Duality Theory:

Every lpp is associated with another lpp known as its dual problem. The original problem in this context is known as primal lpp. The solution of the dual problem which leads to the solution of the primal problem may be helpful in taking a better decision for the given lpp.

Example 1:

A manufacturer produces 3 products, say A, B and C. Each product has to be worked on 2 machines, say M1 and M2. The time required for producing one unit of each product on each machine is given as

	M1	M2
A	5 hrs	3 hrs
B	4 hrs	6 hrs
C	2 hrs	1 hr

The machine M1 can work for a maximum of 80 hrs per week and machine M2 can work for 70 hrs per week. The profit for one unit of the products A, B, and C are 50$, 60$ and 20$ respectively. If we formulate this problem from the manufacturer point of view, then it will be as follows:

Maximize Profit $z = 50x_1 + 60x_2 + 20x_3$

Subject to $5x_1 + 4x_2 + 2x_3 \leq 80$

$$3x_1 + 6x_2 + x_3 \leq 70$$

$$x_1, x_2, x_3 \geq 0$$

Dual Formulation: If we formulate the same problem in terms of an investor point of view who is interested in hiring the machines, it will be as follows:

Minimization of the Rental Cost $= 80\, w_1 + 70\, w_2$ ….. (1)

where w_1 and w_2 are the amounts the investor is ready to give as a rent for 1 hour of M1 and M2 respectively. The constraints associated with this problem are

$$5\, w_1 + 3\, w_2 \geq 50 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

$$4\, w_1 + 6\, w_2 \geq 60 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

$$2\, w_1 + w_2 \geq 20 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

$$w_1 ,\; w_2 \geq 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

Let us assume that w_1 and w_2 are the offers made by the investor for each unit of the hour of M1 and M2 respectively. Then, these offers must satisfy the non-negative constraints also, i.e., $w_1 , w_2 \geq 0$. The objective of the investor here must be to minimize the investment, ie., minimize the rental cost $80w_1 + 70w_2$. The total amount offered by the prospective investor to the resources required to produce one unit of each product must be atleast as high as the profit earned by the manufacturer from that product per unit. Since these resources enable the manufacturer to earn the specified profit corresponding to each product, the manufacturer would not like to sell the resources for anything less. Therefore, these can be written as the constraints for the investor and these can be written as the inequalities as (2) – (4).

Note: The variables in the dual problem are known as dual variables and the constraints are known as dual constraints.

Primal – Dual Relationships:

(i) If the primal lpp is having an objective of Maximization type, then the objective of the dual lpp will be of Minimization type.

(ii) The number of dual variables will be equal to the number of primal constraints and vice-versa.

(iii) The resource levels b_i's in primal lpp will become the cost co-efficients c_i's in the dual and c_i's in the primal will become b_i's in the dual problem.

(iv) If all constraints are of " $\leq$ " (or " $\geq$ ") type in primal, then all constraints will be of " $\geq$ " (or " $\leq$ ") type in dual.

(v) The coefficients of the constraints in the primal problem which appear from left to right are placed from top to bottom in the constraints of the dual problem and vice versa.

Note: 1. The dual of a dual lpp will be primal.

2. The maximization problem with " $\leq$ " type constraints is called a regular lpp.
3. If the constraints are not proper ones, then the corresponding dual variables' nature will change and vice versa.
4. If any constraint is of equality type, then the corresponding dual variable will be unrestricted on and vice versa.

Example 2: Write the dual of the following lpp

Minimize $z = -x_1 + 5x_2 + 2x_3 - 4x_4$

Subject to the constraints

$$2x_1 + 3x_2 - 7x_3 \leq 10$$

$$x_1 + 4x_2 + 6x_3 - x_4 \geq 11$$

$$-2x_2 + 4x_3 = 7$$

x_1 , $x_2 \geq 0$, $x_3 \leq 0$ and x_4 unrestricted.

Solution:

Dual of the above lpp will be as follows:

Maximize $10\,w_1 + 11\,w_2 + 7\,w_3$

Subject to the constraints

$$2\,w_1 + w_2 \leq -1$$

$$3\,w_1 + 4\,w_2 - 2\,w_3 \leq 5$$

$$-7\,w_1 + 6\,w_2 + 4\,w_3 \geq 2$$

$$-w_2 = -4$$

$w_1 \leq 0$, $w_2 \geq 0$, w_3 unrestricted.

Relations between Solutions of the Primal and Dual lpps:

Example 1: Maximize $z = 12x_1 + 3x_2 + x_3$

Subject to $10x_1 + 2x_2 + x_3 \leq 100$

$$7x_1 + 3x_2 + 2x_3 \leq 77$$

$$2x_1 + 4x_2 + x_3 \leq 80$$

$$x_1, x_2, x_3 \geq 0.$$

Analysis:

The Optimal table of the above lpp will be as in the following:

			12	3	1	0	0	0
C_B	x_B	x_{Bi}	x_1	x_2	x_3	x_4	x_5	x_6

12	x_1	73/8	1	0	-1/16	3/16	-1/8	0
3	x_2	35/8	0	1	13/16	-7/16	5/8	0
0	x_6	177/4	0	0	-17/8	11/8	-9/4	1
	$z_j - c_j$	z = 981/8	0	0	11/16	15/16	3/8	0

The optimal solution is $x_1 = 73/8$, $x_2 = 35/8$ and $x_3 = 0$ and the optimal value is z = 981/8.

The dual of the above lpp can be written as

Minimize $\quad f = 100w_1 + 77w_2 + 80 w_3$

Subject to
$$10w_1 + 7w_2 + 2w_3 \geq 12$$
$$2w_1 + 3w_2 + 4w_3 \geq 3$$
$$w_1 + 2w_2 + w_3 \geq 1$$
$$w_1, w_2, w_3 \geq 0.$$

Here, the dual variable values at the optimal level can be identified from the optimal table of the given primal lpp. The value of the net evaluation $z_j - c_j$ corresponding to the dummy variables introduced in the primal lpp will be the values of the dual variables at the optimal level correspondingly. Here,

$w_1 = z_4 - c_4$ (the net evaluation corresponding to the first slack variable in the first constraint in the primal lpp)

$= 15/16.$

$w_2 = z_5 - c_5 = 3/8$ and

$w_3 = z_6 - c_6 = 0.$

The optimal value in the dual lpp will be f = 981/8.

On the same line, we can find out the values of the primal variables by solving the dual lpp. Here , the optimal table corresponding to the dual lpp is as follows:

			-100	-77	-80	0	0	0	- M	-M	-M
c_B	x_B	x_{Bi}	w_1	w_2		w_4	w_5	w_6	A_1	A_2	A_3
0	w_6	11/16	0	0	17/8	-1/16	-13/16	1	-1/16	13/16	1
-77	w_2	3/8	0	1	9/4	1/8	-5/8	0	-1/8	5/8	0
-100	w_1	15/16	1	0	-33/24	-3/16	7/16	0	3/16	-7/16	0
f_j - c_j		$\frac{-981}{8}$	0	0	165/4	73/8	35/8	0	-73/8 + M	-35/8 + M	M

From the above table, the optimal solution to the dual problem can be identified as w_1 = 15/16 , w_2 = 3/8 and w_3 = 0 and the optimal value is f = 981/8. From the optimal dual table, we can identify the values of the primal variable. Consider the initial basic variables introduced in the dual problem and pick up the values of the corresponding f_j - c_j. Omitting the penalty, the rest values are the values of the primal variables. Whenever the objective function is changed for solving the lpp, the net values corresponding to the initial basic variables should be multiplied by -1 after omitting the penalty M, if any. Here, the primal variable values are

x_1 = -(-73/8) = 73/8 , x_2 = -(-35/8) = 35/8 , and x_3 = - (0) = 0.

The optimal value of the primal lpp is z = 981/8 which is equal to the value of the dual lpp.

Note: (i) If the primal (dual) lpp is having an infeasible solution, then the dual (primal) lpp will be having an unbounded solution.

(ii) If the primal (dual) lpp is having an unbounded solution, then the dual (primal) one will be having an infeasible solution.

(iii) If the primal (dual) is having a unique optimal solution, then the dual (primal) also will be having a unique solution and the optimality of both lpps will be equal.

Dual Simplex Method:

Sometimes the lpp cannot be solved by using simplex procedure alone. In such lpps , we can approach the problem from an infeasible point or solution. Let the given lpp be primal. When the primal lpp using simplex method, starts with a feasible point and continues to be feasible until the optimal solution is reached, the dual problem starts with an infeasible point but better than optimum and continues to be infeasible until the true optimum point is reached. Otherwise, when primal is seeking the optimality, the dual is automatically seeking the feasibility. Such a procedure is known as dual simplex method.

Algorithm:

1. Start the simplex iterative procedure from an basic infeasible but better than optimum point. This can be done by suitably modifying the constraints.

2. Identify the exit variable first by picking up the most negative value among the basic variable values, i.e., min $\{x_{Bi}, \ \ x_{Bi} < 0\}$.
3. Take the maximum of $\{(z_j - c_j)/y_{kj}, \ y_{kj} < 0\}$ where k denotes the pivotal row and the corresponding variable x_j will be the entry variable.
4. Do the elementary row operations to make the pivotal element as 1 and all other elements in the pivotal column as 0.
5. Continue the procedure till all the basic solutions will become feasible (positive).

Example 1: Solve the following lpp by dual simplex method

Minimize $w = 2x_1 + 2x_2 + 4x_3$

Subject to
$$2x_1 + 3x_2 + 5x_3 \geq 2$$
$$3x_1 + x_2 + 7x_3 \leq 3$$
$$x_1 + 4x_2 + 6x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0.$$

Solution: Rewriting the given lpp as

Maximize $z = -w = -2x_1 - 2x_2 - 4x_3$

Subject to
$$-2x_1 - 3x_2 - 5x_3 + x_4 = -2$$
$$3x_1 + x_2 + 7x_3 + x_5 = 3$$
$$x_1 + 4x_2 + 6x_3 + x_6 = 5$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

Table 1:

			-2	-2	-4	0	0	0
C_B	x_B	x_{Bi}	x_1	x_2	x_3	x_4	x_5	x_6
0	x_4	-2	-2	$-3^{\#}$	-5	1	0	0
0	x_5	3	3	1	7	0	1	0
0	x_6	5	1	4	6	0	0	1
$z_j - c_j$		z = 0	2	2	4	0	0	0

Here, the initial basic infeasible solution is $x_4 = -2$, $x_5 = 3$, $x_6 = 6$. The exit variable is x_4 . The entry variable is identified from max{$(z_j - c_j)/y_{kj}$, $y_{kj} < 0$}.

i.e., max{$\frac{2}{-2}$, $\frac{2}{-3}$, $\frac{4}{-5}$ } = $-\frac{2}{3}$. This corresponds to the entry variable x_2 .

Table 2:

			-2	-2	-4	0	0	0
C_B	x_B	x_{Bi}	x_1	x_2	x_3	x_4	x_5	x_6
-2	x_2	2/3	2/3	1	5/3	-1/3	0	0
0	x_5	7/3	7/3	0	16/3	1/3	1	0
0	x_6	7/3	-5/3	0	-2/3	4/3	0	1
$z_j - c_j$		z=-4/3	2/3	2	2/3	2/3	0	0

Here, all the basic variable are positive with $z_j - c_j$ being positive. Therefore, the optimal feasible point is reached. The optimal solution is $x_1 = 0$, $x_2 = 2/3$ and $x_3 = 0$ and the optimal value is $w = -z = 4/3$.

Example 2: Solve the following lpp by dual simplex method

Minimize $z = 6x_1 + 7x_2 + 3x_3 + 5x_4$

Subject to the constraints

$$5x_1 + 6x_2 - 3x_3 + x_4 \geq 12$$

$$x_2 + 5x_3 - 6x_4 \geq 10$$

$$2x_1 + 5x_2 + x_3 + x_4 \geq 8$$

$$x_1, x_2, x_3, x_4 \geq 0.$$