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**Subject: Selected Topics in Graph Theory**

**Assignment – 2**

1. **Justify whether the following statement is TRUE or FALSE:**

**Every graph with fewer edges than vertices has a component that is a tree.**

**Ans:** The above statement is True.

By definition a tree is a connected acyclic graph i.e. a graph connected graph with no cycles is called a tree. Let G be a graph with E(G) < V(G) and let G1, . . ., Gk be the components of G. Assume that no Gi is a tree. Then E(Gi) > n(Gi) for all i = 1, . . ., k (because a connected graph H with E(H) < V(H)−1 is not connected, with E(H) = V(H) − 1 is a tree and with E(H) > V(H) is not acyclic). Summing up for all i = 1, . . ., k gives E(G) > V(G), a contradiction.

Hence, above statement is **True.**

1. **Prove that a graph G is a tree if and only if for all x, y ∈ V (G), adding a copy of xy as an edge creates exactly one cycle.**

**Ans:** Assume that G is acyclic, but for any xy does not belong to E(G) the graph G + xy has a cycle. Let’s prove that G is a tree. By definition we know that a tree is connected acyclic graph. here, it is given that G is acyclic, so we only need to prove that G is connected. Assume otherwise that there is no x-y-path in G for some two vertices x and y. Then in particular xy does not belong to E(G). However, G union {xy} has a cycle C and this cycle must contain the edge xy. Thus, there are two edge-disjoint x-y-paths, one of which does not contain the edge xy and thus is a path in G. So, there is an x-y-path in G, a contradiction.

Hence the proof.

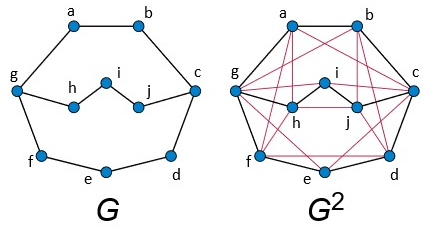
1. **If x and y are adjacent vertices in a connected graph G, then show that |d(x, z) − d(y, z) | ≤ 1 for any vertex z in G.**

**Ans:** We know that, the distance between any two vertices - x, y in a graph is the shortest path between them. Now consider a connected graph G, let the x and y be two adjacent vertices of graph G. Let z some other vertex of graph G and d(x,z) = l. Then while finding d(y,z) will be either l, if we don’t go through edge xy, or it’ll be l+1, if we go through edge xy. These are the only two possibilities. Hence, absolute difference between d(x,z) and d(y,z) is either 0 or 1. This proves the above statement.

**Q4. The square of a simple graph G is a graph G’ where two vertices x and y are adjacent in G’ if and only if dG(x, y) ≤ 2. Show that square of a connected graph G has diameter ceil(diam(G)/2).**

**Ans:** Consider a graph G and it’s square as below:

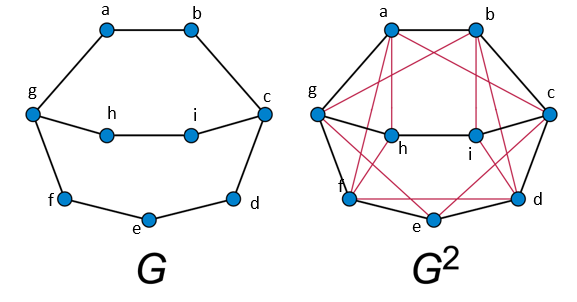
Case 1:



As we know that, diameter of a graph is the maximum distance between a pair of vertices in the graph, so for graph G, the distance between vertices i and e, which is 4, is maximum amongst all pair of vertices.

Therefore, d(i,e) = 4 is the diameter of graph G. Now coming to G’s square i.e. G2, the distance between vertices i and e is 2 and it is maximum amongst all pair of vertices. Therefore, the diameter of G2  is 2, which is diam(G)/2.

Case 2:



As we know that, diameter of a graph is the maximum distance between a pair of vertices in the graph, so for graph G, the distance between vertices i and e, which is 4, is maximum amongst all pair of vertices.

Therefore, d(i,e) = 3 is the diameter of graph G. Now coming to G’s square i.e. G2, the distance between vertices i and e is 2 and it is maximum amongst all pair of vertices. Therefore, the diameter of G2  is 2, which is ceil(diam(G)/2).

Hence, diam(G2) = ceil(diam(G)/2).

**Q5. Prove that if an n-vertex graph G has n − 1 edges and no cycles, then it is connected.**

**Ans:** Let the graph G is disconnected then there exist at least two components G1 and G2 say. Each of the component is circuit-less as G is circuit-less. Now to make a graph G connected we need to add one edge e between the vertices Vi and Vj, where Vi is the vertex of G1 and Vj is the vertex of component G2. Now the number of edges in G = (n – 1) + 1 = n.

Now, G is connected graph and circuit-less with n vertices and n edges, which is impossible because the connected circuit-less graph is a tree and tree with n vertices has (n-1) edges. So, the graph G with n vertices, (n-1) edges and without circuit is connected. Hence the given statement is proved.

Disconnected Graph

( n = 4, e = 2)

Connected Graph

( n = 4, e = 3)

adding an edge

removing an edge

**Q6. Show that every non-trivial tree has at least two maximal independent sets, with equality only for star graphs.**

**Ans:** By definition a tree T is a connected acyclic graph. Also, the tree is non-trivial, so it has at least one edge. Therefore, we can denote T with a bipartite graph. Now we split the vertices V of T into two sets of vertices 𝑉1 and 𝑉2 such that V=𝑉1 U 𝑉2. Since T has no cycles (in turn has no odd cycles), so there is no edge between any two vertices of the same set 𝑉1 or 𝑉2. Therefore, 𝑉1 and 𝑉2 are independent. As V = V1 U V2, therefore 𝑉1 and 𝑉2 are maximal independent sets. Similarly, if we split the V into more sets, we can have more than two maximal independent sets. This proves the first part.

Now for the second part, let’s consider the star graph with V vertices. Split V into two sets 𝑉1 and 𝑉2 where 𝑉1 contains all vertices and 𝑉2 contains center vertex. Therefore, there is no edge between any two vertices of the same set 𝑉1 or 𝑉2. No other combination of vertices would result this scenario. Therefore, 𝑉1 and 𝑉2 are the only possible vertex sets and they are independent and maximal. This proves the second part of the statement.

**Q7. Show that among the trees with n vertices, the star graph has the most independent sets.**

**Ans:** Let us prove this by using Induction. Consider I(T) for the number of independent sets. Let v be a leaf of T, adjacent to a vertex w, and consider I(T−v). Now I(T)=2\*I(T−v) −k, where k is the number of independent sets of T−v that contain w (since these are the sets that would stop being independent if you add v). Note that k≥1, with equality if and only if w is adjacent to every other vertex of T−v, i.e. it is the center vertex of a star. Also, I(T−v) is maximized when T−v is a star (by the induction hypothesis). Putting this together, you get that I(T) is maximized when T−v is a star and v was adjacent to the center of that star, i.e. T is also a star.

Hence among the trees with n vertices, the star graph has the most independent sets.

**Q8. Show that an edge of a connected graph G is a cut-edge (bridge) if and only if it belongs to every**

**spanning tree.**

**Ans:** Suppose e is a bridge in G and T a spanning tree on G not containing e. Then, since T is a tree, it must be connected, but since T is a subgraph of G - {e}, a disconnected graph, it must be disconnected. Therefore, any spanning tree contains every bridge.

Now suppose e is not a bridge. Then G - {e} is still connected, and so has a spanning tree T. However, since G - {e} has the same vertices as G, T is also a spanning tree of G that does not contain e. Therefore, any edge in every spanning tree is a bridge.

**Q9. Show that every tree on even number of vertices has exactly one subgraph in which every vertex has odd degree.**

**Ans:** By definition, a spanning subgraph is a subgraph of a graph consisting of the same vertex set and a subset of the edge set of the graph, which is not necessarily a tree. A spanning tree is a spanning subgraph that is also a tree.

Using induction on the number of vertices = 2k. For k = 1, we have 2 vertices, and so there is only one possible tree, which is just two vertices connected by 1 edge -- the graph itself is a spanning subgraph with odd vertices.

Suppose for k >= 1, every tree with 2k vertices has a unique spanning subgraph with all odd vertices. Now since each leaf has degree 1 in the tree, any such spanning subgraph must include all edges incident to leaves.

Consider a tree with 2(k+1) vertices. Consider a longest possible path in this tree. Suppose 1 endpoint of the path is a vertex v, which would have to be a leaf. Suppose u is the vertex adjacent to u in the path.

Case 1: u has another adjacent leaf (say w).

T - {v, w} has a unique such spanning subgraph by our assumption (we assumed this for any tree with 2k vertices), so use that spanning subgraph and add edges uv and uw -- the degree of u gains 2, so it stays odd, and v and w only have the 1 incident edge, so their degree will be 1, which is odd, and we have the type of subgraph we need.

Case 2: u has no other adjacent leaf.

Since p is the longest path, this means the degree of u is 2, or else there would be a cycle in the graph. So, we know that T - {u, v} has a unique such spanning subgraph by assumption -- add uv to that subgraph and we have the type of subgraph we need (u + uv + v will be a separate component).

So, by induction, the above statement holds for any number of vertices 2n for a positive integer n.

**Q10. Show that a connected graph with n vertices has exactly one cycle if and only if it has exactly n-edges.**

**Ans:** Let G have n vertices and n edges. Since G is a connected graph, it has a spanning tree T with n vertices and n-1 edges. Let e be the edge not in T, with its endpoints x and y. There is a unique path u between x and y in T (since T is a tree). The union of e and u is a cycle.

Suppose that there is some other cycle v. if v does not contain e, then it is contained in T, contradicting that T has no cycles. If v does contain e, write it as the union of e and a path p in T. Then p is a path from x and y. But u is the only path from x and y in T.

Hence, a connected graph with n vertices has exactly one cycle if and only if it has exactly n-edges.