Roll No. 2025d005

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Assignment - 1

Que 1. In terms $P(A)$, $P(B)$, $P(C)$, $P(AB)$, $P(BC)$, $P(AC)$ & $P(ABC)$ express, for $K = 0, 1, 2, 3$ the probability that:

(i) Exactly K of the events A, B & C occur.
(ii) Atleast K of the events A, B & C occur.

→ Solution:-

(i) Let, $P_K = P$[exactly K of the events A, B, C occur] for $K = 0, 1, 2, 3$

then,

$P_0 = P$[None of events A, B, C occur]

$$\Rightarrow P_0 = P[A'B'C'] = P[(A \cup B \cup C)']$$

$$= 1 - P(A \cup B \cup C)$$
$$= 1 - P(A) - P(B) - P(C) + P(AB) + P(BC) + P(AC) - P(ABC) \quad -(1) \quad |K=0$$

Similarly,

$P_1 = P$[Exactly 1 event of the ~~of~~ A, B, C occurs]

$$P_1 = P[(AB'C' \cup A'BC' \cup A'B'C)]$$

$$= P[A(B \cup C)'] + P[B(C \cup A)'] + P[C(A \cup B)']$$

$$= P(A) - P[(AB \cup AC)] + P(B) - P[BC \cup BA] + P(C) - P[CA \cup CB]$$

$$= P(A) - P(AB) - P(AC) + P(ABC)$$
$$+ P(B) - P(BC) - P(BA) + P(ABC)$$
$$+ P(C) - P(CB) - P(CA) + P(ABC)$$

$$= [P(A) + P(B) + P(C)] + 3P(ABC)$$
$$- 2[P(AB) + P(AC) + P(BC)] \quad /K=1$$
$$— ②$$

$P_2 = P$[exactly 2 of events A, B, C occur]

$$= P[ABC' \cup AB'C \cup A'BC]$$

$$= P(ABC') + P(ACB') + P(BCA')$$

$$= P(AB) - P(ABC) + P(AC) - P(ABC)$$
$$+ P(BC) - P(ABC)$$

$$P_2 = [P(AB) + P(BC) + P(AC)] - 3P(ABC) \quad — ③$$
$$/K=2$$

& $P_3 = P$[exactly 3 of events A, B, C occur]

$$\boxed{P_3 = P(ABC)} \quad /K=3$$

$$— ④$$

(ii) P[atleast K of events A, B, C occur]

$$= \sum_{n=K}^{3} P_n \qquad K = 0, 1, 2, 3$$

~~P[a~~None of events A, B, C occur]

$$\cancel{= \sum_{n=0}^{3}}$$

for $K=0$, we have,

P[None of the events A, B, C occur]
+P[1 of events A, B, C occur]
+P[2 of events A, B, C occur]
+P[All of events A, B, C occur]

$= P[A'B'C'] + P[(AB'C') \cup (A'BC') \cup (A'B'C)]$

$+ P[(AB'C) \cup (A'BC) \cup (ABC')]$

$+ P(ABC)$

$\boxed{=1}$

for $k=1$, we have,

P[atleast 1 of the events A, B, C occur]

= 1 − P[Exactly 0 of the events A, B, C occur]

$= 1 - P(A'B'C')$

∴ from eqn ①

$= 1 - 1 + P(A \cup B \cup C)$

$\boxed{= P(A \cup B \cup C)}$

for $k = 2$,

P[atleast 2- of events of A, B, C occur]

= P[exactly 2 of events of A, B, C occur] + P[exactly 3 of events A, B, C occur]

$= P[AB'C \cup A'BC \cup ABC'] + P[ABC]$

$\therefore$ from eq^n (3) & (4)

$= P(AB) + P(AC) + P(BC) - 3P(ABC) + P(ABC)$

$$\boxed{= P(AB) + P(AC) + P(BC) - 2P(ABC)}$$

for $k = 3$;

P[atleast 3- of events A, B, C occur]

$$\boxed{= P(ABC)}$$

Que 2. A secretary goes to work following one of 3-routes A, B, C. Her choice of route is independent of whether. If it rains, the probabilities of arriving late following A, B & C are 0.06, 0.15, 0.12 respectively. The corresponding probabilities if it does not rain are 0.05, 0.10, 0.15 respectively. Given that on a sunny day she arrives late, what is the probability that she took route C? Assumme that on an average in every four days is rainy.

→ Solution:

Consider S be the event that the day is sunny, & A, B, C be events that the secretary follows routes A, B, C resp. & L be the event that secretary arrives late.

then

$$P[C/SL] = \frac{P[CSL]}{P[SL]} = \frac{P[S] \cdot P[CL/S]}{P[S] \cdot P[L/S]}$$

$$\therefore P[C/SL] = \frac{P[C/S] \cdot P[L/CS]}{\left\{ P[A/S] \cdot P[L/AS] + P[B/S]\, P[L/BS] + P[C/S] \cdot P[L/CS] \right\}}$$

Also from given data.

$$P[A/S] = P[B/S] = P[C/S] = 1/3$$

$P[L|AS] = 0.05, \quad P[L|BS] = 0.10$

$\& \; P[L|CS] = 0.15$

$$\therefore P[C|LS] = \frac{(1/3)(0.5)}{(1/3)[0.05 + 0.10 + 0.15]}$$

$$\therefore \boxed{P[C|LS] = 0.5}$$

Hence, probability that the secretary ~~takes~~ took route C on a sunny day & arrived late is 0.5

Que 3. The time taken X by a garage to repair a car is continuous random variable with pdf $f(x) = \begin{cases} \frac{3x}{4}(2-x); & 0 \le x \le 2 \\ 0; & \text{Elsewhere.} \end{cases}$

If, on leaving his car, a motorist goes to keep on an engagement lasting for a time Y, where Y is a continuous rv independent of X with pdf $f(y) = \begin{cases} \frac{1}{2}y; & 0 \le y \le 2 \\ 0; & \text{elsewhere.} \end{cases}$

Determine the probability that the car will not be ready on his return.

→ Solution:- Since X & Y are independent, their joint probability function is given by

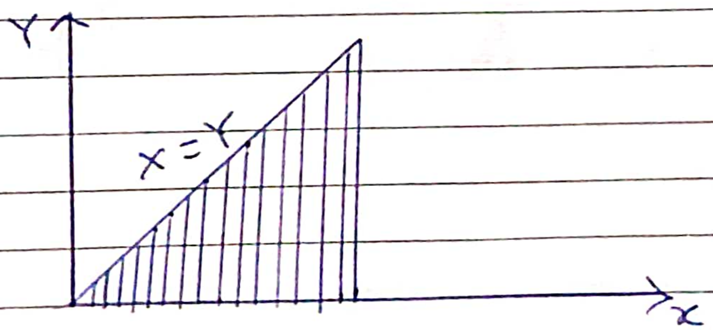
$$f_{X,Y}(x,y) = \frac{3}{4}x(2-x) \times \frac{1}{2}y \qquad 0 \le x, y \le 2$$

$$= \frac{3}{8}xy(2-x) \qquad 0 \le x, y \le 2$$

$$= 0 \qquad \text{elsewhere.}$$

We want to find $P(X > Y)$

i.e.



$$\therefore P(X>Y) = \frac{3}{8}\int_0^2 \int_0^x xy(2-x)\,dy \cdot dx$$

$$= \frac{3}{8}\int_0^2 x(2-x)\left[\frac{y^2}{2}\right]_0^x dx$$

$$= \frac{3}{8}\int_0^2 x(2-x)\frac{x^2}{2}\,dx$$

$$= \frac{3}{16}\int_0^2 (2x^3 - x^4)\,dx$$

$$= \frac{3}{16}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{3}{16}\left[8 - \frac{32}{5}\right]$$

$$\boxed{P(X>Y) = 0.3}$$

$\therefore$ The probability of car not being ready on return of motorist (under given condition/pdf's) is 0.3

Que 4. If a crv X follows a normal distribution such that $P(9.6 \le x \le 13.8) = 0.7008$ & $P(X \geq 0.9) = 0.8159$, find mean & var. of distribution.

→ Solution :-

Given, $P(X \geq 9.6) = 0.8159$

$\Rightarrow \phi(z_1) = 0.8159 \Rightarrow \underline{\underline{z_1 = 0.9}}$

$\phi(-z_1) = 0.1841$

Also, given,

$P(9.6 \le X \le 13.8) = 0.7008$

$P(9.6 \le X \le 13.8) = \phi(-z_1 \le z \le z_2)$

$= \phi(z \le z_2) - \phi(z \le -z_1)$

$0.7008 = \phi(z \le z_2) - 0.1841$

$\Rightarrow \phi(z \le z_2) = 0.7008 + 0.1841$

$= 0.8849$

$\Rightarrow \underline{\underline{z_2 = 1.2}}$

$\therefore -z_1 = -\left(\frac{9.6 - \mu}{\sigma}\right) = 0.9 \Rightarrow 9.6 - \mu = -0.9\sigma$ — ①

& $z_2 = \frac{13.8 - \mu}{\sigma} = 1.2 \Rightarrow 13.6 - \mu = 1.2\sigma$ — ②

solving eqn ① & ② we get

$\mu = 11.4$ & $\sigma = 2$ & variance $= 4$

$\therefore$ mean $= 11.4$ & variance $= 4$

5 (i) Prove that mgf of the sum of 2-independent rvs is the product of their mgfs.

→ We know that, for X, Y rvs, (independent)

$$M_X(t) = E(e^{tx})$$

$$M_Y(t) = E(e^{tY})$$

$$M_{X+Y}(t) = E(e^{t(X+Y)})$$

$$M_{X+Y}(t) = E(e^{tx} \cdot e^{tY})$$

$$M_{X+Y}(t) = E(e^{tx}) \cdot E(e^{tY})$$

... as X & Y are independent

$$\boxed{\therefore M_{(X+Y)}(t) = M_X(t) \cdot M_Y(t)}$$

(ii) Calculate the mgf of continuous rv X whose pdf is $f(x) = \begin{cases} \frac{1}{2} x^2 e^{-x} ; & 0 < x < \infty \end{cases}$

and using the above mgf, obtain mean and variance.

→ Solution :-

Given, continous rv X with pdf.

$$f(x) = \frac{1}{2} x^2 e^{-x} ; \quad 0 < x < \infty$$

~~Mgf~~

$$M_{f(x)}(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$$

$$\therefore M_{f(x)}(t) = \int_{0}^{\infty} e^{tx} \cdot \frac{1}{2} x^2 e^{-x}\, dx$$

$$\therefore M_x(t) = \frac{1}{2}\int_{0}^{\infty} e^{(t-1)x} \cdot x^2\, dx.$$

put $(t-1)x = -z \Rightarrow x = \frac{z}{1-t}$.

$dx = \frac{dz}{1-t}$.

x	0	∞
z	0	∞

$$\Rightarrow \therefore M_x(t) = \frac{1}{2}\int_{0}^{\infty} e^{-z} \cdot \frac{z^2}{(1-t)^2} \times \frac{dz}{(1-t)}$$

$$\therefore M_x(t) = \frac{1}{2}\left[\int_{0}^{\infty} e^{-z} z^2\, dz\right] \times \frac{1}{(1-t)^3}$$

$$\therefore M_x(t) = \frac{1}{2(1-t)^3} \Gamma 3$$

$$\boxed{\therefore M_x(t) = \frac{1}{(1-t)^3}}$$

We know, that, $M_x'(0) = E(x) = \text{mean}$

& $M_x''(0) = E(x^2)$

$$\therefore M'_x(t) = \frac{-3}{(1-t)^{+4}}$$

$$\boxed{M'_x(0) = -3 = \mu}$$

$$M''_x(t) = \frac{-3\times(-4)}{(1-t)^{+5}} = \frac{12}{(1-t)^5}$$

$$\therefore M''_x(0) = 12 = E(x^2)$$

$$\therefore \text{variance} = E(x^2) - [E(x)]^2$$

$$= 12 - 9$$

$$\boxed{\text{var} = 3}$$