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MEEN 602 HW4.

1. a)

$$f(x,y) = 8x^2 + 8xy + 2y^2$$

$$\frac{f(x,y)}{8} = x^2 + xy + \frac{y^2}{4}$$

$$= x^2 + 2x\left(\frac{y}{2}\right) + \left(\frac{y}{2}\right)^2$$

$$= \left(x + \frac{y}{2}\right)^2$$

$$\therefore f(x,y) = 8\left(x + \frac{y}{2}\right)^2$$

b) We know that,

$f(x,y) = ax^2 + 2bxy + cy^2$ for $ac = b^2$ & $a > 0$, then f is positive semi definite.

$$f = 8x^2 + 8xy + 2y^2$$

$c = 2$, $a = 8$, $b = 4$

as $a > 0$, $ac = b^2 \Rightarrow 2 \times 8 = 4^2 = 16$

LHS = RHS $\therefore$ f is positive semi-definite.

c) $f = \vec{x}^T A \vec{x}$

$$ax^2 + 2bxy + cy^2 = [x \; y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \quad R_2 \rightarrow R_2 - \frac{R_1}{2}$$

$$A = \begin{bmatrix} 8 & 4 \\ 0 & 0 \end{bmatrix} = LU$$

After LU decomposition,

$$L = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 8 & 4 \\ 0 & 0 \end{bmatrix}$$

$\text{Rank}(A) = \text{Rank}(U) = 1$

2) $a + c > 2b$

To find $ac < b^2$

Let $a = 2,\ c = -1,\ b = -2$

$a + c = 1,\quad 2b = -4$

$a + c > 2b$

$ac = -2 < b^2 = 4$

$$\therefore\ A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$$

This makes the matrix not positive definite.

3. a) $F = -1 + 4(e^x - x) - 5x\sin y + 6y^2$

at $x = y = 0$.

$$\left.\frac{\partial F}{\partial x}\right|_{x,y=0,0} = \left\{4(e^x - 1) - 5\sin y\right\}\Big|_{\substack{x=0\\y=0}}$$

$$= \left\{4(1-1) - 5\sin 0\right\} = 0$$

$$\left.\frac{\partial F}{\partial y}\right|_{\substack{x=0\\y=0}} = \left\{-5x\cos y + 12y\right\}\Big|_{\substack{x=0\\y=0}} = 0$$

$$\left.\frac{\partial^2 F}{\partial x^2}\right|_{x=0} = \left.4e^{x}\right|_{x=0} = 4$$

$$\left.\frac{\partial^2 F}{\partial y^2}\right|_{y=0} = \left.-5\cos y\right|_{y=0} = -5$$

$$\frac{\partial^2 F}{\partial x^2} > 0$$

$$\frac{\partial^2 F}{\partial x^2} \times \frac{\partial^2 F}{\partial y^2} > \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2$$

$$12 \times 4 > 25$$

$\therefore$ $f(x,y)$ has a minimum at $(x,y) = (0,0)$

b) $f(x,y) = (x^2 - 2x)\cos y$, $(x,y) = (1, \pi)$

$$\left.\frac{\partial F}{\partial x}\right|_{x,y} = \left.(2x-2)\cos y\right|_{x,y} = 0$$

$$\left.\frac{\partial F}{\partial y}\right|_{x,y} = \left.-\sin y\,(x^2 - 2x)\right|_{x,y} = 0$$

$$\left.\frac{\partial^2 F}{\partial x^2}\right|_{x,y} = \left.2\cos y\right|_{x,y} = -2$$

$$\frac{\partial^2 F}{\partial x \partial y} = \left.(2x-2)(-\sin y)\right|_{x,y} = 0$$

Second order derivatives ~~are~~ less than 0.

$\therefore$ (x,y) is a maxima.

4. $f = ax^2 + 2bxy + cy^2$

a) $A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}$

$f(x,y) = \vec{x}^T A_1 \vec{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$= x^2 + 4xy + 9y^2$

b) $A_2 = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

$f(x,y) = \vec{x}^T A_2 \vec{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$= x^2 + 6xy + 9y^2$

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5. A is positive definite.
Let A have λ_1, λ_2 as its eigenvalues, then for A to be positive definite, $\lambda_1, \lambda_2 > 0$ (positive)

A^3 will have λ_1^3 & λ_2^3 as its eigenvalues.
But as $\lambda_1, \lambda_2 > 0$,
$\lambda_1^3, \lambda_2^3 > 0$.
$\therefore$ A^3 will also be positive definite

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6. $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 8 \\ 4 & 8 & 7 \end{bmatrix}$

Both A & B are symmetric matrix, for them to be positive semi definite,

$|A - \lambda I|$, solving for the eigenvalues we get,

$$\lambda_1 = a - 2 > 0$$
$$\lambda_2 = \frac{a}{2} - \frac{\sqrt{a^2+32}}{2} + 2 > 0$$
$$\lambda_3 = \frac{a}{2} + \frac{\sqrt{a^2+32}}{2} + 2 > 0$$

From λ_2, λ_3 we know that $a > 0$,
& for $tr(A) > 0$, $2a + 2 > 0$

$$\therefore \frac{a}{2} + 2 > \frac{\sqrt{a^2+32}}{2}$$

$$\frac{a^2}{\cancel{4}} + 2a + 4 > \frac{a^2}{\cancel{4}} + 8$$

$$2a > 4$$
$$a > 2$$

As there is no sign change, inequality conditions hold true.
These are the 3 conditions of eigen values.
~~As A is~~ But we have from the eigenvalue conditions.

that ~~a~~ $a > 0$.

AS A is symmetric with positive & real eigenvalues,

A is positive & definite.

b) $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{bmatrix}$

$|B - \lambda I| = -\lambda^3 + b\lambda^2 + 8\lambda^2 - 8b\lambda + 77\lambda - 96 + 36 = 0.$

As B is symmetric, we can test for subsymmetric matrices.

$\det|1| = 1$

$\begin{vmatrix} 1 & 2 \\ 2 & b \end{vmatrix} = b - 4 \geq 0$

$|B| = 36 - 9b \geq 0$

Put $b = 4$, we get

$\lambda_1 = -3$,

$\lambda_2 = 0$

$\lambda_3 = 15$

We have one eigenvalue which is negative but real.

$\therefore$ B is NOT positive definite.

7. $x^2 + xy + y^2 = 1$

Here, $A = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$

$$|A - \lambda I| = (\lambda - 1)^2 - 0.25$$
$$\Rightarrow \lambda = \{0.5, 1.5\}$$

Major component half length $= \frac{1}{\sqrt{\lambda_2}} = \sqrt{2} = 1.41$

Minor component half length $= \frac{1}{\sqrt{\lambda_1}} = \sqrt{\frac{2}{3}}$
$= 0.81$

* Eigenvector of λ_1

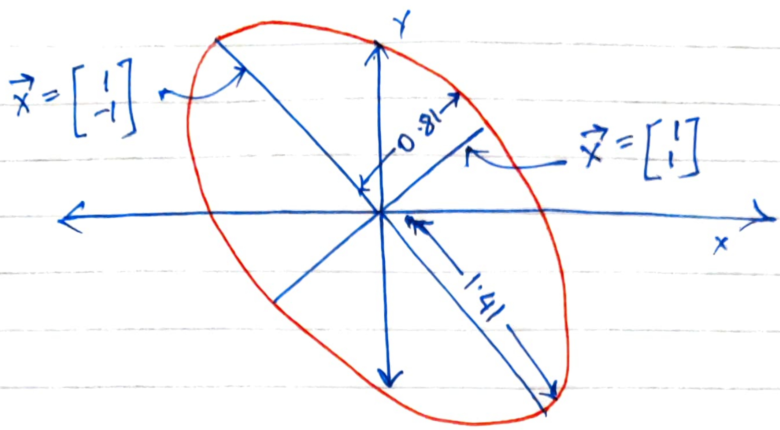
$$\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = 1, x_2 = 1$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

* Eigenvector of λ_2

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{matrix} x_1 = 1 \\ x_2 = -1 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



8) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1/9 \end{bmatrix}$, $A_2 = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

a) For A_1:

$a > 0$, $ac > b^2$

$\therefore$ Positive definite

For A_2:

$a > 0$, $ac < b^2$

$\therefore$ Indefinite

For A_3:

$a > 0$, $ac = b^2$

$\therefore$ Positive semi-definite.

b) (B) & (C) graphs are plotted in MATLAB. code & graphs attached below.

c) The graph in $f(x,y) = 1$ is a slice of the plots seen in (b) part at +1 intercept.