MA859: Selected Topics in Graph Theory

Lecture -12

Representation of Graphs

— Adjacency Matrix.

Although pictorial representation of a graph is convenient for visual study, it is important to find some equivalent representations for a better utilization of this concept, particularly for computer processing.

A matrix is a convenient and useful way of representing a graph to a computer. The matrices lend themselves easily to manipulations.

Furthermore, a number of known results of matrix algebra can be applied to study the structural properties of graphs.

In several applications of Graph Theory, such as electrical networks, operations research etc., matrices turn out to be a natural way of expressing the problems.

Adjacency Matrix

This representation is probably the most frequently used matrix representation of a graph.

Let $G = (V, E, \phi)$ be a graph on n vertices that are labeled $\{v_1, v_2, \ldots, v_n\}$. For each $i, j \in \{1, 2, \ldots, n\}$, define the entry a_{ij} by

$$a_{ij} = |\{e \in E(G) : \phi(e) = \{v_i v_j\}\}|$$

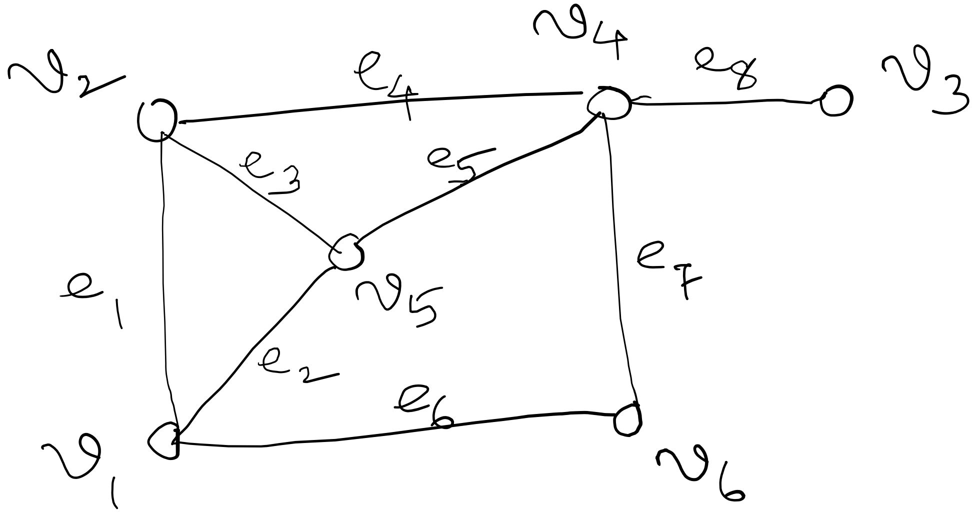
That is, a_{ij} is the number of edges connecting v_i and v_j.

The adjacency matrix of G w.r.t. the labeling of $V(G)$ is an $n \times n$ matrix

$$A(G) = [a_{ij}]_{i,j \in \{1,2,\dots n\}}$$

Ex 1

G:



$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Remarks:

* For any graph G and any listing of vertices, the adjacency matrix $A(G)$ is a symmetric matrix.
* Any symmetric $n \times n$ matrix A with a_{ij} a non-negative integer is an adjacency matrix for some graph G on n vertices.
* For a simple graph G, $A(G)$ is a symmetric binary matrix. In addition, $a_{ii} = 0$ for each i, since there are no loops.

* The no. of edges in G is the sum of all the entries a_{ij}, where $i \geq j$. Since $A(G)$ is symmetric,

$$|E(G)| = \sum_{i \geq j} a_{ij} = \sum_{i \leq j} a_{ij}$$

* Row total / Column total is the degree of the corresponding vertex of G.

$$\deg_G v_i = \sum_{l=1}^{n} a_{il} + a_{ii} = \sum_{l=1}^{n} a_{li} + a_{ii}$$

The notation $A(G)$ for the adjacency matrix of G w.r.t. a given labeling $V(G)=\{v_1, v_2, --, v_n\}$ of vertices suggests that the matrix only depends on the graph. But this is not quite true, since we need a fixed labeling of the vertices.

Suppose we have another labeling $V(G)=\{v_1', v_2', --, v_n'\}$, where $v_i' = v_{\sigma(i)}$ (say) for some fixed permutation

$$\sigma : \{1, 2, --n\} \longrightarrow \{1, 2, ---, n\}.$$

If $A(G)' = [a'_{ij}]_{i,j \in \{1,2,\ldots n\}}$ is the adjacency matrix w.r.t this labeling, then clearly,

$$a'_{ij} = a_{\sigma(i)\,\sigma(j)} \longrightarrow (1)$$

Let $\tilde{\sigma}(i)$ denote the column vector with a one in row $\sigma(i)$ and zero everywhere else. We can define the square permutation matrix

$$P_\sigma = [\, \tilde{\sigma}(1) \,|\, \tilde{\sigma}(2) \,|\, \cdots \,|\, \tilde{\sigma}(n) \,]$$

(vertical bars are shown to show the separate columns)

This matrix is orthogonal and hence

$$P_\sigma^{-1} = P_\sigma^{T}$$

Moreover, if σ and ρ are two permutations on $\{1, 2, \ldots, n\}$, then

$$P_\sigma P_\rho = P_{\sigma\rho}$$

where $\sigma\rho$ is the function composition of the permutations.

From (1), we therefore have

$$A(G) = P_\sigma^{-1} A(G) P_\sigma = P_\sigma^{-1} A(G) P_\sigma$$

Defn: Two matrices X and Y are orthogonally equivalent if there is a permutation matrix P such that $Y = P^{-1} X P$.

Remark: Orthogonal equivalence among $n \times n$ matrices is an equivalence relation (Prove it).

Recall: Two graphs G and H are isomorphic if there exists a bijective vertex map $f: V(G) \to V(H)$ so that there are equally many edges between u and v in G as there are between $f_1(u)$ and $f_1(v)$ in H.

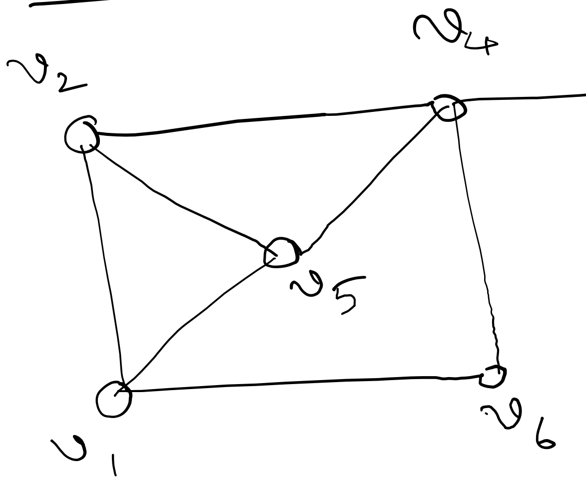
Suppose $V(G) = \{u_1, u_2, \dots u_n\}$ and $V(H) = \{u_1', u_2', \dots, u_n'\}$.
Then there must be a permutation σ such that $f_1(u_i) = u'_{\sigma(i)}$ for each $i \in \{1, 2, \dots, n\}$. This means that the following statements are equivalent:

① The graphs G and H are isomorphic.

② The adjacency matrices $A(G)$ and $A(H)$ are orthogonally equivalent w.r.t. any labeling of their vertices.

The following is an illustration on how various matrix manipulations relate to graph theoretic properties.

Ex 2 Consider the same example as in Ex. 1.



$\leftarrow G$

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A^2(G) = \begin{bmatrix} 3 & 1 & 0 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 1 & 2 & 1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 & 2 & 2 \end{bmatrix}$$

We will now interpret the entries of the above matrix. For each i and j, denote by $a_{ij}^{(2)}$, the entry in row i and column j in $A^2(G)$.

In general, $a_{ij}^{(k)}$ denotes the entry in row i and column j in $A^k(G)$.

By the rules of matrix multiplication,

$$a_{ij}^{(2)} = \sum_{l=1}^{n} a_{il}\, a_{lj} = a_{i1}a_{1j} + a_{i2}a_{2j} + \cdots + a_{in}a_{nj}$$

Note that $a_{il}a_{lj} = 1$ only if both $a_{il} = a_{lj} = 1$

$\Rightarrow v_i$ is adjacent to v_l and v_l is adjacent to v_j.

Therefore, $a_{ij}^{(2)}$ counts the no. of walks of length two between v_i and v_j in G.

Similarly, continuing to obtain $A^3(G)$, we get $a_{ij}^{(3)}$ as the no. of walks from v_i to v_j of length three.

$$A^3(G) = \begin{bmatrix} 2 & 7 & 3 & 2 & 7 & 6 \\ 7 & 4 & 1 & 8 & 5 & 2 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 2 & 8 & 4 & 2 & 8 & 7 \\ 7 & 5 & 1 & 8 & 4 & 2 \\ 6 & 2 & 0 & 7 & 2 & 0 \end{bmatrix}$$

For example: $a_{15}^{(3)} = a_{51}^{(3)} = 7$

$$w_1 = (v_1\, e_1\, v_2\, e_1\, v_1\, e_2\, v_5)$$

$$w_2 = (v_1\, e_2\, v_5\, e_2\, v_1\, e_2\, v_5)$$

$$w_3 = (v_1\, e_6\, v_6\, e_6\, v_1\, e_2\, v_5)$$

$$w_4 = (v_1\, e_2\, v_5\, e_3\, v_2\, e_3\, v_5)$$

$$w_5 = (v_1\, e_6\, v_6\, e_7\, v_4\, e_5\, v_5)$$

$$w_6 = (v_1\, e_2\, v_5\, e_5\, v_4\, e_5\, v_5) \quad \text{and}$$

$$w_7 = (v_1\, e_1\, v_2\, e_4\, v_4\, e_5\, v_5)$$

Generalizing this example, we obtain

<u>**Theorem**</u> Let G be a simple graph with vertex labeling $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let k be a natural number greater than zero. The entry $a_{ij}^{(k)}$ is the number of distinct walks from v_i to v_j of length k in G.

<u>**Proof**</u>: Exercise

(Hint: Use induction on k using the same method as described in the example).

Corollary 1 Let G be a connected graph with vertex labeling $V(G) = \{v_1, v_2, \ldots, v_n\}$. The distance between two distinct vertices v_i and v_j is the smallest natural number k for which the entry $a_{ij}^{(k)}$ in $A^k(G)$ is non-zero.

Corollary 2 Suppose G is a graph on n vertices. Let $Y = A(G) + A^2(G) + \cdots + A^{n-1}(G)$. Then G is connected if and only if all entries of Y are non-zero.

// END OF LECTURE //