First Passage Time:

It is often desirable to make probabilistic statements about number of transitions made by the process in going from state i to state j for the first time. This length of time is called First Passage Time in going from state i to state j.

When j=i, this first passage time is just the number of transitions until the process returns to the initial state i. In this case the first passage time is called the First Return Time or Recurrence Time for state i.

The first passage times are in general are rvs and hence have probability distributions associated with them. These probability distributions depend upon the transition probabilities of the process.

Recurrent State (Persistent State):

A state i is said to be a Recurrent or Persistent if a return to state i is certain. i.e. a process from state i returns to state i with probability 1.

Transient State:

A state i is said to be Transient if a return to state i is uncertain. i.e. once the process is in state i it ever returns to state i.

Let μ_{ij} denote the Expected First Passage Time from state i to state j.

If j=i, then μ_{ii} is the Expected Recurrence Time.

A recurrent state is called Null-recurrent, if $\mu_{ii} = \infty$ and Non-null or Positive recurrent if μ_{ii} is finite ($\mu_{ii} < \infty$).

In a finite state Markov Chain there are no Null-recurrent states.

Periodic State:

A state is to be periodic with period t (t >1) if a return is possible only at t,2t,3t,… etc. steps, else Aperiodic or Non-periodic (say period 1 i.e. t=1).

A recurrent state is said to be Ergodic if it is Non-null and Aperiodic.

Note: A Markov chain is said to be Ergodic if it has Ergodic states.

Regular Matrix:

A stochastic matrix is said to be regular if some positive power of the matrix contains only positive entries. If there are zeros in the matrix they are eliminated during exponentiation.

Whenever a TPM is regular, we say that the chain is regular.

Result: If the TPM of a Markov chain is regular then the Markov chain is ergodic.

Rule to find whether a TPM is regular or not:

Suppose the TPM of a Markov chain is of order 'm' then the highest power to which the TPM must be raised to check whether it is regular or not is $m^2 - 2m + 2$.

Problems

1) Is the following matrix regular?

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution:

Here P is not a stochastic matrix. Hence it is not a tpm. $\therefore$ P is not regular

2) Is the following matrix regular?

$$P = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

Solution:

Here $m^2 - 2m + 2 = 2^2 - 4 + 2 = 2$. So we raise P upto P^2.

Consider $P^2 = P.P = \begin{bmatrix} 1 & 0 \\ 3/4 & 1/4 \end{bmatrix}$ $\therefore$ P is not regular

3) Are the following matrices regular?

a) $P = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

Solve it!

b) $P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$

Solve it!

4) Show that the matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & 0 & 1-a \end{bmatrix}$ is regular for $0 < a < 1$

Solve it!