Transportation Problem:

Let there be m – origin (manufacturing plants, sources etc) from which we have to send transport a particular product to n – different places (demand places or destinations). Let c_{ij} be cost of transportation per unit product from origin i to destination j . Let a_i denote the supply (capacity) available at their origin i and let b_j represent the demand at the destination j. Let x_{ij} denote the amount of quantity to be transported from i to j. The objective here in any transportation problem will generally be to minimize the total transportation cost involved.

i.e., minimize $z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$ ……………….……(1)

The constraints here are (i) the total supply should be exhausted.

(ii)all the demands should be met.

That is, (i) $\sum_{j=1}^{n} x_{ij} = a_i$, i = 1, 2, …., m ………...(2)

(ii) $\sum_{i=1}^{m} x_{ij} = b_j$, j = 1, 2, ….., n ….…….....(3)

Since the quantities to be transported are positive only, we can write,

$x_{ij} \geq 0$ for all (i , j) ……..………..(4)

Equations (1) – (4) represent the linear programming formulation of a transportation problem.

A typical transportation table will be as follows:

Destinations → / Origins ↓	D_1	D_2				D_{n-1}	D_n	Supply
O_1	c_{11}	c_{12}	$\cdots$	$\cdots$	$\cdots$	c_{1n-1}	c_{1n}	a_1
O_2	c_{21}	c_{22}	$\cdots$	$\cdots$	$\cdots$	c_{2n-1}	c_{2n}	a_2
.	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots\cdots$	$\cdots$	..
.	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots\cdots$	$\cdots$	..
O_m	c_{m1}	c_{m2}	$\cdots$	$\cdots$	$\cdots$	c_{mn-1}	c_{mn}	a_m

Demand b_1 b_2 $\ldots\ldots\ldots\ldots\ldots$ b_{n-1} b_n

Initial Allotment:

(i) North West Corner Rule:

Start the allocation from the top left corner cell and go to the next cell in the same row or column depending on the availability of the supply or demand. Here, we not taking the cost of the cell into consideration for the allocation.

Example:

Find the initial allotment to the following transportation problem

	D_1	D_2	D_3	D_4	Supply
O_1	5	8	7	3	50
O_2	4	6	2	9	25
O_3	6	8	1	7	75
O_4	7	5	4	6	100
Demand	50	30	80	90	250

Solution:

	D_1	D_2	D_3	D_4	Supply
O_1	50 5	8	7	3	50
O_2	4	25 6	2	9	25
O_3	6	05 8	70 1	7	75
O_4	7	5	10 4	90 6	100
Demand	50	30	80	90	250

Initial allotment : $x_{11} = 50$, $x_{22} = 25$, $x_{32} = 05$, $x_{33} = 70$, $x_{43} = 10$, and $x_{44} =$ 90. Total transportation cost $z = 1090$.

(ii) Least Coast Method (Matrix Minima Method):

Consider the minimum cost cell in the given matrix and do the allocation in that cell, then go to the next minimum cost in the matrix and allocate and go on till all the supply got exhausted and all the demands are met.

Example: Find an Initial Allotment by Least Cost Method to the TP

	D_1	D_2	D_3	D_4	Supply
O_1	5	8	7	3	50
O_2	4	6	2	9	25
O_3	6	8	1	7	75
O_4	7	5	4	6	100
Demand	50	30	80	90	250

Solution:

	D_1	D_2	D_3	D_4	Supply
O_1	5	8	7	50 3	50
O_2	20 4	6	05 2	9	25
O_3	6	8	75 1	7	75
O_4	30 7	30 5	4	40 6	100
Demand	50	30	80	90	250

Initial Solution: $x_{14} = 50$, $x_{21} = 20$, $x_{23} = 05$, $x_{33} = 75$, $x_{41} = 30$, $x_{42} = 30$, $x_{44} = 40$ and the cost of the transportation $z = 915$.

(iii) Vogel's Approximate Method:

1. Calculate the penalties by taking the differences between the minima and the next minimum unit transportation cost in each row and each column.
2. Choose the largest difference among the row differences and column differences (if there is a tie, choose any one arbitrarily). In that row (or column), identify the minimum cost cell and make the allocation.
3. In case the allocation is made fully to a row (supply is exhausted), then omit that row for further consideration. Similarly, if the allocation is made fully to a column (demand is met), omit the corresponding column for further consideration.
4. Revise the differences again and continue the procedures till all the demands are met and all the capacities are over.

Example: Find an initial allotment by VAM to the following TP.

	D_1	D_2	D_3	D_4	Supply
O_1	5	8	7	3	50
O_2	4	6	2	9	25
O_3	6	8	1	7	75
O_4	7	5	4	6	100
Demand	50	30	80	90	250

Solution:

	D_1	D_2	D_3	D_4	Supply
O_1	5	8	7	50 3	50
O_2	25 4	6	2	9	25
O_3	6	8	75 1	7	75
O_4	25 7	30 5	05 4	40 6	100
Demand	50	30	80	90	250

Initial Allocation: : $x_{14} = 50$, $x_{21} = 25$, $x_{33} = 75$, $x_{41} = 25$,

$x_{42} = 30$, $x_{43} = 05$, $x_{44} = 40$ and the cost of the transportation $z = 910$.

uv-method (To Iterate for Optimality in Transportation Problem):

To check the optimality of any allotment, let us introduce the dummy variables u and v in the transportation table. The number of basic cells should be equal to m+n-1 where m represents the number of rows, n refers the number of columns. Calculate the values of u_i's and v_j's using the costs of the basic cells (allotted cells) by assuming any one of the u_i's (or v_j's) as zero or any constant. Then calculate the values of $u_i + v_j - c_{ij}$ for all non-basic cells (non-allotted cells). If $u_i + v_j - c_{ij} \leq 0$ for all non-basic cells, then that allotment is optimal. If not, we have to iterate further to reach the optimality.

Example: Check the optimality of the previous VAM solution.

Solution:

	D_1	D_2	D_3	D_4	Supply	u_i
O_1	5	8	7	50 3	50	u_1=0
O_2	25 4	6	2	9	25	u_2=0
O_3	6	8	75 1	7	75	u_3=0
O_4	25 7	30 5	05 4	40 6	100	u_4=3
Demand	50	30	80	90	250	
v_j	v_1=4	v_2=2	v_3=1	v_4=3		

For non-basic cells (non-allotted cells),

$u_1 + v_1 - c_{11} = 0 + 4 - 5 = -1$

$u_1 + v_2 - c_{12} = 0 + 2 - 8 = -6$

$u_1 + v_3 - c_{13} = 0 + 1 - 7 = -6$

$u_2 + v_2 - c_{22} = 0 + 2 - 6 = -4$

$u_2 + v_3 - c_{23} = 0 + 1 - 2 = -1$

$u_2 + v_4 - c_{24} = 0 + 3 - 9 = -6$

$u_3 + v_1 - c_{31} = 0 + 4 - 6 = -2$

$u_3 + v_2 - c_{32} = 0 + 2 - 8 = -6$

$u_3 + v_4 - c_{34} = 0 + 3 - 7 = -4$

Here, $u_i + v_j - c_{ij} \leq 0$ for all non-basic cells. Hence, the present allotment is optimum. The optimal solution is $x_{14} = 50$, $x_{21} = 25$, $x_{33} = 75$, $x_{41} = 25$, $x_{42} = 30$, $x_{43} = 05$, $x_{44} = 40$ and the optimal cost is $z = 910$.

Note: 1. If the problem is an unbalanced one, make it as a balanced one by adding a dummy origin (or destination) with the cost of the transportation from this origin (or destination) to various destinations (or origins) being zeroes.

2. Generally, the transportation problem is of minimization type. If the given problem is a maximization one, then construct an equivalent transportation problem of minimization type. This can be done by identifying the maximum element in the given cost matrix and subtracting every given cost from this maximum element. This matrix is called "Opportunity Loss" cost matrix. If we minimize this equivalent transportation problem, that will be equivalent to maximizing the given transportation problem.

Example 1: Solve the following Transportation Problem for minimization:

Destinations

Origins↓	D_1	D_2	D_3	D_4	Supply
O_1	42	48	38	37	160
O_2	40	49	52	51	150
O_3	39	38	40	43	190
Demand	80	90	110	160	500 / 440

Solution: This is an unbalanced transportation problem. Here, the total demand is less than the total supply. So, let us create an artificial demand (imaginary destination) with the cost of transportation from various origins to this destination being zeroes. Then, the balanced transportation problem will be as follows:

Origins↓	D_1	D_2	D_3	D_4	D_5	Supply
O_1	42	48	38	37	0	160
O_2	40	49	52	51	0	150
O_3	39	38	40	43	0	190
Demand	80	90	110	160	60	500 / 500

Table 1: The initial allocation by VAM will be as

Origins↓	D_1	D_2	D_3	D_4	D_5	Supply
O_1	42	48	38	160 37	0	160
O_2	80 40	49	10 52	51	60 0	150
O_3	39	90 38	100 40	43	0	190
Demand	80	90	110	160	60	500 500

Here, the number of basic cells is 6 which is less than the required one of m+n-1. Let us choose one of the non-basic cells in the first row and make it as a basic cell with the allocation of 0. In the above table, let us make the non-basic cell (1,5) as a basic one with 0 quantity. Now, the number of basic cells becomes 7 which is equal to m+n-1. Then, do the optimality checking.

Origins↓	D_1	D_2	D_3	D_4	D_5	Supply	u_i	
O_1	42	48	38	160 37	0 0	160	u_1=0	
O_2	80 40	49	10 52	51	60 0	150	u_2=0	
O_3	39	90 38	100 40	43	0	190	u_3=-12	
Demand	80	90	110	160	60	500		
v_j	v_1=40	v_2=50	v_3=52	v_4=37	v_5=0			

Here, $u_i + v_j - c_{ij} > 0$ for some of the non-basic cells. Hence, this allotment is not optimum. To iterate, we pick up the cell which is having most positive value of $u_i + v_j - c_{ij}$. Here, $u_1 + v_2 - c_{12} = 2$; $u_1 + v_3 - c_{13} = 14$ and $u_2 + v_2 - c_{22} = 1$. Therefore, the maximum positive occurs at the cell (1,3). So, we make this cell as a basic cell through iteration. Starting from the present non-basic cell, make a shortest loop connecting only the basic cells. Assign ' + ' and ' - ' alternatively starting from that non-basic cells with all the joint basic cells. Pick up the minimum allocated quantity of the basic cells corresponding to the – ve signs.

Origins↓	D_1	D_2	D_3	D_4	D_5	Supply	u_i
O_1	42	48	+ 38	160 37	- 0 0	160	u_1=0
O_2	80 40	49	- 10 52	51	+ 60 0	150	u_2=0
O_3	39	90 38	100 40	43	0	190	u_3=-12
Demand	80	90	110	160	60	500	
v_j	v_1=40	v_2=50	v_3=52	v_4=37	v_5=0		

Here, min $\{x_{23}=10,\ x_{15}=0\} = 0$. Add this minimum quantity to all the cells with ' + ' sign and subtract from the allocated quantity of the cells with ' – ' sign. With this modification, the next iterated allotment will be as follows:

Table 2:

Origins↓	D_1	D_2	D_3	D_4	D_5	Supply	u_i
O_1	42	48	0 38	160 37	0	160	u_1=0
O_2	80 40	49	10 52	51	60 0	150	u_2=14
O_3	39	90 38	100 40	43	0	190	u_3=2
Demand	80	90	110	160	60	500	
v_j	v_1=26	v_2=36	v_3=38	v_4=37	v_5=-14		

Again, check the optimality using $u_i + v_j - c_{ij}$ for all non-basic cells. Here, $u_2 + v_2 - c_{22} = 1 > 0$. Hence, this allotment is not optimum. Do the iteration again. Make this cell as a basic cell by doing the allotment. The revised table will be as follows:

Table 3:

Origins↓	D_1	D_2	D_3	D_4	D_5	Supply	u_i
O_1	42	48	0 38	160 37	0	160	u_1=0
O_2	80 40	10 49	52	51	60 0	150	u_2=13
O_3	39	80 38	110 40	43	0	190	u_3=2
Demand	80	90	110	160	60	500	
v_j	v_1=27	v_2=36	v_3=38	v_4=37	v_5=-13		

Here, $u_i + v_j - c_{ij} \leq 0$ for all non – basic cells. Therefore, this allocation is optimum. The optimal solution is $x_{13} = 0$; $x_{14} = 160$; $x_{21} = 80$; $x_{22} = 10$; $x_{32} = 80$; $x_{33} = 110$ and the optimal transportation cost is $z = 17{,}050$.

Example 2: Solve the following transportation problem for maximization:

Destinations

Origins↓	D_1	D_2	D_3	D_4	Supply
O_1	12	8	6	25	200
O_2	8	7	10	18	500
O_3	14	3	11	20	300
Demand	180	320	100	400	1000 / 1000

Solution: This is a maximization problem. So, let us create the "Opportunity Loss" Cost matrix and then we will try to minimize that. The OLC matrix will be as follows:

Destinations

Origins↓	D_1	D_2	D_3	D_4	Supply
O_1	13	17	19	0	200
O_2	17	18	15	7	500
O_3	11	22	14	5	300
Demand	180	320	100	400	1000 / 1000

Make the initial allotment by VAM and check for optimality. If it is not optimal, do the iteration by uv-method. The optimal solution will be

$x_{14} = 200$; $x_{22} = 320$; $x_{23} = 100$; $x_{24} = 80$; $x_{31} = 180$; $x_{34} = 120$:

The optimal (maximal) cost can be found out by multiplying these allotments with the respective given transportation costs. Here, the optimal transportation cost is $z = 14{,}600$.

Example 3: Three electric power plants with capacities of 25, 40 and 30 mkh supply electricity to three cities whose maximum demands are estimated at 30, 35 and 25 mkh. The cost of selling power to the different cities per mkh are as follows (in US$):

	City 1	City 2	City 3	Supply
Plant 1	600	700	400	25
Plant 2	320	300	350	40
Plant 3	500	480	450	30
Demand	30	35	25	

During the month of September, there is a 20% increase in demand at each of the three cities. To meet the excess demand, the power company must purchase additional electricity from another network at a flat price of 1000$ per mkh. This network is however not linked to city 3. Formulate the problem as a Transportation Model for the purpose of establishing the most economical distribution for the power company.

****