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Comparative Analysis of Convergence of Various Numerical Methods

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ABSTRACT

This paper is devoted to the comparison of different numerical methods in respect to their convergence. Any numerical analysis model or method is convergent if and only if a sequence of model solutions with increasingly refined solution domains approaches to a fixed value. Furthermore, a numerical solution method is consistent only if this sequence converges to the solution of the continuous equations which controls the physical phenomenon being modelled. Given that a model is consistent, it is not feasible to apply any method to a problem without testing the size of the time and iterative steps which form the discrete approximation of the solution of the given problem. That is, convergence testing is a required component of any modelling study. In this paper we examined five numerical methods for their convergence. And we can say that for 1st degree based numerical methods Newton Raphson is best among all the discussed methods with 2 as rate of convergence. And for 2nd degree based numerical methods Chebyshev method has a very good rate of convergence i.e. 3.

Keywords: Rate of Convergence, Secant, Muller, Regula-Falsi, Newton-Raphson, Chebyshev.

INTRODUCTION

A numerical method or iterative method is said to be of order p or has the rate of convergence p, if p is the largest positive real number for which there exists a finite constant $C \neq 0$ such that

$|\in_{k+1}| \leq C| \in_k |^p$

Where $\in_k = X_k - \xi$ is the error in the k[th] iterate

The constant C is called the asymptotic error constant and usually depends on derivatives of f(x) at x = ξ.

In other words we can say in numerical analysis, the speed at which a convergent sequence approaches its limit is called the rate of convergence. Although strictly speaking, a limit does not give information about any finite first part of the sequence, this concept is of practical importance if we deal with a sequence of successive approximations for an iterative method, as then typically fewer iterations are needed to yield a useful approximation if the rate of convergence is higher.

Further this paper is organized as section A through E explains the rate of convergence of Secant, Regula-Falsi, Newton-Raphson, Muller, Chebyshev methods respectively. And finally in Conclusion section rate of convergence is compared in the form of table.

A. SECANT METHOD

The secant method is a recursive method used to find the solution to an equation like Newton's Method. The idea for it is to follow the secant line to its *x*-intercept and use that as an approximation for the root. This is like Newton's Method (which follows the tangent line) but it requires two initial guesses for the root.

Recurrence relation

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k \quad \text{for k=1, 2, 3....} \tag{1}$$

Rate of convergence

We assume that ξ is a simple root of f(x) = 0. Substituting $x_k = \xi + \epsilon_k$ in (1) we obtain

$$\varepsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1}) f(\xi + \epsilon_k)}{f(\xi + \epsilon_k) - f(\xi + \epsilon_{k-1})} \tag{2}$$

Expanding $f(\xi + \epsilon_k)$ and $f(\xi + \epsilon_{k-1})$ in Taylor's series about the point ξ and noting that $f(\xi) = 0$ we get,

$$\varepsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1})\left[\epsilon_k f'(\xi) + \frac{1}{2}\varepsilon_k^2 f''(\xi) + \cdots\right]}{(\epsilon_k - \epsilon_{k-1}) f'(\xi) + \frac{1}{2}(\varepsilon_k^2 - \epsilon_{k-1}^2) f''(\xi) + \cdots} \tag{3}$$

$$\varepsilon_{k+1} = \epsilon_k - \left[\epsilon_k + \frac{1}{2}\epsilon_k^2 \frac{f''(\xi)}{f'(\xi)} + \cdots\right]\left[1 + \frac{1}{2}(\epsilon_{k-1} + \varepsilon_k)\frac{f''(\xi)}{f'(\xi)} + \cdots\right]^{-1} \tag{4}$$

or $\varepsilon_{k+1} = \frac{1}{2}\varepsilon_k \varepsilon_{k-1} \frac{f''(\xi)}{f'(\xi)} + O\left(\varepsilon_k^2 \varepsilon_{k-1} + \varepsilon_k \varepsilon_{k-1}^2\right)$

$$\text{or } \varepsilon_{k+1} = C\varepsilon_k \varepsilon_{k-1} \tag{5}$$

where $C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$ and the higher powers of ε_k are neglected.

The relation of the form (5) is called the error equation. Keeping in mind the definition of convergence we want a relation of the form

$\varepsilon_{k+1} = A\varepsilon_k^p$ (6)

Where A and p are to be determined

From (6) we have $\varepsilon_k = A\varepsilon_{k-1}^p$ or $\varepsilon_{k-1} = A^{-1/p}\varepsilon_k^{1/p}$

Substitute the value of ε_{k+1} and ε_{k-1} in (5)

$\varepsilon_k^p = CA^{-(1+\frac{1}{p})}\varepsilon_k^{1+1/p}$ (7)

Comparing the power of ε_k on both sides we get

p=1+1/p which gives p=$\frac{1}{2}\left(1 \pm \sqrt{5}\right)$.

Neglecting the negative sign, we get the rate of convergence for the Secant method (1) is P = 1.618.

B. REGULA-FALSI METHOD

The Regula-Falsi method is also called as Regula-Falsi Method. This is oldest method for computing the real roots of an algebraic equation. With the use of this method we can find two numbers *a* and *b* such that equations f(a) and f(b) are of different sign. Hence the root lies between the point a and b for the graph of y = f(x) and must cross the x axis for the point x=a and x=b.

Recurrence relation

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k$$

Rate of convergence

If the function f(x) in the equation f(x) = 0 is convex in the interval (x_0, x_1) that contains a root then one of the points $x_0, or\ x_1$ is always fixed and the other point varies with k. If the point x_0 is fixed , then the function f(x) is approximated by the straight line passing through the points (x_0, f_0) and (x_k, f_k), k=1, 2, ...

The error equation (5) becomes:

$\varepsilon_{k+1} = C\varepsilon_k\varepsilon_0$

Where $C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$ and $\varepsilon_0 = x_0 - \xi$ is independent of k. Therefore we can Write

$\varepsilon_{k+1} = C'\varepsilon_k$

Where C' = Cε_0 is the asymptotic error constant. Hence, the Regula-Falsi Method has Linear rate of Convergence.

C. NEWTON-RAPHSON METHOD

Newton's method, also called the Newton-Raphson method, is a root-finding algorithm that uses the first few terms of the Taylor series of a function f(x) in the vicinity of a suspected root. Newton's method is sometimes also known as Newton's iteration, although in this work the latter term is reserved to the application of Newton's method for computing square roots.

Recurrence relation

$x_{k+1} = x_k - \frac{f_k}{f'_k}$, k=0, 1, 2... (8)

Rate of convergence

On Substituting $x_k = \xi + \epsilon_k$ in (8) and Expanding $f(\xi + \epsilon_k)$ and $f(\xi + \epsilon_{k-1})$ in Taylor's series about the point ξ, we obtain,

$$\varepsilon_{k+1} = \varepsilon_k - \frac{\left[\varepsilon_k f'(\xi) + \frac{1}{2}\varepsilon_k^2 f''(\xi) + \cdots\right]}{f'(\xi) + \varepsilon_k f''(\xi) + \cdots}$$

$$\varepsilon_{k+1} = \varepsilon_k - \left[\varepsilon_k + \frac{1}{2}\varepsilon_k^2 \frac{f''(\xi)}{f'(\xi)} + \cdots\right]\left[1 + \varepsilon_k \frac{f''(\xi)}{f'(\xi)} + \cdots\right]^{-1}$$

$$\varepsilon_{k+1} = \frac{1}{2}\varepsilon_k^2 \frac{f''(\xi)}{f'(\xi)} + O\left(\varepsilon_k^3\right)$$

On neglecting ε_k^3 and higher powers ofε_k, we get

$\varepsilon_{k+1} = C\varepsilon_k^2$

Where $C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$

Thus the Newton-Raphson Method has Second order Convergence.

D. MULLER METHOD

Muller method is used to find zeros of arbitrary analytic functions (particularly useful in finding roots of polynomials.) It is basically a variation of the secant method. It is also useful when searching for complex roots of analytic functions. (Recall: A function is analytic at a point $x_0$if the function has a convergent Taylor series at x_0. A function is analytic in a region if it is analytic at every point in the region.)

In the secant method we start with 2 initial guesses $x_0$and x_1The next approximation $x_2$is the intersection with the x axis of the line $(x_0, f(x_0)$)and $(x_1, f(x_1)$). Then we continue withx_1 and x_2 etc. Muller's method uses 3 initial guesses x_0, x_1, x_2 and determines the intersection with the x axis of a parabola. Note that this is done by finding the root of an explicit quadratic equation. The case where the roots are not real is handled as well, though the geometric interpretation is more complicated. (This is where the analyticity of the function is important, it makes the value of the function for a complex argument meaningful).

Recurrence relation

$x_{k+1} = x_k + (x_k - x_{k-1})\lambda_{k+1}$ (9)

Where $\lambda_{k+1} = \frac{x-x_k}{x_k-x_{k-1}}$

Alternatively, $x_{k+1} = x_k - \frac{2a_0}{a_1 \pm \sqrt{a_1^2-4a_0a_1}}$ k=2, 3, ... (10)

Where $a_0 = \frac{1}{D}[(x_k - x_{k-2})(f_k - f_{k-1}) - (x_k - x_{k-1})(f_k - f_{k-2})]$ (11)

$a_1 = \frac{1}{D}[(x_k - x_{k-2})^2(f_k - f_{k-1}) - (x_k - x_{k-1})^2(f_k - f_{k-2})]$ (12)

$a_2 = f_k$ (13)

$D = [(x_k - x_{k-1})(x_k - x_{k-2})(x_{k-1} - x_{k-2})]$ (14)

Rate of convergence

On Substituting $x_j = \xi + \varepsilon_j$, j= k-2, k-1, k and Expanding $f(\xi + \varepsilon_j)$ in Taylor's series about the point ξ in 11 to 14, we obtain,

$$D = [(\varepsilon_k - \varepsilon_{k-1})(\varepsilon_k - \varepsilon_{k-2})(\varepsilon_{k-1} - \varepsilon_{k-2})]$$

$$a_2 = \varepsilon_{k0} f'(\xi) + \frac{1}{2}\epsilon_k^2 f''(\xi) + \frac{1}{6}\epsilon_k^3 f'''(\xi) + \cdots$$

$$a_1 = f'(\xi) + f''(\xi)\varepsilon_k + \frac{1}{6}\{2\epsilon_k^2 + \epsilon_k\epsilon_{k-1} + \epsilon_k\epsilon_{k-2} - \epsilon_{k-1}\epsilon_{k-2}\} f'''(\xi) + \cdots$$

$$a_0 = \frac{1}{2} f''(\xi) + \frac{1}{6}\{\epsilon_k + \epsilon_{k-1} + \epsilon_{k-2}\} f'''(\xi) + \cdots$$

So, $a_1^2 - 4a_0a_2 = [f'(\xi)]^2 - \frac{1}{3}(\epsilon_k\epsilon_{k-1} + \epsilon_k\epsilon_{k-2} + \epsilon_{k-1}\epsilon_{k-2}) f'(\xi) f'''(\xi) + \cdots$

$$a_1 + \sqrt{a_1^2 - 4a_0a_2} = 2f'(\xi)\left[1 + \frac{1}{2}\varepsilon_k C_2 + \frac{1}{6}(\varepsilon_k^2 - \varepsilon_{k-1}\varepsilon_{k-2})C_3 + \cdots\right]$$

Where $C_i = \frac{f^{(i)}(\xi)}{f'(\xi)}, i = 2,3, ...$

Hence using (10) we get,

$\varepsilon_{k+1} = \frac{1}{6}(\varepsilon_k\varepsilon_{k-1}\varepsilon_{k-2}C_3) + \cdots$

Therefore, the error equation associated with the Muller method is given by,

$\varepsilon_{k+1} = C\varepsilon_k\varepsilon_{k-1}\varepsilon_{k-2}$ (15)

Where $C = \frac{1}{6}C_3 = \frac{1}{6}\frac{f'''(\xi)}{f'(\xi)}$ (16)

Now we want a relation of the form

$\varepsilon_{k+1} = A\varepsilon_k^p$ (17)

Where A and p are to be determined.

From (17), $\varepsilon_k = A\varepsilon_{k-1}^p$ or $\varepsilon_{k-1} = A^{-1/p}\varepsilon_k^{1/p}$

$\varepsilon_{k-1} = A\varepsilon_{k-2}^p$ or $\varepsilon_{k-2} = A^{-1/p}\varepsilon_{k-1}^{1/p} = A^{-(\frac{1}{p}+\frac{1}{p^2})}\varepsilon_k^{1/p^2}$

Substituting the values of $\varepsilon_{k+1}, \varepsilon_{k-1} and\ \varepsilon_{k-2}$ in (15)

$$\varepsilon_k^p = CA^{-(1+\frac{2}{p}+\frac{1}{p^2})}\varepsilon_k^{1+\frac{1}{p}+\frac{1}{p^2}} \tag{18}$$

On comparing the powers of ε_k on both sides we obtain

$p = 1 + \frac{1}{p} + \frac{1}{p^2}$

Or $F(p) = p^3 - p^2 - p - 1 = 0$ (19)

The equation F(p)=0 has the smallest positive root in the interval (1,2). We use Newton Raphson method to determine this root. Starting with $p_0 = 2$.

$p_1 = 1.8571$, $p_2 = 1.8395$, $p_3 = 1.8393$, ...

therefore the root of the equation (19) is p=1.84(approx.). Hence the rate of convergence of the Muller method is 1.84.

E. CHEBYSHEV METHOD

Chebyshev polynomials, are a sequence of orthogonal polynomials which are related to de Moivre's formula and which can be defined recursively. One usually distinguishes between Chebyshev polynomials of the first kind which are denoted T_n and Chebyshev polynomials of the second kind which are denoted U_n.

The Chebyshev polynomials T_n or U_n are polynomials of degree n and the sequence of Chebyshev polynomials of either kind composes a polynomial sequence.

Recurrence relation

$$x_{k+1} = x_k - \frac{f_k}{f'_k} - \frac{1}{2}\frac{f_k^2}{f'^3_k} f''_k \tag{20}$$

Rate of convergence

Substutuing $x_k = \xi + \varepsilon_k$ and expanding (x_k) , $f'(x_k)$, $f''(x_k)$ about the point ξ in the Chebyshev method,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{1}{2}\left[\frac{f(x_k)}{f'(x_k)}\right]^2 \frac{f''(x_k)}{f'(x_k)} \tag{21}$$

Now, $\frac{f(x_k)}{f'(x_k)} = \frac{f(\xi+\varepsilon_k)}{f'(\xi+\varepsilon_k)} = \frac{\varepsilon_k f'(\xi) + \frac{1}{2}\varepsilon_k^2 f''(\xi) + \frac{1}{6}\varepsilon_k^3 f'''(\xi) + \cdots}{f'(\xi) + \varepsilon_k f''(\xi) + \frac{1}{2}\varepsilon_k^2 f'''(\xi) + \cdots}$

$$\left[\varepsilon_k + \frac{1}{2}C_2\varepsilon_k^2 + \frac{1}{6}C_3\varepsilon_k^3 + \cdots\right] \times \left[1 + \left(C_2\varepsilon_k + \frac{1}{2}C_3\varepsilon_k^2 + \cdots\right)\right]^{-1}$$

$$= \left[\varepsilon_k + \frac{1}{2}C_2\varepsilon_k^2 + \frac{1}{6}C_3\varepsilon_k^3 + \cdots\right] \times \left[1 - C_2\varepsilon_k + \left({C_2}^2 - \frac{1}{2}C_3\right)\varepsilon_k^2 + \cdots\right]$$

$$= \varepsilon_k - \frac{1}{2}C_2\varepsilon_k^2 + \left(\frac{1}{2}{C_2}^2 - \frac{1}{2}C_3\right)\varepsilon_k^3 + \cdots$$

Where $C_i = \frac{f^{(i)}(\xi)}{f'(\xi)}, i = 2,3, \ldots$

Also, $\left[\frac{f(x_k)}{f'(x_k)}\right]^2 = \varepsilon_k^2 - C_2\varepsilon_k^3 + \cdots$

$$\frac{f''(x_k)}{f'(x_k)} = \frac{f''(\xi+\varepsilon_k)}{f'(\xi+\varepsilon_k)} = \frac{f''(\xi)+\varepsilon_k f'''(\xi)+\cdots}{f'(\xi)+\varepsilon_k f''(\xi)+\cdots}$$
$$= \frac{f''(\xi)}{f'(\xi)}\left[1+\frac{C_3}{C_2}\varepsilon_k+\cdots\right][1+(C_2\varepsilon_k+\cdots)]^{-1}$$
$$= C_2\left[1+\frac{C_3}{C_2}\varepsilon_k+\cdots\right][1-C_2\varepsilon_k+\cdots]$$
$$= C_2+(C_3-C_2^2)\varepsilon_k+\cdots$$

Substituting in (21)

$$\varepsilon_{k+1} = \varepsilon_k - \left[\varepsilon_k - \frac{1}{2}C_2\varepsilon_k^2 + \left(\frac{1}{2}C_2^2 - \frac{1}{3}C_3\varepsilon_k^3+\cdots\right)\right]$$
$$-\frac{1}{2}[\varepsilon_k^2 - C_2\varepsilon_k^3+\cdots][C_2+(C_3-C_2^2)\varepsilon_k+\cdots]$$
$$= \left[-\left(\frac{1}{2}C_2^2-\frac{1}{3}C_3\varepsilon_k^3\right)-\frac{1}{2}\{(C_3-C_2^2)-C_2^2\}\right]\varepsilon_k^3+O(\varepsilon_k^4)$$
$$= \left[\frac{1}{2}C_2^2-\frac{1}{6}C_3\right]\varepsilon_k^3+O(\varepsilon_k^4)$$

Hence the rate of Convergence of Chebyshev method is 3.

CONCLUSION

As discussed in above sections the different numerical methods have different rate of convergence. And the rate of convergence is a very important issue in solution of polynomial and transcendental equations, because the rate of convergence of any numerical method determines the speed of the approximation to the solution of the problem.
Following table shows the comparison of rate of convergence.

Method	Based on Equation	Rate of Convergence
Secant	1st degree	1.618
Regula-Falsi	1st degree	1
Newton-Raphson	1st degree	2
Muller	2nd degree	1.84
Chebyshev	2nd degree	3

Table shows that among Secant, Regula-Falsi and Newton-Raphson which are based on 1st degree equations Newton-Raphson has a good rate of convergence i.e. 2. And among Muller and Chebyshev methods which are based on 2nd degree equations Chebyshev is best with the rate of convergence of 3.

REFERENCES

1. David B. Thompson, *"Numerical Methods 101 - Convergence of Numerical Models"*, USGS Staff, University of Nebraska-Lincoln.

2. K. Karthikeyan, "*Comparative study of certain numerical algorithms for solving non linear equations*", Pelagia Research Library, CODEN (USA) ISSN: 0976-8610.
3. Anil V. Rao., "A Survey of Numerical Methods for optimal control", Preprint AAS 09-334.
4. Changbum Chun, "*Computers and mathematics with applications*", 50, pp.1551-1568 (2005).
5. Vinay Kanwar, Sukhjit Singh, J.R.Sharma and Mamta, "*Indian J. pure appl. Math.*", 34(9), pp.1339-1349 (2001).
6. R. Bellman, "*Dynamic Programming. Princeton*", NJ: Princeton University Press.
7. M.K. Jain, S.R.K. Iyengar, R.K. Jain, "*Numerical Methods for Scientific and Engineering Computation*", 5th edition, New Age International Publishers.