**Conditional Probability (Contd…)**

As we have already seen the conditional probabilities may also be used as tools for computing unconditional probabilities.

For ex: $P(A\cap B)=P(A/B).P(B) \rightarrow$ A
i.e. one can compute $P(A\cap B)$ from the knowledge of $P(A/B)$ and $P(B)$.
Also we know that, $P(A)= P(A/B).P(B)+ P(A/\overline{B}).P(\overline{B}) \rightarrow$ B
(i.e. by partitioning S, using B and $\overline{B}$ s.t $B \cap \overline{B}= \phi$)
Finally if $P(A)>0$, we may use A and B to compute $P(B/A)$
i.e. $P(B/A)=[P(A/B).P(B)] / [P(A/B).P(B)+ P(A/\overline{B}).P(\overline{B})] \rightarrow$ C
C is a special case of Bayes' theorem

**BAYES' THEOREM** (Thomas Bayes'- 17th century)

Suppose that $B_1, B_2 \ldots B_k$ are k mutually exclusive and collectively exhaustive events and at least one and not more than one of them must have happened, but it is not known which one. Suppose also that an event A may follow any one of the events $B_i$, with known probability and that A is known to have happened. What is then the probability that it was preceded by a particular event $B_i$?
The answer to this is given by Bayes' theorem.

**Statement:** Let $B_1, B_2 \ldots B_k$ be a partition of the sample space S and let A be any arbitrary event associated with S such that $P(A)>0$. Then we have,

$$P(B_i/A)=P(B_i).P(A/B_i)/ \sum_{j=1}^{k} P(B_j).P(A/B_j)$$

This is called the Bayes' theorem.

**Note**:
1. The probability $P(B_1) \ldots P(B_k)$ are typically subjective probabilities which represent our opinion about the nature prior to any experimentation and are termed as '<u>apriori probabilities</u>'. As they exist before we gain any information from the experiment itself
2. Probabilities $P(A/B_i)$, i=1,2,3….k are called 'Likelihood probabilities'.
3. The probabilities $P(B_i/A)$ i=1,2,3….k are called 'posteriori probabilities' as they are determined after the results of the experiments are known.(after the event A observed to occur)
4. Since $B_i$'s are a partition of S one and only one of the events $B_i$ occurs. Hence Bayes' theorem gives us the probability of a particular $B_i$ given that the event has occurred. In order to apply this theorem we must know the values of $P(B_i)$s. Quite often these values are not known and this limits the applicability of the result.

**Independent Events**:

Quite often we want to know, for what events A and B, it is true that $P(A/B)=P(A)$?

In other words, for what events A and B, it is true that the occurrence of B provides no information about the chance that A will occur?

The answer is, $P(A/B)=P(A)$.

i.e. $P(A\cap B)/P(B)=P(A)$

i.e. $P(A\cap B)=P(A).P(B) \rightarrow$ A .

Thus we say that two events A and B are independent iff A holds.

**Definition:** Two events A and B are said to be independent iff $P(A\cap B)=P(A).P(B)$.

**Note 1:** A & B are disjoint then $P(A\cap B)=P(\phi)=0$, so that A and B cannot be independent unless either $P(A)=0$ or $P(B)=0$.

**Note 2:** If A and B are independent then A and $\bar{B}$ are independent, $\bar{A}$ and B are independent and $\bar{A}$ and $\bar{B}$ are independent.

Prove it!

**Definition:** (**Pair-wise independent events**) A set of events $A_1, A_2, \ldots A_n$. are said to be pair- wise independent if $P(A_i\cap A_j)=P(A_i)P(A_j)\ \forall i\neq j$

**Definition:** (**Mutual independence of n events**) If $A_1, A_2, \ldots A_n$ are n events, then for their mutual independence we should have

$P(A_{i1}\cap A_{i2}\cap \ldots \cap A_{ik})=P(A_{i1})P(A_{i2})\ldots P(A_{ik})\quad k=2,3,4\ldots n.$

i.e. we should have

i) $P(A_i\cap A_j)=P(A_i)P(A_j)$ ; $(i\neq j,\ i,j=1,2,3,4.....n)$

ii) $P(A_i\cap A_j\cap A_k)= P(A_i)P(A_j)P(A_k)$ ; $(i\neq j\neq k;\ i,j,k=1,2,3\ldots n)$

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$P(A_i\cap A_j\cap \ldots \cap A_n)= P(A_1)P(A_2)\ldots P(A_n)$ .

i.e in all there are $2^n-n-1$ conditions. In particular for n=3(say A,B,C)we have $2^3-3-1=4$ conditions for their mutual independence viz,

$P(A\cap B)=P(A).P(B)$, $P(A\cap C)=P(A).P(C)$ , $P(B\cap C)=P(B).P(C)$ and $P(A\cap B\cap C)=P(A).P(B).P(C)$.

**Theorem:** A and B are two events with nonzero probabilities. If they are mutually exclusive then they cannot be independent and conversely.

**Proof:** Given $P(A)>0$ & $P(B)>0 \rightarrow$ 1

a) A and B are mutually exclusive => $P(A\cap B)=0 \rightarrow$ 2
For independence we must have $P(A\cap B)=P(A).P(B)>0$ which contradicts 2.
Hence m.e events cannot be independent.

b) Similarly A and B to be independent $\Rightarrow P(A\cap B)=P(A).P(B)>0 \rightarrow$ 3
Now for A and B to be m.e., we must have $P(A\cap B)=0$, which contradicts 3.
Hence independent events cannot be m.e.