Conditional Probability:

Consider the example: Suppose we have a lot consisting of 100 items of which 80 are good and 20 are defective. Now we choose two items at random from this lot.
We define A= {the first item is defective} and B= {the second item is defective}.
(a) If the selection is With Replacement (WR), what is P(A) and P(B)?
(b) If the selection is Without Replacement (WOR), what is P(A) and P(B)?
Suppose we choose samples WR then $P(A)=20/100=1/5 =P(B)$.
If we chose samples WOR then $P(A)=1/5$. But what is P(B)?
To compute we should know whether A did occur or not. Thus if A and B are two events associated with an experiment E, then P(B/A) denotes the conditional probability of the event B, given that A has occurred, i.e. $P(B/A)=19/99$.

Definition: If A and B are events for which $P(B)>0$, then we define the conditional probability of A given B denoted by P(A/B) as
$P(A/B)=P(A\cap B)/P(B);\ P(B)>0$
Similarly $P(B/A)=P(A\cap B)/P(A);\ P(A)>0$

Note: Whenever we compute P(B/A) , we are essentially computing the P(B) with respect to the reduced sample space A, rather than the originally sample space S.

Ex: Two fair dice are tossed, and the outcomes are recorded

$$S=\left\{\begin{array}{l}(1,1)\ (1,2)\ldots\ldots(1,6)\\ (2,1)\ (2,2)\ldots\ldots(2,6)\\ \quad :\\ \quad :\\ (6,1)\ (6,2)\ldots\ldots(6,6)\end{array}\right\}$$

Let $A=\{(x,y);x+y=10\}$ and $B=\{(x,y);x>y\}$

Then $A=\{(5,5),(4,6),(6,4)\}$ and $B=\{(2,1),(3,1),(3,2)\ldots\ldots(6,5)\}$

$P(A)=3/36$ & $P(B)=15/36$
Now $P(A/B)=? = 1/15$ & $P(B/A)=1/3$
Also, from the above definition we have $P(A/B)= P(A\cap B)/P(B)$
Here $A\cap B$ occurs iff both A and B occur and for this only one outcome is favorable. Hence $P(A\cap B)= 1/36$.

Therefore, $P(A/B)= P(A\cap B)/P(B)= (1/36)/(15/36)=1/15$
Similarly, $P(B/A)= P(B\cap A)/P(A)= (1/36)/(3/36)=1/3$.

Note: P(A/B) satisfies the various axioms,

(i). $0 \le P(A/B) \le 1$

(ii). $P(S/B)=1$

(iii). $P((A_1 \cup A_2)/B)=P(A_1/B)+P(A_2/B)$ if $A_1 \cap A_2=\phi$ i.e A & A are mutually exclusive

(iv). $P(A_1 \cup A_2 \cup \ldots /B)= P(A_1/B)+P(A_2/B)+\ldots$; if $A_i \cap A_j \neq \phi$ for $i \neq j$

(v). If B=S, $P(A/S)=P(A \cap S)/P(S)=P(A)$.

Thus we have two ways of computing P(A/B).

(i). Directly, by considering, P(A) with respect to the reduced sample space B.

(ii). Using the above definition by computing $P(A), P(A \cap B)$ with respect to original sample space S.

Multiplication Theorem:

If A and B are any two events, then

$P(A \cap B) = P(B).P(A/B)$.

$= P(A).P(B/A)$.

Using this theorem, we can compute the probability of the simultaneous occurrence of two events.

Ex: In the above example of a lot consisting of 80 good items and 20 defective items if we were to choose 2 items at random (WOR), what is the probability that both of them are defective?

Solution: Define A={first item is defective}   B={second item is defective}.

$P(A)=1/5$, $P(B/A)=19/99$, $P(A \cap B)=?$

$P(A \cap B)=P(B/A).P(A)=19/495$

Note: Generalization of Multiplication Theorem.

If $A_1, A_2, A_3 \ldots\ldots, A_n$ are any n events then,

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots\ldots \cap A_n)=P(A_1)\, P(A_2/A_1) P(A_3/A_1 A_2) \ldots\ldots P(A_n/ A_1 A_2 \ldots A_{n-1})$$

Definition: The events $B_1, B_2, \ldots\ldots B_k$ are said to partition the sample space S if,

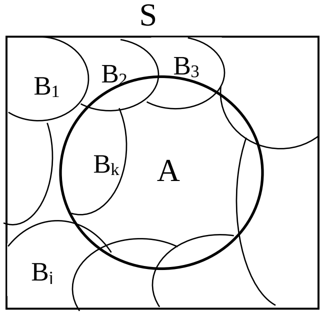
(i). $B_i \cap B_j = \phi \quad \forall\, i \neq j$

(ii). $\bigcup_{i=1}^{k} B_i = S$

(iii). $P(B_i) > 0 \ \forall\, i$

(i.e. when an experiment E is performed, one and only one of the B_i's occurs)

Theorem of Total Probability:



Let A be an event with respect to S and let $B_1, B_2, \ldots\ldots B_k$ be a partition of S. We may decompose and write A as the union of mutually exclusive events.

i.e. $A = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_k)$ where some of $A \cap B_j$ may be $= \phi$

Therefore $P(A) = P[(A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_k)]$

$= P(A \cap B_1) + P(A \cap B_2) + \ldots\ldots + P(A \cap B_k)$

By Multiplication theorem,

$P(A) = P(B_1).P(A/B_1) + P(B_2).P(A/B_2) + \ldots\ldots + P(B_j).P(A/B_j) + \ldots + P(B_k).P(A/B_k)$

$P(A) = \sum_{j=1}^{k} P(B_j).P(A/B_j)$ is called the theorem of total probability.

Ex: Consider the lot of 20 defective and 80 non defective items from which we choose two items WOR. What is the probability that the 2^{nd} item selected is defective?

Let A:{1^{st} selection is defective} and B:{2^{nd} selection is defective}

We have $P(B) = P(B/A)P(A) + P(B/\overline{A})P(\overline{A})$

$= (19/99)(1/5) + (20/99)(4/5) = 1/5.$