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Assignment - 2

GOODLUCK Page No. 1
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Que1) The $n \times n$ Hilbert matrix is defined as

$$H_n = [h_{ij}] \text{ where } h_{ij} = \frac{1}{i+j-1} \quad \begin{matrix} 1 \le i \le n \\ 1 \le j \le n \end{matrix}$$

Fact: H_n is invertible for all $n \geq 1$. They are examples of "Highly ill-conditioned" matrices.

Compute H_n^{-1} for $n=2$ & $n=3$. All the entries in the inverse also to be rational numbers a/b; a & b integers. Use Gauss-Jordan method.

→ Solution:-

Given, $H = [h_{ij}]$ where $h_{ij} = \frac{1}{i+j-1} \quad \begin{matrix} 1 \le i \le n \\ 1 \le j \le n \end{matrix}$

Then, $$H_2 = \begin{bmatrix} \frac{1}{1+1-1} & \frac{1}{1+2-1} \\ \frac{1}{2+1-1} & \frac{1}{2+2-1} \end{bmatrix}$$

$$\therefore H_2 = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \qquad |H_2| = \frac{1}{12} \neq 0$$

$\therefore H_2$ is invertible

Finding H_2^{-1} using Gauss-jordon method,
By property of inverse matrices
$$H_2 \cdot H_2^{-1} = I$$

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \left[H_2^{-1} \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - \frac{1}{2} R_1$$

$$\begin{bmatrix} 1 & 1/2 \\ 0 & 1/12 \end{bmatrix} H_2^{-1} = \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix}$$

$R_2 \to 12R_2$

$$\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} H_2^{-1} = \begin{bmatrix} 1 & 0 \\ -6 & 12 \end{bmatrix}$$

$R_1 \to R_1 - 1/2 R_2$

$$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} H_2^{-1} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$$

$$\therefore H_2^{-1} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$$

Illy, $H_3 = \begin{bmatrix} \frac{1}{1+1-1} & \frac{1}{1+2-1} & \frac{1}{1+3-1} \\ \frac{1}{2+1-1} & \frac{1}{2+2-1} & \frac{1}{2+3-1} \\ \frac{1}{3+1-1} & \frac{1}{3+2-1} & \frac{1}{3+3-1} \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$

$$|H_3| = \begin{vmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{vmatrix} = 1\left[\frac{1}{15} - \frac{1}{16}\right] - \frac{1}{2}\left[\frac{1}{10} - \frac{1}{12}\right] + \frac{1}{3}\left[\frac{1}{8} - \frac{1}{9}\right]$$

$$= 1\left[\frac{1}{16 \times 15}\right] - \frac{1}{2}\left[\frac{2}{120}\right] + \frac{1}{3}\left[\frac{1}{72}\right]$$

$$= \frac{1}{240} - \frac{2}{120 \times 2} + \frac{1}{216}$$

$$= \frac{1}{216} - \frac{1}{240} \neq 0$$

Hence H_3 is invertible.

By property of matrices, $H_3 \cdot H_3^{-1} = I$

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} H_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using $R_2 - 1/2 R_1$ & $R_3 - 1/3 R_1$

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 0 & 1/12 & 4/45 \end{bmatrix} H_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$$

Using $R_2 \rightarrow 12R_2$

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 1/12 & 4/45 \end{bmatrix} H_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -6 & 12 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}$$

Using $R_3 - (1/12)R_2$

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1/180 \end{bmatrix} H_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -6 & 12 & 0 \\ 1/6 & -1 & 1 \end{bmatrix}$$

Using $R_3 \rightarrow 180R_3$

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} H_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -6 & 12 & 0 \\ 30 & -180 & 180 \end{bmatrix}$$

Using $R_2 - R_3$ & $R_1 - \frac{1}{3}R_3$

$$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} H_3^{-1} = \begin{bmatrix} -9 & 60 & -60 \\ -36 & 192 & -180 \\ -30 & -180 & 180 \end{bmatrix}$$

Using $R_1 - 1/2 R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} H_3^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ -30 & -180 & 180 \end{bmatrix}$$

$[\because IA = A]$

$$\therefore H_3^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ -30 & -180 & 180 \end{bmatrix}$$

$$\therefore H_2^{-1} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$$

&

$$H_3^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ -30 & -180 & 180 \end{bmatrix}$$

Q. 2] Write down the 3×3 matrices $A = [a_{ij}]$ $B = [b_{ij}]$ where $a_{ij} = i-j$ & $b_{ij} = i/j$ compute AB, BA & A^2.

→ Sol^n :- From given condition, matrices A & B are

$$A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2-6 & -1-3 & -2/3-2 \\ 1-3 & 1/2-3/2 & 1/3-1 \\ 2+2 & 1+1 & 2/3+2/3 \end{bmatrix}$$

$$AB = \begin{bmatrix} -8 & -4 & -8/3 \\ -2 & -1 & -2/3 \\ 4 & 2 & 4/3 \end{bmatrix} \Leftarrow \text{Ans.}$$

$$BA = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2+2/3 & -1+1/3 & -2-1/2 \\ 1+4/3 & -2+2/3 & -4-1 \\ 3/2+2 & -3+1 & -6-3/2 \end{bmatrix}$$

$$\therefore BA = \begin{bmatrix} 7/6 & -2/3 & -5/2 \\ 7/3 & -4/3 & -5 \\ 7/2 & -2 & -15/2 \end{bmatrix} \Leftarrow \text{Ans}$$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} -5 & -2 & 1 \\ -2 & -2 & -2 \\ 1 & -2 & -5 \end{bmatrix} \Leftarrow \text{Ans.}$$

Q. 3] Give examples of 2x2 matrices, with $a_{12} = 1/2$ for which (a) $A^2 = I$

(b) ⇒ $A^{-1} = A^T$ (c) $A^2 = A$. $b \neq 0$

$a = \frac{1 \pm \sqrt{1-2b}}{2}$ $c = \frac{1 \mp \sqrt{1-2b}}{2}$

→ Solⁿ:-

(a) Let $A = \begin{bmatrix} a & 1/2 \\ b & c \end{bmatrix}$

Then for

$$A^2 = I$$

$$A \cdot A \cdot = \begin{bmatrix} a & 1/2 \\ b & c \end{bmatrix} \begin{bmatrix} a & 1/2 \\ b & c \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + \frac{b}{2} & \frac{a}{2} + \frac{c}{2} \\ ab + bc & \frac{b}{2} + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow a^2 + \frac{b}{2} = 1$ $\frac{a}{2} + \frac{c}{2} = 0 \Rightarrow a = -c$

$ab + bc = 0 \Rightarrow$ $\cancel{a = -c}$ & $\cancel{b \neq 0}$

$$\Rightarrow b = 2(1 - a^2)$$

$\therefore A = \begin{bmatrix} a & 1/2 \\ 2(1-a^2) & -a \end{bmatrix}$ where $a \in R$ (R = set of Real Numbers)

verifying, for $a = 3$ → as a can be any real number.

$$A = \begin{bmatrix} 3 & 1/2 \\ -16 & -3 \end{bmatrix} \begin{bmatrix} 3 & 1/2 \\ -16 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \Rightarrow A^2 = I$$

For $a=-3$ $A = \begin{bmatrix} -3 & 1/2 \\ -16 & 3 \end{bmatrix}$

$\therefore \begin{bmatrix} -3 & 1/2 \\ -16 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1/2 \\ -16 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore$ for $A = \begin{bmatrix} a & 1/2 \\ 2(1-a^2) & -a \end{bmatrix}$ where $a \in R$

$A^2 = I$

(b) $A^{-1} = A^T$;

We know that if for a matrix is to be orthogonal then it has to follows $A^{-1} = A^T$ & Hence it satisfies $AA^T = I$

$\because AA^T = I$ pre-multiplying by A^{-1} we get

$A^{-1}AA^T = A^{-1}I \Rightarrow A^T = A^{-1}$

Hence consider $A = \begin{bmatrix} a & \pm 1/2 \\ b & c \end{bmatrix}$ To find A for $A^{-1} = A^T$

then $AA^T = I \Rightarrow \begin{bmatrix} a & 1/2 \\ b & c \end{bmatrix} \begin{bmatrix} a & b \\ \pm 1/2 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$a^2 + \frac{1}{4} = 1$ $\qquad ab + \frac{c}{2} = 0 \Rightarrow c = -2ab$

$ab + \frac{c}{2} = 0 \Rightarrow c = -2ab$ $\qquad b^2 + c^2 = 1$

$\Rightarrow a = \pm \sqrt{3}/2$ — (1)

$b^2 + 4a^2b^2 = 1 \Rightarrow b^2 = 1/(1+4a^2)$

$b^2 = 1/4 \Rightarrow \boxed{b = \pm 1/2}$

$\boxed{c = -2ab}$ & $b = \pm 1/2$ and $a = \pm \frac{\sqrt{3}}{2}$

Verification:-

for $a = \frac{\sqrt{3}}{2}$, $b = 1/2$, $c = -\frac{\sqrt{3}}{2}$

$\therefore A = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix} \Rightarrow |A| = -\frac{3}{4} - \frac{1}{4} = -1$

$\Rightarrow A^{-1} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$ & $A^T = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$

$\Rightarrow \boxed{A^{-1} = A^T}$

Hence for matrix $A = \begin{bmatrix} a & 1/2 \\ b & c \end{bmatrix}$

$A^{-1} = A^T$ iff $a = \pm\frac{\sqrt{3}}{2}$, $b = \pm 1/2$ & $c = -2ab$

(c) $A^2 = A$ Let $A = \begin{bmatrix} a & 1/2 \\ b & c \end{bmatrix}$

$$\begin{bmatrix} a & 1/2 \\ b & c \end{bmatrix} \begin{bmatrix} a & 1/2 \\ b & c \end{bmatrix} = \begin{bmatrix} a^2 + \frac{b}{2} & \frac{a}{2} + \frac{c}{2} \\ ab + bc & \frac{b}{2} + c^2 \end{bmatrix}$$

$$\begin{bmatrix} a^2 + b/2 & \frac{1}{2}(a+c) \\ ab+bc & c^2 + b/2 \end{bmatrix} = \begin{bmatrix} a & 1/2 \\ b & c \end{bmatrix}$$

$a^2 + \frac{b}{2} = a$ $\qquad a + c = 1 \Rightarrow c = 1 - a.$

$(a+c) \times b = b$ $\qquad c^2 + \frac{b}{2} = c$

$\therefore$ solving $a^2 - a + \frac{b}{2} = 0$

we get $a = \frac{1 \pm \sqrt{1 - 4 \times \frac{b}{2}}}{2} = \frac{1 \pm \sqrt{1 - 2b}}{2}$

& as $c = 1 - a$

$c = 1 - \left(\frac{1 \pm \sqrt{1-2b}}{2} \right)$

$\therefore c = \frac{1 \mp \sqrt{1-2b}}{2}$ ~~& b ≠ 0~~

$$\therefore A = \begin{bmatrix} \frac{1 \pm \sqrt{1-2b}}{2} & \frac{1}{2} \\ b & \frac{1 \mp \sqrt{1-2b}}{2} \end{bmatrix}$$

& ~~b ≠ 0~~ $b \in R$

⇐ Ans

Q. 4 The zero matrix O has all entries $=0$. Let A be $n \times n$. show that it is possible to leave $A \neq O$ but $A^2 = O$. However show that if $A \neq O$ then $A^T A \neq O$ & $A A^T \neq O$

→ Sol^n:- suppose, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then

$$A \cdot A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{bmatrix}$$

for $A^2 = 0$

$a^2+bc=0$ $\quad\quad$ $ab+bd=0$

$\Rightarrow a = \pm i\sqrt{bc}$ $\quad\quad$ $ab=-bd.$

$ac+cd=0$ $\quad\quad$ $d^2+bc=0$

$ac=-cd.$ $\quad\quad$ $d = \pm i\sqrt{bc}$

i.e. either b or c has to be zero.

i.e. A can be $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$

where $b, c \in R.$

$\therefore A \cdot A = 0$ & $A A^T = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$

$$A A^T = \begin{bmatrix} b^2 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

i.e. $A A^T \neq 0$

Also $A^T A = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

$\therefore A^T A = \begin{bmatrix} 0 & 0 \\ 0 & b^2 \end{bmatrix}$

$\therefore \boxed{A^T A \neq 0}$

So for $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$ where, $b, c \in R$

$A^2 = 0$ & $A^T A \neq 0$ & $AA^T \neq 0$

Q. 5 Define $n \times n$ permutation matrix P & show that. $P^{-1} = P^T$

→ ⓐ for $n=2$, $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

for $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $P^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \boxed{P^{-1} = P^T}$

& $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad |A| = -1 \Rightarrow P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

& $P^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\Rightarrow \boxed{P^{-1} = P^T}$

ⓑ For $n=3$ $\quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

or $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

for

$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad |P| = -1[1 \times 1 - 0 \times 0] = -1.$

$\Rightarrow P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ & $P^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore \boxed{P^{-1} = P^T}$

Illy for, for a $n \times n$ permutation matrix there exists $n!$ permutations & it follows, orthogonality condition

i.e. $P^{-1} = P^T$ or $PP^T = I$

& $|P| = \pm 1$

Q. 6 Under what conditions on their entries A & B invertible?

$$A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

→ Solⁿ:-

We know that for a matrix to be invertible it has to be square & non-singular

checking $|A|$ & $|B|$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{vmatrix}$$

$$= a[e\times 0 - 0\times 0] - b[d\times 0 - f\times 0] + c[d\times 0 - e\times f]$$

$$|A| = -cef \neq 0$$

$$\Rightarrow \boxed{c \neq 0,\ e \neq 0 \ \&\ f \neq 0}$$

This is the required condition for matrix A to be invertible.

$$|B| = \begin{vmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{vmatrix}$$

$$\therefore |B| = a[d \times e - 0 \times 0] - b[c \times e - 0 \times 0]$$
$$+ 0[c \times 0 - d \times 0]$$

$$\therefore |B| = ade - bce \neq 0$$

$$\Rightarrow \quad ad\cancel{e} = bc\cancel{e}$$

$$\therefore \boxed{ad \neq bc} \quad \& \quad \boxed{e \neq 0}$$

This is the req condition for matrix B, to be invertible.

Q.7 Suppose elimination fails because there is no ~~point~~ pivot in column 3:

$$A = \begin{bmatrix} 2 & 1 & 4 & 6 \\ 0 & 3 & 8 & 5 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Show that A cannot be invertible. The third row of A^{-1}, multiplying A should give the third row [0 0 1 0] of $A^{-1}A = I$. Why is this impossible?

→ Solⁿ:- Let,

$$A = \begin{bmatrix} 2 & 1 & 4 & 6 \\ 0 & 3 & 8 & 5 \\ 0 & 0 & a & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Now $|A| = 2 \times 3 \times \begin{vmatrix} a & 7 \\ 0 & 9 \end{vmatrix}$

But as $a = 0$ given.

$\therefore$ $|A| = 0$ which meas A is not invertible.

Hence for a 4×4 matrix to be invertible there has to be 4-distinct pivots. But in given numerical the matrix has only 3 pivots. Therefore given matrix is not invertible.

& as inverse does not exists there is no way of getting $A^{-1}A = I$

Q. 8 Find the inverses of matrices.

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}$$

→ Soln :-

(a) Using $A_1 \cdot A_1^{-1} = I$

we have $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix} A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Using $R_1 \leftrightarrow R_4$ & $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A_1^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Using $R_1 \to R_1/4$
$R_2 \to R_2/3$
$R_3 \to R_3/2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A_1^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A_1^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \Leftarrow \underline{\underline{\text{Ans}}}$$

(b) To find A_2^{-1} :

$$A_2 \cdot A_2^{-1} = I$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} A_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using $R_2 \to R_2 + 1/2 R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} A_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using $R_3 \to R_3 + 2/3 R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} A_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/3 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using $R_4 \to R_4 + 3/4 R_3$ we get,

$$A_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/3 & 2/3 & 1 & 0 \\ 1/4 & 1/2 & 3/4 & 1 \end{bmatrix} \Leftarrow \underline{\underline{\text{Ans}}}$$

Q.9 Give examples of A & B such that

(a) A+B is NOT invertible but A & B are invertible

(b) A+B is Invertible but A & B are NOT invertible.

(c) All of A, B & A+B are invertible.

→ Sol^n :-

ⓐ A & B invertible but A+B NOT invertible:-

Suppose $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

then $|A| = 1$ & $|B| = -1$ Hence A^{-1} & B^{-1} exists.

But $A+B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ ⇒ $|A+B| = 0$
∴ $(A+B)^{-1}$ does not exists.

It can be generalized as.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} a & -b \\ -c & -d \end{bmatrix}$

& $ad \neq bc$ & $-ad \neq bc$

(b) A and B are not invertible but A+B invertible

Suppose,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore |A| = 0 \longrightarrow$ A is not invertible

$|B| = 0 \longrightarrow$ B is not invertible

But $|A+B| \neq 0 \rightarrow$ A+B is invertible.

Generalizing,

$$A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \;\&\; B = \begin{bmatrix} 0 & -b \\ 0 & c \end{bmatrix}$$

$$\text{or } A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \;\&\; B = \begin{bmatrix} 0 & 0 \\ -b & c \end{bmatrix}$$

where, $a \neq 0$, $c \neq 0$

(c) All of A, B & A+B are invertible.

Suppose $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\& \therefore A+B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Here, $|A| = 1 \neq 0 \Rightarrow A$ is invertible.

$|B| = 4 \neq 0 \Rightarrow B$ is invertible.

$|A+B| = 9 \neq 0 \Rightarrow A+B$ is invertible.

$$\cancel{A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}} \qquad \cancel{B = \begin{bmatrix} d & f \\ 0 & e \end{bmatrix}}$$

or

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \& \quad B = \begin{bmatrix} x & y \\ z & \omega \end{bmatrix}$$

where, $ad \neq bc$ & $x\omega \neq yz$

also $a + x \neq 0$ & $d + \omega \neq 0$

or $b + y \neq 0$ & $c + z \neq 0$

Q.10 In the last case (c) of Q.9 use identity $A^{-1}(A+B)B^{-1} = B^{-1} + A^{-1}$ to show that $C = B^{-1} + A^{-1}$ is also invertible & find formula for C^{-1}

→ Using $A^{-1}(A+B)B^{-1} = B^{-1} + A^{-1}$

we have, $C = B^{-1} + A^{-1}$

$$C^{-1} = (B^{-1} + A^{-1})^{-1} = (A^{-1}(A+B)B^{-1})^{-1}$$

$$\boxed{C^{-1} = B(A+B)^{-1}A.}$$

To check $CC^{-1} = I$ & $C^{-1}C = I$

$$C \cdot C^{-1} = (B^{-1} + A^{-1})B(A+B)^{-1}A.$$

$$= A^{-1}(A+B)\underbrace{B^{-1}B}_{I}(A+B)^{-1}A$$

$$= A^{-1}\underbrace{(A+B)(A+B)^{-1}}_{I}A$$

$$= A^{-1}A = I$$

$$\therefore \boxed{CC^{-1} = I}$$

$$C^{-1}C = B(A+B)^{-1}A(B^{-1} + A^{-1})$$

$$= B(A+B)^{-1}\underbrace{AA^{-1}}_{I}(A+B)B^{-1}$$

$$= B\underbrace{(A+B)^{-1}(A+B)}_{I}B^{-1}$$

$$= BB^{-1} = I$$

$$\Rightarrow \boxed{C^{-1}C = I}$$

$\therefore \boxed{CC^{-1} = C^{-1}C = I}$

& $\boxed{C^{-1} = B(A+B)^{-1}A}$

where, $C = B^{-1} + A^{-1}$

Q.11 Use Gauss jordon method to invert the following matrices.

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\& \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

→ Soln :-

(a) To find A_1^{-1} :-

$$A_1 \cdot A_1^{-1} = I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using $R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using $R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note:-
i.e. for $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ l & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -l & 1 & -m \\ 0 & 0 & 1 \end{bmatrix}$

(b)

$A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ Using $A_2 \cdot A_2^{-1} = I$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} A_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} -1 & 2 & -1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A_2^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using $R_1 \to (-1)R_1$

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A_2^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix} A_2^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using $R_2 \leftrightarrow (-1)R_3$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 3 & -2 \end{bmatrix} A_2^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

Using $R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix} A_2^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

Using $R_3 \to R_3/4$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} A_2^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}$$

Using $R_2 \to R_2 + 2R_3$ & $R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_2^{-1} = \begin{bmatrix} -1/4 & -3/2 & -3/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}$$

Using $R_1 \to R_1 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_2^{-1} = \begin{bmatrix} 3/4 & +1/2 & 3/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}$$

Q 11

$$\therefore A_2^{-1} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix} \Leftarrow \text{Ans}$$

(c) To Find A_3^{-1} : $A_3 \cdot A_3^{-1} = I$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} A_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A_3^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Using $R_2 \rightarrow R_2 - R_3$
& $R_1 \rightarrow R_1 - R_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_3^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Using $R_1 \rightarrow R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_3^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\therefore A_3^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Leftarrow \text{Ans}$$

Q.12 If P is any permutation matrix, find a non-zero vector "x" such that $(I-P)x=0$ Hence $I-P$ is not invertible.

→ Solution:-

considering a 3x3 permutation matrix,
~~For~~ We want to find a vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ such that for any permutation matrix P

$$(I-P)x=0$$

for $I-P$ we can have following cases:

ⓐ $I=P$:-

In that case, $(I-P)x = 0x = 0$
for any vector x.

ⓑ $P \neq I$:-

In this case, there is atleast one non-zero row in matrix $I-P$.

That meas that in that row we have -1, 1, & 0 in some order.

i.e. $x_1 - x_2 = 0$ or $x_2 - x_3 = 0$

or $x_1 - x_3 = 0$

Since all cases are possible
and

row of matrix $(I-P)$ times vector $x=0$

all three components of vector x must be equal.

$\therefore$ vector of for $x = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. $\alpha \neq 0$

satisfies $(I-P)x=0$ for any permutation of P.

Q.13 For which numbers 'c' is $A = LU$ impossible with 3-pivots?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

→ Sol^n:-

Given,

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Using Gauss-Elimination method

$R_2 \rightarrow R_2 - 3R_1$

$$\therefore A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Using $R_2 \leftrightarrow R_3$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & c-6 & 1 \end{bmatrix}$$

Using $R_3 \to R_3 + (6-c)R_2$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 7-c \end{bmatrix}$$

Hence if $c=7$ then there will only be 2 pivots.

So for $\underline{\underline{c=7}}$, it is impossible to convert A into LU with 3-pivots.

Q.14 Find L & U for non-symmetric matrix

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d, r, s, t to get A = LU with 4-pivots.

→ Soln:- Given

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}$$

Using Gauss elimination
Using $R_2 - R_1$, $R_3 - R_1$ & $R_4 - R_1$

$$A = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & b-r & c-r & t-r \\ 0 & b-r & c-r & d-r \end{bmatrix}$$

Using $R_3 - R_2$, $R_4 - R_2$

$$A = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & c-s & d-s \end{bmatrix}$$

Using $R_4 - R_3$

$$U\cancel{A} = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{bmatrix} \quad \& \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence for $A = LU$ with 4 pivots.

$$\boxed{\begin{matrix} a \neq 0 & \multirow{2}{*}{\&} & c \neq s \\ b \neq r & & d \neq t \end{matrix}}$$

And

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\& \quad U = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{bmatrix}$$

Q.15 Solve $LC = b$, to find 'c'. then solve $Ux = c$ to find x. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

→ Solⁿ:-

$$LC = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$c_1 = 4$; $c_1 + c_2 = 5$; $\Rightarrow c_2 = 1$

$c_1 + c_2 + c_3 = 6 \Rightarrow c_3 = 6 - 4 - 1 = 1$

$$\Rightarrow C = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \Leftarrow \underline{\underline{\text{Ans}}}$$

Now, $Ux = c \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow x_1 + x_2 + x_3 = 4 \quad \Rightarrow x_1 = 3$

$x_2 + x_3 = 1 \quad \Rightarrow x_2 = 0$

$x_3 = 1$

$$\therefore x = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \Leftarrow \underline{\underline{\text{Ans}}}$$

We know that, $Ax = b$

Here $A = LU$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+1 & 1+1 \\ 1 & 1+1 & 1+1+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \Leftarrow \text{Ans.}$$