STOCHASTIC PROCESSES

Introduction

Most of the naturally occurring phenomena show randomness in their behavior. They depend heavily on chance and do not follow the deterministic laws formulated by man. A natural mathematical tool to describe such a phenomena is the Random Function defined on a suitable parameter space. The prediction of the future, though possible only in a statistical sense, is achieved by the study of the evolution of random function with reference to the characteristics of the parameter.

A Stochastic Process is the family of random variables $\{X(t); t \in T\}$ indexed by parameter t varying in an index set T.

In order to describe a stochastic process, one has to specify the probability measure associated with the process that characterizes it.

Usually, the probability measure is specified by joint pdf, $f(x_1, x_2, \ldots, x_n)$.

In a stochastic process $\{X(t);\ t \in T\}$ generally 't' denotes time, but other parameters like space, length, width, area etc. are also used. The values assumed by the process are called the states and the set of all possible values is called the state space denoted by 'S'. Set of possible values of the parameter is called the parameter space denoted by 'T'

Both the Parameter Space (PS) and State Space (SS) may be discrete or continuous. Accordingly, stochastic processes are classified as:

(i) Discrete PS, discrete SS.
Eg: Consumer preference observed on a monthly basis

(ii) Continuous PS, discrete SS
Eg: No. of telephone calls arriving at an exchange over a period of time

(iii) Discrete PS, Continuous SS
Eg: Inventory on hand is observed only at discrete time points

(iv) Continous PS, Continous SS
Eg: Water level/ content of a dam being observed over a period of time

Example:
What are the SS and PS for a stochastic process which is the score during a football match?
Answer: T : {[0,90]}

S : {(x,y) : x,y = 0,1,2,……}

Types of Stochastic processes:

Quite often we find that X_n, the member of the family are not mutually independent. The nature of their dependence varies.

Stochastic processes are classified according the nature of dependence that exists amongst the members of the family.

I. Processes with independent increments

A Stochastic process $\{X(t); t \in T\}$ such that for any $(t_1, t_2, t_3, \ldots t_n) \in T$ where $t_1 < t_2 < t_3 < \ldots < t_n$, the rvs $[X(t_2) - X(t_1)]$; $[X(t_3) - X(t_2)]; \ldots ; [X(t_n) - X(t_{n-1})]$ are all independent is called a process with independent increments.

II. Stationary Processes

A Stochastic process $\{X(t) ; t \in T\}$ is said to be stationary if for any $(t_1, t_2, t_3, \ldots t_n) \in T$ the joint pdf of $\{X(t_1 + h), X(t_2 + h), \ldots X(t_n + h)\}$ and $\{X(t_1), X(t_2), \ldots, X(t_n)\}$ are the same $\forall\, h \in (-\infty, \infty)$

A stationary process is said to be strictly stationary if it is stationary of order n for any integer 'n'. Otherwise the process is said to be weakly or covariance stationary.

A process which is not stationary in any sense is called evolutionary.

III. Markov Processes

A Stochastic process $\{X(t); t \in T\}$ is called a Markov process if for any $t_1 < t_2 < t_3 < \ldots < t_n$, the conditional probability distribution of $X(t_n)$ for given values $X(t_1), X(t_2), \ldots, X(t_{n-1})$ depends only on $X(t_{n-1})$ i.e most recent known value of the process.

$$P[X(t_n) = x_n / X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \ldots, X(t_1) = x_1]$$

$$= P[X(t_n) = x_n / X(t_{n-1}) = x_{n-1}]$$

This property is called the Markov Property.

Markov Process is a Stochastic Process for which the future event or state depends only on

the immediately preceding event or state.

Every Stochastic process with independent increments is a Markov Process.

Markov Chain: A Markov Chain is a Markov process $\{X_n; n \in T\}$ with a Discrete PS and Discrete SS. i.e. the stochastic process $\{X_n; n = 0,1,2,\ldots\}$ is a Markov chain if for $i, j, i_1, i_2, \ldots, i_{n-1} \in$ N(or I)

$$P[X_n = j/X_{n-1} = i, X_{n-2} = i_1, \ldots, X_0 = i_{n-1}] = P[X_n = j/X_{n-1} = i] = p_{ij}$$

where p_{ij} denotes probability that state a process at the n^{th} time point is j given that it was in state i in the (n-1) th time point. These p_{ij}'s are called One Step Transition Probabilities of the process, as p_{ij} refers to states i and j in 2 successive trials.

These p_{ij}'s may or may not be independent of 'n' (the time). p_{ij}'s may depend only on difference in time epochs (m-n) instead of n and m. If the p_{ij}'s are independent of 'n' then the process is said to be time homogeneous or simply homogeneous.

In more general case we are concerned with the states i and j at 2 non-successive trials, say state i at n^{th} trial and state j at $(n+m)^{th}$ trial. The corresponding transition probability is called the m-step transition probability, given by

$$p_{ij}^{(m)} = P[X_{n+m} = j/X_n = i]$$