MA 859: Selected Topics in Graph Theory

LECTURE - 6

COVERINGS AND INDEPENDENCE

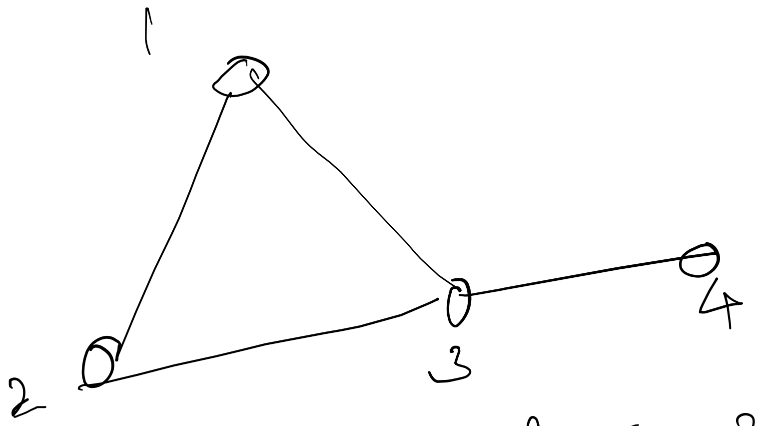
A vertex and an edge are said to <u>cover</u> each other if they are incident.

A set of vertices that covers all the edges of the graph G is called a <u>vertex cover</u> of G.

Likewise, a set of edges that covers all the vertices of G is called an <u>edge cover</u> of G.

The minimum number of vertices in a vertex cover is called the <u>vertex covering number</u> of G and is denoted by $\alpha_0(G)$.

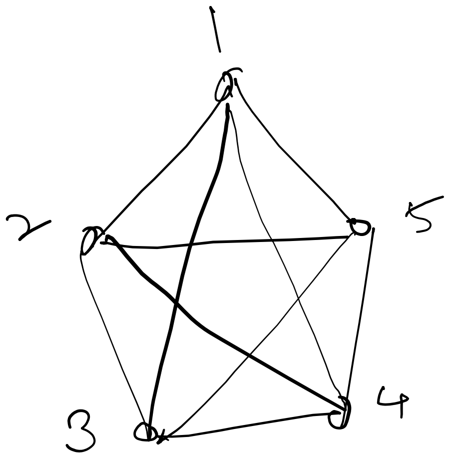
Similarly, the minimum number of edges in an edge cover is called the <u>edge covering number</u> of G and is denoted by $\alpha_1(G)$.



$\{1,3\}$, $\{2,3\}$, $\{1,3,4\}$

vertex covers

$\{(1,2),(3,4)\}$ → edge cover

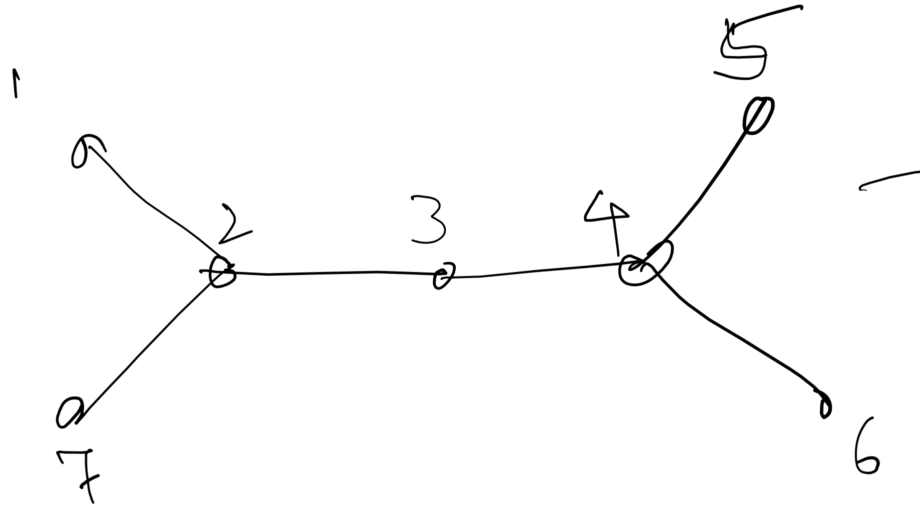


All but one vertices are required to form vertex cover.

$\{(1,2),(3,5),(2,4)\}$

↓

edge cover



$\{2,4\}$ → vertex cover

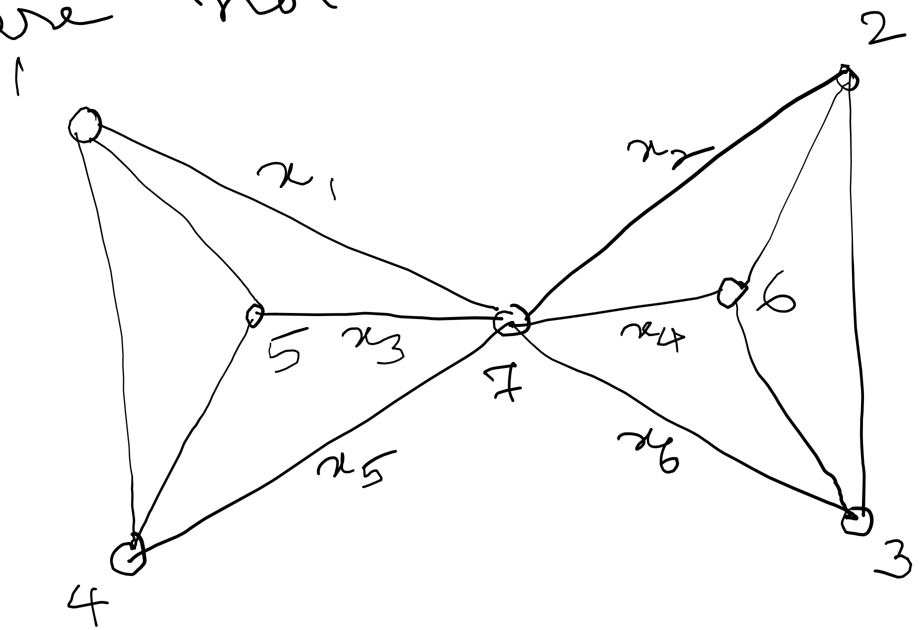
$\{1,5,6,7,3\}$ → vertex cover

$\{(1,2),(2,7),(3,4),(4,5),(4,6)\}$

→ edge cover

A vertex cover or an edge cover is called minimum vertex cover or minimum edge cover if it has α_0 or α_1 elements respectively.

There can be minimal vertex/edge covers which are not minimum.

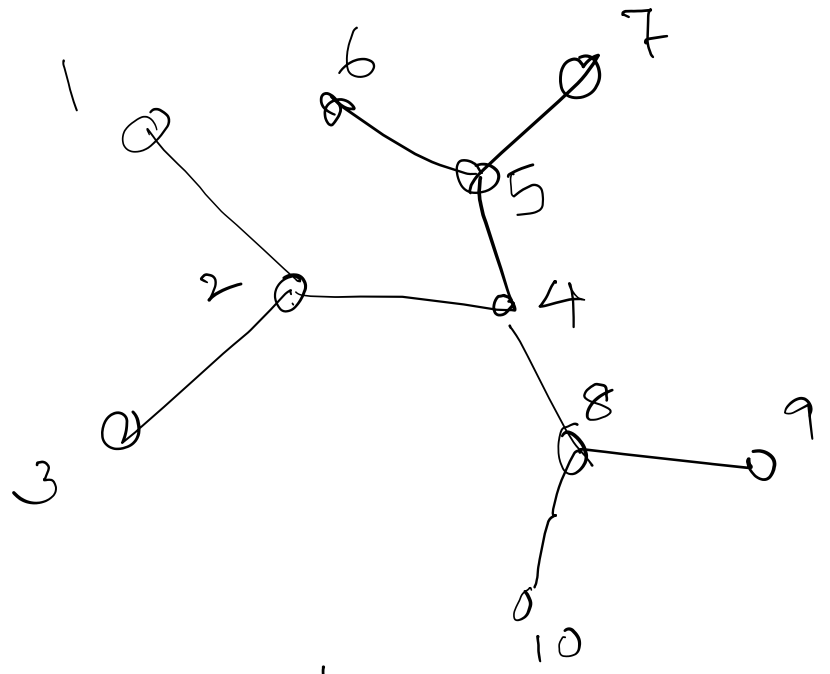


$\{1, 2, 3, 4, 5, 6\}$ — a minimal vertex cover

$\{x_1, x_2, x_3, x_4, x_5, x_6\}$ — minimal edge cover

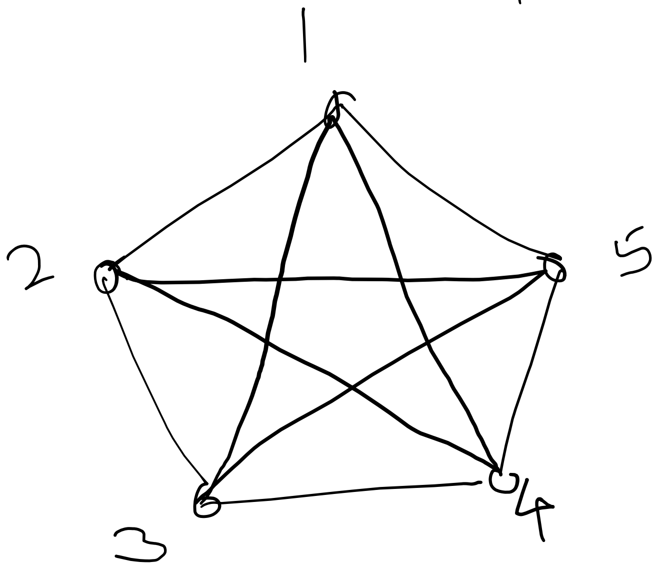
A set of vertices in a graph G is <u>independent</u> if no two of them are adjacent. The maximum number of vertices in such a set is called <u>vertex independence number</u> and is denoted by $\beta_0(G)$.

Analogously, a set of edges of G is <u>independent</u> if no two of them have a common vertex. The maximum number of edges in such a set is called <u>edge independence number</u> and is denoted by $\beta_1(G)$.



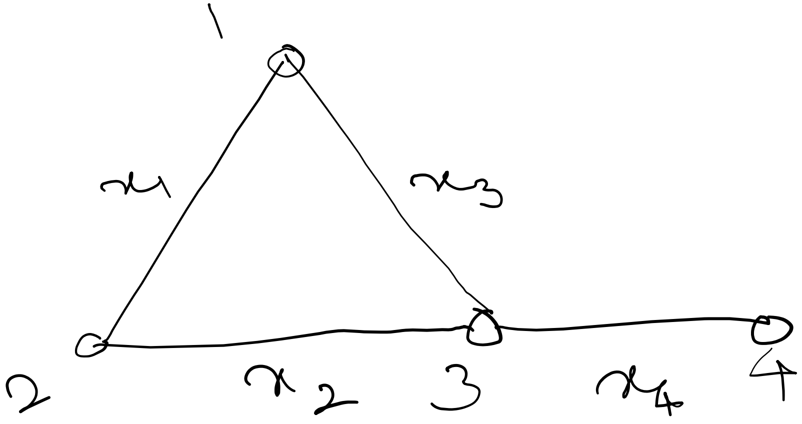
$\{1, 3, 6, 7, 9, 10\} \rightarrow$ independent set of vertices

$\{(1,2), (5,7), (8,9)\} \rightarrow$ independent set of edges.



any single vertex $\rightarrow$ independent set.

$\{(1,2), (3,5)\} \rightarrow$ independent set of edges.

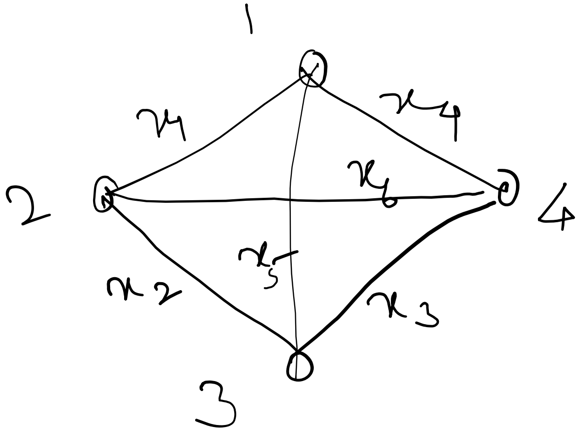


$\alpha_0 =$

$\beta_0 =$

$\alpha_1 =$

$\beta_1 =$

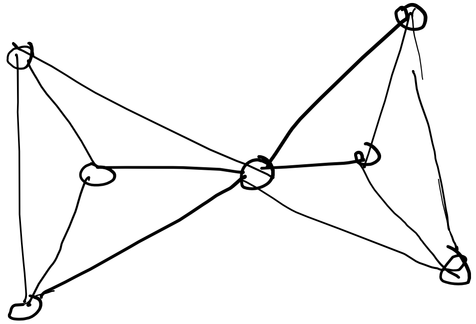


$\alpha_0 =$

$\beta_0 =$

$\alpha_1 =$

$\beta_1 =$



$\alpha_0 =$

$\beta_0 =$

$\alpha_1 =$

$\beta_1 =$

Theorem (Gallai)

For any non-trivial connected graph G on n vertices, $\alpha_0 + \beta_0 = n = \alpha_1 + \beta_1$

Proof: Suppose M_0 is a maximum independent set of vertices. So, $|M_0| = \beta_0$. Note that no two vertices in M_0 are adjacent. So, every edge of G must be incident with some vertex of $n - \beta_0$ vertices. Hence $V - M_0$ must be a vertex cover of G $\Rightarrow |V - M_0| \geq \alpha_0$ or $\alpha_0 \leq n - \beta_0 \rightarrow (1)$

On the other hand, let N_0 be a minimum vertex cover of G. Then $|N_0| = \alpha_0$. Clearly, there can not be any edge which joins any two vertices of $V - N_0$. Hence $V - N_0$ is an independent set. $\Rightarrow \beta_0 \geq |V - N_0|$ or $\beta_0 \geq n - \alpha_0 \rightarrow (2)$

From (1) & (2), we have a dichotomy and hence $\alpha_0 + \beta_0 = n$.

Now, to prove the second equation, consider M_1 to be a maximum independent set of edges.

That is, $|M_1| = \beta_1$.

We construct an edge cover Y by taking the union of M_1 and a set of edges such that we choose one incident each vertex of G not covered by any edge in M_1.

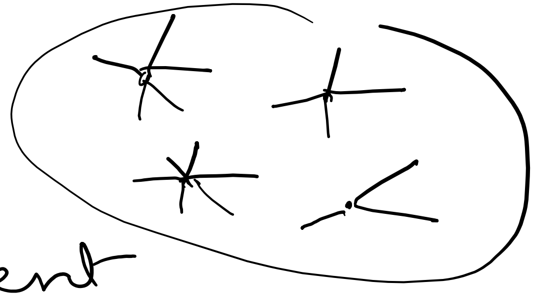
Clearly $|M_1| + |Y| \leq n$ and also, $|Y| \geq \alpha_1$

$$\Rightarrow \alpha_1 + \beta_1 \leq n \longrightarrow (3)$$

On the other hand, let N_1 be a minimum edge cover of G. That is, $|N_1| = \alpha_1$.

Obviously, N_1 can not contain an edge whose both the end vertices are incident with edges which are also in N_1.

Thus, N_1 is a sum of "stars" of G



Select one edge from each of these stars. We get an independent set W of edges and hence $|W| \le \beta_1$

But now, $|N_1| + |W_1| = n$ (since each star covers one more vertex than the number of edges in it).

$$\Rightarrow \alpha_1 + \beta_1 \geq n \longrightarrow (4)$$

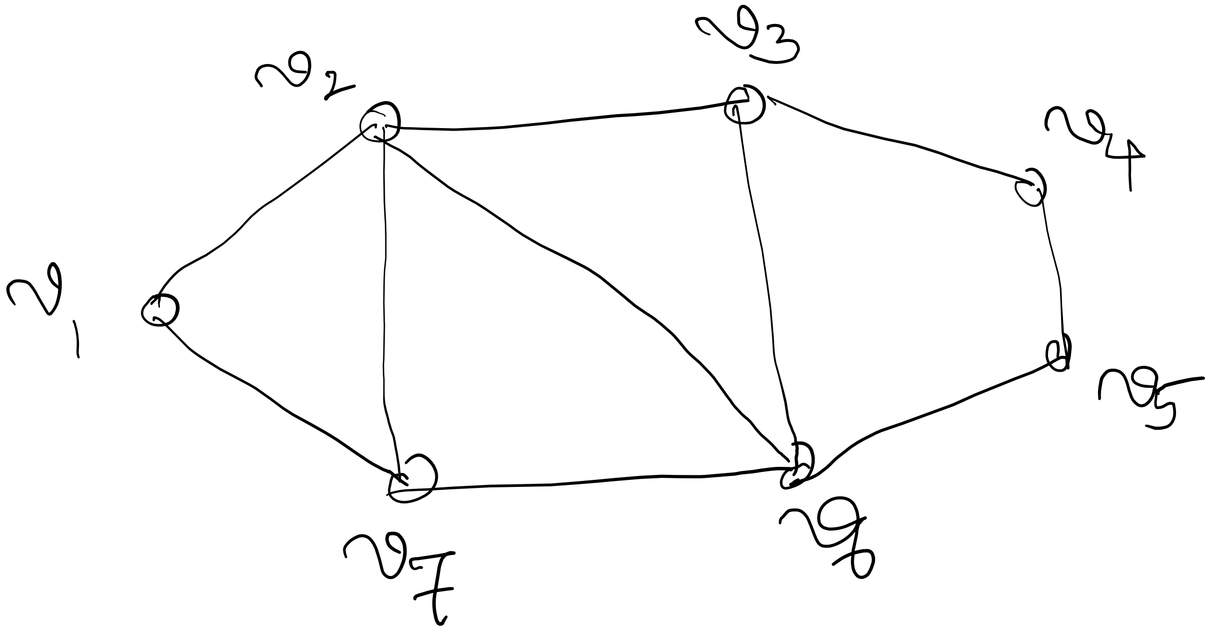
From (3) and (4), we have

$$\alpha_1 + \beta_1 = n. \quad //$$

<u>Exercise</u>: If G is a bipartite graph then

$$\beta_1 = \alpha_0.$$

Find a closed trail of length 7 in the following graph, that is not a cycle:



Ex. Show that the maximum number of edges in a graph on n vertices with w components is $\frac{(n-w)(n-w+1)}{2}$,

Soln Let $G_1, G_2, \dots, G_w$ be the w components of G with n_i no. of vertices in G_i $(i=1,2,\dots w)$. Then if m_i denotes the no. of edges in G_i, clearly $m_i \leq \frac{n_i(n_i-1)}{2}$. So, if m is the no. of edges in G, then $m \leq \sum_{i=1}^{w} \frac{n_i(n_i-1)}{2}$

Now, each $n_i = n - \sum_{j \neq i} n_j = n - w + 1$ (How?).

So, $$\sum_{i=1}^{w} \frac{n_i(n_i - 1)}{2} \le \sum_{i=1}^{w} \frac{(n-w+1)(n_i-1)}{2}$$

$$= \frac{n-w+1}{2} \sum_{i=1}^{w} (n_i - 1) = \frac{(n-w+1)(n-w)}{2}$$ //

Ex. Prove that in a connected graph G on at least 3 vertices, any two longest paths must have a common vertex.

Soln Suppose $P: u_1, u_2, ---, u_k$ and $Q: v_1, v_2, --v_k$ are two longest paths having no vertex in common. Since G is connected, there exists a $u_1 - v_1$ path P' in G. Then there exist u_r' and v_s' in P' $(1 \leq r \leq k$ and $1 \leq s \leq k)$ such that the $u_r - v_s$ section P'' of P has no internal vertex in common with P and Q.

That's all for this lecture!