MA859: Selected Topics in Graph Theory
Lecture 1

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January 26, 2021

Course Plan and Evaluation Plan

Objective of the Course

To expose the students to the basics of Graph Theory and some selected topics such as Domination, Labeling, Coloring, Planarity, Graph Operations, Graph Partitions, Probability on Graphs and Graph Models; thereby giving exposure to some simple applications that are useful to Data Science and related topics.

Skill Development of the student expected from the course

The student should be able to analyze the problem, translate into relevant model and find a solution.

Course Coverage

Graphs –An Introduction, Classes of graphs, Distances in graphs, Domination, Labelling, Coloring –Introduction & Types of coloring –Complete Colorings, Colorings and Distance: -Coloring, (2,1)-Coloring, Radio Coloring, Hamiltonian Coloring, Critical Concepts, Independence, Matching and Covering, Chordal graphs, Perfect graphs, Interval graphs, Planar graphs, Graph Operations, Graph Partition, Probability on graphs –Random graphs, Hypergraphs, Algebraic concepts in graph theory, IP & LP formulation of selected graph problems, Graph Models

Evaluation Plan

1	Quiz/ Assignments/ Tests	60%
2	Mid Semester Examination	15%
3	End Semester Examination	25%

The above scheme of evaluation is recommended by the institute in view of the COVID-19 situation. When the situation improves and the students come into campus for the regular classes, these parameters are likely to change as per the regulations of the institute.

References

1. Douglas B. West, Introduction to Graph Theory, 2nd Edition, PHI Learning Pvt. Ltd., 2012
2. Haynes, T.W., Hedetniemi, S.T. and Slater, P.J., Fundamental of Domination in graphs, Marcel Dekker, Inc.,New York 1998
3. Haynes, T.W., Hedetniemi, S.T. and Slater, P.J., Domination in graphs –Advanced Topics, Marcel Dekker,Inc.,New York 1998
4. Gary Chartrand and Ping Zhang, Chromatic Graph Theory, CRC Press
5. Tommy R. Jensen and Bjarne Toft, Graph Coloring problems, John Wiley & sons
6. Michael Stiebitz, Diego Scheide, Bjarne Toft and Lene M. Favrholdt, Graph Edge Coloring, Wiley
7. Béla Bollobás, Random Graphs, 2nd Edition, Cambridge University Press

There are many other useful books. As the course progresses, the relevant books will be mentioned.

Introduction

There are many practical problems that involve graph theory. For instance:

i) How can n jobs be filled by n people with maximum total utility? (*known famously as "Assignment Problem"*)

ii) What is the maximum flow of internet data per unit time from source to sink in a local area network?

iii) How many layers does a computer chip need so that wires in the same layer don't cross?

iv) How can the season of an IPL be scheduled into the minimum number of weeks?

v) What is the fastest route from the national capital to each state capital?

Definition

The following puzzle is believed to be the birth of Graph Theory and we introduce the basic definition of a graph through this.

Königsberg Bridge Problem

The city of Königsberg was located on the Pregel river in Prussia. The city occupied two islands plus areas on both banks.

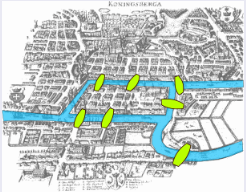


Figure 1: *Königsberg Bridges*

The citizens wondered whether they could leave home, cross every bridge exactly once, and return home.

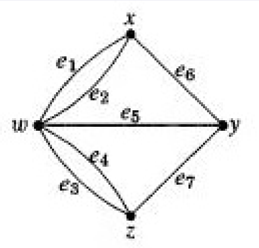


Figure 2: *Graphical representation of the Königsberg Bridge Problem*

The problem reduces to traversing the figure above, with heavy dots representing land masses and curves representing bridges.

The model in the Figure (2) makes it easy to argue that the desired traversal does not exist. Each time we enter and leave a land mass, we use two bridges ending at it. We can also pair the first bridge with the last bridge on the land mass where we begin and end. Thus, existence of the desired traversal requires that each land mass be involved in an even number of bridges. This necessary condition did not hold in Königsberg.

Graph theory is concerned with various types of networks, or really models of networks called graphs. These are not the graphs of analytic geometry, but what are often described as “points connected by line”, for example:

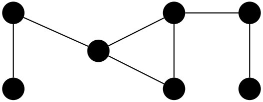


Figure 3: *An example of a graph*

The preferred terminology is ***vertex*** for a point and ***edge*** for a line. The lines need not be straight lines and in fact, the actual definition of a graph is not a geometric definition.

The figure (3) is simply a visualization of a graph; this graph is a more abstract object, consisting of seven vertices, which we might name $\{v_1, \ldots, v_7\}$, and the collection of pairs of vertices that are connected; for a suitable assignment of names v_i to the points in the diagram, the edges could be represented as $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_3, v_5\}$, $\{v_4, v_5\}$, $\{v_5, v_6\}$, $\{v_6, v_7\}$.

Formal Definition of a graph

Definition 1

*A **graph** G is a triple consisting of a **vertex set** $V(G)$, an **edge set** $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its **endpoints**.*

We draw a graph on paper by placing each vertex at a point and representing each edge by a curve joining the locations of its endpoints.

Example 1

In the graph in Figure (2), the vertex set is $\{x, y, z, w\}$, the edge set is $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, and the assignment of endpoints to edges can be read from the picture. Note that edges e_1 and e_2 have the same endpoints, as do e_3 and e_4. Also, if we had a bridge over an inlet, then its ends would be in the same land mass, and we would draw it as a curve with both ends at the same point. We have appropriate terms for these types of edges in graphs.

Definition 2

*A **loop** is an edge whose endpoints are equal. **Multiple edges** are edges having the same pair of endpoints.*
*A **simple graph** is a graph having no loops or multiple edges. We specify a simple graph by its vertex set and edge set, treating the edge set as a set of unordered pairs of vertices and writing $e = uv$ (or $e = vu$) for an edge e with endpoints u and v.*
*When u and v are the endpoints of an edge, they are **adjacent** and are **neighbors**. We write $u \leftrightarrow v$ for "u is adjacent to v".*

In many important applications, loops and multiple edges do not arise, and we restrict our discussions to simple graphs. We sometimes call a simple graph simply as "graph".

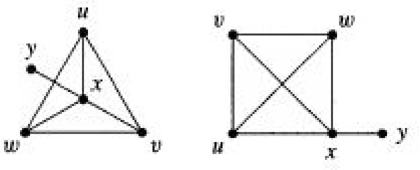


Figure 4: *Same graphs drawn in two different ways*

Remark 1

The null graph is the graph whose vertex set and edge set are empty. We normally ignore this graph for practical problems, as it sometimes causes unnecessary distractions. So, we usually consider graphs with non-empty set of vertices.

Some definitions and terminologies

Definition 3

*The **complement** $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.*
*A **clique** in a graph is a set of pairwise adjacent vertices.*
*An **independent set** (or **stable set**) in a graph is a set of pairwise non-adjacent vertices.*

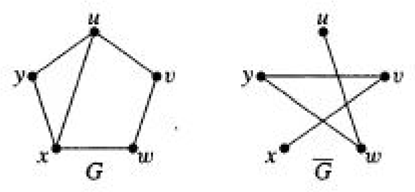


Figure 5: *An example of Complement graph of a graph*

In the Figure (5), $\{u,x,y\}$ is a clique of size 3 and $\{u,w\}$ is an independent set. Note that a clique in G is an independent set in $\overline{G}$.

Definition 4

*A graph G is **bipartite** if $V(G)$ is the union of two disjoint (possibly empty) independent sets called **partite sets** of G.*
Equivalently, a graph $G=(V,E)$ is bipartite if its $V(G)$ can be partitioned into two sets V_1 and V_2 such that every edge of G has its one end vertex in V_1 and the other in V_2.

Note that both V_1 and V_2 are independent sets of G.

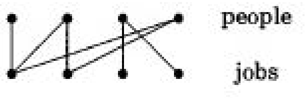


Figure 6: *An example of a bipartite graph*

An institution like NITK has several committees for its activities. The faculty members may be in more than one committees. So, while scheduling meetings, one has to ensure that the members don't get simultaneous meetings of more than one committees.

In a graph theoretic approach, we represent each committee by a vertex and whenever two committees have common members, their respective vertices are made adjacent. We must assign labels (time periods) to the vertices so the endpoints of each edge receive different labels. This labeling leads to partitioning the vertices, the labels need not be numeric. We conveniently label with **colors**.

Definition 5

*The **chromatic number** of a graph G, written $\chi(G)$, is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors. A graph G is k-partite if $V(G)$ can be expressed as the union of k (possibly empty) independent sets.*

This generalizes the idea of bipartite graphs, which are 2-partite. Vertices given the same color must form an independent set, so $\chi(G)$ is the minimum number of independent sets needed to partition $V(G)$. A graph is k-partite if and only if its chromatic number is at most k. We use the term "partite set" when referring to a set in a partition into independent sets.

We can model a road network using a graph with edges corresponding to road segments between intersections. We can assign edge weights to measure distance or travel time. In this context, edges do represent physical links.

If the vertices of the graph represent our houses and other places to visit, then we may want to follow a route that visits every vertex exactly once, so as to visit everyone without overstaying our welcome!

We consider the existence of such a route in due course of the semester.

We need terms to describe the types of routes in graphs.

Definition 6

*A **path** is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A **cycle** is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.*

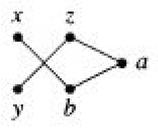


Figure 7: A path

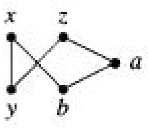


Figure 8: A cycle

Definition 7

*A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. We then write $H \subseteq G$ and say that "G **contains** H".*
*A graph G is **connected** if each pair of vertices in G belongs to a path; otherwise, G is **disconnected**.*

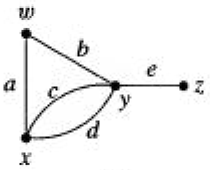
Definition 8

Let G be a loopless graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. The ***adjacency matrix*** *of G, written $A(G)$, is the $n \times n$ matrix in which entry a_{ij} is the number of edges in G with endpoints $\{v_i, v_j\}$. The* ***incidence matrix*** *$M(G)$ is the $n \times m$ matrix in which entry m_{ij} is 1 if u_i is an endpoint of e_j and otherwise is 0.*
If vertex v is an endpoint of edge e, then v and e are ***incident****. The* ***degree*** *of vertex v (in a loopless graph) is the number of incident edges.*

The appropriate way to define adjacency matrix, incidence matrix, or vertex degrees for graphs with loops depends on the application. We will discuss more about it later.

$$
\begin{array}{c|cccc}
 & w & x & y & z \\
\hline
w & 0 & 1 & 1 & 0 \\
x & 1 & 0 & 2 & 0 \\
y & 1 & 2 & 0 & 1 \\
z & 0 & 0 & 1 & 0
\end{array}
$$

$A(G)$



G

$$
\begin{array}{c|ccccc}
 & a & b & c & d & e \\
\hline
w & 1 & 1 & 0 & 0 & 0 \\
x & 1 & 0 & 1 & 1 & 0 \\
y & 0 & 1 & 1 & 1 & 1 \\
z & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$M(G)$

Figure 9: Example of Adjacecy and Incidence Matrices

Remark 2

An adjacency matrix is determined by a vertex ordering. Every adjacency matrix is symmetric. An adjacency matrix of a simple graph G has entries 0 or 1, with O's on the diagonal. The degree of v is the sum of the entries in the row for v in either $A(G)$ or $M(G)$.

Representation of graphs through these matrices will, no doubt, implicitly name the vertices and the edges and thus, makes the graphs unique. However, on aspects such as connectedness, this is not very important. We sometimes need to know whether two given graphs are identical or not. This leads to isomorphism.

Definition 9

*An **isomorphism** from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "G is **isomorphic** to H", written $G \cong H$, if there is an isomorphism from G to H.*

Figure 10: A simple example of isomorphic graphs

Isomorphism is an equivalence relation.

Well; that is it for now!!

Thank you!