1.2.22 A graph G is connected if and only if for every partition of its vertices into two non-empty subsets, there is an edge with an endpoint in each set.

Proof.

($\Longrightarrow$) Suppose G is connected. Take an arbitrary partition $V(G) = X \cup Y$ of $V(G)$ into two non-empty sets. We need to show that G has an edge with one endpoint in X and the other in Y. Select vertices x in X, and y in Y. (Possible because X and Y are non-empty.) Since G is connected, there has to be a path in G that joins x to y. Denote this path as follows.

$$x = x_0, x_1, x_2, x_3, \ldots, x_n = y$$

The first vertex x_0 of this path is in X, and the last vertex x_n is in Y. Any one of the others is either in X or Y. Let i be the smallest index for which $x_i \in Y$. (Such an i exists, because $x_n \in Y$, so i is at most n.) Now we have $x_{i-1} \in X$ and $x_i \in Y$, so $x_{i-1}x_i$ is an edge of G with one endpoint in X and the other in Y.

($\Longleftarrow$) Suppose that for any partition $V(G) = X \cup Y$ of $V(G)$ into two non-empty sets, G has an edge with one endpoint in X and the other in Y. We need to show G is connected. For the sake of contradiction, suppose G is not connected. Let C be one of its components. Now we have a partition

$$V(G) = V(C) \cup (V(G) - V(C))$$

of $V(G)$ into two non-empty sets. By assumption, G has an edge with endpoints in each set in this partition. That is to say G has an edge with one endpoint in one of its components and the other endpoint in another component. This is a contradiction. ■