**M | M | s: FIFO | ∞ | ∞ (Multichannel queueing system)**

In this model it is assumed that the arrivals and departures occur according to Poisson distribution with rates $\lambda$ and $\mu$ respectively. Equivalently the Inter arrival & service times are exponentially distributed with parameters $\lambda$ and $\mu$ respectively. Further it is assumed that there are `s' parallel servers in the system working independently of each other. Also it is assumed that all the servers offer service at the same rate.

Thus, if the no. of customers in the system be n, then

for $n \geq s$ : the combined service rate is $s\mu$

for $n < s$ : the combined service rate is $n\mu$ - since no more than n servers will be busy.

To derive the differential-difference equations we use the following assumptions:

1) For an infinitesimal interval of length 'h'

   P[ one arrival in h ] = $\lambda h + O(h)$

   P[ no arrival in h ] = $1 - \lambda h + O(h)$

   P[ one departure in h ] = $n\mu h + O(h) \quad 0 < n < s$

   $= s\mu h + O(h) \quad n \geq s$

   P[ no departure in h ] = $1 - n\mu h + O(h) \quad 0 < n < s$

   $= 1 - s\mu h + O(h) \quad n \geq s$

   Where $O(h) \to 0$ as $h \to 0$ and $\lim\limits_{h\to 0} \frac{O(h)}{h} = 0$

2) Further, it is assumed that atmost only one event (either an arrival or a departure) can occur in h.

Following the similar line of arguments as in **M|M|1** system we have,

$$p_0(t + h) = p_0(t)\left(1-\lambda h + O(h)\right).1 + p_1(t)(1-\lambda h + O(h))(\mu h + O(h)) + O(h) \text{ ; } n=0$$

$$p_n(t+h) = p_n(t)\left((1-(\lambda+n\mu)h + O(h)\right) + p_{n-1}(t)\left(\lambda h + O(h)\right)$$
$$+p_{n+1}(t)((n+1)\mu h + O(h)) + O(h) \text{ ; } n< s \text{ [ i.e. } n=1,2,3,\ldots..(s-1) \text{ ]}$$

$$p_n(t+h) = p_n(t)\left((1-(\lambda+s\mu)h + O(h)\right) + p_{n-1}(t)\left(\lambda h + O(h)\right)$$
$$+p_{n+1}(t)((s\mu h + O(h)) + O(h) \text{ ; } n\geq s \text{ [ i.e. } n = s,s+1,s+2\ldots..\text{]}$$

Proceeding further as earlier, we obtain the following differential-difference equations:

$$p_0'(t) = -\lambda\, p_0(t) + \mu\, p_1(t) \quad \text{for } n=0 \qquad \text{---------------(1)}$$

$$p_n'(t) = -(\lambda+n\mu)\, p_n(t) + \lambda\, p_{n-1}(t) + (n+1)\,\mu\, p_{n+1}(t) \quad \text{for } n<s \quad \text{-------------(2)}$$

$$p_n'(t) = -(\lambda+s\mu)\, p_n(t) + \lambda\, p_{n-1}(t) + s\mu\, p_{n+1}(t) \quad \text{for } n\geq s \quad \text{----------------(3)}$$

Solving (1), (2) and (3) by Laplace transformation technique a solution to $p_n(t)$, the transient state probabilities can be obtained. But the procedure is quite complex.

Due to the complexity involved in solving the above equations and that our interest lies in obtaining an expression for the steady state probabilities, we consider only the steady state analysis.

we obtain the steady state equations by noting that
as $t\to\infty$ , $p_n'(t) \to 0$ and $p_n(t) \to \pi_n$ for all n.

Thus we get

$$-\lambda\,\pi_0 + \mu\,\pi_1 = 0 \quad \text{for } n=0 \qquad \text{---------------- (4)}$$

$$-(\lambda+n\mu)\,\pi_n + \lambda\,\pi_{n-1} + (n+1)\,\mu\,\pi_{n+1} = 0 \quad \text{for } n< s \quad \text{------------- (5)}$$

$$-(\lambda+s\mu)\,\pi_n + \lambda\,\pi_{n-1} + s\mu\,\pi_{n+1} = 0 \quad \text{for } n\geq s \quad \text{--------------- (6)}$$

A solution $\pi_n$ is obtained by substituting n=1,2.....(s-1) and calling $\rho = \frac{\lambda}{s\mu}$
(the utilization factor for the service facility or the expected fraction of time the individual server is busy)

**To obtain the solution $\pi_n$ :**

Here $\pi_0 = \pi_0$ (initially)

$$\pi_1 = \frac{\lambda}{\mu}\pi_0$$

$$\pi_2 = \frac{\lambda}{2\mu}\pi_1 = \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 \pi_0$$

$$\pi_3 = \frac{\lambda}{3\mu}\pi_2 = \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 \pi_0$$

$$\vdots$$

$$\pi_n = \frac{\lambda}{n\mu}\pi_{n-1} = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \pi_0 \quad \text{for} \quad n < s$$

i.e. $\pi_n = \frac{1}{n!}(s\rho)^n \pi_0$ for $n < s$

Similarly

$$\pi_s = \frac{\lambda}{s\mu}\pi_{s-1} = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \pi_0$$

$$\pi_{s+1} = \frac{\lambda}{s\mu}\pi_s = \frac{1}{s\, s!}\left(\frac{\lambda}{\mu}\right)^{s+1} \pi_0$$

$$\pi_{s+2} = \frac{\lambda}{s\mu}\pi_{s+1} = \frac{1}{s^2\, s!}\left(\frac{\lambda}{\mu}\right)^{s+2} \pi_0$$

$$\vdots$$

$$\pi_n = \pi_{s+n-s} = \frac{1}{s^{n-s}\, s!}\left(\frac{\lambda}{\mu}\right)^n \pi_0 \quad \text{for} \quad n \geq s$$

$$\pi_n = \frac{1}{s!} s^s \rho^n \pi_0 \quad \text{for} \quad n \geq s$$

$$\text{Thus } \pi_n = \begin{cases} \frac{1}{n!}(s\rho)^n \pi_0 \; ; & n < s \\ \frac{1}{s!} s^s \rho^n \pi_0 \; ; & n \geq s \end{cases}$$

where $\pi_0$ = Prob [having no customers in a multi-channel system]

**To obtain $\pi_0$ :**

Since $\pi_n$'s are Probabilities we have $\sum_{n=0}^{\infty} \pi_n = 1$

i.e. $\pi_0 + \sum_{n=1}^{s-1} \pi_n + \sum_{n=s}^{\infty} \pi_n = 1$

$\Rightarrow \pi_0 = \dfrac{1}{1 + \sum_{n=1}^{s-1} \frac{(s\rho)^n}{n!} + \sum_{n=s}^{\infty} \frac{s^s \rho^n}{s!}}$

$\Rightarrow \boldsymbol{\pi_0} = \dfrac{\mathbf{1}}{\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{s^s \rho^s}{s!} \frac{s\mu}{s\mu - \lambda}}$ ; which holds only if $\rho < 1$ , i.e. $s\mu > \lambda$ .

(In all other cases, i.e. $\rho \geq 1 \Rightarrow s\mu \leq \lambda$ , the waiting time increases indefinitely)

Now we obtain $L_s$, $L_q$, $W_s$ and $W_q$ (as earlier using the Little's formula)

i.e. $L_s = \sum_{n=0}^{\infty} n\pi_n$ & $L_q = \sum_{n=s}^{\infty} (n-s)\pi_n$

$L_s = \sum_{n=0}^{\infty} n\pi_n$ , it can be shown that $L_s = \dfrac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!\,(s\mu-\lambda)^2} \pi_0 + \dfrac{\lambda}{\mu}$

$L_q = \dfrac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!\,(s\mu-\lambda)^2} \pi_0$ OR $L_q = \dfrac{\rho(s\rho)^s}{s!\,(1-\rho)^2} \pi_0$

$W_s = \dfrac{\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!\,(s\mu-\lambda)^2} \pi_0 + \dfrac{1}{\mu} = W_q + \dfrac{1}{\mu}$

& $W_q = \dfrac{\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!\,(s\mu-\lambda)^2} \pi_0 \Rightarrow W_q = \dfrac{1}{\lambda} L_q$

P( that an arrival has to wait) = $P(N \geq s) = \sum_{n=s}^{\infty} \pi_n = \dfrac{\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!\,(s\mu-\lambda)} \pi_0$

& P( that an arrival enters service without wait) = $1 - P(N \geq s)$

Average No. of idle servers = s – Average No. of customers served

Utilization factor $\rho = \dfrac{\lambda}{s\mu}$

**Note:** For s = 1, it can be shown that

$\pi_0 = 1 - \rho = 1 - \dfrac{\lambda}{\mu}$ ; $\pi_n = \rho^n (1 - \rho) = \left(\dfrac{\lambda}{\mu}\right)^n \left(1 - \dfrac{\lambda}{\mu}\right)$, n=0,1,2........

$\therefore L_s = \dfrac{\lambda}{(\mu - \lambda)}$ , $L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)}$ , $W_s = \dfrac{1}{(\mu - \lambda)}$ and $W_q = \dfrac{\lambda}{\mu(\mu - \lambda)}$

which is nothing but the **M|M|1** system.

**Problems**

1) Four counters are being run on frontier of a country to check the passports & necessary papers of the tourists. The tourists choose a counter at random, if the arrivals at the frontier is Poisson at the rate $\lambda$ and the service time is exponential with parameter $\frac{\lambda}{2}$, what is the steady state average queue at each counter?

**Solution:**

Here we have to find $L_q$ .

we have $L_q = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!\,(s\mu-\lambda)^2}\ \pi_0$

Here $\lambda = \lambda$ , $\mu = \frac{\lambda}{2}$, s = 4 $\Rightarrow$ $\rho = \frac{\lambda}{s\mu} = \frac{1}{2}$ $\Rightarrow$ $s\rho = 2$

$$\pi_0 = \left(\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!}\frac{s\mu}{s\mu-\lambda}\right)^{-1} = \frac{3}{23} = 0.130$$

$$L_q = \frac{\left(s\,\frac{\lambda}{s\mu}\right)^s\left(\frac{\lambda}{s\mu}\right)}{s!\left(1-\frac{\lambda}{s\mu}\right)^2}\pi_0 = \frac{(\rho)(s\rho)^s}{s!(1-\rho)^2}\ \pi_0 = \frac{4}{23} = 0.174$$