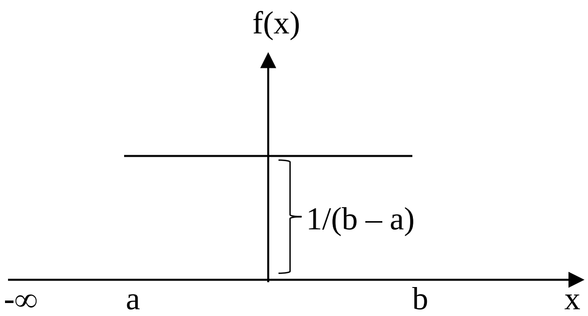
Uniform distribution:

If X is a crv assuming all values in [a,b] ($a < \infty$ and $b < \infty$) and its pdf is given by

$$f(x) = \begin{cases} \frac{1}{b-a}; & a \leq x \leq b \\ 0 \quad ; & \text{Otherwise} \end{cases}$$

then X is said to be Uniformly distributed over the interval [a,b].



Note 1: Uniformly distributed random variables have a pdf constant over the given interval.

Note 2: For any subinterval [c, d] such that $a \leq c < d \leq b$, $P(c \leq X \leq d)$ is the same for all subintervals having the same length.

i.e. $P(c \leq X \leq d) = \int_c^d f(x)\,dx = (d - c)/(b - a)$

So, it depends only on the length of the interval and not on the location of that interval.

Note 3: By choosing a point at random in [a, b], we mean that the x – coordinate of the chosen point, say x, is uniformly distributed over [a, b].

Cumulative Distribution Function (cdf): Let X be a rv. Then the cdf of X denoted by F(x) is given by $F(x) = P(X \leq x)$.

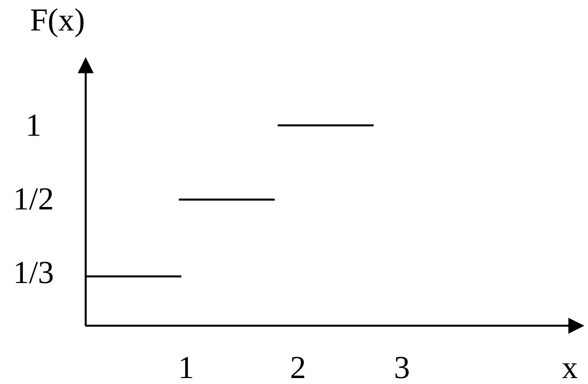
i.e $F(x) = \begin{cases} \sum_{(j;\, x_j \leq x)} p(x_j) & \text{if X is a drv} \\ \int_{-\infty}^{x} f(s)ds & \text{if X is a crv} \end{cases}$

Ex 1: Let X be a rv taking 3 values 0,1,2 with probabilities 1/3, 1/6, and 1/2. Then, we have,

$F(x) = P(X \leq x)$

$= \sum_{(j;\, x_j \leq x)} p(x_j)$ if X is a drv

$$F(x) = \begin{cases} 0\,; & x < 0 \\ \frac{1}{3}\,; & 0 \le x < 1 \\ \frac{1}{2}\,; & 1 \le x < 2 \\ 1\,; & x \ge 2 \end{cases}$$



Note 1: If X is a drv taking values $x_1, x_2, \ldots, x_n$, then we have

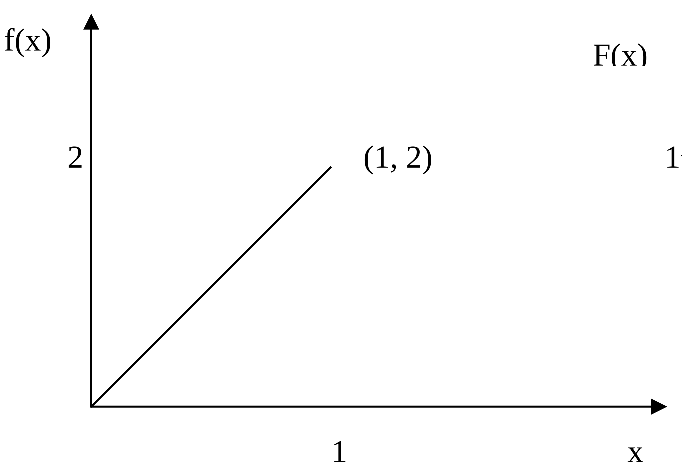
$$F(x) = \begin{cases} 0 & ; \quad x < x_1 \\ p(x_1) & ; \quad x_1 \le x < x_2 \\ p(x_1) + p(x_2) & ; \quad x_2 \le x < x_3 \\ \vdots & \\ p(x_1) + p(x_2) + \cdots + p(x_{n-1}) & ; \quad x_{n-1} \le x < x_n \\ p(x_1) + p(x_2) + \cdots + p(x_n) = 1\,; & \quad x \ge x_n \end{cases}$$

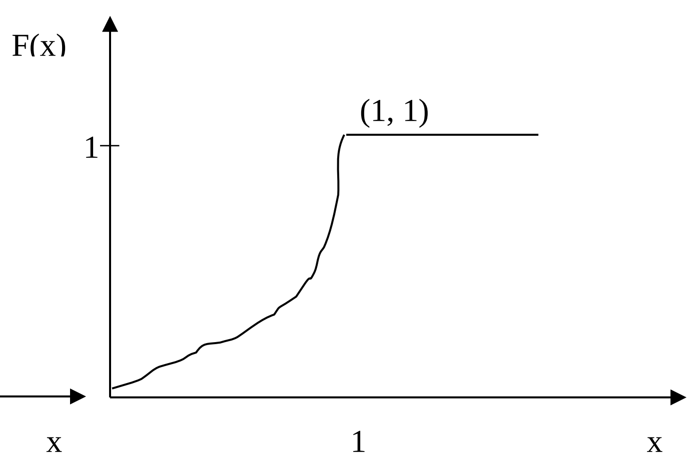
Note 2: For a drv X, F(x) makes jumps at specific values of x i.e. F(x) increases in steps and is called a step function.

Ex 2: Let X be a crv having pdf $f(x) = \begin{cases} 2x; & 0 < x < 1 \\ 0; & \text{Otherwise} \end{cases}$

Then, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(s)ds$

$$F(x) = \begin{cases} 0\,; & x \le 0 \\ \int_0^x 2s\,ds = x^2\,; & 0 < x < 1 \\ 1; & x \ge 1 \end{cases}$$





Remarks:

1. $0 \le F(x) \le 1, -\infty < x < \infty$.

2. F is a non-decreasing function. i.e., if $x_1 < x_2$, then $F(x_1) \le F(x_2)$.

3. $F(-\infty) = \lim_{x\to-\infty} F(x) = 0$; $F(+\infty) = \lim_{x\to\infty} F(x) = 1$.

4. If X is a drv with pmf $p(x_j)$ and cdf $F(x)$, then

$$p(x_j) = P(X = x_j) = F(x_j) - F(x_j - 1).$$

If X is a crv with pdf f(x) and cdf F(x) then,

$$f(x) = \frac{d}{dx}[F(x)] \ \forall \ x \text{ at which F is differentiable,}$$
$$= F'(x)$$

5. Let X be a crv with pdf f(x) and cdf F(x). Then,

$$P(a \le X \le b) = F(b) - F(a)$$

Proof:

$$P(a \le X \le b) = \int_a^b f(x)dx = \int_{-\infty}^b f(x)dx - \int_{-\infty}^a f(x)dx$$
$$= P(X \le b) - P(X \le a)$$
$$= F(b) - F(a).$$

Since, for a crv X, we have $P(X=x_0)=0$, we get

$P(a \le X \le b) = P(a < X < b) = P(a < X \le b) = P(a \le X < b) = F(b) - F(a)$.

However, if X is a drv with pmf $p(x_i)$ and cdf F(x), we have

$P(a < X \le b) = F(b) - F(a)$

$P(a \le X \le b) = P(X = a) + F(b) - F(a)$

$P(a < X < b) = F(b) - F(a) - P(X = b)$

$P(a \le X < b) = F(b) - F(a) + P(X = a) - P(X = b)$.

Prove it!