MA859: Selected Topics in Graph Theory
Lecture - 21
Domination in Graphs

May 19, 2021

Introduction

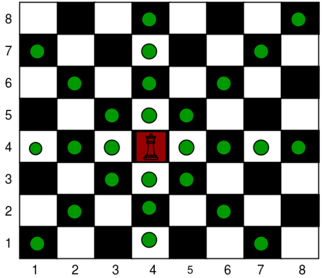
It is natural in any game that the player wants to win the game and his strategy would be to dominate the opponent. Chess is one such ideal example where each player concentrates on dominating over the coins of his opponent. Russians were leading in this game at some point time in the mid of 19th century and perhaps, the origin of study of domination started there.

All of us who know the Chess game, know the movement of the different coins and it is, no doubt, a very complex game to deal with. People started studying this game with mathematical approach and tried to strategise the theory.

What is domination?

Dominating Queens

Consider a standard 8×8 Chessboard. A Queen can, in one move, advance any number of squares horizontally, vertically or diagonally (assuming, of course, that no other chess piece lies in the way).

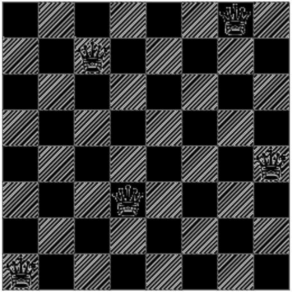


What is the minimum number of queens needed to cover the entire board?

What is domination?

Dominating Queens

Answer is 5:



Other applications

This concept of domination does not limit to only the Chessboard (although it turns out to be the origin of the concept.) There are ample areas where one finds applications starting from Social Networks to Computer Networks to Medical Science (especially in genetic theory). We will, however, not discuss them at the moment.

Definitions:

1. Two vertices u and v are said to *dominate* each other if they are adjacent.
2. A set $S \subseteq V$ of vertices of a graph $G = (V, E)$ is called a *dominating set* if every vertex $v \in V$ is either in S or is adjacent to a vertex in S.
3. For any vertex $v \in V$, the set $N(v) = \{u \in V : uv \in E\}$ is called the *open neighbourhood* of v; while $N[v] = N(v) \cup \{v\}$ is called the *closed neighbourhood* of v.
4. For a set $S \subseteq V$, a vertex v is called an *enclave* of S if $N[v] \subseteq S$.
   A vertex v is called an *isolate* of S if $N(v) \subseteq V - S$.
   A set of vertices $U \subseteq V$ is said to be *enclaveless* if it does not contain any enclaves.

Let $S \subseteq V$ in a graph $G = (V, E)$. S is a dominating set of G if and only if

1. for every $v \in V - S$, there exists a vertex $u \in S$ such that uv is an edge of G.
2. for every $v \in V - S$, $d(v, S) \leq 1$ [$d(v, S)$ is the length of a shortest path from v to a vertex in S].
3. $N[S] = V$.
4. for every vertex $v \in V - S$, $|N(v) \cap S| \geq 1$.
5. for every vertex $v \in V$, $|N[v] \cap S \geq 1$.
6. $V - S$ is enclaveless.

Note that if S is a dominating set of a graph G, then every superset of S is also a dominating set. So, the set V itself is an obvious dominating set of the graph G.
On the other hand, not every subset of S need be a dominating set of G. This fact leads us to define a minimal dominating set.

Minimal Dominating Set

A dominating set S of G is a minimal dominating set of G if no proper subset $S' \subset S$ is a dominating set.

Example:

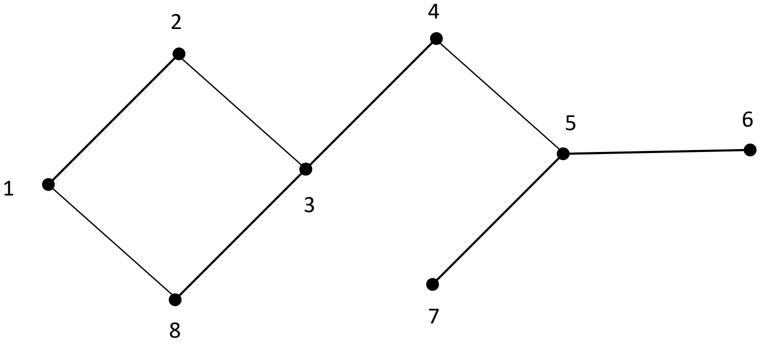


Figure 1: Example for illustrating Dominating Sets

In the figure 1, $\{1, 3, 5\}$, $\{3, 6, 7, 8\}$, $\{2, 4, , 6, 7, 8\}$ are the examples of minimal dominating sets.

Exercise: Are there any more minimal dominating sets of the above graph? If yes, list them out.

Theorem 1 (Ore)

A dominating set S of a graph $G = (V, E)$ is a minimal dominating set if and only if for each $u \in S$, one of the following two conditions holds:

a) *u is an isolate of S or*

b) *there exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$.*

Proof:

Assume that S is a minimal dominating set of G. Then for every vertex $u \in S$, $S \setminus \{u\}$ is not a dominating set. This means that some vertex v in $(V - S) \cup \{u\}$ is not dominated by any vertex in $S \setminus \{u\}$.

There are two possibilities:

1. $v = u$. This means that u is an isolate of S.
2. $v \in V - S$. This means that if v is not dominated by $S \setminus \{u\}$; but is dominated by S, then v is adjacent only $u \in S$, that is, $N(v) \cap S = \{u\}$.

Proof [*Contd...*]

Conversely, suppose that S is a dominating set and for each vertex $u \in S$, one of the conditions (a) or (b) [given in the statement of the theorem] holds. Suppose that S is not a minimal dominating set. Then there exists a vertex $u \in S$ such that $S \setminus \{u\}$ is still a dominating set. This means that u is adjacent to at least one vertex in $S \setminus \{u\}$, that is, condition (a) does not hold.
Also, if $S \setminus \{u\}$ is a dominating set, then every vertex in $V - S$ is adjacent to at least one vertex in $S \setminus \{u\}$, that is, condition (b) does not hold for u.
Thus, neither condition (a), nor condition (b) holds, which contradicts our assumption that at least one these conditions must hold. $\square$

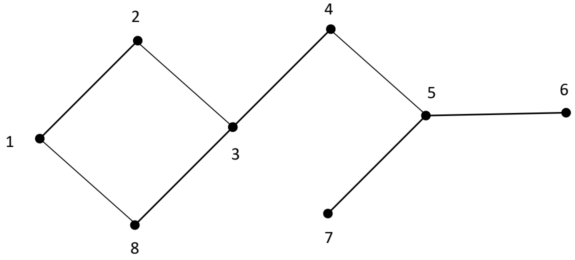
Theorem 1 suggests a definition:

Definition 1

Let S be a set of vertices and $u \in S$. A vertex v is called a private neighbour of u (w.r.t. S) if $N[v] \cap S = \{u\}$. Furthermore, we define the private neighbour set of u w.r.t. S as $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$.

Note that $u \in pn[u, S]$ if u is an isolate in $< S >$, in which case, we say that u is own private neighbour.

Given this terminology, we can say that a dominating set S is a minimal dominating set if and only if every vertex in S has at least one private neighbour, that is, for every $u \in S$, $pn[u, S] \neq \emptyset$.
For example, consider the same graph considered earlier:



$\{3, 6, 7, 8\}$ is a minimal dominating set of the above graph.
3 has 2 and 4 as its private neighbours.
8 has 1 as a private neighbour.
6 and 7 are their own private neighbours.

Theorem 2 (Ore)

Every connected graph G on at least 2 vertices has a dominating set S such that $V - S$ is also a dominating set.

Proof:

Let T be a spanning tree of G and $u \in V$. Let S be the set of vertices consisting of u and all those vertices that are at even distance from u in T. One can easily show that S and $V - S$ are dominating sets of G.

Theorem 3

If a graph G has no isolated vertices, then for every minimal dominating set S of G, $V - S$ is also a dominating set of G.

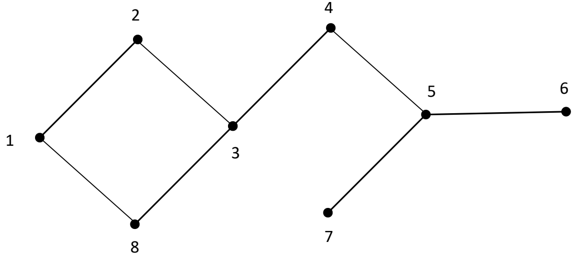
Proof:

Let S be any minimal dominating set of G. Assume that a vertex $u \in S$ is not dominated by any vertex n $V - S$. Since G has no isolated vertices, u must be dominated by at least one vertex in $S \setminus \{u\}$, which means that $S \setminus \{u\}$ is a dominating set of G. This is a contradiction to our assumption that S is a minimal dominating set. Thus, every vertex in S is dominated by at least one vertex in $V - S$. Hence $V - S$ is also a dominating set. $\square$

Definition 2

Let $G = (V, E)$ be any graph. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$.

The maximum cardinality of a minimal dominating set of G is called the upper domination number and is denoted by $\Gamma(G)$.



Here $\gamma(G) = 3$ and $\Gamma(G) = 5$. A dominating set with cardinality $\gamma(G)$ is called a $\gamma - set$ and a dominating set with cardinality $\Gamma(G)$ is called a $\Gamma - set$.

$\{1, 3, 5\}$ and $\{2, 5, 8\}$ are $\gamma - sets$.

$\{2, 4, 6, 7, 8\}$ and $\{1, 2, 4, 6, 7\}$ are $\Gamma - sets$.

Note that the set $\{1, 3, 5\}$ is also an independent set. This prompts us to define......

Definition 3

A minimal dominating set which is independent is called an independent dominating set. The minimum cardinality of an independent dominating set is called independent domination number and is denoted by $i(G)$.

Note: For any graph G, $\gamma(G) \leq i(G)$.

Remark: Like independent domination, there are many versions of domination based on specific properties of the vertices of the dominating set. Connected Domination, Total Domination, Distance Domination etc. are some such examples. We shall discuss about them a little later.

Exercise

Let $\epsilon_F(G)$ denote the maximum number of pendant edges in a spanning forest of a graph G on n vertices. Then show that $\gamma(G) + \epsilon_F(G) = n$.
(This result is due to Nieminen. Easy to prove. You will find this result on some websites too.)