Examples

1. Suppose the joint pmf of X and Y is given by $p(1,1) = 0.5$, $p(1,2) = 0.1$, $p(2,1) = 0.1$, $p(2,2) = 0.3$. Find the pmf of X given $Y = 1$.

   Solution:

   $$p_{X|Y=1}(1) = p(1,1)/p_Y(1) = 0.5/0.6 = 5/6$$
   $$p_{X|Y=1}(2) = p(2,1)/p_Y(1) = 0.1/0.6 = 1/6$$

2. If X and Y are independent Poisson RVs with respective means λ_1 and λ_2, find the conditional pmf of X given $X + Y = n$ and the conditional expected value of X given $X + Y = n$.

   Solution:

   Let $Z = X + Y$. We want to find $p_{X|Z=n}(k)$. For $k = 0, 1, 2, ..., n$

   $$\begin{aligned} p_{X|Z=n}(k) &= \frac{P(X = k, Z = n)}{P(Z = n)} \\ &= \frac{P(X = k, X + Y = n)}{P(Z = n)} \\ &= \frac{P(X = k, Y = n - k)}{P(Z = n)} \\ &= \frac{P(X = k)P(Y = n - k)}{P(Z = n)} \end{aligned}$$

We know that Z is Poisson with mean $\lambda_1 + \lambda_2$.

$$\begin{aligned}
p_{X|Z=n}(k) &= \frac{P(X=k, Z=n)}{P(Z=n)} \\
&= \frac{P(X=k)P(Y=n-k)}{P(Z=n)} \\
&= \frac{e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}}{e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^n}{n!}} \\
&= \binom{n}{k} \cdot \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \cdot \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}
\end{aligned}$$

Hence the conditional distribution of X given $X + Y = n$ is a binomial distribution with parameters n and $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

$$E(X|X+Y=n) = \frac{\lambda_1 n}{\lambda_1 + \lambda_2} \, .$$

3. Consider $n+m$ independent trials, each of which results in a success with probability p. Compute the expected number of successes in the first n trials given that there are k successes in all.

   Solution: Let Y be the number of successes in $n+m$ trials. Let X be the number of successes in the first n trials. Define

$$X_i = \begin{cases} 1 & \text{if the } i\text{th trial is a success} \\ 0 & \text{otherwise} \end{cases}$$