RANDOM VARIABLES

Introduction

While describing the sample space of an experiment, we did not specify that an individual outcome needs to be a number. But in many situations, we are not concerned with all aspect of the outcome of an experiment but only in a particular numerical value of the outcome, such as, the number of red balls in a sample, the height of a randomly selected man, or the income of a randomly selected family. In other words, in many experimental problems (situations), we want to assign a real number x to every element $s \in S$. A random variable is used for this purpose.

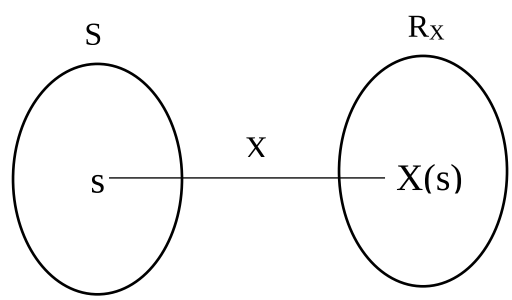
Definition: Let S be the sample space associated with a experiment E. A real valued function X defined on S and taking values in R $(-\infty, \infty)$ is called a **Random Variable** (rv) (or one-dimensional rv). i.e., a function X assigning to every element $s \in S$, a real number X(s) is called a rv and R is called the **Range space.**

Ex: Suppose a coin is tossed twice. Let X=No. of heads appearing. Then X takes values 0, 1, 2 as below:

Outcome	HH	HT	TH	TT
X: No. of Heads	2	1	1	0

Note that with every outcome s, there corresponds a real number X(s)=x. Thus, X is a rv.

One dimensional rvs will be denoted by X, Y, Z,…. The values which X, Y, Z,…… can assume are denoted by x, y, z,…………If x is a real number, the set of all $s \in S$ such that X(s) = x is denoted briefly by X=x.



i.e., $\{X = x\} = \{s \in S : X(s) = x\}$.

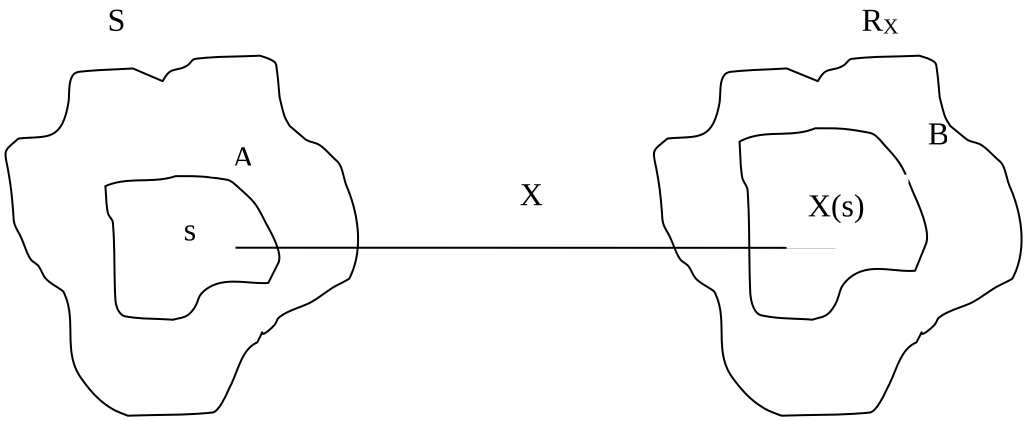
Thus, $P(X = x) = P\{s: X(s) = x\}$

$P(X \leq a) = P\{s: X(s) \in [-\infty,a]\}$

$P(a < X \leq b) = P\{s: X(s) \in [a,b]\}$

Results: If X, X_1 and X_2 are rvs, C, C_1 and C_2 are constants, then CX, X_1X_2, $X_1 + X_2$, $X_1 - X_2$, $C_1X_1 \pm C_2 X_2$, $|X|$, $\text{Max}(X_1, X_2)$, $\text{Min}(X_1, X_2)$, f(x) [where f(.) is a continuous function of X], etc. are all rvs.

Equivalent events: Let E be an experiment and S be the sample space. Let X be a rv defined on S and let R_X be its range space. Let B be an event w.r.t R_X; i.e. B is a subset of R_X.
Suppose that A is defined as $A = \{s \in S: X(s) \in B\}$ i.e. A consists of all outcomes in S for which $X(s) \in B$. Then we say that A and B are equivalent events. i.e. A and B are equivalent events whenever they occur together.



Definition: Let B be an event in the range space R_X. We define P(B) as P(B) = P(A) where $A=\{s \in S : X(s) \in B\}$. i.e. we define P(B) equal to the probability of the event A which is equivalent to B.

Note: We are assuming that probabilities may be associated with events in S. Hence the above definition makes it possible to assign probabilities to events associated with R_X in terms of probabilities defined over S.

Discrete Random Variable (drv): Let X be a rv. If X takes atmost a countable number of values, it is called a Discrete Random Variable. i.e. a real valued function defined on a discrete sample space is called drv.
In Otherwords, if the number of possible values of X is finite or countably infinite, we call X as drv. i.e. the possible values of X may be listed as $x_1, x_2, x_3, \ldots\ldots\ldots, x_n, \ldots\ldots$ In the finite case, the list terminates and in countably infinite case, the list continues indefinitely.

Ex: (i) X = No. of α – particles emitted by a radioactive source in a given period.

(ii) X = No. of telephone calls received at a telephone exchange in a specified time interval.

Definition: Probability Mass Function (pmf) and Probability Distribution of a drv X

Let X be a drv taking atmost a countably infinite number of values $x_1, x_2, \ldots, x_n, \ldots$ with each possible outcome x_i we associate a number $p(x_i) = P(X = x_i)$ called the probability of x_i. The numbers $p(x_i)$, i=1,2,3,………must satisfy the following conditions.

(i) $p(x_i) \geq 0 \ \forall i$

(ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

This function $p(x_i)$ is called the pmf of the drv X and the set $\{p(x_i)\}$ or $\{x_i, p(x_i)\}$, i=1,2,3,… is called the probability distribution of rv X.

Depending on the nature or the functional form of the pmf, we have several Discrete Probability Distributions.

Binomial Distribution:

Consider an experiment E. Let A be an event associated with E. Let P(A) = p so that $P(\bar{A}) = 1-p$. Consider 'n' independent repetitions of E. The sample space consists of all possible sequences $\{a_1, a_2, \ldots\ldots, a_n\}$, where a_i is either A or $\bar{A}$ depending on whether A or $\bar{A}$ occurred on the i^{th} repetition of E.

Let the rv X denote the number of times the event A occurred. We call X a Binomial rv with parameters n and p and its possible values are 0,1,2,3,…..,n (i.e. we say X has a Binomial distribution with parameters n and p). The individual repetitions are called **Bernoulli trials.**

Theorem: Let X be a Binomial rv based on n repetition.

Then $P(X = k) = \binom{n}{k} .p^k. (1-p)^{n-k}$, k = 0,1,2,3,……,n.

Proof: Consider a particular order in an outcome satisfying order in an outcome satisfying X = k as AAAAA…..A (k times) and $\bar{A}\,\bar{A}\,\bar{A}\,\bar{A}$ ……… $\bar{A}$ (n-k times).

i.e., 1st k repetition resulted in the occurrence of A while the last n-k repetition resulted in the occurrence of $\bar{A}$. Since all the repetition are independent, we have,

P(AAAAA…..A (k times) and $\bar{A}\,\bar{A}\,\bar{A}\,\bar{A}$ ……… $\bar{A}$ (n-k times)) = $p^k. (1-p)^k$

But exactly the same probability would be associated with any other outcome for which X = k. The total number of such outcomes in $\binom{n}{k}$ (i.e., we must choose exactly k (out of n) positions for A's). Now these $\binom{n}{k}$ outcomes are all mutually exclusive.

Hence using addition theorem of probability, we get the expression

$P(X = k) = \binom{n}{k}.p^k. (1-p)^{n-k}$, k = 0,1,2,3,……,n.

Continuous Random Variable (crv): A rv X is said to be continuous if it can take all possible values between certain limits. In otherwords, a rv is said to be continuous when its different values cannot be put in one to one correspondence with a set of positive integers.

Definition: Probability Density Function (pdf) of a crv X

Let X be a crv. Then the function f is said to be the probability density function of X if it satisfies the following condition:

(i) $f(x) \geq 0 \;\; \forall x$

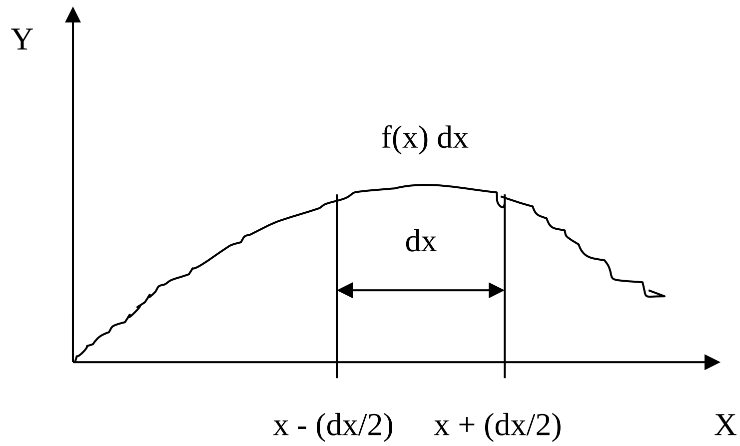
(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

(iii) $P(a \leq X \leq b) = \int_a^b f(x)\,dx \,;\, -\infty < \epsilon\, a < b < \infty.$

Definition: Let X be a crv. Let f(x) be any continuous function of x so that f(x)dx represents the probability that X falls in a small interval (x - (dx/2), x + (dx/2)).

i.e. $f(x)dx = P(x - (dx/2) \leq X \leq x + (dx/2))$.

In the figure, f(x)dx represents the area bounded by the curve. Thus f(x) is called the pdf of X.



It is a consequence of (iii) above that for any specified value of X, say x_0, we have

$P(X = x_0) = 0$, as $P(X = x_0) = P(x_0 \leq X \leq x_0) = \int_{x_0}^{x_0} f(x)dx = 0$

Accordingly, for a continuous rv X, we have

$P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b)$

Again, depending on the nature or the functional form of the pdf, we have several Continuous Probability Distributions.