MA 859: SELECTED TOPICS IN GRAPH THEORY

LECTURE - 7

CONNECTIVITY

This concept is an intuitive area in Graph Theory.

The connectivity (sometimes referred more specifically as vertex-connectivity) $k = k(G)$ of a graph G is the minimum number of vertices whose removal results in either a disconnected graph or a trivial graph.

k(disconnected graph) $= 0$ $\quad k(K_n) = n-1$

k(Graph having a cut vertex) $= 1$

Analogously, the <u>edge-connectivity</u> $\lambda = \lambda(G)$ of a graph G is the minimum number of edges whose removal results in a disconnected graph.

$\lambda(K_1) = 0 \quad \lambda$(disconnected graph) $= 0$

λ(Graph with a bridge) $= 1 \quad \lambda(K_n) = n-1$

Theorem (Whitney)

For any graph G,

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

Proof: We will first show that $\lambda(G) \leq \delta(G)$.

If G has no edges or is disconnected,

then $\lambda = 0$. $\Rightarrow \lambda(G) \leq \delta(G)$.

Otherwise, when we remove all the δ edges incident with the vertex of minimum degree δ, the graph becomes disconnected; hence $\lambda(G) \leq \delta(G)$.

Now, to prove that $\kappa(G) \leq \lambda(G)$, we consider various cases:

* If G is trivial or disconnected, then clearly $\kappa(G) = 0 \leq \lambda(G)$.

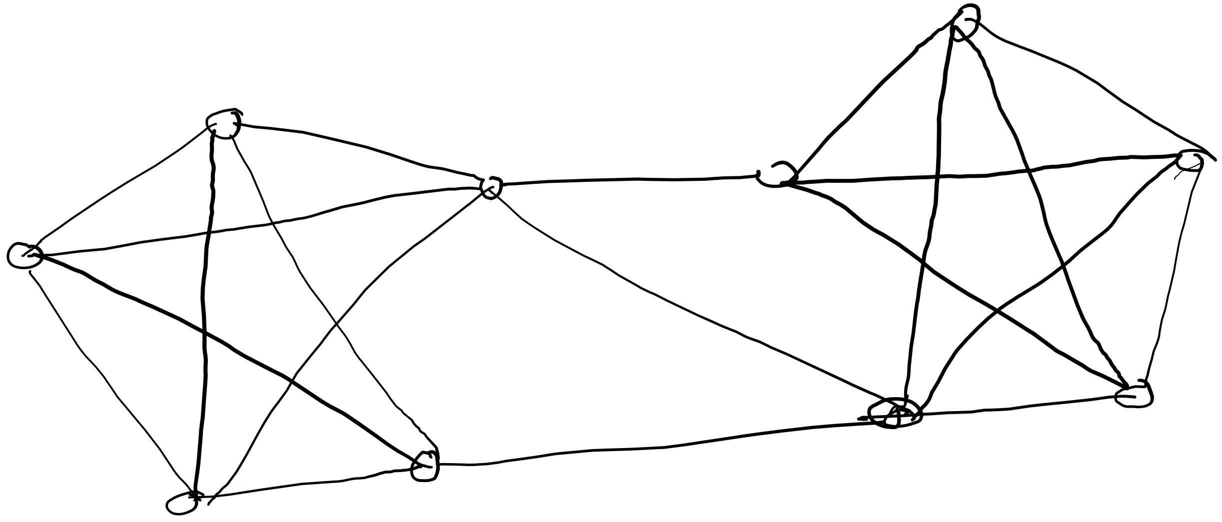
* If G is connected and has a bridge, say x, then clearly, $\lambda(G)=1$. But $\kappa(G)=1$ because removal of any one end vertex of x will remove x too. So, in this case $\kappa(G)=\lambda(G)$.

* Suppose G is a graph for which $\lambda(G)\geq 2$. Then the removal of these $\lambda-1$ edges results in a graph G' with a bridge, say $x=uv$.

Now, consider one end vertex of each of these $\lambda - 1$ edges and when we remove them, the $\lambda - 1$ edges are also certainly removed and possibly more too. So, this might possibly result in a disconnected graph G'! In such case, clearly $K(G) \leq \lambda - 1$ and hence $K(G) < \lambda(G)$.

If G' is not a disconnected graph, then removal of u or v will remove x too and hence the resulting graph becomes disconnected. So, $k(G) = \lambda(G)$.

Thus, in all cases $k(G) \leq \lambda(G) \leq \delta(G)$. //



$k =$ $\lambda =$ $\delta =$

Theorem For all integers a, b, c such that $0 < a \le b \le c$, there exists a graph G with $k = a$, $\lambda = b$ and $\delta = c$

Assignment:
Write an algorithm to construct a graph G for which $k(G) = a$, $\lambda(G) = b$ and $\delta(G) = c$, where a, b, c are integers such that $0 < a \le b \le c$.

That's all

for this lecture!