MA 859: Selected Topics in Graph Theory

Lecture - 9

Hamiltonian Graphs (Contd.)

& Line Graphs.

Prior to Bondy & Chvátal, a number of sufficient conditions for a graph to be Hamiltonian were introduced. However, it turns out that all of these can be deduced from the result due to Bondy & Chvátal.

One of such sufficient conditions was due to Pósa:

Theorem [Pósa]

If G is a graph on $n \geqslant 3$ vertices such that for every integer j $\left(1 \leq j \leq \frac{n}{2}\right)$, the number of vertices of degree not exceeding j is less than j, then G is Hamiltonian.

[Pósa's proof was independent and different from what we can deduce from Bondy & Chvátal.]

Corollary [Dirac]

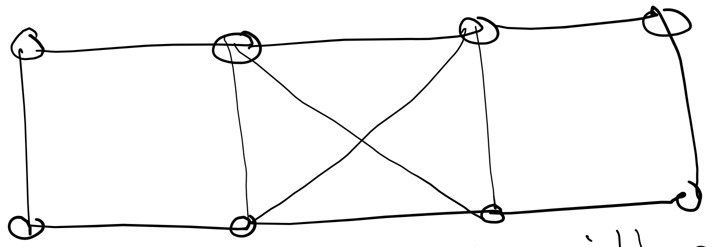
If G is a graph on $n \geq 3$ vertices such that $\deg v \geq \frac{n}{2}$ for every vertex v of G, then G is Hamiltonian.

The results discussed so far, involved the degrees of the vertices.

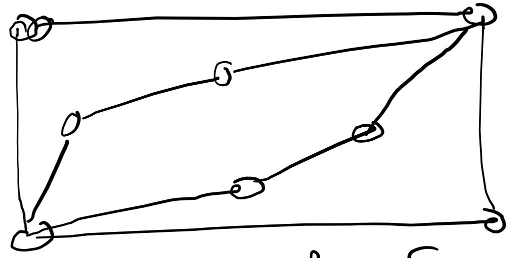
We now mention a different result involving the independence number and vertex connectivity.

Theorem [Chvátal & Erdös]

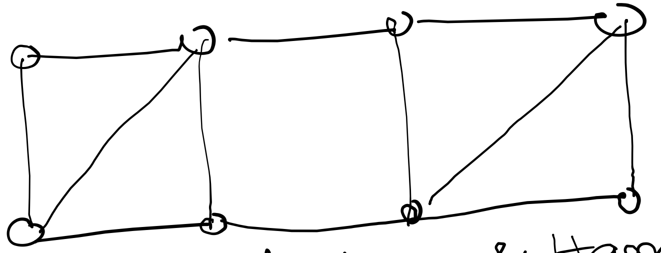
Let G be a graph with $n \geq 3$ vertices. If $\kappa(G) \geq \beta_0(G)$, then G must be Hamiltonian.



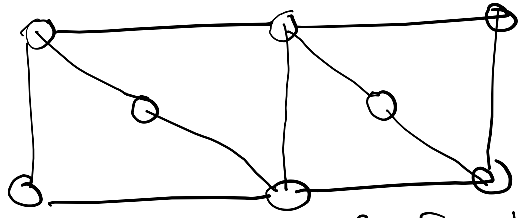
Eulerian & Hamiltonian



Eulerian & Non-Hamiltonian



Non-Eulerian & Hamiltonian



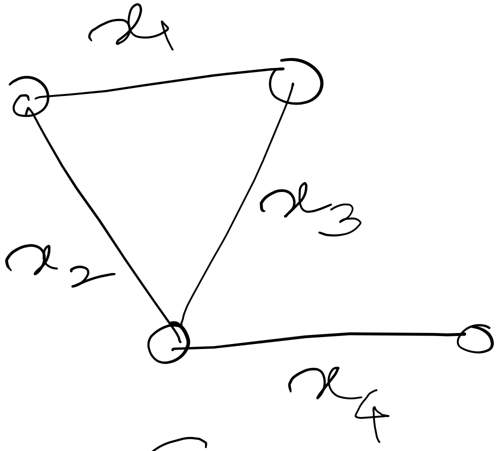
Non-Eulerian & Non-Hamiltonian

LINE GRAPHS

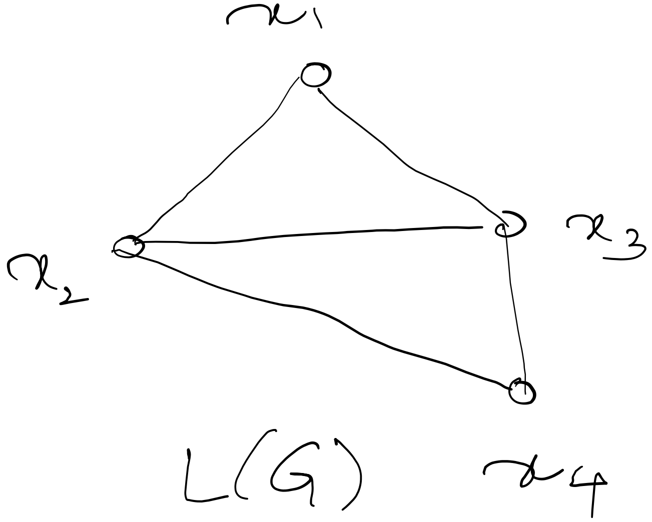
Let $G = (V, E)$ be a graph.
The Line Graph of G, denoted by $L(G)$, is a graph whose vertex set has the same number of vertices as the number of edges in G. Further, we establish a one-to-one correspondence between $E(G)$ and $V(L(G))$ to determine the edge set of $L(G)$ as follows:

Two vertices x and y of $L(G)$, representing the edges x' and y' of G, are adjacent in $L(G)$ if and only if x' and y' are adjacent in G (that is, x' and y' have a common vertex in G).

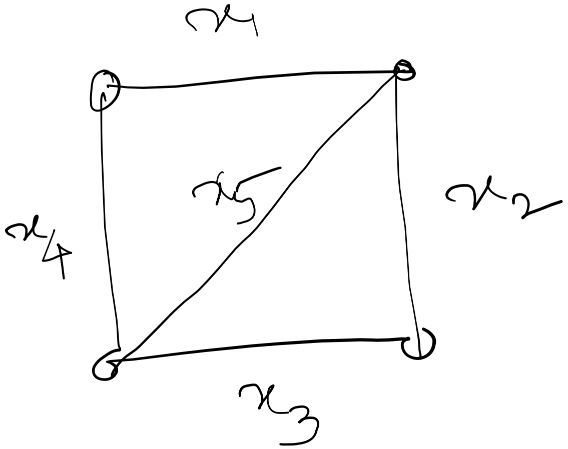
* Ore called it "Interchange Graph"
* Sabidussi called it "Derivative of G".
* Beineke called it "Derived Graph".
* Some people also call it "Edge Graph".



G



L(G)



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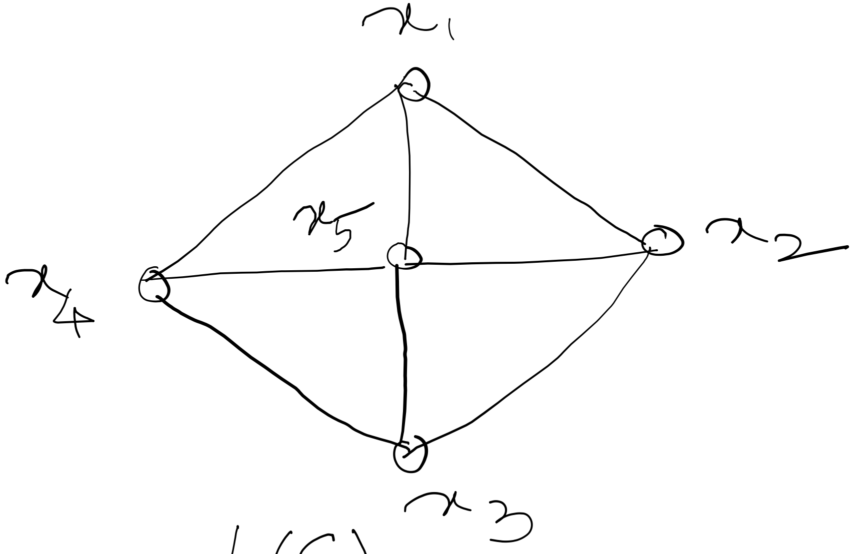


L(G)

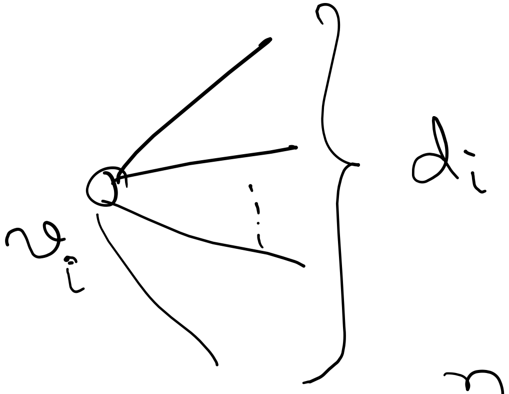
From the definition of $L(G)$, it is an immediate consequence that every cut-vertex of $L(G)$ is a bridge of G, which is not an end edge and conversely.

Theorem: If G is a graph on n vertices and m edges where the vertices have the degrees $d_1, d_2, \ldots, d_n$, then $L(G)$ has m vertices and m_L edges, where

$$m_L = -m + \frac{1}{2} \sum_{i=1}^{n} d_i^2$$

Proof: By definition, it is immediate that $L(G)$ has n vertices.

Let v_i be the vertex of G with degree d_i.



The d_i edges incident with v_i are mutually adjacent in G. So, they contribute $\binom{d_i}{2}$ to m_L.

$$\therefore\ m_L = \sum_{i=1}^{n} \binom{d_i}{2} = \sum_{i=1}^{n} \frac{d_i(d_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{n} d_i^2 - \frac{1}{2}\sum_{i=1}^{n} d_i$$

$$= \frac{1}{2}\sum_{i=1}^{n} d_i^2 - m.$$

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The following result is quite straight forward:

<u>Theorem</u>: A connected graph is isomorphic to its Line Graph if and only if it is a cycle.

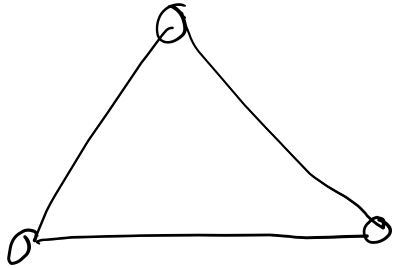
This result can be generalized as follows:

<u>Theorem</u>: A graph G is isomorphic to its Line graph $L(G)$ if and only if G is 2-regular.

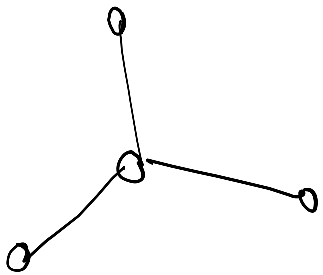
It is an obvious fact that if two graphs G and H are isomorphic, then so are their line graphs.

The converse also _almost_ holds except for the two following graphs:

The line graphs of both K_3 and $K_{1,3}$ are same!



K_3



$K_{1,3}$

Line Graphs & Traversability

Theorem: If G is Eulerian, then $L(G)$ is both Eulerian and Hamiltonian.

Proof: Suppose G is Eulerian. Then every vertex of G is of even degree. Now, consider any edge $x = uv$ of G. Since u and v are both of even degree, the edge x is adjacent with odd number of edges at each of its end vertices.

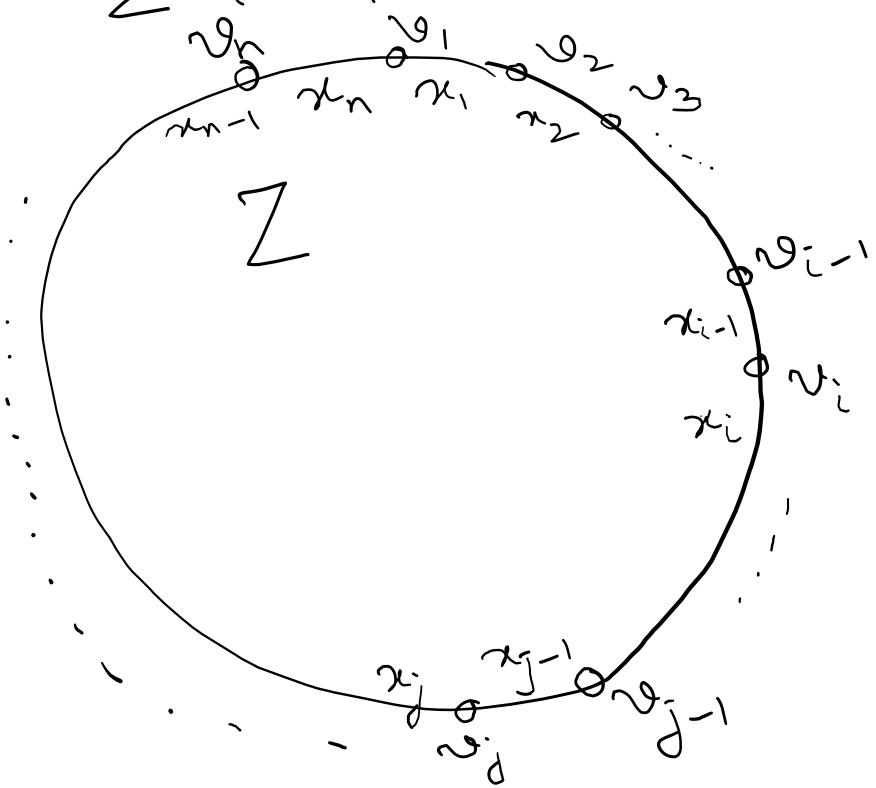
Hence, the vertex x of $L(G)$ corresponding to the edge x of G must be adjacent to even number of vertices in $L(G)$; thus the vertex x in $L(G)$ has even degree.
Since x is arbitrarily chosen, every vertex of $L(G)$ must be of even degree and hence $L(G)$ is Eulerian.

Now, consider an Eulerian Trail Z in G.
By definition, Z passes through every edge of G exactly once. Thus, in $L(G)$, the corresponding vertices of the edges of Z form a spanning cycle in $L(G)$. Hence $L(G)$ is Hamiltonian.

Theorem: If G is Hamiltonian, then so is $L(G)$.

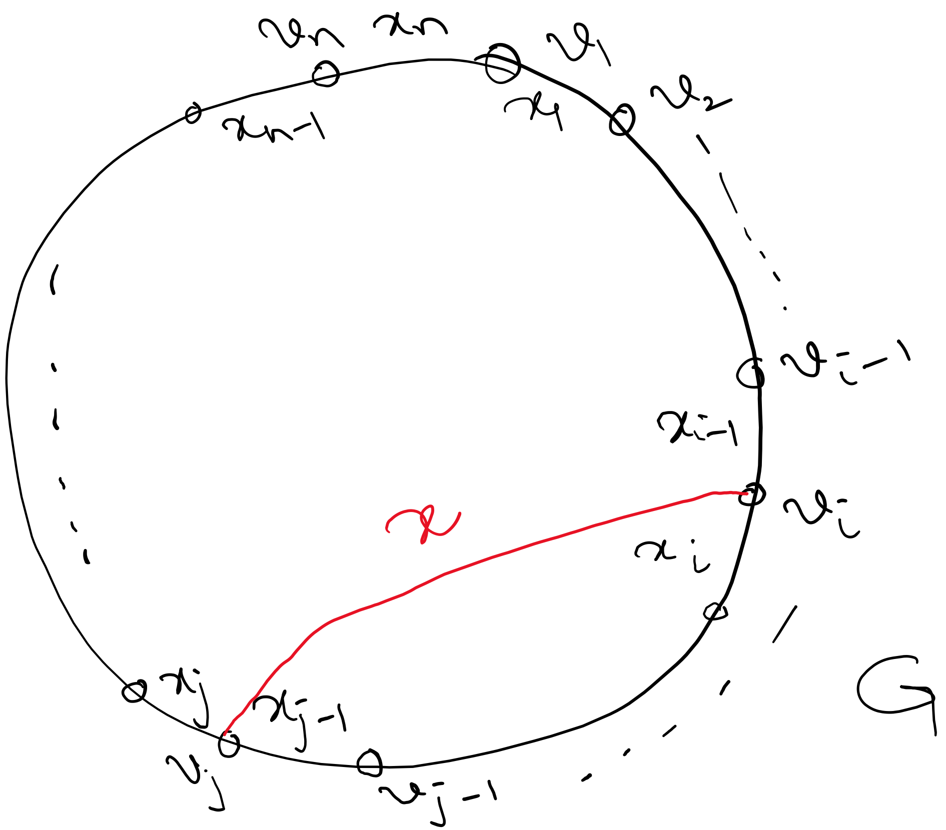
Proof: Let Z be a Hamiltonian cycle of G, where

Z: $v_1 x_1 v_2 x_2 \ldots x_{i-1} v_i x_i \ldots x_{j-1} v_j x_j \ldots x_{n-1} v_n x_n v_1$



Clearly, the vertices of $L(G)$ corresponding to the edges $x_1, x_2 \ldots x_n$ of G will form a cycle in $L(G)$.

If there are only these n vertices in $L(G)$, then the proof is complete; else, we must have some edges in G whose corresponding vertices in $L(G)$ are not on the cycle.



Let $x = v_i v_j$ be one such edge of G whose corresponding vertex in $L(G)$ is not on the cycle corresponding to Z.

We note that in G, the edges x_{i-1}, x_i and x have a common vertex v_i and likewise, the edges x_{j-1}, x_j and x have a common vertex v_j. So, the vertices corresponding to these triples must be mutually adjacent in $L(G)$.

This fact helps us to accommodate the vertex x (of $L(G)$) corresponding to the edge x of G in one of the following two ways:

(1) $x_1 x_2 \cdots x_{i-1}\, x\, x_i x_{i+1} - x_{j-1} x_j x_{j+1} - x_n x_1$

or

(2) $x_1 x_2 \cdots x_{i-1} x_i x_{i+1} \cdots x_{j-1}\, x\, x_j x_{j+1} \cdots x_n x_1$

We note that x is one of the vertices of $L(G)$ which was not on the cycle and it was arbitrarily chosen. Thus, every such vertex can be accommodated into a modified cycle in a similar manner.

We finally get a spanning cycle in $L(G)$. Hence $L(G)$ is Hamiltonian.

That is all for this Lecture.

Let us discuss about Planarity in the next lecture.

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