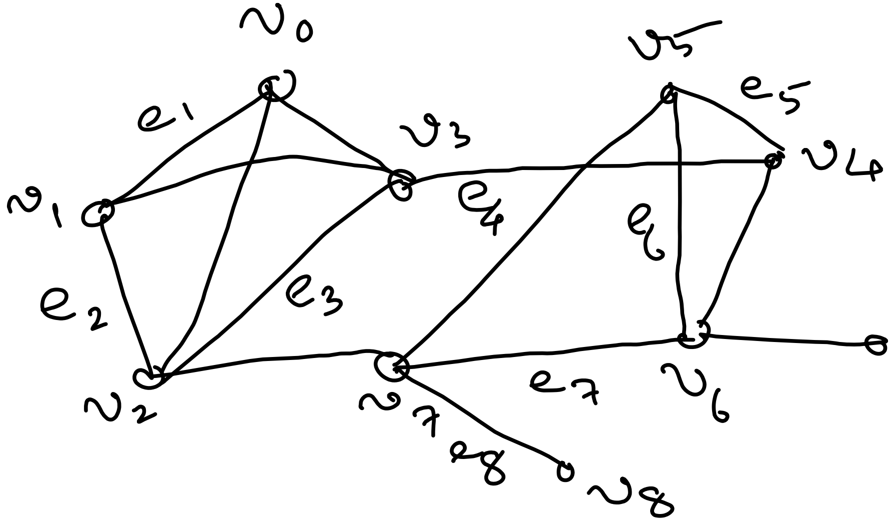
# MA859: Selected topics in Graph Theory

## LECTURE - 2

# Walk in a graph

An alternating sequence $v_0 e_1 v_1 e_2 v_2 \ldots\ldots e_n v_n$ of vertices and edges, beginning and ending with vertices.




When a walk begins and ends at the same vertex, it is a closed walk.

Trail - is a walk in which no edge is repeated.
A closed trail is called a circuit.

Path - is a walk in which no vertex is repeated.
A closed path is called a cycle.

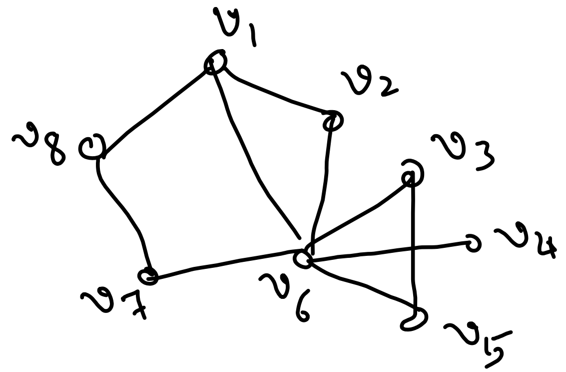
A graph $G$ is connected if and only if there exists a path between any two vertices; otherwise, disconnected.

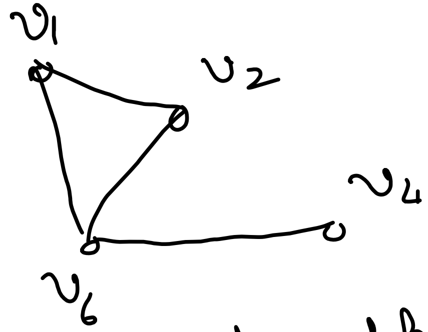
A subgraph $H = (V(H), E(H))$ of a graph $G = (V(G), E(G))$ is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

There are two important types of subgraphs:

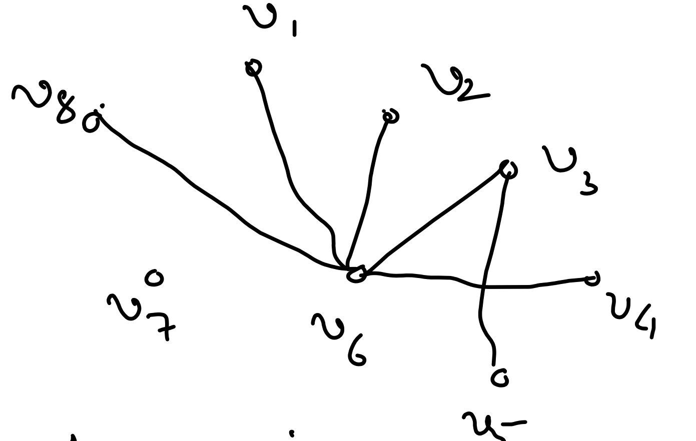
① **Induced subgraph:** Let $G = (V, E)$ be a graph and $S \subset V$. The induced subgraph $\langle S \rangle$ is a maximal subgraph that is drawn on the vertices of $S$.

② **Spanning subgraph:** Let $G = (V, E)$ be a graph. A spanning subgraph has the same vertex set $V$ and the edges of the spanning subgraph are the edges of $G$.




Induced subgraph of $S = \{v_1, v_2, v_4, v_6\}$




A spanning subgraph

**Note:** There are subgraphs other than the induced and the spanning subgraphs.

A maximal connected subgraph of a graph G is called its **component.**

So, if G is a connected graph, then G has only one component, namely, the graph G itself.

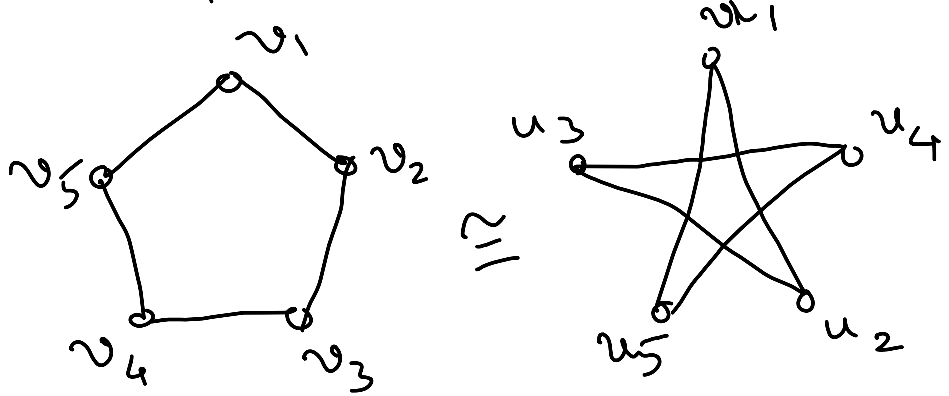
A disconnected graph has at least two components.

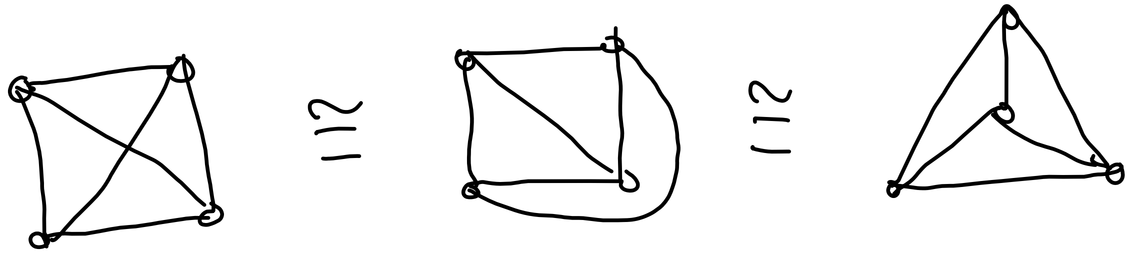
Disconnected graphs
with 2 components each.

# Isomorphism

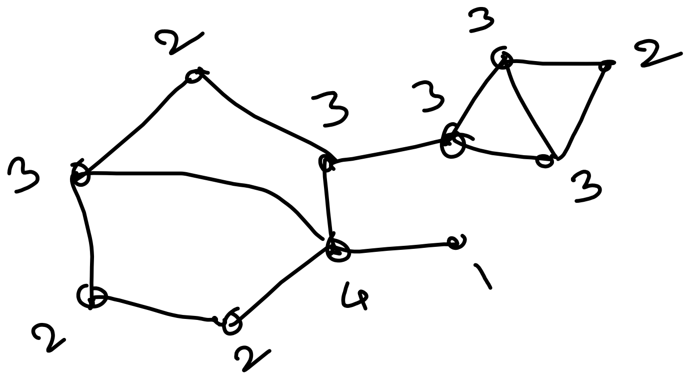
Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be *isomorphic* if there exists a one-to-one correspondence between $V_1$ and $V_2$ that preserves adjacency.

We write $G_1 \cong G_2$



**Degree of a vertex** is the no. of edges incident ~~to~~ it.



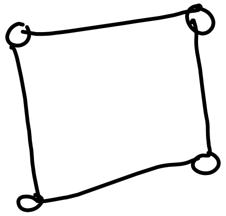

**First result in Graph Theory**

The sum of the degrees of all the vertices is always an even no.

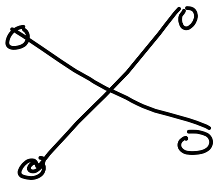
Also known as "**Handshaking Lemma**".

**Cor.** In any graph G, the no. of odd degree vertices is always even.

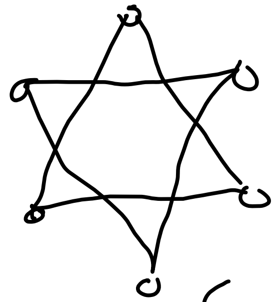
* Length of a walk is the total no. of edges in it.
* Distance between two vertices $u$ and $v$ in a graph $G$, is the length of a shortest path joining $u$ and $v$.
* Length of a path $P_n$ (on $n$ vertices) is $n-1$.
* Length of a cycle $C_n$ (on $n$ vertices) is $n$.
* Complement of a graph $G=(V,E)$ denoted by $\overline{G}$, has the same vertex set $V$ and has all those edges which are not in $E$.



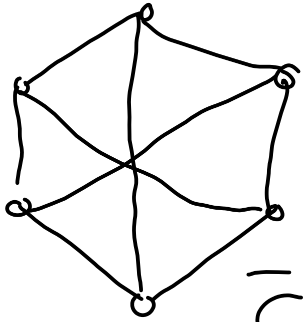
$G_1$



$\overline{G}_1$ ,

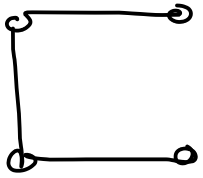


$G_2$



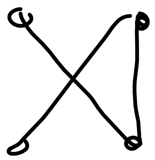
$\overline{G}_2$

* A graph $G$ is self complementary if $G \cong \overline{G}$.

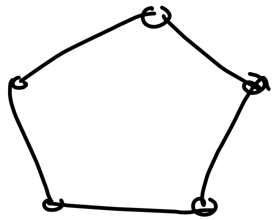


$G$

$\cong$

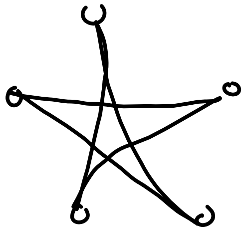


$\overline{G}$

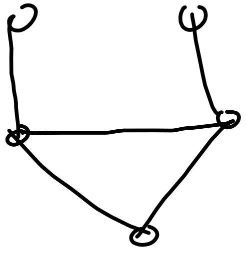


$G$

$\cong$

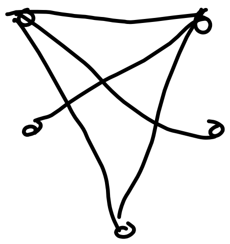


$\overline{G}$



$G$

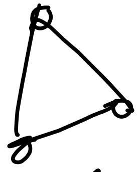
$\cong$



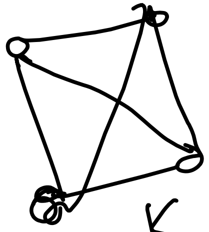
$\overline{G}$

* A graph $G$ on $n$ vertices is called a complete graph (denoted as $K_n$) is a graph in which every pair of vertices is adjacent.

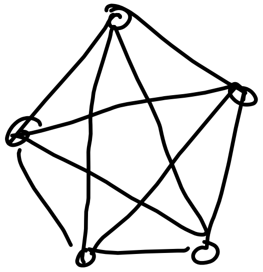
$K_n$ has $n$ vertices and $\binom{n}{2} = {}^{n}C_2$ edges



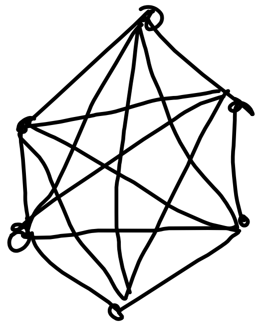
$K_3$



$K_4$



$K_5$



$K_6$

**Theorem** A self complementary graph has $4n$ or $4n+1$ vertices, where $n$ is a positive integer.

**Proof**: Suppose $G$ is self complementary and has $m$ vertices. $G \cong \overline{G}$ and $G \cup \overline{G} = K_m$.

Now $K_m$ has $\binom{m}{2}$ edges. $\Rightarrow$ Edges of $K_m$ are equally divided between $G$ and $\overline{G}$.

$\therefore$ No. of edges in $G = \dfrac{\binom{m}{2}}{2} = \dfrac{m(m-1)}{4}$

Since the no. of edges is always an integer, we must have $m \setminus 4$ or $(m-1) \setminus 4$. $\Rightarrow m = 4n$ or $4n+1$ //

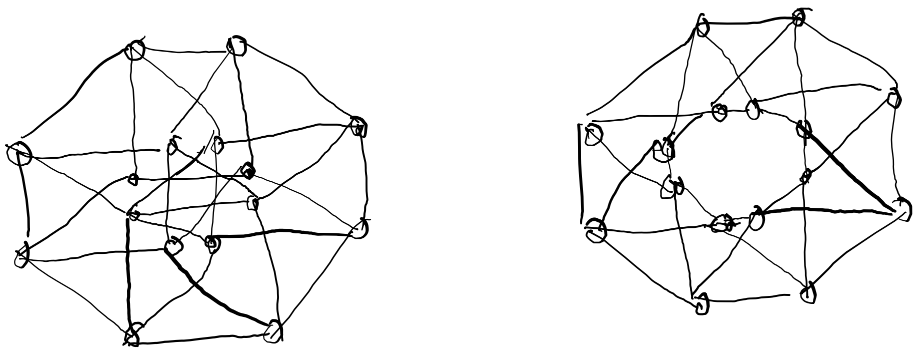
# Assignment

1) Construct a self complementary graph on $4n$ or $4n+1$ vertices, for any given $n$.

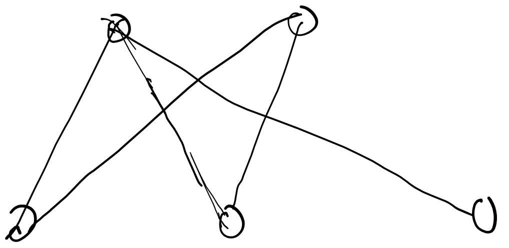
2) Show that a graph $G$ is connected iff for every partition of its vertices into two non-empty sets, there exists an edge with its end vertices in both the sets.

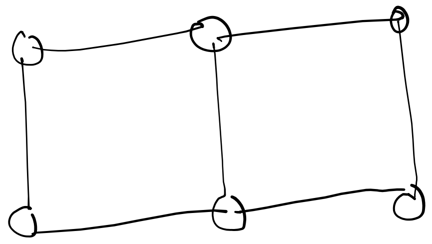
3) If $u$ and $v$ are distinct vertices in a graph $G$, then show that every $u$-$v$ walk in $G$ contains a $u$-$v$ path.

4) Determine whether the following graphs are isomorphic:

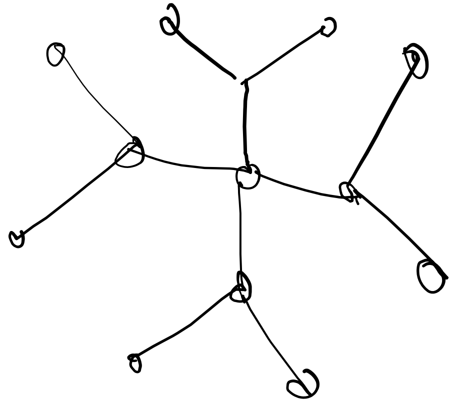


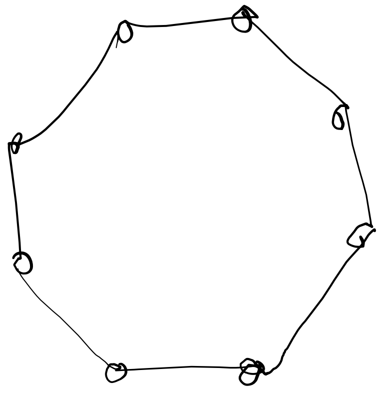
Defn: A graph $G=(V,E)$ is bipartite if $V$ can be partitioned into two sets $V_1$ and $V_2$ such that every edge of $G$ has one end vertex in $V_1$ and the other end in $V_2$.

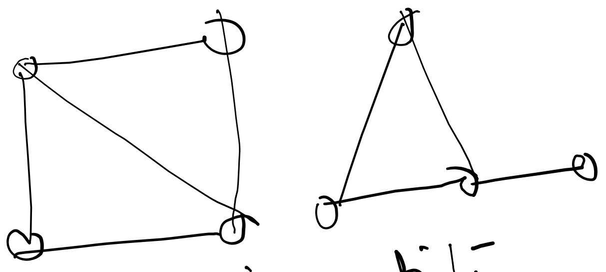




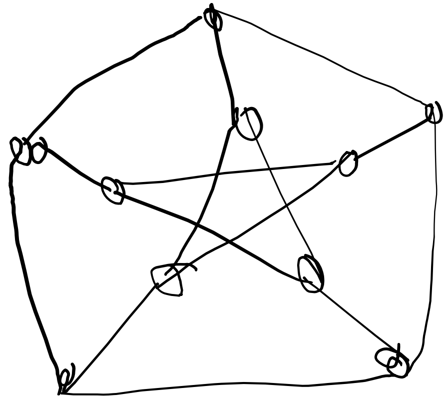
Bipartite







not bipartite



Theorem: A graph G is bipartite iff it has no odd cycles.

Proof:
Necessity: Suppose G is bipartite. Every walk in G alternates between the two partitions of the vertex set; so, every return to the original partition (including to the original vertex) happens after an even no. of steps. Hence G has no odd cycle.

Sufficiency: Suppose G has no odd cycle. We prove that G is bipartite by constructing a bipartition of each non-trivial component.

Let $u$ be a vertex in a non-trivial component $H$ of $G$.
For each $v \in V(H)$, we claim that all $u$-$v$ walks have the same parity. Otherwise, the concatenation of two $u$-$v$ walks of different parity (reverse tracking the second) leads to a closed walk of odd length.

We new claim that every closed odd walk contains an odd cycle:
Suppose $W$ is a closed walk; we use induction on the length $l$ of $W$. If $l = 3$, the closed walk is a cycle by itself.
Assume that the claim holds for walks of length less than $l$.
If $W$ has no repeated vertices (other than the first = last), then $W$ is itself a cycle of odd length.

If a vertex $v$ is repeated in $W$, then we can partition $W$ into two $v$-$v$ walks. Now, since the total length of $W$ is odd, one of the $v$-$v$ walks must be even. The odd one is shorter than $W$; so, by induction hypothesis, it must contain an odd cycle, which appears in order in $W$ as well. Hence the claim.

Now, in view of this claim, the concatenated walk (which is closed) must contain an odd cycle, which is a contradiction.

Thus, we can partition $V(H)$ into disjoint sets $X, Y$ by letting $X = \{v \in V(H) \,/\, u\text{-}v \text{ walks have even length}\}$ and

$Y = \{v \in V(H) \,/\, u\text{-}v \text{ walks have odd length}\}$.

Now, each of $X$ & $Y$ is an independent set, since an edge $vv'$ within $X$ (or $Y$) would create a closed walk of odd length.

Theorem: If $G$ is a graph with no cycle, then $G$ has a vertex of degree 1.

Proof: Let $P$ be a maximal path in $G$ with at least one edge and suppose $u$ is an end vertex of $P$. Because $P$ is maximal, every neighbour of $u$ must be in $P$. To avoid creating a cycle, $u$ must have no neighbour other than its neighbour along $P$ itself. Hence $u$ must have degree 1. //

We know that the maximum no. of edges in a graph $G$ on $n$ vertices is $\binom{n}{2}$.

What is the minimum no. of edges in a connected graph on $n$ vertices?

**Theorem**: The minimum no. of edges in a connected graph $G$ on $n$ vertices is $n-1$.

**Pf**: An $n$-vertex graph with no edges has $n$ components. When an edge is added, the number of components reduces at most by 1. So, if $k$ edges are added, the no. of components is still at least $n-k$. So, every $n$-vertex graph with fewer than $n-1$ vertices has at least two components, and is disconnected. The contrapositive of this is that every connected $n$-vertex graph has at least $n-1$ edges. //

This lower bound is achieved, for instance, on $P_n$.