MA859: Selected Topics in Graph Theory

Lecture 15

Graphical Partitions

A partition of a non-negative integer n is a list of non-negative integers whose sum is n.

For instance:

$$
\begin{aligned}
4 &= 4 \\
&= 3+1 \\
&= 2+2 \\
&= 2+1+1 \\
&= 1+1+1+1
\end{aligned}
$$

$$
\begin{aligned}
5 &= 5 \\
&= 4+1 \\
&= 3+2 \\
&= 3+1+1 \\
&= 2+2+1 \\
&= 2+1+1+1 \\
&= 1+1+1+1+1
\end{aligned}
$$

Consider a graph $G = (V, E)$. Every vertex is associated with a parameter called degree.

By Handshaking Lemma, we know that the sum of all the degrees is an even number.

This idea prompted the people to think about the concept of partitions into determining the partitions of a non-negative integer which represent a graph wherein the vertices have the degrees as the parts of the given integer.

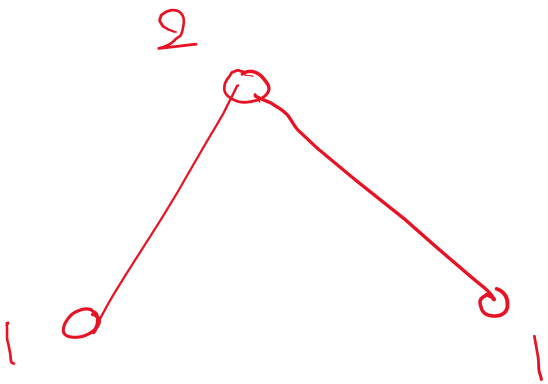
Clearly, we can think of partitioning only the even non-negative integers in this manner.

A partition of an even non-negative integer is called a graphical partition if the parts $d_1, d_2 \ldots$ represent the degrees of the vertices of a graph.

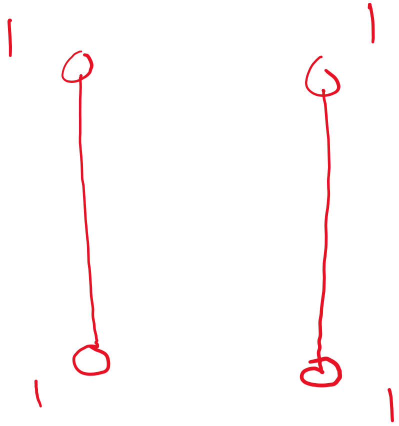
Consider the integer 4 for which we could write 5 partitions, namely

$$
\begin{aligned}
4 &= 4 \\
&= 3+1 \\
&= 2+2 \\
&= 2+1+1 \\
&= 1+1+1+1
\end{aligned}
$$

Among these we notice that only the last two are graphical partitions.



$2+1+1$



$1+1+1+1$

We note that if a partition of an integer $m = \sum_{i=1}^{n} d_i$ is graphical, then

1. m must be an even positive integer.
2. $d_i \le n-1$ for every $i = 1, 2, \ldots, n$.

But these two conditions are not sufficient for a partition to be graphical.

For example: $10 = 3+3+3+1$

Now, two questions arise:

1. How can we tell whether a given partition is graphical?

2. How can we construct a graph for a given graphical partition?

An existential answer to 1 was given by Erdös and Gallai in 1960. But in 1962, Havel and Hakimi independently found a procedure which is constructive in nature and in fact, answers the question 2 as well.

Theorem A partition $\pi = (d_1, d_2, \cdots, d_n)$ of an even number into n parts with $n-1 \geq d_1 \geq d_2 \geq \cdots \geq d_n$ is graphical if and only if the modified partition

$$\pi' = (d_2 - 1, d_3 - 1, \cdots, d_{d_1+1} - 1, d_{d_1+2}, \cdots, d_n)$$ is graphical.

Proof: If π' is graphical, then it is easy to show that π is also graphical. Because from a graph G' with partition π', we can construct a new graph G by adding a new vertex to G' and making it adjacent with the vertices of degrees $d_2-1, d_3-1, \ldots, d_{d_1+1}-1$.

Conversely, let G be a graph representing the partition $\pi = (d_1, d_2, \ldots, d_n)$.

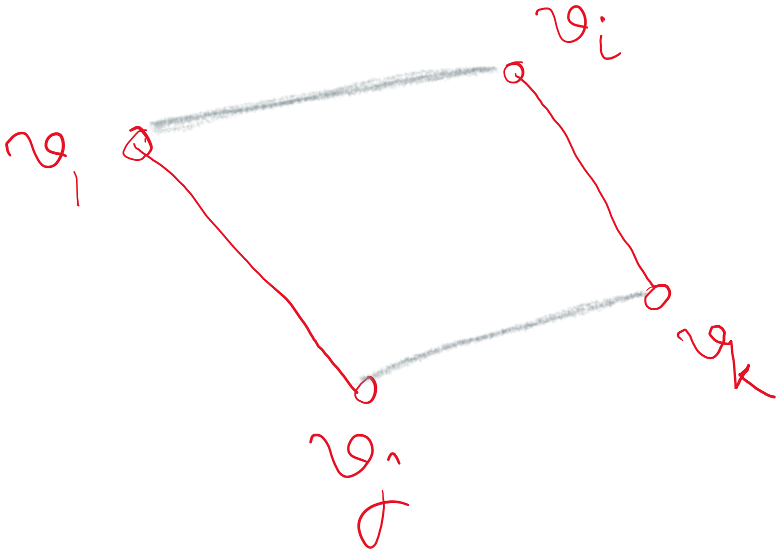
If a vertex of degree d_1 is adjacent to the vertices of degree d_i for $i = 2, 3, \ldots, d_1+1$, then this is a simple case because removal of that vertex from G yields the graph G' which represents π'.

If this is not the case, assume that the vertex v_i has degree d_i, with v_1 being a vertex of degree d_1 for which the sum of the degrees of the adjacent vertices is maximum.

Then there exist vertices v_i and v_j with degrees $d_i > d_j$ such that $v_1 v_j$ is an edge and $v_1 v_i$ is not. Therefore, some vertex v_k is adjacent to v_i, but not to v_j.

Remove $v_1 v_j$ and $v_k v_i$ and add the edges

$v_1 v_i$ and $v_k v_j$.



This results in another graph with the same graphical partition π; but now, the sum of the degrees of the vertices adjacent to v_1 is greater than before.

Continuing in this manner, after a finite number of steps, we eventually get a graph where v_1 has the desired property. //

Algorithm (Havel and Hakimi)

To test whether a given partition $\pi = (d_1, d_2, \dots d_n)$ is graphical or not.

(Each time, we modify the partition as in the theorem, until we end up with every summand zero).

1) Determine the modified partition

$$\pi' = (d_2 - 1,\ d_3 - 1,\ \dots,\ d_{d_1+1} - 1,\ d_{d_1+2},\ \dots d_n)$$

2) Reorder the summands in a non-increasing order, and call the resulting partition π_1

3) If $\pi_1 = (0, 0, \dots 0)$, stop & conclude that π is graphical. Else, repeat the process for π_1 to get π_2 and so on.

Ex

$$\pi = (5, 5, 3, 3, 2, 2, 2)$$

$$\pi' = (4, 2, 2, 1, 1, 2)$$

$$\pi_1 = (4, 2, 2, 2, 1, 1)$$

$$\pi_1' = (1, 1, 1, 0, 1)$$

$$\pi_2 = (1, 1, 1, 1, 0)$$

$$\pi_2' = (0, 1, 1, 0)$$

$$\pi_3 = (1, 1, 0, 0)$$

$$\pi_3' = (0, 0, 0)$$

$\therefore$ π is graphical.

Ex

$$\pi = (4,3,3,3,2,2,2,1)$$
$$\pi' = (2,2,2,1,2,2,1)$$
$$\pi_1 = (2,2,2,2,2,1,1)$$
$$\pi_1' = (1,1,2,2,1,1)$$
$$\pi_2 = (2,2,1,1,1,1)$$
$$\pi_2' = (1,0,1,1,1)$$
$$\pi_3 = (1,1,1,1,0)$$
$$\pi_3' = (0,1,1,0)$$
$$\pi_4 = (1,1,0,0)$$
$$\pi_4' = (0,0,0)$$

$\therefore$ π is graphical.

<u>Ex</u> $\pi = (8, 7, 6, 5, 4, 3, 2, 2, 1)$

Ex $\pi = (5,5,5,3,3,3,3,3)$

Ex

$$\lambda = (5,4,3,2,1,1,1,1,1,1,1,1)$$