PROBLEMS 6.1

1. Consider the matrices

$$H_{10\times 10} = \begin{pmatrix} 1 & & & & \mathbf{0} \\ 1 & 1 & & & \\ 0 & 1 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad H+E$$

where

$$E_{10\times 10} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \frac{1}{2^{10}} \\ 0 & 0 & \cdots & 0 & 0 \\ \hdotsfor{5} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

   (**a**) What are the eigenvalues of H?

   (**b**) Show that $\lambda = \frac{1}{2}$ is an eigenvalue of $H + E$.

   (**c**) Show that $\|E\|_F = 1/2^{10}$.

   (**d**) Why does the stability corollary not apply to H and $H + E$?

2. Calculate the Frobenius norm for the following matrices.

   (**a**) $D = \begin{pmatrix} d_1 & & & \mathbf{0} \\ & d_2 & & \\ & & \ddots & \\ \mathbf{0} & & & d_n \end{pmatrix}$ (**b**) I (**c**) 0 (zero matrix)

   (**d**) $D + I$

3. Regarding Prob. 2, which is larger?

$$\|D + I\|_F \quad \text{or} \quad \|D\|_F + \|I\|_F$$

4. Let A be an $n \times n$ matrix. If $\|A\|_F = 0$, must A be the zero matrix?

5. Let $A = (a_{ij})_{n\times n}$. Define the 1 norm of A by

$$\|A\|_1 = \sum_{1\le i,j\le n} |a_{ij}|$$

Let

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$$

Calculate $\|A\|_1$ and $\|A\|_F$. Which norm is larger?

6. Suppose the eigenvalues of an $n \times n$ symmetric matrix A are to be computed. Because of a data entry error, every entry of A has 0.0001 added to it. What is the error bound for $|\lambda_k - \hat{\lambda}_k|$, as given in the stability corollary? How does the error bound change as n increases? What can you say about the stability of the eigenvalue problem for large versus small matrices?

PROBLEMS 6.2

In Probs. 1 to 5, use the power method to calculate approximations to the dominant eigenpair (if a dominant eigenpair exists). If the method does not work, give a reason.

1. $\begin{pmatrix} 1 & 5 \\ 5 & 6 \end{pmatrix}$

2. $\begin{pmatrix} 3 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 3 \\ -2 & 1 \end{pmatrix}$

4. $\begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}$

5. $\begin{pmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

6. **The power method with scaling**. From the examples in this chapter we saw vectors with large components generated by the power method. To avoid this problem, we can at each step multiply the vector

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{by} \quad \frac{1}{\max\{|x_1|, |x_2|, \dots, |x_n|\}}$$

This is called the scaling of X. For example, the scaling of

$$\begin{pmatrix} 7 \\ 5 \end{pmatrix} \quad \text{is} \quad \frac{1}{5}\begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{7} \end{pmatrix}$$

and the scaling of

$$\begin{pmatrix} 3 \\ -6 \end{pmatrix} \quad \text{is} \quad \frac{1}{6}\begin{pmatrix} 3 \\ -6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix}$$

The power method with scaling proceeds as follows: Choose X_0.
Step 1. Calculate AX_0. Let V_1 = scaled version of AX_0.
Step 2. Calculate AV_1. Let V_2 = scaled version of AX_0.
Step 3. Calculate AV_2. Let V_3 = scaled version of AX_0.
Continue in this way. We then have at step m:

$$\lambda_1 \doteq \frac{A_{m-1} \cdot V_{m-1}}{V_{m-1} \cdot V_{m-1}}$$

and V_m is an approximate eigenvector.
Use the power method with scaling on Probs. 1, 2, and 5.

7. Use the relative error E_{n+1} from Eq. (6.2.8) to estimate the error in the computed dominant eigenvalue in Probs. 1, 2, and 5.