MA 859: Selected Topics in Graph Theory

LECTURE-3

TREES

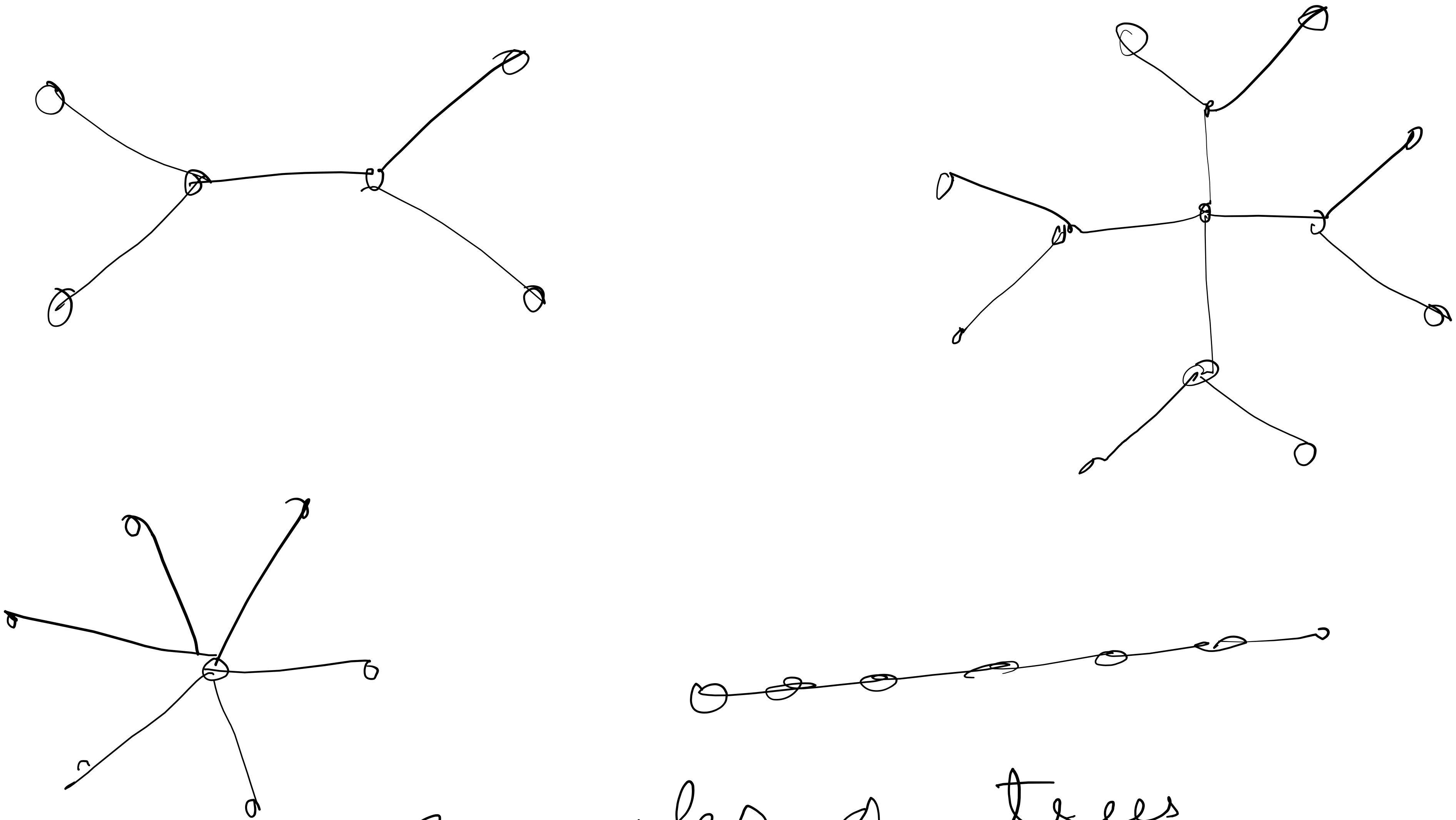
A graph having no cycle is called <u>acyclic</u>.

A <u>forest</u> is an acyclic graph.

A <u>tree</u> is a connected acyclic graph

A <u>leaf</u> is a vertex of degree 1.

↳ Also called 'pendant vertex'



Examples of trees

Spanning Tree

Given a graph G, its spanning subgraph that is a tree is called a spanning tree.

(*) Every tree with at least two vertices has at least two leaves. Deleting a leaf from a tree (on n vertices) produces a smaller tree (on $n-1$ vertices)

Proof: Every connected graph with at least two vertices has an edge. In an acyclic graph, the end points of a maximal non-trivial path have only one neighbour on the path & therefore have degree 1. Hence the end points of a maximal path provide the two desired leaves.

Now, suppose v is a leaf of a tree G and let $G' = G - v$. If $u, w \in V(G')$, then no u-w path P in G can pass through v (because its degree is 1). So, P is also present in G'. Thus G' is connected.

Also, clearly a vertex deletion can not create a cycle $\Rightarrow$ G' is acyclic.

Thus G' is a tree on $n-1$ vertices. //

Theorem

For an n-vertex graph G $(n \geq 1)$, the following are equivalent:

A) G is connected & has no cycles.

B) G is connected & has $n-1$ edges

C) G has $n-1$ edges & no cycles.

D) For $u, v \in V(G)$, G has exactly one u-v path.

Proof :

$A \Rightarrow B, C$: We use induction on n.

For $n=1$, an acyclic 1-vertex graph has no edge.

For the induction step, suppose $n>1$ & assume that the statement holds for graphs with fewer than n vertices.

Now, from $\otimes$ (in Slide 5), for any leaf v, $G' = G - v$ is acyclic. Applying the induction hypothesis on G', we have $n-2$ edges in G' & hence the no. of edges in G must be $n-1$.

$B \Rightarrow A, C$: Delete edges from the cycles of G one by one until the resulting graph G' is acyclic. Since no edge of a cycle has the property that when removed, does not result in a disconnected graph, we have G' to be connected. And in view of the result (established in the previous slide), G' has $n-1$ edges. But this equals the no. of edges in G. So, G itself must be acyclic.

$C \Rightarrow A, B$: Suppose G has k components with $n_1, n_2, \ldots, n_k$ no. of vertices. Since G has no cycles, each component must satisfy (A). Also, in view of the pf in Slide 8, i^{th} component has $n_i - 1$ edges.

Now the no. of edges in $G = \sum_{i=1}^{k} (n_i - 1) = n - k$.

But since G has $n-1$ vertices, $k = 1$

$\Rightarrow$ G is connected.

$A \Rightarrow D$: Let $u, v \in V(G)$. Since G is connected, there must be at least one u-v path. If G has distinct u-v paths P & Q, then some edge $e = xy$ appears in exactly one of them. Together, they form a closed walk in which e occurs once. Deleting e results in an x-y walk W not containing e. Clearly W contains a x-y path R. Now, the cycle $R \cup e$ contradicts the hypothesis that G is acyclic.

$\Rightarrow G$ has exactly one u-v path.

$D \Rightarrow A$: If there is a u-v path for every $u, v \in V(G)$, then G is connected. If G has a cycle C, then G must have two paths between any pair of vertices on C, which is impossible. Hence the result. //