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Department :- Computational and Data science.

Subject :- Probability, statistics & stochastics processes.

Assignment No. :- 2

Q. 1 If X and Y are independent Poisson variates with parameters λ & μ respectively, then show that the conditional probability distribution of $X=r$ given $X+Y=n$ is Binomial with parameters $(n, \lambda/\lambda+\mu)$.

→ Given,

X & Y are independent poisson variates. with parameters λ & μ resp.

then, let ~~λ_1 & λ_2 be mean~~ ~~same~~

$$P(X=r/X+Y=n) = \frac{P(X=r \cap X+Y=n)}{P(X+Y=n)}$$

$$\therefore P(X=r/X+Y=n) = \frac{P(X=r \cap Y=n-r)}{P(X+Y=n)}$$

$$= \frac{\frac{e^{-\lambda}\lambda^r}{r!} \times \frac{e^{-\mu}(\mu)^{n-r}}{(n-r)!}}{\frac{e^{-(\lambda+\mu)}(\lambda+\mu)^n}{n!}}$$

$$= \frac{n!}{(n-r)!\,r!} \times \frac{\lambda^r}{(\lambda+\mu)^r} \times \frac{\mu^{n-r}}{(\lambda+\mu)^{n-r}}$$

$$= \binom{n}{r}\left(\frac{\lambda}{\lambda+\mu}\right)^r\left(\frac{\mu}{\lambda+\mu}\right)^{n-r}$$

⇒ Binomial distribution with parameters $(n, \lambda/\lambda+\mu)$

$$\because \boxed{\frac{\mu}{\lambda+\mu} = 1 - \frac{\lambda}{\lambda+\mu}}$$

Q.2 (a) A factory has 2-machines (identical), but on any given day not more than one is in use. This machine has a constant probability 'p' of failure & if it ~~is~~ fails the breakdown occurs at the end of day's work. A single repairman is employed. It takes him two days to repair a machine & he works on only one machine at a time. Write down TPM of markov chain $\{x_n\}$ which describes the working of the factory, where x_n is number of days that would be needed to get both the machines back in working order & x_n is recorded at end of day n.

→ Soln :-

Here, parameter space is, T,

$$T = \{1, 2, 3, \cdots\}$$

∴ The required stochastic process becomes.

$$\{x_n : n \in T\}$$

where, states space is $S = \{0, 1, 2, 3\}$ such that,

$x_n : 0 \Rightarrow$ Both machine in working order.

$x_n : 1 \Rightarrow$ One machine in working order & other has ~~just~~ ~~failed~~ one day's repair

$x_n : 2 \Rightarrow$ one m/c in working order & other has just broken down

$x_n : 3 \Rightarrow$ One m/c has just broken down & other has a day's repair carried out on it.

These are the only possible cases.
Also, it is given that probability of failure is (p).

Therefore, the required TPM is.

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1-p & 0 & p & 0 \\ 1 & 1-p & 0 & p & 0 \\ 2 & 0 & 1-p & 0 & p \\ 3 & 0 & 0 & 1 & 0 \end{array}$$

Q.2 (b) Three tanks A, B, C fight a duel. Tank A hits its target with probability (2/5), Tank B with (2/3) & Tank C with (3/5) probability. shots are fired simultaneously & once a Tank is hit, it is out of action. When all the tanks are in action each one picks up its target at random. Let the set of Tanks still in action be states of system. obtain TPM of Markov chain.

→ Given,

P(A hitting a target) $= 2/5$

P(B hitting a target $= 2/3$

P(C hitting a target) $= 3/5$

So for given Markov chain possible cases are.

ABC → At the beginning all tanks will be there. & ~~can~~ hit each other.

AB → A & B ~~hit~~ each other. are there.

BC → B & C are there.

~~A~~CA → A & C are there.

A, B, C → ~~at~~ A, B or C are there individually.

& N → None ~~ones~~ there.

∴ The state space becomes.

$S = \{N, A, B, C, AB, BC, CA, ABC\}$

Now, we have, if A is left in Areana then

$P_{AA} = 1$

Illy, $P_{BB} = 1$ & $P_{CC} = 1$.

Now, considering, A & B scenario, 3 conditions are possible.

$P_{AB \to AB}$ ⇒ Both A & B donot hit each other

$P_{AB \to N}$ ⇒ ~~None of them~~ Both hit each other.

$P_{AB \to A}$ ⇒ only A hits & B misses.

$\Rightarrow P_{AB \to AB} = \bar{A}\bar{B} = (1-2/5)(1-2/3) = 3/15$

$P_{AB \to N} = AB = (2/5)(2/3) = 4/15$

$P_{AB \to A} = A\bar{B} = (2/5)(1-2/3) = 2/15$

$P_{AB \to B} = \bar{A}B = (3/5)(2/3) = 6/15$

Illy for BC, $P_{BC \to N} = B \times C = (2/3)(3/5) = 6/15$

$P_{BC \to BC} = \bar{B} \times \bar{C} = (1/3)(2/5) = 2/15$

$P_{BC \to B} = B \times \bar{C} = (2/3)(2/5) = 4/15$

$P_{BC \to C} = \bar{B} \times C = (1/3)(3/5) = 3/15$

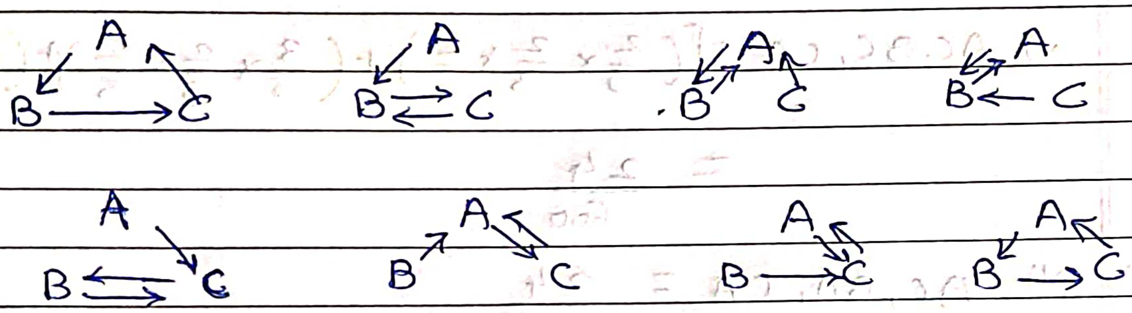
And for CA, $P_{CA \to N} = C \times A = (3/5)(2/5) = 6/25$

$P_{CA \to CA} = \bar{C} \times \bar{A} = (2/5)(3/5) = 6/25$

$P_{CA \to C} = C \times \bar{A} = (3/5)(3/5) = 9/25$

$P_{CA \to A} = \bar{C} \times A = (2/5)(2/5) = 4/25$

Now for ABC, eight cases are possible (i.e. 2^3) which are.



So for $P_{ABC \to N} = \frac{2}{8} \times \frac{2}{5} \times \frac{2}{3} \times \frac{3}{5} = \frac{24}{600}$

$P_{ABC \to ABC} = \bar{A}\bar{B}\bar{C} = \frac{3}{5} \times \frac{1}{3} \times \frac{2}{5} = \frac{48}{600}$

Now, for $P_{ABC \to A} \Rightarrow$

$P_{ABC \to A} = [AB\bar{C}] + [ABC + \bar{A}BC + AB\bar{C}] + [ABC + A\bar{B}C + \bar{A}BC] + [A\bar{B}C]$

$= \left[\frac{2}{5} \times \frac{2}{3} \times \frac{2}{5} \times \frac{1}{8}\right] + \left[\frac{2}{5} \times \frac{2}{3} \times \frac{3}{5} + \frac{3}{5} \times \frac{2}{3} \times \frac{3}{5} + \frac{2}{5} \times \frac{2}{3} \times \frac{2}{5}\right] \times \frac{1}{8}$

$+ \left[\frac{2}{5} \times \frac{2}{3} \times \frac{3}{5} + \frac{2}{5} \times \frac{1}{3} \times \frac{3}{5} + \frac{3}{5} \times \frac{2}{3} \times \frac{3}{5}\right] \times \frac{1}{8}$

$+ \left[\frac{2}{3} \times \frac{1}{3} \times \frac{3}{5} \times \frac{1}{8}\right] = \frac{88}{600}$

Similarly, $P_{ABC \to B} = P_{ABC \to C} = \frac{88}{600}$

Now, For $P_{ABC \to AB}$, we have.

i) $AB, BC, C\bar{A} = \frac{3}{5} \times \frac{2}{3} \times \frac{2}{5} \times \frac{1}{8} = \frac{12}{600}$

ii) $AB, BC, C\bar{B} = \frac{3}{5} \times \frac{2}{3} \times \frac{2}{5} \times \frac{1}{8} = \frac{12}{600}$

iii) $AC, \bar{B}A, \bar{C}A = \frac{2}{5} \times \frac{1}{3} \times \frac{2}{5} \times \frac{1}{8} = \frac{4}{600}$

iv) $AC, \bar{B}A, \bar{C}B = \frac{2}{5} \times \frac{1}{3} \times \frac{2}{5} \times \frac{1}{8} = \frac{4}{600}$

v) $AC, BC, CA = \left[\left(\frac{2}{5} \times \frac{2}{3} \times \frac{2}{5}\right) + \left(\frac{3}{5} \times \frac{2}{3} \times \frac{2}{5}\right) + \left(\frac{2}{5} \times \frac{1}{3} \times \frac{2}{5}\right)\right] \times \frac{1}{8}$

$= \frac{24}{600}$

vi $AC, BA, CA = \frac{24}{600}$

$\Rightarrow P_{ABC \to AB} = \frac{80}{600}$

IIIly, $P_{ABC \to BC} = \frac{120}{600}$

& $P_{ABC \to CA} = 64/600$

∴ The required TPM is.

	N	A	B	C	AB	BC	CA	ABC
N	1	0	0	0	0	0	0	0
A	0	1	0	0	0	0	0	0
B	0	0	1	0	0	0	0	0
C	0	0	0	1	0	0	0	0
AB	4/15	2/15	6/15	0	3/15	0	0	0
BC	6/15	0	4/15	3/15	0	2/15	0	0
CA	6/25	4/25	0	9/25	0	0	6/25	0
ABC	24/600	88/600	88/600	88/600	80/600	120/600	64/600	48/600

Q. 3 Is the following TPM of a markov chain regular? if so, find limiting distribution.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & 0 & 1-a \end{bmatrix} \end{array} \quad ; 0<a<1.$$

$\rightarrow$ Given

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & 0 & 1-a \end{bmatrix}$$

Here, $m^2 - 2m + 2 = 9 - 6 + 2 = 5$, raising P till 5 power.

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & 0 & 1-a \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & 0 & 1-a \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ a & 0 & 1-a \\ a(1-a) & a & (1-a)^2 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0 & 0 & 1 \\ a & 0 & 1-a \\ a(1-a) & a & (1-a)^2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ a & 0 & 1-a \\ a(1-a) & a & (1-a)^2 \end{bmatrix} = \begin{bmatrix} a(1-a) & a & (1-a)^2 \\ a(1-a)^2 & a(1-a) & a+(1-a)^3 \\ a^2+a(1-a)^3 & a(1-a)^2 & a(1-a)+a(1-a)+(1-a)^4 \end{bmatrix}$$

$$\therefore P^5 = P^4 \cdot P = \begin{bmatrix} a(1-a) & a & (1-a)^2 \\ a(1-a)^2 & a(1-a) & a+(1-a)^3 \\ a^2+a(1-a)^3 & a(1-a)^2 & 2a(1-a)+(1-a)^4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & 0 & 1-a \end{bmatrix}$$

$$P^5 = \begin{bmatrix} a(1-a)^2 & a(1-a) & a+(1-a)^3 \\ \cancel{a(1-a)+(1-a)^3} \; a^2+a(1-a)^3 & a(1-a)^2 & 2a(1-a)+(1-a)^4 \\ 2a^2(1-a)+a(1-a)^4 & a^2+a(1-a)^3 & a(1-a)^2+2a(1-a)^2+(1-a)^4 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} a(1-a)^2 & \cancel{a(1-a)}\ \ a(1-a) & a+(1-a)^3 \\ a^2+a(1-a)^3 & a(1-a)^2 & 2a(1-a)+(1-a)^4 \\ 2a^2(1-a)+a(1-a)^4 & a^2+a(1-a)^3 & 3a(1-a)^2+(1-a)^4 \end{bmatrix}$$

As $\forall P_{ij} > 0$ [$\because a \in (0,1)$] Hence TPM or Markov chain is regular. &

To find steady state probabilities, consider steady state eq^{ns}. such that.

$$\pi_j = \sum_{i=0}^{m=2} \pi_i P_{ij} \qquad j = 0, 1, 2$$

$$\therefore \quad \pi_0 = \pi_0 P_{00} + \pi_1 P_{10} + \pi_2 P_{20}$$
$$\pi_1 = \pi_0 P_{01} + \pi_1 P_{11} + \pi_2 P_{21}$$
$$\pi_2 = \pi_0 P_{02} + \pi_1 P_{12} + \pi_2 P_{22}$$

then using P we have,

$$\pi_0 = \pi_0 \times 0 + \pi_1 \times 0 + \pi_2 \times a$$

$$\Rightarrow \boxed{\pi_0 = a\pi_2} \quad -(1)$$

$$\pi_1 = \pi_0 \times 1 + \pi_1 \times 0 + \pi_2 \times 0$$

$$\Rightarrow \boxed{\pi_1 = \pi_0} \quad -(2)$$

$$\& \quad \pi_2 = \pi_0 \times 0 + \pi_1 \times 1 + \pi_2 \times (1-a)$$

$$\Rightarrow \boxed{\pi_2 = \pi_1 / a} \quad -(3)$$

$$\therefore \boxed{\pi_2 = \frac{\pi_1}{a}} \quad \& \quad \boxed{\pi_1 = \pi_0} \quad \text{- from eq}^n \text{ (1), (2) \& (3)}$$

Also, we have steady state eqn as,

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\Rightarrow \quad 2\pi_0 + \frac{\pi_0}{a} = 1$$

$$\Rightarrow \quad \boxed{\pi_0 = \frac{a}{2a+1}}$$

$$\Rightarrow \quad \boxed{\pi_0 = \pi_1 = \frac{a}{2a+1}} \quad \& \quad \boxed{\pi_2 = \frac{1}{2a+1}}$$

This is the required limiting distribution.

Q.4 Let $\{N(t)\}$ be a Poisson process. For $s<t$ find $P[N(s)=k / N(t)=n]$.

→ Sol^n:-

~~Given,~~
let,

$N(s) \rightarrow$ Number of events occuring in interval $[0,s]$

$N(s,t) \rightarrow$ Number of events occuring in interval $[s,t]$

$N(t) \rightarrow$ Number of events occuring in interval $[0,t]$

Using the postulates of poisson process, we can write,

$$P[N(s)=k/N(t)=n] = \frac{P[N(s)=k \cap N(t)=n]}{P[N(t)=n]}$$

$$= \frac{P[N(s)=k] \cdot P[N(s,t]=n-k]}{P[N(t)=n]}$$

$$= \frac{\dfrac{e^{-(s\lambda)}(s\lambda)^k}{k!} \times \dfrac{e^{-(t-s)\lambda}((t-s)\lambda)^{n-k}}{(n-k)!}}{\dfrac{e^{-t\lambda}(\lambda t)^n}{n!}}$$

$$\therefore \boxed{P[N(s)=k/N(t)=n] = \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}}$$

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Q.5 Customers arrive at random at a checkout facility at an average rate of 12/hr. The service time has an exponential distribution with parameter μ. If queing time of atleast 90% of customers should be less than 4 minutes. show that μ must exceed μ_0, where μ_0 satisfies ~~$\mu_0 e^{4\mu_0}$~~ $\mu_0 e^{4\mu_0} = 2e^{4/5}$.

→

~~To find~~, To show that, $\mu_0 e^{4\mu_0} = 2e^{4/5}$

Given, $P(W^* < 4) = 0.9$

we have,

$$P(W^* < 4) = (1-\rho)_{W^*=0} + \int_0^4 \mu_0 \rho (1-\rho) e^{-\mu_0(1-\rho)W^*} dW^*$$

$$= 1-\rho + \rho \left[\frac{e^{-\mu_0(1-\rho)W^*}}{-1} \right]_0^4$$

$$= 1-\rho + \rho \left[-e^{-\mu_0(1-\rho)4} + 1 \right]$$

$$0.9 = 1 - \cancel{\rho} - \rho e^{-(\mu_0-\lambda)4} + \cancel{\rho}$$

$$0.9 = 1 - \rho e^{-(\mu_0-\lambda)4}$$

$$\therefore \quad \rho . e^{-(\mu_0-\lambda)4} = 0.1$$

$$\lambda e^{-\mu_0 \times 4} . e^{\lambda \times 4} = 0.1 \times \mu_0$$

$$\frac{12}{60} \times e^{4/5} = 0.1 \times \mu_0 \times e^{\mu_0 4}$$

$$\therefore \quad \boxed{\mu_0 e^{\mu_0 4} = 2e^{4/5}} \Rightarrow \mu_0 e^{4\mu_0} = 2e^{4/5}$$