MA 859: Selected Topics in Graph Theory

Lecture - 4

Distances in Graphs

The distance between two vertices in a graph is a simple, but surprisingly useful notion. It leads to the definition of several graph parameters such as the diameter, the radius, the average distance and the metric dimension.

We discuss & examine these invariants – how they relate to one another and other graph invariants

A path in a graph is a sequence of distinct vertices, such that adjacent vertices in the sequence are adjacent in the graph.

For an unweighted graph, the <u>length</u> of a path is the no. of edges on the path.

However, for an edge-weighted graph, the length of a path is the sum of the weights of the edges on the path.

We assume that all weights are non-negative and that all graphs are connected.

We shall start with undirected graphs.

The <u>distance</u> between two vertices u and v, denoted by $d(u,v)$, is the length of a shortest path (also called a u-v geodesic).

The distance function is a metric on the vertex set of a (weighted) graph G. In particular, it satisfies the <u>triangle inequality</u>:

$$d(a,b) \leq d(a,c) + d(c,b) \quad \forall\ a,b,c \in V(G).$$

The <u>diameter</u> of a connected graph G, denoted by diam (G), is the maximum distance between two vertices.

The <u>eccentricity</u> of a vertex is the maximum distance from it to any other vertex.

The <u>radius</u>, denoted by rad (G), is the minimum eccentricity among all the vertices of G.

(Of course, the diameter is the maximum eccentricity among all vertices).

For a (weighted) undirected graph G,

$$\mathrm{rad}(G) \le \mathrm{diam}(G) \le 2\,\mathrm{rad}(G).$$

The upper bound follows from the triangle inequality ($d(a,b) \le d(a,c) + d(c,b)$), where c is a vertex of minimum eccentricity.

1. $\mathrm{diam}(K_n) = \mathrm{rad}(K_n) = 1$ (for $n \geq 2$)
2. $\mathrm{diam}(P_n) = n-1$; $\mathrm{rad}(P_n) = \left\lceil \frac{n-1}{2} \right\rceil$
3. $\mathrm{diam}(C_n) = \mathrm{rad}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$

Note that the cycles and complete graphs are vertex transitive; so, the radius and the diameter are automatically the same (every vertex has the same eccentricity).

The <u>centre</u> of a graph is the subgraph induced by the set of vertices of minimum eccentricity.

The graphs G for which $rad(G) = diam(G)$ are called <u>self-centred</u>.

A famous result (originally due to Jordan) is as follows:

<u>Theorem</u>: For trees T, the diameter equals either $\underline{2\,\text{rad}(T)}$ or $\underline{2\,\text{rad}(T) - 1}$.

$\downarrow$ This is when the centre is a single vertex

$\downarrow$ This is when the centre is a pair of adjacent vertices.

The diameter is the most common among the classical distance parameters in Graph Theory. Suppose G is a connected graph on n vertices. Clearly, $1 \leq \operatorname{diam}(G) \leq n-1$. Both the bounds are sharp since $\operatorname{diam}(K_n) = 1$ $\operatorname{diam}(P_n) = n-1$.

We, however, get a better upper bound if we consider the no. of edges in a graph.

Suppose G is a connected graph on n vertices and m edges.

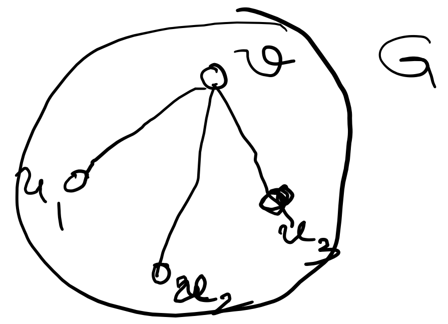
Then $\operatorname{diam}(G) \leq n + \frac{1}{2} - \sqrt{2m - 2n + \frac{17}{4}}$

Now, before moving to more results on Distances in graphs, let us consider some simple; but interesting problems:

1) For any graph G on 6 vertices, G or $\overline{G}$ contains a triangle.

<u>Soln</u> Let G be a graph on 6 vertices and v be any vertex in G. Then clearly, v is adjacent to the remaining 5 vertices either in G or in $\overline{G}$.

WLOG, suppose v is adjacent to 3 vertices u_1, u_2, u_3 in G.



If any two of u_1, u_2 and u_3 are adjacent in G, then we have a triangle in G.

Otherwise, since neither of them are adjacent in G, they must adjacent in $\overline{G}$ mutually; thereby forming a triangle in $\overline{G}$. //

In any graph G, every vertex is associated with a degree. Hence G has a minimum degree and a maximum degree. We denote them by $\delta(G)$ and $\Delta(G)$ respectively.

If $\delta(G) = \Delta(G) = r$, then the degrees of all the vertices of G must be r. We call it <u>r-regular</u> graph.

2) Let G be a graph on n vertices and m edges. Then $\delta(G) \leq \frac{2n}{m} \leq \Delta(G)$.

Soln This result is a simple consequence of the Handshaking Lemma, namely,

$$\sum_{i=1}^{n} \deg v_i = 2m.$$

$\Rightarrow$ $\frac{2m}{n}$ is the average degree. Hence it must lie between $\delta(G)$ and $\Delta(G)$.

3) If G is disconnected, then $\overline{G}$ must be connected.

Sol^n To show that $\overline{G}$ is connected, we have to show that any two vertices in $\overline{G}$ are joined by a path. Let u and v be any two vertices of G (or $\overline{G}$).

* If they belong to two different components of G, then they must be adjacent in $\overline{G}$.
* If they belong to the same component of G, then choose a vertex w that belongs to a different component of G. Then both u and v must be adjacent to w in $\overline{G}$; thereby there exists a u-v path in $\overline{G}$ through w.

Hence $\overline{G}$ must be connected.

4) Let G be a graph on $n \geqslant 2$ vertices. If $\deg v \geqslant \frac{n-1}{2}$ for every vertex v of G, then G must be connected.

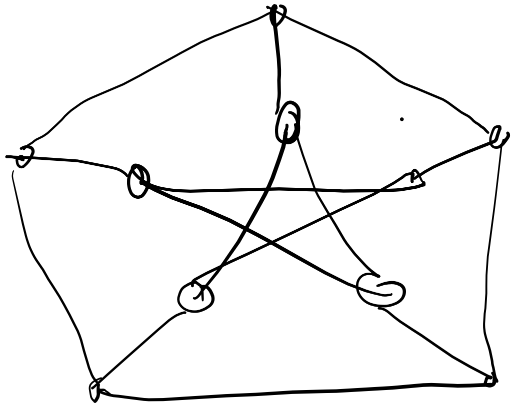
Sol^n $\deg v \geqslant \frac{n-1}{2} \Rightarrow \delta(G) \geqslant \frac{n-1}{2}$.

$\Rightarrow$ There exists a vertex of minimum degree $\geqslant \frac{n-1}{2}$.

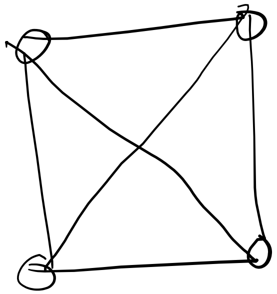
Hence G must have at least one component with at least $\frac{n-1}{2}+1$ vertices, that $\frac{n+1}{2}$ vertices.

So, the remaining components have at most $n-\left(\frac{n+1}{2}\right) = \frac{n-1}{2}$ vertices. This means that the maximum degree of any vertex in those components is $\frac{n-1}{2} - 1 = \frac{n-3}{2} < \frac{n-1}{2}$, which is a contradiction. So, there can not be more than one component in $G \Rightarrow G$ must be connected. //

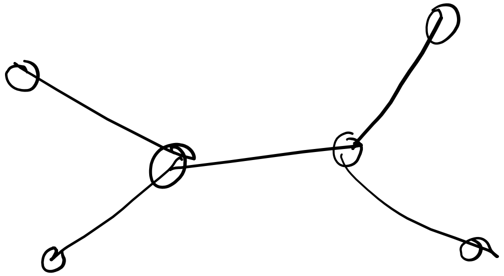
Defn: Girth of a graph G is the length of a shortest cycle in it.



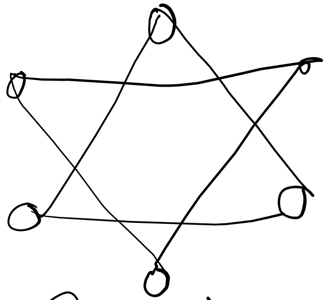
Girth $= 5$



Girth $= 3$

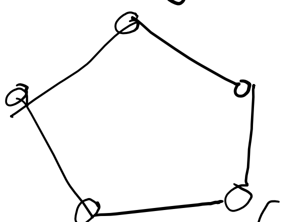


Girth $= 0$

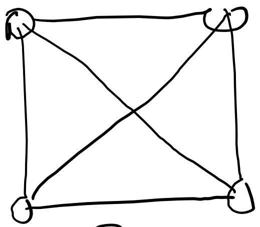


Girth $= 3$

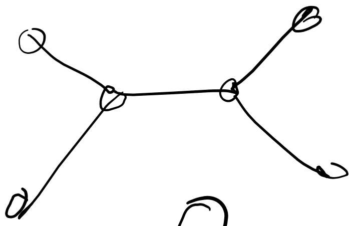
Defn: Circumference of a graph G is the length of any longest cycle in it.



Circumference $= 5$ $=$ Girth



Circumference $= 4$



0

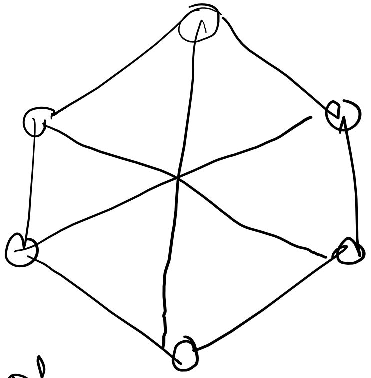
A decomposition of a graph G is a collection of edge-disjoint subgraphs so that every edge of G belongs to exactly one subgraph.

A graph G is called a regular graph if the degrees of the vertices are same.

r-regular graph $\longrightarrow$ degree of every vertex $= r$

K_n is $(n-1)$-regular.

Cycle C_n is 2-regular



$\rightarrow$ 3-regular

Complete bipartite graph is a bipartite graph in which every vertex of one partition is adjacent to every vertex of the other partition. If one part has m vertices & the other, n, then we denote it $K_{m,n}$.

Ex: K_n decomposes into 3 pairwise-isomorphic subgraphs iff $n+1$ is not divisible by 3.

Soln K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. If $(n+1)$ is divisible by 3, then n and $(n-1)$ are not divisible by 3. Hence decomposition into 3 subgraphs of equal size is impossible in this case.

If $(n+1)$ is not divisible by 3, then n or $(n-1)$ is divisible by $3 \implies$ the no. of edges in K_3 is divisible by 3. We construct a decomposition into 3 subgraphs that are pairwise isomorphic. (there are many such decompositions).

// END OF LECTURE - 4 //

If n is a multiple of 3, we partition the vertex set into 3 subsets V_1, V_2, V_3 of equal size. Edges now have two types: within a set or joining two sets.

Let the i^{th} subgraph G_i consist of all the edges within V_i and all the edges joining the two other subsets. Each edge of K_n appears in exactly one of these subgraphs, and each G_i is isomorphic to the disjoint union of $K_{\frac{n}{3}}$ and $K_{\frac{n}{3}, \frac{n}{3}}$.

When $n \equiv 1 \pmod{3}$, consider one vertex w. Since $(n-1)$ is a multiple of 3, we can form the subgraphs G_i as before on the remaining $(n-1)$ vertices.

Modify G_i to form H_i by joining w to every vertex of V_i. Each edge involving w has been added to exactly one of the three subgraphs.

Each H_i is isomorphic to the disjoint union of $K_{1+\frac{(n-1)}{3}}$ and $K_{\frac{n-1}{3}, \frac{n-1}{3}}$.

Ex. If K_n decomposes into triangles, then $(n-1)$ or $(n-3)$ is divisible by 6.

Soln Such a decomposition requires that the degree of each vertex is even and the no. of edges should be divisible by 3.

To have even degree, n must be odd. Also, $\frac{n(n-1)}{2}$ is a multiple of 3; so, 3 divides n or $(n-1)$.

If 3 divides n and n is odd, then $(n-3)$ is divisible by 6.

If 3 divides $n-1$ and n is odd, then $(n-1)$ is divisible by 6. //