MA859: SELECTED TOPICS IN GRAPH THEORY

LECTURE - 11

COLOURING AND PLANARITY

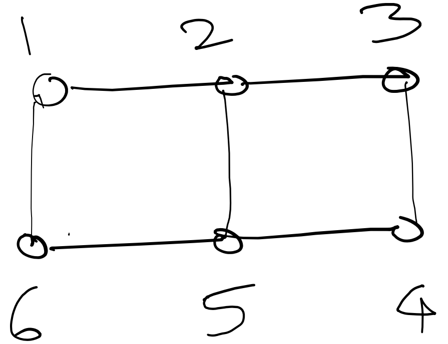
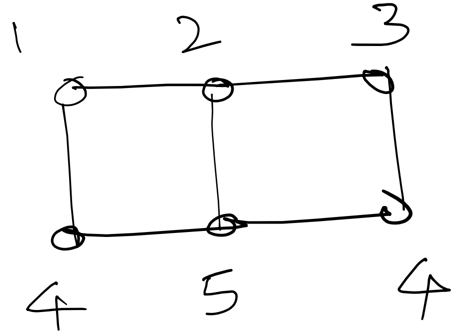
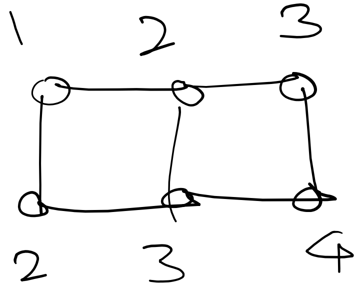
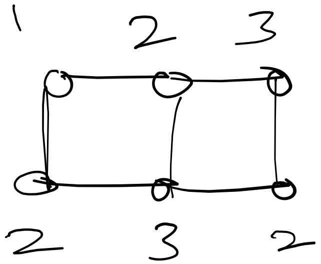
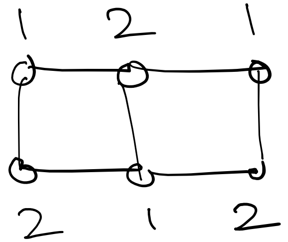
Years back, there was a famous puzzle:
Given a map, what is the minimum number of colours needed?
Interestingly, everyone could perceive that only four colours are needed; but nobody was able to prove it!
Graph Theory could solve this problem with a proof.

A colouring of a graph is an assignment of colours to its vertices so that no two adjacent vertices receive the same colour. Obviously, in such a colouring, the set of all vertices receiving the same colour must be independent. We call it a colour class. An m-colouring of a graph G uses m colours.

The <u>Chromatic Number</u> $\chi(G)$ of a graph G is the minimum m for which G has an m-colouring.

A graph G is <u>m-colourable</u> if $\chi(G) \leq m$ and is m-chromatic if $\chi(G) = m$.

Any graph G on n vertices can have an n-colouring and a χ-colouring. So, it must have an m-colouring where $\chi < m < n$.

$\chi(K_n) = n$ $\chi(K_n - x) = n-1$ $\chi(\overline{K_n}) = 1$

$\chi(K_{m,n}) = 2$ $\chi(C_{2n}) = 2$ $\chi(C_{2n+1}) = 3$

$\chi(\text{Tree}) = 2$ $\chi(\text{Bipartite graph}) = 2$

A graph G is 1-chromatic if and only if it is totally disconnected.

Recall: A graph G is bipartite if and only if it has only even cycles.
This result can be restated in terms of colouring as

Theorem: A graph G is bi-colourable if and only if it has no odd cycles.
So far, no characterization of m-colourable graphs (for $m \geq 3$) is obtained. In fact, such a characterization would have settled the Four colour conjecture!

Also, there is no convenient method to determine the chromatic number of any graph.

However, several bounds for $\chi(G)$ are known.

* For any graph G, $\chi(G) \leq 1 + \max \delta(G')$, where the maximum is taken over all the induced subgraphs G' of G.
* For any graph G, $\chi(G) \leq 1 + \Delta(G)$.

(Exercise: Determine a graph G for which $\chi(G) = 1 + \Delta(G)$)

Theorem: For any graph G on n vertices

$$\frac{n}{\beta_0(G)} \leq \chi(G) \leq n - \beta_0(G) + 1$$

Proof: If $\chi(G) = m$, then $V(G)$ can be partitioned into m colour classes $V_1, V_2, \ldots, V_m$, each of which is an independent set of vertices.

If $|V_i| = n_i$ for $i = 1, 2, \ldots, m$, then clearly

$n_i \leq \beta_0$ and $n = \sum_{i=1}^{m} n_i \leq m\beta_0$

$$\Rightarrow \frac{n}{\beta_0} \leq \chi(G).$$

Now, for the upper bound, let S be a maximal independent set containing β_0 vertices.

Clearly, $\chi(G-S) \geq \chi(G) - 1$.

Since $G-S$ has $n-\beta_0$ vertices,

$$\chi(G-S) \leq n-\beta_0 \implies \chi(G) \leq \chi(G-S)+1 \leq n-\beta_0+1. \quad //$$

The next result is due to Nordhäus and Gaddum.

<u>Theorem</u>: For any graph G on n vertices, let $\chi(G) = \chi$ and $\chi(\overline{G}) = \overline{\chi}$. Then

$$2\sqrt{n} \leq \chi + \overline{\chi} \leq n+1 \longrightarrow (1)$$

$$n \leq \chi\overline{\chi} \leq \left(\frac{n+1}{2}\right)^2 \longrightarrow (2)$$

Proof: Let G be m-chromatic and suppose $V_1, V_2, \ldots, V_m$ are the colour classes of G, where $|V_i| = n_i$. Clearly, $\sum_{i=1}^{m} n_i = n$ and $\max n_i \geq \frac{n}{m}$.

Since each V_i induces a complete subgraph of $\bar{G}$, $\bar{\chi} \geq \max n_i \geq \frac{n}{m}$. $\Rightarrow \chi\bar{\chi} \geq n$.

Also, we know that the Geometric Mean of two positive integers never exceeds their Arithmetic Mean. So, it follows that $\chi + \bar{\chi} \geq 2\sqrt{n}$.

Thus both the lower bounds are established.

Now, to prove the upper bound of (1),
we use induction on n.
Clearly, equality holds for $n=1$ and the result
is true for $n=2$ or 3.
~~Assume~~ that $\chi(G)+\chi(\bar{G}) \leq n$ for all
graphs having $n-1$ vertices.
Suppose H be a graph on n vertices. Let v be
a vertex of H.
Then $H-v$ and $\bar{H}-v$ are clearly complements
of each other and have $n-1$ vertices.

Let $\deg_H v = d$. Then $\deg_{\overline{H}} v = n-d-1$.

Obviously, $\chi(H) \leq \chi(G)+1$ and $\chi(\overline{H}) \leq \chi(\overline{G})+1$.

If either $\chi(H) < \chi(G)+1$ or $\chi(\overline{H}) < \chi(\overline{G})+1$ or both, then clearly $\chi(H)+\chi(\overline{H}) \leq n+1$.

Now, suppose $\chi(H) = \chi(G)+1$ and $\chi(\overline{H}) = \chi(\overline{G})+1$.

This means ~~that~~ ~~the~~ the removal of v from H producing G, decreases the chromatic number; so, $d \geq \chi(G)$. Similarly, $n-d-1 \geq \chi(\overline{G})$

Thus, $\chi(G) + \chi(\overline{G}) \leq n-1$

$\Rightarrow \quad \chi(H) + \chi(\overline{H}) \leq n+1.$

Finally, for the second upper bound, consider the inequality

$$4\chi\overline{\chi} \leq (\chi+\overline{\chi})^2$$

$$\Rightarrow \quad \chi\overline{\chi} \leq \frac{(n+1)^2}{4} = \left(\frac{n+1}{2}\right)^2. \quad //$$

The Chromatic Polynomial

This was introduced by Birkhoff & Lewis while they were attempting to solve 4CC.

Let G be a labeled graph. A colouring of G from t colours is a colouring of G which uses t or fewer colours.

Two colourings of G from t colours are considered to be different if at least one of the labeled vertices is assigned different colours.

Let $f(G, t)$ denote the number of different colourings of a labeled graph G from t colours.

Clearly, $f(G, t) = 0$ if $t < \chi(G)$.

Actually, the smallest t for which $f(G, t) > 0$ is $t = \chi(G)$.

Hence the 4CC asserts that for any planar graph G, $f(G, 4) > 0$.

Consider K_3 for example.
There are t ways of colouring any given vertex of K_3. For the second vertex, any of the $t-1$ colours may be used and for the third, there are $t-2$ ways of colouring.

Thus, $f(K_3, t) = t(t-1)(t-2)$; $t \geqslant 3$

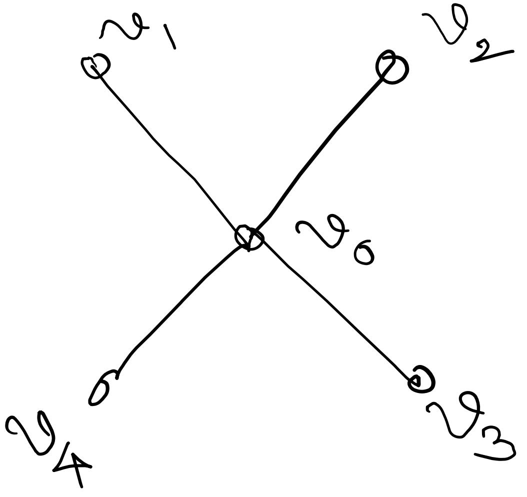
We can generalize this similarly for K_n:

$$f(K_n, t) = t(t-1)\ldots(t-n+1) = t_{(p)}.$$

Now, for $\overline{K_n}$, each vertex can be coloured in t ways independently. So, we may write

$$f(\overline{K_n}, t) = t^n.$$

Consider $K_{1,4}$: v_0 can be coloured in any one of t ways; while the remaining vertices can each be coloured in any one of $t-1$ ways.



$$\therefore f(K_{1,4}, t) = t(t-1)^4.$$

As we see in each of the examples, $f(G, t)$ is a polynomial in t and this is true for any graph G and hence the name <u>Chromatic Polynomial</u>.

The result in the next slide gives a much clearer picture on Chromatic Polynomial.

<u>Theorem</u> If G is a graph on n vertices and m edges,

i) $f(G,t)$ has degree n.

ii) The coefficient of t^n in $f(G,t)$ is 1.

iii) The coefficient of t^{n-1} in $f(G,t)$ is $-m$.

iv) The constant term in $f(G,t)$ is zero.

v) If a graph G has k components, then the smallest exponent of t in $f(G,t)$ with non-zero coefficient is t^k.

Using Chromatic Polynomial, we can give another important characterization of a tree:

<u>Theorem</u>: A graph G on n vertices is a tree if and only if $f(G, t) = t(t-1)^{n-1}$

<u>Proof</u>: First we will show that if G is a tree on n vertices, then $f(G,t) = t(t-1)^{n-1}$.

The result is true for $n=1$ or 2. Assume that the Chromatic Polynomial of all trees with $n-1$ vertices is $t(t-1)^{n-2}$.

Let v be an end vertex of a tree T on n vertices, and let $x = uv$ be the edge incident with v.

By the induction hypothesis, $T' = T - v$ has $t(t-1)^{n-2}$ as its Chromatic Polynomial. Now, the vertex v can be assigned any colour different from the one assigned to u; so, v may be coloured in any one of the $t-1$ ways.

$\therefore f(T, t) = (t-1)\, f(T', t) = t(t-1)^{n-1}$.

Conversely, let G be a graph such that $f(G,t) = t(t-1)^{n-1}$, where n is the number of vertices of G.

Then using the theorem (stated in 3 slides prior to this), we note that the least exponent of t in $f(G,t)$ is 1 and hence G must be connected.

Further, the coefficient of t^{n-1} is $-(n-1)$, which means from the same theorem that G has $n-1$ edges.

Hence G must be a tree. //

Ex. All the students of NITK have to take end semester exam in all the courses they register. Naturally, there can't be concurrent exams when the courses have students in common. How can we organize all the exams in as few parallel sessions as possible?

Soln. When we deploy this problem as a graph, the courses are represented as vertices and whenever two courses have students in common, such vertices are adjacent.

Clearly, the independents sets of the representing graph correspond to conflict-free courses.

Hence the required minimum number of parallel sessions is the chromatic number of the graph. //

<u>Def^n</u>: A graph G is called <u>k-degenerate</u> if every subgraph of G has a vertex of degree less than or equal to k.

The next result connects degeneracy to $\chi(G)$.

<u>Lemma</u>: If G is k-degenerate, then $\chi(G) \leq k+1$.

<u>Proof</u>: We prove this by induction on the number of vertices. The statement is clearly true for all graphs G with at least $k+1$ vertices. Find a vertex of degree $\leq k$ in G. The graph $G - v$ is k-degenerate; so, it can be properly coloured in $k+1$ colours. Then colour the vertex v into the colour that does not appear among the colours of its neighbours. This gives a proper colouring of G. //

Remark: It was mentioned in the beginning that $\chi(G) \leq \Delta(G)+1$.

Here we may note that any graph G is Δ-degenerate.

// END OF LECTURE //