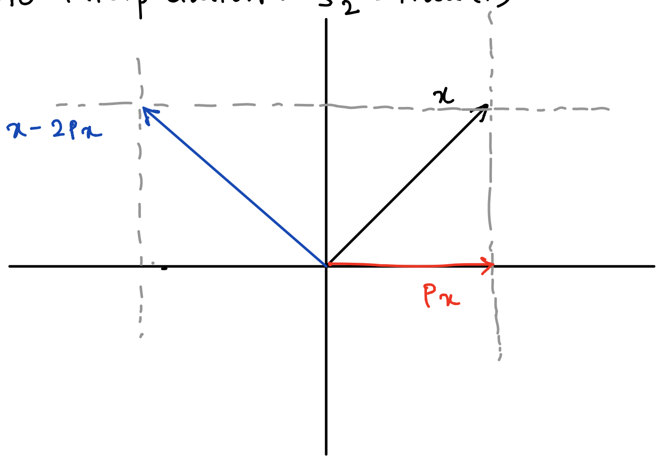
We have, $(I-2P)^*(I-2P)$

$$= I - 2P - 2P^* + 4P^2$$

$$= I - 4P + 4P = I \quad [\text{since } P^2 = P = P^*]$$

$\therefore$ $I-2P$ is unitary.

Geometric interpretation: $S_2 = \text{Null}(P)$



$S_1 = \text{Range}(P)$

$$(I-2P)x = x - 2Px$$

$\therefore$ It is the mirror image of x w.r.t S_2 which is the null space of P. Since length of x remains the same under this transformation, this means $I-2P$ is unitary.

6.2 $Ex = \dfrac{(x+Fx)}{2}$, where F flips $(x_1, \dots, x_m)^*$ to $(x_m, \dots, x_1)^*$.

$$= \frac{(I+F)}{2}x \qquad \therefore \quad E = \frac{1}{2}(I+F)$$

$$E^2 = \frac{1}{4}(I+F)^2 = \frac{1}{4}(I + 2F + F^2)$$

applying F twice to x gives x itself, so $F^2 = I$

$$\Rightarrow \quad E^2 = \frac{1}{4}(2I+2F) = E$$

$\therefore$ E is a projector.

To check if E is orthogonal/oblique, we need to check if $E^* = E$.

$$E^* = \frac{1}{2}(I+F)^* = \frac{1}{2}(F^*+I)$$

We can write out F as:

$$F = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & \cdots & 1 & 0 \\ & & \iddots & \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$ which means that $F^* = F$ since F is symmetric.

This means that $E^* = E$. So E is an orthogonal projector.

$$E = \frac{1}{2}\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 1 & 0 \\ \vdots & & 2 & & \vdots \\ \vdots & \iddots & & \ddots & \\ 1 & \cdots & & \cdots & 1 \end{bmatrix}$$

6.3 $A \in \mathbb{C}^{m \times n}$ $\boxed{\text{Full rank} \equiv \text{Invertible} \equiv (Ax = 0 \Rightarrow x = 0)}$

($\Rightarrow$) Given A has full rank. To show that A^*A is non singular, i.e., it has a matrix inverse. Alternatively

$$A^*Ax = 0 \Rightarrow x = 0 \quad \text{(To show)}$$

Since A is full rank, $Ax = 0 \Rightarrow x = 0$ — ①

Now, suppose $A^*Ax = 0$. This means that Ax is in the null space of A^*. However, since A is full rank, A^* is also full rank, which means $A^*v = 0 \Rightarrow v = 0$, and so $Ax = 0$. But from ①, this means $x = 0$, which proves A^*A is invertible.

($\Leftarrow$) Given $\underbrace{A^*A \text{ is nonsingular}}$, to show that A is full rank.

i.e $A^* A x = 0 \Rightarrow x = 0 \Rightarrow Ax = 0$

$\Rightarrow$ A^* is invertible

$\Rightarrow$ A is invertible i.e., A is full rank.

6.4 (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Range(A) = column space of A.

We can write an orthonormal basis for range(A) as

$$\hat{Q} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix}$$

A projector on an orthonormal basis is given as $P = \hat{Q}\hat{Q}^*$

$$\therefore \quad P = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Image under P $= Px = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = (2,2,2)^*$$

(b) We can either find an orthogonal basis for B as in (a), but that requires more computation, so we instead find projection under arbitrary basis

$$P = B(B^*B)^{-1}B^*$$

Rest of computation is simple so we skip.

6.5 $P \in \mathbb{C}^{m \times m}$ is a nonzero projector, i.e., $P^2 = P$

$$\|P\|_2 = \max_{x \in \mathbb{C}^m \setminus \{0\}} \frac{\|Px\|_2}{\|x\|_2}$$

We have, $P(Px) = P^2 x = Px$ (since $P^2 = P$)

$\Rightarrow \quad Px = x$

if $x \in \text{range}(P)$

$\therefore \quad \dfrac{\|Px\|_2}{\|x\|_2} = 1$ and so $\|P\|_2 \geq 1$ since it is a supremum

Now to show that P is orthogonal $\Leftrightarrow$ $\|P\|_2 = 1$

Let $P = U \Sigma V^*$.

$\|P\|_2 = \sigma_{max} = \|\Sigma\|_2$

So $\|P\|_2 = 1 \Leftrightarrow \|\Sigma\|_2 = 1 \Leftrightarrow \Sigma = I$, since Σ is diagonal.

$P^2 = P$

$\Leftrightarrow \quad U \Sigma V^* U \Sigma V^* = U \Sigma V^*$

$\Leftrightarrow \quad V^* U = I$

$\Leftrightarrow \quad V^* = U^{-1}$

$\therefore$ $V = U$, so P is orthogonal.

(since U and V are unitary)

Flow of proof