Poisson Process

Here we shall study some stochastic processes in continuous time with discrete state space. One such process is the Poisson Process. Let N(t) denote the number of events occurring in the interval of length t, say (0, t]. Then, the process {N(t)} is a stochastic process with state space S = {0,1, 2,….} and parameter space T = {t; t ≥ 0}. Then the process {N(t)} is called a counting process.

Let $p_n(t)$ = P[N(t)=n]. We proceed to show that under certain conditions N(t) follows the Poisson distribution with parameter λt, where λ is a constant.

i.e. N(t) ~ P(λt) ⇒ $p_n(t) = P(N(t)=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$; n = 0,1,2,3………..; $\lambda > 0$

The postulates for Poisson Process are:

1) Independence: N(t) is independent of the number of occurrences of the event in an interval prior to (0,t] i.e. {N(t)} has independent increments.

2) Homogeneity in time: $p_n(t)$ depends only on the length t of the time interval and is independent of where this time interval is located.

3) Regularity: In an interval of infinitesimal (very small) length 'h',
   P (exactly one occurrence) = λh + O(h)
   P (more than one occurrence) = O(h)

where O(h) → 0 more rapidly than h i.e. as h→0, $\frac{O(h)}{h} \to 0 \equiv \lim_{h\to 0}\frac{O(h)}{h} = 0$

Accordingly, we have $p_1(h) = P[N(h) = 1] = \lambda h + O(h)$

$p_0(h) = 1 - \lambda h + O(h)$

$p_n(h) = O(h)$; n > 1

Under the above postulates it can be shown that N(t) follows a Poisson distribution with parameter λt

i.e. N(t) ~ P(λt) ⇒ $p_n(t) = P(N(t)=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$; n = 0,1,2,3………..; $\lambda > 0$

i) Consider, when n ≥ 1

$p_n(t+h)$ = P[N(t+h)=n] = P[n occurrences by epoch t+h starting from t=0]

This is the probability that n events occurring by epoch t and no events occurring in (t,t+h) OR (n-1) events occurring by epoch t and one event occurring in (t,t+h) etc. etc. OR no events occurring by epoch t and n events occurring in (t,t+h)

i.e $p_n(t+h)$ = P[n occurrences by epoch t and no occurrence during h]

+P[n-1 occurrences by epoch t and one occurrence during h]

+ …………

$= p_n(t)\, p_0(h) + p_{n-1}(t)\, p_1(h) + \ldots + p_0(t)\, p_n(h)$

$= p_n(t)[1-\lambda h+O(h)]+p_{n-1}(t)[\lambda h+O(h)]+O(h)$

$p_n(t+h) = p_n(t)[1-\lambda h]+p_{n-1}(t)[\lambda h]+O(h)$

$$\frac{(p_n(t+h) - p_n(t))}{h} = \frac{\lambda h(p_{n-1}(t) - p_n(t))}{h} + \frac{O(h)}{h} = \lambda(p_{n-1}(t) - p_n(t)) + \frac{O(h)}{h}$$

As $h \to 0$, $p_n'(t) = \lambda(p_{n-1}(t) - p_n(t))$; $n \geq 1$ ----------- **(1)**

(ii) When n = 0 we have,

$p_0(t+h) = P[N(t+h)=0] = p_0(t)\, p_0(h)$

$= p_0(t)[1-\lambda h+O(h)]$

$$\frac{(p_0(t+h) - p_0(t))}{h} = -\lambda\, p_0(t) + \frac{O(h)}{h}$$

As $h \to 0$, $p_0'(t) = -\lambda p_0(t)$; n = 0 ----------- **(2)**

Equations **(1)** and **(2)** are called differential difference equations which along with the given initial conditions completely specify the process.

Initial Conditions:

Assuming that the process starts at t=0 i.e. N(0) = 0, we have

$p_0(0) = P[N(0)=0] = 1$

$p_n(0) = 0$; $n \neq 0$

These differential difference equations may be solved by Laplace transformation technique or generating functions technique etc. and the solution is given by

$$p_n(t) = P(N(t)=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad ; \quad n = 0,1,2,3\ldots\ldots ; \ \lambda > 0$$

which is the Poisson distribution with mean and variance λt i.e. if the mean number of occurrences in an interval of length t is λt then the mean number of occurrences per unit time is λ. λ is called the rate of occurrence.

Problems

1. Suppose that the customers arrive at a bank according to a Poisson process at a rate 3 per minute. What is the probability that in an interval of two minutes, the number of customers arriving is (i) exactly 4 (ii) greater than 4 (iii) less than 4?

   Solve it!

2. Let vehicles arrive at a junction according to a Poisson process at a rate (λ) 5 per hour. What is the probability that

   (i) Exactly 3 vehicles arrive at the traffic junction in an hour?

   (ii) more than 10 vehicles arrive in 2 hours?

   Solve it!

A Note on Differential-Difference Equations:

Difference Equations

Let f(n) be a function defined only for non-negative integral values of the argument n.
The first difference of f(n) is defined by the increment of f(n) and is denoted by $\Delta f(n)$, i.e. $\Delta f(n) = f(n+1) - f(n)$
The second and higher differences are defined by,
$\Delta^{k+1} f(n) = \Delta^k f(n+1) - \Delta^k f(n); k > 0$
By a Difference Equation we mean an equation involving a function evaluated at the arguments which differ by any of a fixed number of values.
Ex: 1) f(n+2)-f(n+1)-f(n) = 0
2) $a_0 u_x + a_1 u_{x+1} + \ldots + a_k u_{x+k} = g(x)$

Differential-Difference Equations

Suppose that $u_n(t)$, n=1,2,3,…, is a function of t having a derivative $\frac{du_n(t)}{dt} = u_n'(t)$
An equation involving $u_n'(t)$, $u_n(t)$, $u_{n+1}(t)$, etc. is called a Differential-Difference Equation.
Ex: 1) $u_n'(t) = u_{n-1}(t)$; $t \geq 0$; n=1,2,3…
2) $p_n'(t) = -\lambda[p_n(t) - p_{n-1}(t)]$

To arrive at a solution for a set of differential difference equation several techniques exist, such as 1) Generating Functions Technique 2) Laplace Transforms Technique etc.