Higher Transition Probabilities: Chapman – Kolmogorov Equations (C-K Equations)

The m- step transition probability is given by $p_{ij}^{(m)}$ = P [X_{n+m} = j/ X_n = i] where $p_{ij}^{(m)}$ gives the probability that from state i at the n^{th} trial the state j is reached at $(n+m)^{th}$ trial in m-steps. i.e it is the probability of transition from state i to state j in exactly m- steps. Since p_{ij} is time homogeneous, consider,

$$p_{ij}^{(2)} = P [X_{n+2} = j / X_n = i]$$

Let k be some intermediate state. Consider a fixed value of k, then we have

$$p_{ij}^{(2)} = P[X_{n+2} = j / X_n = i] = P [X_{n+2} = j, X_{n+1} = k / X_n = i]$$

$$= P [X_{n+2} = j / X_{n+1} = k, X_n = i]. P [X_{n+1} = k / X_n = i]$$

$$= P [X_{n+2} = j / X_{n+1} = k]. P [X_{n+1} = k / X_n = i]$$

$$= p_{kj} . p_{ik}$$

$$p_{ij}^{(2)} = p_{ik} . p_{kj}$$

Since, k= 1,2,3,…… are all mutually exclusive we have

$$p_{ij}^{(2)} = \sum_{k\in S} p_{ik} p_{kj}$$

Thus, by mathematical induction we have;

$$p_{ij}^{(m+1)} = P [X_{n+m+1} = j / X_n = i]$$

$$= \sum_k P [X_{n+m+1} = j / X_{n+m} = k] P [X_{n+m} = k / X_n = i]$$

$$= \sum_k p_{kj}\, p_{ik}^{(m)} = \sum_k p_{ik}^{(m)}\, p_{kj}$$

Thus in general we have,

$$p_{ij}^{(m+n)} = \sum_k p_{ik}^{(m)}\, p_{kj}^{(n)}$$

$$\equiv \; p_{ij}^{(n)} = \sum_{k\in S} p_{ik}^{(n-1)}\, p_{kj}$$

This is called the Chapman-Kolmogorov (CK) equation which is satisfied by the transition probabilities of a Markov Chain.

We can write these results in terms of transition probability matrices.

Let $P = (p_{ij})$ denote the one step tpm

$P^{(m)} = \left(p_{ij}^{(m)}\right)$ denote the m-step tpm

$P^{(2)} = \left(p_{ij}^{(2)}\right)$, where $p_{ij}^{(2)} = \sum_k p_{ik} p_{kj}$; i.e the elements of $P^{(2)}$ are the elements of the matrix obtained by multiplying P by itself.

$P^{(2)} = P.\ P$

Similarly, $P^{(m)} = P.\ P^{(m-1)} = P^{(m-1)}.\ P = P^m$

Thus, the m-step tpm is obtained by multiplying the one step tpm itself m-times. In other words, the probability of finding m-step tpm is one of finding the powers of the given one step tpm.

The CK equation are the basic equations in the study of Markov Process as they provide an efficient means of studying the m- step transition probabilities. i.e they enable us to build convenient relationship for transition probabilities between two points in the parameter space at which the process exhibits the Markov dependence property.

Problems

1) Let $\{ X_n ; n \geq 0 \}$ be a Markov Chain with states 0,1,2 and tpm

$$P = \begin{array}{c} \\ i=0 \\ i=1 \\ i=2 \end{array} \begin{array}{c} \begin{array}{ccc} j=0 & j=1 & j=2 \end{array} \\ \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \end{array}$$

and with initial probability distribution $P(X_0 = i) = 1/3, \forall\, i$

Find (i) $P[X_1 = 1/ X_0 = 2]$

(ii) $P[X_2 = 1/ X_1 = 1]$

(iii) $P[X_2 = 2/ X_1 = 1]$

(iv) $P[X_2 = 2, X_1=1/ X_0 = 2]$

(v) $P[X_2 = 2, X_1=1, X_0 = 2]$

(vi) $P[X_3 =1, X_2 = 2, X_1=1, X_0 = 2]$

Solution:

(i) $P[X_1 = 1/ X_0 = 2] = p_{21} = 3/4$

(ii) $P[X_2 = 1/ X_1 = 1] = p_{11} = 1/2$

(iii) $P[X_2 = 2/ X_1 = 1]= p_{12} = 1/4$

(iv) $P[X_2 = 2, X_1=1/ X_0 = 2] = P[X_2 = 2/ X_1=1, X_0 = 2]. P[X_1=1/ X_0 = 2]$

$= P[X_2 = 2/ X_1=1]. P[X_1=1/ X_0 = 2]$

$= p_{12}. p_{21} = (3/4)(1/4) = 3/16$

(v) $P[X_2 = 2, X_1=1, X_0 = 2] = P[X_2 = 2/ X_1=1, X_0 = 2]. P[X_1=1/ X_0 = 2]. P[X_0 = 2]$

$= p_{12}. p_{21}.(1/3) = (3/4)(1/4)(1/3) = 1/16$

(vi) $P[X_3 =1, X_2 = 2, X_1=1, X_0 = 2]$

$= P[X_3 =1, X_2 = 2, X_1=1/ X_0 = 2]. P[X_0 = 2]$

$= P[X_3 =1, X_2 = 2/ X_1=1].P[X_1 = 1/ X_0 = 2]. P[X_0 = 2]$

$= P[X_3 =1/ X_2 = 2].P[X_2 = 2/ X_1 = 1]. P[X_1 = 1/ X_0 = 2]. P[X_0 = 2]$

$= p_{21}.p_{12}.p_{21}.(1/3) = (3/4)(1/4)(3/4)(1/3) = (3/64)$

2) A Markov Chain $\{X_n ; n \geq 0\}$ has the tpm $P = \begin{bmatrix} x & y \\ 2x & y/3 \end{bmatrix}$ with initial probability distribution $p^0 = \{y,x\}$ Find p^0 ?

Solution:

Since P is a Markov matrix, we have $x+y = 1$ and $2x+ y/3=1$. Solving we get $y= 3/5$ and $x=2/5$