Transition Probability Matrix (TPM):

The transition probabilities p_{ij} satisfies:

$p_{ij} \geq 0 \; \forall \; i, j$

$\sum_{j \in s} p_{ij} = 1 \; \forall \; i$

These probabilities may be conveniently written in the matrix form as,

$$P = \begin{array}{c} \\ i=0 \\ i=1 \\ \vdots \\ i=n \end{array} \begin{array}{c} \begin{array}{cccc} j=0 & j=1 & \ldots & j=n \end{array} \\ \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0n} \\ p_{10} & p_{11} & \cdots & p_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n0} & p_{n1} & \cdots & p_{nn} \end{bmatrix} \end{array}$$

This 'P' is called the homogenous Transition Probability Matrix or simply TPM. This P is a Stochastic Matrix or Markov Matrix. [i.e. a square matrix with non-negative elements and unit row sums]. In addition, if the column sum is also unity, it is then called a Doubly Stochastic Matrix.

Similarly, for m- step transition probability we have.

$$P^{(m)} = \begin{array}{c} \\ i=0 \\ i=1 \\ . \\ . \\ i=n \end{array} \begin{array}{c} \begin{array}{cccc} j=0 & j=1 & \ldots & j=n \end{array} \\ \begin{bmatrix} p_{00}^{(m)} & p_{01}^{(m)} & \ldots & p_{0n}^{(m)} \\ p_{10}^{(m)} & p_{11}^{(m)} & \ldots & p_{1n}^{(m)} \\ \vdots & \vdots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \vdots \\ p_{n0}^{(m)} & p_{n1}^{(m)} & \ldots & p_{nn}^{(m)} \end{bmatrix} \end{array}$$

Order of a Markov Chain

A Markov Chain $\{ X_n ; n \in T \}$ is said to be of order s; s=1,2,3…..n,

if $\forall$ n, $P[X_n=j / X_{n-1}=i, \ldots, X_{n-s}=i_{s-1}, \ldots] = P[X_n=j / X_{n-1}=i, \ldots, X_{n-s}=i_{s-1}]$

And a Markov Chain is said to be order 1 if above is $P[X_n=j / X_{n-1}=i]$ i.e s=1

Finite State Markov Chain

A Stochastic process {Xn ; n $\in$ T } is said to be a finite state Markov Chain if it has the following properties.

1. A finite number of states
2. Markovian property
3. Stationary or time homogeneous transition probabilities
4. Set of initial probabilities $P(X_0 = i) \; \forall \; i$

Thus, a tpm together with the initial probabilities associated with the states completely describes a Markov chain.

Note: The joint probability distribution of the stochastic process is given by,

$P\{X_0= i_0, X_1= i_1,\ldots., X_n= i_n\}$

$= P(X_0= i_0)\, P(X_1= i_1/ X_0= i_0)\, P(X_2= i_2/ X_0= i_0, X_1= i_1)\ldots.\, P(X_n= i_n/ X_0= i_0\ldots., X_{n-1}= i_{n-1})$

$= P(X_0= i_0)\, P(X_1= i_1/ X_0= i_0)\, P(X_2= i_2/X_1= i_1)\ldots.\, P(X_n= i_n/X_{n-1}= i_{n-1})$ ($\because$ Markovian Property)

The first factor on RHS i.e. $P(X_0= i_0)$ is called the initial probability distribution of the chain and the remaining factors are the conditional probabilities which are the one step transition probabilities of the Markov Chain.

Hence, to be able to write down probability law, we need to know initial probability distribution of the chain as well as all one step transition probabilities.

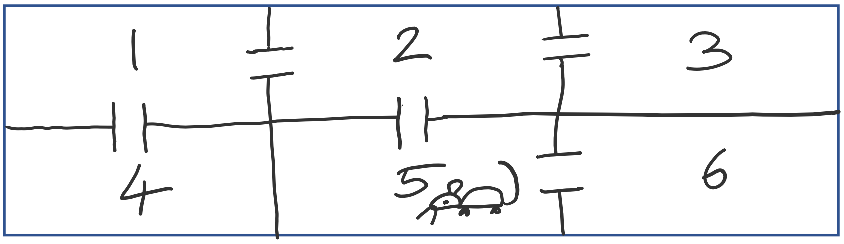
Problems

1) What is the TPM of a Markov Chain $\{Z_n\}$, where each Z_n is independently distributed as the rv Z, which has the distribution $P(Z=k) = p_k$?

Solution: Let $p_{ij} = P[Z_1=j/ Z_0=i] = P(Z_1=j) = p_j$

$$\therefore\ T = \begin{array}{c} \\ i=0 \\ i=1 \\ . \\ . \\ i=n \end{array} \begin{array}{c} \begin{array}{cccc} j=0 & j=1 & \ldots & j=n \end{array} \\ \begin{bmatrix} p_0 & p_1 & \cdots & p_n \\ p_0 & p_1 & \cdots & p_n \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ p_0 & p_1 & \cdots & p_n \end{bmatrix} \end{array}$$

2) A rat is putting a maze as shown in the figure. At each time instant it changes its room choosing its exit at random. What is the TPM of the Markov Chain $\{Z_n\}$, where Z_n is the room the rat is occupying during the interval (n,n+1) ?



Solution:

$$\therefore\ T = \begin{array}{c} \\ i=1 \\ i=2 \\ i=3 \\ i=4 \\ i=5 \\ i=6 \end{array} \begin{array}{c} \begin{array}{cccccc} j=1 & j=2 & j=3 & j=4 & j=5 & j=6 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

3) A factory has two machines, but on any given day not more than one is in use. This machine has a constant probability 'p' of failure and if it fails the breakdown occurs at the end of the day's work. A single repairman is employed. It takes him two days to repair a machine and he works on only one machine at a time. Construct a stochastic process which will describe the working of the factory (i.e determine the state space and parameter space). List the possible transitions between the states of the factory and obtain the tpm of the Markov Chain $\{X_n\}$ where X_n is the number of days that would be needed to get both the machines back in working order and X_n is recorded at the end of day n. Also write the transition diagram.

Solve it!