MA859: Selected Topics in Graph Theory

Lecture - 16

Spectral Graph Theory

Spectral Graph Theory is a study of the relationship between the topological properties of a graph with the spectral (algebraic) properties of the matrices associated with the graph.

The most common matrix that is studied here is the adjacency matrix.

L. Collatz and U. Sinogowitz were the pioneers to begin the exploration of Spectral Graph Theory in 1957.

Originally, spectral graph theory studies analyzed the eigenvalues of the adjacency matrix of a graph. Recent developments, however, have been leaning towards geometric aspects, such as random walks and Markov Chains, Centrality, etc.

The eigen values are strongly connected to most of the key invariants of a graph. They hold a wealth of information about graphs. This is what spectral graph theory concentrates on.

We consider simple graphs.

Spectrum of a graph

The spectrum of a graph G is the set of eigenvalues of G, together with their algebraic multiplicities, or the number of times that they occur.

Lemma 1 A graph with n vertices has n eigenvalues.

Proof: This is a direct consequence of the Fundamental Theorem of Algebra. The characteristic polynomial of G with n vertices is a polynomial of degree n, and hence it must have exactly n roots, of course, counting multiplicity and the field of complex numbers. //

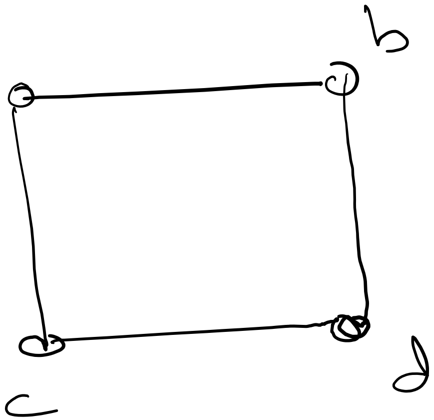
If a graph G has k distinct eigen values $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ with multiplicities $m(\lambda_1), m(\lambda_2), \ldots, m(\lambda_k)$, then the spectrum of G is given by

$$\mathrm{Spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m(\lambda_1) & m(\lambda_2) & \cdots & m(\lambda_k) \end{pmatrix},$$ where

$$\sum_{i=1}^{k} \lambda_i = n$$

Ex

G :

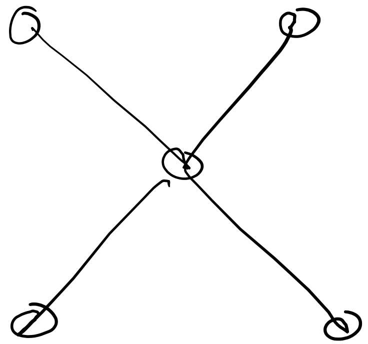


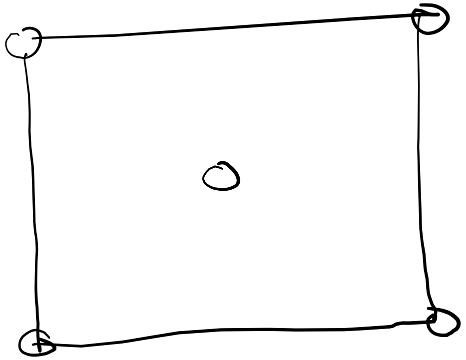
$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The characteristic polynomial is $4\lambda^4 - 4\lambda^2$ with
(verify!)
or $\lambda^4 - 4\lambda$?
eigen values $\{0, 0, 2, -2\}$.
So, the graph G has 3 distinct eigen values $-2, 0, 2$.
Hence $\operatorname{Spec}(G) = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$.

One consistent question in Graph Theory is —
When is a graph characterized by its spectrum?
Properties that can not be determined spectrally, can be determined by comparing two non-isomorphic graphs with the same spectrum.

The following two graphs have the same spectrum.
(verify).

G: 

H: 

Obviously, the above two graphs are structurally different, with different adjacency matrices. Yet, they have the same spectrum

$$\mathrm{Spec}(G) = \mathrm{Spec}(H) = \begin{pmatrix} -2 & 0 & 2 \\ 1 & 3 & 1 \end{pmatrix}.$$

It can be proved that the graphs with the same spectrum have the same number of triangles.
In this example, neither graph has a triangle. So, it also shows that graphs with the same spectrum do not necessarily have the same number of rectangles.
Also, our example exhibits that the graphs with the same spectrum need not have the same degree sequence.
It may be noted also that connectedness can not be determined by the spectrum.

There is, however, a correlation between the degrees of the vertices of a graph and the largest eigen value λ_1

For a k-regular graph, $\lambda_1 = k$.

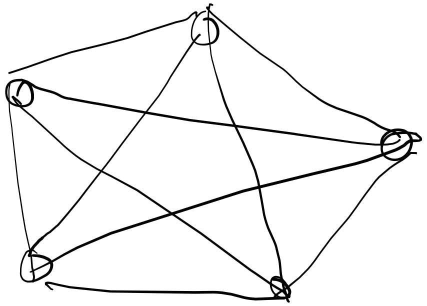
For K_n, $\lambda_1 = n-1$

If G is a connected graph, then $\lambda_1 \leq \Delta(G)$.

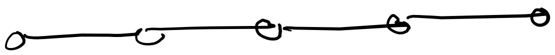
λ_1 increases with graphs that contain vertices of higher degree.

Further, the degrees of the vertices adjacent to the vertex of degree $\Delta(G)$ affect the value of λ_1.

Ex

G: 

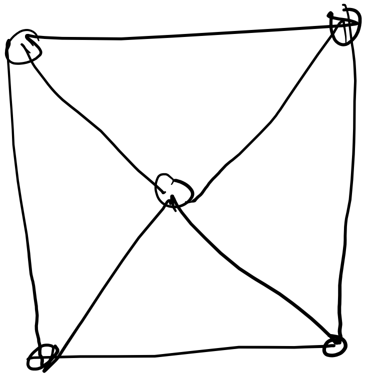
$\lambda_1(G) = 4$

H: 

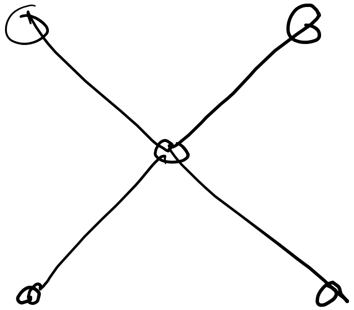
$\lambda_1(H) = \sqrt{3}$

Ex Compare the following graphs G and H:

This example illustrates how the degrees of the vertices adjacent to the vertex of degree Δ affect the value of λ_1.

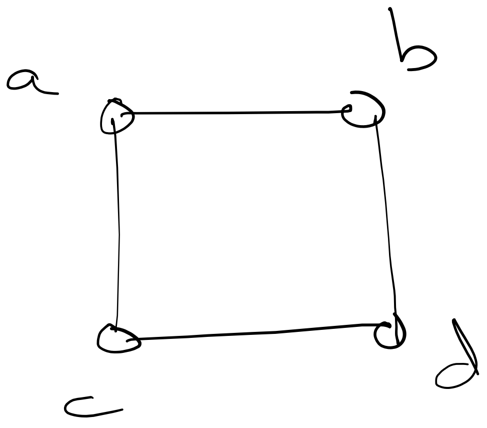
G: 

$\lambda_1 = 1 + \sqrt{5}$
≈ 3.236

H: 

$\lambda_1 = 2$

Bipartite graphs

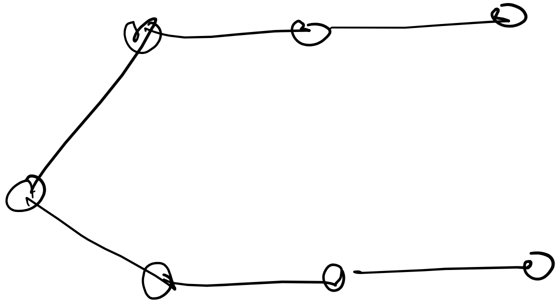


Bipartite graph

$U = \{a, d\}$ $W = \{b, c\}$

Characteristic Polynomial is $\lambda^4 - 4\lambda$

Eigenvalues are $-2, 0, 0, 2$.



Also a bipartite graph

Spectrum:

$\left(-\sqrt{2+\sqrt{2}},\ -\sqrt{2},\ -\sqrt{2-\sqrt{2}},\ 0,\ \sqrt{2-\sqrt{2}},\ \sqrt{2+\sqrt{2}}\right)$

The eigenvalues of bipartite graphs have some special properties:

<u>Theorem</u> 1 If G is a bipartite graph and λ is an eigenvalue, then $-\lambda$ is also an eigenvalue.

<u>Proof</u>: Suppose G is a bipartite graph with bipartition $U = \{u_1, u_2, \ldots, u_n\}$ and $W = \{w_1, w_2, \ldots, w_m\}$.
Clearly, all the edges are of the form $u_i w_j$, where $u_i \in U$ and $w_j \in W$. This makes the adjacency matrix of G, $A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$, where B is an $n \times m$ matrix.

Because λ is an eigenvalue, we know that

$$A(G)\vec{v} = \lambda\vec{v}.$$

So, $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$

Using matrix multiplication, we get $By = \lambda x$ (as one of the eqns)

Multiply both sides by -1, we get $-By = -\lambda x$

The second equation is $B^T x = \lambda y$. Also, $\lambda y = (-\lambda)(-y)$.

Therefore, $B^T x = (-\lambda)(-y)$.

$\Rightarrow \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} = -\lambda \begin{bmatrix} x \\ -y \end{bmatrix} \Rightarrow -\lambda$ is also an eigenvalue.

//

Corollary The spectrum of a bipartite graph is symmetric around zero. In other words, the eigen values of a bipartite graph occur in pairs of additive inverses.

The general form of a characteristic polynomial is

$$\lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_n$$

Theorem 2 If G is a bipartite graph, then $c_{2n-1} = 0$ for $n \geq 1$.

Proof: Since G is bipartite, there are no odd cycles in G.

Thus $c_{2n-1} = 0$ for $n \geq 1$ //

Walks and the Spectrum

Recall the defn of a $v_i v_j$ walk in a graph G. We know that $A^k_{i,j}$ represents the number of walks of length k from v_i to v_j.

$\Rightarrow$ Given vertices v_i and v_j, where $d(v_i, v_j) = t$ in a graph G with adjacency matrix A, we have $A^k_{i,j} = 0$ for $0 \leq k < t$.

A minimal polynomial of a graph G is the monic polynomial $q(x)$ of the smallest degree such that $q(G) = 0$.

For ex. if $f(x) = x^3 (x-5)^2 (x+4)^4$, the minimal polynomial for $f(x)$ is $x(x-5)(x+4)$.

Theorem 3 The degree of the minimal polynomial is larger than the diameter.

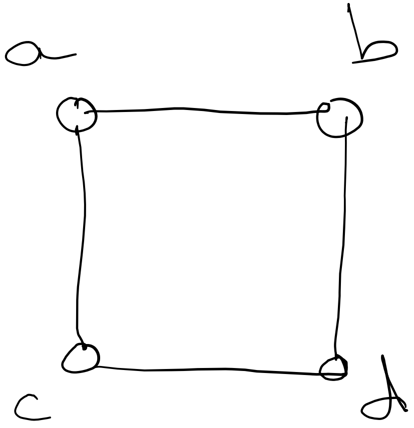
Proof: Since A is symmetric, the minimal polynomial is given by $q(G:\lambda) = \prod(\lambda-\lambda_k)$, for $1 \leq k \leq m$, where λ_k's are all distinct.

If we have m distinct eigen values, then the degree of the minimum polynomial is m. Suppose $m \leq d$, where d is the diameter of G and let i and j be two vertices of G whose distance is m. Then the $(i,j)^{th}$ entry of A^m is positive, while the $(i,j)^{th}$ entry of A^k is zero for all $k<m$.

$\Rightarrow q(G:A) \neq 0$ which is a contradiction. Hence $m > d$. //

Ex

G



$$\operatorname{diam}(G) = 2$$

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Characteristic Polynomial is $4\lambda^4 - 4\lambda^2 = 4\lambda^2(\lambda^2 - 1)$.

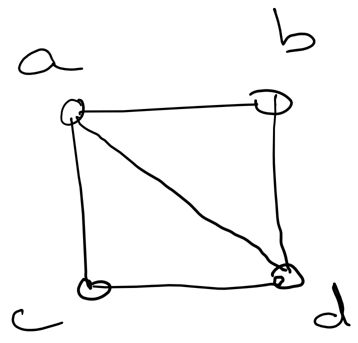
So, the minimal polynomial is $4\lambda(\lambda^2 - 1)$, which is of degree $3 > \operatorname{diam}(G) = 2$.

Theorem 4: If a graph G has diameter d and has m distinct eigen values, then $m \geq d+1$

Proof is a consequence of the previous result.

Ex

G



$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Characteristic Polynomial $\lambda^4 - 5\lambda^2 - 4\lambda$

(verify)

Eigen values $\{-1.56, -1, 0, 2.56\}$.

$\Rightarrow$ 4 distinct eigenvalues. $\operatorname{diam}(G) = 2$.

$\therefore \quad m > d+1$

Theorem 5: The complete graph is the only connected graph with exactly two distinct eigenvalues.

Proof: If a graph has exactly 2 distinct eigenvalues, then its diameter should be 1 or 0.

Two distinct eigenvalues $\Rightarrow$ minimal polynomial of degree 2.

Also, the degree must be greater than the diameter.

If the diameter is zero, then the graph must be K_1.

Otherwise, the graph is K_n for $n > 1$. //

The degree of the Characteristic Polynomial of K_n is n, giving us n eigen values.

Using matrix algebra, we can determine the characteristic Polynomial of K_n as $(\lambda+1)^{n-1}(\lambda-n+1)$.

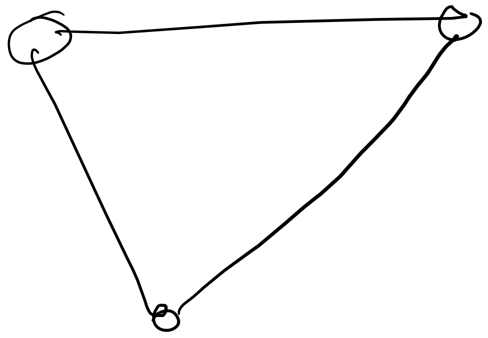
$\Rightarrow$ -1 is a root with multiplicity $n-1$ and the other root, $n-1$ occurs once.

<u>Theorem 6</u>: The complete graph K_n is determined by its spectrum.

<u>Proof</u>: This follows from Theorems 3 and 5. If there are exactly 2 eigenvalues in the spectrum, then the graph must be complete. Based on the multiplicities of the eigenvalues, we know the number of vertices. //

Ex $\quad \mathrm{Spec}(G) = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$

$\Rightarrow$ G is complete with 3 vertices.



// End of Lecture //