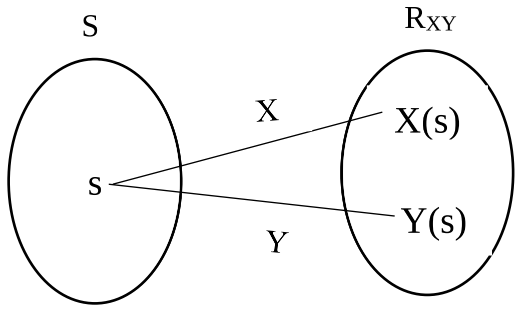
TWO AND HIGHER DIMENSIONAL RANDOM VARIABLES

Definition:
Let S be the sample space associated with a given random experiment E. Let X=X(s) and Y=Y(s) be two functions each assigning a real number to each outcome s ∈ S. Then we call the pair (X,Y) a 2-dimensional random variable.



Ex: Height and weight of a randomly chosen person.

Definition:
If $X_1=X_1(s)$, $X_2=X_2(s)$……$X_n=X_n(s)$ are n functions each assigning a real number to every outcome s ∈ S, then we call $(X_1,X_2….X_n)$ a n-dimensional random variable.

Definition:

Discrete case: A 2-dimensional rv (X,Y) is said to be discrete if the possible values of (X,Y) are finite or countably infinite.

Continuous case: A 2-dimensional rv (X,Y) is said to be continuous if it can take all values in some non-countable set R of the Euclidean plane.

Probability distribution:

Discrete case: Let (X,Y) be a 2-dimensional drv. With each possible outcome (x_i,y_j) we associate a No. $p(x_i,y_j) = P(X=X_i, Y= Y_j)$, satisfying

1. $p(x_i, y_j) \geq 0 \; \forall \; i, j$
2. $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} p(x_i, y_j) = 1$

The function $p(x_i,y_j)$ defined above is called the joint probability mass function (pmf) of the 2-dimensional drv (X,Y) and the set of triplets$(x_i, y_j, p(x_i,y_j))$, where i,j=1…n is called the joint probability distribution of the 2-dimensional drv (X,Y).

Continuous case: Let (X,Y) be a 2-dimensional crv. Then there exists a function f(x,y) called the joint pdf of the 2-dimensional crv (X,Y) satisfying

1. $f(x, y) \geq 0 \; \forall \; x, y \in R.$
2. $\iint_R f(x, y)dxdy = 1$ or equivalently $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1.$

Also, 3. $P(a \leq X \leq b, c \leq Y \leq d) = \int_c^d \int_a^b f(x, y)dxdy$

Joint Cumulative distribution function:

Let (X,Y) be a 2-dimensional random variable then the joint cdf F of (X,Y) is defined as

$F(x,y) = P(X \le x, Y \le y) = P(-\infty < X \le x , -\infty < Y \le y)$

$$= \sum_{\{(i,j)\,:\,(x_i,y_j) \le (x,y)\}} \sum p(x_i,y_j) \quad \text{if (X,Y) is a 2-dimensional drv}$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\, ds\, dt \quad \text{if (X,Y) is a 2-dimensional crv}$$

Marginal probability distribution:

Discrete case: Let us associate with each 2-dimensional drv (X,Y), two one dimensional random variables, say, X and Y individually. Our interest may be in finding the probability distribution of X and the probability distribution of Y respectively. We have the joint probability distribution of X and Y given by $p_{ij} = p(x_i,y_j) = P(X=x_i, Y=y_j)\ \forall\, i,j$.

This is usually represented in the tabular form as shown below:

X\Y	y_1	y_2	…………	…………	y_m	Total
x_1	p_{11}	p_{12}	…………..	…………	p_{1m}	$P_{1.}$
x_2	p_{21}	p_{22}	…………	………..	p_{2m}	$P_{2.}$
:	:	:	………..	…………	:	:
:	:	:	………..	…………	;	:
x_n	p_{n1}	p_{n2}	………..	…………	p_{nm}	$P_{n.}$
Total	$P_{.1}$	$P_{.2}$	………..	………..	$P_{.m}$	1

Note: $\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$.

The probability distribution of X called the Marginal probability distribution of X is given by

$p(x_i) = P(X=x_i) = P(X=x_i,Y=y_1) + P(X=x_i,Y=y_2) + \ldots\ldots + P(X=x_i,Y=y_j) + \ldots$

$= \sum_{j=1}^{n} p_{ij}$

$= \sum_{j=1}^{\infty} p(x_i, y_j)\ \forall\, i.$

(since $X=x_i$ must occur with $Y=y_j$ for some j and can occur with $Y=y_j$ for only one j)

Similarly, $q(y_j) = P(Y=y_j) = \sum_{i=1}^{\infty} p(x_i, y_j)\ \forall\, j$ is the Marginal probability distribution of Y.

Continuous case: Let f (x,y) be the joint pdf of a 2-dimensional crv (X,Y). Then g(x) and h(y), the marginal probability density functions of X and Y, are respectively given by

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Note: These pdfs g(x) and h(y) correspond to the basic pdfs of one-dimensional random variables X and Y respectively.

Also, $P(a \le X \le b) = P(a \le X \le b; -\infty < Y < \infty)$

$$= \int_{a}^{b} \int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_{a}^{b} g(x)\,dx$$

Similarly, we obtain, $P(a \le Y \le b) = \int_{a}^{b} h(y)\,dy$.

Conditional probability distribution:

Discrete case: Let (X,Y) be a 2-dimensional drv with joint probability distribution $p(x_i,y_j)$. Let $p(x_i)$ and $q(y_j)$ be the marginal probability distributions of X and Y respectively. Then the conditional probability distribution of $X=x_i$ given $Y=y_j$ is defined as

$p(x_i/y_j) = P(X=x_i/Y=y_j)$
$= P(X=x_i,Y=y_j) / P(Y=y_j)$
$= p(x_i,y_j)/q(y_j);\ q(y_j)>0$

Similarly, the conditional probability distribution of $Y=y_j$ given $X=x_i$ is defined as

$q(y_j /x_i)= P(Y=y_j / X=x_i)$
$= p(x_i ,y_j) / p(x_i);\ p(x_i)> 0$

Continuous case: Let (X,Y) be a 2-dimensional crv with joint pdf f(x,y). Let g(x) and h(y) be the marginal pdfs of X and Y respectively. Then the conditional pdf of X given Y is defined as
$g(x/y) = f(x,y) / h(y)\ ;\ h(y)>0$

Similarly, the conditional pdf of Y given X is defined as
$h(y/x) = f(x,y) / g(x)\ ;\ g(x)>0.$

Independent random variables:

Discrete case: Let (X,Y) be a 2-dimensional drv with joint probability distribution $p(x_i,y_j)$. Let $p(x_i)$ and $q(y_j)$ be the marginal probability distributions of X and Y respectively. Then X and Y are said to be independent random variables iff

$p(x_i,y_j) = p(x_i).\ q(y_j)\ \ \forall\ i,j.$
i.e. $P(X=x_i,Y=y_j) = P(X=x_i)\ P(Y=y_j)\ \ \forall\ i,j.$

Continuous case: Let (X,Y) be a 2-dimensional crv with joint pdf f(x,y). Let g(x) and h(y) be the marginal pdfs of X and Y respectively. Then X and Y are said to be independent random variables iff
$f(x,y) = g(x).h(y)\ \ \forall\ x,y.$

In other words

(a) Let (X,Y) be a 2-dimensional drv. Then X and Y are said to be independent iff
$p(x_i/y_j) = p(x_i)\ \forall\ i,j.$ [or≡ iff $q(y_j/x_i) = q(y_j), \forall i, j$]

(b) Let (X,Y) be a 2-dimensional crv. Then X and Y are said to be independent iff
$g(x/y) = g(x)\ \forall\ x,y.$ [or≡ iff $h(y/x) = h(y), \forall x, y$]