MA859: Selected Topics in Graph Theory

Lecture - 14

Matrix Tree Theorem

In this lecture, we shall discuss an important interpretation of the adjacency matrix of a simple graph on n vertices.

We assume that G has m edges unless stated otherwise.

As discussed in the previous lecture, when a graph G has no loops, each row of the incidence matrix $B(G)$ has precisely two 1's. If, in addition, G has no multiple edges, then the adjacency matrix $A(G)$ is also a binary matrix.

Hence, given a graph G, it is important to know about the relationship between the incidence matrix and the adjacency matrix.

Defn Let G be a graph and $V(G) = \{u_1, u_2, \dots u_n\}$ be a labeling of its vertex set. The diagonal matrix $D(G)$, where the i^{th} diagonal entry is the degree of u_i, is called the degree matrix of G with respect to the labeling $V(G)$.

Recall $B_{-1}(G)$ described in the previous lecture:

- a matrix obtained from $B(G)$ where we have arbitrarily replaced one of the 1's in each column by -1.

Viewing this $B_{-1}(G)$ as a row matrix

$$B_{-1}(G) = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{bmatrix}$$

we see that the entry in row i and column j in the $n \times n$ matrix $B_{-1}(G) \cdot B_{-1}(G)^T$ is given by the dot product $\tilde{b}_i \cdot \tilde{b}_j$.

In particular, the i^{th} diagonal entry equals $\deg_G v_i$ for each vertex v_i of G.

Also, if $i \neq j$, then $\tilde{b}_i \cdot \tilde{b}_j = 0$ if the vertices v_i and v_j are not connected. However, $\tilde{b}_i \cdot \tilde{b}_j = 1 \cdot (-1) = -1$ if they are connected.

From this observation, we have the following relation between $A(G)$ and $B(G)$:

For a simple graph G,

$$A(G) + B_{-1}(G) \cdot B_{-1}(G)^T = D(G).$$

The matrix $D(G) - A(G)$ has some interesting properties. We need the following theorem to prove some of them.

<u>Theorem</u> (Binet - Cauchy)

Let m and n be non-negative integers, where $m \geq n$. Let X be a $n \times m$ matrix and Y, an $m \times n$ matrix, where their entries are elements of a given ring. For each $S = \{i_1, \ldots i_n\} \subseteq \{1, 2 \ldots m\}$, let X_S (Y_S) be the $n \times n$ square matrix obtained by choosing the columns (respectively rows) number i_1 through i_n from X (respectively Y).

In this case, we have

$$\det(XY) = \sum_{S \subseteq \{1, 2, \ldots m\}} \det(X_S) \det(Y_S)$$

where the sum is taken over all the $\binom{m}{n}$ subsets S of $\{1, 2, \ldots m\}$ that contain precisely n elements.

<u>Proof omitted</u>

We will now discuss an important result :

Matrix Tree Theorem

If G is a simple graph, then all the cofactors of the matrix $D(G) - A(G)$ are equal to a number $\tau(G)$, the number of distinct spanning trees of G.

<u>Proof</u>: By an observation made in one of the previous slides, we have

$$D(G) - A(G) = \underline{B}_1(G) \cdot \underline{B}_1(G)^T.$$

From this, we note that the $(i,i)^{th}$ cofactor of $D(G) - A(G)$ is obtained by the matrix product

$B_{-1;i}(G) \cdot B_{-1;i}(G)^T$, where $B_{-1;i}(G)$ is obtained by removing the i^{th} row.

By the Binet-Cauchy Theorem, $\det(B_{-1;i}(G) \cdot B_{-1;i}(G)^T)$ is the sum of all summands $B' \cdot B'^T$, where B' is a non-singular $(n-1) \times (n-1)$ submatrix of $B_{-1}(G)$.

By a theorem (discussed in the previous lecture

Thm: Let G be a connected loopless graph on n vertices and m edges. Let $B(G)$ denote $n \times m$ incidence matrix. A $(n-1) \times (n-1)$ submatrix B' of $B(G)$ is non-singular if and only if the $n-1$ edges corresponding to the $n-1$ columns of this matrix constitute a spanning tree of G. In this case, $\det(B') = \pm 1$. This also holds for the matrix $B_{-1}(G)$ where we have arbitrarily replaced one of the 1's in each column of $B(G)$ by -1.)

there are precisely $\tau(G)$ such summands and each summand is equal to $(\pm 1)^2 = 1$. Therefore, we have that each $(i,i)^{th}$ cofactor of $D(G) - A(G)$ equal to $\tau(G)$.

Now, to show that every $(i,j)^{th}$ cofactor is equal to $\tau(G)$, we first note that the sum of all the rows in $X = D(G) - A(G)$ is zero vector. Consider the submatrices $X_{i,j}$ and $X_{j,j}$ from X by removing row i and column j and row j & column j respectively.

The corresponding cofactors are

$$C_{i,j} = (-1)^{i+j} \det(X_{i,j}) \quad \text{and}$$

$$C_{j,j} = (-1)^{2j} \det(X_{j,j}) = \det(X_{j,j}).$$

One of the properties of a determinants is that it remains unaltered when we add a multiple of one row to a given row. Hence adding $1, 2, \ldots, i-1, i+1, \ldots, n$ to i^{th} row in $X_{i,j}$ will not change $C_{i,j}$.

Since the rows in X sum upto zero, the i^{th} row in $X_{i,j}$, after this alteration, will be the j^{th} row in $X_{j,j}$.

Finally, by moving this altered i^{th} row to j^{th} place, by means of $|j-i|-1$ interchanges of rows, we get the matrix $X_{j,j}$, where row i is multiplied by -1. Since each interchange reverses the sign of the determinant, we finally have

$$C_{ij} = (-1)^{i+j+|j-i|-1} \left(-C_{j,j}\right) = C_{jj}.$$

We call the equation implied by the above theorem as Matrix-Tree Formula.

Ex For $n > 2$,

$$D(K_n) - A(K_n) = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix}$$

By the Matrix Tree Theorem, $\tau(K_n)$ is the cofactor of the preceding matrix. Consider the cofactor $C_{1,1}$ and then subtracting the first row from all the remaining rows, we note that

$$\tau(K_n) = \det \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \\ -1 & -1 & \cdots & n-1 \end{bmatrix}$$

$$= \det \begin{bmatrix} n-1 & -1 \cdots\cdots -1 \\ -n & \\ \vdots & n \cdot I_{n-2} \\ -n & \end{bmatrix},$$ where I_{n-2} is the identity matrix of order $n-2$.

Adding columns $2, \ldots, n-1$ to the first column, we finally get

$$\tau(K_n) = \det \begin{bmatrix} 1 & -1 \cdots\cdots -1 \\ 0 & \\ \vdots & n \cdot I_{n-2} \\ 0 & \end{bmatrix} = n^{n-2}.$$

(This example is also known with the name Cayley's Theorem.

An application in Electrical Circuits

The objective is to evaluate the total resistance between two points in a network.

Consider the following circuit:

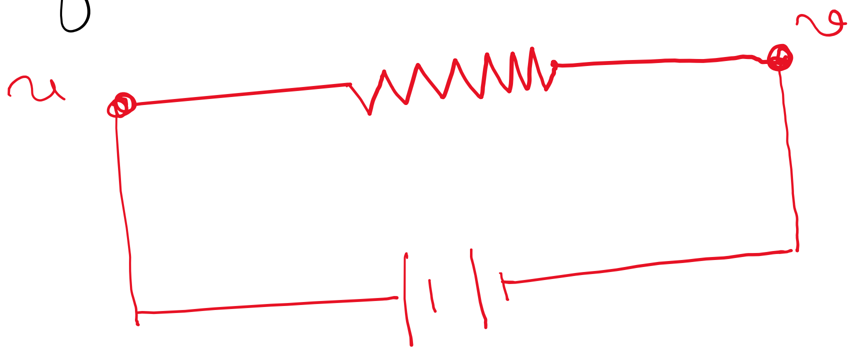


Fig. 1:

An electrical circuit with a given potential and a system of resistors (shown as one combined resistor).

The electrical circuits considered here are provided with a power supply having a given potential

between two nodes u and v, denoted by V_{uv} (or simply V, when we have no scope for any ambiguity).
These electrical devices are connected to a power supply in an arbitrary fashion with combined resistance R.
In our case here, the ~~total~~ electrical current I flowing from u to v is given by

$$I = \frac{V_{uv}}{R}.$$

Hence, if we can compute R, we can calculate I.

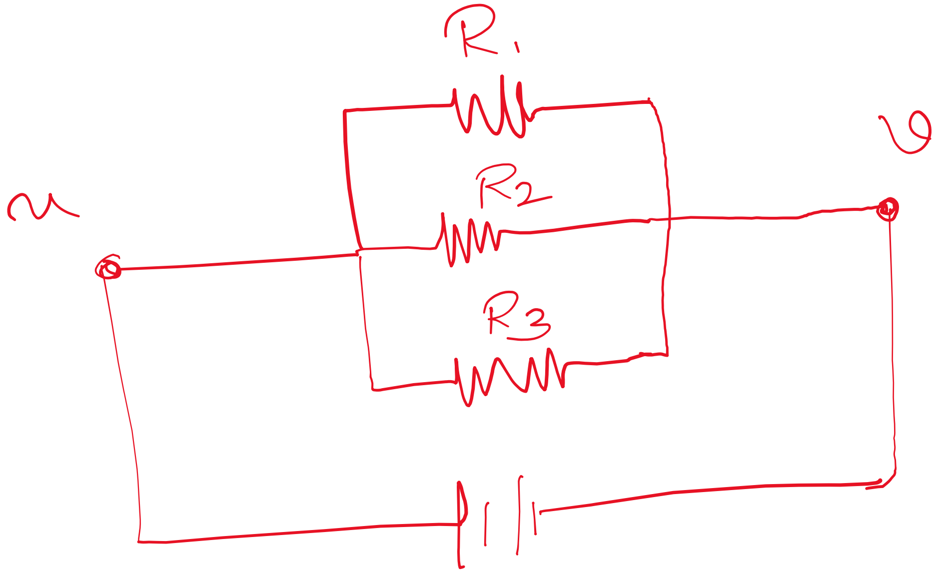
The problem we are concerned with is, to compute the total combined resistance of the network in question.

If we have the electrical devices that are connected as in the following figure (Fig. 2):

Fig 2:

Then the total resistance $R_t = R_1 + R_2 + \cdots + R_n \longrightarrow (1)$

On the other hand, if the resistors are connected in parallel (as in Fig. 3):



then R_t is obtained from $$\frac{1}{R_t} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n} \quad (2)$$

There are many other physical phenomena that behave in a similar fashion; but for the sake of simplicity, we restrict ourselves to networks involving only a potential, electric current and resistance.

The salient point to notice is that the total resistance of many resistors network systems can be calculated by the repeated use of the formulas ① & ②.

The following theorem is useful to compute the total resistance in a completely mechanical way between two nodes u and v in many more cases without any prior knowledge of physics.

<u>Theorem</u> (*) Let G be a simple graph and u, v be two neighbouring vertices of G. Assume that each edge in G corresponds to the resistance of one ohm ($1\,\Omega$). In this case, the total resistance of the network between u and v is given by

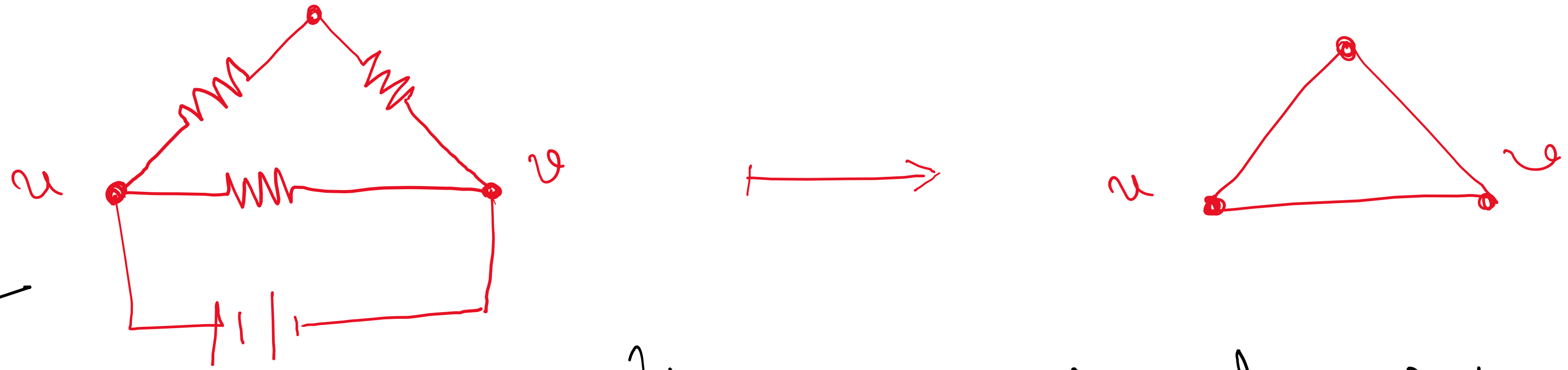
$$R_t = \frac{\tau_{uv}(G)}{\tau(G)} \quad \text{where}$$

τ_{uv} is the no. of spanning trees that include the edge uv.

<u>Note:</u> The number $\tau(G \cdot e)$ is precisely the no. of spanning trees of G containing e. We can write R_t as

$$R_t = \frac{\tau(G \cdot uv)}{\tau(G)} = \frac{\tau(G \cdot uv)}{\tau(G \cdot uv) + \tau(G - uv)}$$

<u>Ex</u>



<u>Fig 3</u>

In the above figure, each edge represents a resistance of $1\,\Omega$.

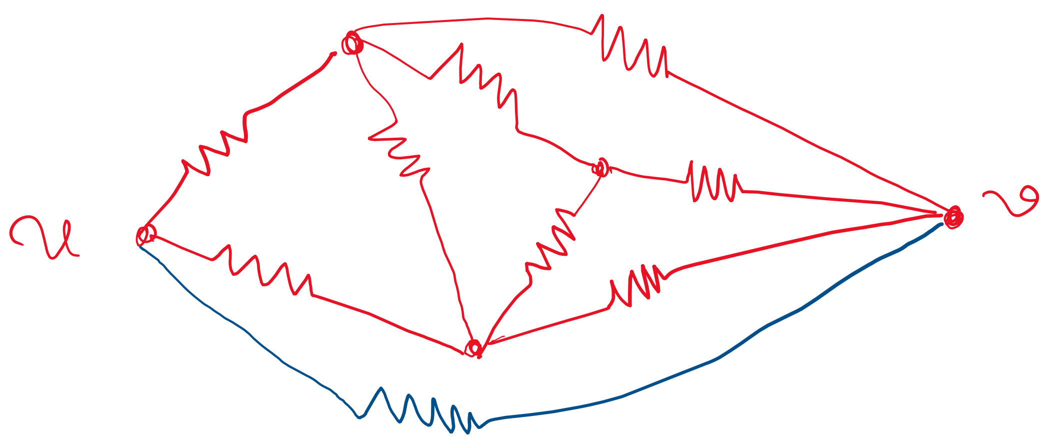
We can easily compute the total resistance between u and v using ① & ②. Here

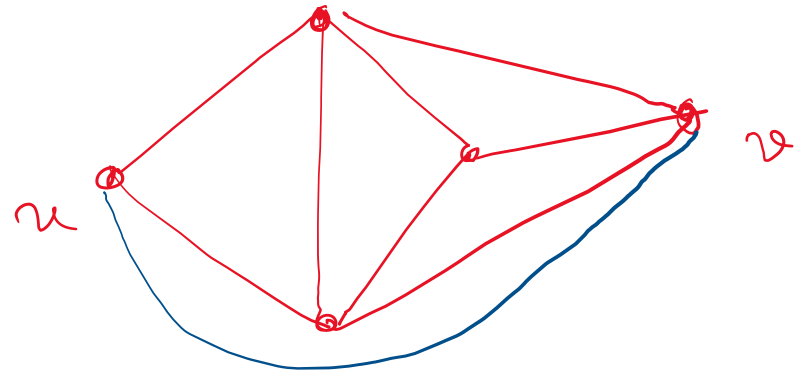
$$\frac{1}{R_t} = \frac{1}{1+1} + \frac{1}{1} \implies R_t = \frac{2}{3}\,\Omega$$

Since u and v are neighbours in G, using the theorem described just now, we get

$$R_t = \frac{\tau_{uv}(G)}{\tau(G)} = \frac{2}{3}.$$

Ex Consider the following circuit & its graph:





Here u and v are not neighbours; so the direct application of the formulas will not be sufficient.

Adding the edge $e=uv$, we obtain a graph G'.

Let G'' be the graph we obtain from G by collapsing u and v into one vertex.

Now, if R_t is the total resistance of the network corresponding to G and R_t' is the total resistance corresponding to G', we have by formula (2) that

$$\frac{1}{R_t'} = \frac{1}{R_t} + \frac{1}{1}$$

Since $R_t' = \dfrac{\tau_{uv}(G')}{\tau(G)}$, we have

$$R_t = \frac{\tau_{uv}(G')}{\tau(G') - \tau_{uv}(G')} = \frac{\tau_{uv}(G')}{\tau(G' - uv)} = \frac{\tau(G'')}{\tau(G)}.$$

Thus, we observe that the previous theorem (*) reduces the tricky problem of determining total resistance between given two points of a large class of electrical networks to that of computing the no. of spanning trees of corresponding graphs.

The Matrix-Tree Theorem reduces the problem of computing the no. of spanning trees of graphs to the problem of evaluating a determinant.

Since there are many efficient ways of evaluating the determinants of square matrices, we have a mechanical and efficient way of computing the total resistance of general electrical networks, where each edge of the corresponding graphs represents $1\,\Omega$.

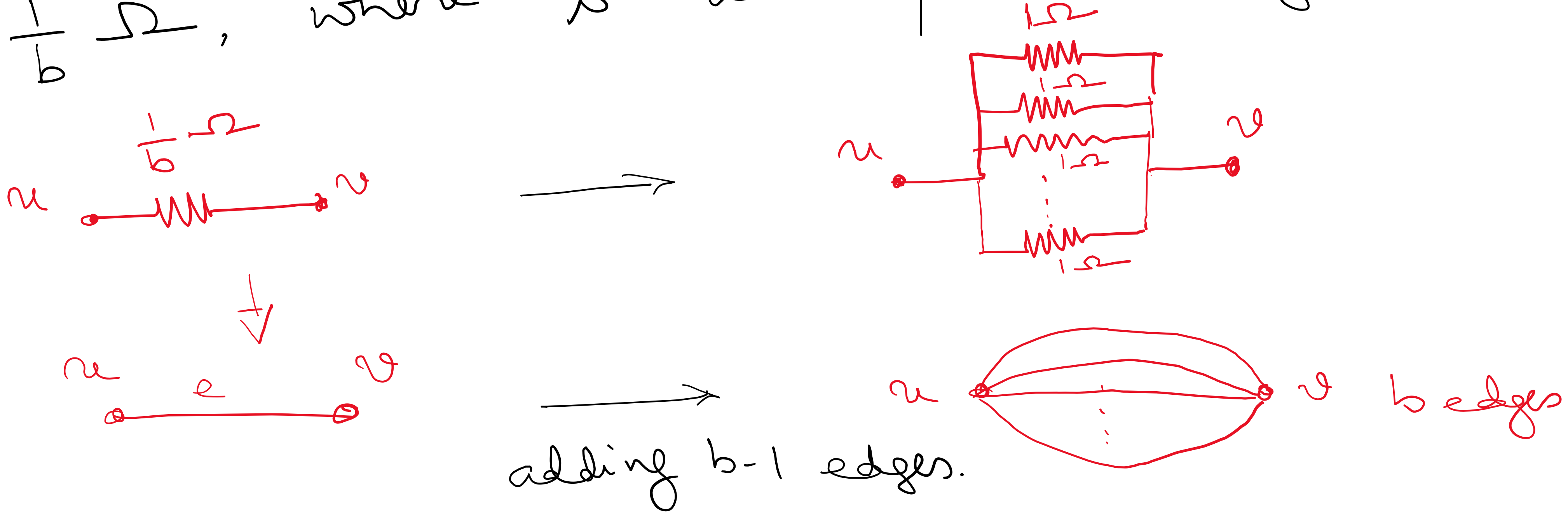
Now, what about general networks where the edges represent different resistance?

For this, there is a partial algorithmic solution.

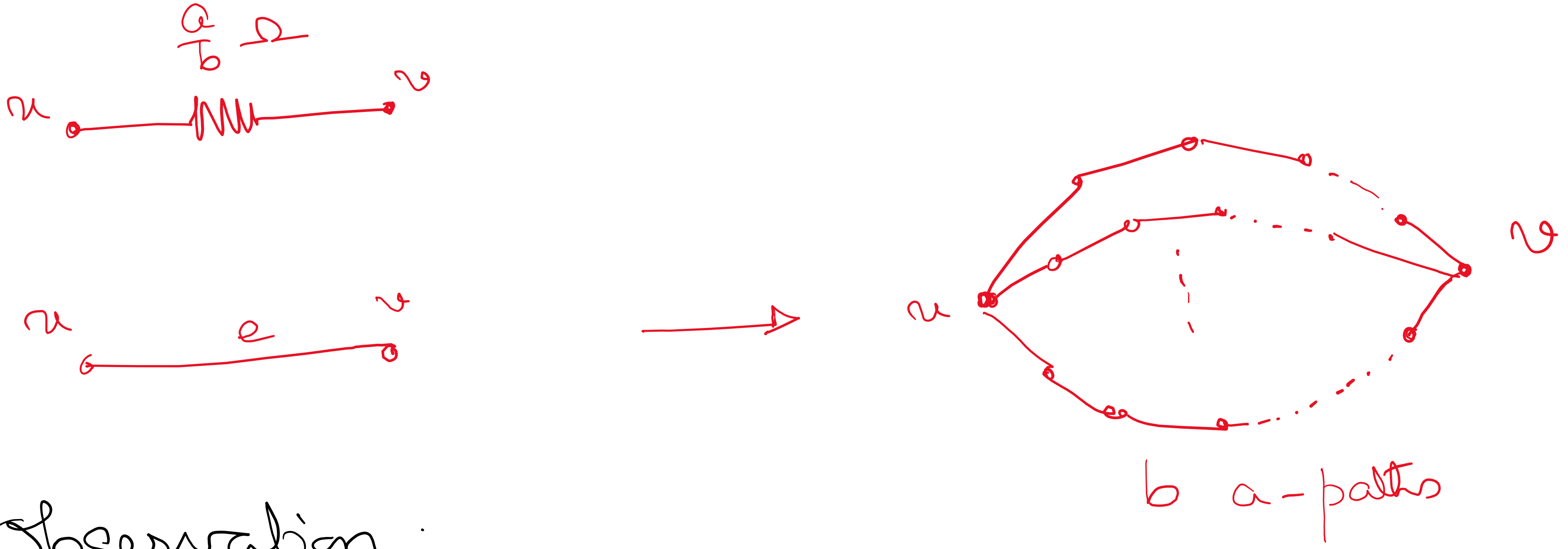
Suppose, we have an electrical network of resistors, where each resistor can represent a $a\,\Omega$, where a is a positive integer.

Here, looking at the corresponding graph of the network, we replace each edge representing $a\,\Omega$ with a simple path of length a. By this subdivision, we have that all edges represent a resistance of $1\,\Omega$.

Suppose now that we have an electrical network of resistors where each resistor can represent any $\frac{1}{b}\Omega$, where b is a positive integer.



We now combine these two methods. In a network of resistors, where each resistor can represent any rational resistance $\frac{a}{b}\Omega$, replacing the corresponding edge by b parallel a-paths, using the previous procedure yields a graph where each edge corresponds to $1\,\Omega$.



Observation:

The total resistance of an electrical network of rational resistors can be computed by applying the Matrix-Tree theorem to the "altered" corresponding graph.

<u>Note</u>: The procedure described for computing the total resistance of a given electrical network of resistors does not provide us with an exact solution if some resistors have irrational resistance $x\,\Omega$, where $x \in \mathbb{R}/\mathbb{Q}$.

However, from a practical point of view, our method does provide a computational method for any real world network since every resistor is given by a floating number of finitely many decimal (or binary) digits that does indeed represent a rational number.

// End of Lecture //