PROBABILITY DISTRIBUTION

Discrete Probability Distributions

Bernoulli Distribution:

A drv X is said to follow the Bernoulli distribution with parameter p, if it assumes only non-negative values 0 and 1 and its probability function is given by

$P(X=x) = p^{x} q^{1-x}$; x=0,1 and p+q= 1

Now, to obtain the mean and variance of the Bernoulli distribution, consider

$E(X) = \sum_x xp(x) = \sum_0^1 xp^x q^{1-x} = p$ and

$V(X)=E(X^2)-(E(X))^2$

where $E(X^2)= \sum_x x^2 p(x) = \sum_0^1 x^2 p^x q^{1-x} = p$

$V(X)=p-p^2=p(1-p)=pq$

Binomial Distribution:

A drv X is said to follow the Binomial distribution with parameters n and p, if it assumes only non-negative values 0,1,2,3,……..n and its probability function is given by

$P(X=x)=\binom{n}{x} p^x q^{n-x}$; x=0,1,2,3,4,……,n and p+q=1

Note: $\sum_{x=0}^{n} P(X = x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = [p + (1-p)]^n = 1$

Hence P(X=x) is the probability distribution.

Now, to obtain mean and variance of Binomial distribution:

$E(X)= \sum_x xp(x) = np$

and $V(X)=E(X^2)-(E(X))^2=npq$

Poisson Distribution:

A discrete random variable X is said to follow the Poisson distribution with parameter λ, (λ > 0) if it assumes only non-negative values 0,1,2,3,4,… and its probability function is given by

$P(X=x)= e^{-\lambda}\lambda^x / x!$, x=0,1,2,3,4,…

Note: $\sum_{x=0}^{\infty} P(X = x) = \sum_{x=0}^{\infty} e^{-\lambda}\lambda^x / x! = 1$

Hence P(X=x) is the probability distribution.

Now, to obtain mean and variance of Poisson distribution:

$E(X)= \sum_x xp(x) = \sum_{x=0}^{\infty} xe^{-\lambda}\lambda^x / x! = \ldots = \lambda$ (Prove it!)

and $V(X)=E(X^2)-(E(X))^2 = \ldots = \lambda$ (Prove it!)

Geometric Distribution:

A drv X is said to follow the geometric distribution with parameter p if it takes values 1,2,3,4,… and its probability function is given by

$P(X=x) = p\, q^{x-1}$; x=1,2,3,4,… and p+q=1

Now, to obtain mean and variance of Geometric distribution:
$E(X)=\sum_x xp(x) = \sum_{x=1}^{\infty} xp\, q^{x-1}$ =……= 1/p (Prove it!)
and $V(X)= E(X^2)-(E(X))^2$ =…= q/p^2 (Prove it!)

Uniform Distribution:
A drv X is said to follow the Uniform distribution if it takes values 1,2,3,…,n and its probability function is given by
P(X=x) = 1/n; x=1,2,3,…,n
We have, E(X) = (n+1)/2 and V(X) = $(n^2-1)/12$.

Poisson Distribution as an approximation to the Binomial Distribution / Poisson Distribution as a Limiting case of the Binomial Distribution

Now what happens to the binomial probability $\binom{n}{x} p^x q^{n-x}$, x=0,1,2,3,4,…n, when n becomes large i.e. $n \to \infty$? For this consider a rv X~B(n,p) we have,
$P(X=x) = \binom{n}{x} p^x q^{n-x}$, x=0,1,2,3,4,…n
Now we put the following conditions:
1) n, the number of trials is indefinitely large i.e. $n \to \infty$
2) p, the constant probability of success for each trial is indefinitely small i.e. $p \to 0$.
3) $np = \lambda$ say, is finite, so that $p = \lambda/n$ so that $q = 1-(\lambda/n)$
Under these conditions it can be shown that, (Prove it!)
$$\lim_{n\to\infty} P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!};\ x = 0,1,2, \dots n;\ n \to \infty$$
Or $P(X=x)= \frac{e^{-\lambda}\lambda^x}{x!};\ x = 0,1,2, \dots$, which is nothing but the Poisson distribution.

Problems:

1) What is the probability of getting a 6 atleast once in two throws of fair die?

Solution: Let X=No. of times 6 appears.
Given n=2 and p=P(getting 6 in a single throw of a die)=1/6
Clearly X~B(n,p) with n=2; p=1/6
$P(X=x) = \binom{n}{x} p^x q^{n-x}$, x=0,1,2…,n
P(getting a 6 atleast once in 2 throws) = P(X=1) + P(X=2)
$= \binom{2}{1}(1/6)(5/6) + \binom{2}{2}(1/6)^2$
= 11 / 36.

2) 6 coins are tossed. Find the probability of getting **a)** exactly 3 heads **b)** atmost 3 heads **c)** atleast 3 heads **d)** atleast 1 head

Solution: Let X = No. of heads appearing; Given n=6, p=P(getting a H)= 1/2.
Clearly X~B(n,p) with n=6; p=1/2.
$P(X=x) = \binom{n}{x} p^x q^{n-x}$; x=0,1,2,…n.

a) P(exactly 3 heads) = P(X =3) = $\binom{6}{3}$(1/64) = 20/64 = 5/16.

b) P(atmost 3 heads) = P(X=0) + P(X=1) + P(X=2) + P(X=3)
$= \binom{6}{0}(1/64) + \binom{6}{1}(1/64) + \binom{6}{2}(1/64) + \binom{6}{3}(1/64)$
= 21 / 32.

c) P(atleast 3 heads) = P(X=3) + P(X=4) + P(X=5) + P(X=6)
$= \binom{6}{3}(1/64)+ \binom{6}{4}(1/64) + \binom{6}{5}(1/64) + \binom{6}{6}(1/64)$
= 21 / 32.

d) P(atleast one head) = P(X=1) + P(X=2) + ------ +P(X=6)
= (1/64) (6+15+20+15+6+1)
= 63 / 64

3) If X has a Poisson probability distribution with parameter λ, show that
i) P(X is even) = $\frac{1}{2}(1 + e^{-2\lambda})$
ii) P(X is odd) = $\frac{1}{2}(1- e^{-2\lambda})$

Solution: Since X~ P(λ) , we have
$P(X=k) = e^{-\lambda} \lambda^k / k!$; k=0,1,2,…
Now P(X is even) = P(X=0)+P(X=2)+P(X=4)+…

$$= \sum_{k=0}^{\infty} P(X = 2k)$$
$$= \sum_{k=0}^{\infty} e^{-\lambda} \lambda^{2k}/(2k)!$$
$$= e^{-\lambda} \sum_{k=0}^{\infty} \lambda^{2k}/(2k)! \longrightarrow (1)$$

Now Consider, $(e^{\lambda} + e^{-\lambda})/2$.

$$=1/2\left(\sum_{k=0}^{\infty} \lambda^k/k! + \sum_{k=0}^{\infty}(-\lambda)^k/k!\right)$$
$$=1/2\left(\sum_{k=0}^{\infty} 2\lambda^{2k}/(2k)!\right)$$
$$=\sum_{k=0}^{\infty} \lambda^{2k}/(2k)! \longrightarrow (2)$$

Using (2) in (1), we get
P(X is even) = $e^{-\lambda} [(e^{\lambda} + e^{-\lambda})/2]$

$=1/2[1 + e^{-2\lambda}]$

Similarly show that P(X is odd) = $\frac{1}{2}(1 - e^{-2\lambda})$

4) If X has a Poisson distribution with parameter λ and if P(X=0) = 0.2 , evaluate P(X>2).

Solve it!

5) Suppose that X has a Poisson distribution with parameter λ. If P(X=2) = $\frac{2}{3}$ P(X=1), evaluate P(X=0) and P(X=1)

Solve it!

6) 6 coins are tossed 6400 times. Using Poisson distribution obtain the approximate probability of getting 6 heads 100 times.

Solution: Let X= number of times 6 heads appearing
Given n=6400 and p= P(getting 6 heads in a throw of 6 coins)= $1/2^6$ = 1/64
Clearly X~B(n,p) with n=6400 p=1/64
$P(X=x)=\binom{n}{x}p^x q^{n-x}; x = 0,1,2,\ldots,n$
But since n is large and p is small we may use Poisson approximation, with λ=np=100
Thus we have, with $X\sim P(\lambda)$,
$P(X=x)=e^{-\lambda}\lambda^x/x!$; x=0,1,2,…
$\therefore$ P(getting 6 heads 100 times)=P(X=100) = $e^{-100}(100)^{100}/100!$

7) Suppose that the probability that an item produced by a particular machine is defective equals 0.2. If 10 items produced from this machine are selected at random. What is the probability that not more than one defective is found? Use the binomial and Poisson distributions and compare the answers.

Solve it!

8) Suppose that a container contains 10,000 particles. The probability that such a particle escapes from the container equals 0.0004. What is the probability that more than 5 such escapes occur? (assume that the various escapes are independent of one another).

Solve it!