Mathematical Expectation or Expected Value of a random variable

Let X be a rv. Then the mathematical expectation of X is defined as

$$E(X) = \begin{cases} \sum_{i=1}^{\infty} x_i p(x_i) \ ; \ \text{if x is a drv.} \\ \int_{-\infty}^{\infty} x f(x) dx \ ; \text{if x is a crv} \end{cases}$$

Properties:

(i). $E(C) = C$
(ii). $E(CX) = CE(X)$
(iii). $E(X+Y) = E(X)+E(Y)$
Generalization:
$E(X_1+ X_2 \ldots..+ X_n) = E(X_1)+ E(X_2)+\ldots+ E(X_n)$
i.e. $E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$
(iv). $E(XY) = E(X).E(Y)$ iff X and Y are independent

Variance:
Let X be a rv. Then the variance of X, denoted by σ^2, is given by
$V(X) = E(X - E(X))^2$,
$= E(X^2) - (E(X))^2$

Properties:
(i). $V(C)=0$
(ii). $V(CX) = C^2V(X)$
(iii). $V(X+Y) = V(X) + V(Y)$ iff X and Y independent

$$= E(X+Y)^2 - (E(X+Y))^2$$
$$= E(X^2) + E(Y^2) + 2E(XY) - \left(E(X)\right)^2 - \left(E(Y)\right)^2 - 2E(X)E(Y)$$
$$= E(X^2) - \left(E(X)\right)^2 + E(Y^2) - \left(E(Y)\right)^2 + 2[E(XY) - E(X)E(Y)]$$
$$= V(X)+V(Y)+0$$

$\therefore V(X+Y) = V(X) + V(Y)$
Also $V(aX+bY) = a^2V(X)+b^2V(Y)$ iff X and Y are independent

Covariance of (XY)
$Cov(XY) = E[(X-E(X))(Y-E(Y))]$
$= E(XY)-E(X).E(Y)$

Note 1: If X and Y are independent then $Cov(XY)=0$. However, $Cov(XY)=0$ does not necessarily mean that the rvs are independent.

Note 2: If X and Y are any two rvs then we have
$V(X+Y)=V(X)+V(Y)+2[E(XY)-E(X)E(Y)]=V(X)+V(Y)+2Cov(XY)$

Also, $V(aX + bY) = a^2V(X) + b^2V(Y) + 2ab\ Cov(XY)$

Mean and Variance of Binomial distribution:
Let X~B(n,p)
$P(X=x)=\binom{n}{x}p^x q^{n-x}$; x=0,1,2,…,n
$E(X)=\sum_{x=0}^{n} xp(x)=\cdots=np$
Thus mean of B.D is np
Now, V(X)= $E(X^2)-(E(X))^2$
where, $E(X^2)=\sum_{x=0}^{n} x^2\binom{n}{x}p^x q^{n-x}=\cdots=n(n-1)p^2+np$
$\therefore V(X)=n(n-1)p^2+np-(np)^2$
$= npq$
Thus for the B.D. mean > variance; for 0<p<1

Mean and variance of Uniform distribution:
Let X~U[a,b]
We have $f(x)=\begin{cases}\frac{1}{b-a}; & a\le x\le b\\ 0 \quad ; & \text{elsewhere}\end{cases}$
The mean of uniform distribution is
E(X) = $\int_{-\infty}^{\infty} x.f(x)dx=\int_a^b x\frac{1}{b-a}dx=\frac{a+b}{2}$
Now, V(X) = $E(X^2)-(E(X))^2$
where, $E(X^2)=\int_a^b x^2\frac{1}{b-a}dx=\frac{1}{3}(b^2+ab+a^2)$
Thus V(X) = $\frac{1}{3}(b^2+ab+a^2)-\frac{(a+b)^2}{4}$
V(X) = $\frac{1}{12}(b-a)^2$