# MA859: Selected Topics in Graph Theory

## LECTURE - 13

Representation of graphs

- Incidence Matrix

We discuss about another important matrix representation of a graph.
Unlike the adjacency matrix, the incidence matrix is not always a square matrix.
Further, in addition to labeling the vertices, we label the edges as well here.
Therefore, the incidence matrix contains more information than the adjacency matrix, since it distinguishes between the multiple edges.

**Defn:** Let $G = (V, E, \phi)$ be a graph with vertex labeling $V(G) = \{u_1, u_2, \ldots, u_n\}$ and edge labeling $E(G) = \{e_1, e_2, \ldots, e_m\}$. For each $i \in \{1, 2, \ldots n\}$ and $j \in \{1, 2, \ldots, m\}$, we define

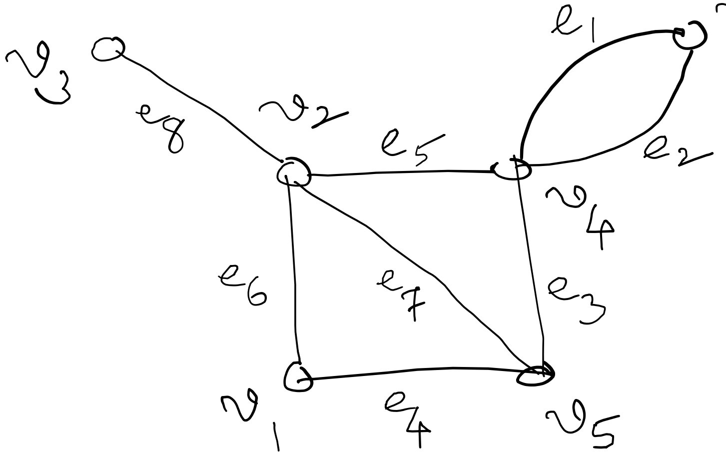
$$b_{ij} = |\{u_i\} \cap \phi(e_j)|.$$

That is, $b_{ij} = 1$ if $u_i$ is an end vertex of $e_j$; zero otherwise.

The incidence matrix $B(G)$ of a graph $G$ is a binary matrix of order $n \times m$

$$B(G) = [b_{ij}]_{i \in \{1,2,\ldots,n\},\, j \in \{1,2,\ldots,m\}}$$

## Example




$$B(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

## Remarks:

- Each column has either one or two 1's. Column $j$ has exactly two 1's if the edge $e_j$ has two distinct end vertices, and one 1 if $e_j$ is a loop.
- The number of 1's in row $i$, where the 1's corresponding to a loop are counted twice, is equal to $\deg_G v_i$. In particular, if $G$ has no loops, the number of 1's in row $i$ is precisely $\deg_G v_i$.
- Any $n \times m$ binary matrix such that each column has one or two 1's is an incidence matrix for a graph on $n$ vertices and $m$ edges.

As in the case of the adjacency matrix, the incidence matrix $B(G)$ depends on the labelings of the vertices and the edges. So, two incidence matrices for a given graph $G$ only differ by a permutation of the rows and the columns.

Suppose $G$ and $G'$ are two graphs, where

$V(G) = \{u_1, u_2, \ldots u_n\}$ and $E(G) = \{e_1, e_2, \ldots e_m\}$

$V(G') = \{u'_1, u'_2, \ldots, u'_n\}$ and $E(G') = \{e'_1, e'_2, \ldots, e'_m\}$

Assume that $f = (f_1, f_2) : G \to G'$ is an isomorphism with $f_1(u_i) = u'_{\sigma(i)}$ and $f_2(e_j) = e'_{\rho(j)}$, where $\sigma$ and $\rho$ are some permutations on $\{1, 2, \ldots, n\}$ and $\{1, 2, \ldots, m\}$ respectively.

Suppose $P_\sigma$ and $P_\rho$ are the corresponding $n \times n$ and $m \times m$ permutation matrices of $\sigma$ and $\rho$ respectively. Then the incidence matrices satisfy

$$B(G') = P_\sigma^T B(G) P_\rho$$

<u>Summary</u>

Two graphs $G$ and $G'$ on $n$ vertices and $m$ edges are isomorphic if and only if there are $n \times n$ and $m \times m$ permutation matrices $P$ and $Q$, respectively such that

$$B(G') = P^T B(G) Q.$$

If we look at the incidence matrix as a matrix of the Galois field $Z_2$, the row space and the column space of $B(G)$ have the same dimension over $Z_2$. This dimension is called the rank of the matrix. It can be shown that if $X$ and $X'$ are two $n \times m$ matrices such that $X' = P^T X Q$, then they have the same rank.

In view of the above, we define:

**Defn**: The rank of a graph $G$, denoted by $\mathrm{rank}(G)$, is the rank of its incidence matrix $B(G)$ with respect to any labeling of $V(G)$ and $E(G)$.

In view of this definition, we can compute the rank of any connected graph $G$ that has no loops.

**Theorem** If $G$ is a connected graph on $n$ vertices without loops, then $\text{rank}(G) = n-1$.

**Proof**: Since $G$ has no loops, there are exactly two 1's in each column of $B(G)$. Therefore, the sum of all row vectors of $B(G)$ is the zero vector in $\mathbb{Z}_2^m$, provided that $G$ has $m$ edges. Hence the dimension the row space is $n-1$ or less.

$$\therefore \text{rank}(G) \leq n-1 \longrightarrow (1)$$

Assume, we have $n_1$ less than $n-1$ rows that add up to the zero vector in $\mathbb{Z}_2^m$. The remaining $n_2 = n - n_1$ rows must also add up to the zero vector in $\mathbb{Z}_2^m$ since together, they all add up to the zero vector.

In this case, we can permute the vertices of $G$ in such a way that we first list the $n_1$ vertices that correspond to the rows adding upto the zero vector in $\mathbb{Z}_2^m$, and then the remaining $n_2$ rows that also add up to the zero vector. Since each column has exactly two 1's, each pair of 1's must either both be among the first $n_1$ entries of that column, or in the last $n_2$ ones. This means, there is a permutation of the columns such that the first $m_1$ columns contain all the pairs of 1's among the first $n_1$ rows, and the last $m_2 = m - m_1$ columns contain all the pairs of 1's among the last $n_2$ rows.

$\Rightarrow$ The incidence matrix of $G$ with respect to the corresponding labelings of $V(G)$ and $E(G)$ has the form

$$B(G) = \begin{bmatrix} B(G_1) & 0 \\ 0 & B(G_2) \end{bmatrix}$$

where $B(G_1)$ is the incidence matrix of the subgraph of $G$ induced by the first $n_1$ vertices and $B(G_2)$ is the incidence matrix induced by the last $n_2$ vertices of $G$.

Since $B(G)$ has this form, $G = G_1 \cup G_2$ must be disconnected. This is a contradiction to our assumption. Since we are working in the Galois field $\mathbb{Z}_2$, we can conclude the only equation that the rows vectors satisfy is that their sum is the zero vector in $\mathbb{Z}_2^m$. $\Rightarrow$ rank of $B(G) \geq n-1$. $\longrightarrow (2)$

From (1) & (2), we have rank $(G) = n-1$. //

Corollary: A graph $G$ on $n$ vertices that has $k$ components has $\text{rank}(G) = n-k$.

If we remove any one row from the incidence matrix of a connected graph, the remaining $(n-1) \times m$ submatrix is of rank $n-1$ (by the previous theorem).

In other words, the remaining $n-1$ row vectors sum up to the removed row. Thus, we need only $n-1$ rows of an incidence matrix to specify the corresponding graph completely, as the $n-1$ rows contain the same amount of information as the entire matrix.

**Defn** Any $(n-1) \times m$ submatrix $B_f(G)$ of an $n \times m$ incidence matrix $B(G)$ of a connected graph $G$ with no loops is called a reduced incidence matrix of the graph $G$. The vertex corresponding to the deleted row of $B(G)$ is called the reference vertex with respect to this reduced incidence matrix.

**Remarks:**

→ Any vertex of a connected graph can be made as the reference vertex.

→ Since a tree is a connected loopless graph on $n$ vertices and $n-1$ edges, its reduced incidence matrix is a square matrix of order and rank $n-1$.

**Corollary** A reduced incidence matrix of a loopless connected graph $G$ is non-singular if and only if $G$ is a tree.

**Proof:** Clearly, a graph with $n$ vertices and $n-1$ edges and is not a tree must be disconnected. The rank of the incidence matrix of such a graph is less than $n-1$. $\Rightarrow$ The $(n-1)\times(n-1)$ reduced incidence matrix of such a graph must be singular. //

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Let $\det(M)$ denote the determinant of a matrix $M$.
Suppose $G'$ is a subgraph of a graph $G$, where $G$ has $n$ vertices and $m$ edges.

Let $B(G')$ and $B(G)$ denote the incidence matrices of $G$ and $G'$ respectively. By permuting rows and columns, $B(G')$ is a submatrix of $B(G)$.

There exists a one-to-one correspondence between each $n \times k$ submatrix of the $n \times m$ incidence matrix $B(G)$ and each subgraph of $G$ having $k$ edges. Submatrices of $B(G)$ corresponding to special types of subgraphs such as cycles, spanning trees, bridges in $G$ exhibit special properties as the following theorem shows:

**Theorem** Let $G$ be a connected loopless graph on $n$ vertices and $m$ edges. Suppose $B(G)$ denotes its $n \times m$ incidence matrix. An $(n-1) \times (n-1)$ submatrix $B'$ of $B(G)$ is non-singular if and only if the $n-1$ edges corresponding to the $n-1$ columns of this matrix constitute a spanning tree of $G$. In this case, we also have that $\det(B') = \pm 1$. This also holds for the matrix $B_{-1}(G)$, where we have arbitrarily replaced one of the $1$'s in each column of $B(G)$ by $-1$.

Proof: Every square submatrix of size $n-1$ in $B(G)$ is the reduced incidence matrix of the ~~same~~ subgraph in $G$ with $n-1$ edges, and vice versa. By the corollary (discussed just prior to the definition), an $(n-1) \times (n-1)$ square submatrix $B'$ of $B(G)$ is non-singular if and only if the corresponding subgraph is a tree, because it contains $n-1$ edges of the graph $G$ on $n$ vertices. Since every tree has a leaf, there is a row in $B'$ with exactly one 1 in it. Using induction on $n$, and expanding the determinant along that row completes the proof. //

**Theorem** Let $G$ be a graph on $n$ vertices and $m$ edges. Then

$$A(L(G)) = B^T B - 2I_m$$

where $A(L(G))$ is the adjacency matrix of $L(G)$, $B$ is the incidence matrix of $G$ and $I_m$ is the unit square matrix of order $m$.

**Proof omitted.**

// End of Lecture //