Chapter 3.

For Unevenly Spaced Data
- → Lagranges Interpolation
- → Newton's divided difference method → [Generalizing Newton's forward & Backward Interpolⁿ]

★ Lagranges Interpolation

→ Data points given ~~(x,y)~~ (x, y), if they are not in ascending or descending order ~~not~~ ~~sorted~~ then also it's fine

$$P_n(x) = y_0 L_0(x) + y_1 L_1(x) + \dots + y_n L_n(x)$$

$$P_n(x_i) = y_i \Rightarrow L_i(x_i) = 1 \qquad L_j(x_i) = 0, \; j \neq i$$

e.x.

$$P_n(x_0) = y_0 \Rightarrow y_0 L_0(x_0) + \cancel{y_1 L_1(x_0)}^{0} + \dots + \cancel{y_n L_n(x_n)}^{0} = y_0$$

$$L_0(x_0) = 1$$

$$L_j(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{j-1})(x-x_{j+1})\dots(x-x_n)}{(x_j-x_0)(x_j-x_1)\dots(x_j-x_{j-1})(x_j-x_{j+1})\dots(x_j-x_n)}$$

Ex.

$x =$	1	3	4	6	10
$y = f(x) =$	0	18	48	180	900

$$f(x) \approx P_4(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

↑ 4^{th} degree polynomial

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$$= L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2) + L_3(x) f(x_3)$$

$$= \frac{(x-3)(x-4)(x-6)(x-10)}{(1-3)(1-4)(1-6)(1-10)} \times 0 + \underbrace{\frac{(x-1)(x-4)(x-6)(x-10)}{(3-1)(3-4)(3-6)(3-10)} \times 12}_{-3/7}$$

$$+ \underbrace{\frac{(x-1)(x-3)(x-6)(x-10)}{(4-1)(4-3)(4-6)(4-10)} \times 48}_{4/3} + \underbrace{\frac{(x-1)(x-3)(x-4)(x-10)}{(6-1)(6-3)(6-4)(6-10)} \times 180}_{-3/2}$$

$$+ \underbrace{\frac{(x-1)(x-3)(x-4)(x-6)}{(10-1)(10-3)(10-4)(10-6)} \times 900}_{25/42}$$

$$(x-a)(x-b)(x-c)(x-d) = x^4 + (a+b+c+d)x^3 + \{a(b+c+d) + b(c+d) + cd\}x^2 + \{(a+b)cd + (c+d)ab\}x + abcd$$

$$-\frac{3}{7} + \frac{4}{3} - \frac{3}{2} + \frac{25}{42} = 0$$

→ This shows we will not get 4th degree polynomial

After calculation

$P_4(x) = x^3 - x^2$ → Interpolation

$f(5) \approx P_4(5) = 5^3 - 5^2 = 100$

We can use this for less than 1 or greater than 10 but not very far from these boundaries → Extrapolation

→ Here the drawback if we miss any data point initially for e.x. if we (7.5, 300) in initial data then we have to do rework again to calculate polynomial

~~Lj(5)~~ P_5

$x_0=1$ $x_1=2$ $x_2=7$ $x_3=8$

Ex. $f(1)=4$ $f(2)=5$ $f(7)=5$ $f(8)=4$

find 'x' such that f(x) is minimum/maximum

→ ~~$f'(x)=0 \Rightarrow x$~~ Construct polynomial

then $f'(x)=0$

then find $f''(x) < 0$ or > 0

n-3

$$P_3(x) = \frac{(x-2)(x-7)(x-8)}{(1-2)(1-7)(1-8)} \times 4 + \frac{(x-1)(x-7)(x-8)}{(2-1)(2-7)(2-8)} \times 5$$

$(-2/21)$ $(1/6)$

$$+ \frac{(x-1)(x-2)(x-8)}{(7-1)(7-2)(7-8)} \times 5 + \frac{(x-1)(x-2)(x-7)}{(8-1)(8-2)(8-7)} \times 4$$

$(-1/6)$ $(2/21)$

$$\left[-\frac{2}{21} + \frac{2}{21} + \frac{1}{6} - \frac{1}{6} = 0 \Rightarrow x^3 \text{ terms will be zero}\right)$$

$$= 0 \times x^3 - \frac{1}{6}x^2 + \frac{3}{2}x + \frac{8}{3} \quad (2^{nd} \text{ degree})$$

find stationary point $P_3'(x) = 0 \Rightarrow -\frac{2x}{6} + \frac{3}{2} = 0$

$x = 9/2$ approx

$P_3''(x) = -2/6 < 0$ ($\approx f''(x)$ –) giving maximum

Calculation of coefficient of x^2, x and constant

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Constant →

$$-2\times-7\times-8\times\frac{-2}{21}+-1\times-7\times-8\times\frac{1}{6}$$

$$+-1\times-2\times-8\times\frac{-1}{6}+-1\times-2\times-7\times\frac{2}{21}$$

$$=\frac{32}{3}+\left(-\frac{28}{3}\right)+\frac{16}{6}+\left(\frac{-28}{21}\right)$$

$$=8/3$$

~~Coefficient of~~ x

$\cancel{(a+b)c+}$

$$(x-a)(x-b)(x-c)=x^3+(a+b+c)x^2+\{a(b+c)+bc\}x+abc$$

Coefficient of x^2

$$(-2-7-8)\times\frac{-2}{21}+(-1-7-8)\times\frac{1}{6}+(-1-2-8)\left(\frac{-1}{6}\right)+(-1-2-7)\times\frac{2}{21}$$

$$=-17\times\frac{-2}{21}+-16\times\frac{1}{6}+-11\times\frac{-1}{6}+-10\times\frac{2}{21}$$

$$=\frac{34}{21}+\frac{-16}{6}+\frac{11}{6}+\frac{-20}{21}=\frac{-1}{6}$$

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coefficient of x

$$-\{(-1)(-7-8)+(-7)(-8)\}\left(\frac{-2}{21}\right)$$

$$+\{(-1)(-7-8)+(-7)(-8)\}\left(\frac{1}{6}\right)$$

$$+\{(-1)(-2-8)+(-2)(-8)\}\left(-\frac{1}{6}\right)$$

$$+\{(-1)(-2-7)+(-2)(-7)\}\left\{\frac{2}{21}\right\}$$

$$= 3/2$$

(Notes from moodle recording)

Proving uniqueness of $P_n(x)$ →
Is it possible to have another polynomial $q_n(x)$
for degree $\leq n$ so that $q_n(x_i) = y_i$
but $P_n \neq q_n$

→ Assume that there is $q_n(x) \Rightarrow q_n(x_i) = y_i$
Now we have $P_n(x_i) = y_i$ — ①
$q_n(x_i) = y_i$ — ② $\quad i = 0, 1, \ldots, n$
(n+1) points

$$r_n(x) = P_n(x) - q_n(x)$$

$$r_n(x_i) = 0 \quad , i = 0, 1, \ldots, n.$$

r_n → ~~at~~ utmost 'n' degree polynomial
↳ (n+1) roots ~~are~~ there

$f_n(x_i) = 0$ → n+1 roots → i.e. for $x_0, x_1, \vdots, x_n$ } n+1 points

$i = 0, 1, \dots, n$

P_n → utmost 'n' degree polynomials

Generally n'th degree polynomial has $\leq n$ roots

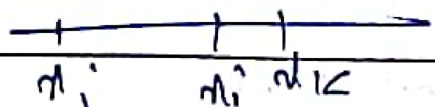
⇒ but it's not possible
because n degree polynomial can't have n+1 roots
∴ it is not true mathematically
violating fundamental theorem of algebra

∴ ∄ another polynomial $q_n(x)$ so that
q_n also interpolate the data

∴ P_n is unique

* Newton's divided Difference interpolation

→ Newton's forward & Backward interpolation are special case of Newton's divided difference interpolation



→ In newton's forward & Backward interpolation order of these points matter. like ~~it should be~~ in descending or ascending order

~~but in case of lagrange & newton's divided~~

~~difference order does not matter~~

Theorem 4

-> Divided differences are symmetric about its arguments

$$f[x_i, x_j] = \frac{f(x_j)-f(x_i)}{x_j - x_i} = \frac{f(x_i)-f(x_j)}{x_i - x_j} = f[x_j, x_i]$$

These are arguments

-> Though we swap values [changed order] still result is same

$\therefore$ Divided differences are symmetric about its arguments

=> Newton's (forward) divided difference formula

$$P_n(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + \dots + b_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

$$P_n(x) = f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \dots + f[x_0,x_1,\dots,x_n](x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

=> Newton's (Backward) divided difference formula

$$P_n(x) = b_0 + (x-x_n)b_1 + \dots + (x-x_n)\dots(x-x_{n-1})\cdots(x-x_1)\,b_n$$

$$= f(x_n) + (x-x_n)f[x_n,x_{n-1}] + \dots + f[x_n,x_{n-1},\dots,x_1,x_0](x-x_n)(x-x_{n-1})\cdots(x-x_1)$$

Ex.

X	$f(x)$
3	3
2	12
1	15
-1	-21

find corresponding polynomial using newton's divided diffⁿ interpolatⁿ

→

x	$f(x)$	1st D.D $f[x_i, x_j]$	2nd D.D $f[x_i, x_{i+1}, x_{i+2}]$	3rd D.D
3	3			
		-9		
2	12		-3	
		-3		1
1	15		-7	
		+18		
-1	-21			

→ values will use for forward diffⁿ

values will be used for backward diffⁿ

-1, -1, -2

un-even placed

1st D.D

$$\frac{12-3}{2-3} \quad \frac{15-12}{1-2} \quad \frac{-21-15}{-1-1}$$

$$\frac{9}{-1} \quad \frac{3}{-1} \quad \frac{-36}{-2}$$

$$-9 \quad -3 \quad 18$$

2nd D.D $= \dfrac{-3-(-9)}{1-3} = \dfrac{6}{-2} = -3$

$$\frac{18+3}{-1-2} = \frac{21}{-3} = -7$$

3rd D.D. $\dfrac{-7-(-3)}{-1-3} = \dfrac{-4}{-4} = 1$

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$n=3$ $\quad f(x) \approx P_3(x) = f_0 + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$

$= 3 - 9(x-3) - 3(x-3)(x-2) + 1(x-3)(x-2)(x-1)$

$= x^3 - 9x^2 + 17x + 6$

-> Easier to find coefficients here compared to lagrangian.

-> Now if miss the data while interpolating, & now we want to include those data pts them.

e.x. (0, -10) data points we missed here
(2.5, 8)

Here we won't have to do rework i.e. we don't have to start the work from begining like we did in lagrange

x	$f(x)$	1st	2nd	3rd	4th	5th
3	3					
2	12					
1	15					
-1	-21					
0	-10					
2.5	8					

new values to be calculated

we used these 3 values above to calculate polynomial, & no change happened to them

Two more terms will be added above without touching above terms

$$\left\{\begin{array}{l} + f[x_0, x_1, x_2, x_3, x_4](x-x_0)(x-x_1)(x-x_2)(x-x_3) \\ + f[x_0, \dots, x_5](x-x_0) \dots (x-x_4) \end{array}\right.$$

Ex. $f(0) = -18$ $f(1) = 0$ $f(3) = 0$ $f(5) = -248$
$f(6) = 0$ $f(9) = 13104$

$\to$ ~~$P_5(x) = (x-1)(x-3)(x-6)\,\phi(x) = f(x)$~~

(as ~~we~~ at point $x = 1, 3, 6$ $y = 0$)

$\phi(x) = x^2 + x + 1 \to$ [don't know how she calculated]

↳ ~~we can do regul~~
we can solve it by regular method, like we did for earlier one. because this will be lengthy

☆ Newton's D.D formula (forward)

$$P_n(x) = f(x_0) + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) + \dots + f[x_0, x_1, \dots x_n](x-x_0)(x-x_1) \dots (x-x_{n-1})$$

if $x_0, x_1, \dots x_n$ are equally spaced
$$x_{i+1} - x_i = h, \quad i = 0, 1, \dots n-1$$

$$f[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}}{2h}$$

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$$= \frac{\Delta y_1 - \Delta y_0}{2h^2} = \frac{\Delta^2 y_0}{2!\,h^2} \rightarrow \text{forward difference coefficients}$$

~~$f[x_0, x_1, x_2, x_3] = f[x_1$~~

Assume it's ~~true~~ for $k = n-1$

$$f[x_0, \cdots, x_{n-1}] = \frac{\Delta^{n-1} y_0}{(n-1)!\,h^{n-1}}$$

$$f[x_1, x_2, \cdots, x_n] = \frac{\Delta^{n-1} y_1}{(n-1)!\,h^{n-1}}$$

$$f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots x_n] - f[x_0, \cdots x_{n-1}]}{x_n - x_0}$$

$$= \frac{\dfrac{\Delta^{n-1} y_1}{(n-1)!\,h^{n-1}} - \dfrac{\Delta^{n-1} y_0}{(n-1)!\,h^{n-1}}}{nh} = \frac{\Delta^n y_0}{n!\,h^n}$$

similarly we can do it for Divided difference of backward difference coefficient

$$f[x_n, x_{n-1}] = \frac{\nabla y_n}{h}$$

Ex. Suppose the function $f(x) = \sin x$ is approximated by a polynomial of degree 9 (that interpolates f using 10 pts) in this interval [0, 1]. Then how large is the error on this interval?

$\rightarrow$ $$|f(x) - P_9(x)| = \left| \frac{f^{(10)}(\psi(x))}{10!} \prod_{i=0}^{9} (x - x_i) \right|$$

Error

Approx

$x_0 \; x_1 \; \ldots \; x_9$

0 ——— 1

$y = \sin(x_0) \; \ldots \; \sin(x_9)$

we won't be able to find exact error as we don't know $\psi(x)$, but we can find upper bound (maxm error)

$$|f(x) - P_9(x)| \leq$$

for maximizing error, we have to bound two things $f^{(10)}$ & $\prod_{i=0}^{9} (x - x_i)$

$$\left| f^{(10)}(x) \right| = \left| \frac{d^{10}}{dx^{10}} \sin x \right| = |-\sin x| \leq \sin 1$$

$$|f(x) - P_9(x)| \leq \frac{|\sin(1)|}{10!} \times \prod_{i=0}^{9} (x - x_i)$$

our point are ranging from 0 to 1

so max^m is $(x_9 - x_0) = 1 - 0 = 1^9$
at extreme — Assuming for every difference $(x - x_i)$

$\therefore |f(x) - P_9(x)| \le \frac{\sin(1)}{10!} \times 1^9 = 2.319 \times 10^{-7}$

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ using newton's forward difference formula

Define $f(x) = \sum_{k=1}^{n} k^2$ $f(1) = 1^2 = 1$, $f(2) = 1^2 + 2^2 = 5$

$\Delta f_n = f(n+1) - f(n) = \sum_{k=1}^{n+1} k^2 - \sum_{1}^{n} k^2 = (n+1)^2$

$\Delta^2 f_n = \Delta f_{n+1} - \Delta f_n = (n+2)^2 - (n+1)^2 = 2n+3$

$\Delta^3 f_n = \Delta^2 f_{n+1} - \Delta^2 f_n = 2(n+1)+3 - (2n+3) = 2$

$\Delta^4 f_n = \Delta^3 f_{n+1} - \Delta^3 f_{n+1} = 2 - 2 = 0$

$\Delta^5 f_n = \Delta^6 f_n = \dots = 0$

k	f_k	Δf	$\Delta^2 f$	$\Delta^3 f$
$x_0 = 1$	1			
		4		
2	5		5	
		9		2
3	14		7	
		16		2
4	30		9	
		25		
5	55			

For any 'n' (natural no) $h = 1$ $s = \frac{x - x_0}{h} = \frac{n-1}{1} = n - 1$

$f(n) = f(1) + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0$
$+ \dots + \frac{s(s-1)\dots(s-k+1)}{k!}\Delta^k f_0$

$= 1 + (n-1)4 + \frac{5}{2}(n-1)(n-2) + \frac{2}{6}(n-1)(n-2)(n-3)$

$= \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} = \frac{n(n+1)(2n+1)}{6}$

[Next Topics are in Diary]

Numerical methods [continuing from notebook]

Error

$$|f(x) - P_n(x)| = \left| \frac{f^{(n+1)}(\varphi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \right|$$

Let $f(x) = x^3 + 3x^2 + 5x + 2$

$(x_0, f(x_0))$
$(x_1, f(x_1))$
$(x_2, f(x_2))$
$(x_3, f(x_3))$
$(x_4, f(x_4))$

These are set of points what is error

$n = 4$

$|f(x) - P_4(x)| = ?$

here $f^5(x) = \frac{d^5(f(x))}{dx^5} = 0$

$\therefore |f(x) - P_n(x)| = 0$

→ If data is coming from quadratic functⁿ and if we have more than 3 data points, naturally we will have zero error

→ In this example if $n \geq 3$ then we will get back same polynomial from which we took the data

Ex. $\sum_{i=0}^{n} L_i(x) = 1 \quad , \forall x$ $\quad L_i(x) \to$ Lagrange polynomial

$$L_i(x) = \prod_{j=0}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

Consider $f(x) = 1 \quad \forall x$

$x =$	$x_0,$	$x_1,$	$\ldots$	x_n
$y =$	1	1		1

$$\sum_{i=1}^{n} \prod_{j=0}^{n} \frac{(x-x_j)}{(x_i-x_j)} = 1$$

since $f(x)$ is a polynomial of degree zero & $n \geq 1$

Using lagrange's Interpolation formula

$$f(x) \approx P_n(x) = \sum_{i=0}^{n} y_i L_i(x)$$

$$= \sum_{i=0}^{n} 1 \, L_i(x)$$

$$1 \approx P_n(x) = \sum_{i=0}^{n} L_i(x)$$

Ex

$x_0 = 2$, $x_1 = 2.75$, $x_2 = 4$

$f(x) = 1/x$ $\varphi \in [2,4]$

Max error?

$\rightarrow$ $y_0 = 1/2$, $y_1 = \frac{1}{2.75}$, $y_2 = 1/4$

$n = 2$ (3pts)

$$\text{Error} = \left| \frac{f^3(\varphi(x))}{3!} (x-2)(x-2.75)(x-4) \right|$$

$f(x) = \frac{1}{x}$ $f'(x) = -\frac{1}{x^2}$ $f''(x) = \frac{2}{x^3}$ $f''' = \frac{-6}{x^4}$

$f^{(3)}(x) = \frac{-6}{x^4}$ is max^m 2 in interval $[2,4]$

because it's decreasing f^n

$(x-2)(x-2.75)(x-4)$

$= (x^2 - 2.75x - 2x + 5.5)(x-4)$

$= (x^3 - 2.75x^2 - 2x^2 + 5.5x) - (4x^2 - 11x - 8x + 22)$

$g(x) = x^3 - 8.75x^2 + 24.5x - 22$

Find max^m of $g(x)$

$g'(x) = 3x^2 - 8.75 \times 2 \times x + 24.5 = 0$

$x = 3.5, 2.32$

@ $x = 3.5$	$g(x) = -0.5625$	$\lvert g(x)\rvert = 0.5625$
$x = 2.32$	$g(x) = 0.2314$	$\lvert g x\rvert = 0.2314$

$$\text{Error} = \frac{6}{2^4 \times 6} \times 0.5625$$

$= 0.03516$

Ex.

x	y
2	8
3	27
4	64
5	125

y data value $y = x^3$

calculate the cuberoot of ~~3.5~~ 10 correct to 3 decimals using lagrange's/ Newton's Interpolⁿ formula

→ Here we to find x given y.

We will have interchange role of x & y as y is not equally spaced so we cant use Newton's forward & backward

Lagrange =

$x \approx P_3(y)$

$= x_0 L_0(y) + x_1 L_1(y) + x_2 L_2(y) + x_3 L_3(y)$

= Newton's DD

$= x_0 + f[y_0, y_1](y-y_0) + f[y_0, y_1, y_2](y-y_0)(y-y_1)$

$+ f[y_0, y_1, y_2, y_3](y-y_0)(y-y_1)(y-y_2)$

we will use Lagrange

$$x = \frac{(y-27)(y-64)(y-125)}{(8-27)(8-64)(8-125)} \times 2 + \quad ($$

$$\frac{(y-8)(y-64)(y-125)}{(27-8)(27-64)(27-125)} \times 3$$

$$+ \frac{(y-8)(y-27)(y-125)}{(64-8)(64-27)(64-125)} \times 4$$

$$+ \frac{(y-8)(y-27)(y-64)}{(125-8)(125-27)(125-64)} \times 5$$

we have to find @ $y=10$

$$= \frac{345}{494} + 0.54083 + \left(\frac{-1955}{15799}\right)$$

$$\frac{5865}{6916} \times 2 + \frac{510}{38857}$$

↳ ~~something wrong in calculation~~

$= 2.1263$

↳ this was inverse interpolation

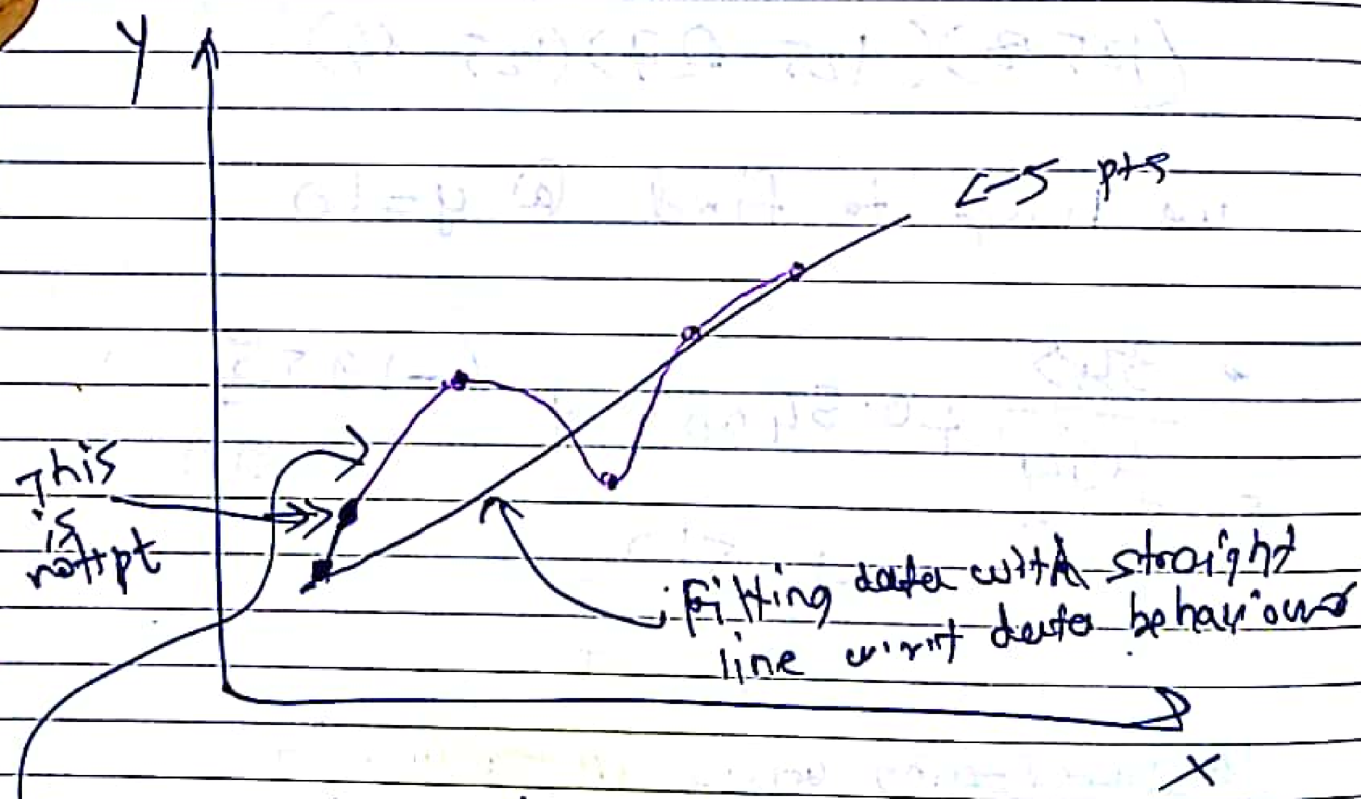
20/4/21

Least square method → Curve fitting

→ Data is given → fit a curve

poly, expon, log, power fn

general term (can be suitable for interpoln also)



Interpolation polynomial $P_4(x)$

In interpoln req. condn was that $P_n(x_i) = y_i$, $i = 0, 1, \ldots, n$

but in least square method it is not necessary, here we try to minimize the error

[sometimes in data there is noise, so interpolation may not give exact behaviour of data, so we use Least square method]

$x_0 \quad y_0$

$x_1 \quad y_1$

$\vdots \quad \vdots$

$x_n \quad y_n$

min error?

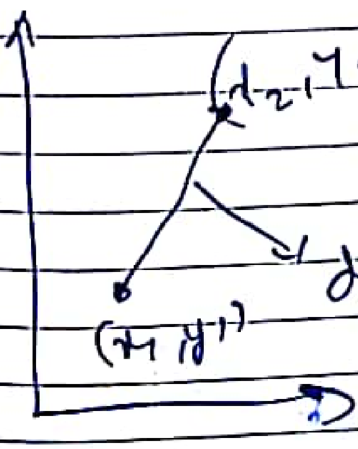
Let $g(x) \rightarrow$ approximate curve

Find g such that error is min^m

$y_i - g(x_i)$, $i = 0, 1, \dots, n$

$[y_0 - g(x_0), \; y_1 - g(x_1) \dots, y_n - g(x_n)]$

n dimensional error vector



dist^n bet^n two points

euclidian dist^n

$$\sqrt{(x_1 - x_2)^2 + (y_2 - y_1)^2}$$

Least square method

Least square Error:

$$E = E(a_0, a_1, \cdots a_n)$$

$$= \sum_{i=1}^{m} (y_i - g(x_i))^2$$

[n polym degre → $(x_1, y_1) \ldots (x_m, y_m)$]

Linear Least square (fitting a straight line)

$$g(x) = a_0 + a_1 x \qquad a_0, a_1 ?$$

Find a_0, a_1 so that the error

$$E = \underbrace{E(a_0, a_1)}_{\text{(Two variable fn)}} = \sum_{i=1}^{m} \left[y_i - \{a_0 + a_1 x_i\} \right]^2$$

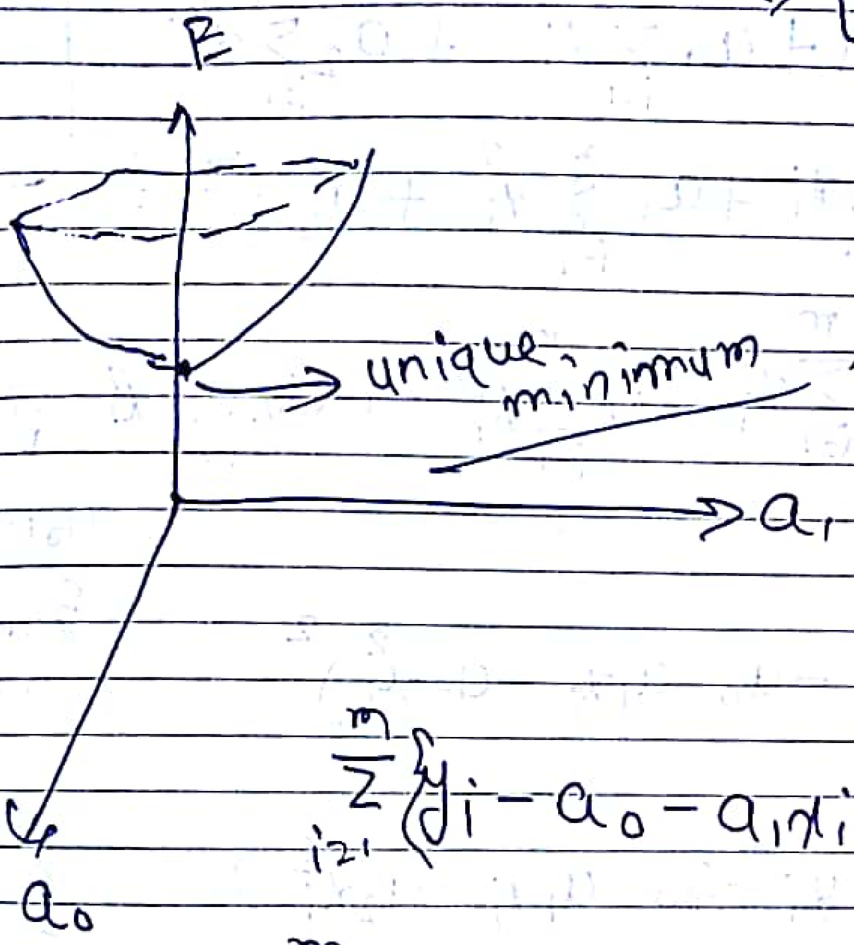
$$\frac{\partial E}{\partial a_0} = 0 \qquad \frac{\partial E}{\partial a_1} = 0$$

$$\sum_{i=1}^{m} 2\left[y_i - \{a_0 + a_1 x_i\}\right](-1) = 0 \rightarrow (1)$$

$$\sum_{i=1}^{m} 2\left[y_i - \{a_0 + a_1 x_i\}\right](-x_i) = 0 \rightarrow (2)$$

solving these two will give stationary points

→ [we will get unique minimum]



$$\sum_{i=1}^{m} \{y_i - a_0 - a_1 x_i\} = 0 \rightarrow \text{from } ①$$

$$\sum_{i=1}^{m} \{y_i - a_0 - a_1 x_i\} x_i = 0 \rightarrow \text{from } ②$$

$$\sum_{1}^{m} (a_0 + a_1 x_i) = \sum y_i$$

$$\boxed{m a_0 + \left(\sum_{i=1}^{m} x_i\right) a_1 = \sum_{i=1}^{m} y_i} \quad — ③$$

$$a_0 \left(\sum_{1}^{m} x_i\right) + a_1 \left(\sum_{i=1}^{m} x_i^2\right) = \sum_{i=1}^{m} x_i y_i \quad — ④$$

normal equations

$g(x) = a_0 + a_1 x$

Derive Normal Eqⁿ

$g(x) = a_0 + a_1 x + a_2 x^2$

$① \rightarrow \sum_{i=1}^{m} y = a_0 m + a_1 \sum_{i=1}^{m} x_i + a_2 \sum_{i=1}^{m} x_i^2$

$② \rightarrow \sum_{i=1}^{m} x_i y_i = a_0 \sum_{i=1}^{m} x_i + a_1 \sum_{i=1}^{m} x_i^2 + a_2 \sum_{i=1}^{m} x_i^3$

$③ \rightarrow \sum_{i=1}^{m} x_i^2 y_i = a_0 \sum_{i=1}^{m} x_i^2 + a_1 \sum_{i=1}^{m} x_i^3 + a_2 \sum_{i=1}^{m} x_i^4$

Normal Equations

Derivation

$$E = \sum_{i=1}^{m} \left(y_i - a_0 - a_1 x_i - a_2 x_i^2 \right)^2$$

$$\frac{\partial E}{\partial a_0} = -2 \sum_{i=1}^{m} \left(y_i - a_0 - a_1 x_i - a_2 x_i^2 \right) \quad — ①$$

$$\frac{\partial E}{\partial a_1} = -2 \sum_{i=1}^{m} x_i \left(y_i - a_0 - a_1 x_i - a_2 x_i^2 \right) \quad — ②$$

$$\frac{\partial E}{\partial a_2} = -2 \sum_{i=1}^{m} x_i^2 \left(y_i - a_0 - a_1 x_i - a_2 x_i^2 \right) \quad — ③$$

after solving ①, ② & ③ we will get normal equations.

$$\begin{array}{ccc} A & \bar{x} & b \end{array}$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ \vdots \end{bmatrix}$$

we are minimizing $\|b - A\bar{x}\|_2^2$

$\to$ n^{th} degree poly $g(x) = a_0 + a_1 x + \dots + a_n x^n$

$$E(a_0, a_1, \dots a_n) = \sum_{i=1}^{m} \left[y_i - (a_0 + a_1 x_i + \dots + a_n x_i^n) \right]^2$$

$$m a_0 + \sum x_i a_i + \dots + \sum x_i^n a_n = \sum y_i$$

$$\sum x_i a_0 + (\sum x_i^2) a_1 + \dots + (\sum x_i^{n+1}) a_n = \sum x_i y_i$$

$$\vdots$$

$$\left(\sum x_i^n\right) a_0 + \left(\sum x_i^{n+1}\right) a_1 + \dots + \left(\sum x_i^{2n}\right) a_n = \sum x_i^n y_i$$

$n \times n$ system

It will given unique sol^n } provided $x_1, x_2 \dots x_m$ are distinct.

$\Sigma x_i = 2.5$

x	y
0	1
0.25	1.2840
0.5	1.6487
0.75	2.1170
1	2.7183

$\Sigma x_i^2 = 1.875$

$\Sigma x_i^3 = 1.5625$

$\Sigma x_i^4 = 1.3828$

$\Sigma y_i = 8.768$

$\Sigma x_i y_i = 5.45$

$m=5$

$\Sigma x_i^2 y_i = 4.4015$

fit polynomials of degree 1 & 2.

→ st. line $y = a_0 + a_1 x$

$$\begin{bmatrix} 5 & \sum_1 x_i \\ \sum_1 x_i & \sum_1 x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum_1^5 y_i \\ \sum_{i=1}^5 x_i y_i \end{bmatrix}$$

Quadratic $y = b_0 + b_1 x + b_2 x^2$

$$\begin{bmatrix} 5 & \sum_1^5 x_i & \Sigma x_i^2 \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 \\ \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \\ \Sigma x_i^2 y_i \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2.5 \\ 2.5 & 1.875 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 8.768 \\ 5.45 \end{bmatrix}$$

$a_0 = 0.90079$

$a_1 = 1.705607$

$$\begin{bmatrix} 5 & 2.5 & 1.875 \\ 2.5 & 1.875 & 1.5625 \\ 1.875 & 1.5625 & 1.3828 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 8.768 \\ 5.45 \\ 4.4015 \end{bmatrix}$$

$b_0 = 1.0051$ $b_1 = 0.8647$ $b_2 = 0.8432$

we ~~try~~ tried to minimize

$$Error = \sqrt{\sum_{i=1}^{m} (y_i - (a_0 + a_1 x_i))^2}$$

$$Error = \sqrt{\sum_{i=1}^{m} (y_i - \{b_0 + b_1 x_i + b_2 x_i^2\})]^2}$$

Ex.

$y = ae^{bx}$ → How to use least square method for exponential

Least sqr Error $= E(a,b) = \sum_{i}^{n} \{y_i - a e^{bx_i}\}^2$

$\frac{\partial E}{\partial a} = 0$ $\frac{\partial E}{\partial b} = 0$ ⇒ stationary points

$\sum_{1}^{m} (y_i - a e^{bx_i}) e^{bx_i} = 0$ — ① ← $\frac{\partial E}{\partial a} = 0$

$\sum_{1}^{m} (y_i - a e^{bx_i})(a x_i e^{bx_i}) = 0$ → ② ← $\frac{\partial E}{\partial b} = 0$

Non linear system [$Ax = b$ not possible here]

→ Iterative solving is the only ~~correct~~ method for non linear system (difficult part to solve)

(we can't say it will have unique, infinite or no sol^n for non linear system ∴ we try to avoid non-linearity so we convert it to linear

ae^{bx} → Two paramters a & b to be found

→ convert this to straight line

$y = a e^{bx}$

$\underbrace{\log y}_{} = \underbrace{\log a}_{} + b\underbrace{x}_{}$

$Y = A + bx$

$(x,y) \to (x,Y)$

now normal eq^n we know for straight line

$$\begin{bmatrix} m & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} \begin{bmatrix} A \\ b \end{bmatrix} = \begin{bmatrix} \Sigma Y_i \\ \Sigma x_i Y_i \end{bmatrix}$$

solve for A & b

then find $\log a = A$ $\log y = Y$

but when ~~y~~ any one of $y_i = 0$ or $-ve$

this can't be solved

power f^n

$y = ax^b$ $a, b \in R$

solve similarly like exponential case

$\ln y = \ln a + b \ln x$

$Y = A + bX$

$X = \ln x$

$Y = \ln y$

$A = \ln a$

$(x,y) \to (X,Y)$

$$\begin{bmatrix} m & \Sigma X_i \\ \Sigma X_i & \Sigma X_i^2 \end{bmatrix} \begin{bmatrix} A \\ b \end{bmatrix} = \begin{bmatrix} \Sigma Y_i \\ \Sigma X_i Y_i \end{bmatrix}$$

Pressure Vs Volume at various instances during the expansion of steam in a cylinder

$PV^n = C$

{ Using least square method
{ str. line m & c

P	V
200	1
100	1.70
50	2.89
30	4.80
20	5.88
10	10.00

$\ln P + n \ln V = \ln C$

or

$\underbrace{\log_{10} P}_{y} + n \underbrace{\log_{10} V}_{x} = \underbrace{\log_{10} C}_{b}$

$y + nx = b$

$y = b - nx = b + (-n)x$

$x = \log_{10} V \quad y = \log_{10} P$

Normal eq^n

$$\begin{bmatrix} 6 & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} \begin{bmatrix} b \\ -n \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix}$$

const^n

coeff^n of x

$\Sigma x_i = 3.1419$

$\Sigma x_i^2 = 2.3215$

$\Sigma y_i = 9.7781$

$\Sigma x_i y_i = 4.2512$

$$\begin{bmatrix} 6 & 3.1419 \\ 3.1419 & 2.3215 \end{bmatrix} \begin{bmatrix} b \\ -n \end{bmatrix} = \begin{bmatrix} 9.7781 \\ 4.2512 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} b \\ -n \end{bmatrix} = \frac{1}{4.058} \begin{bmatrix} 2.3215 & -3.1419 \\ -3.1419 & 6 \end{bmatrix} \begin{bmatrix} 9.77 \\ 4.25 \end{bmatrix}$$

$b = 2.3027$

$-n = -1.2855$ $\quad n \Rightarrow 1.2855$

$b = \log_{10} C \Rightarrow C = 200.77705$

Ex. X: 1 2 3 4 5 6

y: 2.6 5.4 8.7 12.1 16 20.2

Fit the data using the law of form $y = ax + bx^2$ using a st. line fit.

$\rightarrow \frac{y}{x} = a + bx$

$Y = a + bx \qquad (x_i, y_i) \rightarrow (x_i, y_i/x_i)$

$\Sigma x_i = 21$

$\Sigma x_i^2 = 91$

$\Sigma Y_i = 427/24 = 17.7917$

$\Sigma x_i Y_i = 65$

$a = 2.4194$

$b = 0.156$

or

$$E(a,b) = \sum_{1}^{m} \{ y_i - (ax_i + bx_i^2) \}^2$$

$$\frac{\partial E}{\partial a} = \sum_{1}^{m} [2(y_i - (ax_i + bx_i^2))(-x_i)] = 0$$

$$\frac{\partial E}{\partial b} = \sum_{1}^{m} [2(y_i - (ax_i + bx_i^2))(-x_i^2)] = 0$$

after simplification we get

$$-\sum_{1}^{m} x_i y_i + a \sum_{i=1}^{m} x_i^2 + b \sum_{i=1}^{m} x_i^3 = 0$$

$$-\sum_{i=1}^{m} x_i^2 y_i + a \sum_{i=1}^{n} x_i^3 + b \sum_{i=1}^{n} x_i^4 = 0$$

$$\begin{bmatrix} \Sigma x_i^2 & \Sigma x_i^3 \\ \Sigma x_i^3 & \Sigma x_i^4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \Sigma x_i y_i \\ \Sigma x_i^2 y_i \end{bmatrix}$$

Numerical Integration: (Based on Interpolation)

→ ~~Gaussian Quadrate~~

What we did in Interpol^n

$y = f(x)$: (x_i, y_i) $i = 0, 1, \ldots, n$

$P_n(x) \approx f(x)$ $[a, b]$

$$\int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx$$

Trapezoidal & Simpson's Rules

Lagrange Interpola^n formula?

(x_0, y_0) (x_1, y_1) $\ldots$ (x_n, y_n)

$$f(x) \approx P_n(x) = \sum_{i=0}^{n} y_i L_i(x)$$

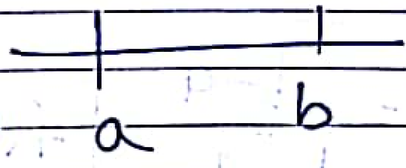
$$= \sum_{i=0}^{n} y_i \left[\prod_{\substack{j=0 \\ j \neq i}}^{n} \cancel{L_i(x)}\, \frac{(x - x_j)}{(x_i - x_j)} \right]$$

$$\int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx = \int_a^b \left\{ \sum_{i=0}^{n} L_i(x)\, y_i \right\} dx$$

$$= \sum_{i=0}^{n} y_i \int_a^b L_i(x)\,dx$$

$n=1$ (2 points x_0, x_1) $[a, b]$

$x_0 = a \quad x_1 = b$



$P_1(x) = y_0 L_0(x) + y_1 L_1(x)$

$= y_0\left(\frac{x-b}{a-b}\right) + y_1\left(\frac{x-a}{b-a}\right)$

$= \frac{1}{b-a}\left[y_0(x-b) + y_1(x-a)\right]$

$\int_a^b f(x)\,dx \cancel{\approx \frac{1}{b-a}\int_a^b y_0 (b-x)}$

$\approx \frac{1}{b-a}\left[\int_a^b f(a)(b-x) + f(b)(x-a)\right]dx$

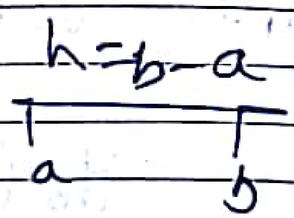
$\approx \frac{1}{b-a}\left[f(a)\left(bx - \frac{x^2}{2}\right)_a^b + f(b)\left(\frac{x^2}{2} - ax\right)_a^b\right]$

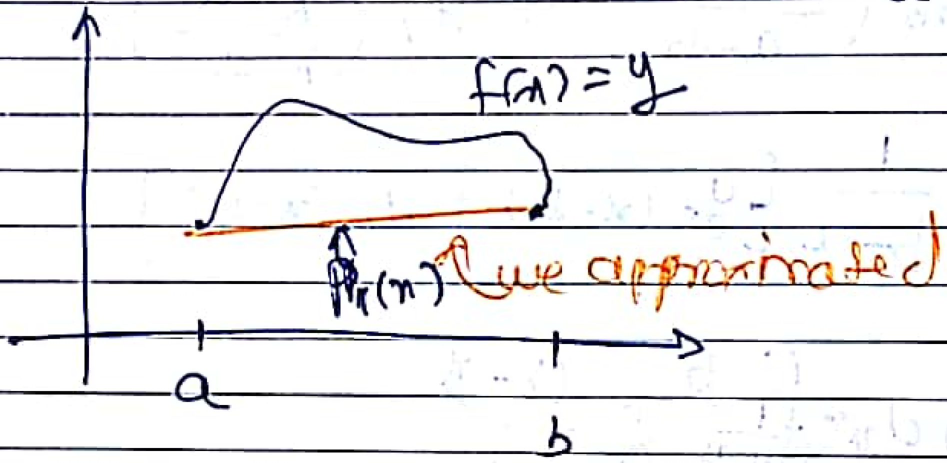
$\approx \frac{1}{b-a}\left[f(a)\left(b^2 - \frac{b^2}{2} - \left(ab - \frac{a^2}{2}\right)\right) + f(b)\left(\frac{b^2}{2} - ab - \left(\frac{a^2}{2} - a^2\right)\right)\right]$

$\approx \frac{1}{b-a}\left[f(a)\left[\frac{b^2}{2} - ab + \frac{a^2}{2}\right] + f(b)\left[\frac{b^2}{2} - ab + \frac{a^2}{2}\right]\right]$

$$= \frac{1}{2(b-a)}\left[f(a)(b-a)^2 + f(b)(b-a)^2\right]$$

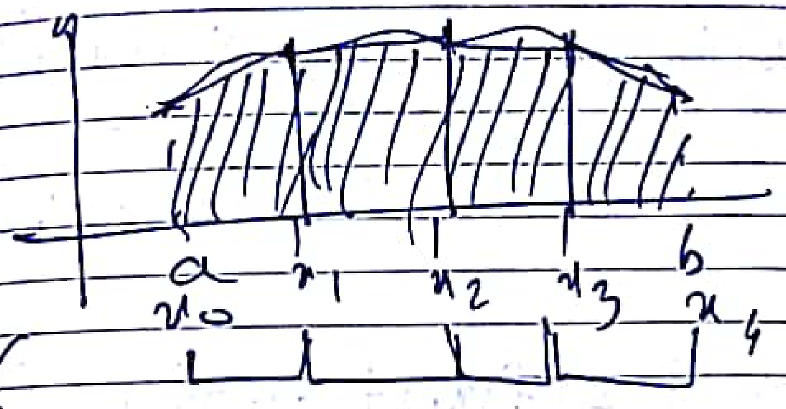
$$= \boxed{\frac{b-a}{2}\left[f(a)+f(b)\right]}$$





$$P_1(x) \quad \begin{cases} P_1(a) = f(a) \\ P_1(b) = f(b) \end{cases}$$

Instead of only a, b, we will go for more points



This is better approximⁿ for more points with Trapezoidal rule for each interval.

$$\int_a^b f(x)\,dx = \int_{a=x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx$$

This is not approximⁿ [exact value]

$$+ \int_{x_2}^{x_3} f(x)\,dx + \int_{x_3}^{x_4} f(x)\,dx \quad — (1)$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left[f(a)+f(b)\right]$$

Composite Trapezoidal Rule

$x_0, x_1, \ldots x_n$ $n+1$ points/ n sub intervals

eqⁿ (1) — 11 — Assume equally spaced

$x_{i+1} - x_i = h$

$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right] + \frac{h}{2}\left[f(x_1) + f(x_2)\right] + \cdots \frac{h}{2}\left[f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{2}\left[f(x_0) + 2\left(f(x_1) + \cdots + f(x_{n-1})\right) + f(x_n)\right]$$

Simpon's $\frac{1}{3}$rd Rule

$x_0 = a \quad x_1 = \frac{a+b}{2} \quad x_2 = b$

$h = \frac{b-a}{2}$

$$f(x) \approx P_2(x) = y_0 + s\Delta y_0 + s(s-1)\frac{\Delta^2 y_0}{2!}$$

$$\int_a^b f(x)\,dx \approx \int_a^b P_2(x)\,dx \qquad \left[s = \frac{x - x_0}{h}\right]$$

$$= \int_a^b \left(y_0 + s\Delta y_0 + s(s-1)\frac{\Delta^2 y_0}{2!}\right) dx$$

$$\left[\frac{ds}{dx} = \frac{1}{h}\right] \rightarrow ds = \frac{dx}{h}$$

$dx = h \cdot ds$

$ds = \frac{dx}{h}$ $\quad h\,ds = dx$

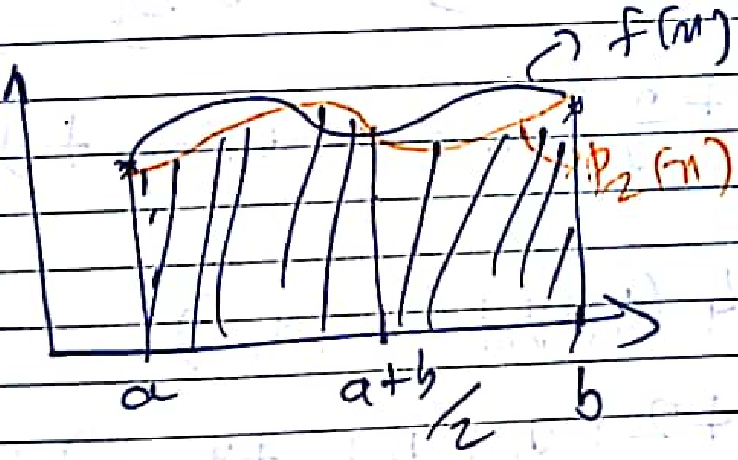
$x = a \quad s = 0$

$x = b \quad s = 2$

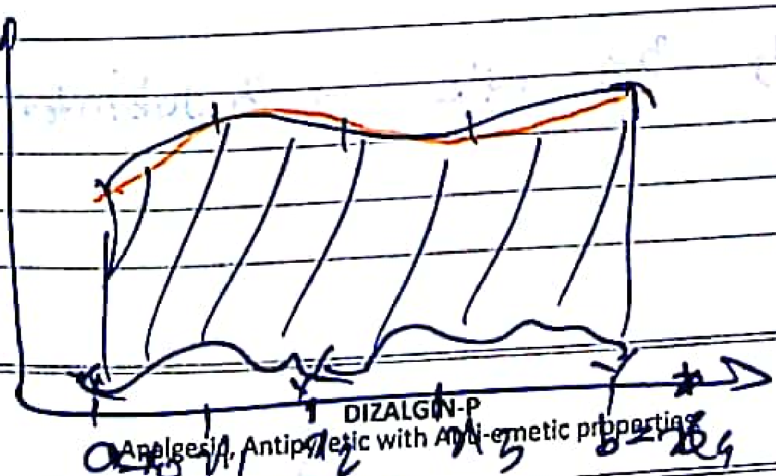
$$= h\int_0^2 \left[y_0 + s\Delta y_0 + s(s-1)\frac{\Delta^2 y_0}{2!} \right] ds$$

$\Delta y_0 \to y_1 - y_0$ $\quad \Delta^2 y_0 \to y_2 - 2y_1 + y_0$

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + 4y_1 + y_2 \right]$$



We can improve further



Composite rule for simpson.

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad \dots \quad x_{n-1} \quad x_n$

but we take two intervals together
so restriction is n should be even

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \dots \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2] + \frac{h}{3}[y_2 + 4y_3 + y_4] \quad \boxed{h = \frac{b-a}{n}}$$

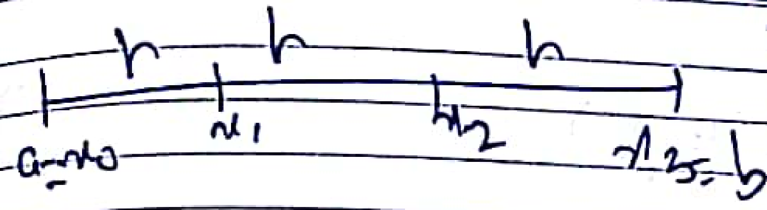
$$+ \dots + \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

$$= \frac{h}{3}[y_0 + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2}) + y_n]$$

$\Rightarrow$ Simpson's $3/8$ rule $\rightarrow$ 3 subintervals

Scanned by CamScanner

simpson's 3/8th rule



$n \Rightarrow 3 \rightarrow$ 3 subintervals (4 pts)

$\left[a, a+\frac{b-a}{3}\right]$ $\left[a+\frac{b-a}{3}, a+2\frac{b-a}{3}\right]$,

$\left[a+2\left(\frac{b-a}{3}\right), \underbrace{a+3\frac{b-a}{3}}_{b}\right]$

$y = f(x) \approx P_3(x) = y_0 + s\Delta y_0 + s(s-1)\frac{\Delta^2 y_0}{2!}$

$+ s(s-1)(s-2)\frac{\Delta^3 y_0}{3!}$

$\int_a^b f(x)\,dx \approx \int_a^b \left[y_0 + \frac{s\Delta y_0}{1!} + \dots + \frac{s(s-1)(s-2)\Delta^3 y_0}{3!} \right] dx$

$[hs = x - x_0]$

$h\,ds = dx$

$\left[s = \frac{x - x_0}{h}\right]$

$a = x_0$

$b = x_3$

$= h\int$

$\boxed{s=0}$

$s = \frac{x_3 - x_0}{h}$

$s = \frac{x_0 + 3h - x_0}{h}$

$\boxed{s=3}$

change $dx \to ds$

$$= h\int_0^3 [y_0 + s(y_1 - y_0) \; . \; . \; . \quad] \, ds$$

$$= h\left[y_0 (s)_0^3 + \frac{\Delta y_0}{1!}\left(\frac{s^2}{2}\right)_0^3 + \frac{\Delta^2 y_0}{2!}\left(\frac{s^3}{3} - \frac{s^2}{2}\right)_0^3 \right.$$

$$\left. + \frac{\Delta^3 y_0}{2!}\left(\frac{s^4}{4!} - s^3 + s^2\right)_0^3 \right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \; . \; . \; . \; . \right.$$

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0) + 3(y_1 + y_2) + y_3 \right]$$

For Trapezoidal rule $\to$ poly degree ≤ 1 (2pts)

$1/3$ S.R $\to$ ≤ 2 (3pts)

$3/8$ S.R $\to$ ≤ 3 (4pts)

quadratic

cubic polynomial

subintervals

Composite Simpson's $\frac{3}{8}$th rule [$n \to$ multiply of 3]

for T.R x_0 —— x_1

$\frac{1}{3}$ S.R x_0 —— x_1 —— x_2 $\to n \to$ even

$\frac{3}{8}$ S.R x_0 —— x_1 —— x_2 —— x_3 $\to n \to$ multiple of 3

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_{n-1} \quad x_n$

$[x_0, x_3] \; [x, x_6] \; \cdots \; [x_{n-3}, x_n]$

$$\frac{3h}{8}\Big[\{y_0 + 3(y_1 + y_2) + y_3\} +$$

$$\{y_3 + 3(y_4 + y_5) + y_6\} +$$

$$\{y_6 + 3(y_7 + y_8) + y_9\} +$$

$$\vdots$$

$$\{y_{n-3} + 3(y_{n-2} + y_{n-1}) + y_n\}$$

$\int_a^b f(x)\,dx$

$$\approx \int_a^b f(x)\,dx = \sum_{j=0}^{n/3} \int_{x_i}^{x_{i+3}} f(x)\,dx$$

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[y_0 + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 + \cdots + 2(y_3 + y_6 + \cdots + y_{n-3}) + y_n\right]$$

$$= \frac{3h}{8}\left[y_0 + 3\sum_{i=1}^{n/3}(y_{3i-2} + y_{3i-1}) + 2\sum_{i=1}^{n/3 - 1} y_{3i} + y_n\right]$$

$n \to$ multiple of 3

Ex. $\int_0^1 \frac{1}{1+x}\,dx$ Use both Trapezoidal & Simpson's Rule ($\frac{1}{3}$rd) with $h = 0.5$

$= \log(1+x)\Big|_0^1 = \log_e 2 = 0.69314$ | ~~8~~ $h = 0.25$

T.R. → $h = 0.5$ x_0, x_1, x_2: 0, 0.5, 1

T.R. $\int_0^1 \frac{1}{1+x}\,dx = \frac{h}{2}\left[y_0 + 2y_1 + y_2\right]$

$= \frac{0.5}{2}\left[\frac{1}{1+0} + \frac{2}{1+0.5} + \frac{1}{1+1}\right] = \frac{17}{24}$

S.R. $\int_0^1 \frac{1}{1+x}\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$ $= 0.708$

$= \frac{0.5}{3}\left[\frac{1}{1+0} + \frac{4}{1+0.5} + \frac{1}{1+1}\right]$

$\cancel{y = \frac{1}{1+x}}$

$= \frac{25}{36}$

$= 0.6945$

S.R. is better accurate than T.R.

$h = 0.25$

$n = \frac{b-a}{h}$ $\quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

$\quad\quad\quad 0 \quad 0.25 \quad 0.5 \quad 0.75 \quad 1$

$= \frac{1-0}{0.25} = 4$

$T.R. = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right]$

$= \frac{0.25}{2}\left[\frac{1}{1+0} + 2\left[\frac{1}{1+0.25} + \frac{1}{1+0.5} + \frac{1}{1+0.75}\right] + \frac{1}{1+1}\right]$

$= \cancel{0.6968}$

$= 0.69702$

$Sim.\ \frac{1}{3} = \frac{h}{3}\left[y_0 + 4(y_1 + y_3) + 2y_2 + y_4\right]$

$= \frac{0.25}{3}\left[\frac{1}{1+0} + 4\left(\frac{1}{1+0.25} + \frac{1}{1+0.75}\right) + 2\left(\frac{1}{1+0.5}\right) + \frac{1}{(1+1)}\right]$

$= \frac{0.25}{3}\left[1 + \frac{192}{35} + \frac{4}{3} + \frac{1}{2}\right]$

$= 0.69325$

Error

$h = 0.5 \quad |\log_e 2 - 0.70833| = 0.015\ldots$

$|\log_e 2 - 0.69315| = 0.0013\ldots$

$h = 0.25$ $\quad |\log 2 - 0.697021| = 0.0038$

$|\log 2 - 0.69325| = 0.00005$

Ex.

$\int_4^{5.2} \log_e x \, dx \rightarrow$ use all $\begin{cases} T.R \\ S.R\ 1/3 \\ S.R\ 3/8 \end{cases}$ $n = 6$

$[4, 5.2]$

$h = ?$

→ We have to use multiplication rule of integrating $\log x \, dx$

$\int_4^{5.2} \log x \cdot 1 \, dx$

Exact $= 5.2 \log 5.2 - 4 \log 4 - 1.2$
$= 1.8278474\cancel{0}8574$

$h = \frac{5.2 - 4}{6}$

$= \frac{1.2}{6} = 0.2$

4 4.2 4.4 4.6 4.8 5 5.2

T.R =

$\frac{h}{2} [y_0 + 2(y_1 + y_2 + \dots + y_5) + y_6] = 1.82765$

$$SR \tfrac{1}{3} = \frac{h}{3}\left[y_0 + 4(y_1+y_3+y_5) + 2(y_2+y_4) + y_6\right]$$

$$= 1.82784726$$

$$SR(\tfrac{3}{8}) = \frac{3h}{8}\left[y_0 + 3(y_1+y_2+y_4+y_5) + 2y_3 + y_6\right]$$

$$= 1.82784707$$

Error for all 3 value,

1) $1.9226857\ 4 \times 10^{-4}$ → T.R

2) 1.48574×10^{-7} → $\tfrac{1}{3}$ rule

3) 3.38574×10^{-7} → $\tfrac{3}{8}$ rule

★ Error in Numerical Integration

MV Theorem for Integrals; f is continuous in $[a, b]$

$\exists\ c \in [a, b]$. $f(c) = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{avg of } f \text{ in } [a,b]}$

↳ avg of f in $[a, b]$

weighted MVT →

f is contn in [a,b] & w is integrable on [a,b]

Also 'w' never changes its sign on [a,b]

Then ∃ $c \in [a,b]$

$$f(c) = \frac{\int_a^b w(x) f(x)\,dx}{\int_a^b w(x)\,dx}$$ — ☆ [we will use this

T.R. ~~Erroror~~

$$\text{Error} = \int_a^b f(x)\,dx - \underbrace{\left\{ \frac{b-a}{2} \left(f(a) + f(b) \right) \right\}}_{①}$$

Two pts : (a, f(a)) (b, f(b))

polynomial interpoln error if n = 1 (2 pts used)

$$f(x) - P_1(x) = \frac{f^{(2)}(\varphi)}{2!} (x-a)(x-b) \quad ②$$

$$\int_a^b f(x)\,dx - \int_a^b P_1(x)\,dx =$$

$$\int_a^b \left\{ f(x) - P_1(x) \right\} dx =$$

$$= \int_a^b \frac{f^2(\psi(x))}{2!} (x-a)(x-b)\, dx$$

Now we will weighted MVT rule

we assum $f^2(\psi(x))$ is continuous

$w(x) = (x-a)(x-b)$ $\cancel{f(x)}$

$\cancel{f(x)}$ $\frac{f^2(\psi(x))}{2!} \approx$

$$\text{Error} = \int_a^b \frac{f^2(\psi(x))}{2!} (x-a)(x-b)\, dx$$

using MVT [$\cancel{A}$ eq^n] $\{\psi(x)$ vary w.r.t. x

$[\cancel{f(x)}$

$$= \frac{f^2(c)}{2!} \int_a^b (x-a)(x-b)\, dx$$

$$\text{Error} = \frac{f^2(c)}{2!} \frac{(a-b)^3}{6} = -\frac{f^2(c)}{12}(b-a)^3$$

$$\text{Error} = \frac{-f^2(c)}{12}(b-a)^3$$

Composite rule

$$\int_a^b f(x)\,dx - \int_a^b P_1(x)\,dx = \int_{x_0}^{x_1}\left(f(x)\,dx - P_1(x)\right) + \dots + \int_{x_{n-1}}^{x_n}\left[f(x)\,dx - P_1(x)\right]$$

$$= \frac{-f^2(c_1)}{12}h^3 + \frac{-f^2(c_2)}{12}h^3 + \dots$$

$$+ \frac{-f^2(c_n)}{12}h^3$$

$$= \frac{-h^3}{12}\sum_{i=1}^{n} f^2(c_i)$$

$$= \frac{-h^3}{12} \times n \times \frac{1}{n}\sum_{i=1}^{n} f^2(c_i) \to (3)$$

Since

[f^2 is continuous in $[a,b]$

$\Rightarrow f^2$ is bounded in $[a,b]$]

So it attains max or min in $[a,b]$

$\min_{a \le x \le b} f^2 \qquad \max_{a \le x \le b} f^2$

Then $\exists\ \eta \in [a,b]$ $f''(\eta) = \underbrace{\frac{1}{n}\sum_{j=1}^{n} f''(\varphi_j)}_{\text{put in } (3)}$

(3) becomes

$$\text{Error} = \frac{-h^3}{12} \times n \times f''(\eta)$$

$$= -\frac{h^3}{12} \times \frac{b-a}{h} \times f''(\eta)$$

$$\boxed{\text{Error} = -\frac{(b-a)}{12} h^2 f''(\eta)}$$ $\eta \in [a,b]$

Error in T.R.

$$\boxed{\text{Error in } S\ 1/3 \text{ rule} = -\frac{(b-a)}{180} h^4 f^{iv}(\eta_1)}$$ $\eta_1 \in [a,b]$

$$\boxed{\text{Simpson } 3/8 \text{ Error} = -\frac{(b-a)}{80} h^4 f^4(\eta_2)}$$ $\eta_2 \in [a,b]$

$\eta, \eta_1, \eta_2 \in [a,b]$

$h=0.5$ | error will reduced | and
$h=0.25$
$h=0.1$

error $\to 0$
as $h \to 0$

(faster if order is max)

Ex $\int_4^{5.2} \log x \, dx =$ $h=0.2$

$n=6$

calculate max possible error in each of these methods

~~$\eta, \eta_1, \eta_2 \in [a,b]$~~

$\to$ $f(x)=\log x$ $f'(x)=1/x$ $f''(x) = -1/x^2$

$f'''(x) = 2/x^3$ $f^{iv}(x) = -6/x^4$.

$[a,b]=[4,5.2]$

T.R. Rule

Absolute Error $= \left| -\dfrac{(b-a)}{12} h^2 f''(\eta) \right|, \eta \in [a,b]$

$= \dfrac{1.2}{12} \times 0.2^2 \times |f''(\eta)|$

$\leq \dfrac{1.2 \times 0.2^2}{12} \times \max_{[4,5.2]} \dfrac{1}{x^2}$

while calculating previous error [1.1922 -- x 10^{-7}] we didn't round off much so we got very good accuracy ~~error is less than upper bound~~
if we round off very badly then it may cross upper bound

$$= \frac{1.2 \times 0.2^2}{12} \times \frac{1}{4^2} = \frac{1}{4000} = 2.5 \times 10^{-4}$$

[previously we got error $1.92268574 \times 10^{-4}$]

[Here round of errors are not considered] ~~so error is some i.e. we have not rounded off, so error is more~~]

[$1.92268574 \times 10^{-4} \leq \underbrace{2.5 \times 10^{-4}}_{\text{upper bound}}$]

Simpsons 1/3 rd :

$$\text{Abs. error} = \left| -\frac{(b-a)}{180} h^4 f^{(iv)}(x) \right|$$

$$\leq \frac{(5.2-4)}{180} \times 0.2^4 \times \max_{[4,5.2]} \frac{6}{x^4}$$

decreasing f^n ↳ max at 4

$$= \frac{1.2 \times 0.2^4}{180} \times \frac{6}{4^4}$$

$$= 2.5 \times 10^{-7} \rightarrow \text{upper bound}$$

[Earlier error was 1.48574×10^{-7}]

simpons $3/8^{th}$ rule

$$= \left| \frac{1.2 \times 0.2^4}{80} \times \frac{6}{4^6} \right|$$

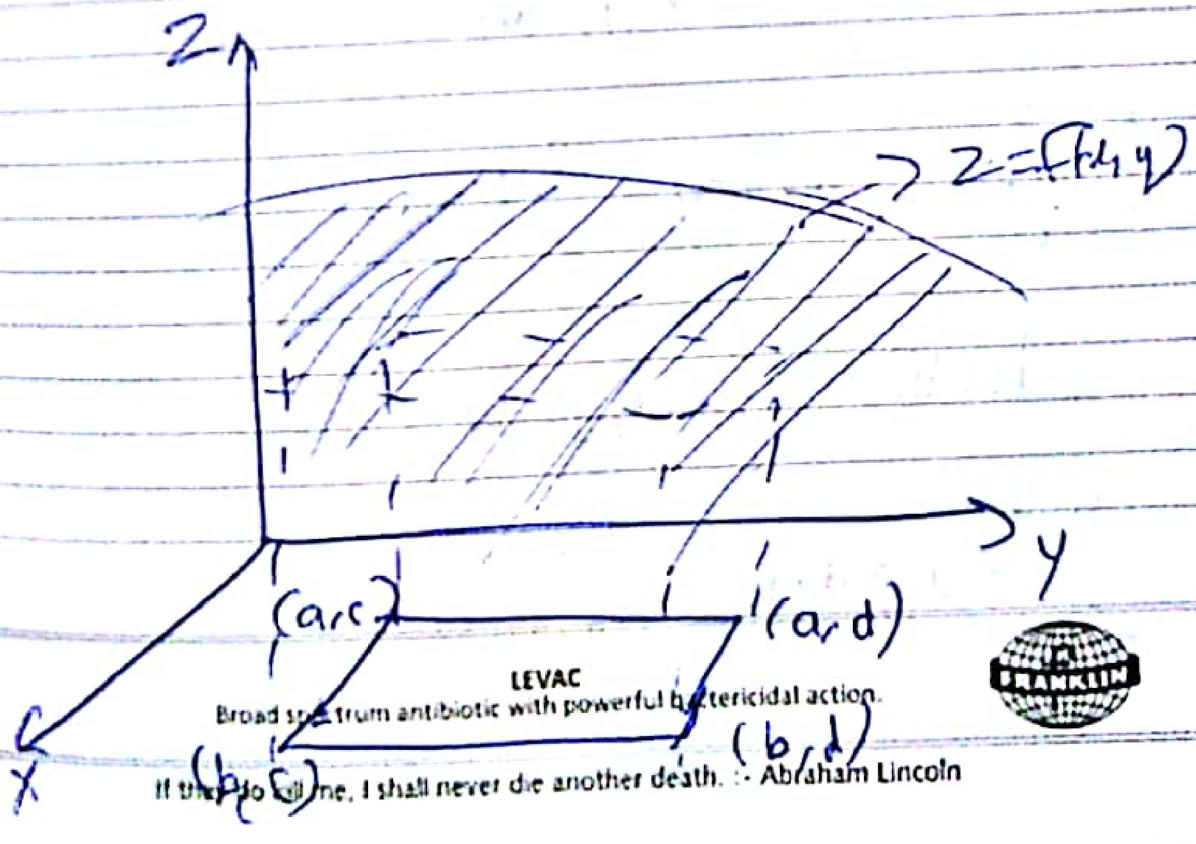
$$= \underline{5.625 \times 10^{-7}} \rightarrow \text{upper bound}$$

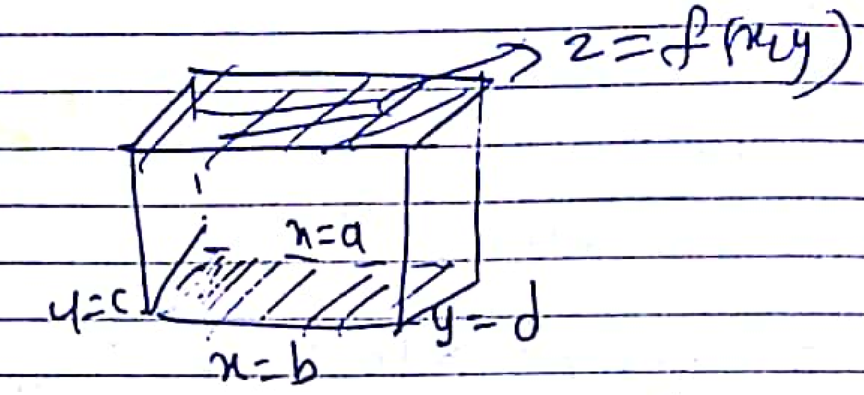
[Earliear error is 3.38574×10^{-7}]

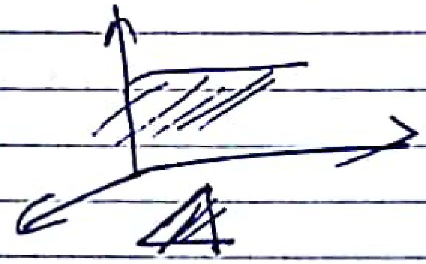
✱ Double Integration

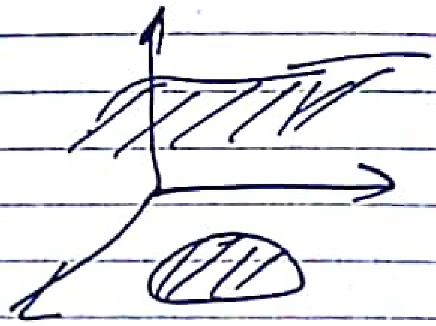
$$\int_c^d \int_a^b \underbrace{f(x,y)}_{\text{surface } z=f(x,y)} \, dx \, dy$$

$[a,b] \times [c,d]$ = Rectangle in xy-plane









$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_a^b g(x)\,dx$$

$$g(x) = \int_c^d f(x,y)\,dy.$$

$$\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_c^d h(y)\,dy$$

$$h(y) = \int_a^b f(x,y)\,dx$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Trapezoidal Rule:

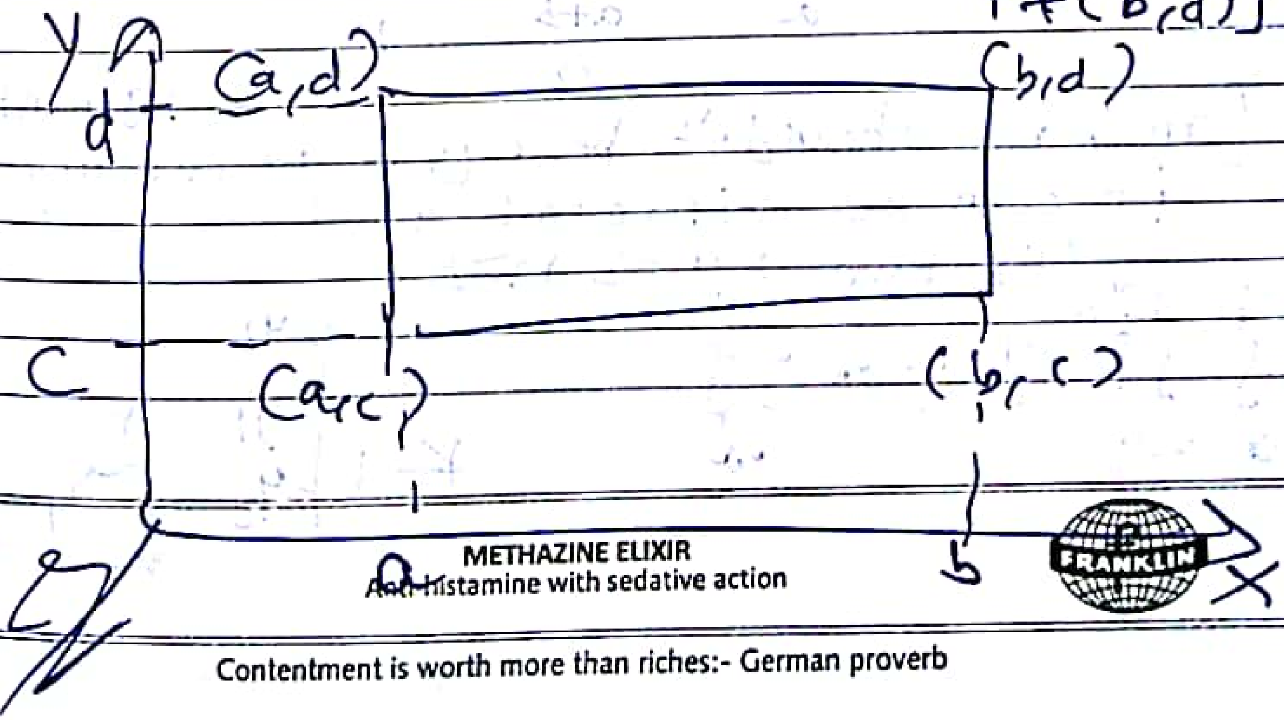
$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left(\underbrace{\int_c^d f(x,y)\,dy}_{} \right) dx$$

$$= \int_a^b \frac{d-c}{2} \left[f(x,c) + f(x,d) \right] dx$$

$$= \frac{d-c}{2} \left[\int_a^b f(x,c)\,dx + \int_a^b f(x,d)\,dx \right]$$

$$= \frac{d-c}{2} \left[\frac{b-a}{2} \left(f(a,c) + f(b,c) \right) + \frac{b-a}{2} \left(f(a,d) + f(b,d) \right) \right]$$

$$= \frac{(b-a)(d-c)}{4} \left[f(a,c) + f(a,d) + f(b,c) + f(b,d) \right]$$



$[a,b]$ $n=$

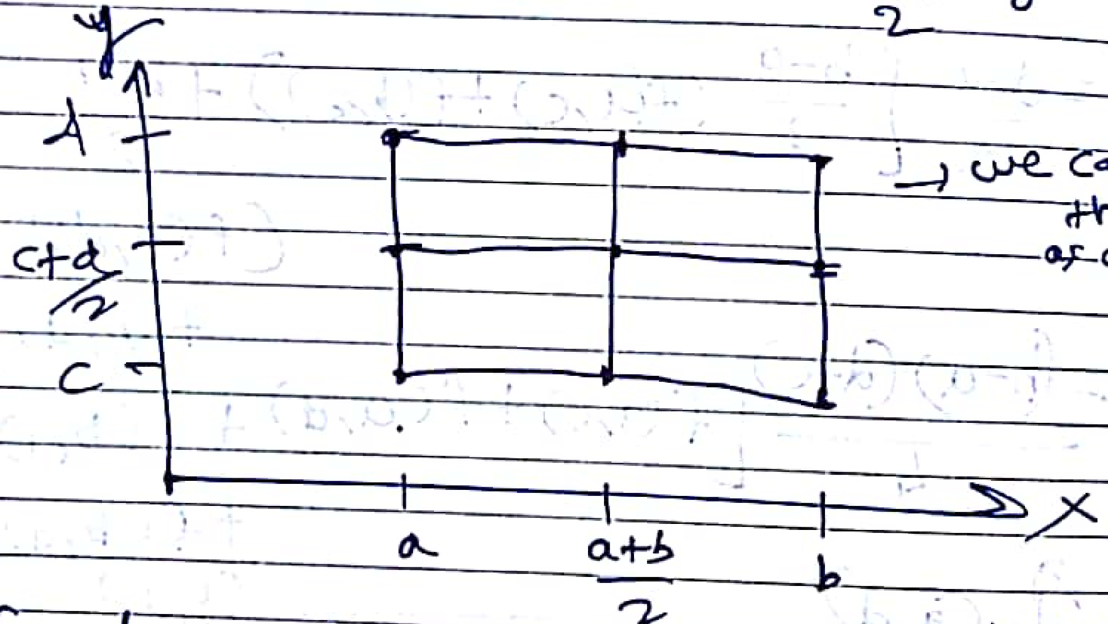
$[c,d]$ $m=$

$m=n=1$

$x_0 \to a$, $x_1 \to b$, $y_0 \to c$, $y_1 \to d$

$m=2$ $n=2$

x_0	x_1	x_2	a	$\frac{a+b}{2}$	b
y_0	y_1	y_2	c	$\frac{c+d}{2}$	d



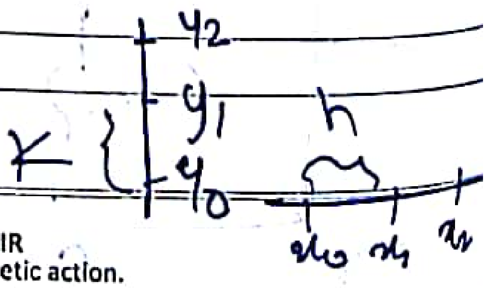
→ we call this as grid

→ 9 points

[m & n shouldn't be equal necessarily it can be different also]

T.R. Rule → m=2 n=2

$$\int_a^b \int_{c=y_0}^{d=y_2} f(x,y)\,dy\,dx$$



$$= \int_a^b \left[\int_{y_0}^{y_1} f(x,y)\,dy + \int_{y_1}^{y_2} f(x,y)\,dy \right] dx$$

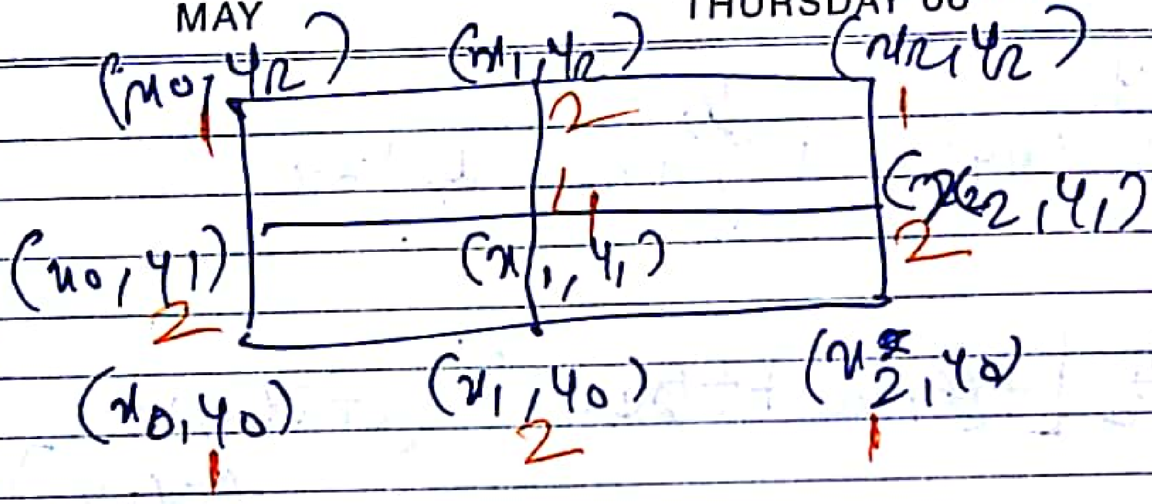
$$= \int_a^b \frac{h}{2} \left\{ \left(f(x,y_0) + f(x,y_1) \right) + \left(f(x,y_1) + f(x,y_2) \right) \right\} dx$$

$$= \frac{k}{2} \left[\frac{h}{2} \underbrace{\int_a^b f(x,y_0)\,dx}_{g_1(x)} + 2\frac{h}{2} \underbrace{\int_a^b f(x,y_1)\,dx}_{g_2(x)} + \frac{h}{2} \underbrace{\int_a^b f(x,y_2)\,dx}_{g_3(x)} \right]$$

$$= \frac{k}{2} \left[\frac{h}{2} \left(f(x_0,y_0) + 2 f(x_1,y_0) + f(x_2,y_0) \right) + \frac{h}{2} \cdot 2 \left(f(x_0,y_1) + 2 f(x_1,y_1) + f(x_2,y_1) \right) + \frac{h}{2} \left(f(x_0,y_2) + 2 f(x_1,y_2) + f(x_2,y_3) \right) \right]$$

$$= \frac{hk}{4} \left[f(x_0,y_0) + 2 \left\{ f(x_0,y_1) + f(x_1,y_0) + f(x_1,y_2) + f(x_2,y_1) \right\} + 4 f(x_1,y_1) + f(x_0,y_2) + f(x_2,y_0) + f(x_2,y_2) \right]$$

[orange no. → coeffⁿ]



$m \rightarrow$ no. of pts in $\underbrace{[c,d]}_{y}$

$n \rightarrow$ no of pts in $\underbrace{[a,b]}_{x}$

Try writing general formula