Non-Linear Programming

A general non-linear programming problem can be described as follows:
Let z be a real valued function of n variables defined by
$z = f(x_1, x_2, \ldots, x_n)$.
Let $(b_1, b_2, \ldots, b_n)$ be a set of constraints such that
$g^1(x_1, x_2, \ldots, x_n) <=> b_1$
$g^2(x_1, x_2, \ldots, x_n) <=> b_2$
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$g^m(x_1, x_2, \ldots, x_n) <=> b_m$
where g^i's are real valued functions of n variables, $x_1, x_2, \ldots, x_n$.
Finally, let $x_j \geq 0$ for $j = 1, 2, 3, \ldots, n$. If either $f(x_1, x_2, \ldots, x_n)$ or some $g^i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, 3, \ldots, m$; or both are non-linear, then the problem of determining the n-type $(x_1, x_2, \ldots, x_n)$ which makes z a maximum or minimum and satisfies the given constraints is called a non-linear programming problem (nlpp).

Constrained Maxima and Minima:

Equality Constraints:
If the non-linear programming problem is composed of some differentiable objective function and equality constraints, the optimization may be achieved by the use of Lagrange Multipliers. Let the nlpp be
Maximize (or minimize) $z = f(x_1, x_2, \ldots, x_n)$
Subject to the constraints $g^i(x_1, x_2, \ldots, x_n) = b_i$
and $x_j \geq 0$ for $j = 1, 2, 3, \ldots, n$.
Let $h^i(x_1, x_2, \ldots, x_n) = g^i(x_1, x_2, \ldots, x_n) - b_i$

Then the given nlpp can be written in the matrix form as

Optimize z = f($\mathbf{x}$) , where $\mathbf{x} \in \mathbf{R}^n$

Subject to the constraints $h^i(\mathbf{x}) = \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$.

To find the necessary conditions for a maximum or minimum of f($\mathbf{x}$), a new function, Lagrangian function L($\mathbf{x}$, λ) , is formed by introducing Lagrangian multipliers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. This function is defined as $L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i h^i$. Assuming that L, f and h^i are all differentiable partially with respect to $x_1, x_2, \ldots, x_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_m$, the necessary conditions for a maximum or minimum of f($\mathbf{x}$) are given as

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^i}{\partial x_j} = 0 \ , \quad j = 1, 2, \ldots, n \qquad (1)$$

$$\frac{\partial L}{\partial \lambda_i} = - h^i(\mathbf{x}) = \mathbf{0} \ , \qquad i = 1, 2, \ldots, m \qquad (2)$$

These m + n necessary conditions also become sufficient for a maximum (minimum) of the objective function if the objective function is concave (convex) and the constraints are all equality ones. If the concavity (convexity) of the function is not known, the method of Lagrangian multipliers can be generalized to determine a set of sufficient conditions for a maximum (minimum) of the objective function. Let us assume that the function L($\mathbf{x}$, λ), f($\mathbf{x}$) , and h($\mathbf{x}$) all possess partial derivatives of order one and two with respect to the decision variables.

Let $V = \left(\frac{\partial^2 L(x, \lambda)}{\partial x_i \partial x_j} \right)_{nxn}$ be the matrix of second order partial derivatives of L with respect to decision variables and let $U = (h^i_j)_{mxn}$ where

$$h^i_j(\mathbf{x}) = \frac{\partial h^i(x)}{\partial x_j}, \ i = 1, 2, \ldots, m \ \text{ and } j = 1, 2, \ldots, n.$$

Define the square matrix $H^B = \begin{bmatrix} 0 & U \\ U^T & V \end{bmatrix}_{(m+n)x(m+n)}$ where O is an (m x m) null matrix. The matrix H^B is called the bordered Hessian Matrix. Then, the sufficient conditions for maximum and minimum stationary points are given below:

Let $(\mathbf{x}_0, \lambda_0)$ for the function L $(\mathbf{x}, \lambda)$ be its stationary point. Let H_0^B be the corresponding bordered Hessian matrix computed at this stationary point. Then, $\mathbf{x}_0$ is (i) a maximum point, if starting with principal minor of order (2m + 1), the last (n-m) principal minors of H_0^B form an alternating sign pattern starting with $(-1)^{m+n}$ and (ii)minimum point, if starting with the principal minor of order (2m+1), the last (n-m) principal minors of H_0^B have the sign of $(-1)^m$.

Example 1: Solve the following non-linear programming problem:

Optimize $z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

Subject to the constraints

$$x_1 + x_2 + x_3 = 15,$$

$$2x_1 - x_2 + 2x_3 = 20.$$

Solution: Here, we have

$$f(\mathbf{x}) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$$

$$h^1(\mathbf{x}) = x_1 + x_2 + x_3 - 15,$$

$$h^2(\mathbf{x}) = 2x_1 - x_2 + 2x_3 - 20 .$$

Construct the Lagrangina function

$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda_1 h^1(\mathbf{x}) - \lambda_2 h^2(\mathbf{x})$

$= (4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2) - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20).$

The stationary point ($\mathbf{x}_0$, λ_0) is thus given from the following necessary conditions:

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + x_3 - 15) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -(2x_1 - x_2 + 2x_3 - 20) = 0 .$$

The solution to these simultaneous equations yields

$\mathbf{x}_0 = (x_1, x_2, x_3) = (33/9 , 10/3 , 8)$ and $\lambda_0 = (\lambda_1 , \lambda_2) = (40/9 , 52/9)$.

The bordered Hessian matrix at this solution is given by

$$H_0^B = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{bmatrix}$$

Here, since n = 3 and m = 2 , we have n – m = 1 , (2m + 1 = 5). This means that one need to check the determinant of H_0^B only and it must have the sign of $(-1)^2$. Now, since det (H_0^B) = 72 > 0 , $\mathbf{x}_0$ is a minimum point.

Problem 1: Solve the following non-linear programming problem, using the method of Lagrange multipliers:

Maximize $z = x_1^2 + x_2^2 + x_3^2$

Subject to the constraints

$$x_1 + x_2 + 3x_3 = 2\,; \quad 5x_1 + 2x_2 + x_3 = 5\,; \quad x_1\,,\; x_2\,,\; x_3 \;\geq 0.$$

Inequality Constraints:

When the constraints are of inequality type, then the necessary conditions can be derived as follows.

Let us consider the nlpp:

Maximize $z = f(\mathbf{x})$, where $\mathbf{x} \in \mathbf{R}^n$

Subject to the constraints $h^i(\mathbf{x}) \;\leq \mathbf{0}$, $-\mathbf{x} \leq \mathbf{0}$; $i = 1, 2, \ldots, m$.

Clearly, there are m + n inequality constraints, and thus we add the square of (m + n) slack variables $S_1, S_2, \ldots, S_m, S_{m+1}, \ldots, S_{m+n}$ in the inequalities so as to convert them in the equations:

$h^i(\mathbf{x}) + S_i^2 = 0$ for $i = 1, 2, \ldots, m$

$-\;x_j + S_{m+j}^2 = 0$ for $j = 1, 2, \ldots, n$.

To find the necessary conditions for maximum of $f(\mathbf{x})$, we construct the associated Lagrangian function

$$L(\mathbf{x}, \mathbf{S}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \,[h^i(\mathbf{x}) + S_i^2] - \sum_{j=1}^{n} \lambda_{m+j}\, [-x_j + S_{m+j}^2]$$

where $\mathbf{S} = (S_1, S_2, \ldots, S_{m+n})$; and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m+n})$ are the Lagrangian multipliers. The necessary conditions for $f(\mathbf{x})$ to be a maximum are :

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^i}{\partial x_j} + \lambda_{m+j} = 0 \;, \quad j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial S_i} = -2\,\lambda_i S_i = 0 \qquad , \quad i = 1, 2, \ldots, m$$

$$\frac{\partial L}{\partial S_{m+j}} = 2\ \lambda_{m+j} S_{m+j} = 0 \qquad , \quad j = 1, 2, \ldots , n$$

$$\frac{\partial L}{\partial \lambda_i} = -\ (h^i + S_i^2) = 0 \qquad , \quad i = 1, 2, \ldots , m$$

$$\frac{\partial L}{\partial \lambda_{m+j}} = -\ [\ -x_j + S_{m+j}^2\ ] = 0 \qquad , \quad j = 1, 2, \ldots , n.$$

Upon simplifications, these conditions can be rewritten as

$$\frac{\partial f}{\partial x_j} = \sum_{i=1}^{m} \lambda_i \frac{\partial h^i}{\partial x_j} - \lambda_{m+j} \qquad (j = 1, 2, \ldots, n)$$

$$\lambda_i [\ h^i(\mathbf{x})] = 0 \qquad (i = 1, 2, \ldots, m)$$

$$-\ \lambda_{m+j}\ x_j = 0 \qquad (j = 1, 2, \ldots, n)$$

$$h^i(\mathbf{x}) \le 0 \qquad (i = 1, 2, \ldots, m)$$

$$\lambda_i,\ \lambda_{m+j}\ ,\ x_j \ge 0 \qquad (i = 1, 2, \ldots, m\ ;\ j = 1, 2, \ldots, n)$$

These necessary conditions are known as Kuhn-Tucker conditions. These conditions are sufficient also if $f(\mathbf{x})$ is concave and all $h^i(\mathbf{x})$ are convex in $\mathbf{x}$. Similarly, the Kuhn-Tucker conditions for nlpp of minimization type can be derived as follows:

Let the given nlpp be

Minimize $z = f(\mathbf{x})$, where $\mathbf{x} \in \mathbf{R}^n$

Subject to the constraints $h^i(\mathbf{x}) \ \geq \ \mathbf{0}$, $-\mathbf{x} \le \mathbf{0}$; $i = 1, 2, \ldots , m$.

The Kuhn-Tucker Conditions are :

$$\frac{\partial f}{\partial x_j} = \sum_{i=1}^{m} \lambda_i \frac{\partial h^i}{\partial x_j} - \lambda_{m+j} \qquad (j = 1, 2, \ldots, n)$$

$$\lambda_i [\ h^i(\mathbf{x})] = 0 \qquad (i = 1, 2, \ldots, m)$$

$- \lambda_{m+j}\, x_j = 0$ (j = 1, 2, …, n)

$h^i(\mathbf{x}) \geq 0$ (i = 1, 2, ..., m)

$\lambda_i,\ \lambda_{m+j},\ x_j \geq 0$ (i = 1, 2, ..., m ; j = 1, 2, …, n) .

These conditions are sufficient if f(**x**) is convex and all h^i(**x**) are concave in **x** .

Example 1: Determine x_1 , x_2 so as to

Maximize $z = - x_1^2 - x_2^2 + 4 x_1 + 6 x_2$

Subject to the conditions $x_1 + x_2 \leq 2$; $2x_1 + 3x_2 \leq 12$; $x_1 , x_2 \geq 0$.

Solution:

Here, $f(\mathbf{x}) = - x_1^2 - x_2^2 + 4 x_1 + 6 x_2$

$h^1(\mathbf{x}) = x_1 + x_2 - 2$

$h^2(\mathbf{x}) = 2x_1 + 3x_2 - 12$

Clearly, f(**x**) is concave and $h^1(\mathbf{x})$ and $h^2(\mathbf{x})$ are convex in **x**. Thus, the Kuhn-Tucker conditions will be necessary and sufficient conditions for a maximum. Let

$$L(\mathbf{x}, \mathbf{S}, \lambda) = f(\mathbf{x}) - \lambda_1 [h^1(\mathbf{x}) + S_1^2] - \lambda_2 [h^2(\mathbf{x}) + S_2^2] - \lambda_3 [- x_1 + S_3^2]$$

$$- \lambda_4 [- x_2 + S_4^2]$$

where $\mathbf{S} = (S_1 , S_2 , S_3 , S_4)$, $\lambda = (\lambda_1 , \lambda_2 , \lambda_3 , \lambda_4)$. S_1 , S_2 , S_3 , S_4 are slack variables and $\lambda_1 , \lambda_2 , \lambda_3 , \lambda_4$ are the Lagrangian multipliers. The Kuhn-Tucker conditions are given by

$$- 2 x_1 + 4 = \lambda_1 + 2\lambda_2 - \lambda_3 \quad (1)$$

$$- 2 x_2 + 6 = \lambda_1 + 3\lambda_2 - \lambda_4 \quad (2)$$

$\lambda_1(x_1 + x_2 - 2) = 0$ (3)

$\lambda_2 (2x_1 + 3x_2 - 12) = 0$ (4)

$- \lambda_3 x_1 = 0$ (5)

$- \lambda_4 x_2 = 0$ (6)

$x_1 + x_2 - 2 \leq 0$ (7)

$2x_1 + 3x_2 - 12 \leq 0$ (8)

$\lambda_1 , \lambda_2 , \lambda_3 , \lambda_4 \geq 0$; (9)

$x_1 , x_2 \geq 0$. (10)

We try for non-trivial solution, ie., $x_1 , x_2 \neq 0$. Then, in that case, the Lagrange multipliers $\lambda_3 = 0$, $\lambda_4 = 0$ from equations (5) and (6). Now, there arise four cases.

Case 1: $\lambda_1 = 0$ and $\lambda_2 = 0$.

Then, from equations (1) and (2), we get $x_1 = 2$, $x_2 = 3$. However, this solution violates equations (7) and (8). Hence, we discard this solution.

Case 2: $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

Here, from equations (1) , (2) and (4), we get $x_1 = 24/13$, $x_2 = 36/13$ and $\lambda_2 = 2/13$. However, this solution violates equation (7). So, this solution is also discarded.

Case 3: $\lambda_1 \neq 0$ and $\lambda_2 = 0$.

Here, from equations (1), (2) and (4), we get $x_1 = 1/2$, $x_2 = 3/2$ and $\lambda_1 = 3$ and this solution does not violate any of the Kuhn-Tucker conditions.

Case 4: $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

From equations (3) and (4), we get $x_1 = -6$ and $x_2 = 8$. Further, using equations (1) and (2), we get $\lambda_1 = 68$ and $\lambda_2 = -26$. The values of x_1 violates the equation (10) and the value of λ_2 violates the equation (9). So, we discard this solution.

Hence, the optimum (maximum) solution to the given nlpp is

$x_1 = 1/2$, $x_2 = 3/2$ and $\lambda_1 = 3$, $\lambda_2 = 0$ and the maximum value of the objective function is $z = 17/2$.

Problem 1:

Use the Kuhn-Tucker conditions to solve the following non-linear programming problems:

1. Maximize $z = 8x_1^2 + 2x_2^2$

   Subject to $x_1^2 + x_2^2 \leq 9$; $x_1 \leq 2$; $x_1, x_2 \geq 0$.

2. Minimize $z = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$

   Subject to $x_1 + 3x_2 \leq 6$; $5x_1 + 2x_2 \leq 10$; $x_1, x_2 \geq 0$.

3. Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 5)^2$

   Subject to $-x_1^2 + x_2 \leq 4$; $-(x_1 - 2)^2 + x_2 \leq 3$; $x_1, x_2 \geq 0$.