Post Optimal Analysis (or Sensitivity Analysis):

This analysis is to study the effect of changes in the parameters of the given lpp on the current optimal solution. Some typical situations are the study of the impact of changes in the profit (cost) coefficients in the objective function or increase/decrease of the resource levels in the constraints of the lpp. This analysis can be carried out from the optimal simplex table of the given lpp.

Variation/Changes in the cost coefficients (c_i's) in the objective function:

Example 1: Maximize $z = 12x_1 + 3x_2 + x_3$

$$\begin{aligned} \text{Subject to} \quad 10x_1 + 2x_2 + x_3 &\leq 100 \\ 7x_1 + 3x_2 + 2x_3 &\leq 77 \\ 2x_1 + 4x_2 + x_3 &\leq 80 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Analysis:

The Optimal table of the above lpp will be as in the following:

			12	3	1	0	0	0
C_B	x_B	x_{Bi}	x_1	x_2	x_3	x_4	x_5	x_6
12	x_1	73/8	1	0	-1/16	3/16	-1/8	0
3	x_2	35/8	0	1	13/16	-7/16	5/8	0
0	x_6	177/4	0	0	-17/8	11/8	-9/4	1
	$z_j - c_j$	z = 981/8	0	0	11/16	15/16	3/8	0

The optimal solution is $x_1 = 73/8$, $x_2 = 35/8$ and $x_3 = 0$ and the optimal value is $z = 981/8$. We will do the sensitivity analysis for the cost coefficients corresponding to the non-basic variables and that of basic variables separately.

Case (i) : Change of cost co-efficients corresponding to non-basic variables

Here, the non-basic (decision) variable in the optimal table is x_3 and its value is 0. Let c_3 be the cost or profit coefficient of this variable x_3 and let c_3' be the new cost coefficient of x_3. Then, the present optimal solution will remain unchanged as long as $z_3 - c_3' \geq 0$. That is,

$$z_3 - c_3 + c_3 - c_3' \geq 0$$

i.e., $11/16 + c_3 - c_3' \geq 0$

i.e., $11/16 + 1 - c_3' \geq 0$

i.e., $c_3' \leq 27/16$.

That is, till the new cost of x_3 (c_3') $\leq 27/16$, the present optimal solution will remain unchanged. It means that the cost of c_3 can be increased maximum upto the level of 27/16.

Case (ii) : Change of cost co-efficients corresponding to basic variables

Let us take the cost co-efficient corresponding to the basic variable x_1 and let the changes in the corresponding cost coefficient be $\Delta_1 = c_1' - c_1 = c_1' - 12$. Let us divide each value of $z_j - c_j$ of non-basic variable by the corresponding coefficients in the x_1 row. Here, it is $\{ (z_j - c_j)/y_{1j}$, j=3,4,5$\}$. Then, the present optimal solution will remain unchanged so long as

$$- \min_j \{ (z_j - c_j)/y_{1j} \;,\; y_{1j} > 0\} \leq \Delta_1 \leq \min_j \{ (z_j - c_j)/-y_{1j} \;,\; y_{1j} < 0\}.$$

$$- \min \{(z_4 - c_4)/y_{14} \;, y_{14} > 0\} \leq \Delta_1 \leq \min \{(z_3 - c_3)/-y_{13} \;, (z_5 - c_5)/-y_{15} \}$$

$-\ (15/16)\,/\,(3/16) \le \Delta_1 \le \min\{(11/16)\,/\,(1/16)\ ,\ (3/8)\,/\,(1/8)\}$

$-\ 5 \le \Delta_1 \le \min\{11, 3\}$

$-\ 5 \le \Delta_1 \le 3$ i.e., $-5 \le c_1' - 12 \le 3$ i.e., $7 \le c_1' \le 15.$

It means that the cost coefficients of the basic variable x_1 (c_1) can be decreased upto the level of 7 and increased upto 15 without changing the present optimality. Similarly, the analysis can be done for the cost coefficient corresponding to another basic variable x_2. The range of c_2 can be calculated from

$-\min j\ \{(z_j - c_j)/y_{2j}\ ,\ y_{2j} > 0\} \le \Delta_2 \le \min j\ \{(z_j - c_j)/-y_{2j}\ ,\ y_{2j} < 0\}$ where $\Delta_2 = c_2' - c_2 = c_2' - 3$. The range for c_2' will be $(12/5) \le c_2' \le (36/7)$. That is, within this range, the cost corresponding to the basic variable x_2 can be varied so that the present optimal solution will remain optimal.

Example 2: Solve the following lpp and find the sensitivity of the cost coefficients corresponding to the decision variables

Maximize $z = 2x_1 + 4x_2 + 3x_3$

Subject to

$$x_1 + 4x_2 + 3x_3 \le 240$$

$$2x_1 + x_2 + 5x_3 \le 300$$

$$x_1, x_2, x_3 \ge 0.$$

Case (ii): Changes in the resource levels (b_i's):

Whenever there is change in the resource level in a lpp, then it may affect the feasibility of the present optimal solution of that lpp. Therefore, we have to find the range of the resource levels in which the current optimum will remain feasible.

Example 1: Solve the following lpp and find the ranges of b_1 and b_2 for which the optimum solution will remain feasible.

Maximize $z = 2x_1 + 4x_2 + 3x_3$

Subject to $x_1 + 4x_2 + 3x_3 \le 240$

$2x_1 + x_2 + 5x_3 \le 300$

$x_1, x_2, x_3 \ge 0.$

Solution: The optimal simplex table to the above lpp will be as given here:

			2	4	3	0	0
C_B	x_B	x_{Bi}	x_1	x_2	x_3	x_4	x_5
4	x_2	180/7	0	1	1/7	2/7	-1/7
2	x_1	960/7	1	0	17/7	-1/7	4/7
	$z_j - c_j$	z = 2640/7	0	0	17/7	6/7	4/7

Please note that the basic variables are always taken as $x_{Bi} = B\,\bar{b}$ where B is matrix consisting of columns corresponding to the initial basic feasible variables.

Here, $x_{Bi} = B\,\bar{b} = \begin{pmatrix} 2/7 & -1/7 \\ -1/7 & 4/7 \end{pmatrix} \begin{pmatrix} 240 \\ 300 \end{pmatrix} = \begin{pmatrix} 180/7 \\ 960/7 \end{pmatrix}$. The optimal solution is $x_2 = 180/7$, $x_1 = 960/7$, $x_3 = 0$ and the optimal value is $z = 2640/7$. If the initial resource level is changed to $\bar{b} + \Delta\bar{b}$, then the present optimal solution may change as $x_{Bi} = B\,(\bar{b} + \Delta\bar{b})$ where $B = \begin{pmatrix} 2/7 & -1/7 \\ -1/7 & 4/7 \end{pmatrix}$.

But, the change in $\bar{b}$ does not affect the nature of $z_j - c_j$. Here, let us take the new resource level as, say, $\begin{pmatrix} 300 \\ 400 \end{pmatrix}$. Then, the new optimal solution is $x_{Bi} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} =$

$$\begin{pmatrix} 2/7 & -1/7 \\ -1/7 & 4/7 \end{pmatrix} \begin{pmatrix} 300 \\ 400 \end{pmatrix} = \begin{pmatrix} 200/7 \\ 1300/7 \end{pmatrix}$$ and the optimal value is z = 3400/7.

The ranges for which the new optimal solution will remain feasible can be calculated using the following relation

$$\max \{- x_{Bi} / b_{ik} \,,\ b_{ik} > 0\} \le \Delta b_k \le \min \{ - x_{Bi} / b_{ik} \,,\ b_{ik} < 0 \}$$

where b_{ik} are the elements in the matrix B.

<u>For changes in b_1</u>:

$$\max \{- x_{Bi} / b_{i1} \,,\ b_{i1} > 0\} \le \Delta b_1 \le \min \{ - x_{Bi} / b_{i1} \,,\ b_{i1} < 0 \} \quad \text{----------} \ (1)$$

Here, $B = \begin{pmatrix} 2/7 & -1/7 \\ -1/7 & 4/7 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

Using equation (1), we can write

$$\max \{-x_{B1}/b_{11}\} \le \Delta b_1 \le \min \{ -x_{B2}/b_{21} \}$$

i.e., $\max \ \{-(180/7) / (2/7)\} \le \Delta b_1 \le \min \{ -(960/7) / -(1/7)\}$

i.e., $- 90 \le \Delta b_1 \le 960$

i.e., $- 90 \le$ new b_1 - old $b_1 \le 960$

i.e., $- 90 \le$ new $b_1 - 240 \le 960$

i.e., $150 \le$ new $b_1 \le 1200$

Hence, within this range of [150 , 1200] for the resource level b_1, the optimal solution will remain feasible.

Similarly, we can find out the range for the resource level b_2 within which the optimal solution will remain feasible.

Example 2: Solve the following lpp and determine the ranges of the components b_1, b_2 and b_3 in which the feasibility of the current optimal solution will be maintained. Also, find the new optimal solution when b_2 is changed to 8 and b_3 is changed to 5:

$$\text{Maximize} \quad z = -x_1 + 2x_2 - x_3$$

$$\begin{aligned} \text{Subject to} \quad 3x_1 + x_2 - x_3 &\leq 10 \\ -x_1 + 4x_2 + x_3 &\geq 6 \\ x_2 + x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Example 3: Solve Maximize $z = 50x_1 + 60x_2 + 20x_3$

$$\begin{aligned} \text{Subject to} \quad 5x_1 + 4x_2 + 2x_3 &\leq 80 \\ 3x_1 + 6x_2 + x_3 &\leq 70 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

and (i) do the post optimal analysis corresponding to the changes in the cost coefficients and the changes in the resource levels. (ii)If the cost coefficient c_3 is changed to 30, what will be the effect on the optimality? (ii)If the resource levels b_1 is changed 100 and b_2 is changed to 40 . what will be the new optimal solution , if it exists?