Waiting time distribution of a customer in the system:

Let $W(t)$ denote the waiting time of a customer in the queuing system for the customer joining the system at time t. In steady state, $\lim_{t\to\infty} W(t) = W$.

W is the waiting time for the customer joining the system at any time.

Note: Although derivation of π_n is completely independent of service discipline and so also the expected waiting times. The probability distribution of waiting times depends on service discipline adopted. This means that while the expected value of waiting times remain the same their variance differ.

Let the service discipline be FCFS $(M/M/1 \text{ Model})$.

Let there be n customers in the system of which one be undergoing service. Let 'W' be the waiting of $(n+1)^{th}$ customer joining the system. Let $t_2, t_3,, t_4 \dots . t_n$ be the service times of $(n-1)$ customers in the queue and let t_1' be the remaining service time of the customer being served and t_{n+1} be the service time of arriving customer.

Clearly, W = $t_1' + t_2 + t_3 + \dots + t_n + t_{n+1}$

Let $f(w/n)$ be the conditional pdf of W given that there are n customers in the system ahead of the arriving customer. Since the variables t_i; i =2,3,4….,n+1 each has exponential distribution with parameter μ and further due to forgetfulness property of exponential distribution t_1' also has exponential distribution with parameter μ . Thus, $f(w/n)$ is a gamma distribution with parameters μ & n+1, with pdf

$$f(w/n) = \frac{\mu(\mu w)^n}{n!} \; e^{-\mu w} \; ; \; w \geq 0$$

Hence the density function of W is,

$$f(w) = \sum_{n=0}^{\infty} f(w/n)\, \pi_n = \sum_{n=0}^{\infty} \frac{\mu(\mu w)^n}{n!} \; e^{-\mu w} \; \rho^n (1-\rho)$$

$$= \mu(1-\rho)\, e^{-\mu w} \sum_{n=0}^{\infty} \frac{(\mu w \rho)^n}{n!}$$

$$= \mu(1-\rho)\, e^{-\mu w} \; e^{\mu w \rho}$$

$$f(w) = \mu(1-\rho) e^{-\mu(1-\rho)w} \; ; \quad w \geq 0 \; , \; \rho < 1$$

Which is an exponential distribution with parameter $\mu(1-\rho)$.

∴ Expected waiting time of the customer in the system: $W_s = \frac{1}{\mu(1-\rho)}$

Similarly, we can obtain the **Waiting time distribution of an arriving customer in the queue.** (i. e. before he receives service)

Let $W^*(t)$ denote the waiting time of a customer in the queue for the customer joining the system at time t. In steady state, $\lim_{t\to\infty} W^*(t) = W^*$.

W^* is the waiting time for the customer joining the Queue at any time.

The Probability distribution of W^* has 2 components:

1) Customer starts receiving the service immediately upon his arrival, if there is no customer in the system.

   Thus $P(W^* = 0) = \pi_0 = 1 - \rho = 1 - \frac{\lambda}{\mu}$; $w^* = 0$

2) If there are n customers in the system $(n \geq 1)$, when a new customer arrives, following similar line of arguments, the waiting time before the service starts is the sum of n independently exponentially distributed r.vs.

   Clearly, $W^* = t_1' + t_2 + t_3 + \ldots + t_n$

   Thus, $f(w^*/n)$ is a gamma distribution with parameters μ & n, with pdf

$$f(w^*/n) = \frac{\mu(\mu w^*)^{n-1}}{(n-1)!} e^{-\mu w^*} \; ; \; w^* > 0$$

Hence the density function of W^* is,

$$f(w^*) = \sum_{n=1}^{\infty} f(w^*/n)\, \pi_n = \sum_{n=1}^{\infty} \frac{\mu(\mu w^*)^{n-1}}{(n-1)!} e^{-\mu w^*} \pi_n \quad ; \; w^* > 0$$

$$= \sum_{n=1}^{\infty} \frac{\mu(\mu w^*)^{n-1}}{(n-1)!} e^{-\mu w^*} \rho^n (1-\rho)$$

$$= \mu\rho(1-\rho) e^{-\mu w^*} \sum_{n=1}^{\infty} \frac{(\mu w^* \rho)^{n-1}}{(n-1)!}$$

$$= \mu\rho(1-\rho) e^{-\mu w^*} e^{\mu\rho w^*}$$

$$f(w^*) = \mu\rho(1-\rho)\; e^{-\mu(1-\rho)w^*}$$

$$\therefore \quad f(w^*) = \begin{cases} \mu\rho(1-\rho)\, e^{-\mu(1-\rho)w^*} & ; \; w^* > 0 \\ (1-\rho) & ; \; w^* = 0 \end{cases}$$

By elementary integration methods, we obtain,

Expected waiting time of a customer in the queue: $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu(1-\rho)}$

Problems

1) For a M/M/1 queuing system with an arrival rate 5/hr and mean service time 10 minutes. Find
    (i). Probability that customer directly gets the service.
    (ii). Probability that a customer has to wait.
    (iii). Expected No. of customers in the system, L_s.
    (iv). Expected No. of customers in the queue, L_q.
    (v). Expected waiting time of the customer in the system, W_s
    (vi). Expected waiting time of the customer in the queue, W_q
    (vii). Probability that 5 or more customers in the system at any time.

Solution:

$\lambda = 5$ arrivals per hour, $\mu = 1$ departure per 10 minutes = 6 departures per hour

$\rho = \frac{\lambda}{\mu} = \frac{5}{6} < 1$

$\therefore \pi_n = \rho^n (1 - \rho)$

(i). $\pi_0 = 1 - \rho = \frac{1}{6}$

(ii). $\rho = \frac{5}{6}$

(iii). $L_s = \frac{\rho}{1-\rho} = 5$

(iv). $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{25}{6}$

(v). $W_s = \frac{L_s}{\lambda} = 1$ hr

(vi). $W_q = \frac{L_q}{\lambda} = \frac{5}{6}$ hr $=$ 50 minutes

(vii). $P(N \geq 5) = 1 - P(N \leq 4) = 1 - (\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4)$

$$= 1 - \pi_0 (1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \left(\frac{\lambda}{\mu}\right)^4)$$

$$= 1 - \frac{1}{6}\left(1 + \frac{5}{6} + \frac{25}{36} + \frac{125}{216} + \frac{625}{1296}\right)$$

$$= \left(\frac{5}{6}\right)^5$$

2) Consider a M/M/1 situation where the mean arrival rate (λ) is 1 customer every 4 minutes and the mean service time $\left(\frac{1}{\mu}\right)$ is 2.5 minutes. Calculate L_q, L_s, W_s & W_q.

Solution:

$\lambda = \frac{1}{4}$ = 0.25 arrivals per minute = 15 arrivals per hour,

$\mu = \frac{1}{2.5} =$ 0.4 departures per minute = 24 departures per hour

$\rho = \frac{\lambda}{\mu} = \frac{15}{24} < 1$

$\therefore\ \pi_n = \rho^n(1 - \rho\,)$

(i). $L_s = \frac{\rho}{1-\rho} = \frac{5}{3} \simeq 2$ customers

(ii). $L_q = \frac{\rho^2}{1-\rho} = \frac{25}{24} \simeq 1$ customer

(iii). $W_s = \frac{L_s}{\lambda}$ = 6.67 minutes

(iv). $W_q = \frac{L_q}{\lambda}$ = 4.16 minutes

3) Arrival at a telephone booth is considered to be Poisson with an average time of 10 minutes between 2 arrivals. The length of a phone call is assumed to be distributed exponentially with mean 3 minutes.

(i). What is the Probability that person will have to wait?

(ii). The telephone dept. will install a 2nd booth when convinced that an arrival would expect to wait at least 3 minutes for the phone. By how much must the flow of arrivals be increased in order to justify a second booth?

(iii). Find the average No. of customers in the system?

(iv). Estimate the fraction of the day that the phone will be in use.

(v). What is the probability that it will take a person more than 10 minutes to wait for the phone and complete his call?

Solution:

$\lambda = 1$ arrival per 10 minutes = 6 arrivals per hour,

$\mu = \frac{1}{3}$ departures per minute = 20 departures per hour

$\rho = \frac{\lambda}{\mu} = \frac{6}{20} < 1$

$\therefore \pi_n = \rho^n(1-\rho)$

(i). $\mathrm{P}(\text{an arrival has to wait}) = 1 - \pi_0 = \rho = \frac{6}{20}$

(ii). The expected waiting time in the queue for an arrival before he gets the service is

$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$.

Here $\mu = 0.33$. To find the new value of λ, say $\lambda^{|}$, for which $W_q = 3$.

$3 = \frac{\lambda^{|}}{0.33(0.33-\lambda^{|})} \Rightarrow \lambda^{|} = 0.16$ arrivals per minute $\simeq$ 10 arrivals per hour

So we must increase the flow of arrivals from 6 per hr to 10 per hr to justify the installation of a 2nd booth

(iii). $L_s = \frac{\rho}{1-\rho} = \frac{0.3}{0.7} = \frac{3}{7}$

(iv). $\rho = 0.3$

(v). $P(W \geq 10) = \int_{10}^{\infty} f(w)\,dw = \int_{10}^{\infty} \mu(1-\rho)\,e^{-\mu(1-\rho)\,w}\,dw = 0.1$

4) Customers arrive at a ticket window according to a Poisson process with rate 30 per hour. A single person is appointed to serve the customers. The service time is exponential with mean 90 seconds. Find the average waiting time of the customer in the system and in the queue.

Solution:

$\lambda = 30$ arrivals per hour

$\mu = \frac{1}{90}$ departures per second = 40 departures per hour

$\rho = \frac{\lambda}{\mu} = \frac{3}{4} < 1$

$\therefore \pi_n = \rho^n(1 - \rho)$

$W_s = \frac{\rho}{\lambda(1-\rho)} = 60 \text{ min} = 360 \text{ seconds}$

$W_q = \frac{\rho^2}{\lambda(1-\rho)} = 270 \text{ seconds}$

5) Arrival at a counter in a bank occur is in accordance with a Poisson process at an Average rate of 8 per hour .The duration of service of a customer has an exponential distribution with a mean of 6minutes. Find the probability that an arriving customer

(i). has to wait on arrival.

(ii). finds 4 customers in the queue .

(iii). has to spend less than 15 minutes in the bank?

(iv). Also estimate fraction of the total time the counter busy.

Solution:

$\lambda = 8$ arrivals per hour

$\mu = \frac{1}{6}$ departures per minute = 10 departures per hour

$\rho = \frac{\lambda}{\mu} = \frac{4}{5} < 1$

$\therefore \pi_n = \rho^n(1 - \rho)$

(i). P(arriving customer has to wait) = $1 - \pi_0 = \rho$ = 0.8

(ii). P(arriving customer finds 4 customers in the queue)

$= P(N = 4) = \pi_4 = \rho^4 \pi_0 = (0.8)^4(1 - 0.8) = 0.8192$

(iii). $P(W < 15) = 1 - P(W \geq 15) = 1 - \int_{15}^{\infty} (\mu - \lambda).\ e^{-(\mu-\lambda)\,w}\, dw = 0.3935$

(iv). $\rho = 0.8$

6) A TV repairman finds that the time spent on the TVs has an exponential distribution with mean 20mins. If he repairs the sets in the order in which they arrive and if the arrival of the sets is approximately Poisson with an average rate of 10 per 8 hour a day. What is the repairman's expected idle time each day? How many jobs are ahead on an average of the set just brought in?

Solve it!

7) In a post office there is only one window and a stationary employee performs all the services required. The window remains open continuously from 7AM to 1PM. It has been observed that average number of clients arriving is 54 and average service time is 5 minutes per person. Assuming Poisson arrival and Poisson departure determine L_S, L_q, W_S, W_q.

Solve it!

8) Customers arrive at random at a checkout facility at an average rate of 12/hr. The service time has an exponential distribution with parameter μ. If the queing time of atleast 90% of the customers should be less than 4 minutes, show that μ must exceed μ_0, where μ_0 satisfies $\mu_0 e^{4\mu_0} = 2e^{4/5}$.

Solve it!

9) In a railway marshalling yard goods train arrive at a rate 30 trains per day. Assuming that inter-arrival time is exponentially distributed and the service time distribution is also exponential with an average of 36 minutes. Calculate
   (a). Mean queue size.
   (b). Probability that queue size exceeds 10.

If the input of trains exceeds an average of 33 per day, what is the change in (a) and (b) above.

Solve it!

10) At what average rate must a clerk at a supermarket work in order to ensure a probability of 0.9 that customers will not wait longer than 12 minutes. It is assumed that there is one counter at which customers arrive at average rate of 15/hour. Assume that the length of service by the clerk has an exponential distribution.

Solve it!