Further results associated with Poisson process

Result 1: The interval between two successive occurrences of a Poisson process {N(t)} having a parameter λt has a negative exponential distribution with mean $\frac{1}{\lambda}$

Result 2: Further it can be shown that X_1, X_2, X_3,….., the intervals between successive occurrences of events E_i and E_{i+1}, i= 1,2,3,….., for a Poisson process {N(t)} are all independent and have identical exponential distribution with mean $\frac{1}{\lambda}$.

Result 3: If the intervals between successive occurrences of events E_i are independently distributed with common exponential distribution with mean $\frac{1}{\lambda}$, then the events E_i form a Poisson process with mean λt.

Memoryless property or Forgetfulness property of exponential distribution:

The exponential distribution has the property that the time until the next occurrence of event is independent of the time that elapsed since the occurrence of the last event i.e. the future is independent of past i.e. the process forgets its past history.

Mathematically, if T is the random variable (Inter occurrence time which is exponentially distributed) then $P(T>t+s/T>s) = P(T > t)$, where s is the occurrence time of the last event.

Proof : Given $T \sim \text{Exp}(\alpha) \Longrightarrow f(t) = \alpha e^{-\alpha t}$; $t > 0$

and $F(t) = P(T \le t) = 1 - e^{-\alpha t} \Longrightarrow P(T > t) = e^{-\alpha t}$

Consider $P(T>t+s/T>s) = \frac{P(T>t+s, T>s)}{P(T>s)} = \frac{P(T>t+s)}{P(T>s)} = \frac{e^{-\alpha(t+s)}}{e^{-\alpha s}} = e^{-\alpha t} = P(T > t)$

$\therefore \; P(T>t+s/T>s) = P(T > t)$

This property demonstrates that the process is completely random as it shows that the time that has elapsed since the occurrence of the last event has no effect on the time of occurrence of the next event.

Pure Birth Process

Consider situations where only arrivals take place i.e. the **customers join the queuing system but never leave**. In this case the arrivals may be thought of as the occurrence of events. Such a process is called a **Pure birth process or an Arrival process** (Here birth refers to the arrival of a new customer).

Example: State health department records the birth of new babies effective from a given date. i.e. Birth information for each baby is recorded.
Our objective is to obtain an expression for $P_n(t)$, the probability of n arrivals during an interval of length t (say (0, t]). It can be shown that,
$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$; n = 0,1,2…. ; $\lambda > 0$.
(where λ = arrival rate or rate of occurrence of events)

i.e. The distribution of $P_n(t)$ is Poisson with parameter λt.
It can be shown that the inter arrival times are exponentially distributed with parameter λ.

The above expression for $P_n(t)$ may be obtained under the assumptions or postulates made earlier. We may arrive at differential - difference equations, which may be solved using Laplace-transform techniques, to obtain $P_n(t)$.

Pure Death Process

Here we assume that the system starts with a given No. of customers, say N, who leave the facility at the rate μ after being serviced. **But no new customer is allowed to join the system.** Such a process is called a **pure death process** or **a departure process** (Here death refers to departure of a customer).

Example: Inventory situations may be modeled as pure death process. Say, the inventory consists of N items to start with and the items are withdrawn from the stock at a rate μ (units per unit time).
Let $q_n(t)$ = Probability of n departures during an interval of length t.
Analogous to pure birth process (with λ=μ here), solving the differential- difference equations we may show that

$q_n(t) = \frac{(\mu t)^n e^{-\mu t}}{n!}$; $\mu > 0$; n= 0,1,2,3,…..$N-1$

$q_N(t) = 1-\sum_{n=0}^{N-1} q_n(t)$; n = N

= P[N(all) customers departed during t]

i.e. The distribution of $q_n(t)$ is Poisson with parameter μt.
Also the inter departure times or service times are exponentially distributed with parameter μ.

Birth and Death Process

For a Pure Birth process we have

$p_k(h)$ = P[No. of births between t and t+h is k, given that the No. of births by epoch t is n]

= P[N(h)=k/N(t)=n]

$$= \begin{cases} \lambda h + O(h) & ; \ k = 1 \\ O(h) & ; \ k \geq 2 \\ 1 - \lambda h + O(h) & ; \ k = 0 \end{cases} \qquad (\lambda \text{ is the arrival rate}) \qquad \text{---------} \ (A)$$

Similarly, for a Pure Death process we have

$q_k(h)$ = P[No. of deaths between t and t+h is k, given that the No. of deaths by epoch t is n]

$$= \begin{cases} \mu h + O(h) & ; \ k = 1 \\ O(h) & ; \ k \geq 2 \\ 1 - \mu h + O(h) & ; \ k = 0 \end{cases} \qquad (\mu \text{ is the departure rate}) \qquad \text{-----------} \ (B)$$

With (A) and (B) above together we have a **birth and death process.**

Let N(t) = No. of occurrences by epoch t starting from t = 0

Let $p_n(t) = P[N(t)=n]$

Consider $p_n(t+h) = P[N(t+h)=n]$ = P[n occurrences by epoch t+h starting from t=0]

This can happen in the following mutually exclusive ways.

$p_n(t+h)$ = P[n occurrences by t and no births and no deaths during h]

+P[n-1 occurrences by t and one birth and no death during h]

+ P[n+1 occurrences by t and no birth and one death during h]

$= p_n(t)(1-\lambda h)(1-\mu h)+p_{n-1}(t)\lambda h+p_{n+1}(t)\mu h+O(h)$

for n > 0 we have $p_n(t+h)= p_n(t)(1-\lambda h-\mu h)+p_{n-1}(t)\lambda h+p_{n+1}(t)\mu h+O(h)$ --------------(1)

and for n=0, we have {Note: P[zero deaths during h] = 1},

$p_0(t+h)= p_0(t)(1-\lambda h).1+p_1(t)\, \mu h+O(h)$ ----------------(2)

From (1) and (2) we have,

$$\frac{p_n(t+h) - p_n(t)}{h} = -(\lambda + \mu)p_n(t) + \mu\, p_{n+1}(t) + \lambda\, p_{n-1}(t) + \frac{O(h)}{h}$$

$$\frac{p_0(t+h) - p_0(t)}{h} = -\lambda p_0(t) + \mu\, p_1(t) + \frac{O(h)}{h}$$

Taking limits as $h \to 0$ we get

$$p'_n(t) = -(\lambda + \mu) p_n(t) + \mu\, p_{n+1}(t) + \lambda p_{n-1}(t) \;;\; n > 0 \quad \text{-------------(3)}$$

$$p'_0(t) = -\lambda p_0(t) + \mu\, p_1(t) \;;\; n = 0 \quad \text{------------(4)}$$

Equations (3) and (4) are the differential-difference equations of the Birth and Death process which play a vital role in the study of queuing theory.