ANSWERS TO ODD-NUMBERED PROBLEMS

PROBLEMS 1.1

1. (**a**) Nonlinear. (**b**) Nonlinear. (**c**) Linear. (**d**) Linear.
(**e**) Nonlinear. (**f**) Nonlinear. (**g**) Nonlinear. (**h**) Linear.

3. Let L be the number of batches of linear, E be the number of batches of extra lean. The equations are

$$\text{Total fat} = 10 = 1.5L + E$$
$$\text{Total red} = 80 = 8.5L + 9E$$

The solution is $L = 2, E = 7$.

5. $x = 2.1, y = -.9$ **7.** $x = 76/13, y = -5/13$

9. $B = 400C + 0.35T + 120H$

PROBLEMS 1.2

1. (**a**) $x_1 = 1, x_2 = -1, x_3 = 4$
(**b**) $x_1 = (14 + 2s)/5, x_2 = (1 + 3s)/5, x_3 = s, s$ arbitrary
(**c**) No solution. (**d**) $x_1 = 11/3, x_2 = -1/3$.
(**e**) $x_1 = (30 + s)/8, x_2 = (11s - 6)/16, x_3 = s, s$ arbitrary
(**f**) No solution.

3. (**a**), (**b**), (**d**), (**g**), and (**h**) are in row echelon form.
(**c**) Fails because $a_{13} \neq 0$, (**e**) Fails because $a_{13} \neq 0$.
(**f**) Fails because $a_{44} \neq 1$ or 0.

5. In each case, eliminate variables.

(**a**) Only one solution regardless of choice of a and b.

(**b**) If $-2a + b \neq 0$, then no solution; if $-2a + b = 0$, then an infinite number of solutions.

(**c**) Only one solution regardless of the choice of a and b.

(**d**) If $-a - 3b + c \neq 0$, then no solution; if $-a - 3b + c = 0$, then an infinite number of solutions.

7. Row reduction leads to

$$\left(\begin{array}{cc|c} a & b & r \\ c & d & s \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & b/a & r/a \\ c & d & s \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & b/a & r/a \\ 0 & d - cb/a & s - cr/a \end{array}\right).$$

Therefore if $d - cb/a \neq 0$ we have a unique solution. Equivalently, the condition is $ad - bc \neq 0$.

9. Row reduction as in Prob. 7 leads to

$$\left(\begin{array}{cccc|c} 1 & b/a & 0 & 0 & r/a \\ 0 & ad - bc & 0 & 0 & as - cr \\ 0 & 0 & 1 & f/e & t/e \\ 0 & 0 & 0 & eh - fg & eu - gt \end{array}\right)$$

If $ad - bc \neq 0$ and $eh - fg \neq 0$, then we have a unique solution.

11. Suppose first that $a \neq 0$. Then we can row reduce

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \begin{pmatrix} 1 & b/a \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & b/a \\ 0 & d - bc/a \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & b/a \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} 1 & b/a \\ 0 & 0 \end{pmatrix}$$

This is the reduced row echelon form. Suppose on the other hand that $a = 0$. Then we have $ad - bc = -bc = 0$, so either $b = 0$ or $c = 0$.
If $b = 0$ **and** $c = 0$, we have the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

Which row reduces to $\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}$. If $d \neq 0$ we have reduced row echelon form of

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

If $d = 0$ we have

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

If $b = 0$ and $c \neq 0$, we have the matrix

$$\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$$

The reduced row echelon form is

$$\begin{pmatrix} 1 & d/c \\ 0 & 0 \end{pmatrix}$$

The $b \neq 0$ and $c = 0$, we have the matrix

$$\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$$

The reduced row echelon form is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

13. Substitute:

$$a(u + r) + b(v + s) = au + ar + bv + bs = (au + bv) + (ar + bs) = m + 0 = m$$
$$c(u + r) + d(v + s) = cu + cr + dv + ds = (cu + dv) + (cr + ds) = n + 0 = n$$

15. $x_1 = t, x_2 = \sqrt{2}t, x_3 = t, t$ arbitrary.

17. (**a**) See the solution to Prob. 11.

(**b**) Simply consider all possible pairings. For instance compare

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix}$$

No row operations can introduce a nonzero entry in the first row and first column of the first matrix. Thus these two matrices cannot be row equivalent.

(**c**) Suppose a 2×2 matrix had two distinct reduced row echelon forms. These forms would have to come from the list in b. Let A be the original matrix and B and C be the two reduced row echelon forms. Then by Prob. 16, since C is row equivalent to A and A is row equivalent to B, then C has to be row equivalent to B, a contradiction.

19. The last row is all zeros. Perform the row operations $-R1+R2$, $-R1+R3, \dots, -R1+Rn$. Then perform

$$\frac{1}{n}R2, \frac{1}{2n}R3, \dots, \frac{1}{(n-1)n}Rn$$

to obtain

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \hdotsfor{5} \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Now it is easy to obtain all zeros in rows 3 through n.

21. $x = \overline{2 - is} = 2 + is, y = s, s$ arbitrary.

23. For part 1 suppose $s_1, \dots, s_n$ is a solution to the system. Therefore

$$\begin{aligned} a_{11}s_1 + a_{12}s_2 + \cdots + a_{1n}s_n &= b_1 \\ \cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n &= b_i \\ \cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{m1}s_1 + a_{m2}s_2 + \cdots + a_{mn}s_n &= b_m \end{aligned} \tag{1}$$

Now multiply equation i by r to obtain for the ith equation

$$ra_{i1}x_1 + ra_{i2}x_2 + \cdots + ra_{in}x_n = rb_i$$

Now factor:

$$r(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n) = rb_i$$

From (1), $a_{i1}s_1 + \cdots + a_{in}s_n = b_i$; therefore,

$$rb_i = rb_i$$

All the other equations being unchanged have $(s_1, \ldots, s_n)$ as a solution also. Therefore $(s_1, \ldots, s_n)$ is a solution of the new system. This argument is reversible, by multiplying by $1/r$.

For part 2, note that interchanging two equations simply reorganizes the system; no algebraic operation has occurred.

PROBLEMS 1.3

1. (**a**) .0008. (**b**) $.5480 \times 10^7$. (**c**) .7930. (**d**) $.5000 \times 10^4$

3. $\dfrac{a/b}{c/d} = \dfrac{-1}{(.0003)/2} = \dfrac{-1}{.0002} = -5000 \quad \dfrac{ad}{bc} = \dfrac{-6}{.0009} = -6667$

5. (**a**) $x_1 = 10, x_2 = 1$ (**b**) $x_1 = -2.571, x_2 = .9997$
(**c**) $x_1 = -2.582, x_2 = 1.001$
The actual solution is $x_1 = -2.580979\ldots, x_2 = 1.000396\ldots$ Sample calculations for a

$$\left(\begin{array}{cc|c} .0001 & 3.172 & 3.173 \\ .6721 & 4.227 & 2.494 \end{array}\right) @> \frac{1}{.0001}R1 >> \left(\begin{array}{cc|c} 1 & 31720 & 31730 \\ .6721 & 4.227 & 2.494 \end{array}\right)$$

Now using $-.6721R1 + R2$ we have

$$\begin{aligned} (-.6721)(31720) + 4.227 &= (-.6721 \times 10^0)(.3172 \times 10^5) + (.4227 \times 10^1) \\ &= -(.2131901 \cdots \times 10^5) + (.4227 \times 10^1) \\ &\overset{\text{Round}}{=} -(.2132 \times 10^5) + .4227 \times 10^1 \\ &= -.2132 \times 10^5 + .00004 \times 10^5 \\ &\overset{\text{Round}}{=} -.2132 \times 10^5 + 0 = -21320 \end{aligned}$$

So we have

$$\left(\begin{array}{cc|c} 1 & 31720 & 31730 \\ 0 & -21320 & -21330 \end{array}\right)$$

Thus after rounding, $x_2 = 1$ and $x_1 = 10$.

7. (**a**) $x_1 = 99.9, x_2 = .999, x_3 = .1$ (**b**) $x_1 = 100, x_2 = 1, x_3 = .1$
(**c**) $x_1 = 100, x_2 = 1, x_3 = .1$
Note: in **b** after partial pivoting and back substitution the equation for x_1 is

$$x_1 = 99.1 + x_2 - x_3 = 99.1 + 1 - .1$$

If we calculate $(99.1+1)-.1$, we get 99.9. If we calculate $99.1+(1-.1)$, we get 100. In the first instance the subtraction of a small number from a large one leads to inaccuracy. The exact solution is $x_1 = 100$, $x_2 = 1$, $x_3 = .1$.

9. (**a**) $(x_1, x_2, x_3) = (1, 1, 1)$. (**b**) $(x_1, x_2, x_3) = (1.09, .49, 1.5)$. Pivoting gives no advantage.

PROBLEMS 1.4

1. (**a**) Row one, column two. (**b**) Row three, column two.
(**c**) Row two, column one. (**d**) Row four, column one.
(**e**) Row two, column three. (**f**) Row six, column one.

3. (**a**) $\left(\begin{array}{rrr|r} 1 & -1 & 1 & 6 \\ 1 & 1 & 2 & 8 \\ 2 & -3 & -1 & 1 \end{array}\right), 3 \times 4$ (**b**) $\left(\begin{array}{rrr|r} 2 & -3 & 1 & 5 \\ 1 & 1 & -1 & 3 \\ 4 & -1 & -1 & 11 \end{array}\right), 3 \times 4$

(**c**) $\left(\begin{array}{rrr|r} 1 & -3 & 1 & 6 \\ 2 & 1 & -3 & -2 \\ 1 & 4 & -4 & 0 \end{array}\right), 3 \times 4$ (**d**) $\left(\begin{array}{rr|r} 1 & -1 & 4 \\ 2 & 1 & 7 \\ 5 & -2 & 19 \end{array}\right), 3 \times 3$

(**e**) $\left(\begin{array}{rrr|r} 3 & -2 & 1 & 12 \\ 2 & -6 & 4 & 6 \end{array}\right), 2 \times 4$ (**f**) $\left(\begin{array}{rr|r} 1 & 2 & 6 \\ 3 & 6 & 8 \\ 5 & 10 & 12 \end{array}\right), 3 \times 3$

5. For example $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and any $B_{3\times 3}$. **7.** $AB = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = BC$.

9. No. $\begin{pmatrix} i & i \\ i & i \end{pmatrix} \begin{pmatrix} i & 2i \\ 2i & i \end{pmatrix} = \begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix}$; other examples are possible.

11. Set $a = GK$ and $b = L$. **13.** All the entries of B are zero.

15. No. Let

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

Other examples are possible.

17. (**a**) Not possible. (**b**) $\begin{pmatrix} 3 & 0 & 9 \\ -2 & 6 & 0 \end{pmatrix}$ (**c**) $\begin{pmatrix} 15 & 60 \\ -4 & 49 \end{pmatrix}$ (**d**) $\begin{pmatrix} -5 \\ 0 \\ 12 \\ 3 \end{pmatrix}$

PROBLEMS 1.5

1. (**a**) $A + B = \begin{pmatrix} 1 & 6 \\ -1 & 7 \end{pmatrix}$ $B + A = \begin{pmatrix} 1 & 6 \\ -1 & 7 \end{pmatrix}$ $A + B = B + A$

(**b**) $A + (B + C) = \begin{pmatrix} 1 & 2 \\ -3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 7 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 4 & 9 \end{pmatrix}$

$(A + B) + C = \begin{pmatrix} 1 & 6 \\ -1 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 4 & 9 \end{pmatrix}$

$A + (B + C) = (A + B) + C$

(**c**) $A(B + C) = \begin{pmatrix} 1 & 2 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 7 & 7 \end{pmatrix} = \begin{pmatrix} 15 & 18 \\ 11 & 2 \end{pmatrix}$

$AB + BC = \begin{pmatrix} 4 & 14 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 11 & 4 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 15 & 18 \\ 11 & 2 \end{pmatrix}$

$A(B + C) = AB + AC$

(**d**) $(B + C)A = \begin{pmatrix} 1 & 4 \\ 7 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -11 & 10 \\ -14 & 28 \end{pmatrix}$

$BA + CA = \begin{pmatrix} -12 & 8 \\ -13 & 14 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -1 & 14 \end{pmatrix} = \begin{pmatrix} -11 & 10 \\ -14 & 28 \end{pmatrix}$

$(B + C)A = BA + CA$

(**e**) $(AB)C = \begin{pmatrix} 4 & 14 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 74 & 28 \\ -6 & -4 \end{pmatrix}$

$A(BC) = \begin{pmatrix} 1 & 2 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 20 & 8 \\ 27 & 10 \end{pmatrix} = \begin{pmatrix} 74 & 28 \\ -6 & -4 \end{pmatrix}$

$(AB)C = A(BC)$

(**f**) $r(A+B) = 4\begin{pmatrix} 1 & 6 \\ -1 & 7 \end{pmatrix} = \begin{pmatrix} 4 & 24 \\ -4 & 28 \end{pmatrix}$

$rA + rB = \begin{pmatrix} 4 & 8 \\ -12 & 8 \end{pmatrix} + \begin{pmatrix} 0 & 16 \\ 8 & 20 \end{pmatrix} = \begin{pmatrix} 4 & 24 \\ -4 & 28 \end{pmatrix}$

$r(A+B) = rA + rB$

(**g**) $(r+s)A = -3\begin{pmatrix} 1 & 2 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -3 & -6 \\ 9 & -6 \end{pmatrix}$

$rA + sA = \begin{pmatrix} 4 & 8 \\ -12 & 8 \end{pmatrix} + \begin{pmatrix} -7 & -14 \\ 21 & -14 \end{pmatrix} = \begin{pmatrix} -3 & -6 \\ 9 & -6 \end{pmatrix}$

$(r+s)A = rA + sA$

(**h**) $(rs)A = -28\begin{pmatrix} 1 & 2 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -28 & -56 \\ 84 & -56 \end{pmatrix}$

$r(sA) = 4\begin{pmatrix} -7 & -14 \\ 21 & -14 \end{pmatrix} = \begin{pmatrix} -28 & -56 \\ 84 & -56 \end{pmatrix}$

$s(rA) = -7\begin{pmatrix} 4 & 8 \\ -12 & 8 \end{pmatrix} = \begin{pmatrix} -28 & -56 \\ 84 & -56 \end{pmatrix}$

$(sr)A = -28\begin{pmatrix} 1 & 2 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -28 & -56 \\ 84 & -56 \end{pmatrix}$

All are equal.

3. Yes. Let $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ with $a_{ij} = 0$, $b_{ij} = 0$, whenever $i \neq j$ (that is, off the diagonal). $A + B = (a_{ij} + b_{ij})$. Now whenever $i \neq j$, $a_{ij} + b_{ij} = 0 + 0 = 0$. Thus $A + B$ is a diagonal matrix.

5. Yes. Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ be symmetric. That is, suppose $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$ for all $i, j = 1, 2, \dots, n$. $A + B = (a_{ij} + b_{ij}) = (c_{ij})$. Now $c_{ij} = a_{ij} + b_{ij} = a_{ji} + b_{ji} = c_{ji}$. Thus $A + B$ is symmetric.

7. (**a**) The product of diagonal matrices is diagonal. Let A and B be diagonal and set $C = AB$. Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Now if $i \neq j$ the sum expands as

$$\sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{lj} + \cdots + a_{ii} b_{ij} + \cdots + a_{ij} b_{jj} + \cdots + a_{ij} b_{nj}$$

and it is seen that every summand is zero because at least one factor of each summand has unequal subscripts. Thus $c_{ij} = 0$ when $i \neq j$ and AB is diagonal.

(**b**) No. Consider $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}$.

(**c**) No. Consider $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ 1 & -2 \end{pmatrix}$.

(**d**) Yes. Let $A_{n\times n}$ and $B_{n\times n}$ be upper triangular. That is $a_{ij} = 0$, $b_{ij} = 0$, whenever $i > j$. Let $AB = C$. Consider

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad \text{for } i > j$$

In the expansion of the sum, when $i > k, a_{ik} = 0$. So

$$c_{ij} = \sum_{k=i}^{n} a_{ik} b_{kj}$$

Now when $k > j, b_{kj} = 0$; in the sum $k \geq i$ and $i > j$, so $k > j$ and

$$c_{ij} = \sum_{k=1}^{n} a_{ik}(0) = 0$$

Thus when $i > j, c_{ij} = 0$.

9. (**a**) $(A+B)^2 = \begin{pmatrix} -5 & 48 \\ -8 & 43 \end{pmatrix}$ (**b**) $A^2 + 2AB + B^2 = \begin{pmatrix} 11 & 54 \\ 9 & 27 \end{pmatrix}$

11. (**a**) $\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$

(**b**) Yes. Let A and B be skew-symmetric, so that $A^T = -A, B^T = -B$. Now

$$(A+B)^T = A^T + B^T = -A + (-B) = -(A+B)$$

so $A + B$ is skew-symmetric.

(**c**) No. Let A be skew-symmetric, so that $A^T = -A$. $(A^2)^T = (AA)^T = A^T A^T = (-A)(-A) = A^2$. Therefore A^2 is actually symmetric. (Note: $A = 0$ is both symmetric and skew-symmetric.)

(**d**) Yes. Let A be skew-symmetric so that $A^T = -A$. Now $(A^3)^T = (A^T)^3 = (-A)^3 = -A^3$, so A^3 is skew-symmetric.

(**e**) Not much unless $AB = BA$. Let A be symmetric, B be skew-symmetric so that $A^T = A$, $B^T = -B$. Now $(AB)^T = B^T A^T = -BA$. If $AB = BA$, then $(AB)^T = -BA = -AB$ and AB is skew-symmetric.

(**f**) The diagonal must consist of all zeros. After all, if $a_{ij} = -a_{ji}$ then when $i = j$ we have $a_{ii} = -a_{ii}$ which can be satisfied if and only if $a_{ii} = 0$.

13. If and only if $AB = BA$. Calculating, we have

$$(A+B)(A-B) = A(A-B) + B(A-B) = A^2 - AB + BA - B^2$$

15. Calculate.

17. (**a**) No. Consider $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$.

$$AD = \begin{pmatrix} 3 & 8 \\ 5 & 12 \end{pmatrix} \neq \begin{pmatrix} 3 & 4 \\ 10 & 12 \end{pmatrix} = DA.$$

(**b**) If D is a scalar matrix $AD = DA$ by Theorem 1.5.2 parts g and h together.

19. Let $A = (a_{ij}), a_{ij} \in R$. Because a_{ij} is real $\overline{a_{ij}} = a_{ij}$. Thus $\bar{A} = (\overline{a_{ij}}) = (a_{ij}) = A$.

21. $C = \begin{pmatrix} 3+i & 1+i \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} i & i \\ 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}}_{A} + i \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}_{B}$

$D = \begin{pmatrix} 4+4i \\ 2i \end{pmatrix} = \underbrace{\begin{pmatrix} 4 \\ 0 \end{pmatrix}}_{R} + i \underbrace{\begin{pmatrix} 4 \\ 2 \end{pmatrix}}_{S}$.

Using Prob. 20 system is

$$\begin{pmatrix} 3 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 4 \\ 2 \end{pmatrix}$$

23. (**a**) $A^{**} = (A^*)^* = (\overline{a_{ji}})^* = (\overline{\overline{a_{ij}}}) = (a_{ij}) = A.$

(**b**) $(A+B)^* = (\overline{A+B})^T = (\overline{A} = \overline{B})^T + \overline{A}^T + \overline{B}^T = A^* + B^*.$

(**c**) $(AB)^* = (\overline{AB})^T = (\bar{A}\overline{B})^T = \overline{B}^T A^T = B^* A^*.$

25. $(A^*A)^* = A^*A^{**} = A^*A.$

27. $(A+A^T)^T = A^T + A^{T^T} = A^T + A = A + A^T.$

29. $A = \underbrace{\tfrac{1}{2}(A+A^T)}_{\text{Symmetric}} + \underbrace{\tfrac{1}{2}(A-A^T)}_{\text{Skew-symmetric}}$

31. (**a**) and (**b**) simply calculate.

(**c**) A looks like $\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ & & & & \ddots & 1 \\ 0 & 0 & 0 & & \cdots & 0 \end{pmatrix}$. As powers of A are computed the diagonal of 1's moves toward the upper right. $A^n = 0$.

33. Let $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $V^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and

$$V - V^T = \begin{pmatrix} 0 & b-c \\ -(b-c) & 0 \end{pmatrix}$$

(**a**) Solve
$\begin{pmatrix} 0 & b-c \\ -(b-c) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{array}{r} b-c=1 \\ -(b-c)=1 \end{array} \Rightarrow$ No solution.

(**b**) Solve
$\begin{pmatrix} 0 & b-c \\ -(b-c) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{array}{r} b-c=-1 \\ -(b-c)=1 \end{array} \Rightarrow b = c - 1$. a and d are unrestricted.

$$V = \begin{pmatrix} a & c-1 \\ c & d \end{pmatrix}$$

(**c**) V can be any symmetric matrix.

35. Because there are no zero rows, and the number of columns equals the number of rows, the reduced row echelon form at least looks like

$$\begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & ? & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Now by definition of reduced row echelon form only zeros may lie above each leading 1 in a row. Therefore the form is I_n.

37. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
A and $C$are symmetric
B is skew-symmetric
B is Hermitian.

39. Use induction. For $n = 1$, $(AB)^1 = AB = A^1B^1$. On the other hand, by commutativity, $AB = BA = B^1A^1$, so $A^1B^1 = B^1A^1$. The induction hypothesis is that (for $n = k-1$) $(AB)^{k-1} = A^{k-1}B^{k-1} = B^{k-1}A^{k-1}$. Now to prove $(AB)^k = A^kB^k = B^kA^k$ we have $(AB)^k = (AB)^{k-1}(AB) = A^{k-1}B^{k-1}AB = A^{k-1}BB\cdots BBAB$. Using $BA = ABk-1$ times the last term is equal to $A^{k-1}A\ BB\cdots BBB = A^kB^k$. The argument for $(BA)^k$ works the same way.

PROBLEMS 1.6

1. (**a**) -7. (**b**) 3. (**c**) -543. **3.** (**a**) 0. (**b**) 56. (**c**) 0.

5. (**a**) $x = \frac{16}{11} + \frac{7i}{11}, y = \frac{24}{11} - \frac{28}{11}i$. (**b**) $(x, y, z) = \left(-\frac{5}{3}, -\frac{8}{3}, -\frac{2}{3}\right)$.

7. $\det(AB) = \det A \det B$; $\det(BA) = \det B \det A$; Now $\det A, \det B$ are just complex numbers. So $\det A \det B = \det B \det A$. Therefore $\det(AB) = \det(BA)$.

9. $\det(AB) = \det \begin{pmatrix} 16 & 19 \\ 9 & 51 \end{pmatrix} = 645 \quad \det A \det B = (15)(43) = 645.$

11. Use induction. $P(n)$ is: If A is $n \times n$, then $\det A^T = \det A$.

1. $P(2)$ is true. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \det A = ad - bc$$
$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \det A = ad - bc$$

Therefore $\det A = \det A^T$.

2. Suppose $P(k-1)$ is true and prove $P(k)$ is true. Consider

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{11} & \cdots & a_{k1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{kk} \end{pmatrix}$$

Let A_{ij} = submatrix of A generated by deleting row i and column j.

Let B_{ij} = submatrix of A^T generated by deleting row i and column j.

Note that $A_{ij} = {B_{ji}}^T$ and since A_{ij} and B_{ji} are $k-1 \times k-1$, we have by the induction hypothesis that $\det B_{ji} = \det {B_{ji}}^T = \det A_{ij}$. Now using the last row for A

$$\det A = (1)^{k+1} a_{k1} \det A_{k1} + \cdots + (-1)^{2k} a_{kk} \det A_{kk}$$

and using the last column for B,

$$\begin{aligned} \det A^T &= (-1)^{k+1} a_{k1} \det B_{1k} + \cdots + (-1)^{2k} a_{kk} \det B_{kk} \\ &= (-1)^{k+1} a_{k1} \det A_{k1} + \cdots + (-1)^{2k} a_{kk} \det A_{kk} \\ &= \det A \end{aligned}$$

13. $\det A^* = \det(\overline{A})^T = \det \overline{A} \underset{\substack{\uparrow \\ \text{Prob.} \\ 12}}{=} \overline{\det A}$

15. cI_n is certainly upper triangular, so its determinant is the product of the diagonal elements. So $\det cI_n = c \cdot c \cdots c = c^n$.

17. Because $A^2 = A, \det(A^2) = \det A$. Thus

$$\begin{aligned}
\det(AA) &= \det A \\
\det A \det A &= \det A \\
(\det A)^2 - \det A &= 0 \\
\det A(\det A - 1) &= 0 \\
\det A &= 0 \quad \text{or} \quad 1
\end{aligned}$$

19. The determinant must be 0. Because

$$A^n = 0, \quad \det(A^n) = \det 0$$

So

$$\begin{aligned}
\det(AA \cdots A) &= 0 \\
(\det A)(\det A) \cdots (\det A) &= 0 \\
(\det A)^n &= 0 \\
\det A &= 0
\end{aligned}$$

21. No. For example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

both have determinant zero, but are not equal.

23. The determinant is zero **if n is odd**. We have $A^T = -A$ so

$$\det A^T = \det(-A) = (-1)^n \det A$$

On the other hand, $\det A^T = \det A$ (always) so

$$\det A = (-1)^n \det A$$

If n is even there is no information. However if n is odd we have $\det A = -\det A$ and therefore $\det A = 0$.

25. $\det H_2 = \dfrac{1}{12} \quad \det H_3 = \dfrac{1}{2160} \quad \det H_4 = \dfrac{1}{6048000}$

PROBLEMS 1.7

1. (**a**) $A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$ (**b**) $A^{-1} = \begin{pmatrix} \frac{2}{5} - \frac{1}{5}i & \frac{4}{5} - \frac{2}{5}i \\ 0 & -\frac{1}{3} \end{pmatrix}$

(**c**) $A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A$ (**d**) $A^{-1} = \dfrac{1}{111} \begin{pmatrix} 14 & -1 & -9 \\ 11 & 23 & -15 \\ 40 & 13 & 6 \end{pmatrix}$

(**e**) $A^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 3 & \frac{5}{2} \end{pmatrix}$

3. $\dfrac{1}{3}\begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix}$ **5.** Reader's choice!

7. $\det(AB) = \det A \det B = (\det A)0 = 0$. Because $\det(AB) = 0$, AB is not invertible.

9. A is nonsingular so A^{-1} exists. Multiply $AB = AC$ on both sides by A^{-1}.

$$\begin{aligned} A^{-1}(AB) &= A^{-1}(AC) \\ (A^{-1}A)B &= (A^{-1}A)C \\ IB &= IC \\ B &= C \end{aligned}$$

Note that the multiplication was from the left—this is called **premultiplying**.

11.

$$\begin{aligned} AA^T &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \end{aligned}$$

Because the inverse of a matrix is unique, $A^T = A^{-1}$.

13. Because $A^{-1} = A^*$, $\det A^{-1} = \det A^* = \overline{\det A}$, so $1/\det A = \overline{\det A}$ and $1 = \det A\overline{\det A} = |\det A|^2$. $|\det A|$ being positive then must be equal to 1.

15. No. Let A be 3×3 and skew-symmetric, $A^T = -A$. Now $\det A^T = \det(-A) = (-1)^3 \det A = -\det A$. Also $\det A^T = \det A$, so $\det A = -\det A$ and we must have $\det A = 0$. Generalization: If n is odd, $A_{n\times n}$ is skew-symmetric, then $\det A = 0$ and A is not invertible (see Prob. 23 in Prob. Set 1.6).

17. If A is nilpotent of exponent k, $A^k = 0$. Now $0 = \det 0 = \det(A^k) = (\det A)^k$, so $\det A = 0$ and A is singular.

19. If AB is invertible $\det(AB) \neq 0$. Thus $(\det A)(\det B) \neq 0$ and neither $\det A$ or $\det B$ can be zero. Hence both A and B are invertible.

21. We have $A^{-1} = B^{-1}$. Multiply both sides by B (from the right):

$$\begin{aligned} A^{-1}B &= B^{-1}B \\ A^{-1}B &= I \end{aligned}$$

Now multiply both sides by A (from the left):

$$\begin{aligned} A(A^{-1}B) &= AI \\ (AA^{-1})B &= A \\ IB &= A \\ B &= A \end{aligned}$$

23. In the last section each determinant was found to be nonzero so H_2, H_3, and H_4 are invertible. Using the values from the last section

$$\begin{aligned} \det H_2^{-1} &= \frac{1}{\det H_2} = 12 \\ \det H_3^{-1} &= \frac{1}{\det H_3} = 2160 \\ \det H_4^{-1} &= \frac{1}{\det H_4} = 6,048,000 \end{aligned}$$

PROBLEMS 1.8

1. (**a**) $L = \begin{pmatrix} 1 & 0 \\ 6721 & 1 \end{pmatrix}$ $\quad U = \begin{pmatrix} 0.0001 & 3.172 \\ 0 & -21314.785 \end{pmatrix}$

(**b**) $L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ \frac{1}{6} & -\frac{10}{17} & 1 \end{pmatrix}$ $\quad U = \begin{pmatrix} 3 & -2 & 4 \\ 0 & -\frac{17}{30} & \frac{1}{3} \\ 0 & 0 & -\frac{75}{51} \end{pmatrix}$

(**c**) $L = \begin{pmatrix} 1 & 0 & 0 \\ -100 & 1 & 0 \\ 0 & \frac{1}{99} & 1 \end{pmatrix}$ $\quad U = \begin{pmatrix} 0.01 & -1 & 0 \\ 0 & -99 & -1 \\ 0 & 0 & \frac{991}{99} \end{pmatrix}$

3. (**a**) Is not. For the second row $2 \not> |-1| + |-1|$. (**b**) Is.

(**c**) Is not. For a skew-symmetric matrix, the diagonal elements are all zero.

(**d**) Is.

5. Not necessarily. Consider

$$A = \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & 3 \\ 2 & 3 \end{pmatrix}$$

7. $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ $\quad U = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

9. (**a**) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

These are not positive definition because they are symmetric with negative determinant.

(**b**) $\begin{pmatrix} 1 & 2 \\ 2 & -8 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 2 & 0 \\ 2 & -8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

These are symmetric and the required "subdeterminants" are positive, so they are positive definite, but the first row of each precludes **SRD**.

11. $L = I, \quad U = I. \quad L = rI \quad U = \frac{1}{r}I, \quad r \neq 0.$

ADDITIONAL PROBLEMS (CHAPTER 1)

1. Yes, adjoin an equation inconsistent with a previous equation. For example

$$\begin{aligned} x + y &= 2 \\ x - y &= 0 \end{aligned}$$

has a unique solution. Adjoin $x + y = 3$.

3. If A is invertible the answer is yes, because $X = A^{-1}B$ and both A^{-1} and B have only real entries. If A is not invertible, then complex solutions are possible unless for some reason they have been removed from the discussion. For instance

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s \\ s \end{pmatrix}$$

where s is completely arbitrary and could be assigned any complex value.

5. Not necessarily. Consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$
$$A + iB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

If A and B are real $A + iB$ still need not be invertible. Consider

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$A + iB = \begin{pmatrix} 1+i & 2 \\ 1 & 1-i \end{pmatrix} \quad \text{and} \quad \det(A + iB) = 0$$

7. Not necessarily.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \geq \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = B$$

but $\det A = -1 < 1 = \det B$.

9.

$$\begin{aligned}(A+I)(A-I) &= A(A-I) + I(A-I) \\ &= A^2 - A + A - I \\ &= A^2 - I \\ (A-I)(A+I) &= (A-I)A + (A-I)I \\ &= A^2 - IA + AI - I^2 \\ &= A^2 - A + A - I \\ &= A^2 - I.\end{aligned}$$

11. The dominance matrix is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} \quad A^3 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

$$A + A^2 + A^3 = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 2 \end{pmatrix}$$

The powers for competitors 1 through 4 are 7, 6, 4, and 8, respectively. Predict competitor number 4 to win. (But there are various possibilities—what if competitors 1 and 2 decide to work together until competitor 4 is eliminated? Then competitor 2 leaks the plan to competitor 3 and makes the deal that once competitor 4 is eliminated, competitors 2 and 3 work to eliminate competitor 1.)

13. $BA = \begin{pmatrix} 1 & R_1 \\ 1/R_2 & 1 + R_1/R_2 \end{pmatrix} \quad AB \neq BA$
The combined circuits are not equivalent.

15. $A+B=\begin{pmatrix} 2 & R_1 \\ 1/R_2 & 2 \end{pmatrix}$
In parallel circuits, the placement above and below of the boxes is immaterial.

PROBLEMS 2.1

3. (**a**) (2,6); (**b**) $(-6,-9)$; (**c**) $(3,-2)$; (**d**) $(6,-9)$; (**e**) $(-1,-1,-8)$; (**f**) (4,1,1).

5. (**a**) $\sqrt{5}$; (**b**) $\sqrt{10}$; (**c**) $\sqrt{5}$; (**d**) $\sqrt{\pi^2+1}$. **7.** $k=\pm\sqrt{26}/26$.

11. In R^2, let $A=(a,b)$. So

$$|A|=\sqrt{a^2+b^2} \quad \text{and} \quad \frac{1}{|A|}A=\left(\frac{a}{\sqrt{a^2+b^2}},\frac{b}{\sqrt{a^2+b^2}}\right)$$

The length of $(1/|A|)A$ is

$$\sqrt{\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}}=\sqrt{1}=1$$

13. Let $\mathbf{v}=(a,b)$. Then $r\mathbf{v}=(ra,rb)$.

$$(r\mathbf{v})_s=\begin{pmatrix} ra \\ rb \end{pmatrix}$$
$$r\mathbf{v}_s=r\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} ra \\ rb \end{pmatrix}$$

15. Let $\mathbf{v}=(v_1,v_2,v_3)$ and $A=(a_{ij})$. The entries of $A\mathbf{v}_s$ are

$$c_i=\sum_{k=1}^{3}a_{ik}v_k$$

The entries of $(\mathbf{v}A^T)_s$ are

$$d_i=\sum_{k=1}^{3}v_k a_{ik}=\sum_{k=1}^{3}a_{ik}v_k$$

Thus $c_i=d_i$, $i=1,2,3$, and $A\mathbf{v}_s=(\mathbf{v}A^T)_s$.

PROBLEMS 2.2

1. (**a**) 17; (**b**) -3; (**c**) 4; (**d**) -1.

3. (**a**) $17\sqrt{290}/290$; (**b**) $-\sqrt{145}/145$; (**c**) $2\sqrt{410}/205$; (**d**) $-\sqrt{2}/22$.

5. (**a**) $\frac{17}{5}(1,2)$; (**b**) $\frac{-3}{29}(-5,2)$; (**c**) $\frac{4}{41}(5,4,0)$; (**d**) $-\frac{1}{2}(0,1,-1)$.

7. By direct calculation $\mathbf{i}\cdot\mathbf{j} = (1,0,0)\cdot(0,1,0) = 0$ Similarly $\mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0$.

9. Place the cube so that one corner is at the origin and the three edges coming from the corner lie in the x, y, and z axes. If the length of each side is s, then the diagonal forms a vector (s,s,s). The angle between the edge in the x axis and the diagonal is the angle between $(s,0,0)$ and (s,s,s).

$$\cos\theta = \frac{(s,0,0)\cdot(s,s,s)}{|(s,0,0)||(s,s,s)|} = \frac{s^2}{\sqrt{s^2}\sqrt{3s^2}} = \frac{\sqrt{3}}{3}$$
$$\theta = \cos^{-1}\frac{\sqrt{3}}{3} \doteq .955 \text{ radius } \doteq 54.7^\circ$$

11.

$$\begin{aligned} |A+B|^2 &= |A|^2 + |B|^2 - 2|A||B|\cos(\pi-\theta) \\ &= |A|^2 + |B|^2 + 2|A||B|\cos\theta \qquad (1) \\ |A-B|^2 &= |A|^2 + |B|^2 - 2|A||B|\cos\theta \qquad (2) \end{aligned}$$

(**a**) Take (1) $-$ (2) and use $A\cdot B = |A||B|\cos\theta$. (**b**) Take (1) + (2).

13. $B_{\text{proj } A} = \dfrac{A\cdot B}{|A|^2}A$ so

$$\begin{aligned} [B - B_{\text{proj } A}]\cdot A &= \left(B - \frac{A\cdot B}{|A|^2}A\right)\cdot A = B\cdot A - \frac{(A\cdot B)(A\cdot A)}{|A|^2} \\ &= B\cdot A - A\cdot B = 0 \end{aligned}$$

PROBLEMS 2.3

1. (**a**) Dilation, constant 3; (**b**) contraction, constant $\frac{1}{2}$; (**c**) dilation, constant 2; rotation, π radians.

$$R_\pi D_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

The order can be reversed.

3. (**a**) One possibility is rotation of $3\pi/2$ radians and then projection onto x_1 axis:

$$P_{x_1} R_{3\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Note that $R_{3\pi/2} P_{x_1}$ does not work. Another possibility is to project on the x_2 axis first and then rotate $3\pi/2$ radians:

$$R_{3\pi/2} P_{x_2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(**b**) Rotation of $\pi/2$ radians clockwise.

5. (**a**) $D_k C_{1/k} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} 1/k & 0 \\ 0 & 1/k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(**b**) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$

$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(**c**) $P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; $\det P_x = 0$. So P_x is not invertible.

7. If $x_1 > 0, x_2 > 0$ then

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and f is the identity. If $x_1 < 0$ and $x_2 < 0$,

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = R_\pi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Similar statements can be made for the other cases: $x_1 < 0$, $x_2 > 0$ (reflection) and $x_1 > 0, x_2 < 0$ (reflection). That is, in each quadrant, f is represented by matrix multiplication; f is "linear on each quadrant."

PROBLEMS 2.4

1. (**a**) $x + 2y + 3x = 0$; (**b**) $3x - y = 0$; (**c**) $x + 2 = 0$;
(**d**) $x + y + z = 1$.

3. (**a**) $(-11, 4, 1) \cdot (x - 2, y - 1, z - 3) = 0; -11x + 4y + z = -15$;
(**b**) $(1, 1, 1) \cdot (x + 1, y, z) = 0$; $x + y + z = -1$;
(**c**) $(1, 0, 0) \cdot (x - 2, y - 4, z - 6) = 0; x = 2$.

5. (**a**) $x = t + 1, y = -t, z = 0$; (**b**) $x = t, y = t, z = t$;
(**c**) $x = t, y = t, z = 1 - t$; (**d**) $x = 1 + t, y = -t, z = 1$.

7. $z = 0, y = 0, x = 0$.

9. (**a**) $N = (3, -3, -1)$ is normal to the plane and $V = (2, 1, 3)$ is parallel to the line. $N \cdot V = 0$.

(**b**) Substitution of the expressions for x, y, and z into $3x - 3y - z = 6$ leads to $6t - 18 - 3t - 12 - 3t + 6 = 6$ or $-24 = 6$.

ADDITIONAL PROBLEMS (CHAPTER 2)

1. $B = (0, \frac{3}{2}), A = (\sqrt{3}, 1)$. Let $C = (c_1, c_2)$. $A + B + C = (\sqrt{3}, \frac{5}{2}) + (c_1, c_2) = 0$. $c_1 = -\sqrt{3}, c_2 = -\frac{5}{2}$.

3. The homogeneous equations are

$$\begin{aligned} 7w_1 - 6w_2 - 11w_3 &= 0 \\ 2w_2 - 3w_3 &= 0 \\ 7w_1 - 8w_2 - 8w_3 &= 0 \end{aligned}$$

The reduced equations are

$$\left(\begin{array}{ccc|c} 7 & -6 & -11 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right)$$

The solution is $(\frac{20}{7}t, \frac{3}{2}t, t)$.

5. The system of equations can be written

$$\begin{pmatrix} 1 & 1 & -1 \\ -2 & k & 0 \\ 7 & 5-k & -5 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The system has a nontrivial solution if the determinant of the coefficient matrix is zero. The determinant is zero regardless of the value of k. Thus the system can be balanced for any k.

7. If A and B are component forces and

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{then} \quad R_\theta A + R_\theta B = R_\theta(A+B)$$

The right-hand side represents the rotation of the resultant vector.

9. Let A and B be the component forces and let C_k be the contraction matrix. Then $C_kA+C_kB = C_k(A+B)$. The right-hand side represents the contraction of the resultant vector.

PROBLEMS 3.1

1. $B = E, A = C$.

3. Choose for example

$$\mathbf{x} = (1,1), \mathbf{y} = (1,2), \mathbf{z} = (2,1)$$
$$(\mathbf{x}-\mathbf{y})-\mathbf{z} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$
$$\mathbf{x}-(\mathbf{y}-\mathbf{z}) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

5. $(3,-2,1) = 3(1,0,0) - 2(0,1,0) + 1(0,0,1)$.

7. $\det\begin{pmatrix} 1 & -2 & 3 \\ 3 & 1 & 0 \\ 4 & -1 & 3 \end{pmatrix} = 0.$ **9.**. $\det\begin{pmatrix} 1 & -2 & 3 \\ 3 & 1 & 0 \\ a+3b & -2a+b & 3a \end{pmatrix} = 0$

11. The equation is equivalent to

$$\begin{aligned} 2c_1 - c_2 + c_3 &= 1 \\ 3c_1 + 3c_3 &= -1 \\ 5c_1 + 6c_2 + 11c_3 &= 4 \end{aligned}$$

These reduce to

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 1 \end{array}\right)$$

13. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

$$\begin{aligned} \mathbf{x} + \theta &= (x_1, x_2, \ldots, x_n) + (0, 0, \ldots, 0) = (x_1 + 0, x_2 + 0, \ldots, x_n + 0) \\ &= (x_1, x_2, \ldots, x_n) = \mathbf{x} \end{aligned}$$

Example from E^4: Let $\mathbf{x} = (1, -6, 4, 2)$.

$$(1, -6, 4, 2) + (0, 0, 0, 0) = (1 + 0, -6 + 0, 4 + 0, 2 + 0) = (1, -6, 4, 2)$$

15. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$

$$\begin{aligned} (r+s)\mathbf{x} &= (r+s)(x_1, x_2, \ldots, x_n) = ((r+s)x_1, (r+s)x_2, \ldots, (r+s)x_n) \\ &= (rx_1 + sx_1, rx_2 + sx_2, \ldots, rx_n + sx_n) \\ &= (rx_1, rx_2, \ldots, rx_n) + (sx_1, sx_2, \ldots, sx_n) \\ &= r(x_1, x_2, \ldots, x_n) + s(x_1, x_2, \ldots, x_n) = r\mathbf{x} + s\mathbf{x} \end{aligned}$$

Example in E^5:

$$\begin{aligned} (7+4)(1, -2, 0, 6, -3) &= 11(1, -2, 0, 6, -3) \\ &= (11, -22, 0, 66, -33) \\ 7(1, -2, 0, 6, -3) + 4(1, -2, 0, 6, -3) &= (7, -14, 0, 42, -21) \\ &\quad + (4, -8, 0, 24, -12) \\ &= (11, -22, 0, 66, -33) \end{aligned}$$

PROBLEMS 3.2

1. V is a vector space. **3.** V is a vector space.
5. V is a vector space. **7.** V is a vector space.
9. V is a vector space. **11.** V is a vector space.

13. V is not a vector space. Closure for addition fails to hold. For example I and $-I$ are in V, but $I + -I = 0$ is not in V.

15. V is not a vector space. Closure for addition fails to hold. For example consider in the case $n = 2$:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Now $A^2 = B^2 = 0$ so A and B are nilpotent. However, $(A+B)^2 = I \neq 0$ and no power of $A + B$ is 0.

17. V is not a vector space. Closure for scalar multiplication fails to hold. Let A be in V so that $A^2 = I$. Consider $2A$: $(2A)^2 = 4A^2 = 4I \neq I$.

19. V is a vector space. The verification is virtually the same as in example 7 of this section.

21. V is a vector space,

23. $$\begin{aligned} -\mathbf{x} &= (-\mathbf{x}) + \theta = (-\mathbf{x}) + \mathbf{0x} = (-\mathbf{x}) + (\mathbf{1} + (-\mathbf{1}))\mathbf{x} \\ &= (-\mathbf{x}) + 1\mathbf{x} + (-1)\mathbf{x} = (-\mathbf{x}) + \mathbf{x} + (-1)\mathbf{x} \\ &= \mathbf{x} + (-\mathbf{x}) + (-1)\mathbf{x} = \theta + (-1)\mathbf{x} = (-1)\mathbf{x} \end{aligned}$$

25. V is a vector space.

27. No. The existence of vector θ in 4 is given in 3.

PROBLEMS 3.3

1. W is not subspace because W is not closed under scalar multiplication. For example, if A is in W, $(-1)A$ is not in W.

3. W is a subspace.

5. W is not a subspace because W is not closed under vector addition. For example

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 2 & 3 \end{pmatrix}$$

are in W but the sum is not.

7. W is a subspace.

9. W is not a subspace. Note that scalars can have nonzero imaginary part. So, for example, if A is in W, iA is not in W and W fails to be closed under scalar multiplication.

11. W is a subspace. **13.** W is a subspace.

15. Since the set of solutions is a subset W of C^n, the usual operations are those of C^n (remember a sub**space** is a sub**set** which is a vector space). Now if X and Y are any solutions, then

$$A(X+Y) = AX + AY = 0 + 0 = 0 \quad \text{and} \quad A(cX) = c(AX) = c0 = 0$$

by laws of matrix algebra. Thus W is a subspace.

17. $\text{tr } A = \sum_{i=1}^{n} a_{ii} \qquad \text{tr } B = \sum_{i=1}^{n} b_{ii}$

$$\text{tr } (A+B) = \sum_{i=1}^{n} (a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \text{tr } A + \text{tr } B$$

Also

$$r \text{ tr } (A) = r \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} r a_{ii} = \text{tr } (rA)$$

For the last part, let $AB = C, BA = D$, so that

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad d_{ij} = \sum_{q=1}^{n} b_{iq} a_{qj}$$

Now

$$\text{tr }(AB) = \text{tr }C = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} a_{ik}b_{ki}\right) \quad \text{and} \quad \text{tr }(BA)$$
$$= \text{tr }D = \sum_{p=1}^{n} d_{pp} = \sum_{p=1}^{n}\left(\sum_{q=1}^{n} b_{pq}a_{qp}\right)$$

Working with the last series we interchange the order of summation and use commutativity in R to obtain

$$\text{tr }(BA) = \sum_{q=1}^{n}\left(\sum_{p=1}^{n} a_{qp}b_{pq}\right)$$

Now the subscripts are only indices which can be renamed without changing the value of the sum. Put $q = i, p = k$ to find

$$\text{tr }(BA) = \sum_{i=1}^{n} a_{ik}b_{ki} = \text{tr }(AB)$$

19. Let (x_1, x_2) and (y_1, y_2) be in W. Then $x_2 = mx_1, y_2 = my_1$. Now $c(x_1, x_2) = (cx_1, cx_2)$ and $(cx_2) = cmx_1 = m(cx_1)$ and (cx_1, cx_2) is in W. Also $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $(x_2 + y_2) = mx_1 + my_1 = m(x_1 + y_1)$ and $(x_1 + y_1, x_2 + y_2)$ is in W. Now in the second case if $x_2 = mx_1 + b$, $b \neq 0$, then $2(x_1, x_2)$ is not in W because $2x_2 = 2(mx_1 + b) = m(2x_1) + 2b$ and $2b \neq 0$.

21. If W is a subspace, then $n(0, 0, 0)$ is in W and the equations in the definition must hold for some t. If the equations hold for some $t = T$ we have $k_1 = -aT$, $k_2 = -bT$ and $k_3 = -cT$. Therefore

$$W = \{(x_1, x_2, x_3) | (x_1, x_2, x_3) = (a(t-T), b(t-T), c(t-T)) \quad t \text{ in } R\}$$
$$= \{(x_1, x_2, x_3) | (x_1, x_2, x_3) = (a\tau, b\tau, c\tau) \quad \tau \text{ in } R\}$$

Now the closure is not hard to show. If $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are in W then

$$\begin{aligned} x_1 + y_1 &= a\tau + a\sigma \\ x_2 + y_2 &= b\tau + b\sigma \qquad \sigma, \tau \in R \\ x_3 + y_3 &= c\tau + c\sigma \end{aligned}$$

and $(x_1 + y_1, x_2 + y_2, x_3 + y_3) = (a(\tau + \sigma), b(\tau + \sigma), c(\tau + \sigma))$, $\tau + \sigma \in R$. Also $(rx_1, rx_2, rx_3) = (a(r\tau), b(r\tau), c(r\tau))$, $r\tau \in R$.

PROBLEMS 3.4

1. (**a**) (a, b, c) is in span (S) if and only if $c = \frac{2}{3}b - \frac{4}{3}a$;

(**b**) S is linearly independent.

3. (**a**) (a, b, c) is in span (S) if and only if $c = a + ib$;

(**b**) S is linearly dependent; (**c**) $(i, i-1, -1) = i(1, i, 0) + i(0, 1, i)$.

5. (**a**) Span(S) is E^2; (**b**) S is linearly dependent;

(**c**) $(7, 12) = \frac{9}{2}(1, 1) + \frac{5}{2}(2, 6)$.

7. (**a**) (a, b, c) is in span (S) if and only if (a, b, c) is a scalar multiple of $(1, -1, 2)$;

(**b**) S is linearly dependent; (**c**) $(0, 0, 0) = 0(1, -1, 2)$.

9. (**a**) Span (S) is all 2×2 real matrices with trace zero;
(**b**) S is linearly independent.

11. Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be two vectors from E^3. Try to solve $c_1\mathbf{x} + c_2\mathbf{y} = (a, b, c)$. The resulting equations are represented by the augmented matrix

$$\left(\begin{array}{cc|c} x_1 & y_1 & a \\ x_2 & y_2 & b \\ x_3 & y_3 & c \end{array}\right)$$

Row reduction leads to a third equation of $0 =$ (expression in a, b, c) and (a, b, c) cannot be arbitrary.

13. Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2), \mathbf{z} = (z_1, z_2)$ and consider $c_1\mathbf{x}+c_2\mathbf{y}+c_3\mathbf{z} = (0, 0)$. The resulting equations are represented by the matrix

$$\left(\begin{array}{ccc|c} x_1 & y_2 & z_1 & 0 \\ x_2 & y_2 & z_2 & 0 \end{array}\right)$$

Because there are fewer equations than unknowns, a nontrivial solution exists.

15. The elements of span (S) are of the form $a + bx^2$ where a and b are arbitrary.

17. The elements of span (S) are real matrices of the form

$$A = \begin{pmatrix} b & a \\ a & -b \end{pmatrix}$$

where a and b are arbitrary. Clearly A is symmetric and has trace zero.

19. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be linearly dependent and let $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1}, \ldots, \mathbf{v}_n\}$. Consider

$$c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_n + c_{m+1}\mathbf{v}_{m+1} + \cdots + c_n\mathbf{v}_n = \theta$$

There exists a nontrivial solution to this equation. In fact, put $c_{m+1} = 0,\ c_{m+2} = 0, \ldots, c_n = 0$ and we have

$$c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m = \theta$$

which has a nontrivial solution by the linear dependence of S.

21. $[A, I] = 0 \quad [A, B] = \begin{pmatrix} -1 & 15 \\ -12 & 1 \end{pmatrix} \quad [B, A] = \begin{pmatrix} 1 & -15 \\ 12 & -1 \end{pmatrix}$
$[A, A] = 0$

23. If $AB = BA$, then $AB - BA = 0$ and $[A, B] = 0$; If $[A, B] = 0$, then $AB - BA = 0$ and $AB = BA$.

PROBLEMS 3.5

1. (**a**) Is a basis; (**b**) is not a basis; linearly dependent set;

(**c**) is not a basis; does not span E^2;

(**d**) is not a basis; linearly dependent set;

(**e**) is not a basis; linearly dependent set.

3. $\{(1, -1, 0), (0, 1, -1), (0, 0, 1)\}$ is a basis containing S. There are other possible solutions.

5. $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is a basis containing S. There are other possible solutions.

7. $\{(i,1,0),(0,i,1),(0,0,1)\}$ is a basis containing S. There are other possible solutions.

9. S is linearly dependent and any vector in S can be written as a nontrivial linear combination of the others. Deletion of any vector leads to a basis for span (S).

11. Delete $x-x^2$. If $x+x^2$ is deleted the remaining set is linearly dependent.

13. The set

$$\{(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1)\}$$

is a basis for E^5; $\dim E^5 = 5$.

15. $S = \left\{\begin{pmatrix}1&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&0&0&0\end{pmatrix},\right.$ $\left.\begin{pmatrix}0&0&0&0\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&0&0\\0&0&0&1\end{pmatrix}\right\}$ is a basis for M_{24}; $\dim M_{24} = 8$.

17. $S = \left\{\begin{pmatrix}i&0\\0&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\-i&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&0\\1&0\end{pmatrix},\right.$ $\left.\begin{pmatrix}0&0\\0&0\\0&i\end{pmatrix}\right\}$ is a basis for C_{32}; $\dim C_{32} = 6$. The standard basis for M_{32} is also a basis for C_{32}.

19. Form the basis $S = \{M^{ij}, i,j = 1,2,\ldots,n$, such that the entry in the ith row and jth column is 1 and all other entries are $0\}$. S has n^2 elements, so $\dim M_{nn} = n^2$. A basis for the symmetric matrices is, using the notation above

$$T = \{M^{11}, M^{22}, \ldots, M^{nn}, M^{ij} + M^{ji}, i > j\}.$$

For example if $n = 3$

$$T = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

The number of elements of T is $n + [(n^2 - n)/2] = (n^2 + n)/2$. The dimension of the subspace of symmetric matrices is $(n^2 + n)/2$.

21. The dimension of both is mn. Construct bases using Probs. 15, 17, and 19 as a guide.

23. (**a**) rank $A = 2$; (**b**) rank $A = 4$; (**c**) rank $A = 2$.

25. A basis is $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$

27. (**a**) $(A|B)$ reduces to

$$\left(\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & -1 \\ 0 & 0 & 6 & 6 \end{array} \right)$$

so rank $(A) =$ rank $((A|B)) = 3$ and a solution exists.

(**b**) $(A|B)$ reduces to

$$\left(\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 0 & -2 & 6 & 5 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

So rank $(A) =$ rank $((A|B)) = 2$ and a solution exists.

(**c**) $(A|B)$ reduces to

$$\left(\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 0 & -2 & 6 & 5 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

So rank $(A) \neq$ rank $((A|B))$ and no solution exists.

PROBLEMS 3.6

1. (**a**) Is an inner product,

(**b**) Is not an inner product; $\langle \mathbf{x}, \mathbf{x} \rangle = 2x_1$, so that $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ need not mean $\mathbf{x}$ is the zero vector.

(**c**) Is not an inner product; $\langle \mathbf{x}, \mathbf{x} \rangle = {x_1}^2 - {x_2}^2$, so that $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ need not mean $\mathbf{x}$ is the zero vector.

(**d**) Is not an inner product; $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle \neq \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$.

(**e**) Is not an inner product; note that $\langle (1, 0), (1, 0) \rangle = 0$, but the vector is not the zero vector.

(**f**) Is not an inner product; note that $\langle (1, 0), (1, 0) \rangle = 0$ but the vector is not the zero vector.

(**g**) Is not an inner product; note that $\langle (1, -1), (1, -1) \rangle = 0$, but the vector is not the zero vector.

(**h**) Is not an inner product; $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle \neq \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$.

3. (1) $\langle \mathbf{y}, \mathbf{x} \rangle = ay_1x_1 + by_2x_2 = ax_1y_1 + bx_2y_2 = \overline{ax_1y_1 + bx_2y_2} = \overline{\langle \mathbf{x}, \mathbf{y} \rangle}$

(2) $$\begin{aligned} \langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle &= \langle (x_1 + z_1, x_2 + z_2), (y_2, y_2) \rangle \\ &= a(x_1 + z_1)y_1 + b(x_2 + z_2)y_2 \\ &= ax_1y_1 + bx_2y_2 + az_1y_1 + bz_2y_2 = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle \end{aligned}$$

(3) $$\begin{aligned} \langle r\mathbf{x}, \mathbf{y} \rangle &= \langle (rx_1, rx_2), (y_1, y_2) \rangle = arx_1y_1 + brx_2y_2 \\ &= r(ax_1y_1 + bx_2y_2) = r\langle \mathbf{x}, \mathbf{y} \rangle \end{aligned}$$

(4) $\langle \mathbf{x}, \mathbf{x} \rangle = a{x_1}^2 + b{x_2}^2 \geq 0$ because $a > 0, b > 0$. $a{x_1}^2 + b{x_2}^2 = 0$ if and only if $x_1 = x_2 = 0$. If either a or b is less than zero then property (4) fails to hold and we do not have an inner product.

5. (1) $\langle \mathbf{y}, \mathbf{x} \rangle_k = k\langle \mathbf{y}, \mathbf{x} \rangle = k\overline{\langle \mathbf{x}, \mathbf{y} \rangle} = k\overline{\langle \mathbf{x}, \mathbf{y} \rangle} = \overline{\langle \mathbf{x}, \mathbf{y} \rangle_k}$

(2) $\langle \mathbf{x}+\mathbf{z},\mathbf{y}\rangle_k = k\langle \mathbf{x}+\mathbf{z},\mathbf{y}\rangle = k\langle \mathbf{x},\mathbf{y}\rangle + k\langle \mathbf{z},\mathbf{y}\rangle = \langle \mathbf{x},\mathbf{y}\rangle_k + \langle \mathbf{x},\mathbf{z}\rangle_k$

(3) $r\langle \mathbf{x},\mathbf{y}\rangle_k = rk\langle \mathbf{x},\mathbf{y}\rangle = kr\langle \mathbf{x},\mathbf{y}\rangle = k\langle r\mathbf{x},\mathbf{y}\rangle = \langle r\mathbf{x},\mathbf{y}\rangle_k$

(4) $\langle \mathbf{x},\mathbf{x}\rangle_k = k\langle \mathbf{x},\mathbf{x}\rangle$. Since $k > 0$, $k\langle \mathbf{x},\mathbf{x}\rangle > 0$ unless $\mathbf{x} = \theta$ and then $k\langle \mathbf{x},\mathbf{x}\rangle = 0$. Therefore $\langle \mathbf{x},\mathbf{x}\rangle_k > 0$ and $\langle \mathbf{x},\mathbf{x}\rangle_k = 0$ if and only if $\mathbf{x} = \theta$. Note that $\|\mathbf{x}\|_k = \sqrt{k\langle \mathbf{x},\mathbf{x}\rangle} = \sqrt{k}\|\mathbf{x}\|$. If $k < 1$, $\|\mathbf{x}\|_k < \|\mathbf{x}\|$; if $k > 1$, $\|\mathbf{x}\|_k > \|\mathbf{x}\|$.

7. (**a**) $r(-3,-9,7), r \neq 0$.

(**b**) For $\mathbf{x}$ choose (1,1,0) because it is clearly orthogonal to $(1,-1,2)$. Now let $\mathbf{y} = (-1,1,1)$; it is clearly orthogonal to $(1,-1,2)$ and it is not a multiple of (1,1,0). Of course you could proceed as follows. After choosing $\mathbf{x} = (1,1,0)$, let $\mathbf{y} = (y_1,y_2,y_3)$, we want $\langle (y_1,y_2,y_3), (1,-1,2)\rangle = 0$ which implies $y_1 - y_2 + 2y_3 = 0$. Thus we have $y = (r-2s, r, s)$. Now we want $c_1\mathbf{x} + c_2\mathbf{y} = \theta$ to have only the trivial solution. The equations are

$$\left(\begin{array}{cc|c} 1 & r-2s & 0 \\ 1 & r & 0 \\ 0 & s & 0 \end{array}\right)$$

which reduce to

$$\left(\begin{array}{cc|c} 1 & r-2ss & 0 \\ 0 & s & 0 \\ 0 & 0 & 0 \end{array}\right)$$

We have only the trivial solution if and only if $s \neq 0$, while r can take any value. Thus $\mathbf{y} = (r-2, r, 1)$ will do.

9. (1) $\overline{\langle \mathbf{y},\mathbf{x}\rangle} = \overline{y_1x_1 + \cdots + y_nx_n} = \overline{x_1y_1} + \cdots + \overline{x_ny_n} = x_1y_1 + \cdots + x_ny_n$ because $x_1,\cdots,x_n$, and $y_1,\cdots,y_n$ are real numbers. Thus $\overline{\langle \mathbf{y},\mathbf{x}\rangle} = \langle \mathbf{x},\mathbf{y}$

(2) $\langle \mathbf{x}+\mathbf{z},\mathbf{y}\rangle = (x_1+z_1)y_1 + \cdots + (x_n+z_n)y_n = x_1y_1 + z_1y_1 + \cdots + x_ny_n + z_ny_n = x_1y_1 + \cdots + x_ny_n + z_1y_1 + \cdots + z_ny_n = \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{z},\mathbf{y}\rangle$.

(3) $r\langle \mathbf{x},\mathbf{y}\rangle = r(x_1y_1 + \cdots + x_ny_n) = ((rx_1)y_1 + \cdots + (rx_n)y_n) = \langle r\mathbf{x},\mathbf{y}\rangle$

(4) $\langle \mathbf{x},\mathbf{x}\rangle = {x_1}^2 + \cdots + {x_n}^2 \geq 0$, and can equal zero if and only if $x_1 = x_2 = \cdots = x_n = 0$.

11. **(b)** $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. From the definition of inner product $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \theta$. Therefore $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \theta$.

(c) $\langle \mathbf{x}, r\mathbf{y} \rangle = \overline{\langle r\mathbf{y}, \mathbf{x} \rangle} = \overline{r\langle \mathbf{y}, \mathbf{x} \rangle} = \bar{r}\overline{\langle \mathbf{y}, \mathbf{x} \rangle} = \bar{r}\langle \mathbf{x}, \mathbf{y} \rangle \cdot \langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \overline{\langle \mathbf{y} + \mathbf{z}, \mathbf{x} \rangle}$
$= \overline{\langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{x} \rangle} = \overline{\langle \mathbf{y}, \mathbf{x} \rangle} + \overline{\langle \mathbf{z}, \mathbf{x} \rangle} = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$.

13. $\|\mathbf{x} + \mathbf{y}\|^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{x} + \mathbf{y}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle +$
$\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle = \|\mathbf{x}\|^2 + 0 + 0 + \|\mathbf{y}\|^2$

15. $X^TY = (1 \quad 2 \quad -1)\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} = -2 + 8 - 3 = 3$

17. **(a)** $m = 2, n = 3$

(b) (1) $\langle B, A \rangle = \text{tr } (B^TA)$. Now for a square matrix C, $\text{tr } (C) = \text{tr } (C^T)$, therefore $\text{tr } (B^TA) = \text{tr } ((B^TA)T) = \text{tr } (A^TB) = \langle A, B \rangle$. Because the entries of matrices from M_{mm} are real $\overline{\langle B, A \rangle} = \langle B, A \rangle$. Therefore $\overline{\langle B, A \rangle} = \langle A, B \rangle$.

(2) $\langle A + C, B \rangle = \text{tr } ((A + C)^TB) = \text{tr } ((A^T + C^T)B)$
$= \text{tr } (A^TB + C^TB) = \text{tr } (A^TB) + \text{tr } (C^TB)$
$= \langle A, B \rangle + \langle C, B \rangle$

(3) $r\langle A, B \rangle = r\text{tr } (A^TB) = \text{tr } (r(A^TB)) = \text{tr } ((rA^T)B)$
$= \text{tr } ((rA)^TB) = \langle rA, B \rangle$.

(4) $\langle A, A \rangle = \text{tr } \langle A^TA \rangle$. Let $A^TA = C$. Now

$$c_{ii} = \sum_{k=1}^{m} {a_{ki}}^2 \qquad i = 1, 2, \ldots, n$$

so

$$\text{tr } (A^TA) = \sum_{i=1}^{n} \sum_{k=1}^{m} {a_{ki}}^2$$

which is a sum of squares of real numbers. Therefore $\langle A, A \rangle \geq 0$ and equals zero if and only if $a_{ki} = 0$, $k = 1, 2, \ldots, m$; $i = 1, 2, \ldots, n$. That is, $\langle A, A \rangle = 0$ if and only if $A = 0$.

PROBLEMS 3.7

1. To verify orthogonality show that all possible inner products are zero. To obtain orthonormal basis, normalize all vectors.

(**a**) $\left\{\frac{1}{\sqrt{2}}(1,1,0),(0,0,1),\frac{1}{\sqrt{2}}(-1,1,0)\right\}$ is an orthonormal basis of E^3.

(**b**) $\left\{\frac{1}{\sqrt{3}}(1,1,1),\frac{1}{\sqrt{2}}(0,1,-1),\frac{1}{\sqrt{6}}(2,-1,-1)\right\}$ is an orthonormal basis of E^3.

3. (**c**) and (**d**) are orthogonal matrices.

5. $$\begin{aligned}\|\mathbf{x}\|^2 = \langle \mathbf{x},\mathbf{x}\rangle &= \left\langle \sum_{k=1}^{n} c_k\mathbf{v}_k, \sum_{j=1}^{n} c_j\mathbf{v}_j \right\rangle = \sum_{k=1}^{n} c_k \left\langle \mathbf{v}_k, \sum_{j=1}^{n} c_j\mathbf{v}_j \right\rangle \\ &= \sum_{k=1}^{n} c_k \sum_{j=1}^{n} c_j\langle \mathbf{v}_k,\mathbf{v}_j\rangle = \sum_{k=1}^{n} c_k \left(\sum_{j=1}^{n} c_j\delta_{kj}\right) = \sum_{k=1}^{n} {c_k}^2\end{aligned}$$

7. The dimension of M_{22} is four. The set O is orthonormal, which is shown by calculating all possible inner products. Because O is a set of four linear independent vectors and $\dim(M_{22}) = 4, O$ is basis. Using $v = \langle v,v_1\rangle v_1 + \cdots + \langle v,v_n\rangle v_n$, we have

$$\begin{aligned}\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} &= 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + (-\sqrt{2})\begin{pmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} \\ &+ (-2\sqrt{2})\begin{pmatrix} 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}\end{aligned}$$

9. Suppose A^T is orthogonal. Then $(A^T)(A^T)^T = I$ and $(A^T)^T(A^T) = I$. That is $A^TA = I$ and $AA^T = I$. Therefore A is orthogonal.

11. $O = \left\{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right),\left(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},0\right),(0,0,1)\right\}$

13. (**a**) $A^*A = I$, so $1 = \det I = \det(A^*A) = \det A^* \det A = \overline{(\det A)}(\det A) = |\det A|^2$. Therefore $|\det A| = 1$.

(**b**) Consider

$$A = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$$

from Prob. 4. A is unitary and $\det A = i$.

(**c**) $A^TA = I$, so $1 = \det I = \det(A^TA) = \det A^T \det A = (\det A)^2$. Since A has real entries only, $\det A = \pm 1$.

15. Yes. $(A^2)^T(A^2) = (A^T)^2A^2 = A^TA^TAA = A^TA = 1$
Yes. $(A^2)^*(A^2) = \overline{(A^2)}^TA^2 = (\bar{A}\bar{A})^TAA = \bar{A}^T\bar{A}^TAA = A^*A^*AA = A^*A = I$.

17. (**a**) Since $U^TU = I$. We have $\det U^T = 1/(\det U)$. Now $\det B = \det(U^TAU) = \det U^T \det A \det U = \det U^T \det U \det A = 1/(\det U)$ $\det U \det A = \det A$.

(**b**) Since $U^*U = I$, we have $\det U^* = 1/(\det U)$. Now proceed as in part a of this problem.

19. A is involutory so $A^2 = I$. If n is odd then $n = 2m+1$, m, a nonnegative integer, and $A^n = A^{2m+1} = A^{2m}A = (A^2)^mA = I^mA = A$.

21. An idempotent matrix need not be symmetric. For example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

satisfies $A^2 = A$ but A is not symmetric.

23. When inspecting A^*A remember that the entries of A^* have been conjugated. Otherwise the steps are exactly the same.

25. Because the columns must be orthonormal in C^4, we have
$|a^2| + |b|^2 = 1 \quad \bar{a}(ib) + (\overline{ib})a = 0 \quad \text{or} \quad |a|^2 + |b|^2 = 1 \quad i\bar{a}b - ia\bar{b} = 0$
If a and b are real, this reduces to the condition that $a+ib$ be a complex number of magnitude one.

PROBLEMS 3.8

1. (**a**) $\begin{pmatrix} -\sqrt{2} & -3\sqrt{2}/2 \\ 2\sqrt{2} & -\sqrt{2}/2 \end{pmatrix} = P_{T\leftarrow S}$ (**b**) $\begin{pmatrix} -2\sqrt{2}/7 & -\sqrt{2}/7 \\ -\sqrt{2}/14 & -3\sqrt{2}/14 \end{pmatrix} = P_{U\leftarrow T}$

(**c**) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = P_{U\leftarrow S}$
$P_{U\leftarrow T}P_{T\leftarrow S} = P_{U\leftarrow S}$
$P_{T\leftarrow S}P_{U\leftarrow T} \neq P_{U\leftarrow S}$

3. $P_{Z\leftarrow T} = \dfrac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{3}-1 & \sqrt{3}+1 \\ -\sqrt{3}-1 & \sqrt{3}-1 \end{pmatrix}$ $\quad P_{T\leftarrow Z} = \dfrac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{3}-1 & -\sqrt{3}-1 \\ \sqrt{3}+1 & \sqrt{3}-1 \end{pmatrix}$

5. (**a**) $P_{T\leftarrow S} = \begin{pmatrix} 0 & -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2} & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 1 & 1 \end{pmatrix}$ (**b**) $P_{U\leftarrow T} = \begin{pmatrix} \sqrt{2}/2 & 0 & \frac{1}{2} \\ -\sqrt{2}/2 & 0 & \frac{1}{2} \\ 0 & \sqrt{2}/2 & \frac{1}{2} \end{pmatrix}$

(**c**) $P_{U\leftarrow S} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$P_{U\leftarrow S} = P_{U\leftarrow T}P_{T\leftarrow S}$

7. $P_{Z\leftarrow T} = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\sqrt{2}/2 \\ -\frac{1}{2} & \frac{1}{2} & \sqrt{2}/2 \end{pmatrix}$ $\quad P_{T\leftarrow Z} = \begin{pmatrix} \sqrt{2}/2 & -\frac{1}{2} & -\frac{1}{2} \\ \sqrt{2}/2 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$

9. $\mathbf{x} = 3(1,1,0) - 2(0,1,1) + 4(1,0,1) = (7,1,2)$. **11.** $\mathbf{x}$ is the zero vector.

13. Follow the indicated steps. **15.** Let $\mathbf{x}$ be in V. $P(\mathbf{x})_S = (\mathbf{x})_T$, $Q(\mathbf{x})_T = (\mathbf{x})_U$. Thus $(\mathbf{x})_U = Q(\mathbf{x})_T = Q(P(\mathbf{x})_S) = (QP)(\mathbf{x})_S$.

17. Because any basis has n vectors in it, each column has n entries.

19. $P = \begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -\frac{1}{2} \\ \frac{1}{2} & \sqrt{3}/2 \end{pmatrix}$

PROBLEMS 3.9

1. (**a**) $\frac{2}{5}$; (**b**) 0; (**c**) 1; (**d**) $\frac{1}{12}$ $\quad$ **3.** $\|f\| = \sqrt{3}/3$

5. (1) $\displaystyle \langle g, f\rangle = \int_a^b g(x)f(x) + g'(x)f'(x)dx$

$\displaystyle = \int_a^b f(x)g(x) + f'(x)g'(x)dx = \langle f, g\rangle$

Since f and g are real valued functions, $\langle f, g\rangle = \overline{\langle g, f\rangle}$.

(2) $\displaystyle \langle f + h, g\rangle = \int_a^b (f(x) + h(x))g(x) + (f'(x) + h'(x))g'(x)dx$

$\displaystyle = \int_a^b f(x)g(x) + h(x)g(x) + f'(x)g'(x) + h'(x)g'(x)dx$

$$= \int_a^b f(x)g(x)+f'(x)g'(x)dx + \int_a^b h(x)g(x)+h'(x)g'(x)dx$$
$$= \langle f, g\rangle + \langle h, g\rangle$$

(3) $r\langle f, g\rangle = r\int_a^b f(x)g(x)+f'(x)g'(x)dx = \int_a^b (rf(x))g(x)+(rf'(x))g'(x)dx$
$= \langle rf, g\rangle$

(4) $\langle f, f\rangle = \int_a^b (f(x))^2 + (f'(x))^2 dx$
Because f and f' are continuous on $[a, b]$, this integral can be zero if and only if f is identically zero.

7. (**a**) Linearly independent; (**b**) linearly independent;
(**c**) linearly independent.

9. The function $f(x) = x^{2/3}$ has no derivative at $x = 0$. Consider $c_1x^{2/3} + c_2x^2 = 0$. If the equation is to have nontrivial solution for all x, it must have nontrivial solution for any two values of x. Choose $x = 1$, $x = 8$, to obtain

$$c_1 + c_2 = 0 \qquad 4c_1 + 64c_2 = 0$$

which has only the trivial solution. Thus the equation $c_1x^{2/3}+c_2x^2 = 0$ cannot have nontrivial solution for all x and S must be linearly independent.

11. $\cos\theta = \dfrac{\int_0^1 (x)(x^3)dx}{(\int_0^1 x^2\ dx)^{1/2}(\int_0^1 x^6\ dx)^{1/2}} = \dfrac{\sqrt{21}}{5} \qquad \theta = \cos^{-1}\left(\dfrac{\sqrt{21}}{5}\right)$

13. The set of solutions is a subspace. If f and g are solutions then $f'' = f$ and $g'' = g$. So $(f + g)'' = f'' + g'' = f + g$ and $(rf)'' = rf'' = rf$. By the closure then the solution set is a subspace. Now e^x and e^{-x} are two linearly independent solutions of $y'' = y$. Clearly dim span $(\{e^x, e^{-x}\}) = 2$.

ADDITIONAL PROBLEMS (CHAPTER 3)

1. V is not a vector space; Vfails to be closed under scalar multiplication.

3. Yes. The space V is n dimensional and the equation $c_1 i\mathbf{v}_1 + c_2 i\mathbf{v}_2 + \cdots + c_n i\mathbf{v}_n = \theta$ is equivalent to $i(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = \theta$ or $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \theta$. Thus $c_1 = c_2 = \cdots = c_n = 0$. Thus the set $\{i\mathbf{v}_1, \ldots, i\mathbf{v}_n\}$ is a linearly independent set of n vectors in an n-dimensional space and must be a basis.

5. The set $S = \{c_1\mathbf{v}_1, c_2\mathbf{v}_2, \ldots, c_n\mathbf{v}_n\}$ is a basis for V. Since V is n-dimensional, all that needs to be shown is that S, having n elements spans V. Let $\mathbf{x}$ be in V. Since $T = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis, there exist constants $d_1, d_2, \ldots, d_n$ such that $\mathbf{x} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n$. Now we can rewrite $\mathbf{x} = (d_1/c_1)c_1\mathbf{v}_1 + (d_2/c_2)c_2\mathbf{v}_2 + \cdots + (d_n/c_n)c_n\mathbf{v}_n$ to see that S spans V.

7.
$$\begin{aligned}
P^T &= (A(A^TA)^{-1}A^T)^T = A^{T^T}((A^TA)^{-1})^T A^T \\
&= A((A^TA)^T)^{-1}A^T = A(A^T(A^T)^T)^{-1}A^T = A(A^TA)^{-1}A^T = P \\
P^2 &= (A(A^TA)^{-1}A^T)(A(A^TA)^{-}A^T) = A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T \\
&= A(A^TA)^{-1}IA^T = A(A^TA)^{-1}A^T = P
\end{aligned}$$

9. The set of orthogonal $n \times n$ matrices is not a subspace of M_{nn}. For example, when $n = 2$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is orthogonal but $2I$ is not orthogonal.

11. Let X and Y satisfy $AX = cX$, $AY = cY$. Now $A(X + Y) = AX + AY = cX + cY = c(X + Y)$ so V is closed under addition. Also $A(rX) = rAX = rcX = c(rX)$ so V is closed under scalar multiplication. Therefore V is a subspace of C_{n1}.

13. With the given inner product an orthonormal basis is $O = \{1, \sqrt{3}(1 - 2x), \sqrt{5}(6x^2 - 6x + 1)\}$. The best approximation is

$$\begin{aligned}
&\left(\int_0^1 e^x \cdot 1\, dx\right) 1 + \left(\int_0^1 e^x(\sqrt{3}(1-2x))\, dx\right)\sqrt{3}(1-2x) \\
&\quad + \left(\int_0^1 e^x\sqrt{5}(6x^2 - 6x + 1)\, dx\right)\sqrt{5}(6x^2 - 6x + 1) \\
&= (e-1) + 3(e-3)(1-2x) + 5(7e-19)(6x^2 - 6x + 1) \\
&\doteq 1.013 + .851x + .839x^2 = p(x)
\end{aligned}$$

Note that

$$p(0) \doteq 1.013, e^0 = 1.0 \quad p(.5) \doteq 1.64825, e^{.5} = 1.64872\cdots$$
$$p(1) \doteq 2.703, e^1 = 2.71828\cdots$$

15. $$\langle \mathbf{x}, \mathbf{y}\rangle = \left\langle \sum_{k=1}^{n} a_k \mathbf{v}_k, \sum_{j=1}^{n} b_j \mathbf{v}_j \right\rangle = \sum_{j=1}^{n} a_k \left\langle \mathbf{v}_k, \sum_{j=1}^{n} b_j \mathbf{v}_j \right\rangle = \sum_{k=1}^{n} a_k \sum_{j=1}^{n} b_j \langle \mathbf{v}_k, \mathbf{v}_j\rangle$$

$$\begin{aligned}
(\mathbf{x})_S{}^T G(\mathbf{y})_S &= (a_1, a_2, \ldots, a_n)(\langle \mathbf{v}_i, \mathbf{v}_j\rangle)_{n\times n} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\
&= (a_1, a_2, \ldots, a_n) \left(\sum_{j=1}^{n} \langle \mathbf{v}_i, \mathbf{v}_j\rangle b_j \right)_{n\times 1} \\
&= \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{n} b_j \langle \mathbf{v}_i, \mathbf{v}_j\rangle \right)
\end{aligned}$$

Now substitute k for i as the index of summation.

PROBLEMS 4.1

1. (**a**) Linear

(**b**) Not linear. $T(c\mathbf{x}) = (cx_1, cx_2, c^2x_2x_3); cT(\mathbf{x}) = (cx_1, cx_2, cx_2x_3)$.

(**c**) Linear.

(**d**) Not linear. $T(\mathbf{x}+\mathbf{y}) = (1,0,0) \neq (2,0,0) = T(\mathbf{x}) + T(\mathbf{y})$.

(**e**) Not linear. $T(-\mathbf{x}) \neq -T(\mathbf{x})$. (**f**) Linear.

3. (**a**) Linear. (**b**) Not linear. $T(0) \neq 0$ unless $c = 0$.

(**c**) Not linear. $T(p+q) = 2; T(p) + T(q) = 2+2 = 4$.

(**d**) Not linear. T is not even defined at $p \equiv 0$.

(**e**) Linear.

5. (**a**) ket $T = \{\theta\}$. range $(T) = E^3$, so $\eta(T) = 0$. $R(T) = 3$. dim(domain(T)) = $\dim E^3 = 3$. ket T has no basis, the standard basis suffices for range (T).

(**c**)

$$\begin{aligned}
\ker T &= \{x \mid x_1 = 0, x_2 = r, x_3 = s, r, s \text{ in } R\}\\
\text{range}(T) &= \{y \mid y_1 = t, y_2 = y_3 = 0, t \text{ in } R\}\\
\text{Basis for } \ker T &= \{(0,1,0), (0,0,1)\}\\
\text{Basis for range } (T) &= \{(1,0,0)\}\\
\eta(T) &= 2, R(T) = 1
\end{aligned}$$

(**f**) ket $T = \{\theta\}$. range $(T) = E^3, \eta(T) = 0, R(T) = 3$. $\ker T$ has no basis. The standard basis suffices for range (T).

7. (**a**)

$$\begin{aligned}
\ker T &= \{a + bx + cx^2 \mid a = b = 0, c \text{ arbitrary}\}\\
\text{range } T &= \{\text{degree two polynomials with constant}\\
&\qquad \text{term zero}\}\\
\text{Basis for } \ker T &= \{x^2\}\\
\text{Basis for range } (T) &= \{x, x^2\}\\
\eta(T) &= 1, R(T) = 2. \dim(\text{domain}) = \dim P_2 = 3
\end{aligned}$$

(**e**) ket $T = \{\theta\}$. range$(T) = P_2$, so $\eta(T) = 0, R(T) = 3$. ket T has no basis. The standard basis suffices for range (T).

9. Let $T\colon\ V \to V$ be defined by $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in V. Let c be any scalar. $T(c\mathbf{x}) = c\mathbf{x}$, $cT(\mathbf{x}) = c\mathbf{x}$. $T(\mathbf{x}+\mathbf{y}) = \mathbf{x}+\mathbf{y}$, $T(\mathbf{x}) + T(\mathbf{y}) = \mathbf{x}+\mathbf{y}$. Therefore $T(c\mathbf{x}) = cT(\mathbf{x})$ and $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$.

11. Let $T\colon\ V \to V$ be defined by $T(\mathbf{x}) = k\mathbf{x}$ for all $\mathbf{x}$ in V, where $k > 1$. Let c be any scalar. $T(c\mathbf{x}) = k(c\mathbf{x}) = c(k\mathbf{x}) = cT(\mathbf{x})$. $T(\mathbf{x}+\mathbf{y}) = k(\mathbf{x}+\mathbf{y}) = k\mathbf{x} + k\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y})$.

15. Use the hint. $pT(\mathbf{x}) = T(p\mathbf{x}) = T(q(p/q)\mathbf{x})$. Now $T(q(p/q\mathbf{x})) = qT(p/q\mathbf{x})$ by Prob. 14. Therefore $pT(\mathbf{x}) = qT(p/q\mathbf{x})$. Divide both sides by q.

17. To show range (T) is a subspace of W, we need to show closure. Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be in range (T). This means there exist $\mathbf{x}_1$ and $\mathbf{x}_2$ in V such that $T(\mathbf{x}_1) = \mathbf{w}_1$, $T(\mathbf{x}_2) = \mathbf{w}_2$. Now we ask the closure questions: For any scalar c is $c\mathbf{w}_1$ in range (T) and is $\mathbf{w}_1+\mathbf{w}_2$ in range (T)? Consider $c\mathbf{w}_1$. We have $c\mathbf{w}_1 = cT(\mathbf{x}_1) = T(c\mathbf{x}_1)$. Now V is a vector space, so $c\mathbf{x}_1$ is in V. Thus $T(c\mathbf{x}_1)$ is in range (T). Consider now $\mathbf{w}_1 + \mathbf{w}_2$. We have $\mathbf{w}_1 + \mathbf{w}_2 = T(\mathbf{x}_1) + T(\mathbf{x}_2) = T(\mathbf{x}_1 + \mathbf{x}_2)$. Because V is a vector space $\mathbf{x}_1 + \mathbf{x}_2$ is in V, so $T(\mathbf{x}_1 + \mathbf{x}_2)$ is defined and $T(\mathbf{x}_1 + \mathbf{x}_2)$ is in range (T).

19.

$$\begin{aligned} T(cf)(x) &= x(cf)(x) = c(xf)(x) = cT(f) \\ T(f+g)(x) &= x(f+g)(x) = xf(x) + xg(x) = T(f)(x) + T(g)(x) \end{aligned}$$

21. $T((8,3,2,)) = T(2(1,0,1) + 3(2,1,0)) = 2T(1,0,1) + 3T(2,1,0))$
$= 2(1,-1,3) + (3(0,2,1)) = (2,4,9)$

23. (3,0,4) is not in span $(\{(1,0,1),(2,1,0)\})$. This is found by showing that $(3,0,4) = c_1(1,0,1) + c_2(2,1,0)$ leads to inconsistent linear equations.

25. Let the line be given by $R = X + tY, t$ in R, where X and Y are in M_{21}. Now $M(X+tY) = MX + M(tY) = MX + tMY$. Now the points satisfying $S = MX + tMY, T$ in R form a line because the equation is a vector equation for a time.

PROBLEMS 4.2

1. (**a**) $(A \mid 0)$ reduces to

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus rank $(A) = 2$. The solutions are $\{(x,y,z) = (t,-2t,t)\}$. A basis for the solution space is

$$S = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

Thus dim (solution space) + rank (A) = $1 + 2$ = number of columns of A.

(**b**) $(A \mid 0)$ reduces to

$$\left(\begin{array}{rrr|r} 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus rank $(A) = 2$. The solutions are $\{(x, y, z) = (-3t, -t, t)\}$. A basis for the solution space is

$$S = \left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Thus dim (solution space) + rank $(A) = 1 + 2 = 3$.

(**c**) $(A \mid 0)$ reduces to

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 0 \\ 0 & -10 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus rank $(A) = 2$. The solutions are $\{(x, y, z, w) = (-4s, -2t, 7s + t, 10s, 10t)\}$. A basis for the solution space is

$$S = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 10 \end{pmatrix}, \begin{pmatrix} -4 \\ 7 \\ 10 \\ 0 \end{pmatrix} \right\}$$

Thus dim (solution space) + rank (A) = $2 + 2$ = number of columns of A.

3. (**a**) $M = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -1 & 1 \end{pmatrix}$

(**b**) Directly: $T(1, -1, 2) = (4, -3)$. Using M:

$$\begin{aligned} [T(1,-1,2)]_s &= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -1 & 1 \end{pmatrix} [(1,-1,2)]_{\mathrm{STD}} \\ &= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{7}{2} \end{pmatrix} \end{aligned}$$

Therefore $T(1,-1,2) = \frac{1}{2}(1,1) + \frac{7}{2}(1,-1) = (4,-3)$.

5. Give the domain $I = \{(1,1),(1,-1)\}$ for a basis and let the range have the standard basis. In this case

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

On the other hand, if the range is equipped with I a basis,

$$M = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

7. Let $Z\colon\ V \to W$ be the zero transformation. That is for all $\mathbf{x}$ in $V, Z(\mathbf{x}) = \theta$, the zero vector of W. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be bases for V and W respectively. Now $Z(\mathbf{v}_i) = \theta = 0\mathbf{w}_1 + 0\mathbf{w}_2 + \cdots + 0\mathbf{w}_m$ (uniquely). Thus every column of M_Z consists of all zeros.

9. For each $k, k = 0,1,2,\ldots,n$, $T(x^k) = x^{k+1} = 0 + 0x + \cdots + 1x^{k+1} + \cdots + 0x^{n+1}$. Thus column $k+1$ of M has $n+2$ entries, all being zero except for the $k+2$ entry, which is 1.

11. (**a**) $\begin{pmatrix} 1 & 1 \\ 0 & i \end{pmatrix}$ (**b**) $\begin{pmatrix} 0 & i & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}$ (**c**) $\begin{pmatrix} 1+i & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 1+i \end{pmatrix}$.

PROBLEMS 4.3

1. (**a**) $M_I = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(**b**) $P_{I\leftarrow T} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad P_{T\leftarrow I} = P_{I\leftarrow T}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$$M_T = P_{T\leftarrow I} M_I P_{I\leftarrow T} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

3. (**a**) $M_I = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}$

(**b**) Transition matrices are as in $1b$.

$$M_T = P_{T\leftarrow I} M_I P_{I\leftarrow T} = \begin{pmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{pmatrix}$$

5. (**a**) $M_I = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

(**b**) $P_{I\leftarrow T} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $P_{T\leftarrow T} = P_{I\leftarrow T}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$$M_T = P_{T\leftarrow I} M_I P_{I\leftarrow T} = \frac{\sqrt{2}}{2}\begin{pmatrix} -2 & -5 \\ 2 & 4 \end{pmatrix}$$

7. (**a**) $M_I = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(**b**) $P_{I\leftarrow T} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ $P_{T\leftarrow I} = P_{I\leftarrow T}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

$$M_T = P_{T\leftarrow I} M_I P_{I\leftarrow T} = \frac{1}{2}\begin{pmatrix} 5 & 3 & 2 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

9. (**a**) $M_I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(**b**) $P_{I\leftarrow T} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

$$P_{T\leftarrow I} = P_{I\leftarrow T}^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$M_T = P_{T\leftarrow I} M_I P_{I\leftarrow T} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

11. (**a**) The traces are not equal.

(**b**) Neither the traces nor the determinants are equal.

(**c**) We have tr A = tr B, $\det A = \det B$; we need to work a little harder. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and suppose $PB = AP$. Writing out the products and equating elements of the matrices we get equations

$$2c = a + 2c \qquad 2d = b + 2d$$

which means $a = b = 0$. But then

$$P = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$$

which is not invertible. Thus A cannot be similar to B.

13. Identify

$$\begin{array}{rcl} E^2 & & P_1 \\ (a,b) & \longleftrightarrow & a + bx \\ I = \{(1,0), & \longleftrightarrow & \{1 + 0x \\ (0,1)\} & \longleftrightarrow & 0 + 1x\} \\ (a,b) & \longleftrightarrow & a + bx \\ T\downarrow & \longleftrightarrow & T\downarrow \\ (a+b) + (2a - 3b) & \longleftrightarrow & (a+b) + (2a-3b)x \end{array}$$

That is, identify the first component of an element in E^2 with the constant coefficient of the degree one polynomial; identify the second component of an element of E^2 with the coefficient of x in the degree one polynomial.

PROBLEMS 4.4

1. The standard matrix for T is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

M is not invertible, but $M^2 = M$. Therefore T is not invertible, but T is idempotent.

3. The standard matrix for T is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

M is invertible and $M^2 = I$. Therefore T is invertible but T is not idempotent.

5. The standard matrix for T is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

M is invertible.

$$M^2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \qquad M^3 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 3 & 1 \end{pmatrix}$$

$$M^4 = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 10 & 4 & 1 \end{pmatrix}, \ldots, M^n = \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ (n+1)n/2 & n & 1 \end{pmatrix}$$

Thus M is not nilpotent. Actually we don't have to work this hard. Recall in an earlier chapter we had a problem: "Show that if A is nilpotent, then $\det A = 0$." As soon as we know M is invertible we know that M cannot be nilpotent. Anyway T is invertible and not nilpotent.

7. The standard matrix for a rotation of 120° is

$$M = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\sqrt{3}/2 \\ \sqrt{3}/2 & -\frac{1}{2} \end{pmatrix}$$

Now

$$M^2 = \begin{pmatrix} -\frac{1}{2} & \sqrt{3}/2 \\ -\sqrt{3}/2 & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad M^3 = I$$

9. The standard matrix for T is

$$M = \begin{pmatrix} 0 & 0 & 0 & \cdots & & 0 \\ 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \ddots & & \\ \vdots & \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Since $\det M = 0, M$ is not invertible. As higher powers of M are calculated, the diagonal of 1's moves to the lower left corner; $M^n = 0$. Thus T is nilpotent. T is not idempotent, because $M^2 \neq M$.

11. If A is involutory and B is similar to A, then B is involutory. Suppose $A^2 = I$, and $B = P^{-1}AP$. Then $B^2 = P^{-1}APP^{-1}AP = P^{-1}A^2P = P^{-1}P = I$. Since $B^2 = I$, B is involutory.

13. $B^T = (P^T\ AP)^T = P^T\ A^T P^{T^T} = P^T\ AP = B$, so B is symmetric.

15. Consider the equation to determine independence

$$c_1L(\mathbf{v}_1) + c_2L(\mathbf{v}_2) + \cdots + c_kL(\mathbf{v}_k) = \theta$$

Using the linearity of L we have

$$\theta = L(c_1\mathbf{v}_1) + L(c_2\mathbf{v}_2) + \cdots + L(c_k\mathbf{v}_k) = L(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k)$$

Now since the kernel of L is only the zero vector, we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \theta$$

The linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ means that the last equation has only the solution $c_1 = c_2 = \cdots = c_k = 0$. Therefore $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_k)\}$ is a linearly independent set.

PROBLEMS 4.5

1. Because V is a subset of the vector space of all functions defined on R, we need only show closure. Let f and g be functions with

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = M$$

Now does $f+g$ have a limit at a? From calculus we have the theorem which tells us

$$\lim_{x\to a}(f+g)(x) = L + M$$

Thus $f+g$ has limit at a and $f+g$ is in V. Similarly

$$\lim_{x\to a}(rf)(x) = r\lim_{x\to a}(f)(x) = rL \quad \text{and} \quad rf \text{ is in } V$$

Note that the theorems we used from calculus concerning limits at a point a can be written in the shorthand from the statement of the problem

$$L_a(f+g) = L_a(f) + L_a(g) \qquad L_a(rf) = rL_a(f)$$

That is L_a is a linear transformation from V to R.

3. $L_f(g(x)) = \displaystyle\int_0^x \sin x \sin nx \; dx \qquad n > 1$

By trigonometric identities $\cos(nx - x) - \cos(nx + x) = \cos nx \cos x + \sin nx \sin x - (\cos nx \cos x - \sin nx \sin x) = 2 \sin nx \sin x$. Thus

$$\begin{aligned}\int_0^\pi \sin x \sin nx \; dx &= \frac{1}{2}\int_0^\pi \cos(n-1)x - \cos(n+1)x \; dx \\ &= \frac{1}{2}\left(\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right)\Bigg|_0^x = 0\end{aligned}$$

5. From calculus we know that $(f+g)' = f' + g'$ and $(rf)' = rf'$. Thus $(f+g)'' = ((f+g)')' = (f'+g')' = f'' + g''$ and $(rf)'' = (rf')' = rf''$. The standard matrix for D^2 is

$$N = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is equal to M^2.

7. We could try

$$I\colon\ a_0 + a_1 x + a_2 x^2 + \cdots \ \longrightarrow c + a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots$$

But we see that the "constant of integration" c is a problem. If we restrict our attention to the subset of functions in V with $f(0) = 0$ then $c = 0$ and I could invert the action of D.

9.

$$\begin{aligned}
\text{curl }(r\mathbf{f}) &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ rf_1 & rf_2 & rf_3 \end{pmatrix} = r\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{pmatrix} \\
&= r \text{ curl }(\mathbf{f}) \cdot \ \text{curl }(f+g) = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1+g_1 & f_2+g_2 & f_3+g_3 \end{pmatrix} \\
&= \left(\frac{\partial}{\partial x}(f_2+g_2) - \frac{\partial}{\partial y}(f_1+g_1)\right)\mathbf{i} \\
&\quad + \left(\frac{\partial}{\partial z}(f_1+g_1) - \frac{\partial}{\partial x}(f_3+g_3)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}(f_2+g_2)\right. \\
&\quad \left. - \frac{\partial}{\partial y}(f_1+g_1)\right)\mathbf{k}
\end{aligned}$$

Now use the linearity of the partial derivative.

11. **(a)**

$$\begin{aligned}
T(rf)(x) &= x(rf)(x) = r(xf(x)) = r(T(f)(x)) \\
(T(f+g))(x) &= x(f+g)(x) = xf(x) + xg(x) = (T(f))(x) + (T(g))(x)
\end{aligned}$$

(**b**) $(T(g))(x) = xg(x) = \begin{cases} x \cdot 1 & x = 0 \\ x \cdot 0 & x \neq 0 \end{cases} = \begin{cases} 0 & x = 0 \\ 0 & x \neq 0 \end{cases}$

(**c**) Because g is not the zero vector but is in the kernel of T, $\ker T \neq \{\theta\}$ and T is not invertible.

ADDITIONAL PROBLEMS (CHAPTER 4)

1. $T(A+B) = (A+B)-(A+B)^T = A+B-A^T-B^T = A-A^T+B-B^T = T(A)+T(B)$. $T(rA) = rA-(rA)^T = rA-rA^T = r(A-A^T) = rT(A)$. The kernel of T is all $A_{n\times n}$ with $A - A^T = 0$ or $A = A^T$; the kernel is the subspace of symmetric matrices.

3.

$$\begin{aligned} \text{dim(symmetric } n \times n \text{ matrices} &= \frac{n(n+1)}{2} \\ \text{dim(upper triangular } n \times n \text{ matrices} &= \frac{n(n+1)}{2} \end{aligned}$$

The dimensions are equal.

5. $T(A+B) = P(A+B) = PA+PB = T(A)+T(B)$. $T(rA) = P(rA) = rPA = rT(A)$. To see whether T is one-to-one we look at $T(A) = 0$, which is $PA = 0$. Now we can't just multiply by P^{-1} because P may not be invertible. For example if $n = 2$,

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

satisfies $P^2 = P$ but P is not invertible. Note that choosing

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$PA = 0$ but $A \neq 0$. Thus in general we cannot say that T is one-to-one

7.

$$\begin{aligned} B(I-A) &= (I+A)(I-A) = I - A^2 \\ B(I-A+A^2) &= (I+A)(I-A+A^2) \\ &= I - A^2 + (I+A)A^2 = I + A^3 \\ B(I-A+A^2-A^3) &= I + A^3 - (I+A)A^3 = I - A^4 \end{aligned}$$

If $A^k = 0$ then $(I+A)(I-A+A^2-\cdots+(-1)^{k-1}A^{k-1}) = I+(-1)^{k-1}A^k = 1+0 = I$. Therefore if A is nilpotent exponent $k, (I+A)^{-1}$ exists and

$$(I+A)^{-1} = I + \sum_{n=1}^{k-1}(-1)^n A^n$$

9. $(L \circ T)(A) = L(T(A)) = L(A - A^T) = (A - A^T) + (A - A^T)^T = A - A^T + A^T - A = 0$. Therefore $L \circ T$ is the zero transformation. Now $(T \circ L)(A) = T(L(A)) = T(A + A^T) = (A + A^T) - (A + A^T)^T = A + A^T - A^T - A = 0$. Therefore $T \circ L$ is also the zero transformation and $L \circ T = T \circ L$.

11. $\det A(t) = \cos^2(\omega t) + \sin^2(\omega t) = 1$ for all t, so $A(t)$ is invertible for all t. Also

$$(A(t))^T(A(t)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for all } t$$

so $A(t)$ is orthogonal for all t.

13. Let a line have the vector equation $R = tU + V$, $t \in R$. Now $MR = M(tU+V) = t(MU)+MV$. The points described by R are transformed to points $S = MR = t(MU) + MV$. This equation is an equation of a line.

15.

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} 2x_n & 2y_n \\ 1 & -1 \end{pmatrix} \begin{pmatrix} {x_n}^2 + {y_n}^2 - 2 \\ x_n - y_n \end{pmatrix}$$

Now

$$\begin{pmatrix} 2x_n & 2y_n \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2(x_n + y_n)} \begin{pmatrix} 1 & 2y_n \\ 1 & -2x_n \end{pmatrix}$$

so we have

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \frac{1}{2(x_n + y_n)} \begin{pmatrix} -1 & -2y_n \\ -1 & 2x_n \end{pmatrix} \begin{pmatrix} {x_n}^2 + {y_n}^2 - 2 \\ x_n - y_n \end{pmatrix}$$

or

$$x_{n+1} = x_n + \frac{1}{2(x_n + y_n)}(2 - {x_n}^2 - {y_n}^2 + 2{y_n}^2 - 2x_n y_n)$$
$$y_{n+1} = y_n + \frac{1}{2(x_n + y_n)}(2 - {x_n}^2 - {y_n}^2 + 2{x_n}^2 - 2x_n y_n)$$

which reduces to

$$x_{n+1} = \frac{2 + {x_n}^2 + {y_n}^2}{2(x_n + y_n)} \qquad y_{n+1} = \frac{2 + {x_n}^2 + {y_n}^2}{2(x_n + y_n)}$$

Therefore using $x_1 = 1, y_1 = 0$, we have $x_2 = y_2 = \frac{3}{2}$, $x_3 = y_3 = \frac{13}{12}$. Using a calculator, $x_4 = y_4 = 1.0032051\ldots$, $x_5 = y_5 = 1.000051\ldots$, $x_6 = y_6 = 1.00000000$ to calculator accuracy. Note that if you start with $\binom{-1}{0}$, Newton's method will lead to the solution $\left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]$. Thus where you **start** in Newton's method affects which solution the method gives. Just as in calculus, Newton's method does not always work, but when it does work it works well.

PROBLEMS 5.1

1. $$A^{10} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}^{10} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1/2^{10} & 0 \\ 0 & 2^{10} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1/2^9 - 2^{10} & 1/2^9 - 2^{11} \\ 2^{10} - 1/2^{10} & 2^{11} - 1/2^9 \end{pmatrix}$$

3. Let S_1 = state "liquid detergent user." Let S_2 = state "dry detergent user." The transition matrix is

$$\begin{array}{cc} & \text{This week} \\ & \begin{array}{cc} S_1 & S_2 \end{array} \\ & \begin{pmatrix} .2 & .4 \\ .8 & .6 \end{pmatrix} \begin{array}{c} S_1 \\ S_2 \end{array} \quad \text{Next week} \end{array}$$

The initial state is

$$\begin{pmatrix} .5 \\ .5 \end{pmatrix} \quad \text{and} \quad M^4 \begin{pmatrix} .5 \\ .5 \end{pmatrix} = \begin{pmatrix} .3336 \\ .6664 \end{pmatrix}$$

After 4 weeks liquid detergent will have $\frac{1}{3}$ of the market. If the advertising campaign leads to a market share of more than $\frac{1}{3}$ then the agency did a good job.

5. Sunday's state is

$$M^5S = \begin{pmatrix} .143482 \\ .856518 \end{pmatrix}$$

The probability of a dry day is not appreciably greater than Saturday's probability.

PROBLEMS 5.2

1. In each answer, the given eigenvector can of course be replaced by any nonzero multiple of it.

(**a**) Eigenpairs $\left(i, \begin{pmatrix} 1 \\ i \end{pmatrix}\right), \begin{pmatrix} -1 \\ i \end{pmatrix}$ (**b**) Eigenpair $\left(1, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

(**c**) Eigenpairs $\left(1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right), \left(-1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

(**d**) Eigenpairs $\left(1, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right), \left(-1, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}\right), \left(2, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}\right)$

(**e**) Eigenpairs $\left(0, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \left(4, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$

(**f**) Eigenpairs $\left(1, \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}\right), \left(-2, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right), \left(3, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right)$

(**g**) Eigenpairs $\left(1, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right), \left(-1, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}\right), \left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}\right), \left(2, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right)$

(**h**) Eigenpairs $\left(i, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \left(-i, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$ (**i**) Eigenpair $\left(i \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

(**j**) Eigenpairs $\left(1, \begin{pmatrix} 1 \\ i \end{pmatrix}\right), \left(-1, \begin{pmatrix} 1 \\ -i \end{pmatrix}\right)$

3. $E = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

5. Put
$$a = \frac{27}{100} \quad r = \frac{1}{100} \quad \frac{a}{1-r} = \frac{27/100}{99/100} = \frac{27}{99}$$

7. $AX = \lambda X$ so $k(AX) = k(\lambda X)$ and thus $(kA)X = (k\lambda)X$

9. We know that $AX = \lambda X$. Multiply by A^{-1}: $A^{-1}(AX) = A^{-1}(\lambda X)$. Now $(A^{-1}A)X = \lambda(A^{-1}X)$ and $X = \lambda(A^{-1}X)$. Now divide by λ to get $A^{-1}X = (1/\lambda)X$.

11. The determinant of a triangular matrix is just the product of the diagonal entries. Now $(A - \lambda I)$ is still triangular and the diagonal entries are $a_{ii} - \lambda$, $i = 1, 2, \ldots, n$. Therefore $\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$. Clearly the solutions of $\det(A - \lambda I) = 0$ are $a_{11}, a_{22}, \ldots, a_{nn}$, the diagonal entries of A.

13. $\det(A^T - \lambda I) = \det(A^T - \lambda I^T)$ because $I^T = I$. Now $\det(A^T - \lambda I^T) = \det(A - \lambda I)^T$; but $\det B^T = \det B$ for any square matrix B, so $\det(A^T - \lambda I) = \det(A - \lambda I)$ and the eigenvalues for A and A^T satisfy the same characteristic equation. Thus A and A^T have the same eigenvalues.

15. Let $AX = \lambda X$. Then $\overline{AX} = \overline{\lambda X}$ or $\bar{A}\bar{X} = \bar{\lambda}\bar{X}$. Therefore $\bar{\lambda}$ is an eigenvalue of $\bar{A}$ whenever λ is an eigenvalue of $\bar{A}$.

17. We have $AX = \lambda X, A^2X = \lambda^2 X$ by Prob. 8. Now suppose $A^kX = \lambda^k X$ and show $A^{k+1}X = \lambda^{k+1}X$:

$$A^{k+1}X = A(A^kX) = A(\lambda^kX) = \lambda^kAX = \lambda^k\lambda X = \lambda^{k+1}X$$

Therefore, by induction, (λ^n, X) is an eigenpair of A^n.

19. Let λ be an eigenvalue of A so that $AX = \lambda X$, $X \neq 0$. Now by the nilpotency, for some $k > 0$, $A^kX = OX = 0$. On the other hand, by Prob. 17, $A^kX = \lambda^kX$, so that $\lambda^kX = 0$. Now because $X \neq 0$, we must have $\lambda = 0$.

PROBLEMS 5.3

1. (**a**) $D = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $P = \begin{pmatrix} -i & i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $P^{-1} = \begin{pmatrix} -i/2 & \frac{1}{2} & 0 \\ i/2 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(**b**) $D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{pmatrix}$ $P = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$ $P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$

(**c**) $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $P = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{pmatrix}$
$P^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 2 & 0 \\ -1 & -1 & -1 \end{pmatrix}$

(**d**) $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $P = \begin{pmatrix} 1 & 4 & 29 \\ 0 & 1 & 12 \\ 0 & 0 & 2 \end{pmatrix}$ $P^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -8 & 19 \\ 0 & 2 & -12 \\ 0 & 0 & 1 \end{pmatrix}$

(**e**) The eigenvalues are 1,2, and 3, with 3 having multiplicity 2. For $\lambda = 3$, we have solution to $(A - \lambda I)X = 0$ as

$$\begin{pmatrix} t \\ 2t \\ s \\ s \end{pmatrix}.$$

Thus we can choose linearly independent eigenvectors as

$$\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

So we have

$$D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 2 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

(**f**) $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad P = \begin{pmatrix} -1 & -1 & -1 \\ 4 & 1 & -2 \\ 1 & 1 & -1 \end{pmatrix}$

$P^{-1} = \dfrac{1}{6} \begin{pmatrix} -1 & 2 & -3 \\ -2 & -2 & 6 \\ -3 & 0 & -3 \end{pmatrix}$

(**g**) Not diagonalizable; $\lambda = 1$ is an eigenvalue of multiplicity 3 with eigenspace of dimension 1.

(**h**) $D = \begin{pmatrix} (3+i\sqrt{15})/2 & 0 \\ 0 & (3-i\sqrt{15})/2 \end{pmatrix} \quad P = \frac{1}{4} \begin{pmatrix} 4 & 4 \\ 1+i\sqrt{15} & 1-i\sqrt{15} \end{pmatrix}$

$P^{-1} = \dfrac{1}{30} \begin{pmatrix} 15+i\sqrt{15} & -4i\sqrt{15} \\ 15-i\sqrt{15} & 4i\sqrt{15} \end{pmatrix}$

(**i**) $D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad P^{-1} = \dfrac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(**j**) Not diagonalizable; $\lambda = i$ is an eigenvalue of multiplicity 2 with an eigenspace of dimension 1.

(**k**) $D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \quad P^{-1} = \dfrac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$

3. $A = PDP^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Therefore

$$\begin{aligned} A^{12} &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2^{12} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1/2^{-11} \\ 0 & 1/2^{12} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2-1/2^{11} \\ 0 & 1/2^{12} \end{pmatrix} \end{aligned}$$

5. Yes. $A = PDP^{-1}$ (with $d_{ii} \neq 0$, $i = 1, 2, \ldots, n$, by the invertibility of A). (Remember Prob. 10 of the last section.) Now D^{-1} exists; in fact the diagonal entries of D^{-1} are just $1/d_{ii}$, $i = 1, 2, \ldots, n$. So $A^{-1} = (PDP^{-1})^{-1} = P^{-1^{-1}}D^{-1}P^{-1} = PD^{-1}P^{-1}$. Therefore A^{-1} is diagonalizable.

7. A, having all positive entries, must have trace greater than zero. Since A is diagonalizable and the trace is invariant under similarity, trace $A =$ trace $D = \lambda_1 + \lambda_2 + \cdots + \lambda_n > 0$. Therefore since $\lambda_1, \ldots, \lambda_n$ are real, one of the eigenvalues must be positive in order for the sum to be positive.

9. Let $A = PDP^{-1}$. Because $A^2 = I$ we have $I = A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$.
Thus

$$\begin{aligned} P^{-1}IP &= P^{-1}(PD^2P^{-1})P \\ I &= D^2 \end{aligned}$$

Therefore any eigenvalue λ must have the property $\lambda^2 = 1$. The only possibilities are 1 or -1.

11. D_1D_2 is not necessarily a diagonalization of AB. $AB = P^{-1}D_1PQ^{-1}D_2Q$ and unless $P = Q$ we will not have $AB = P^{-1}D_1D_2P$. $D_1 + D_2$ is not necessarily a diagonalization of $A + B$. Consider

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Both are diagonalizable (see Examples 2 and 11) but their sum

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

is not diagonalizable.

PROBLEMS 5.4

1. $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad P = \begin{pmatrix} 1/\sqrt{2} & \frac{2}{3} & 1/\sqrt{6} \\ 0 & -\frac{1}{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & \frac{2}{3} & 1/\sqrt{6} \end{pmatrix}$

3. $D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad P = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

5. $D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad P = \dfrac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

7. Use Gram-Schmidt process on the eigenspace corresponding to $\lambda = 2$.

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 20 \end{pmatrix} \qquad P = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/6 & \frac{2}{3} \\ -\sqrt{2}/2 & \sqrt{2}/6 & \frac{2}{3} \\ 0 & 2\sqrt{2}/3 & \frac{1}{3} \end{pmatrix}$$

9. $\det(A - \lambda I) = \lambda^2 - 2\cos\theta\lambda + 1 = 0$. The roots are $\lambda = [2\cos\theta \pm \sqrt{4(\cos^2\theta - 1)}]/2$ which are complex unless $\theta = 0$ or π, which we have disallowed. If complex eigenvalues are allowed we have $\lambda = \cos\theta + i\sin\theta$ and

$$D = \begin{pmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos - i\sin\theta \end{pmatrix} \quad P = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

Thus A is diagonalizable if complex eigenvalues are admissible.

11. $D = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix} \quad P = \dfrac{\sqrt{3}}{3}\begin{pmatrix} i & 1+i \\ 1-i & -1 \end{pmatrix}$

13. $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & (1+\sqrt{5})/2 & 0 \\ 0 & 0 & (1-\sqrt{5})/2 \end{pmatrix}$

$$P = \begin{pmatrix} 0 & \sqrt{2/(5-\sqrt{5})} & \sqrt{2/(5+\sqrt{5})} \\ 0 & \sqrt{2/(5-\sqrt{5})}[(1-\sqrt{5})/2]i & \sqrt{2/(5+\sqrt{5})}[(1+\sqrt{5})/2)]i \\ 1 & 0 & 0 \end{pmatrix}$$

15. Calculate all possible standard inner products of distinct rows.

17. Since A is real symmetric it is diagonalizable and the diagonal of D consists of the eigenvalues of A. The determinant is invariant under similarity, so $\det A = \det D = \lambda_1 \cdot \lambda_2 \bullet \cdots \bullet \lambda_n$.

19. A is diagonalizable with

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

where $\lambda_1 > 0, \ldots, \lambda_n > 0$. Now

$$\begin{aligned} A = PDP^T &= P \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} P^T \\ &= P\sqrt{D}\,\sqrt{D}\ P^T = P\sqrt{D}(P^T P)\sqrt{D}\ P^T \\ &= (P\sqrt{D}\ P^T)(P\sqrt{D}\ P^T) \end{aligned}$$

Therefore
$$\sqrt{D} = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}$$
is a square root of S. So is

$$-\sqrt{D} = \begin{pmatrix} -\sqrt{\lambda_1} & & \\ & \ddots & \\ & & -\sqrt{\lambda_n} \end{pmatrix}$$

PROBLEMS 5.5

1. $Y(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + c_3 e^{-3t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

3. $Y(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

5. $Y(t) = c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} + c_4 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

7. Repeated eigenvalues.

9. (3) $Y(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow c_1 = -1$, $c_2 = 0$

11. The rate of change of concentration of salt in a cell is proportional to the difference in the concentrations of salts in the adjoining cells. Thus

$$\begin{aligned} \frac{y_1'}{v} &= K\left(\frac{y_2}{v} - \frac{y_1}{v}\right) + K\left(\frac{y_3}{v} - \frac{y_1}{v}\right) \\ \frac{y_2'}{v} &= K\left(\frac{y_1}{v} - \frac{y_2}{v}\right) + K\left(\frac{y_3}{v} - \frac{y_2}{v}\right) \end{aligned}$$

and

$$\frac{y_3'}{v} = K\left(\frac{y_1}{v} - \frac{y_3}{v}\right) + K\left(\frac{y_2}{v} - \frac{y_3}{v}\right)$$

Multiplying by volume v;

$$\begin{aligned} y_1'(t) &= K(y_2(t) - y_1(t)) + K(y_3(t) - y_1(t)) \\ y_2'(t) &= K(y_1(t) - y_2(t)) + K(y_3(t) - y_2(t)) \\ y_3'(t) &= K(y_1(t) - y_3(t)) + K(y_2(t) - y_3(t)) \end{aligned}$$

$$Y' = AY \qquad A = \begin{pmatrix} -2K & K & K \\ K & -2K & K \\ K & K & -2K \end{pmatrix}$$

The eigenvalues are $\lambda = 0,\ 3K$; $3K$ being of multiplicity 2.

ADDITIONAL PROBLEMS (CHAPTER 5)

1. No. The sum of diagonalizable matrices need not be diagonalizable, as was seen in an earlier problem. Thus the set of diagonalizable matrices is not closed under addition.

3. $\lambda^2 + a_2\lambda + a_1 \quad (-1)(\lambda^3 + a_3\lambda^2 + a_2\lambda + a_1)$

5. Form $A - \lambda I$ and do the following row and column operations $-R_n + R_{n-1},\ -R_n + R_{n-2}, \ldots, -R_n + R_2,\ -R_n + R_1$, then $C_1 + C_n,\ C_2 + C_n, \ldots, C_{n-1} + C_n$. The resulting matrix is lower triangular and the product of the diagonal elements is $(-\lambda)^{n-1}(n - \lambda)$.

7. $A = P^T DP, A^n = P^T D^n P$. If A is to be nilpotent, some power of D must be zero. This is only possible if $D = 0$. Now trace $A =$ trace $D = 0$.

9. $A - \lambda I = A - 0I = A$

$$A\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad A^2 = \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$A^3 = 0$ so

$$A^3\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

11. Yes

$$A = PDP^{-1} \quad \text{and} \quad D = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$$

with $\lambda_1 > 0, \ldots, \lambda_n > 0$. By the positivity of the eigenvalues we can form

$$\widetilde{D} = \begin{pmatrix}\sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n}\end{pmatrix}$$

and so $A = P\widetilde{D}\widetilde{D}P^{-1} = (P\widetilde{D}P^{-1})(P\widetilde{D}P^{-1})$. Thus $P\widetilde{D}P^{-1}$ is a square root of A.

13. The characteristic equation is $\lambda^2 + (k/m)\lambda + (c/m) = 0$. The solutions are

$$\lambda = \frac{-\dfrac{k}{m} \pm \sqrt{\left(\dfrac{k}{m}\right)^2 - 4\left(\dfrac{k}{m}\right)\left(\dfrac{c}{m}\right)}}{2}$$

We will have complex solutions if $(k/m)^2 - 4(kc/m^2) < 0$. This yields $k - 4c < 0$ or $k < 4c$.

15. The characteristic equation is $\lambda^2 + (1/(RC)\lambda + 1/(LC) = 0$ which has solutions

$$\frac{-1/(RC) \pm \sqrt{1/(R^2C^2) - 4/(LC)}}{2}$$

Complex solutions exist if $1/(R^2C^2) - 4/LC < 0$. This leads to $R^2 > L/(4C)$.

PROBLEMS 6.1

1. (**a**) H has eigenvalue 1, of multiplicity 10.

(**b**) The characteristic equation for $H + E$ is

$$(1-\lambda)^{10} - \frac{1}{2^{10}} = 0 \qquad \lambda = \frac{1}{2}$$

satisfies the equation.

(**c**) $\|E\|_F = (\Sigma |e_{ij}|^2)^{1/2} = \left(\left(\frac{1}{2^{10}} \right)^2 \right)^{1/2}$

Note that $\|E\|_F = 1/2^{10}$, but $|1 - \frac{1}{2}|$ is not less than $\|E\|_F$.

(**d**) H and E are not symmetric.

3. Consider a matrix A as a vector $\tilde{A} = (a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn})$ in E^{n^2}. Now $\|A\|_F$ is just the standrad form of $\tilde{A}$ in E^{n^2}. Reasoning this way $\|A + B\|_F = \|\tilde{A} + \widetilde{B}\|$ and $\|\tilde{A} + \widetilde{B}\| \leq \|\tilde{A}\| + \|\widetilde{B}\|$ by the triangle inequality. Thus $\|A+B\|_F \leq \|A\|_F + \|B\|_F$ for any two $n \times n$ matrices. Thus $\|D\|_F + \|I\|_F \geq \|D + I\|_F$.

5. $\|A\|_1 = 5, \|A\|_F = \sqrt{4+4+1} = 3$. The 1 norm is larger.

PROBLEMS 6.2

1. After the step with A^6 calculated the approximate dominant eigenpair is

$$\left(9.09, \begin{pmatrix} 1 \\ 1.62 \end{pmatrix} \right)$$

3. No dominant eigenpair. A has all complex eigenvalues.

5. After the step with A^6 calculated the approximate dominant eigenpair is

$$\left(7.16, \begin{pmatrix} 1 \\ 1.39 \\ 0 \end{pmatrix}\right)$$

7. As described in this section, the power method will not find complex eigenvalues. However the method can be modified. For example consider

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 3i \end{pmatrix}$$

The dominant eigenpair is

$$\left(2i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right)$$

Choose

$$X_0 = \begin{pmatrix} i \\ 0 \end{pmatrix}$$

and compute $AX_0, A^2X_0, \ldots$ as before. We find, using the trick of scaling that the scaled versions of A^6X_0 and A^7X_0 are

$$\begin{pmatrix} -.984i \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -.992i \\ 1 \end{pmatrix}$$

so we can choose

$$\begin{pmatrix} -.992i \\ 1 \end{pmatrix}$$

as an approximate eigenvector.

Then we calculate

$$A\begin{pmatrix} -.992i \\ 1 \end{pmatrix}$$

and see if it is nearly a multiple of

$$\begin{pmatrix} -.992i \\ 1 \end{pmatrix}$$

We have

$$A\begin{pmatrix} -.992i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2.008i \end{pmatrix}$$

Now $2/.992i = 2.016i$; $2.008i/1 = 2.008i$. These ratios are nearly equal. We could use their average $2.012i$ as an approximate eigenvalue.

So we see that a power rule can work for matrices with complex eigenvalues. However if A has all real entries and complex eigenvalues, they must occur in conjugate pairs which is trouble as far as dominance is concerned. In any case complex arithmetic must be used on the computer.

9. (1) $\left(-2.09, \begin{pmatrix} -1.611 \\ 1 \end{pmatrix}\right)$

(2)

$$(\lambda_2, X_2) = \left(2, K\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

(λ_3, X_3) are not found; lack of symmetry causes this difficulty.

(5) $(\lambda_2, X_2) = \left(6, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \quad (\lambda_3, X_3) = \left(.841, \begin{pmatrix} -1.41 \\ 1 \\ 0 \end{pmatrix}\right)$

PROBLEMS 6.3

1. $QR = \begin{pmatrix} -.1960 & -.9806 \\ -.9806 & .1960 \end{pmatrix}\begin{pmatrix} -5.099 & -6.864 \\ 0 & -3.727 \end{pmatrix}$

3. $QR = \begin{pmatrix} -.7072 & .7072 \\ .7072 & .7071 \end{pmatrix}\begin{pmatrix} -2.8288 & -1.4144 \\ 0 & 2.8287 \end{pmatrix}$

5. $QR = \begin{pmatrix} -.7072 & -.7072 & 0 \\ -.7072 & .7071 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -4.2432 & -5.6576 & 0 \\ 0 & 1.4139 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

7. $A_1 = A_7 = A_{13} = \cdots = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}$

$A_2 = A_8 = A_{14} = \cdots = \begin{pmatrix} 1.6 & -2.8 \\ 1.2 & .4 \end{pmatrix}$

The method fails to converge; it cycles.

ADDITIONAL PROBLEMS (CHAPTER 6)

1. The eigenvalues are -15.88 and -15.92.

3. The eigenvalues are -1 and $-\frac{1}{3}$.

5. The eigenvalues are 0, $-1.2324k_1$, and $-2.4342k_1$, the last two being approximate.

7. None of the circles
$|z - i| \le \frac{1}{2}, \quad |z - (1 + i)| < .4, \quad |z + 3i| < .1$
has circumference or interior intersecting the real axis.

PROBLEMS 7.1

1. Maximum is 12, achieved at (4,0). Minimum is -8, achieved at (0,4).

3. Maximum is 0, achieved at (0,0). Minimum is -38, achieved at (3,7).

5. The graph of $xy = 16$ touches the feasible region at (4,4). If $xy = k > 16$, then $xy = k$ does not intersect the feasible region at all.

7. Consider $T = -xy$ with the same feasible region as in Prob. 5.

9. Producing 80 basic models and 120 self-propelled models maximizes the revenue at \$44,400.

11. The lines $x + y = k$ intersect the feasible region for all $k, k \ge \frac{1}{2}$.

PROBLEMS 7.2

1. $A_0 = \begin{pmatrix} 5 \\ 8 \\ 4 \end{pmatrix} \quad A_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$A_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$y_1 = x \quad y_2 = y \quad y_3 = z_1 \quad y_4 = z_2 \quad y_5 = z_3$

$\mathrm{P} = \begin{pmatrix} 3 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad N = n + m = 5$

Maximize $\quad P^T Y \Rightarrow$ maximize $3y_1 - 2y_2$

Subject to $\quad Y \geq 0 \Rightarrow x \geq 0, y \geq 0, z_1 \geq 0, z_2 \geq 0, z_3 \geq 0$

$$A_0 = \sum_{j=1}^{3} y_j A_j \Rightarrow \begin{pmatrix} 5 \\ 8 \\ 4 \end{pmatrix} = y_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = y_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$+ y_4 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + y_5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} y_1 & +y_3 & & \\ y_2 & & +y_4 & \\ y_1 + y_2 & & & +y_5 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & +z_3 & & \\ x_2 & & +z_2 & \\ x_1 + x_2 & & & +z_3 \end{pmatrix}$$

3. $A_0 = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix} \quad A_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$A_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$y_1 = x \quad y_2 = y \quad y_3 = z_1 \quad y_4 = z_2 \quad y_5 = z_3$

$$P = \begin{pmatrix} -1 \\ -5 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad N = 5$$

Maximize $\quad P^T Y \Rightarrow$ maximize $-1y_1 + 5y_2$
Subject to $\quad Y \geq 0 \Rightarrow x \geq 0 \quad y \geq 0 \quad z_1 \geq 0 \quad z_2 \geq 0 \quad z_3 \geq 0$

$$A_0 = \sum_{j=1}^{5} y_j A_j \Rightarrow \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix} = y_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + y_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$+ y_4 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + y_5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} y_1 & & +y_3 & & \\ & y_2 & & +y_4 & \\ y_1 + y_2 & & & & +y_5 \end{pmatrix}$$

$$= \begin{pmatrix} x & +z_1 & & \\ y & & +z_2 & \\ x + y & & & +z_3 \end{pmatrix}$$

5. Maximize $\quad 2y_1 + y_2 + 3y_3$
Subject to $\quad y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0, y_5 \geq 0, y_6 \geq 0$ and

$$\begin{pmatrix} 450 \\ 1200 \\ 900 \end{pmatrix} = \begin{pmatrix} 3y_1 + 2y_2 & & +y_4 & & \\ 3y_1 + 2y_2 + 2y_3 & & & +y_5 & \\ y_2 + 2y_3 & & & & +y_6 \end{pmatrix}$$

PROBLEMS 7.3

1. **(a)** $\underline{B_1 = \{A_3, A_4, A_5\}} \Rightarrow y_1 = 0 \; y_2 = 0 \; y_3 = 5 \; y_4 = 8 \; y_5 = 4$;
$\underline{\underline{P^T Y}} = 0$
$\{A_2, A_4, A_5\} \Rightarrow$ Not a basis
$\{A_1, A_4, A_5\} \Rightarrow y_1 = 5 \; y_2 = 0 \; y_3 = 0 \; y_4 = 8 \; y_5 = -1$:
Not feasible
$\{A_2, A_3, A_5\} \Rightarrow y_1 = 0 \; y_2 = 8 \; y_3 = 5 \; y_4 = 0 \; y_5 = -4$:
Not feasible
$\{A_1, A_3, A_5\} \Rightarrow$ Not a basis
$\{A_2, A_3, A_4\} \Rightarrow y_1 = 0 \; y_2 = 4 \; y_3 = 5 \; y_4 = 0 \; y_5 = 4$:
$P^T Y = 8$
$\{A_1, A_3, A_4\} \Rightarrow y_1 = 4 \; y_2 = 0 \; y_3 = 1 \; y_4 = 8 \; y_5 = 0$:

$P^TY = 12$

$\underline{B_2 = \{A_1, A_3, A_4\}} \Rightarrow \underline{\underline{P^TY = 12}}$

$\{A_1, A_2, A_4\} \Rightarrow y_1 = 5\ y_2 = -1\ y_3 = 0\ y_4 = 9\ y_5 = -0$:

Not feasible

$\{A_1, A_2, A_3\} \Rightarrow y_1 = 4\ y_2 = 8\ y_3 = 9\ y_4 = 0\ y_5 = 0$:

Not feasible

$\text{Max} = 12 \qquad \text{Basis } \{A_1, A_3, A_4\} \Rightarrow y_1 = 4, y_2 = 0.$

(b)
$$\left(\begin{array}{ccc|ccc|c} 1 & -3 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 & 0 & 8 \\ 0 & 1 & 1 & 0 & 0 & 1 & 4 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc|c} 1 & 0 & 5 & 0 & 0 & 3 & 12 \\ \hline 0 & 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 8 \\ 0 & 1 & 1 & 0 & 0 & 1 & 4 \end{array}\right)$$

$\text{Max} = 12$

$y_1 = 4 \qquad y_2 = 0$

3. **(a)** $\underline{B_1 = \{A_3, A_4, A_5\}} \Rightarrow \underline{\underline{P^TY = 0}}$

$\{A_1, A_4, A_5\} \Rightarrow (5, 0, 0, 7, 6) \qquad P^TY = -5$

$\{A_1, A_3, A_5\} \Rightarrow$ Not basis

$\{A_1, A_3, A_4\} \Rightarrow (10, 0, -5, 7, 0) \qquad$ Not feasible

$\{A_2, A_4, A_5\} \Rightarrow$ Not basis

$\{A_2, A_3, A_5\} \Rightarrow (0, 7, 5, 0, 3) \qquad P^TY = -35$

$\{A_2, A_3, A_4\} \Rightarrow (0, 10, 5, -3, 0) \qquad$ Not feasible

$\text{Max} = 0 \qquad y_1 = 0 \qquad y_2 = 0$

(b)
$$\left(\begin{array}{ccc|ccc|c} 1 & 1 & 5 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 & 0 & 7 \\ 0 & 1 & 1 & 0 & 0 & 1 & 10 \end{array}\right)$$

$\text{Max} = 0$

$y_1 = 0 \qquad y_2 = 0$

5.
$$\left(\begin{array}{cccc|ccc|c} 1 & -2 & \overset{\downarrow}{-4} & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 2 & 1 & 1 & 0 & 0 & 400 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 100 \\ 0 & 1 & 3 & 0 & 0 & 0 & 1 & 200 \end{array}\right)$$

$$\left(\begin{array}{cccc|ccc|c}
1 & -\frac{2}{3} & 0 & \overset{\downarrow}{-1} & 0 & 0 & \frac{4}{3} & \frac{800}{3} \\
\hline
0 & \frac{1}{3} & 0 & 1 & 1 & 0 & -\frac{2}{3} & \frac{800}{3} \\
0 & -\frac{1}{3} & 0 & 1 & 0 & 1 & -\frac{1}{3} & \frac{100}{3} \\
0 & \frac{1}{3} & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{200}{3}
\end{array}\right)$$

$$\left(\begin{array}{cccc|ccc|c}
1 & \overset{\downarrow}{-1} & 0 & 0 & 0 & 1 & 1 & 300 \\
\hline
0 & \frac{2}{3} & 0 & 0 & 1 & -1 & -\frac{1}{3} & \frac{700}{3} \\
0 & -\frac{1}{3} & 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{100}{3} \\
0 & \frac{1}{3} & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{200}{3}
\end{array}\right)$$

$$\left(\begin{array}{cccc|ccc|c}
1 & 0 & 3 & 0 & 0 & 1 & 2 & 500 \\
\hline
0 & 0 & -2 & 0 & 1 & -1 & -1 & 100 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 100 \\
0 & 1 & 3 & 0 & 0 & 0 & 1 & 200
\end{array}\right)$$

$\text{Max} = 500 \qquad y_1 = 200 \qquad y_2 = 0 \qquad y_3 = 100$