Assignment - 1

Submitted by: Gavali Deshabhakt Nagnath

Roll No.: 202CD005

Department: Mathematics and Computational Sciences

Specialization: Computational and Data Science

Subject: Selected Topics in Graph Theory

Instructor: Prof. Dr. Shyam Kamath

Submitted on: 28/02/2021

Q1. Construct a self-complementary graph on 4n or 4n+1 vertices, for any given positive integer n. Describe the procedure step-by-step.

Answer:

A self-complementary graph is a graph which is isomorphic to its complement.

A complement of graph A is a graph $\bar{A}$ on the same vertices such that two distinct vertices of $\bar{A}$ are adjacent if and only if they are not adjacent in A. That is, to generate the complement of a graph, one fills in all the missing edges required to form a complete graph, and removes all the edges that were previously there.

So now keeping these two definitions we can construct a self-complementary graph on 4n or 4n+1 vertices.

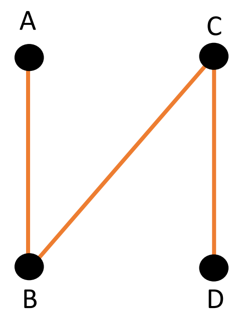
We will consider n = 1 for constructing self-complementary graph:

Steps:

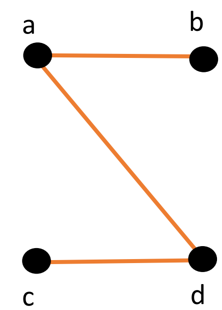
1) Calculate 4n (or 4n+1)

$$\therefore Total\ number\ of\ vertices = 4 * 1 = 4$$

2) Constructing graph, A and $\bar{A}$:



Graph: A



Graph: $\bar{A}$

3) Checking for isomorphism:
   For a two graphs to be isomorphic they have to obey following conditions,

   i. Equal number of vertices
   ii. Equal number of edges
   iii. Same degree sequence
   iv. Same number of circuits of particular length.
   (Note: Circuit is closed trail)

   Now in our case,

   i. Graph A and $\bar{A}$ has same number of vertices i.e., 4 (also $\bar{A}$ is A's complement so they must have same number of vertices)

ii. Number of edges in A are 3 and in Ā its also 3. So second condition is also satisfied.
iii. The degree sequence of graph A and Ā are: {1,1,1} and {1,1,1} respectively. Therefore 3^{rd} condition is also satisfied.
iv. From graphs it can be seen that neither of them has a circuit, so 4^{th} condition is also satisfied.
    Therefore, the two graphs are isomorphic. Also they are complement of each other.
    As the two graphs are isomorphic and complement of each other, therefore graph A is a self-complementary graph.

Similarly, using above steps we can construct self-complementary graphs from 4n+1 vertices.

Q2. Show that a graph is connected if and only if for every partition of its vertices into two non-empty sets, there exists an edge with its end vertices in both the partitions.

Answer:

Let's assume a connected graph G such that, V(G) = {x_0, x_1, x_2, …., x_n}. Now divide these vertices into two non-empty sets X and Y such that V(G) = X U Y.

We need to show that G has an edge with one endpoint in X and the other in Y.

Select vertices x in X, and y in Y. Since G is connected, there has to be a path in G that joins x to y. Denote this path as follows:

$x = x_0, x_1, x_2, \ldots., x_n = y$

The first vertex x_0 of this path is in X, and the last vertex x_n is in Y. All other vertices are either in X or in Y. Let i be the smallest index for which $x_i \in Y$, (such an i exists, because $x_n \in Y$, so i is at most n).

Therefore, we have $x_{i-1} \in X$ and $x_i \in Y$ so $x_{i-1}\, x_i$ is an edge of G with one endpoint in X and the other in Y.

Hence the proof.

Q3. If u and v are distinct vertices of a graph G, then show that every u-v walk in G contains a u-v path.

Answer: By definition,

1) A walk in a graph is an alternating sequence of vertices and edges, beginning and ending with vertices. Therefore, in a walk, there can be repeated vertices.
2) A path in a graph is a walk in which no vertices are repeated.

To prove the statement, we consider a graph such that
V(G) = {u_1, u_2, u_3, …., u_n} and V(E) = {e_1, e_2, e_3, …, e_n}.

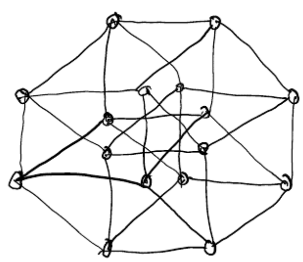
Let us consider a walk u_1 u_n, since graph G has distinct vertices the u_1u_n walk is a u_1u_n path.

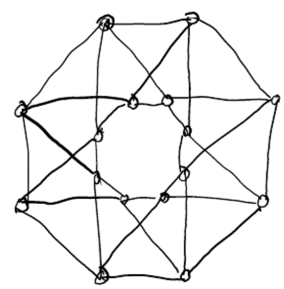
Now let's assume a walk u_1u_n in which a vertex (say u_j) is repeated, then we can have a u_1u_n path by just excluding the portion of walk because of which we arrive at same vertex i.e. u_j and arrive at u_n.

This shows that, every u_1u_n walk has a u_1u_n path, and in turn, every u-v walk has a u-v path.
Hence the proof

Note: If a vertex is repeated in walk then length of the walk will not be same as length of path.
(Length of path will be lesser in that case)

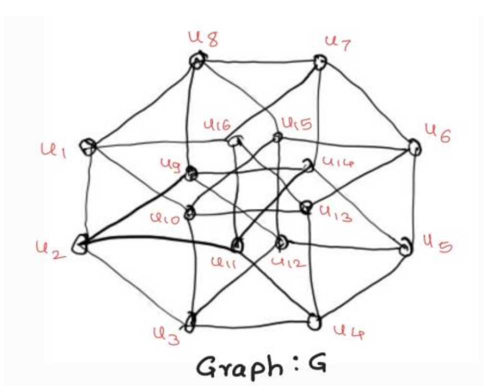
Q.4 Determine whether the following graphs are isomorphic:

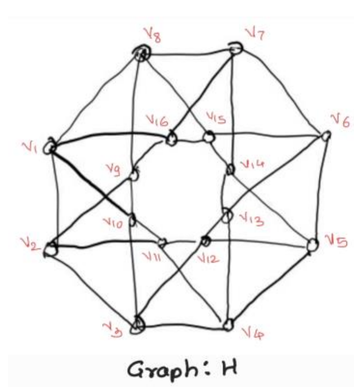




Justify your answer.

Answer: Let us label the graphs G and H and vertices as u_1, u_2, …, u_{16} and as v_1, v_2, …, v_{16} respectively.





The adjacency matrices of these to graphs are as follows:

	u1	u2	u3	u4	u5	u6	u7	u8	u9	u10	u11	u12	u13	u14	u15	u16
u1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1
u2	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
u3	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0
u4	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0
u5	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0
u6	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0
u7	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1
u8	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0
u9	0	1	0	0	0	0	0	1	0	0	0	1	0	1	0	0
u10	1	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0
u11	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0	1
u12	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0
u13	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1
u14	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	0
u15	0	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0
u16	1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0

Adjacency matrix of graph G

	v1	v2	v3	v4	v5	v6	v7	v8	v9	v10	v11	v12	v13	v14	v15	v16
v1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1
v2	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
v3	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0
v4	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0
v5	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0
v6	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0
v7	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1
v8	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0
v9	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1
v10	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
v11	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0
v12	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0
v13	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0
v14	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0
v15	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1
v16	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0

Adjacency matrix of graph H

Following are the conditions for isomorphism:

1. Equal number of vertices
2. Equal number of edges
3. Same degree sequence
4. Same number of circuits of particular length.

Checking conditions one by one:

1. The number of vertices in graph G are 16 and in graph H also 16.
2. The number of edges in graph G are 32 and in graph H also 32
3. From adjacency matrices it can be seen that both the graphs have same degree sequence which is {4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4}
4. In the graph G, every vertex appears in only six 4-cycles, whereas in the graph H, every vertex appears in eight 4-cycles.

Thus as 4th condition is not mate the graphs are **not isomorphic**.

Also, by assuming a bijection function, f: V(G) -> V(H) which maps edges in two graphs as follows,

$u_1 \rightarrow v_1$
$u_2 \rightarrow v_2$
$u_3 \rightarrow v_3$
.
.
.
$u_{16} \rightarrow v_{16}$

From adjacency matrix we can see that, the edges from vertices u_1 to u_8 perfectly maps with the edges from vertices v_1 to v_8, but from u_9 onwards the edges in both graphs do not map with each other with the current bijection function. Also, even if we change the bijection then also the edges won't map to each other. Hence the given two graphs are not isomorphic.