ECEN 743: Reinforcement Learning

Markov Chains Review

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References

- D. Bertsekas, and J. Tsitsiklis, *Introduction to Probability*, Athena Scientific, 2008.

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Definition (Markov Chain)

The sequence $\{X_0, X_1, X_2, \ldots\}$ is a Markov chain if it satisfies the Markov condition

$$\mathbb{P}(X_t = y | X_{t-1} = x_{t-1}, X_{t-2} = x_{t-2}, \ldots, X_0 = x_0) = \mathbb{P}(X_t = y | X_{t-1} = x_{t-1})$$

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for all $t \geq 1$ and for all $y, x_0, \ldots, x_{t-1} \in \mathcal{X}$

- Conditional distribution of any future state given the past states and current state is independent of the past states, and depends only on the current state

Transition Probability Matrix

- Homogeneous Markov chain: A Markov chain is homogeneous if

$$\mathbb{P}(X_t = y | X_{t-1} = x) = \mathbb{P}(X_1 = y | X_0 = x), \quad \forall t \geq 1, \forall x, y \in \mathcal{X}$$

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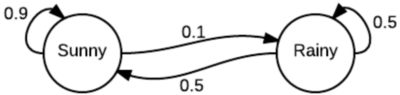
- Properties of transition probability matrix P:
  - P has non-negative entries
  - P has row sums equal to one, $\sum_j p_{ij} = 1$

Markov Chain Representation

- Markov chains with small state space are often represented by directed graphs

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$



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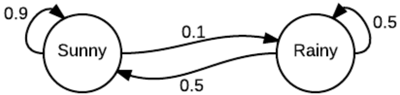
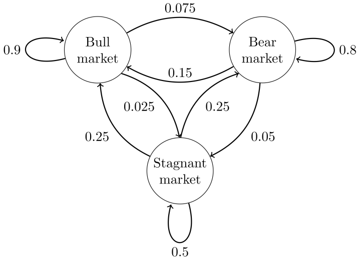


$$P = \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$



[Figures are taken from Wikipedia]

One Step to Many Steps

- Let $p(t)$ be the row vector such that $p_i(t) = \mathbb{P}(X_t = i)$
- Given $p(t)$, how do we compute $p(t+1), p(t+2), \ldots, p(t+m)$?

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Proposition

Let P be the transition probability matrix of a Markov chain with state space $\mathcal{X} = \{1, 2, \ldots, n\}$. Let $p(t)$ be the row vector such that $p_i(t) = \mathbb{P}(X_t = i)$. Then,

$$p(t+1) = p(t)\ P, \quad p(t+m) = p(t) P^m$$

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Proof:

$$p_j(t+1) = \mathbb{P}(X_{t+1} = j) = \sum_{i \in \mathcal{X}} \mathbb{P}(X_{t+1} = j | X_t = i)\mathbb{P}(X_t = i)$$

$$= \sum_{i \in \mathcal{X}} p_{ij}\ p_i(t) = p(t)P$$

$\square$

Classification of States

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- Reachable states: State j is reachable from state i if $\mathbb{P}(X_t = j | X_0 = i) > 0$ for some $t \geq 1$. Equivalently, if $P_{ij}^t > 0$ for some $t \geq 1$.

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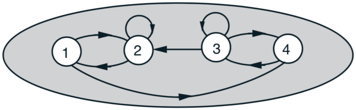
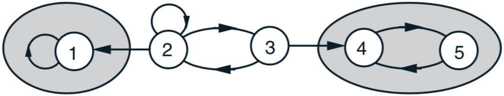
- Irreducible Markov chain: A Markov chain is irreducible if its state space is a single communicating class
- Absorbing state: State i is absorbing if $P_{ij} = 0$ for all $j \in \mathcal{X}$
- Transient and recurrent state: State i transient if starting at state i there is a nonzero probability of never returning to i. State i is called recurrent if it is not transient.

Classification of States

Proposition

(i) A Markov chain can be decomposed into one or more communicating (recurrent) classes, and possibly some transient states.
(ii) A recurrent state is accessible from all states in its class, but is not accessible from recurrent states in other classes.
(iii) A transient state is not accessible from any recurrent state.
(iv) At least one, possibly more, recurrent states are accessible from a given transient state.





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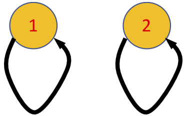
- $P = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$. Does this Markov chain have a stationary distribution? Is it unique?



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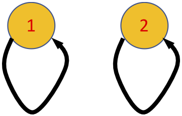

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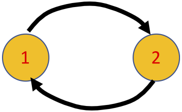

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Proposition

A finite state irreducible Markov chain has a unique stationary distribution, i.e., there exists a unique vector π such that $\pi = \pi P$.

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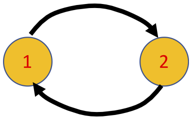
- $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $\pi = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$
- Assume $P(X_0 = 1) = 1.0$
- What is $\lim_{t\to\infty} p(t)$?



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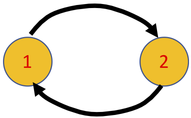

$$p(t) = \begin{cases} (0,1) & \text{if } t \text{ is odd} \\ (1,0) & \text{if } t \text{ is even} \end{cases}$$

- Limiting distribution may not exist

Limiting Distribution

- Period of a state: State i is said to have period $d(i)$ if it is only possible to return to state i in $t \geq 1$ steps, where t is a multiple of $d(i)$. Formally,

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Proposition

For an irreducible and aperiodic MC, the stationary probability π satisfy $\pi_j = \lim_{t \to \infty} \frac{n_{ij}(t)}{t}$, where $n_{ij}(t)$ is the expected value of the number of visits to state j within the first t transitions, starting from state i.