ECEN 743: Reinforcement Learning

Markov Decision Process (MDP)

Dileep Kalathil
Assistant Professor
Department of Electrical and Computer Engineering
Texas A&M University

References

- [SB, Chapter 3-4]
- [DB, Chapter 2]
- [AJKS, Chapter 1]

- Acknowledgment: Some figures for this lecture are taken from UC Berkeley CS188 course, with permission.

Markov Decision Process (MDP)

Definition (Markov Decision Process (MDP))

A (discounted) Markov Decision Process (MDP) is a tuple $(\mathcal{S}, \mathcal{A}, P, r, \gamma)$, where,

- $\mathcal{S}$ is a set of states (state space)
- $\mathcal{A}$ is a set of actions (action space)
- P is a transition probability function, $P(s'|s,a) = \mathbb{P}(s_{t+1} = s'|s_t = s, a_t = a), \forall s, s' \in \mathcal{S}, a \in \mathcal{A}$
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- A stationary policy $\pi : \mathcal{S} \to \Delta(\mathcal{A})$ specifies the control action for each state

Policy Evaluation

MDP Questions

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- How do we compute the value of a policy π?

$$V_\pi(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) | s_0 = s, \pi\right], \text{ where, } s_{t+1} \sim P(\cdot|s_t, a_t),\ a_t \sim \pi(s_t, \cdot)$$

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- How do we compute the optimal policy?

$$\pi^\star = \arg\max_{\pi \in \Pi} \ V_\pi$$

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- V_π is also called the value function of policy π

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- We will use the vector form V_π, r_π and the matrix form P_π

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- Intuition

$$V_\pi(s) = \underbrace{r_\pi(s)}_{\substack{\text{Expected reward for state } s \\ \text{under policy } \pi}} + \gamma \underbrace{\sum_{s'} P_\pi(s, s') V_\pi(s')}_{\substack{\text{Expected value of the next state} \\ \text{under policy } \pi}}$$

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&= r_\pi(s) + \gamma (P_\pi V_\pi)(s)
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We can show that $(I - \gamma P_\pi)$ is invertible as follows.

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Let $V \in \mathbb{R}^{|\mathcal{S}|}$ be an arbitrary non-zero vector.

$$\begin{aligned}\|(I - \gamma P_\pi)V\|_\infty = \|(V - \gamma P_\pi V)\|_\infty &\overset{(a)}{\geq} \|V\|_\infty - \gamma \|P_\pi V\|_\infty \\ &\overset{(b)}{\geq} \|V\|_\infty - \gamma \|V\|_\infty = (1-\gamma)\|V\|_\infty > 0\end{aligned}$$

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So, $(I - \gamma P_\pi)$ does not have 0 as eigenvalue. □

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- We are solving a linear system of equations $(I - \gamma P_\pi) V_\pi = r_\pi$

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- Policy evaluation using the linear system of equations approach has $O(|\mathcal{S}|^3)$ complexity

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Let $T_\pi V = r_\pi + \gamma P_\pi V$. Define the policy evaluation iteration $V_{k+1} = T_\pi V_k$. Then $\lim_{k\to\infty} V_k = V_\pi$

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- How many iterations until convergence?

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- Recall that the value function of policy π is defined as

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Bellman Optimality Equation

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Theorem (Bellman Optimality Equation)

Let V^* be the optimal value function. Then V^* satisfies the equation

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- **Reading assignment:** Proof of the above theorem

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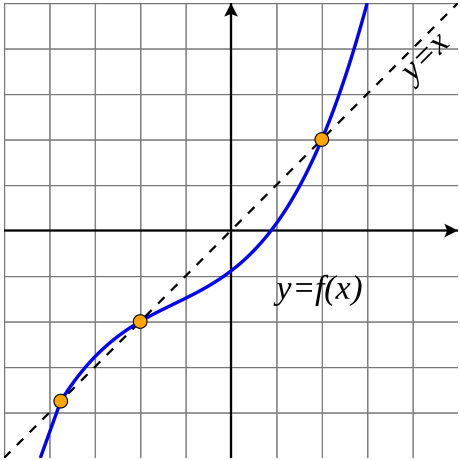
- So, we are taking the optimal action in the current state and in all following states. This should give the optimal value corresponding to the current state

Fixed Point and Contraction Mapping

Fixed Point of a Function

Definition (Fixed Point)

Let $f : \mathcal{U} \to \mathcal{U}$. u^* is a fixed point of $f(\cdot)$ if $f(u^*) = u^*$.



Contraction Mapping

Definition

Let $f : \mathcal{U} \to \mathcal{U}$ and let $\|\cdot\|$ be a norm defined on $\mathcal{U}$. We say $f(\cdot)$ is a contraction mapping (or simply, contraction) if $\|f(u) - f(v)\| \leq \alpha\|u - v\|$ for an $\alpha \in (0, 1)$ and for all $u, v \in \mathcal{U}$.

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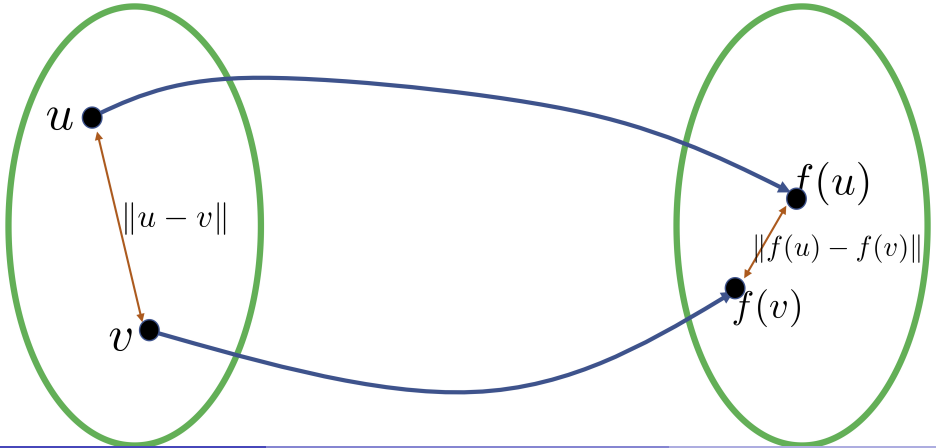

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- Example: $f : [\sqrt{a}, \infty] \to [\sqrt{a}, \infty], f(u) = \frac{1}{2}\left(u + \frac{a}{u}\right)$. Then $f(\cdot)$ is a contraction mapping.

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- Example: $f : \mathbb{R}^n \to \mathbb{R}^n, f(u) = Au$, where $A \in \mathbb{R}^n \times \mathbb{R}^n$. Assume that the row sum of A is strictly less than 1, i.e., $\sum_j |a_{ij}| \leq \alpha < 1$. Then $f(\cdot)$ is a contraction mapping with respect to $\|\cdot\|_\infty$

Banach Fixed Point Theorem

Theorem (Banach Fixed Point Theorem)

Let $\mathcal{U}$ be a closed and bounded subset of $\mathbb{R}^n$. Let $f : \mathcal{U} \to \mathcal{U}$ be a contraction mapping, i.e, $\|f(u) - f(v)\| \leq \alpha \|u - v\|$, $\alpha \in (0, 1)$, for all $u, v \in \mathcal{U}$. Then,

1. *$f(\cdot)$ has a unique fixed point u^**
2. *The iteration $u_{k+1} = f(u_k)$ with an arbitrary initial element u_0 will converge to the unique fixed point, i.e., $\lim_{k \to \infty} u_k = u^*$*

Banach Fixed Point Theorem: Proof

Proof: **Existence:** Let $u_0 \in \mathcal{U}$ be arbitrary. Define the iteration $u_{k+1} = f(u_k)$.

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Since $f(\cdot)$ is a contraction, it is also a continuous function.
Then, $u^* = \lim_{k \to \infty} u_k = \lim_{k \to \infty} u_{k+1} = \lim_{k \to \infty} f(u_k) = f(\lim_{k \to \infty} u_k) = f(u^*)$

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This is a contradiction.

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This is a contradiction.
So, $\bar{u}^* = u^*$.

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Value Iteration

Bellman Operator

- Bellman operator $T : \mathbb{R}^{|\mathcal{S}|} \to \mathbb{R}^{|\mathcal{S}|}$ is defined as

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Bellman operator T is a contraction with respect to $\|\cdot\|_\infty$, i.e, for any $V_1, V_2 \in \mathcal{R}^{|\mathcal{S}|}$, $\|TV_1 - TV_2\|_\infty \leq \gamma \|V_1 - V_2\|_\infty$.

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Lemma

Let g, h be any two real-valued function. Then,

$$\left|\max_a \ g(x,a) - \max_a \ h(x,a)\right| \leq \max_a \ |g(x,a) - h(x,a)|$$

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Proof: (of contraction property)

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&|(TV_1)(s) - (TV_2)(s)| \\
&= |\max_{a \in \mathcal{A}} (r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_1(s')]) - \max_{a \in \mathcal{A}} (r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_2(s')])|
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&\overset{(a)}{\leq} \max_{a\in\mathcal{A}} |(r(s,a) + \gamma\mathbb{E}_{s'\sim P(\cdot|s,a)}[V_1(s')]) - (r(s,a) + \gamma\mathbb{E}_{s'\sim P(\cdot|s,a)}[V_2(s')])|
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\begin{aligned}
&|(TV_1)(s) - (TV_2)(s)| \\
&= |\max_{a \in \mathcal{A}} (r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_1(s')]) - \max_{a \in \mathcal{A}} (r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_2(s')])| \\
&\overset{(a)}{\leq} \max_{a \in \mathcal{A}} |(r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_1(s')]) - (r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_2(s')])| \\
&= \max_{a \in \mathcal{A}} |\gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_1(s')] - \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_2(s')]|
\end{aligned}
$$

□

Bellman Operator

Proof: (of contraction property)

$$
\begin{aligned}
&|(TV_1)(s) - (TV_2)(s)| \\
&= |\max_{a \in \mathcal{A}} (r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_1(s')]) - \max_{a \in \mathcal{A}} (r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_2(s')])| \\
&\overset{(a)}{\leq} \max_{a \in \mathcal{A}} |(r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_1(s')]) - (r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_2(s')])| \\
&= \max_{a \in \mathcal{A}} |\gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_1(s')] - \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V_2(s')]| \\
&\leq \max_{a \in \mathcal{A}} \ \gamma \ \mathbb{E}_{s' \sim P(\cdot|s,a)}[|V_1(s') - V_2(s')|]
\end{aligned}
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&\leq \max_{a\in\mathcal{A}} \ \gamma \ \mathbb{E}_{s'\sim P(\cdot|s,a)}[\|V_1 - V_2\|_\infty] = \gamma \|V_1 - V_2\|_\infty
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\end{aligned}
$$

This implies

$$\|TV_1 - TV_2\|_\infty \leq \gamma \|V_1 - V_2\|_\infty$$

□

Optimal Value Function

- Bellman operator: $(TV)(s) = \max_a \left(r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V(s')]\right)$

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- **Reading assignment:** Proof of the above theorem

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- Part (1) of the above theorem is essentially stating the Bellman optimality equation, $V^*(s) = \max_a \left(r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[V^*(s')]\right)$

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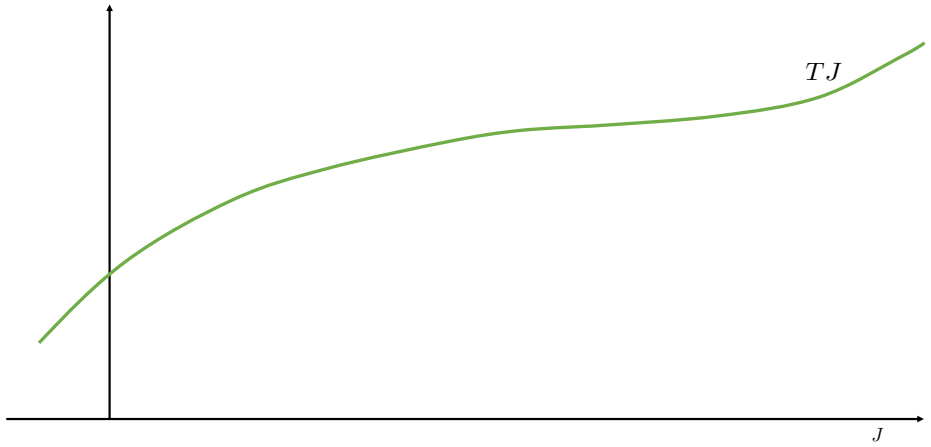
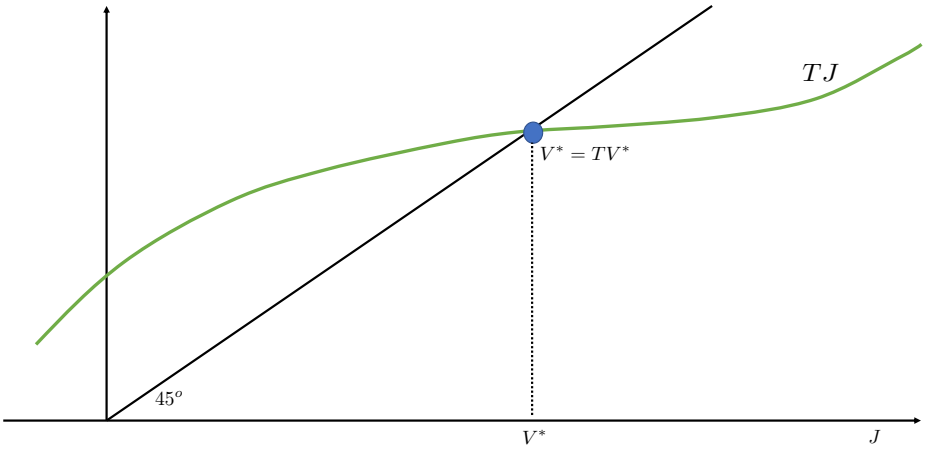
- Part (2) follows from the fact that T is a contraction mapping and V^* is its unique fixed point

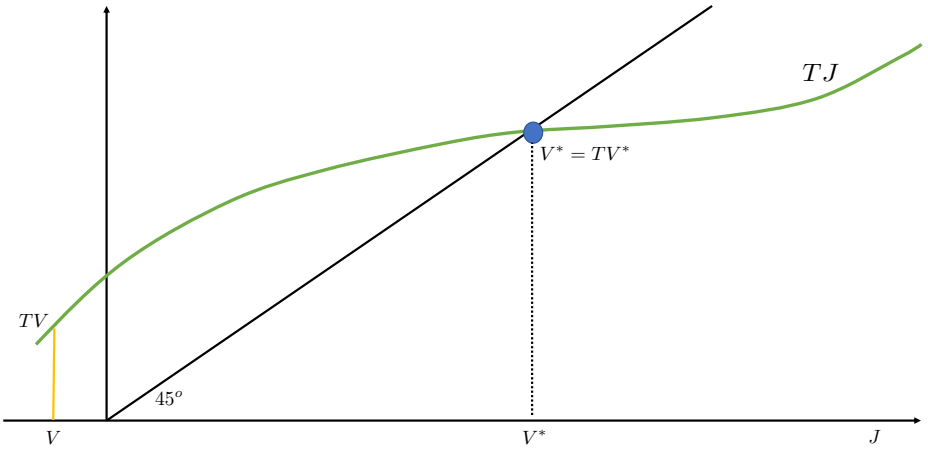
Contraction, Fixed Point, Convergence



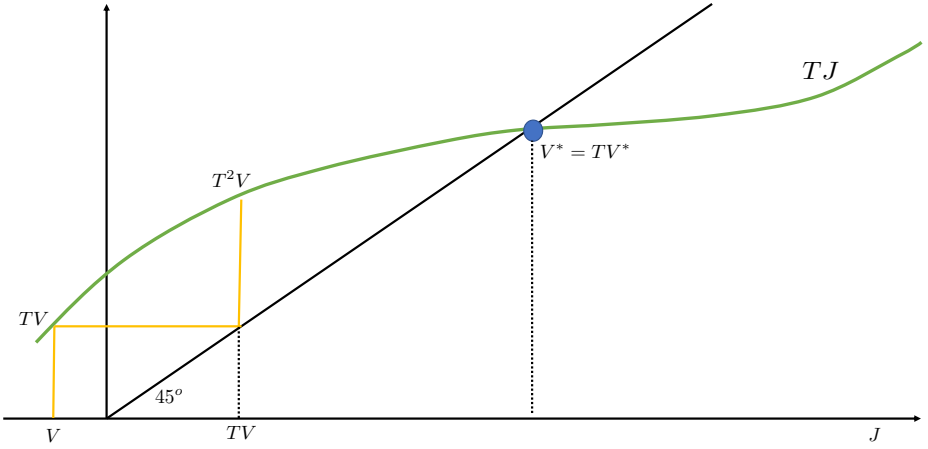
Contraction, Fixed Point, Convergence



Contraction, Fixed Point, Convergence



Contraction, Fixed Point, Convergence



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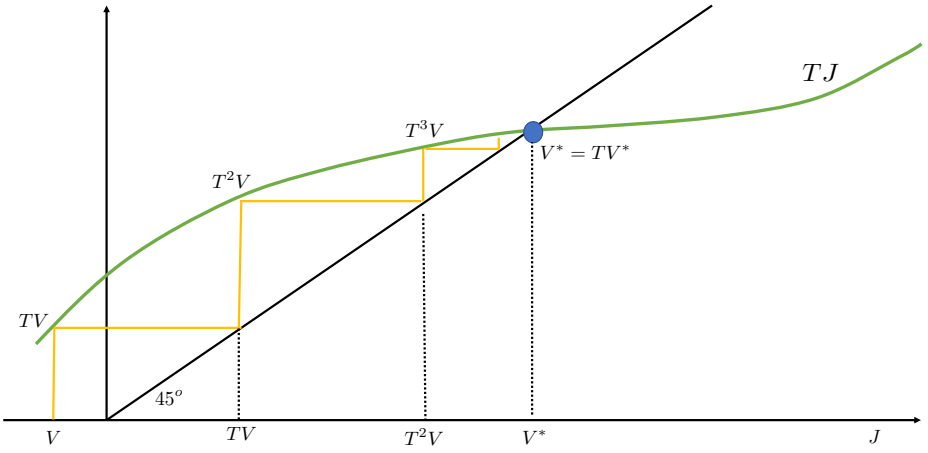


Value Iteration

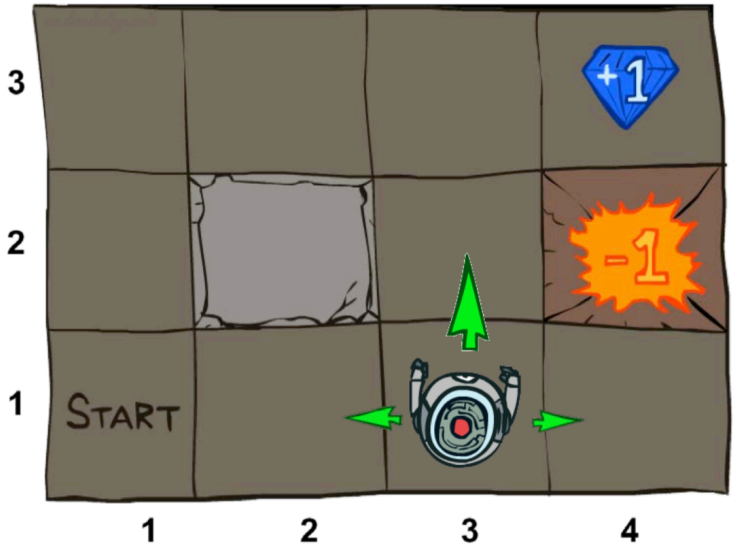
- How do we find the optimal value function V^*?

$$V^*(s) = \max_{\pi} V_\pi(s)$$

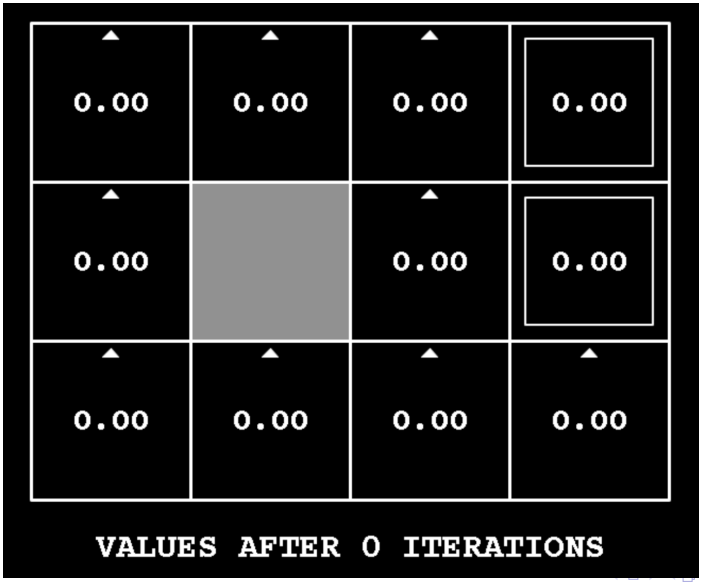
- Value Iteration: $V_{k+1} = TV_k$
- How do we find the optimal policy?

$$\pi^*(s) = \arg\max_{a} \left(r(x,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^*(s')\right)$$

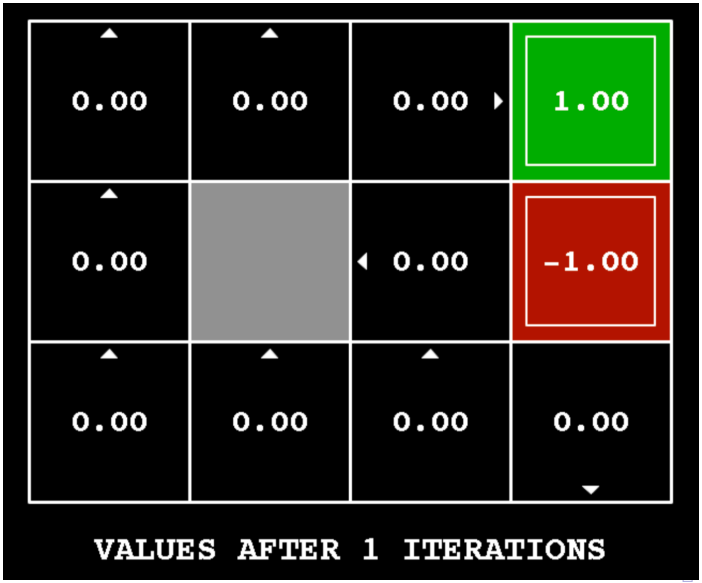
Value Iteration: Example



Value Iteration: Example



Value Iteration: Example



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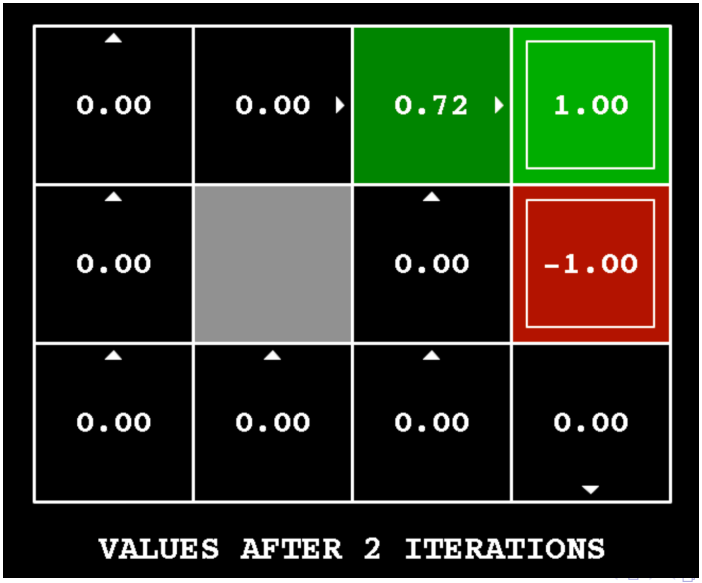

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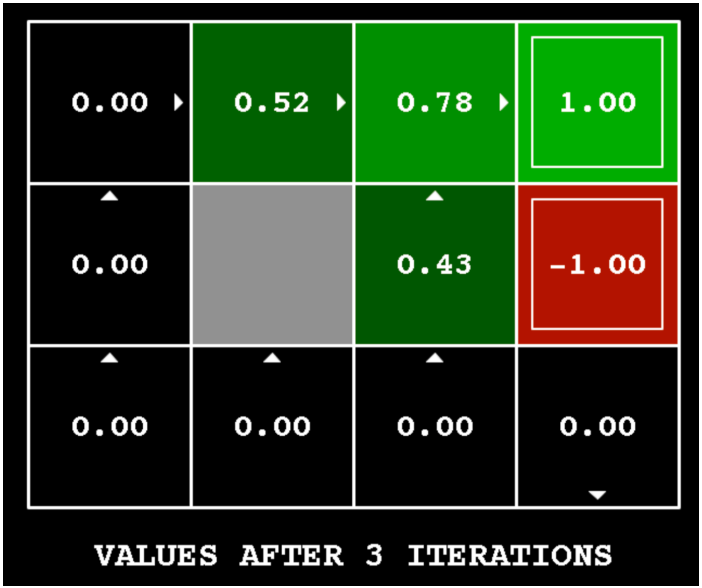

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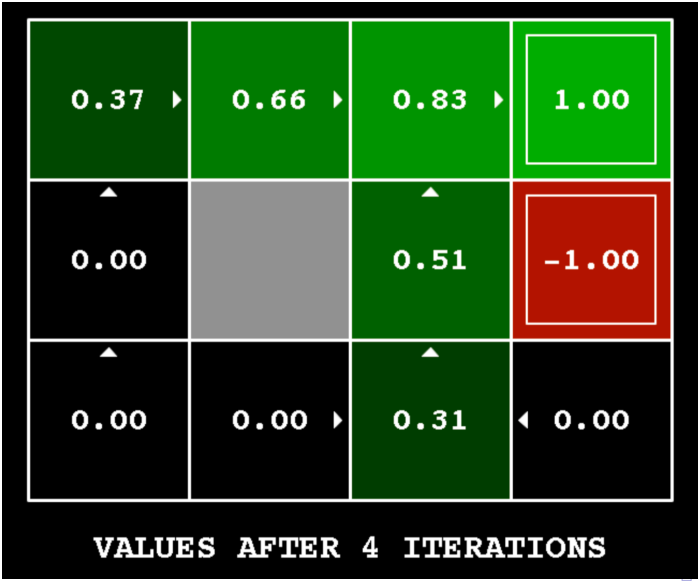

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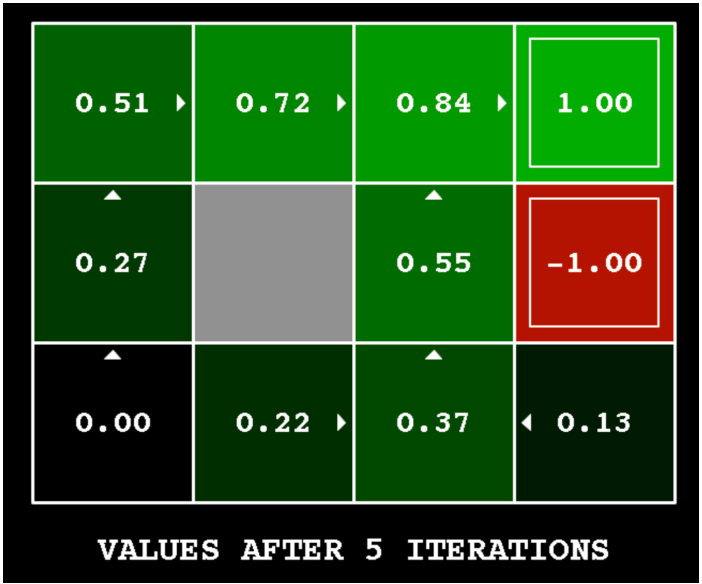

Value Iteration: Example

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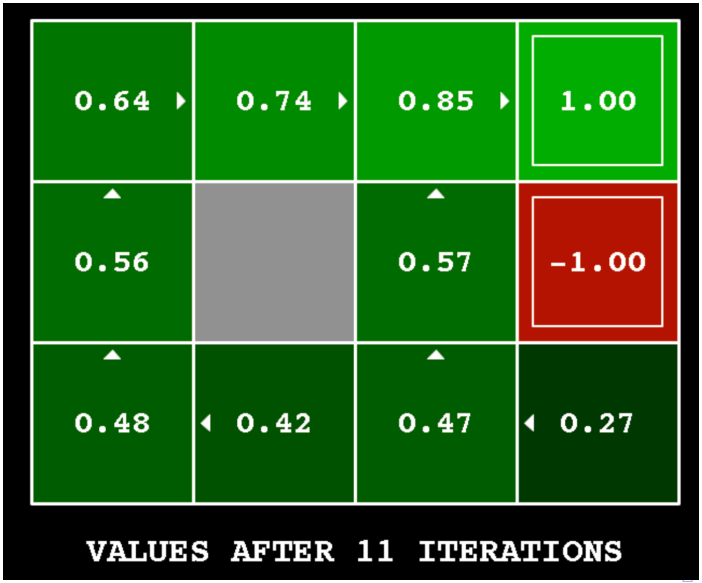

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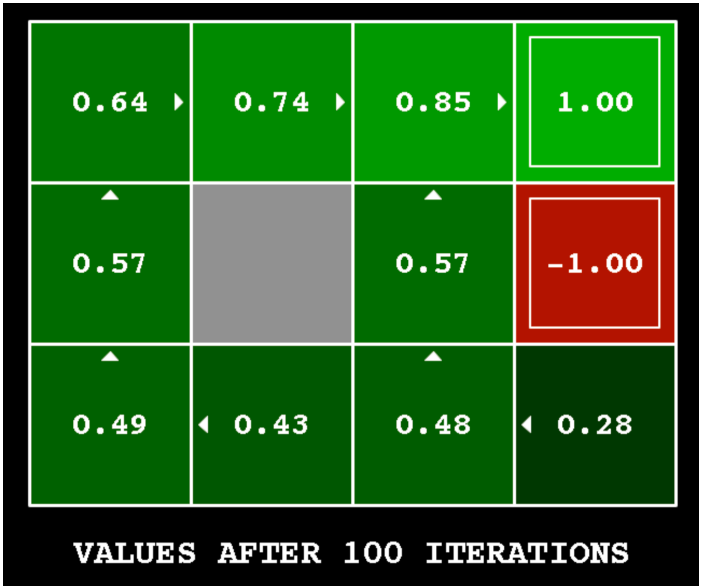

Optimal Q-value Function

- Q-value function of a policy π

$$Q_\pi(s,a) = \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) | s_0 = s, a_0 = a]$$

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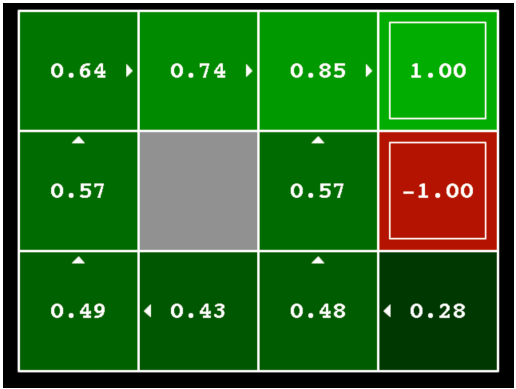
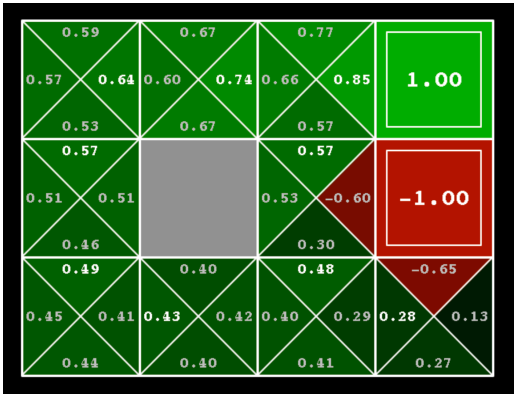
$$\pi^*(s) = \arg\max_a \; Q^*(s,a)$$

Policy Computation from V^*/Q^*





$$V^*(s) = \max_a Q^*(s,a), \qquad Q^*(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} V^*(s'),$$

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Computing Q^* Directly

- Optimal Q-value function Q^* satisfies the Bellman equation

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- Define the mapping $F : \mathbb{R}^{|\mathcal{S}||\mathcal{A}|} \to \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$ as

$$(FQ)(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[\max_b Q(s',b)]$$

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Proposition

1. *F is a contraction mapping w.r.t $\|\cdot\|_\infty$*
2. *Q^* is the unique fixed point of F*
3. *Define Q-value iteration, $Q_{k+1} = FQ_k$. Then, $\lim_{k\to\infty} Q_k = Q^*$*

Policy Iteration

Policy Evaluation: Recap

- Value function V_π satisfies the equation $V_\pi = R_\pi + \gamma P_\pi V_\pi$

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Proposition

1. T_π *is a contraction mapping w.r.t* $\|\cdot\|_\infty$
2. V_π *is the unique fixed point of* T_π
3. *Define the policy evaluation iteration,* $V_{k+1} = T_\pi V_k$. *Then* $\lim_{k\to\infty} V_k = V_\pi$

Policy Iteration

- Can we find the optimal policy by 'cleverly' searching over the policy space?

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  1. Policy evaluation: compute the value of the policy to get V_{π_k}:

$$\text{Solve} \quad T_{\pi_k} V_{\pi_k} = V_{\pi_k}$$

Policy Iteration

- Can we find the optimal policy by 'cleverly' searching over the policy space?
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Proposition (Convergence of Policy Iteration)

1. $V_{\pi_{k+1}} \geq V_{\pi_k}$
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Policy Iteration

Lemma (Monotone Property of T_π)

Let $V_1 \geq V_2$. *Then,* $T_\pi V_1 \geq T_\pi V_2$.

Policy Iteration

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Proof: (of convergence of policy iteration):
Monotone property: Note that

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Rate of convergence:

$$\|V^* - V_{\pi_{k+1}}\|_\infty \leq \|V^* - T V_{\pi_k}\|_\infty \leq \gamma \|V^* - V_{\pi_k}\|_\infty$$

□

Dynamic Programming

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Reinforcement Learning Questions

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