Theorem1:

If ϕ is the Null event, then $P(\phi)=0$

Proof: For any event A, $A=A\cup\phi$, where A and ϕ are mutually exclusive.

$P(A)=P(A\cup\phi)=P(A)+P(\phi)$

$\Rightarrow P(A)=P(A)+P(\phi)$

$\Rightarrow P(\phi)=0$

Theorem2:

If $\bar{A}$ is the complementary event of A, then $P(A) = 1 - P(\bar{A})$

Proof:

If S is the Sample Space, with A and $\bar{A}$ being disjoint events then

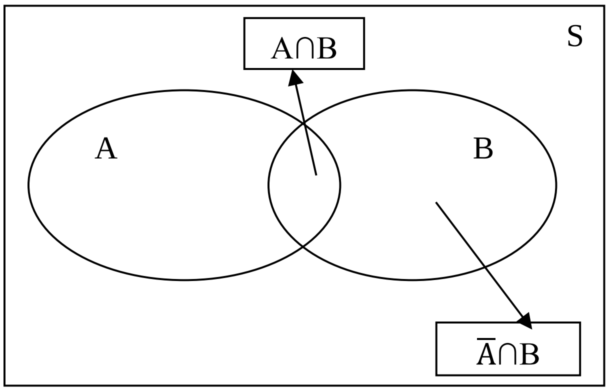
$S = A\cup\bar{A}$

$P(S) = P(A\cup\bar{A}) \Rightarrow 1=P(A)+P(\bar{A})$

Or $P(A)= 1-P(\bar{A})$

Cor: If A & B are any two events, then $P(\bar{A}\cap B)=P(B)-P(A\cap B)$

Proof:



Since $\bar{A}\cap B$ and $A\cap B$ are disjoint events, we can write

$B=(A\cap B)\cup(\bar{A}\cap B)$

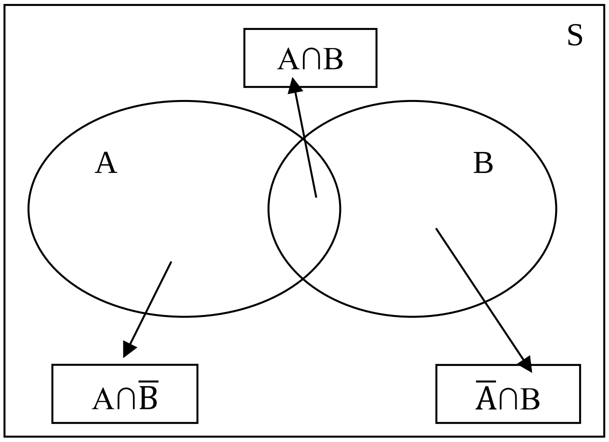
$P(B)=P(A\cap B)+P(\bar{A}\cap B)$

Or $P(\bar{A}\cap B)=P(B)-P(A\cap B)$

Theorem 3: (Addition Theorem)

If A and B are any 2 events, then $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

Proof:



We can write

$A\cup B=A\cup(B\cap\bar{A})$

$P(A\cup B)=P(A)+P(B\cap\bar{A}) \longrightarrow (1)$

$B=(A\cap B)\cup(B\cap\bar{A})$

$P(B)=P(A\cap B)+P(B\cap\bar{A}) \longrightarrow (2)$

Subtracting (2) from (1), we get

$P(A\cup B)-P(B)=P(A)-P(A\cap B) \Rightarrow P(A\cup B)=P(A)+P(B)-P(A\cap B)$

Aliter:

Let $A\cup B=A\cup(B\cap\bar{A})$

$P(A\cup B)=P(A)+P(B\cap\bar{A})$

Add and Subtract $P(A\cap B)$, we get

$P(A\cup B)=P(A)+(P(B\cap\bar{A})+P(A\cap B))-P(A\cap B)$

$\Rightarrow P(A\cup B)=P(A)+P(B)-P(A\cap B)$

Theorem 4:

For any three events A, B and C, prove that

$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap C)-P(A\cap B)-P(B\cap C)+P(A\cap B\cap C)$

Prove it !

Theorem 5:

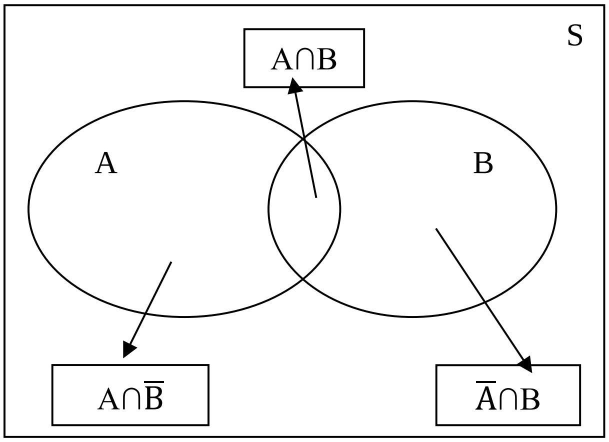
If $A\subset B$ then $P(A)\leq P(B)$

Prove it !

Results:

1) Show that the probability (that exactly one of the events A or B occurs) is given by $P(A)+P(B)-2P(A\cap B)$

Proof:



For any two events A & B,

Probability (that exactly one of the events A or B occurs) = $P((A\cap \bar{B})U(B\cap \bar{A})]$

Since $A\cap \bar{B}$ and $B\cap \bar{A}$ are mutually exclusive, we have

$P((A\cap \bar{B})U(B\cap \bar{A})]=P(A)+P(B)-2P(A\cap B)$

$\Rightarrow$ $P((A\cap \bar{B})U(B\cap \bar{A})]= P(A\cap \bar{B})+P(B\cap \bar{A})$ --------------(a)

But $A\cap B$ is disjoint with both these sets and the union of the events $A\cap \bar{B}$ and $B\cap \bar{A}$ and $A\cap B$ is nothing but AUB.

Add & Subtract $P(A\cap B)$ in (a) above, we get,

$$
\begin{aligned}
P[(A\cap \bar{B})U\ (B\cap \bar{A})] &= P(A\cap \bar{B})+P(B\cap \bar{A})+ P(A\cap B)- P(A\cap B)\\
&= [P(A\cap \bar{B})+P(B\cap \bar{A})+ P(A\cap B)]- P(A\cap B)\\
&= P(AUB)-P(A\cap B)\\
&= P(A)+P(B)-P(A\cap B)-P(A\cap B)
\end{aligned}
$$

$\therefore$ The required probability = $P(A)+P(B)-2P(A\cap B)$

2) For any two events A & B, $P(A\cap B)\leq P(A)\leq P(AUB)\leq P(A)+P(B)$

Prove it !