Assignment Problem:

Let there be m jobs (operations) and m workers (operators). Let c_{ij} be the cost of assigning the job i to j. Then, assigning these m jobs to m workers in such a way that the total assignment cost is minimum, is called an assignment problem. A typical assignment table will be as follows:

Operators → / Operations ↓	M_1	M_2	…………….			M_{m-1}	M_m
O_1	c_{11}	c_{12}	…	…	…	c_{1m-1}	c_{1m}
O_2	c_{21}	c_{22}	…	…	…	c_{2m-1}	c_{2m}
.	…	…	…	…	…	……	…
.	…	…	…	…	…	……	…
O_m	c_{m1}	c_{m2}	…	…	…	c_{mm-1}	c_{mm}

Let us define the decision variables x_{ij} as follows:

$x_{ij} = 1$, if i^{th} job is assigned to j^{th} operator.

$= 0$, other wise.

Then, the lp form of an assignment problem can be written as

Minimize $z = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_{ij}$ ………… …….(1)

The constraints here are

$\sum_{j=1}^{m} x_{ij} = 1$, $i = 1, 2, …., m$ ………...(2)

$\sum_{i=1}^{m} x_{ij} = 1$, $j = 1, 2, ….., m$ …..…....(3)

$x_{ij} \geq 0$ for all (i , j) …….………….(4)

Solution of an Assignment Problem (Hungarian Metthod):

Algorithm To find an Initial Assignment:

1. Find the minimum cost in each row and subtract that element from every element in the respective row.
2. Find the minimum cost in each column and subtract that element from every element in the respective column.
3. Starting from rows, identify a singly ‘0’ in any row and encircle it as a possible assignment and cross out all the other zeroes in that particular column where the assignment is made. Do this for all the possible rows.
4. Does this process column wise. If any assignment is done (encircling) in any particular column, strike off all the other zeroes in that particular row where the assignment is made.

Note: Whenever more than one zero is available (row or column), choose anyone arbitrarily and make the assignment. At that time, strike off all the other zeroes in that particular row as well as in that particular column.

Optimality Checking: If the number of assignments is equal to the order of the matrix(number of operators or operations), then that particular allotment is optimal. If not, we have to iterate further to reach the optimal assignment.

Iteration towards Optimality:

1. Mark the unassigned row. If any striken off zero is there in that row, mark that corresponding column. Again in that particular column, if any assignment (encircled zero) is there, mark that corresponding row. Repeat this process till there is no more possibility of marking.
2. Cross the unmarked rows and marked columns. In this process, all zeroes (encircled or striken off) will be covered by some lines. Pick up the elements

which are not covered by any line and identify the minimum among these elements.

3. Subtract this element from every element in that group and add this minimum to those elements which are crossed by two lines. All other elements will remain same. Again do the assignment as described in the initial assignment process.
4. Check for the optimality and if not, iterate again till you reach the optimal assignment.

Example 1: Solve the following Assignment Problem for minimization.

	Operator 1	Operator 2	Operator 3	Operator 4
Operation 1	5	8	7	10
Operation 2	10	11	9	12
Operation 3	8	10	11	13
Operation 4	11	13	10	11

Solution: Initial Assignment will be as follows:

After the row and column operations,

	Operator 1	Operator 2	Operator 3	Operator 4
Operation 1	$\boxed{0}$	1	2	4
Operation 2	1	$\cancel{0}$	$\boxed{0}$	2
Operation 3	$\cancel{0}$	$\boxed{0}$	3	4
Operation 4	1	1	$\cancel{0}$	$\boxed{0}$

Here, the number of assignments are 4 which is equal to the order of the cost matrix. Therefore, this assignment is optimal. The optimal solution is

Operation 1 → Operator 1

Operation 2 → Operator 3

Operation 3 → Operator 2

Operation 4 → Operator 4

And the optimal assignment cost is 5 + 9 + 10 + 11 = 35.

Example 2: Find the optimal assignment to the following AP

	Operator 1	Operator 2	Operator 3	Operator 4	Operator 5
Operation 1	15	18	10	20	13
Operation 2	25	30	28	35	32
Operation 3	10	13	15	17	12
Operation 4	20	25	23	28	30
Operation 5	5	10	8	11	7

Solution:

The initial assignment is

	Operator 1	Operator 2	Operator 3	Operator 4	Operator 5
Operation 1	5	5	$\boxed{0}$	4	1
Operation 2	$\boxed{0}$	2	3	4	5
Operation 3	$\cancel{0}$	$\boxed{0}$	5	1	$\cancel{0}$
Operation 4	$\cancel{0}$	2	3	2	8
Operation 5	$\cancel{0}$	2	3	$\boxed{0}$	$\cancel{0}$

Here, the number of assignments are 4 which is less than the order of the matrix 5. Hence, this assignment is not optimal. So, we have to iterate using Marking-

Unmarking process. Using this technique, we can get the next iterated assignment as follows:

	Operator 1	Operator 2	Operator 3	Operator 4	Operator 5
Operation 1	7	5	$\boxed{0}$	4	1
Operation 2	$\boxed{0}$	$\cancel{0}$	1	2	3
Operation 3	2	$\boxed{0}$	5	1	$\cancel{0}$
Operation 4	$\cancel{0}$	$\cancel{0}$	1	$\boxed{0}$	6
Operation 5	2	2	3	$\cancel{0}$	$\boxed{0}$

Here, the number of assignments are 5 which is equal to the order of the matrix. Hence, this is the optimal assignment. The optimal assignment cost is 83.

Note:

1. If the objective of an assignment problem is of maximization type, then construct an "Opportunity Loss" cost matrix from the given matrix and minimize the opportunity loss cost. This will be equivalent to maximizing the given cost.
2. If the given assignment problem is an unbalanced one (number of operations $\neq$ number of operators), then create an artificial operator or operation (depending on the problem) with zero assignment costs against those operator or operation and then apply regular Hungarian method.

Example 3: AIT has decided to build 4 new buildings to house 4 new schools. The tenders have been called for constructing the buildings. The building committee felt that the quotations of contractors A, B, C and D are reasonable and decided to reject the remaining tenders. What decision the committee should take so that the amount to be spent by AIT in construction is minimal? The following table gives the quotations of 4 contractors (in millions of bahts) for constructing the respective building.

	School 1	School 2	School 3	School 4
Contractor A	10	24	30	15
Contractor B	16	22	28	12
Contractor C	19	20	32	10
Contractor D	9	26	34	16

Example 4: A student has to select one and only one elective in each semester and the same elective should not be selected again, once it is chosen. Due to various reasons, the expected grades in each semester if selected in different semester vary and they are give below:

	Elective 1	Elective 2	Elective 3	Elective 4
Semester I	F	E	D	C
Semester II	E	E	C	C
Semester III	C	D	C	A
Semester IV	D	A	S	S

The grade points are S = 10, A = 9, B = 8, C = 7, D = 6, E = 5, F = 4. How a student should select the electives in order to maximize the total expected grade.

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