Problems:

1. A coin is known to come up heads three times as often as tails. This coin is tossed three times. Let X be the number of heads that appear. Writeout the probability distribution of the rv X. Also obtain the cdf of X.

Solution: Given that P(H) = ¾ =p $\Rightarrow$ P(T) = ¼ = q and n=3,

Let X = No. of heads; (taking values {0,1,2,3}).

Clearly, X~B(n,p). Thus we have,

$P(X = x) = \binom{n}{x}.p^x.\ q^{n-x}$; x=0,1,2,3…n; p+q=1

$P(X = 0) = \binom{n}{0}.p^0.\ q^n = 1(1/4)^3 = 1/64$

$P(X = 1) = \binom{n}{1}.p^1.\ q^{n-1} = 3(3/4)(1/4)^2 = 9/64$

$P(X = 2) = \binom{n}{2}.p^2.\ q^{n-2} = 3(3/4)^2(1/4) = 27/64$

$P(X = 3) = \binom{n}{3}.p^3.\ q^0 = 1(3/4)^3(1) = 27/64$

Now, $F(x) = P(X \le x)$

$= \sum_{(j;\ x_j \le x)} p(x_j)$

$$F(x) = \begin{cases} 0\ ; & x < 0 \\ \frac{1}{64}; & 0 \le x < 1 \\ \left(\frac{1}{64}\right) + \left(\frac{9}{64}\right) = \frac{10}{64}; & 1 \le x < 2 \\ \left(\frac{10}{64}\right) + \left(\frac{27}{64}\right) = \frac{37}{64}; & 2 \le x < 3 \\ \left(\frac{37}{64}\right) + \left(\frac{27}{64}\right) = 1; & x \ge 3 \end{cases}$$

2. Given $f(x) = \begin{cases} kx^3; & 0 < x < 1 \\ 0; & \text{elsewhere} \end{cases}$

   Find k so that the above is a pdf and hence find

   (i) P(1/4 < X < 3/4)

   (ii) P(X < 1/2)

   (iii) P(X > 0.8)

   (iv) CDF of X.

Solution: We have $\int_{-\infty}^{\infty} f(x)dx = 1$.

Here $\int_0^1 f(x)dx = 1 \Rightarrow \int_0^1 k\,x^3 dx = 1 \Rightarrow [k\,x^4/4]_0^1 = 1 \Rightarrow k/4 = 1$ or k=4.

(i). $P(1/4 < X < 3/4) = \int_{1/4}^{3/4} 4x^3\, dx = [4\,x^4/4]_{1/4}^{3/4} = (3/4)^4 - (1/4)^4 = 80.$

(ii). $P(X < 1/2) = \int_0^{1/2} 4x^3\, dx = [4\,x^4/4]_0^{1/2} = (1/2)^4 = 1/16.$

(iii). $P(X > 0.8) = \int_{0.8}^{1} 4x^3 dx = [4\,x^4/4]_{0.8}^{1} = 1-(0.8)^4 = 0.5904.$

(iv). $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$

For $x \leq 0$, $f(x) = 0 \Rightarrow F(x) = 0$.

For $0 < x < 1$, $F(x) = \int_{-\infty}^{0} f(t)dt + \int_0^x f(t)dt = 0 + \int_0^x 4t^3\, dt = \frac{4t^4}{4}]_0^x = x^4$

For $x \geq 1$, $F(x) = \int_{-\infty}^{0} f(t)dt + \int_0^1 f(t)dt + \int_1^x f(t)dt = 0 + \int_0^1 4t^3 + 0 = [t^4]_0^1 = 1.$

Hence, $F(x) = \begin{cases} 0; & x \leq 0 \\ x^4; & 0 < x < 1 \\ 1; & x \geq 1 \end{cases}$

3. A rv X assumes 4 values with probabilities (1+3x)/4, (1-x)/4, (1+2x)/4 and (1-4x)/4. For what range of values of x is this a probability distribution?

   Solve it!

4. Suppose that the rv X has possible values 1,2,3,4,……..and

   $P(X = j) = 1/2^j$, j=1,2,3,…………

   Compute: a) P(X is even) b) $P(X \geq 5)$ c) P(X is divisible by 3)

   Solve it!

5. Let X be a crv with pdf given by

$$f(x) = \begin{cases} ax\,; & 0 \le x \le 1 \\ a; & 1 \le x \le 2 \\ -ax + 3a; & 2 \le x \le 3 \\ 0; & \text{elsewhere} \end{cases}$$

(a) Determine the constant ‘a’

(b) Obtain the cdf F(x).

Solve it!

6. The diameter on an electric cable, say X, is assumed to be a crv with pdf,

$f(x) = 6x(1-x)\ ;\ 0 < x < 1$.

(a) Check whether the above f(x) is a pdf

(b) Obtain the cdf of X

(c) Determine a number ‘b’ such that $P(X > b) = 2\,P(X > b)$

(d) Compute $P[(X \le 1/2) / (1/3 < X < 2/3)]$.

Solve it!

7. Suppose that X is a uniformly distributed rv, over the interval (-a, +a) where a > 0. Determine ‘a’, wherever possible, so that the following are satisfied:

(a) $P(X > 1) = 1/3$

(b) $P(X < 1) = 1/2$

(c) $P(X < 1/2) = 0.7$

Solve it!

8. Let the rv K be uniformly distributed over the interval [0, 5]. What is the probability that the roots of the $4x^2 + 4xk + k + 2 = 0$ are real?

Solution: Since K is uniformly distributed,

We have $f(k) = \begin{cases} \frac{1}{5}; & 0 \le k < 5 \\ 0; & \text{elsewhere} \end{cases}$

To find: P(the roots of the equation $4x^2 + 4xk + k + 2 = 0$ are real)=?

Now, the roots of the equation $4x^2 + 4xk + k + 2 = 0$ are given by $\frac{-4 \pm \sqrt{16k^2 - 16(k+2)}}{8}$

For the roots to be real, we must have, $16k^2 - 16(k+2) \ge 0 \Rightarrow k^2 - k - 2 \ge 0$

$\Rightarrow (k - 2)(k + 1) \ge 0$

$\therefore$ Required probability = $P((k - 2)(k + 1) \ge 0)$=?

Now, we have 2 possibilities:

(i) $k-2 \ge 0$ and $(k + 1) \ge 0$ implies $k \ge 2$ and $k \ge -1 \Rightarrow k \ge 2 \Rightarrow k \in (2, 5)$.

Hence, probability of $k \in (2, 5) = P(2 \le k \le 5) = \int_2^5 1/5 \, dk = 3/5$.

OR

(ii) When $k-2 \le 0$ and $k+1 \le 0$ implies $k \le 2$ and $k \le -1. \Rightarrow k \le -1 \Rightarrow k \in (-\infty,-1)$.

Hence, $P(k \in (-\infty,-1)) = P(-\infty \le k \le -1) = 0$

$\therefore$ Required probability = 3/5.