Roll No. 202cd005

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Assignment - 1

[INEQUALITIES]

Que1. If $a_1, a_2, \ldots, a_n$ are n distinct odd natural numbers. not divisible by any prime greater than 5. show that

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n} < 2.$$

→ As it is given that, for give odd natural numbers, ~~so~~ should have only prime factors less than 5 i.e. 3 & 5 then, for any integer. x we have,

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < \left(\frac{1}{3^0} + \frac{1}{3^1} + \cdots + \frac{1}{3^x} + \cdots \infty\right) * \left(\frac{1}{5^0} + \frac{1}{5^1} + \cdots + \frac{1}{5^x} + \cdots \infty\right)$$

This is a product of 2- G.P. & their summation is.

$$\Rightarrow \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < \left(\frac{1}{1-1/3}\right)\left(\frac{1}{1-1/5}\right)$$

$$< \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{8} < 2$$

$$\therefore \boxed{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < 2}$$

Hence proved.

Que2 Let a, b, c denote sides of a triangle show that the quantity.

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$ lies betn $\frac{3}{2}$ & 2.

Can equality hold at either limit?

→ Given,

$$\frac{a}{c+b} + \frac{b}{c+a} + \frac{c}{a+b}$$

$$= \frac{ac+a^2+a^2+ab+\cancel{a}bc+b^2+ab+b^2 + c^2+bc+c^2+ac}{(c+b)(c+a)(a+b)}$$

$$= \frac{2a^2+2ab+2b^2+2c^2+2bc+2ac}{(a+b)(b+c)(c+a)}$$

$$= \frac{1}{2}(a+b+c)\left\{\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right\} - 3$$

$$= \frac{1}{2}\left[[(a+b)] + (b+c) + (c+a)\right]\left\{\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right\} - 3$$

then by AM≥HM inequality we have for $a=b=c$

$$= \frac{1}{2}\cdot 9 - 3 = \frac{3}{2} \quad - (1)$$

let a, b & c be arranged such that

$$a \le b \le c$$

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$$\Rightarrow \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

$$\leq \frac{a}{a+c} + \frac{c}{c+a} + \frac{c}{a+b}$$

$$\leq 1 + \frac{c}{a+b}$$

$$< 1 + 1 = 2$$

...... By property of triangle $c < a+b$.

$\therefore$

$$\boxed{\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 2}$$

$\therefore$ Inequality holds at $3/2$ only if $a=b=c$ & holds at 2 only if $a \leq b \leq c$.

Que3 Determine the largest number in the infinite sequence

$$1, \sqrt{2}, \sqrt[3]{3}, \ldots \sqrt[n]{n}, \ldots$$

→ The given sequence can be represented by following f^n $\sqrt[x]{x}$.

from this we have,

$$\sqrt[1]{1} = 1, \quad \sqrt[2]{2} = 1.4142.$$

$$\sqrt[3]{3} = 1.4422 \ \& \ \sqrt[4]{4} = 1.4142$$

$$\sqrt[5]{5} = 1.3797$$

from this it is clear that $\sqrt[3]{3}$ is the largest number in the sequence.

But we need to show that,

$$\sqrt[n]{n} > \sqrt[n+1]{n+1} \text{ for } n \geq 3$$

$$\Rightarrow \left(1 + \frac{1}{n}\right)^n < n \text{ for } n \geq 3$$

from binomial theorem, we have

$$\left(1+\frac{1}{n}\right)^n = 1 + {}^nC_1 \times \frac{1}{n} + {}^nC_2 \times \frac{1}{n^2} + \ldots$$

$$+ {}^nC_k \times \frac{1}{n^k} + \ldots + \frac{1}{n^2}$$

Also,

$$^nC_k \times \frac{1}{n^k} = \frac{n!}{(n-k)!\,k!} \times \frac{1}{n^k}$$

$$\therefore {}^{n}C_{k} \times \frac{1}{n^k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k! \times n^k}$$

$$= \frac{1}{k!} \times \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k+1}{n}\right)$$

Because each term is 1 - something, hence,

$$\therefore {}^{n}C_{k} \times \frac{1}{n^k} < \frac{1}{k!}$$

$\Rightarrow$ for $n \geq 3$

$$\left(1+\frac{1}{n}\right)^n < 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \dots + \frac{1}{\cancel{2^{n-1}}\, n!}$$

[since $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^{x}$]

$$\cancel{= 1 + e\left(1 - \frac{1}{2^n}\right)} \quad \cancel{< (1 + 2.7) = 3.7}$$

$$< 1 + e = 3.7$$

$\therefore$ for $n \geq 3$,

$$\boxed{\left(1+\frac{1}{n}\right)^n < 3 \leq n}$$

Que 4. If a & b are positive real numbers such that $a+b=1$, prove that

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \geq \frac{25}{2}.$$

→ ~~Gi~~ Given,

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 = a^2+\frac{1}{a^2}+b^2+\frac{1}{b^2}+4$$

$$= (a+b)^2 - 2ab + \left(\frac{1}{a}+\frac{1}{b}\right)^2 - \frac{2}{ab} + 4$$

$$= 1-2ab+\frac{1-2ab}{a^2b^2}+4$$

But according to AM-GM inequality.

$$\frac{a+b}{2} \geq \sqrt{ab}$$

$$\therefore \boxed{\frac{1}{4} \geq ab} \quad /\text{as } 'a+b=1' \text{ given}$$

∴ we get

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \geq \left(1-\frac{1}{2}\right) + \frac{1-1/2}{1/16}+4$$

$$\geq 1/2 + 8 + 4$$

$$\boxed{\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \geq \frac{25}{2}}$$

Que 5. If $a_0, a_1, \dots, a_{50}$ are coefficients of the polynomial

$$(1+x+x^2)^{25}$$

prove that the sum $a_0+a_2+\dots+a_{50}$ is even

→ Given polynomial is $(1+x+x^2)^{25}$

for $x = 1$.

$$(1+x+x^2)^{25} = (1+1+1^2)^{25} = 3^{25}$$

i.e. $a_0 + a_1x + a_2x^2 + \dots + a_{50}x^{50} = 3^{25}$

$$\therefore a_0 + a_1 + a_2 + \dots + a_{50} = 3^{25} \quad -(1)$$

for $x = -1$,

$$a_0 - a_1 + a_2 + \dots + a_{50} = 1^{25} \quad -(2)$$

adding eq^n (1) & (2) we get,

$$2[a_0 + a_2 + a_4 + \dots + a_{50}] = 3^{25} + 1^{25}$$

⇒ By cyclicity we will get 4 at unit digit on R.H.S. & ~~after~~ it is divisible by 2. Hence R.H.S. is even.

⇒ $\boxed{a_0 + a_2 + \dots + a_{50} = \text{even}}$

Que 6. Let a, b, c be real numbers with $0 < a < 1$, $0 < b < 1$, $0 < c < 1$ & $a + b + c = 2$ prove that

$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8$$

→ Given

$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8$$

put $a = 1-x$, $b = 1-y$, $c = 1-z$
such that, $0 < x < 1$, $0 < y < 1$
$0 < z < 1$ & $x + y + z = 1$.

$$\Rightarrow \frac{1-x}{x} \cdot \frac{1-y}{y} \cdot \frac{1-z}{z} \geq 8$$

$$\Rightarrow (1-x)(1-y)(1-z) \geq 8xyz.$$

$$(1 - y - x + xy)(1-z) \geq 8xyz.$$

$$1 - y - x + xy - z + yz + zx - xyz \geq 8xyz.$$

$$\Rightarrow xy + yz + zx \geq 9xyz.$$

⇒ It can be rewritten as,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9.$$

which is AM - HM inequality & it equality holds iff $a = b = c$.

Hence proved.

Que 7. Prove that

$$1 < \frac{1}{1001} + \frac{1}{1002} + \dots + \frac{1}{3001} < \frac{4}{3}$$

→ Consider, for 2001 terms,

$$f(x) = \frac{1}{k}, \quad \text{for } 1001 \le k \le 3001$$

Using AM-HM inequality,

$$\left(\sum_{k=1001}^{3001} k\right)\left(\sum_{k=1001}^{3001} 1/k\right) > (2001)^2$$

But,

$$\sum_{k=1001}^{3001} k = (2001)^2$$

$$\Rightarrow \boxed{\sum_{k=1001}^{3001} \frac{1}{k} > 1.} \quad - (1)$$

Also,

Grouping 500 terms at a time, we have

$$\sum_{k=1001}^{3001} \frac{1}{k} < \frac{500}{1000} + \frac{500}{1500} + \frac{500}{200} + \frac{500}{2500} + \frac{1}{3000}$$

$$< \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{3000}$$

$$\lessapprox \frac{3851}{3000} < \frac{4}{3}$$

$$\Rightarrow \boxed{\sum_{k=1001}^{3001} \frac{1}{k} < \frac{4}{3}} \quad - (2)$$

From eq^n (1) & (2)

$$\boxed{1 < \frac{1}{1001} + \frac{1}{1002} + \dots + \frac{1}{3001} < \frac{4}{3}}$$

Hence proved

Que 8 If $x, y, \& z$ are 3-real numbers such that

$$x+y+z=4 \quad \& \quad x^2+y^2+z^2=6$$

then show that each of x, y, z lie in the closed interval $[(2/3), 2]$. Can x attain the extreme values $2/3$ & 2?

→

from $QM \geq AM$

we have, $\sqrt{\frac{y^2+z^2}{2}} \geq \frac{z+y}{2}$

$$\Rightarrow \boxed{y^2+z^2 \geq \frac{1}{2}(y+z)^2} \quad -(1)$$

Given eq^{ns} can be rewritten as,

$$y+z = 4-x \quad \& \quad y^2+z^2 = 6-x^2 \quad -(2)$$

from eq^n (1) & (2) we get.

$$6-x^2 \geq \frac{1}{2}(4-x)^2$$

$$12-2x^2 \geq 16-8x+x^2$$

$$\Rightarrow (3x-2)(x-2) \leq 0$$

$$\Rightarrow x \geq 2/3 \;\&\; x \leq 2$$

$$\Rightarrow \boxed{2/3 \leq x \leq 2}$$

As given relations are symmetric. we can show the same for y & z.
for y,

$$x^2+z^2 \geq \frac{1}{2}(x+z)^2 \;\&\; x+z=4-y$$
$$\& \; x^2+z^2=6-y^2$$

$$\Rightarrow 6-y^2 \geq \frac{1}{2}(4-y)^2$$

$$\Rightarrow (3y-2)(y-2) \leq 0$$

$$\Rightarrow \boxed{y \geq 2/3 \;\&\; y \leq 2}$$

$$\therefore \boxed{2/3 \leq y \leq 2}$$

Similarly, $\boxed{2/3 \leq z \leq 2}$

And yes x can attain extreme valuees $2/3$ & 2.

Que. 9 For positive real numbers a, b, c, d satisfying $a+b+c+d \leq 1$. Prove the following inequality:

$$\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c} \leq \frac{1}{64abcd}$$

→ Simplifying given inequality,

$$a^2cd + b^2cd + c^2ab + d^2ab \leq 1/64$$

$$\Rightarrow \underbrace{(ac+bd)}_{x}\underbrace{(ad+bc)}_{y} \leq 1/64$$

By using AM-GM inequality.

i.e. $xy \leq \frac{(x+y)^2}{4}$

$$(ac+bd)(ad+bc) \leq \frac{(ac+bd+ad+bc)^2}{4} \quad \text{—①}$$

But $ac+bd+ad+bc = (a+b)(c+d)$

using same inequality i.e. AM-GM again we get.

$$(a+b)(c+d) \leq \frac{(a+b+c+d)^2}{4}$$

$$\leq \frac{1}{4}$$

putting this in eqn ① we get.

$$a^2cd + b^2cd + c^2ab + d^2ab \le \frac{(1/4)^2}{4}$$

$$\therefore\ a^2cd + b^2cd + c^2ab + d^2ab \le \frac{1}{64}$$

$$\Rightarrow \boxed{\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c} \le \frac{1}{64abcd}}$$

Hence proved.

Que 10 Given positive real numbers $a_1, a_2, \ldots, a_n$. Let $b_1, b_2, \ldots, b_n$ be any rearrangement of $a_1, a_2, \ldots, a_n$. Show that.

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n$$

when can equality hold?

→ Given, $a_1, a_2, \ldots, a_n$ are real & positive number & $b_1, b_2, \ldots, b_n$ is there any rearrangement then

$$\frac{a_1 \cdot a_2 \cdots a_n}{b_1 \cdot b_2 \cdots b_n} = 1.$$

We know that AM ≥ GM.

$$\Rightarrow \frac{1}{n} \times \left[\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}\right]$$

$$\geq \left[\frac{a_1}{b_1} \times \frac{a_2}{b_2} \times \cdots \times \frac{a_n}{b_n}\right]^{1/n}$$

$$\therefore \frac{1}{n}\left[\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}\right] \geq (1)^{1/n}$$

$$\Rightarrow \boxed{\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n}$$

Hence proved & equality holds iff $\frac{a_i}{b_i} = 1$ for $i = 1, 2, \ldots, n$.