Moments and Moment Generating Function (mgf)

Moments:

Let X be a rv. Then its k^{th} moment about the mean is defined as

$$\mu_k = E(X - E(X))^k$$

$$= \begin{cases} \sum_x \left(x - E(X)\right)^k p(x) & ; \text{ if x is a drv} \\ \int_{-\infty}^{\infty} \left(x - E(X)\right)^k f(x)dx & ; \text{ if x is a crv} \end{cases}$$

The k^{th} moment about the origin is defined as

$$\mu'_k = E(X)^k$$

for k=2; $\mu_2 = V(X)$

Moment Generating Function (mgf)

Let X be a rv then the Moment Generating Function (mgf) is defined as

$M_X(t) = E(e^{tX})$ where t is a real variable

$$= \begin{cases} \sum_{x=0}^{\infty} e^{tx} p(x) ; & \text{ if x is a drv} \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx & ; \text{ if x is a crv} \end{cases}$$

$$M_X(t) = E(e^{tX})$$

$$= E\left[1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots + \frac{t^nX^n}{n!} + \cdots\right]$$

$$= E(1) + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^n}{n!}E(X^n) + \cdots$$

$$\therefore M_X(t) = 1 + t\mu'_1 + \frac{t^2}{2!}\mu'_2 + \cdots + \frac{t^n}{n!}\mu'_n + \cdots$$

i.e. the n^{th} order moment about the origin of any distribution is obtained as coefficient of $\left(\frac{t^n}{n!}\right)$ is to expansion of its mgf.

Differentiating w.r.t ‘t’ we get,

$$M'_X(t) = \mu'_1 + t\mu'_2 + \frac{t^2}{2!}\mu'_3 + \cdots + \frac{t^{n-1}}{(n-1)!}\mu'_n + \cdots$$

put t=0; $M'_X(0) = \mu'_1 = E(X)$

Differentiating again

$$M_X''(t) = \mu_2' + t\mu_3' + \cdots + \frac{t^{n-2}}{(n-2)!}\mu_n' + \cdots$$

put t=0; $M_X''(0) = \mu_2'$

$$\vdots$$

$$M_X^n(0) = \mu_n'$$

i.e. the n^{th} moment about the origin can be obtained by differentiating its mgf n times and putting $t = 0$.

Theorem:

If the rv X has the mgf M_X and Y is the rv $Y = \alpha X + \beta$, then mgf of Y is given by

$M_Y(t) = e^{\beta t} M_X(\alpha t)$, where α, β are constant.

Prove it!

Result:

If X and Y are two rvs with mgfs $M_X(t)$ and $M_Y(t)$ and if $M_X(t) = M_Y(t)\ \forall\ t$. Then their probability distribution must be the same.

Theorem:

Let X and Y be the two independent rvs and let Z=X+Y, (sum of two independent rvs) then mgf of Z is given by

$M_Z(t) = M_X(t)M_Y(t)$

Prove it!

Generalization: Let $X_1, X_2 \ldots, X_n$ be n independent rvs. Let $Z = \sum_{i=1}^{n} X_i$, then

$M_Z(t) = \prod_{i=1}^{n} M_{X_i}(t)$

Now, obtain the MGFs of the following distributions:

1. Binomial distribution
2. Poisson distribution
3. Normal distribution
4. Chi Square distribution
5. Gamma distribution

Also obtain the mean and the variance of all the above distributions using their mgfs.

Problems

1. Let X be the outcome when a fair die is rolled. Find the mgf of X and hence obtain its mean and variance.

   Solve it!

2. Suppose that the mgf of a rv X is of the form $M_X(t) = (0.4e^t + 0.6)^8$. Find the mgf of the rv Y=3X+2 and also obtain E(X) using the mgf.

   Solve it!

3. Let the rv X has the pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda(x-\alpha)}; & x \geq a \\ 0; & \text{Otherwise} \end{cases}$$

   find mgf of X and hence mean and variance.

   Solve it!

4. A rv X has a pdf $f(x) = \frac{1}{2}e^{-|x|}; -\infty < x < \infty$. Obtain the mgf of X and hence the mean and the variance.

   Solve it!

5. Find the mgf of a rv X which is uniformly distributed over [-a,a] and hence evaluate $E(X^{2n})$?

   Solve it!

6. A rv X follows $X \sim N(\mu, \sigma^2)$. Show that $E(X-\mu)^{2n} = 1.3.5. \ldots (2n-1)\sigma^{2n}\ ;\ n = 1,2, \ldots.$

   Solve it!

7. A rv X takes values 0,1,2,3 etc.. with $p(x)=ab^x$ where a and b are positive and a+b=1. Find the mgf of X. If $E(X)=m_1$ and $E(X^2)=m_2$, show that $m_2=m_1(2m_1+1)$

   Solve it!

Reproductive property (Additive property)

If two or more independent rvs having certain distribution are added, then the resulting rv also will have a distribution of the same type as its summands. This is known as the reproductive property.

Some of the probability distribution exhibit this property and some do not.

Check whether the following distributions exhibit the reproductive property

1. Binomial distribution
2. Poisson distribution
3. Normal distribution
4. Chi Square distribution
5. Exponential distribution