MA859: SELECTED TOPICS IN GRAPH THEORY

LECTURE - 8

TRAVERSABILITY

The perambulatory problem of crossing the bridges can be abstracted graph theoretically as follows:

Given a graph G, is it possible to find a walk that traverses through each edge exactly once, goes through all the vertices and ends at the starting vertex?

A graph for which this is possible is known as <u>Eulerian</u> (named after Euler). An Eulerian graph has a closed trail consisting of all the edges (exactly once), which we call, <u>Eulerian Trail</u>.

Theorem: The following statements are equivalent for a connected graph G. [The result holds for multigraphs as well].

1. G is Eulerian
2. Every vertex of G has even degree.
3. The set of edges of G can be partitioned into cycles.

Proof:

(1) $\Rightarrow$ (2): Let T be an Eulerian Trail in G. Each occurrence of a given vertex in T contributes 2 to the degree of that vertex, and since each edge of G appears exactly once in T, every vertex must have even degree.

(2) $\Rightarrow$ (3): Since G is connected and non-trivial, every vertex has degree at least 2, so G contains a cycle Z.

The removal of the edges of Z results in a spanning subgraph G_1 in which every vertex still has even degree. (of course, some vertices may be of zero degree).
If G_1 has no edges, then (3) already holds.
Otherwise, repeat the argument on G_1; we get another subgraph G_2, in which again, all the vertices are of even degree. Continuing in this manner, we finally end up in a

totally disconnected subgraph and we thus have a partition of all the edges into cycles.

(3) $\Rightarrow$ (1): Let Z_1 be one of the cycles of this partition. If G consists of this cycle alone, then G is obviously Eulerian. Otherwise, there is another cycle Z_2 with a vertex v in common with Z_1 (because G is connected).

The walk beginning at v and consisting of the cycles Z_1 and Z_2 in succession, forms a closed trail containing the edges of Z_1 and Z_2.

Continuing this process, we finally obtain a closed trail containg all the edges of G.

Hence G must be Eulerian. //

<u>Corollary 1</u>: If G is a connected graph with exactly two vertices of odd degree, then it has an open trail containing all the edges of G (and hence all the vertices of G), which begins at one odd degree vertex and ends at the other.

<u>Corollary 2</u> If G is a connected graph with exactly $2n$ vertices of odd degree $(n \geq 1)$, then the set of edges of G can be partitioned into n open trails.

Theorem: An undirected graph G is Eulerian if and only if it is connected and all its vertices are of even degree.

Proof:

Necessity: Suppose G is Eulerian. Then obviously G must be connected. When an Eulerian Trail passes through a vertex, it traverses exactly two distinct edges – one reaching the vertex; other exiting it.

Since this happens every time when it passes through the vertex, that vertex must have even degree. Hence every vertex of G must be of even degree.

<u>Sufficiency</u>: Assume that G is connected and every vertex of G is of even degree. We shall use induction on the number of edges.

The result is obvious when the number of edges $m = 3$.

Suppose that the result is true for all connected graphs with all its vertices of even degree and having fewer than m edges.

Let G be such a graph with m edges.

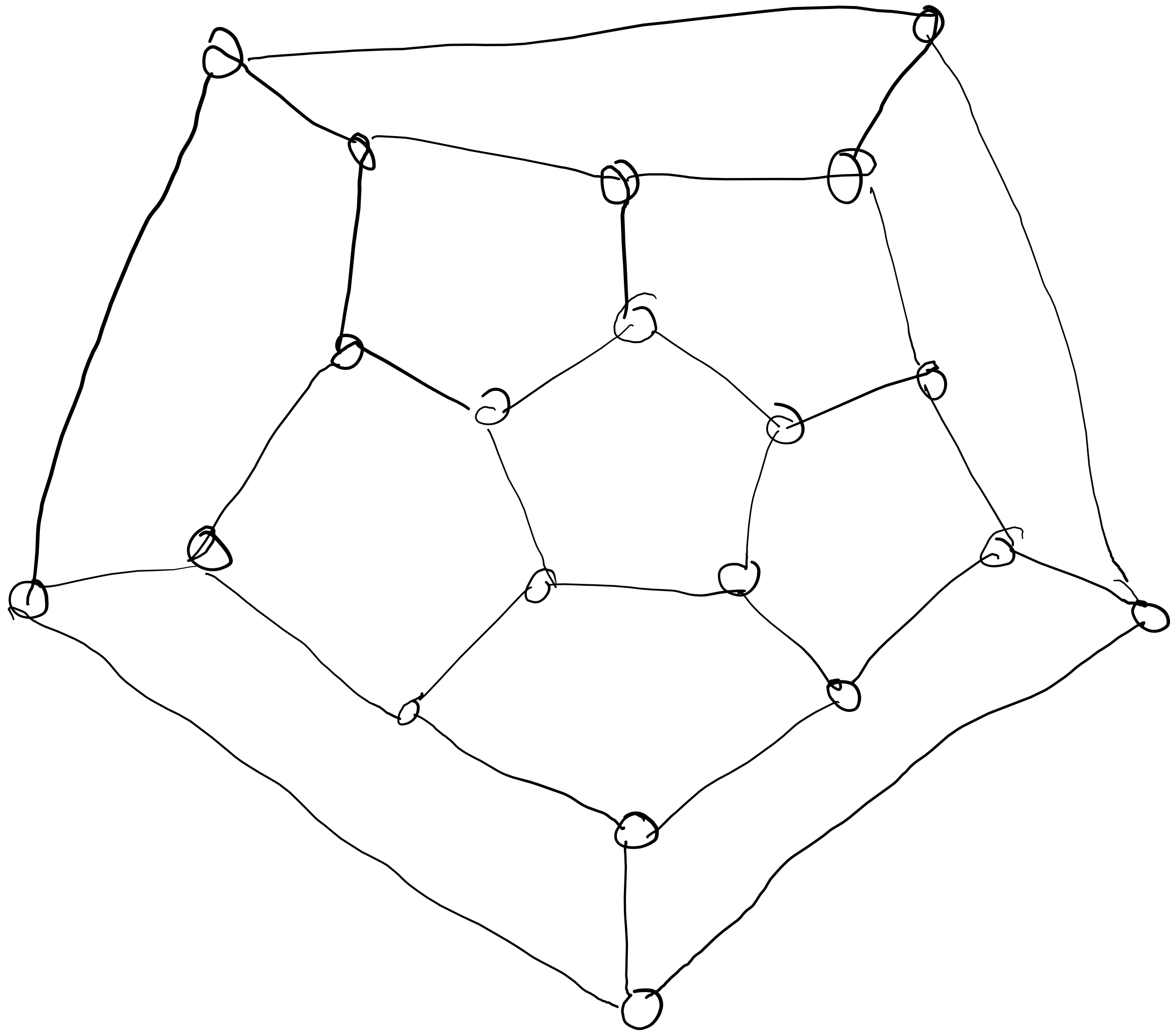
Start at any vertex of G and construct a closed trail C which finishes at the same vertex.

This is always possible since G is connected and each vertex has even degree.

If all the edges have been traced, then
the proof is complete.
If not, remove from G, all the edges
on C. The resulting subgraph H has
less edges than m; but all its vertices
are still of even degree. Therefore, by
the induction hypothesis, each component
of H has an Eulerian trail.

Also, each component of H has at least one vertex on C.
Therefore, since G is connected, an Eulerian Trail in G can be obtained by tracing C until a vertex in H is reached and then tracing the Eulerian Trail of the corresponding component of H before returning to C and so on, until we return to the

starting vertex. Thus, the induction
is complete. Hence G must be
Eulerian. //



Dodecahedron
(20 vertices)

A graph G in which there exists closed path spanning all its vertices is called Hamiltonian.

Such closed path is actually a spanning cycle and is often called Hamiltonian cycle.

Unlike the Eulerian graphs, there is no elegant characterization of Hamiltonian graphs.

Theorem [Ore]

If G is a graph on $n \geq 3$ vertices such that for all distinct non-adjacent vertices u and v, $\deg u + \deg v \geq n$, then G is Hamiltonian.

Proof: We prove this result by contradiction. Suppose there exists a maximal non-Hamiltonian graph G on $n \geq 3$ vertices that satisfies the hypothesis / condition of the theorem.

[By maximality here, we mean, G is non-Hamiltonian and for every two non-adjacent vertices w_1 and w_2, $G + w_1 w_2$ is Hamiltonian].

Clearly, if G is non-Hamiltonian, then it is not complete. Suppose u and v are two non-adjacent vertices of G.
$G+uv$ is Hamiltonian (in view of maximality property), and furthermore, every Hamiltonian cycle of $G+uv$ contains the edge uv.
Hence, there is a u-v path
$P: u = u_1 u_2 \cdots u_n = v$ in G containing every vertex of G.

If $u_1 u_i \in E(G)$ for some $2 < i \leq n$, then $u_{i-1} u_n \notin E(G)$; for otherwise, $u_1 u_i u_{i+1} \cdots u_n u_{i-1} u_{i-2} \cdots u_1$ will be a Hamiltonian cycle in G

$\therefore$ For each vertex in $\{u_2, u_3, \ldots, u_n\}$ adjacent to u_1, there is a vertex in $\{u_1, u_2, \ldots, u_{n-1}\}$ not adjacent to u_n.

Thus, $\deg u_n \leq (n-1) - \deg u_1$

$\Rightarrow \deg u + \deg v \leq n-1$ $\times\!\!\times$

$\therefore$ G must be Hamiltonian.

If G is Hamiltonian, then certainly so is $G+uv$ for any pair u, v of non-adjacent vertices in G.

Conversely, suppose G is a graph on n vertices, with non-adjacent vertices u and v such that $G+uv$ is Hamiltonian. Also, suppose that $\deg_G u + \deg_G v \geq n$.

If G is non-Hamiltonian, then as in the proof of the previous result, we would arrive at a contradiction, namely

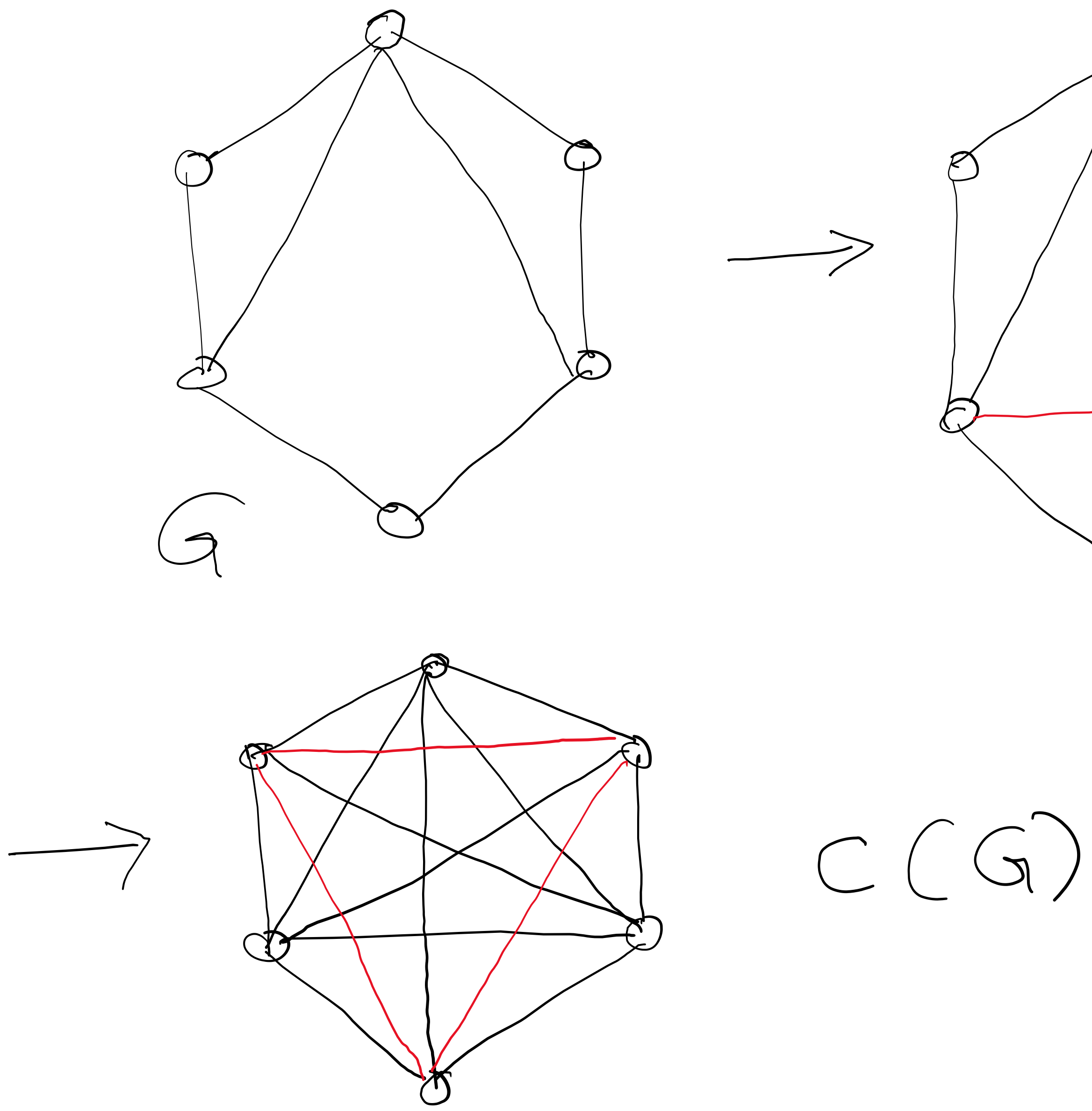
$$\deg_G u + \deg_G v \leq n-1.$$

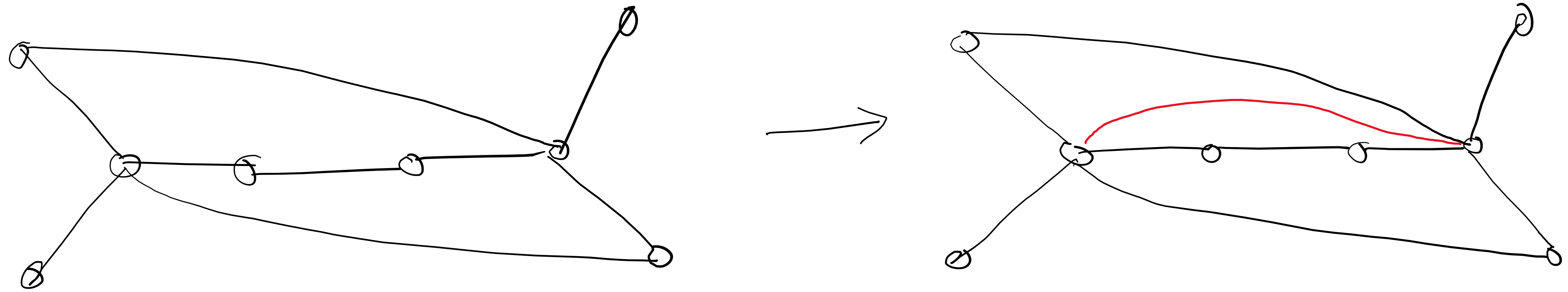
Thus, we have the following result:

Theorem: Let u and v be distinct non-adjacent vertices of a graph G on n vertices such that $\deg u + \deg v \geq n$. Then $G + uv$ is Hamiltonian if and only if G is Hamiltonian.

This result motivates a new definition:

The <u>closure</u> of a graph G on n vertices, denoted by $C(G)$, is the graph obtained from G by recursively joining pairs of non-adjacent vertices whose degree sum is at least n, until no such pair remains.


G
C(G)



$G \longrightarrow C(G).$

Theorem: If G_1 and G_2 are two graphs obtained from a graph G on n vertices by recursively joining the pairs of non-adjacent vertices whose degree sum is at least n, then $G_1 \cong G_2$.

[This theorem exhibits that closure is well defined].

Proof: Let $e_1, e_2, \ldots, e_j$ and $f_1, f_2, \ldots f_k$ be the sequences of edges added to G to obtain G_1 and G_2 respectively.

It suffices to show that each e_i is an edge of G_2 $(1 \leq i \leq j)$ and each f_i is an edge of G_1 $(1 \leq i \leq k)$.

Suppose this is not true.
WLOG, we may assume that for some m $(0 \le m \le j-1)$, the edge $e_{m+1} = uv$ does not belong to G_2, and furthermore, suppose that $e_i \in E(G_2)$ for $i < m$.
Let G_3 be the graph obtained from G by adding the edges $e_1, e_2, \ldots, e_m$.

It follows from the definition of G_3 that

$\deg_{G_3} u + \deg_{G_3} v \geq n.$

Further, since $G_3 \subset G_2$, we have

$\deg_{G_2} u + \deg_{G_2} v \geq n.$ ###

Thus, each e_i is an edge of G_2 and each f_i is an edge of G_1.

$\therefore G_1 \cong G_2$ //

<u>Theorem</u>: A graph G is Hamiltonian if and only if its closure c(G) is Hamiltonian.

Proof is an immediate consequence of the definition of closure and the previous result.

Theorem (Bondy and Chvátal)

Let G be a graph with at least 3 vertices. If $C(G)$ is complete, then G is Hamiltonian.

Proof: Since each complete graph $(n \geq 3)$ is Hamiltonian, the proof is obvious in the light of the previous result.

This result, indeed, helps us in developing an algorithm to determine a Hamiltonian cycle of G, given a Hamiltonian cycle of $C(G)$.

[If $C(G)$ is complete, then it is trivial to find a Hamiltonian cycle of $C(G)$].

The key step of the algorithm is the following 'modification process':

Suppose C is a Hamiltonian cycle of some graph H on $n \geq 4$ vertices, and e, an edge of C.

Label the vertices of H so that

$C : v_1, v_2, \dots, v_n, v_1$ and $e = v_1 v_n$.

If there is an integer i $(3 \leq i \leq n-1)$ such that $v_1 v_i$ and $v_{i-1} v_n$ are edges in H, then

$C' : v_1 v_2 \cdots v_{i-1} v_n v_{n-1} \cdots v_i v_1$ is also a Hamiltonian Cycle of H, and

$$E(C') = E(C) - \{v_1 v_n, v_{i-1} v_i\} \cup \{v_1 v_i, v_{i-1} v_n\}.$$

We say that $v_1 v_i$ and $v_{i-1} v_n$ are modifying edges w.r.t. (C, e), and that C' is obtained by modifying C via $(v_1 v_n, v_1 v_i, v_{i-1} v_n)$.

Algorithm (Bondy & Chvátal)

In this algorithm, we assume that in constructing $C(G)$, each edge e of G is labeled $l(e) = 1$, and that if f is the k^{th} edge added to G, f is labeled $l(f) = k+1$.

[Of course, this labeling need not be unique).

Given the [edge-labeled] closure $C(G)$ of a graph G, and a Hamiltonian cycle C of $C(G)$,

1. Set $m = \max_{e \in E(C)} \{ l(e) \}$

   If $m = 1$, then stop;
   else, let e be the unique edge of C labeled m
   and go to step 2.

2. Select edges e_1 and e_2 of $C(G)$ such that

   (a) $l(e_1), l(e_2) < m$

   (b) e_1 and e_2 are modifying edges w.r.t. (C, e).

3. Modify C via (e, e_1, e_2).

4. Go to Step 1.

Theorem Bondy & Chvátal algorithm terminates with a Hamiltonian cycle C of G.

Proof: We first show that Step 2 can always be completed.

Let $n = n(G)$ and label the vertices of $C(G)$ so that $C: v_1, v_2, \ldots, v_n v_1$ and $e = v_1 v_n$.

Let G' denote the spanning subgraph of $C(G)$ with edge set $E(G) \cup \{f \in E(C(G)) / l(f) < m\}$.

Then $C - e$ is a subgraph of G' (why?)

And in view of the process of the construction of $C(G)$, $\deg_{G'} v_1 + \deg_{G'} v_n \geq n$.

It follows from an argument similar to the proof of Ore's result, that there is an integer i $(3 \leq i \leq n-1)$ such that $v_1 v_i$ and $v_{i-1} v_n$ are edges of G' (i.e., e_1 and e_2 are modifying edges w.r.t. (C, e) with $l(e_1), l(e_2) < m$).

To complete the proof, we observe that since the value of m decreases by at least 1 each time when Step 1 is repeated, this algorithm naturally terminates with $m = 1$, whereby we get a Hamiltonian Cycle of G.

<u>Remark</u>: If a graph G satisfies the conditions of the theorem [Ore], then $C(G)$ is complete, and so, by the theorem [Bondy & Chvátal], Ore's result actually becomes a corollary. Nevertheless, it is a fact that Ore discovered his result long before Bondy & Chvátal.

More on Hamiltonian graphs will be covered in the next lecture, which will also introduce a new concept, called Line Graphs.

// End of Lecture - 8 //