Assignment No.:- 2

Numerical Calculation of Eigenvalues

* Problem set - A

① Consider matrices

$$H_{10\times10} = \begin{bmatrix} 1 & & & & 0 \\ 1 & 1 & & & \\ 0 & 1 & 1 & & \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \quad \& \quad H+E$$

Where,

$$E = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1/2^{10} \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{10\times10}$$

ⓐ What are eigen values of H?

→ Here,

$$|H-\lambda I| = \begin{bmatrix} 1-\lambda & & & & & & & & \\ 1 & 1-\lambda & & & & & & 0 & \\ 0 & 1 & 1-\lambda & & & & & & \\ 0 & & 1 & 1-\lambda & & & & & \\ 0 & & & 1 & 1-\lambda & & & & \\ \vdots & & & & 1 & 1-\lambda & & & \\ \vdots & & 0 & & & 1 & 1-\lambda & & \\ \vdots & & & & & & 1 & \ddots & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1-\lambda \end{bmatrix}_{10\times10}$$

It has $(1-\lambda)$ on diagonal

$\therefore |H - \lambda I| = (1-\lambda)^{10} = 0$

$\therefore (1-\lambda)^{10} = 0$

$\therefore \boxed{\lambda = 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}$

$\therefore$ matrix H has eigen value of 1 with multiplicity of 10.

ⓑ Show that $\lambda = 1/2$ is an eigen value of $H+E$.

→ Here,

$$|(H+E) - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & \cdots & 0 & \frac{1}{2^{10}} \\ 1 & 1-\lambda & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & 0 \\ 0 & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1-\lambda \end{vmatrix}_{10 \times 10}$$

$\therefore |(H+E) - \lambda I| = (1-\lambda)^{10} - \frac{1}{2^{10}} = 0$

$\therefore \boxed{\lambda = 1/2}$

Hence the proof.

ⓒ Show that $\|E\|_F = 1/2^{10}$.

→ We know that.

$$\|A\|_F = \sqrt{\Sigma a_{ij}^2}$$

$$\therefore \|E\|_F = \sqrt{\left(\frac{1}{2^{10}}\right)^2 + 99\times 0^2}$$

$$\therefore \boxed{\|E\|_F = \frac{1}{2^{10}}}$$

(d) Why does the stability corrollary not apply to H & H+E?

→ As the stability corollary follows Hypothesis made /used for stability theorem, ~~that stat~~ which states the matrices should be real & symmetric.
But Here, neither H nor E are not symmetric & also H + E is not symmetric.
Therefore the stability corollary does not apply here.

(2) Calculate Frobenius norm of the following matrices.

(a) $D = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots & d_n \end{bmatrix} \Rightarrow \|D\|_F = \sqrt{\sum_{i=1}^{n} d_i^2}$

(b) $I_{n\times n} \Rightarrow \|I\|_F = \sqrt{n\times 1^2} = \sqrt{n}.$

(c) 0 (zero matrix) $\Rightarrow \|0\|_F = \sqrt{(n\times n)\times 0^2} = \underline{\underline{0}}$

ⓓ $D+I = \begin{bmatrix} d_1+1 & & & O \\ & d_2+1 & & \\ O & & \ddots & \\ & & & d_n+1 \end{bmatrix} \Rightarrow \|D+I\|_F = \sqrt{\sum_{i=1}^{n} (d_i+1)^2}$

③ Regarding Prob. 2, which is larger?

$$\|D+I\|_F \quad \text{or} \quad \|D\|_F + \|I\|_F$$

→ Let us consider a matrix A, as a vector $\tilde{A} = (a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots a_{n1}, \dots, a_{nn})$ in E^{n^2}

Now $\|A\|_F$ is just the standard norm of A in E^{n^2}.

$\therefore$ Illy, $\|A+B\|_F = \|\tilde{A}+\tilde{B}\|_F$ &

By triangle inequality $\|\tilde{A}+\tilde{B}\| \leq \|\tilde{A}\| + \|\tilde{B}\|$.

Thus,
$\|A+B\|_F \leq \|A\|_F + \|B\|_F$ for any 2 - $n \times n$ mat.

Thus, we same reasoning,
$\|D+I\|_F \leq \|D\|_F + \|I\|_F$.

(4) Let A be an $n \times n$ matrix. If $\|A\|_F = 0$, must A be the zero matric?

→ As we know that for a matrix $A = [a_{ij}]_{n \times n}$

the frobenius norm is

$$\|A\|_F = \sqrt{\Sigma a_{ij}^2}$$

Hence as, $\|A\|_F$ is square root of sum of square of all the elements of matrix A,

Therefore, the matrix A has to be zero matrix.

So, Yes. if $\|A\|_F = 0 \Rightarrow A = 0$

(5) Let $A = [a_{ij}]_{n \times n}$. Define the 1 norm of A by

$$\|A\|_1 = \sum_{1 \le i,j \le n} |a_{ij}|$$

Let, $A = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$; calculate $\|A\|_1$ & $\|A\|_F$. which norm is larger?

→ Here,

$$\|A\|_1 = 1 + 2 + 2 + 0 = 5$$

$$\|A\|_F = \sqrt{1 + 4 + 4 + 0} = \sqrt{9} = 3$$

$$\therefore \boxed{\|A\|_1 > \|A\|_F}$$

⑥ Suppose eigen values of an $n \times n$ symmetric matrix A are to be computed. Because of a data entry error, every entry of A has 0.0001 added to it. What is the error bound for $|\lambda_k - \tilde{\lambda}_k|$ as given in stability corollary? How does the error bound change as n increases? What can you say about the stability of eigen value problem for large vs. small matrices?

→ Given $A_{n \times n}$ & $\varepsilon = 0.0001$

let $E_{n \times n}$ with $[e_{ij}] = 0.0001$ ε $\forall\, i,j$

Then by stability corollary,

$$|\lambda - \tilde{\lambda}_k| \leq \sqrt{(n \times n) \times \varepsilon^2}$$

$$\leq n\varepsilon$$

(i) $\therefore$ $\boxed{|\lambda - \tilde{\lambda}_k| \leq n \times 0.0001}$

(ii) Now, as we go on increasing "n", the error bound goes on increasing.

(iii) For smaller matrices, the approxiamate eigen values do not differ much from actual eigen values. (as n is small, $\therefore |\lambda - \tilde{\lambda}_k|$ is small)

But for large matrices the difference betn actual| correct| theoretical eigen values ~~is much more than~~ & approxiamate eigen values is much more.

Thus the eigen values of smaller matrix tends to be more stable than eigen values of larger matrix.

* Problem set - 8

Q. In Prob. 1 to 5 use power method to calculate approximations to the dominant eigen pair (if a dominant eigenpair exists). If the method does not work give reason:

① $\begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix}$

→

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 5 \\ 5 & 6-\lambda \end{vmatrix} = (1-\lambda)(6-\lambda) - 25 = 0$$

$$6 - \lambda - 6\lambda + \lambda^2 - 25 = 0$$

$$\therefore \cancel{\lambda =}$$

$$\lambda^2 - 7\lambda - 19 = 0 \Rightarrow \lambda = 9.090169, -2.09016.$$

Using power method,

Let $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$

$$A x_0 = \begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

$$A^2 x_0 = A(Ax_0) = \begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 6 \\ 11 \end{bmatrix} = \begin{bmatrix} 61 \\ 96 \end{bmatrix}$$

~~Ratio = 61/96 =~~

$$A^3 x_0 = A(A^2 x_0) = \begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 61 \\ 96 \end{bmatrix} = \begin{bmatrix} 541 \\ 881 \end{bmatrix}$$

$$A^4 x_0 = A(A^3 x_0) = \begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 541 \\ 881 \end{bmatrix} = \begin{bmatrix} 4946 \\ 7991 \end{bmatrix}$$

$$A^5 x_0 = A(A^4 x_0) = \begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 4946 \\ 7991 \end{bmatrix} = \begin{bmatrix} 44901 \\ 72676 \end{bmatrix}$$

$$A^6 x_0 = A(A^5 x_0) = \begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 44901 \\ 72676 \end{bmatrix} = \begin{bmatrix} 408281 \\ 660561 \end{bmatrix}$$

Now, using formula.

$$\frac{A^{m+1} x_0 \cdot A^m x_0}{A^m x_0 \cdot A^m x_0} = \lambda_1$$

$$\therefore \lambda_1 = \frac{\begin{bmatrix} 408281 & 660561 \end{bmatrix} \begin{bmatrix} 44901 \\ 72676 \end{bmatrix}}{\begin{bmatrix} 44901 & 72676 \end{bmatrix} \begin{bmatrix} 44901 \\ 72676 \end{bmatrix}}$$

$$\therefore \lambda_1 = 9.090169$$

& corresponding eigen vector is $\begin{bmatrix} 408281 \\ 660561 \end{bmatrix}$

i.e. dominant eigen value & eigen vector pairs ~~are~~ is

$$\boxed{\lambda = 9.090169 \ \& \ X = \begin{bmatrix} 1 \\ 1.6179 \end{bmatrix}}$$

(2) $\begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

→ Sol^n:- Using Power Method,

let $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\therefore A x_0 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix}$$

$$A^2 x_0 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 37 \\ 19 \\ 4 \end{bmatrix}$$

$$A^3 x_0 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 37 \\ 19 \\ 4 \end{bmatrix} = \begin{bmatrix} 187 \\ 94 \\ 8 \end{bmatrix}$$

$$A^4 x_0 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 187 \\ 94 \\ 8 \end{bmatrix} = \begin{bmatrix} 937 \\ 469 \\ 16 \end{bmatrix}$$

$$A^5 x_0 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 937 \\ 469 \\ 16 \end{bmatrix} = \begin{bmatrix} 4687 \\ 2344 \\ 32 \end{bmatrix}$$

$$A^6 x_0 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4687 \\ 2344 \\ 32 \end{bmatrix} = \begin{bmatrix} 23437 \\ 11719 \\ 64 \end{bmatrix}$$

$$A^7 x_0 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 23437 \\ 11719 \\ 64 \end{bmatrix} = \begin{bmatrix} 117187 \\ 58594 \\ 128 \end{bmatrix}$$

Now,
Using

$$\frac{A^{m+1}x_0 \cdot A^m x_0}{A^m x_0 \cdot A^m x_0} = \lambda$$

$$\therefore \lambda = \frac{\begin{bmatrix} 117187 & 58594 & 128 \end{bmatrix} \begin{bmatrix} 24437 \\ 11719 \\ 64 \end{bmatrix}}{\begin{bmatrix} 24437 & 11719 & 64 \end{bmatrix} \cdot \begin{bmatrix} 24437 & 11719 & 64 \end{bmatrix}^T}$$

$\therefore \lambda = 5.000033.$

& corresponding eigen vector $\begin{bmatrix} 117187 \\ 58594 \\ 128 \end{bmatrix}$

$\therefore$ dominant pair $\Rightarrow$ $\boxed{\lambda = 5 \text{ \& } X = \begin{bmatrix} 1 \\ 0.5 \\ 0.00109 \end{bmatrix}}$

③ $\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$

$\rightarrow$ Solⁿ:- let $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Before applying power method let's first check the nature of eigen values of given matrix.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ -2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) + 6 = 0$$

$$\therefore 2 - 2\lambda - \lambda + \lambda^2 + 6 = 0$$

$$\therefore \lambda^2 - 3\lambda + 8 = 0$$

$$\therefore \lambda = \frac{3 \pm \sqrt{9 - 32}}{2} = \frac{3 \pm \sqrt{-23}}{2}$$

Here we have $\sqrt{-23}$ in numerator.
Therefore $\forall$ λ's of A are complex.

Thus there is no dominant eigen value & eigen vector pair for given matrix.

(4) $\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$

→ Sol^n:- let $x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$

$\therefore A x_0 = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

$A^2 x_0 = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 42 \\ 58 \end{bmatrix}$

$A^3 x_0 = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 42 \\ 58 \end{bmatrix} = \begin{bmatrix} 300 \\ 416 \end{bmatrix}$

$A^4 x_0 = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 300 \\ 416 \end{bmatrix} = \begin{bmatrix} 2148 \\ 2980 \end{bmatrix}$

$A^5 x_0 = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2148 \\ 2980 \end{bmatrix} = \begin{bmatrix} 15384 \\ 21344 \end{bmatrix}$

$$A^6 x_0 = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 15384 \\ 21344 \end{bmatrix} = \begin{bmatrix} 110184 \\ 152872 \end{bmatrix}$$

$$A^7 x_0 = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 110184 \\ 152872 \end{bmatrix} = \begin{bmatrix} 789168 \\ 1094912 \end{bmatrix}$$

Using following formula.

$$\frac{A^7 x_0 \cdot A^6 x_0}{A^6 x_0 \cdot A^6 x_0} = \lambda$$

$$\therefore \lambda = \frac{\begin{bmatrix} 789168 & 1094912 \end{bmatrix} \begin{bmatrix} 110184 \\ 152872 \end{bmatrix}}{\begin{bmatrix} 110184 & 152872 \end{bmatrix} \begin{bmatrix} 110184 \\ 152872 \end{bmatrix}}$$

$$\therefore \boxed{\lambda = 7.162276 \ \& \ x = \begin{bmatrix} 789168 \\ 1094912 \end{bmatrix}}$$

$\therefore$ Dominant pair is $\lambda = 7.162276$ & $x = \begin{bmatrix} 1 \\ 1.3874 \end{bmatrix}$

(5) $\begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

→ Sol^n:-

let, $x_0 = [1 \ 1 \ 1]^T$

Using power method

$$Ax_0 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 6 \end{bmatrix}$$

$$A^2x_0 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 42 \\ 58 \\ 36 \end{bmatrix}$$

$$A^3x_0 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 42 \\ 58 \\ 36 \end{bmatrix} = \begin{bmatrix} 300 \\ 416 \\ 216 \end{bmatrix}$$

$$A^4x_0 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 300 \\ 416 \\ 216 \end{bmatrix} = \begin{bmatrix} 2148 \\ 2980 \\ 1296 \end{bmatrix}$$

$$A^5 x_0 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 2148 \\ 2980 \\ 1296 \end{bmatrix} = \begin{bmatrix} 15384 \\ 21344 \\ 7776 \end{bmatrix}$$

$$A^6 x_0 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 15384 \\ 21344 \\ 7776 \end{bmatrix} = \begin{bmatrix} 110184 \\ 152872 \\ 46656 \end{bmatrix}$$

$$A^7 x_0 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 110184 \\ 152872 \\ 46656 \end{bmatrix} = \begin{bmatrix} 789168 \\ 1094912 \\ 279936 \end{bmatrix}$$

Using following formula.

$$\frac{A^7 x_0 \cdot A^6 x_0}{A^6 x_0 \cdot A^6 x_0} = \lambda$$

$$\therefore \boxed{\lambda = 7.\cancel{0}65145 \quad \& \quad x = \begin{bmatrix} 2.82 \\ 3.91 \\ 1 \end{bmatrix}}$$

This is the dominant pair of eigen value & eigen vector.

6 Power Method With Scaling.

6.1 (1) $\begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix}$

→ Solⁿ:- let $x_0 = [1 \quad 1]^T$

$$Ax_0 = \begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix} \quad \& \quad V_1 = \frac{Ax_0}{11} = \begin{bmatrix} 0.545 \\ 1 \end{bmatrix}$$

$$AV_1 = \begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0.545 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.545 \\ 8.725 \end{bmatrix} \quad \& \quad V_2 = \begin{bmatrix} 0.635 \\ 1 \end{bmatrix}$$

$$AV_2 = \begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0.635 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.6354 \\ 9.177 \end{bmatrix} \quad \& \quad V_3 = \begin{bmatrix} 0.614 \\ 1 \end{bmatrix}$$

$$AV_3 = \begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0.614 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.614 \\ 9.0704 \end{bmatrix} \quad \& \quad V_4 = \begin{bmatrix} 0.619 \\ 1 \end{bmatrix}$$

$$AV_4 = \begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0.619 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.6189 \\ 9.0947 \end{bmatrix} \quad \& \quad V_5 = \begin{bmatrix} 0.618 \\ 1 \end{bmatrix}$$

$$AV_5 = \begin{bmatrix} 1 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 0.618 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.6178 \\ 9.089 \end{bmatrix} \quad \& \quad V_6 = \begin{bmatrix} 0.618 \\ 1 \end{bmatrix}$$

$$\lambda = \frac{AV_5 \cdot V_5}{V_5 \cdot V_5} = \frac{[5.6178 \quad 9.089][0.618 \quad 1]^T}{[0.618 \quad 1][0.618 \quad 1]^T}$$

$$\therefore \boxed{\lambda = 9.08935 \quad \& \quad x = \begin{bmatrix} 0.618 \\ 1 \end{bmatrix}}$$

This is dominant Pair.

6.2 (2)
$$\begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

→ sol^n: let $x_0 = [1 \quad 1 \quad 1]^T$

$$Ax_0 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix} \quad \& \quad V_1 = \begin{bmatrix} 1 \\ 0.571 \\ 0.233 \end{bmatrix}$$

$$AV_1 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.571 \\ 0.233 \end{bmatrix} = \begin{bmatrix} 5.286 \\ 2.714 \\ 0.5174 \end{bmatrix} \quad \& \quad V_2 = \begin{bmatrix} 1 \\ 0.514 \\ 0.108 \end{bmatrix}$$

$$AV_2 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.514 \\ 0.108 \end{bmatrix} = \begin{bmatrix} 5.054 \\ 2.540 \\ 0.2162 \end{bmatrix} \quad \& \quad V_3 = \begin{bmatrix} 1 \\ 0.503 \\ 0.043 \end{bmatrix}$$

$$AV_3 = \begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.503 \\ 0.043 \end{bmatrix} = \begin{bmatrix} 5.0106 \\ 2.508 \\ 0.0855 \end{bmatrix} \quad \& \quad V_4 = \begin{bmatrix} 1 \\ 0.501 \\ 0.017 \end{bmatrix}$$

Using formula, $\lambda = \dfrac{A V_3 \cdot V_3}{V_3 \cdot V_3}$

$$\therefore \lambda = \frac{\begin{bmatrix} 5.0106 & 2.508 & 0.0855 \end{bmatrix} \begin{bmatrix} 1 \\ 0.503 \\ 0.043 \end{bmatrix}}{\begin{bmatrix} 1 & 0.503 & 0.043 \end{bmatrix} \begin{bmatrix} 1 \\ 0.503 \\ 0.043 \end{bmatrix}}$$

$$\therefore \boxed{\lambda = 5.002 \ \ \& \ \ x = \begin{bmatrix} 1 \\ 0.503 \\ 0.043 \end{bmatrix}}$$

This is the dominant pair.

6.3 (5) $\begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$\rightarrow$ Sol^n:- let $x_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$

$$A x_0 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 6 \end{bmatrix} \ \ \& \ \ v_1 = \begin{bmatrix} 0.75 \\ 1 \\ 0.75 \end{bmatrix}$$

$$AV_1 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0.75 \\ 1 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 5.25 \\ 7.25 \\ 4.5 \end{bmatrix} \quad \& \quad V_2 = \begin{bmatrix} 0.724 \\ 1 \\ 0.621 \end{bmatrix}$$

$$AV_2 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0.724 \\ 1 \\ 0.621 \end{bmatrix} = \begin{bmatrix} 5.1724 \\ 7.1724 \\ 3.7241 \end{bmatrix} \quad \& \quad V_3 = \begin{bmatrix} 0.721 \\ 1 \\ 0.519 \end{bmatrix}$$

$$\cancel{AV_3 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0.721 \\ 1 \\ 0.519 \end{bmatrix} = \begin{bmatrix} 0.721 \\ 1 \\ 0.435 \end{bmatrix} \quad \& \quad V_4 = }$$

$$AV_3 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0.721 \\ 1 \\ 0.519 \end{bmatrix} = \begin{bmatrix} 5.1634 \\ 7.1634 \\ 3.1153 \end{bmatrix} \quad \& \quad V_4 = \begin{bmatrix} 0.721 \\ 1 \\ 0.435 \end{bmatrix}$$

$$AV_4 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0.721 \\ 1 \\ 0.435 \end{bmatrix} = \begin{bmatrix} 5.1624 \\ 7.1624 \\ 2.6093 \end{bmatrix} \quad \& \quad V_5 = \begin{bmatrix} 0.721 \\ 1 \\ 0.364 \end{bmatrix}$$

$$AV_5 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0.721 \\ 1 \\ 0.364 \end{bmatrix} = \begin{bmatrix} 5.1622 \\ 7.1622 \\ 2.1859 \end{bmatrix} \quad \& \quad V_6 = \begin{bmatrix} 0.721 \\ 1 \\ 0.305 \end{bmatrix}$$

$$AV_6 = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0.721 \\ 1 \\ 0.305 \end{bmatrix} = \begin{bmatrix} 5.1622 \\ 7.1622 \\ 1.8312 \end{bmatrix} \quad \& \quad V_7 = \begin{bmatrix} 0.721 \\ 1 \\ 0.255 \end{bmatrix}$$

$\therefore$ Here, $\boxed{\lambda = 7.1623 \ \& \ x = \begin{bmatrix} 0.721 \\ 1 \\ 0.256 \end{bmatrix}}$

is ~~are~~ a dominant eigen value & eigen vector pair.

(7) Use the relative error formula E_{n+1} & estimate error in computed dominant eigen value in Probs. 1, 2 & 5

(1) Relative error in λ_1 for problem (1)

$$E_2 = \frac{|\lambda_1^1 - \lambda_1^2|}{|\lambda_1^2|} = \frac{|1-11|}{|11|} = 0.909$$

Illy.

$$E_2 = \frac{|11-8.727|}{|8.727|} = 0.2604$$

$$E_3 = \frac{|8.727-9.177|}{|9.177|} = 0.04901$$

$$E_4 = \frac{|9.177-9.0703|}{|9.0703|} = 0.01176$$

$$E_5 = \frac{|9.0703 - 9.09473|}{|9.09473|} = 0.002678$$

$$E_6 = \frac{|9.09473 - 9.08912|}{|9.08912|} = 0.0006172$$

$$E_7 = \frac{|9.08912 - 9.0904|}{|9.0904|} = 0.0001418$$

② Relative error in λ_1 for problem ②

→

$$E_{n+1} = \frac{|\lambda_1^n - \lambda_1^{n+1}|}{|\lambda_1^{n+1}|} \; ; \; \therefore E_2 = \frac{|1-7|}{|7|} = 0.8571$$

$$\therefore E_3 = \frac{|\lambda_1^2 - \lambda_1^3|}{|\lambda_1^3|} = \frac{|7 - 5.2857|}{|5.2857|} = 0.3243$$

$$E_4 = \frac{|5.2857 - 5.0540|}{|5.0540|} = 0.0458$$

$$E_5 = \frac{|5.054 - 5.0106|}{|5.0106|} = 0.008653$$

$$E_6 = \frac{|5.0106 - 5.0021|}{|5.0021|} = 0.001711$$

(3) Relative error in λ_1 for problem (5)

→

$$E_{n+1} = \frac{|\lambda_1^{n} - \lambda_1^{n+1}|}{|\lambda_1^{n+1}|}$$

$$E_2 = \frac{|1-8|}{|8|} = 0.875$$

$$E_3 = \frac{|8-7.25|}{|7.25|} = 0.10345$$

$$E_4 = \frac{|7.25-7.1724|}{|7.1724|} = 0.01082$$

$$E_5 = \frac{|7.1724-7.1635|}{|7.1635|} = 0.00125$$

$$E_6 = \frac{|7.1635-7.16242|}{|7.16242|} = 0.00015$$

$$E_7 = \frac{|7.16242-7.16229|}{|7.16229|} = 1.7069 \times 10^{-5}$$