Long-run behavior / Limiting behavior of Markov Chains:

Suppose a Markov system is in operation for a sufficiently long period of time and suppose that a large number of transitions take place during this period of time.

As the number of transitions increase the absolute probability (i.e. the probability that the system will be in state j) becomes independent of the initial state i or initial distribution.

In other words the system reaches the condition when the initial disturbances or start-up effects die out and the system may be regarded as having reached some kind of STEADY STATE OR EQUILIBRIUM STATE / DISTRIBUTION.

This called the Long-run behavior / limiting behavior of Markov Chains.

For an irreducible ergodic Markov Chain it can be shown than $\lim_{n\to\infty} p_{ij}^{(n)}$ exists and is independent of initial state i.

i.e. $\lim_{n\to\infty} p_{ij}^{(n)}=\pi_j \; \forall\, j; j = 0,1,2, \ldots.$, where π_j exists uniquely satisfying following steady state equations:

$$\pi_j = \sum_{i=0}^{\infty} \pi_i \, p_{ij} \;\; \forall\, j; j = 0,1,2, \ldots$$

$$\sum_{j=0}^{\infty} \pi_j = 1$$

$$\pi_j \geq 0 \;\forall\, j$$

The π_j's are called the steady state probabilities of the Markov Chain and are equal to reciprocal of expected recurrence time.

$$\pi_j = \frac{1}{\mu_{jj}} \;\forall\, j$$

Here, the term Steady State Probability means that the probability of finding the process in a certain state j after a large number of transitions tends to the value π_j which is independent of initial probability distribution defined over the states.

Note: Steady state probability does not mean that the process settles down into one of the states. In fact, the process continues to make transitions from state to state and at any step n, the transition probability from state i to state j is still p_{ij}.

Suppose j=0,1,2,…,m ; then we have the steady state equations with (m+2) equations in (m+1) unknowns. Since, it has a unique solution one of the equations must be redundant and hence can be deleted. This cannot be $\sum_{j=0}^{m} \pi_j = 1$, since $\pi_j = 0 \;\forall\, j$ will satisfy the other m+1 equations.

Thus, one of the equations $\pi_j = \sum_{i=0}^{m} \pi_i \, p_{ij}$ is redundant.

Thus, we have a set of linear equations which may be solved by any of the methods to obtain π_j's, the steady state probabilities.

We may also write these steady state equations in the matrix form as $\Pi = \Pi.\ P$, where Π is a matrix with 'm' identical rows, each represented by a vector. $\Pi = (\pi_0, \pi_1, \pi_2, \ldots \ldots, \pi_m)$ and P is the tpm of the irreducible ergodic Markov Chain with (m+1) states.

Problems

1) Consider the tpm $P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$ Obtain the steady state probabilities.

Solution:

Calculating the powers of P we get

$P^2 = \begin{bmatrix} 0.54 & 0.30 & 0.16 \\ 0.28 & 0.48 & 0.24 \\ 0.20 & 0.46 & 0.34 \end{bmatrix}$; $P^4 = \begin{bmatrix} 0.4076 & 0.3796 & 0.2128 \\ 0.3336 & 0.4248 & 0.2416 \\ 0.3048 & 0.4372 & 0.2580 \end{bmatrix}$; $P^8 =$……..; $P^{16} =$…….; $P^{32}=$…….;

Note that although the rows of P are quite different, the rows of P^3, P^4, P^6, etc. are more and more similar. In fact if we were to calculate the higher powers of P we would see that P^n approaches L $= \begin{bmatrix} 0.353 & 0.412 & 0.235 \\ 0.353 & 0.412 & 0.235 \\ 0.353 & 0.412 & 0.235 \end{bmatrix}$, called the limiting matrix of P. Note that the rows of L are identical. This implies that, if n is large the probability of the process being in state 1 at time n does not depend upon weather the chain was initially in state 1,2 or 3.

For an ergodic Markov Chain such a limit exists. i.e $\lim_{n\to\infty} P^n = L = \Pi$ This Π is called the limiting distribution or equilibrium distribution or stationary distribution of the Markov Chain or the steady states of the chain or the vector of stable probabilities.

Note:

1) For a Markov Chain with μ_{jj}, the mean recurrence time, it can be shown that $\mu_{jj} = \frac{1}{\pi_j}$. This $\mu_{jj} < \infty$ if $\pi_j \neq 0$ and if $\pi_j = 0$ then $\mu_{jj} = \infty$, then state j is a Null-recurrent state and hence the Markov Chain is not Ergodic.

2) If the TPM of a Markov chain is doubly stochastic and has 'r' states, then the steady state distribution is $\Pi = (1/r, 1/r, 1/r, \ldots \ldots, 1/r)$