PROBLEMS

1. Suppose that the joint pdf of the 2-dimensional crv (X,Y) is given by

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3}; & 0 \le x \le 1, 0 \le y \le 2 \\ 0 \quad ; & \text{elsewhere} \end{cases}$$

Find (i). g(x)
(ii). h(y)
(iii). g(x/y)
(iv). h(y/x)
(v). P(X>$\frac{1}{2}$)
(vi). P(Y<X)
(vii).P[(Y<$\frac{1}{2}$)/(X<$\frac{1}{2}$)]

Solution:

(i). $g(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^2 (x^2 + \frac{xy}{3})dy = 2x^2 + \frac{2x}{3}$, $0 \le x \le 1$

(ii). $h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 (x^2 + \frac{xy}{3})dx = \frac{y}{6} + \frac{1}{3}$, $0 \le y \le 2$

(iii). g(x/y) = f(x,y) / h(y) ; h(y)>0

$$= \frac{x^2 + \frac{xy}{3}}{\frac{1}{3} + \frac{y}{6}} = \frac{6x^2 + 2xy}{2+y}, \quad 0 \le x \le 1 \text{ and } 0 \le y \le 2$$

(iv). h(y/x) = f(x,y)/g(x) ; g(x)>0.

$$= \frac{x^2 + \frac{xy}{3}}{2x^2 + \frac{2x}{3}} = \frac{3x+y}{6x+2}, \quad 0 \le x \le 1 \text{ and } 0 \le y \le 2$$

(v). $P(X>\frac{1}{2}) = P(X>\frac{1}{2}, 0 \le Y \le) = \int_0^2 \int_{\frac{1}{2}}^1 f(x,y)dxdy$

$$P\left(X > \frac{1}{2}\right) = \int_0^2 \int_{\frac{1}{2}}^1 \left(x^2 + \frac{xy}{3}\right) dxdy = \int_0^2 [(\frac{x^3}{3} + \frac{x^2 y}{6})]_{\frac{1}{2}}^1 dy$$

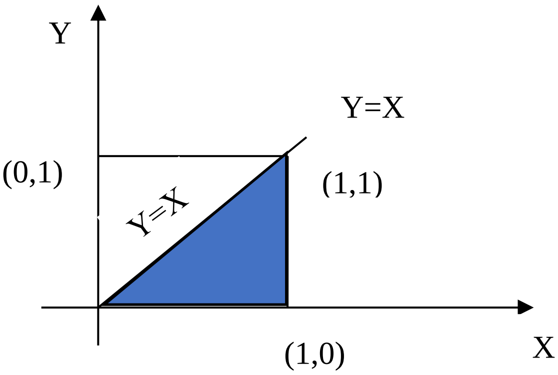
$$= \int_0^2 (\frac{1}{3} + \frac{y}{6} - \frac{1}{24} - \frac{y}{24})dy = \int_0^2 \left(\frac{7}{24} + \frac{3y}{24}\right) dy$$

$$= \left[\frac{7}{24}y\right]_0^2 + \left[\frac{y^2}{16}\right]_0^2 = \frac{5}{6}$$

(vi). $P(Y < X) = \int_0^1 \int_0^x \left[x^2 + \frac{xy}{3}\right] dydx$

$$= \int_0^1 \left[x^3 + \frac{x^3}{6}\right] dx = \left[\frac{x^4}{4} + \frac{x^4}{24}\right]_0^1$$

$$= \frac{7}{24}$$



(vii). $P(Y < \frac{1}{2}/X < \frac{1}{2}) = \frac{P(Y < \frac{1}{2} \cap X < \frac{1}{2})}{P(X < \frac{1}{2})}$

$$P\left(Y < \frac{1}{2} \cap X < \frac{1}{2}\right) = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(x^2 + \frac{xy}{3}\right) dxdy = \int_0^{\frac{1}{2}} \left(x^2 y + \frac{xy^2}{6}\right)_0^{\frac{1}{2}} dx$$

$$= \left[\frac{x^3}{6} + \frac{x^2}{48}\right]_0^{\frac{1}{2}} = \frac{5}{19}$$

$$P(X < \frac{1}{2}) = \int_0^2 \int_0^{\frac{1}{2}} \left[x^2 + \frac{xy}{3}\right] dxdy = \int_0^2 \left[\frac{x^3}{3} + \frac{x^2 y}{6}\right]_0^{\frac{1}{2}} dy$$

$$= \left[\frac{y}{24} + \frac{y^2}{48}\right]_0^2 = \frac{1}{6}$$

$$\therefore\ P(Y < \frac{1}{2}/X < \frac{1}{2}) = \frac{\frac{5}{192}}{\frac{1}{6}} = \frac{5}{32}$$

2. Suppose that the 2-dimensional crv (X,Y) has the pdf

$$\mathrm{f(x,y)} = \begin{cases} \mathrm{Cx(x-y)}\ ;\ 0 < \mathrm{x} < 2, -\mathrm{x} < \mathrm{y} < \mathrm{x} \\ \quad 0 \quad ; \qquad \text{elsewhere} \end{cases}$$

a) Evaluate C
b) Find the marginal pdf of X
c) Find the marginal pdf of Y

Solve it!

3. Let $f(x,y) = \begin{cases} \frac{2}{a^2}\ ; & 0 \le x < y \le a \\ 0\ ; & \text{elsewhere} \end{cases}$ be the joint pdf of the 2-dimensional crv (X,Y). Find g(x/y) and h(y/x)?

Solution:

We have $g(x/y) = \frac{f(x,y)}{h(y)}\ ; h(y) > 0$ and $h(y/x) = \frac{f(x,y)}{g(x)}\ ; g(x) > 0$

$$g(x) = \int_x^a f(x,y)dy = \int_x^a \frac{2}{a^2}dy = \left[\frac{2y^2}{a}\right]_x^a = \frac{2}{a^2}(a-x), 0 \le x \le a$$
$$= 0, \text{elsewhere}$$

$$h(y) = \int_0^y \frac{2}{a^2}dx = \left[\frac{2}{a^2}x\right]_0^y = \frac{2y}{a^2}, 0 \le y \le a$$
$$= 0, elsewhere$$

$$g(x/y) = \frac{f(x,y)}{h(y)} = \frac{\frac{2}{a^2}}{\frac{2y}{a^2}} = \frac{1}{y}, o \le y \le a$$
$$= 0, \text{elsewhere}$$

$$h(y/x) = \frac{f(x,y)}{g(x)} = \frac{\frac{2}{a^2}}{\frac{2(a-x)}{a^2}} = \frac{1}{a-x}; 0 \le x \le a$$
$$= 0, \text{elsewhere}$$

4. For what values of 'k', is $f(x,y)=ke^{-(x+y)}$ a joint pdf of (X,Y) over the region $0<x<1;0<y<1$?

Solve it!

5. Two rvs X and Y have their joint pdf given by
$f(x,y) = \begin{cases} 6(e^{-2x-3y}); & x,y \ge 0 \\ 0\ ; & \text{elsewhere} \end{cases}$
Find
a) $P(1<X<2, 2<Y<3)$
b) $P(0<X<2, Y>2)$
c) Marginal pdfs of X and Y. Also the Conditional pdfs of X given Y and Y given X

Solve it!

6. Test for the independence of the rvs X and Y, given the joint pdf
$f(x,y) = \begin{cases} 2(e^{-x-y}); & 0 < x < y < \infty \\ 0\ ; & \text{elsewhere} \end{cases}$

Solution:

X and Y are independent iff $f(x,y)=g(x).h(y)\ \forall\ x,y$

Consider,

$$g(x) = 2\int_x^\infty (e^{-x-y})dy = 2e^{-2x}$$
$$h(y) = 2\int_0^y (e^{-x-y})dx = -2e^{-2y} + 2e^{-2y}$$
$$g(x).h(y) = 2e^{-2x}(-2e^{-2y} + 2e^{-2y}) \ne f(x,y)$$

Hence X and Y are not independent.

7. Test for the independence of the rvs X and Y, given the joint pdf
$$f(x,y) = \begin{cases} 8xy \ ; & 0 < x < y < 1 \\ 0 \ ; & \text{elsewhere} \end{cases}$$

Solve it!

8. Test for independence of the rvs X and Y whose joint probability function is given below:

X/Y	1	2	3
1	1/8	1/8	2/8
2	3/8	0	0
3	0	1/8	0

Solution:

X/Y	1	2	3	$P(X=x_i)$
1	1/8	1/8	2/8	4/8
2	3/8	0	0	3/8
3	0	1/8	0	1/8
$P(Y=y_j)$	4/8	2/8	4/8	1

$$P(Y=1) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2}, \qquad P(X=1) = \frac{4}{8} = \frac{1}{2}$$
$$P(Y=2) = \frac{2}{8}, \qquad P(X=2) = \frac{3}{8}$$
$$P(Y=3) = \frac{3}{8}, \qquad P(X=3) = \frac{1}{8}$$

Now for X and Y to be independent we must have $P(X=x_i, Y=y_j) = P(X=x_i)\ P(Y=y_j) \quad \forall$ i,j.
Consider,
$$P(X=1, Y=1) = \frac{1}{8} \text{ and } P(X=1).P(Y=1) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$$
$$P(X=1, Y=1) \neq P(X=1).P(Y=1)$$

Hence X and Y are not independent.

9. Test for the independence of the rvs X and Y whose joint probability function is given below:

Y/X	-1	0	1
-1	1/12	1/12	2/12
0	0	0	0
1	2/12	2/12	4/12

Solve it!

10. Suppose X and Y are independent rvs and X takes values 2, 5 and 7 with probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$ respectively while Y takes values 3 and 5 with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ respectively. Determine the joint pmf of X and Y. Also determine the probability distribution of Z=X+Y?

Solution:

Since X and Y are independent, we have $P(X=x_i,Y=y_j) = P(X=x_i)\, P(Y=y_j)\ \ \forall\, i,j.$

Y/X	2	5	7	$P(Y=y_j)$
3	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$
5	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
$P(X=x_i)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	1

To find the probability distribution of Z=X+Y

Note that Z takes values 5,8,10,7,12

$P(Z=5) = P(X=2,Y=3) = \frac{1}{6}$

$P(Z=8) = P(X=5,Y=3) = \frac{1}{12}$

$P(Z=10) = P(X=5,Y=5) + P(x=7,Y=3) = \frac{1}{6}+\frac{1}{12}=\frac{1}{4}$

$P(Z=7) = P(X=2,Y=5) = \frac{1}{3}$

$P(Z=12) = P(X=7,Y=5) = \frac{1}{6}$

Joint pmf: $P(X=x_i,Y=y_j) = P(X=x_i).P(Y=y_j)$; since X and Y are independent

$$P_{11} = P(X=2, Y=3) = P(X=2).P(Y=3) = \frac{1}{2}.\frac{1}{3} = \frac{1}{6}$$

$$P_{12} = P(X=2, Y=5) = P(X=2).P(Y=5) = \frac{1}{2}.\frac{2}{3} = \frac{1}{3}$$

$$P_{21} = P(X=5, Y=3) = P(X=5).P(Y=3) = \frac{1}{4}.\frac{1}{3} = \frac{1}{12}$$

$$P_{22} = P(X=5, Y=5) = P(X=5).P(Y=5) = \frac{1}{4}.\frac{2}{3} = \frac{1}{6}$$

$$P_{31} = P(X=7, Y=3) = P(X=7).P(Y=3) = \frac{1}{4}.\frac{1}{3} = \frac{1}{12}$$

$$P_{32} = P(X=7, Y=5) = P(X=7).P(Y=5) = \frac{1}{4}.\frac{2}{3} = \frac{1}{6}$$