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Subject: Selected Topics in Graph Theory

Assignment – 2

Q1. Justify whether the following statement is TRUE or FALSE:
Every graph with fewer edges than vertices has a component that is a tree.

Ans: The above statement is True.

By definition a tree is a connected acyclic graph i.e. a graph connected graph with no cycles is called a tree. Let G be a graph with E(G) < V(G) and let $G_1, \ldots, G_k$ be the components of G. Assume that no G_i is a tree. Then $E(G_i) > n(G_i)$ for all i = 1, . . ., k (because a connected graph H with E(H) < V(H)−1 is not connected, with E(H) = V(H) − 1 is a tree and with E(H) > V(H) is not acyclic). Summing up for all i = 1, . . ., k gives E(G) > V(G), a contradiction.

Hence, above statement is **True.**

Q2. Prove that a graph G is a tree if and only if for all x, y ∈ V (G), adding a copy of xy as an edge creates exactly one cycle.

Ans: Assume that G is acyclic, but for any xy does not belong to E(G) the graph G + xy has a cycle. Let's prove that G is a tree. By definition we know that a tree is connected acyclic graph. here, it is given that G is acyclic, so we only need to prove that G is connected. Assume otherwise that there is no x-y-path in G for some two vertices x and y. Then in particular xy does not belong to E(G). However, G union {xy} has a cycle C and this cycle must contain the edge xy. Thus, there are two edge-disjoint x-y-paths, one of which does not contain the edge xy and thus is a path in G. So, there is an x-y-path in G, a contradiction.

Hence the proof.

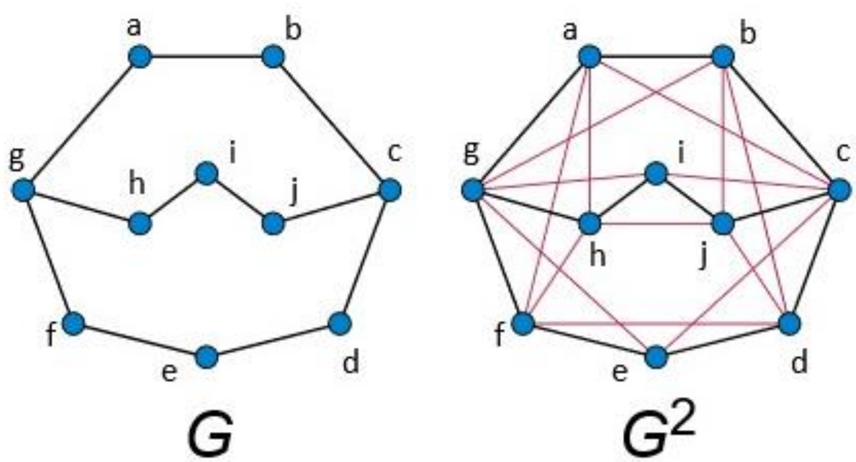
Q3. If x and y are adjacent vertices in a connected graph G, then show that |d(x, z) − d(y, z) | ≤ 1 for any vertex z in G.

Ans: We know that, the distance between any two vertices - x, y in a graph is the shortest path between them. Now consider a connected graph G, let the x and y be two adjacent vertices of graph G. Let z some other vertex of graph G and d(x,z) = l. Then while finding d(y,z) will be either l, if we don't go through edge xy, or it'll be l+1, if we go through edge xy. These are the only two possibilities. Hence, absolute difference between d(x,z) and d(y,z) is either 0 or 1. This proves the above statement.

Q4. **The square of a simple graph G is a graph G' where two vertices x and y are adjacent in G' if and only if $d_G(x, y) \le 2$. Show that square of a connected graph G has diameter ceil(diam(G)/2).**

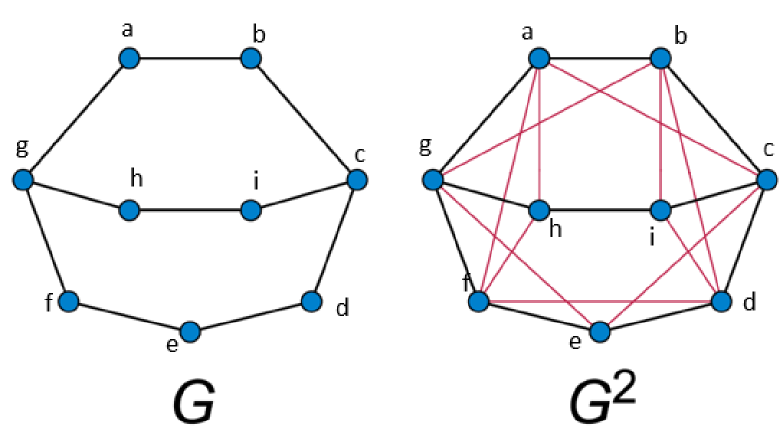
Ans: Consider a graph G and it's square as below:

Case 1:



As we know that, diameter of a graph is the maximum distance between a pair of vertices in the graph, so for graph G, the distance between vertices i and e, which is 4, is maximum amongst all pair of vertices.
Therefore, d(i,e) = 4 is the diameter of graph G. Now coming to G's square i.e. G^2, the distance between vertices i and e is 2 and it is maximum amongst all pair of vertices. Therefore, the diameter of G^2 is 2, which is diam(G)/2.

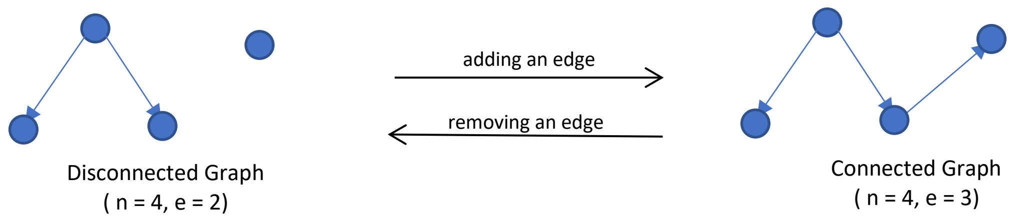
Case 2:



As we know that, diameter of a graph is the maximum distance between a pair of vertices in the graph, so for graph G, the distance between vertices i and e, which is 4, is maximum amongst all pair of vertices.
Therefore, d(i,e) = 3 is the diameter of graph G. Now coming to G's square i.e. G^2, the distance between vertices i and e is 2 and it is maximum amongst all pair of vertices. Therefore, the diameter of G^2 is 2, which is ceil(diam(G)/2).
Hence, diam(G^2) = ceil(diam(G)/2).

Q5. Prove that if an n-vertex graph G has n – 1 edges and no cycles, then it is connected.

Ans: Let the graph G is disconnected then there exist at least two components G_1 and G_2 say. Each of the component is circuit-less as G is circuit-less. Now to make a graph G connected we need to add one edge e between the vertices V_i and V_j, where V_i is the vertex of G_1 and V_j is the vertex of component G_2. Now the number of edges in G = (n – 1) + 1 = n.
Now, G is connected graph and circuit-less with n vertices and n edges, which is impossible because the connected circuit-less graph is a tree and tree with n vertices has (n-1) edges. So, the graph G with n vertices, (n-1) edges and without circuit is connected. Hence the given statement is proved.



Q6. Show that every non-trivial tree has at least two maximal independent sets, with equality only for star graphs.

Ans: By definition a tree T is a connected acyclic graph. Also, the tree is non-trivial, so it has at least one edge. Therefore, we can denote T with a bipartite graph. Now we split the vertices V of T into two sets of vertices V_1 and V_2 such that V=V_1 U V_2. Since T has no cycles (in turn has no odd cycles), so there is no edge between any two vertices of the same set V_1 or V_2. Therefore, V_1 and V_2 are independent. As V = V_1 U V_2, therefore V_1 and V_2 are maximal independent sets. Similarly, if we split the V into more sets, we can have more than two maximal independent sets. This proves the first part.
Now for the second part, let's consider the star graph with V vertices. Split V into two sets V_1 and V_2 where V_1 contains all vertices and V_2 contains center vertex. Therefore, there is no edge between any two vertices of the same set V_1 or V_2. No other combination of vertices would result this scenario. Therefore, V_1 and V_2 are the only possible vertex sets and they are independent and maximal. This proves the second part of the statement.

Q7. Show that among the trees with n vertices, the star graph has the most independent sets.

Ans: Let us prove this by using Induction. Consider I(T) for the number of independent sets. Let v be a leaf of T, adjacent to a vertex w, and consider I(T−v). Now I(T)=2*I(T−v) −k, where k is the number of independent sets of T−v that contain w (since these are the sets that would stop being independent if you add v). Note that k≥1, with equality if and only if w is adjacent to every other vertex of T−v, i.e. it is the center vertex of a star. Also, I(T−v) is maximized when T−v is a star (by the induction hypothesis). Putting this together, you get that I(T) is maximized when T−v is a star and v was adjacent to the center of that star, i.e. T is also a star.

Hence among the trees with n vertices, the star graph has the most independent sets.

Q8. Show that an edge of a connected graph G is a cut-edge (bridge) if and only if it belongs to every spanning tree.

Ans: Suppose e is a bridge in G and T a spanning tree on G not containing e. Then, since T is a tree, it must be connected, but since T is a subgraph of G - {e}, a disconnected graph, it must be disconnected. Therefore, any spanning tree contains every bridge.

Now suppose e is not a bridge. Then G - {e} is still connected, and so has a spanning tree T. However, since G - {e} has the same vertices as G, T is also a spanning tree of G that does not contain e. Therefore, any edge in every spanning tree is a bridge.

Q9. Show that every tree on even number of vertices has exactly one subgraph in which every vertex has odd degree.

Ans: By definition, a spanning subgraph is a subgraph of a graph consisting of the same vertex set and a subset of the edge set of the graph, which is not necessarily a tree. A spanning tree is a spanning subgraph that is also a tree.

Using induction on the number of vertices = 2k. For k = 1, we have 2 vertices, and so there is only one possible tree, which is just two vertices connected by 1 edge -- the graph itself is a spanning subgraph with odd vertices.

Suppose for k >= 1, every tree with 2k vertices has a unique spanning subgraph with all odd vertices. Now since each leaf has degree 1 in the tree, any such spanning subgraph must include all edges incident to leaves.

Consider a tree with 2(k+1) vertices. Consider a longest possible path in this tree. Suppose 1 endpoint of the path is a vertex v, which would have to be a leaf. Suppose u is the vertex adjacent to u in the path.

Case 1: u has another adjacent leaf (say w).

T - {v, w} has a unique such spanning subgraph by our assumption (we assumed this for any tree with 2k vertices), so use that spanning subgraph and add edges uv and uw -- the degree of u gains 2, so it stays odd, and v and w only have the 1 incident edge, so their degree will be 1, which is odd, and we have the type of subgraph we need.

Case 2: u has no other adjacent leaf.

Since p is the longest path, this means the degree of u is 2, or else there would be a cycle in the graph. So, we know that T - {u, v} has a unique such spanning subgraph by assumption -- add uv to that subgraph and we have the type of subgraph we need (u + uv + v will be a separate component). So, by induction, the above statement holds for any number of vertices 2n for a positive integer n.

Q10. Show that a connected graph with n vertices has exactly one cycle if and only if it has exactly n-edges.

Ans: Let G have n vertices and n edges. Since G is a connected graph, it has a spanning tree T with n vertices and n-1 edges. Let e be the edge not in T, with its endpoints x and y. There is a unique path u between x and y in T (since T is a tree). The union of e and u is a cycle.

Suppose that there is some other cycle v. if v does not contain e, then it is contained in T, contradicting that T has no cycles. If v does contain e, write it as the union of e and a path p in T. Then p is a path from x and y. But u is the only path from x and y in T.

Hence, a connected graph with n vertices has exactly one cycle if and only if it has exactly n-edges.