MA859: SELECTED TOPICS IN GRAPH THEORY

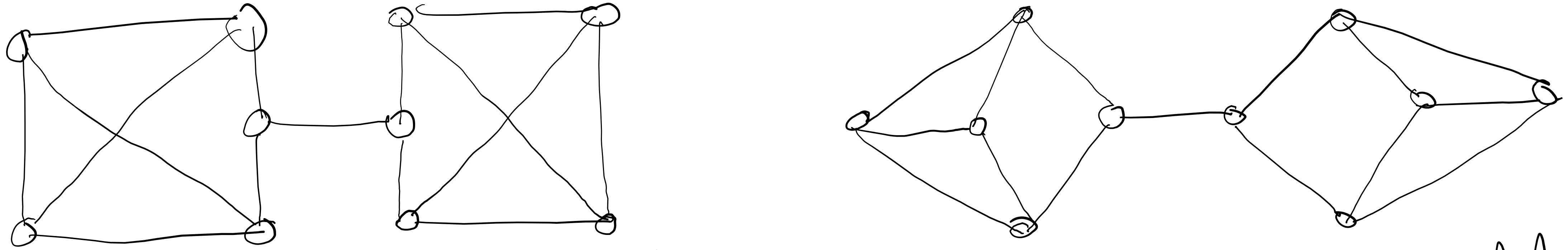
LECTURE-10

PLANARITY

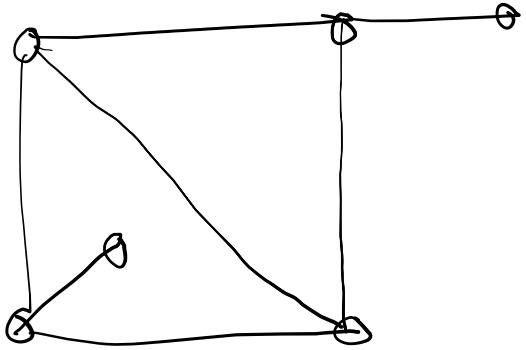
A graph is said to be <u>embedded</u> in a surface if it can be drawn on the surface without any edges overlapping.

A graph G is called <u>planar</u> if it can be embedded in a plane.

A graph that is already embedded in a plane is called a <u>plane graph</u>.



Planar Graph and its plane embedding

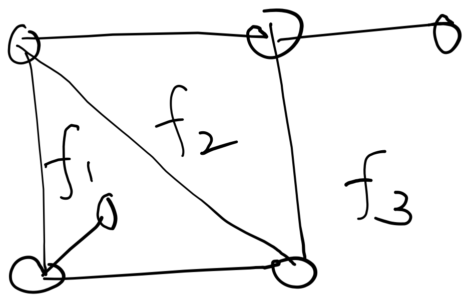


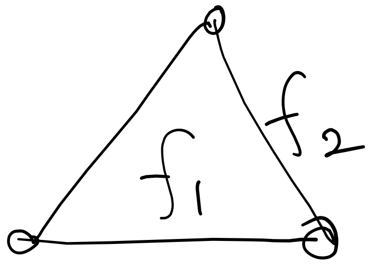
Plane Graph

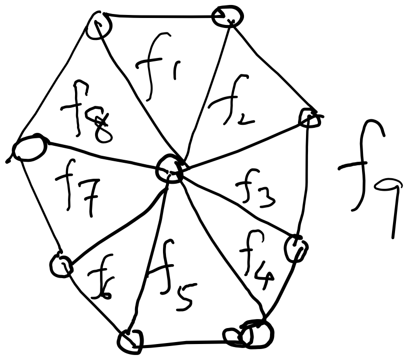
The regions defined by a plane graph (plane embedding of a planar graph) are called its <u>faces</u>.

The bounded region is called <u>interior face</u>.

The <u>exterior face</u> is the unbounded region







The subject of planar graphs was also discovered by Euler during his works on Polyhedrons.

It is possible to associate a graph with every polyhedron consisting of its vertices and edges, which we call, its <u>1-skeleton</u>.

Euler's Polyhedron formula is one of the classical results in Mathematics.

Euler's Polyhedron Formula (EPF)

For any spherical polyhedron with V vertices, E edges and F faces,

$$V - E + F = 2$$

This problem is obviously geometrical. Let us recast it graph theoretically first:

A plane map is a connected plane graph together with all its faces. Then

For a plane map with n vertices, m edges and r faces

$$n - m + r = 2$$

__Proof:__ We prove this by induction on the number of edges, with the result being true for $m = 2$ or 3 or 4.

So, suppose that the result is true for all plane maps with fewer than m edges. Consider a plane map G that has n vertices, m edges and r faces.
Let $x = uv$ be an edge in G. Then $G - x$ has clearly n vertices, $m-1$ edges and $r-1$ faces. So, by the induction hypothesis,

$$n - (m-1) + (r-1) = 2 \Rightarrow n - m + r = 2$$

Thus, the induction is complete.

Theorem: Let G be a planar graph on $n \geq 3$ vertices. Then $|E(G)| \leq 3|V(G)| - 6$

Proof: Assume that G is maximally planar (that is, adding any edge will result is a non-planar graph).

If the bound holds for such G, then it must hold for any planar graph.

Clearly, G is connected. Every edge is in the boundary of two faces and every face has 3 edges on its boundary (otherwise, we could add an edge).

Double counting the pairs (e, f), where the edge e is on the boundary of the face f, we get $2|E(G)| = 3|F_D(G)|$ where D is the planar drawing of G.

Putting this in the EPF,

$$2 = |V(G)| - |E(G)| + \frac{2}{3}|E(G)|$$

$$\Longrightarrow |E(G)| = 3|V(G)| - 6$$

Hence the result. //

Corollary: If G is planar, then it has a vertex of degree at most 5.

Proof: From the theorem proved just now and the Handshaking Lemma, we get

$$\sum_{v \in V(G)} \deg v = 2|E(G)| \leq 6|V(G)| - 12$$

$$\Rightarrow \frac{1}{|V(G)|} \sum_{v \in V(G)} \deg v \leq 6 - \frac{12}{|V(G)|}$$

$\Rightarrow$ Average degree is strictly less than 6. Hence the result. //

Corollary: If G is a planar graph on n vertices, m edges and no triangles, then $m \leq 2n - 4$.

// That's all for this lecture! //