M/M/1: GD/∞/ ∞

This is the basic model widely used in queuing system. Here there is one server with no limit on the capacity on queuing system or calling source. Arrivals & departures occur according to a Poisson process with rates λ and μ respectively. Here $\lambda < \mu$ so that, there is no unending queue and all the customers are served. Service discipline being general discipline.(π_n is independent of service discipline). Thus, we use GD in the above notation. Our objective is to obtain an expression for π_n.

We first derive the differential- difference equations for $p_n(t)$, i.e. Probability of having `n' customers in the system during time 't'. Then under appropriate conditions, we take limits as t→∞ to obtain steady state probability π_n .

To derive the differential- difference equations, we make the following assumptions:

1) For an infinitesimal interval of length 'h'
   P[one arrival in `h'] = λh + O(h).

   P[no arrival in `h'] = 1- λh + O(h).

   P[one departure in `h'] = μh + O(h).

   P[no departure in `h'] = 1- μh + O(h).
   Where $O(h) \to 0$ as $h \to 0$ and $\lim_{h\to 0} \frac{O(h)}{h} = 0$

2) Further, it is assumed that atmost only one event (either an arrival or a departure) can occur in h. Consider,
   (i) When n>0 ; $p_n(t+h) = P[N(t + h) = n]$

   The above probability is the sum of the following probabilities:

$$\begin{aligned} p_n(t+h) = {} & P[N(t) = n]\ P(\text{no arrival in h})\ P(\text{no departure in h}) \\ & + P[N(t) = n-1]\ P(\text{one arrival in h})\ P(\text{no departure in h}) \\ & + P[N(t) = n+1]\ P(\text{no arrival in h})\ P(\text{one departure in h}) \\ = {} & p_n(t)\,(1 - \lambda h + O(h))(1 - \mu h + O(h)) \\ & + p_{n-1}(t)\,(\lambda h + O(h))(1 - \mu h + O(h)) \\ & + p_{n+1}(t)\,(1 - \lambda h + O(h))(\mu h + O(h)) \end{aligned}$$

$$p_n(t+h) = p_n(t)\,(1 - \lambda h - \mu h) + p_{n-1}(t)\,\lambda h + p_{n+1}(t)\,\mu h + O(h)$$

$$p_n(t+h) - p_n(t) = \lambda h\,(p_{n-1}(t) - p_n(t)) + \mu h\,(p_{n+1}(t) - p_n(t)) + O(h)$$

$$\frac{p_n(t+h) - p_n(t)}{h} = \lambda\,(\,p_{n-1}(t) - \;p_n(t)) + \;\mu\,(p_{n+1}(t) - p_n(t)) + \frac{O(h)}{h}$$

As $h \to 0$: $p_n'(t) = \;\lambda(\,p_{n-1}(t) - \;p_n(t)) + \mu(\,p_{n+1}(t) - p_n(t))$

Or $p_n'(t) = \;\lambda\,p_{n-1}(t) + \mu\,p_{n+1}(t) + (\lambda + \mu)p_n(t) \quad ; \quad n > 0$ --------- (1)

(ii) When n = 0 ; $p_0(t\, +\, h) \; = \; P[N(t\, +\, h) = 0]$

Noting that, for n=0, probability of occurrence of 0 departures during h is one,

we have,

$$p_0(t\, +\, h) \; = \; p_0(t)\,(1 \,-\, \lambda h + \,O(h)\,) \; + \; p_1(t)\,(1 - \,\lambda h \,+ O(h))(\,\mu h \,+ \,O(h))$$

$$p_0(t\, +\, h) = p_0(t)\,(1 - \lambda h + O(h)) + p_1(t)\,(\,\mu h + O(h))$$

$$p_0(t\, +\, h) - p_0(t) = p_1(t)\mu h \,- p_0(t)\,\lambda h \,+\, O(h)$$

$$\frac{p_0(t+h) - p_0(t)}{h} = \;\mu\,p_1(t) - \;\lambda\,p_0(t) \quad + \frac{O(h)}{h}$$

As $h \to 0$: $p_0'\,(t) \; = \mu\,p_1(t) - \;\lambda\,p_0(t) \quad ; \quad$ n=0 ------------ (2)

Solving (1) and (2) by Laplace transformation technique a solution to $p_n(t)$, the transient state probabilities can be obtained. But the procedure is quite complex.

Due to the complexity involved in solving the above equations and that our interest lies in obtaining an expression for the steady state probabilities, we consider only the steady state analysis.

We obtain the steady state equations by noting that
as $t \to \infty$, $p_n'\,(t) \;\to 0$ and $p_n(t) \to \pi_n$ for all n.

Thus as $t \to \infty$, (1) and (2) reduce to

$\lambda\,(\pi_{n-1} - \pi_n) + \mu\,(\pi_{n+1} - \pi_n) = 0\,; \;\; n > 0$ ------------------- (3)

$\mu\pi_1 - \;\lambda\,\pi_0 = 0 \;;\; n = 0$ --------------------------- (4)

Solution for π_n : from (4) we get, $\pi_1 = \frac{\lambda}{\mu}\pi_0 = \rho\pi_0$

In (3) put n=1 : $\lambda(\pi_0 - \pi_1) + \mu(\pi_2 - \pi_1) = 0$

$\mu\pi_2 = (\lambda + \mu)\pi_1 - \lambda\pi_0$

or $\pi_2 = \frac{\lambda^2}{\mu^2}\pi_0 = \rho^2\pi_0$

Similarly, put n=2 : $\lambda(\pi_1 - \pi_2) + \mu(\pi_3 - \pi_2) = 0$

$\mu\pi_3 = (\lambda + \mu)\pi_2 - \lambda\pi_1$

or $\pi_3 = \frac{\lambda^3}{\mu^3}\pi_0 = \rho^3\pi_0$

Thus we may arrive at: $\boldsymbol{\pi_n = \frac{\lambda^n}{\mu^n}\pi_0 = \rho^n\pi_0}$

To obtain π_0 :

Since π_n's are Probabilities, we have

$$\sum \pi_n = 1 \Rightarrow \sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n \pi_0 = 1$$

$$\pi_0 = \frac{1}{\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n} = \frac{1}{\frac{1}{1-\frac{\lambda}{\mu}}} \quad ; \text{(since } \frac{\lambda}{\mu} < 1)$$

$$\therefore \ \pi_0 = \frac{\mu - \lambda}{\mu} = 1 - \frac{\lambda}{\mu} = 1 - \rho$$

$\Rightarrow \boldsymbol{\pi_n = \rho^n(1 - \rho)}$

Now $\pi_0 =$ Probability that server is idle or no queue = $\left(1 - \frac{\lambda}{\mu}\right) = 1 - \rho$

$\therefore$ Probability that server is busy = $1 - \pi_0 = \rho$

Consider,

$$L_s = \sum_{n=0}^{\infty} n\,\pi_n = \sum_{n=0}^{\infty} n\left(\left(\frac{\lambda}{\mu}\right)^n - \left(\frac{\lambda}{\mu}\right)^{n+1}\right)$$

$$= \left(1-\frac{\lambda}{\mu}\right)\sum_{n=0}^{\infty} n\left(\frac{\lambda}{\mu}\right)^n = \left(1-\frac{\lambda}{\mu}\right)\left(0 + 1.\left(\frac{\lambda}{\mu}\right) + 2.\left(\frac{\lambda}{\mu}\right)^2 + \cdots\right)$$

$$L_s = \left(1-\frac{\lambda}{\mu}\right)\frac{\frac{\lambda}{\mu}}{\left(1-\frac{\lambda}{\mu}\right)^2} \quad ; \text{ Since } \frac{\lambda}{\mu} < 1 \quad \left[s = \frac{a}{(1-a)^2}\right]$$

$$L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$$

$$L_q = L_s - \rho = \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

Or $L_q = \sum_{n=1}^{\infty} (n-1)\,\pi_n$

$$W_s = \frac{L_s}{\lambda} = \frac{\lambda}{\lambda(\mu-\lambda)} = \frac{1}{(\mu-\lambda)} = \frac{1}{\mu(1-\rho)} = \frac{\rho}{\lambda(1-\rho)}$$

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu^2(1-\rho)} = \frac{\rho^2}{\lambda(1-\rho)}$$