MA859: Selected Topics in Graph Theory

Lecture - 17

Laplacian Matrix

The Laplacian

We have discussed about the Laplacian Matrix when we discussed the Matrix Tree Theorem. The Laplacian Matrix $L = D - A$, where D is the diagonal matrix, wherein the diagonal elements are the degrees of the vertices of the graph G and A is the adjacency matrix of G.

We discuss here some more properties of the Laplacian Matrix.

The Laplacian of a connected graph has eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

The algebraic connectivity is defined to be λ_2, the second smallest eigenvalue. (The name is a result of its connection to the vertex connectivity and the edge connectivity of a graph.)

A positive semidefinite matrix is one that is Hermitian $\left[A = \bar{A}^T\right]$ (also known as self-adjoint matrix), and whose eigenvalues are all non-negative.

Ex $B = \begin{bmatrix} 1 & 2+4i \\ 2-4i & 3 \end{bmatrix}$ → a Hermitian matrix. (not symmetric)

[This is true when the elements are complex]

A real matrix is symmetric if and only if it is Hermitian.

The characteristic function is a function for which every subset N of X, has a value of 1 at points of N and 0 at points of X − N.

In other words, it takes value 1 for numbers in the set, 0 for numbers not in the set.

Theorem 7 The smallest eigen value of L is zero.

Proof: This is the direct result of L being a positive semidefinite matrix. It will have n real Laplace eigenvalues, namely, $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. //

Theorem 8 The multiplicity of 0 as an eigenvalue of L is the number of components in the graph G.

(Proof is almost immediate from Theorem 7)

Theorem 9 The algebraic connectivity is positive if and only if the graph is connected.

Proof: If $\lambda_2 > 0$ and $\lambda_1 = 0$, then G must be connected, because the eigenvalues of the Laplacian are $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Conversely, this is a direct consequence of Theorem 7. If G is connected, then zero is the smallest eigenvalue. λ_2 must be greater than zero and hence positive. //

In the case of a k-regular graph, there is a linear relationship between the eigenvalues of the Laplacian and the eigenvalues of the adjacency matrix A.

If $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_n$ are the eigenvalues of A, then $\theta_i = k - \lambda_i$. This is a result of graphs being regular. $\Rightarrow \quad L = kI - A$.

However, no such relationship exists between the eigen values of L and of A for a non-regular graph.

//End of Lecture//