Assignment - 2 :

21 NOV 2020

Answer all the questions neatly and show the details of your work.

1. The $n \times n$ Hilbert matrix is defined as $H_n = [h_{ij}]$ where

$$h_{ij} = \frac{1}{i+j-1}, \quad \begin{matrix} 1 \le i \le n \\ 1 \le j \le n \end{matrix}.$$

Fact :- H_n is invertible for all $n \geq 1$. They are examples of "highly ill-conditioned" matrices.

Compute H_n^{-1} for $n=2$ and $n=3$.

All the entries in the inverse also to be rational numbers $\frac{a}{b}$, 'a' and 'b' integers (do not round off and use decimals). Use Gauss-Jordan method.

2. Write down the 3×3 matrices $A = [a_{ij}]$, $B = [b_{ij}]$ where $a_{ij} = i - j$ and $b_{ij} = \frac{i}{j}$.

Compute the products AB, BA and A^2.

3. Give examples of 2×2 matrices A with $a_{12} = \frac{1}{2}$ for which:

(a) $A^2 = I$ (b) $A^{-1} = A^T$ (c) $A^2 = A$.

__Bonus__ if you can find __all__ __such__ 'A'.

4. The zero matrix O has all entries $= 0$. Let A be $n \times n$. Show that it is possible to have $A \neq O$ but $A^2 = O$. However, show that if $A \neq O$, then $A^T A \neq O$ and $A A^T \neq O$.

[Try with A as 2×2 first. Then try the general case of $n \times n$].

5. Define the $n \times n$ Permutation matrices P and show that $P^{-1} = P^T$.

6. Under what conditions on their entries are A and B invertible?

$$A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

7. Suppose elimination fails because there is no pivot in column 3:

$$A = \begin{bmatrix} 2 & 1 & 4 & 6 \\ 0 & 3 & 8 & 5 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 9 \end{bmatrix}.$$

Show that A cannot be invertible. The third row of A^{-1}, multiplying A, should give the third row $[0\ 0\ 1\ 0]$ of $A^{-1}A = I$. Why is this impossible?

8. Find the inverses (in any legal way).

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}$$

i.e. $$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}.$$

9. Give examples of A and B such that

(a) $A+B$ is not invertible although A and B are invertible.

(b) $A+B$ is invertible although A and B are not invertible

(c) all of A, B, and $A+B$ are invertible.

10. In the last case (c) of the previous question, use the identity

$$A^{-1}(A+B)B^{-1} = B^{-1} + A^{-1}$$ to show

that $C = B^{-1} + A^{-1}$ is also invertible -

and find a formula for C^{-1}.

11. Use Gauss-Jordan method to invert the following matrices,

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \text{ and}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

12. If P is a any permutation matrix, find a nonzero vector "X" such that $(I-P)X=0$. Hence $I-P$ is not invertible.

13. For which numbers 'c' is $A = LU$ impossible – with 3 pivots?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

14. Find L and U for the non-symmetric matrix

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on a, b, c, d, r, s, t to get $A = LU$ with four pivots.

15. Solve $Lc = b$ to find 'c'. Then solve $Ux = c$ to find 'x'. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}; \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

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