# ECEN 743: Reinforcement Learning

## Policy Gradient Theorem

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# References

- [SB, Chapter 13]
- [AJKS, Section 3]

# Reinforcement Learning Questions

- How do we learn the value of a policy $\pi$?
- How do we learn the optimal action-value function $Q^*$?
- How do we learn the optimal policy $\pi^*$?

... without the knowledge of the model $P$

- Need to learn from the observed sequence of states, actions, and rewards

# Finding the Optimal Policy Parameter

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- Can we develop (stochastic) gradient algorithms for solving the above problem?

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- Gaussian policies

$$\pi(s,a) = \mathcal{N}(\mu_\theta(s), \Sigma_\theta(s))$$

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Policy Gradient Theorem!

# Probability of a Trajectory

- Let $\tau$ be a trajectory generated using $\pi_\theta$,

$$\tau = (s_0, a_0, s_1, a_1, \ldots, s_T, a_T), \;\; a_t \sim \pi_\theta(s_t, \cdot), s_{t+1} \sim P(\cdot | s_t, a_t), s_0 \sim \mu_0(\cdot)$$

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- We can extend this to arbitrary long trajectories, $\tau = (s_t, a_t)_{t\geq 0}$,

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# Gradient of the Value Function: Form 1

- Let $\tau = (s_t, a_t)_{t \geq 0}$, $s_0 \sim \mu_0(\cdot)$. Denote the cumulative discounted reward of trajectory $\tau$ as

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**Theorem (Policy Gradient Theorem - Form 1)**

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- Gradient above does not depend on the gradient of the (stationary) distribution induced by the policy!

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$$\begin{aligned}\nabla V_{\pi_\theta} &= \nabla \mathbb{E}_{\tau \sim P_{\mathrm{traj},\theta}(\cdot)}\left[R(\tau)\right] = \nabla \sum_{\tau} R(\tau) P_{\mathrm{traj},\theta}(\tau) \\ &= \sum_{\tau} R(\tau)\ \nabla P_{\mathrm{traj},\theta}(\tau) = \sum_{\tau} R(\tau)\ P_{\mathrm{traj},\theta}(\tau)\ \nabla \log P_{\mathrm{traj},\theta}(\tau)\end{aligned}$$

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$$\begin{aligned}\nabla V_{\pi_\theta} &= \nabla \mathbb{E}_{\tau \sim P_{\mathrm{traj},\theta}(\cdot)}\left[R(\tau)\right] = \nabla \sum_{\tau} R(\tau) P_{\mathrm{traj},\theta}(\tau) \\ &= \sum_{\tau} R(\tau)\ \nabla P_{\mathrm{traj},\theta}(\tau) = \sum_{\tau} R(\tau)\ P_{\mathrm{traj},\theta}(\tau)\ \nabla \log P_{\mathrm{traj},\theta}(\tau) \\ &= \sum_{\tau} R(\tau)\ P_{\mathrm{traj},\theta}(\tau)\ \nabla \log\left(\mu_0(s_0)\ \prod_{t=0}^{\infty} \pi_\theta(s_t, a_t)\ P(s_{t+1}|s_t, a_t)\right)\end{aligned}$$

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&= \sum_{\tau} R(\tau)\ P_{\mathrm{traj},\theta}(\tau)\ \nabla \log \left(\mu_0(s_0)\ \prod_{t=0}^{\infty} \pi_\theta(s_t, a_t)\ P(s_{t+1}|s_t, a_t)\right) \\
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- This approach, however, is unreliable due very high variance
- This issue is avoided by exploiting the temporal structure and introducing a baseline
  - Variance reduction for PG is an active area of research

# Visitation Distribution

- For a given policy $\pi$, (discounted) state-action visitation distribution is defined as
$$\rho_{\pi,\mu_0}(s,a) = (1-\gamma)\sum_{t=0}^{\infty}\gamma^t \mathbb{P}(s_t = s, a_t = a),$$
where $(s_t, a_t)_{t\geq 0}$ is the trajectory generated according to $\pi$, and $s_0$ is sampled according to a given distribution $\mu_0$.

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- Similarly, for a given policy $\pi$, (discounted) state visitation distribution is defined as
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where $(s_t)_{t\geq 0}$ is the state-trajectory generated according to $\pi$ (we are *overloading* the notation)
- It is straight forward to show that
$$\rho_\pi(x,a) = \rho_\pi(x)\pi(x,a)$$

# Visitation Distribution and Value Function

**Lemma**

$$V_\pi = \frac{1}{(1-\gamma)}\mathbb{E}_{(s,a)\sim\rho_\pi(\cdot,\cdot)}\left[r(s,a)\right] = \frac{1}{(1-\gamma)}\sum_{(s,a)}\rho_\pi(s,a)r(s,a)$$

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*Proof:*

$$V_\pi = \mathbb{E}_{\tau\sim P_{\mathrm{traj},\pi}(\cdot)} [\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t)]$$

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&= \sum_s \sum_a r(s,a) (\sum_{t=0}^{\infty} \mathbb{P}(s_t = s, a_t = a)\ \gamma^t)
\end{aligned}$$

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&= \sum_s \sum_a r(s,a) (\sum_{t=0}^{\infty} \mathbb{P}(s_t = s, a_t = a)\ \gamma^t) \\
&= \frac{1}{(1-\gamma)} \sum_x \sum_a r(s,a) \rho_\pi(s,a)
\end{aligned}$$

$\square$

# Gradient of the Value Function: Form 2

**Theorem (Policy Gradient Theorem - Form 2)**

$$\nabla V_{\pi_\theta} = \frac{1}{(1-\gamma)} \mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)} \left[ Q_{\pi_\theta}(s,a) \; \nabla \log \pi_\theta(s,a) \right]$$

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$$= \nabla \mathbb{E}_{s_0 \sim \mu_0}[V_{\pi_\theta}(s_0)] = \mathbb{E}_{s_0 \sim \mu_0}[\nabla \sum_{a_0} \pi_\theta(s_0, a_0) Q_{\pi_\theta}(s_0, a_0)]$$

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&= \mathbb{E}_{s_0 \sim \mu_0}[\sum_{a_0} \nabla \pi_\theta(s_0, a_0)\ Q_{\pi_\theta}(s_0, a_0) + \sum_{a_0} \pi_\theta(s_0, a_0)\ \nabla Q_{\pi_\theta}(s_0, a_0)]
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&= \mathbb{E}_{s_0 \sim \mu_0}[\sum_{a_0} \nabla \pi_\theta(s_0, a_0)\ Q_{\pi_\theta}(s_0, a_0) + \sum_{a_0} \pi_\theta(s_0, a_0)\ \nabla Q_{\pi_\theta}(s_0, a_0)] \\
&= \mathbb{E}_{s_0 \sim \mu_0}[\sum_{a_0} \pi_\theta(s_0, a_0)\ \nabla \log \pi_\theta(s_0, a_0)\ Q_{\pi_\theta}(s_0, a_0)] + \mathbb{E}_{s_0 \sim \mu_0}[\sum_{a_0} \pi_\theta(s_0, a_0)\ \nabla Q_{\pi_\theta}(s_0, a_0)]
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&= \mathbb{E}_{s_0 \sim \mu_0}[\sum_{a_0} \nabla \pi_\theta(s_0, a_0)\ Q_{\pi_\theta}(s_0, a_0) + \sum_{a_0} \pi_\theta(s_0, a_0)\ \nabla Q_{\pi_\theta}(s_0, a_0)] \\
&= \mathbb{E}_{s_0 \sim \mu_0}[\sum_{a_0} \pi_\theta(s_0, a_0)\ \nabla \log \pi_\theta(s_0, a_0)\ Q_{\pi_\theta}(s_0, a_0)] + \mathbb{E}_{s_0 \sim \mu_0}[\sum_{a_0} \pi_\theta(s_0, a_0)\ \nabla Q_{\pi_\theta}(s_0, a_0)] \\
&= \mathbb{E}_{s_0 \sim \mu_0} \mathbb{E}_{a_0 \sim \pi_\theta(s_0, \cdot)}[\nabla \log \pi_\theta(s_0, a_0)\ Q_{\pi_\theta}(s_0, a_0)] + \mathbb{E}_{s_0 \sim \mu_0} \mathbb{E}_{a_0 \sim \pi_\theta(s_0, \cdot)}[\nabla Q_{\pi_\theta}(s_0, a_0)]
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&= \mathbb{E}_{s_0 \sim \mu_0} \mathbb{E}_{a_0 \sim \pi_\theta(s_0, \cdot)}[\nabla \log \pi_\theta(s_0, a_0)\ Q_{\pi_\theta}(s_0, a_0)] + \mathbb{E}_{s_0 \sim \mu_0} \mathbb{E}_{a_0 \sim \pi_\theta(s_0, \cdot)}[\nabla Q_{\pi_\theta}(s_0, a_0)] \\
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&\qquad \mathbb{E}_{s_0 \sim \mu_0} \mathbb{E}_{a_0 \sim \pi_\theta(s_0, \cdot)}[\nabla (r(s_0, a_0) + \gamma \mathbb{E}_{s_1 \sim P(\cdot | s_0, a_0)}[V_{\pi_\theta}(s_1)])
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&= \mathbb{E}_{s_0 \sim \mu_0}[\sum_{a_0} \pi_\theta(s_0, a_0)\ \nabla \log \pi_\theta(s_0, a_0)\ Q_{\pi_\theta}(s_0, a_0)] + \mathbb{E}_{s_0 \sim \mu_0}[\sum_{a_0} \pi_\theta(s_0, a_0)\ \nabla Q_{\pi_\theta}(s_0, a_0)] \\
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&\qquad \gamma \mathbb{E}_{s_0 \sim \mu_0} \mathbb{E}_{a_0 \sim \pi_\theta(s_0, \cdot)} \mathbb{E}_{s_1 \sim P(\cdot | s_0, a_0)}[\nabla V_{\pi_\theta}(s_1)]
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&= \mathbb{E}_{s_0 \sim \mu_0}[\sum_{a_0} \nabla \pi_\theta(s_0, a_0)\ Q_{\pi_\theta}(s_0, a_0) + \sum_{a_0} \pi_\theta(s_0, a_0)\ \nabla Q_{\pi_\theta}(s_0, a_0)] \\
&= \mathbb{E}_{s_0 \sim \mu_0}[\sum_{a_0} \pi_\theta(s_0, a_0)\ \nabla \log \pi_\theta(s_0, a_0)\ Q_{\pi_\theta}(s_0, a_0)] + \mathbb{E}_{s_0 \sim \mu_0}[\sum_{a_0} \pi_\theta(s_0, a_0)\ \nabla Q_{\pi_\theta}(s_0, a_0)] \\
&= \mathbb{E}_{s_0 \sim \mu_0} \mathbb{E}_{a_0 \sim \pi_\theta(s_0, \cdot)}[\nabla \log \pi_\theta(s_0, a_0)\ Q_{\pi_\theta}(s_0, a_0)] + \mathbb{E}_{s_0 \sim \mu_0} \mathbb{E}_{a_0 \sim \pi_\theta(s_0, \cdot)}[\nabla Q_{\pi_\theta}(s_0, a_0)] \\
&= \mathbb{E}_{s_0 \sim \mu_0} \mathbb{E}_{a_0 \sim \pi_\theta(s_0, \cdot)}[\nabla \log \pi_\theta(s_0, a_0)\ Q_{\pi_\theta}(s_0, a_0)] + \\
&\qquad \mathbb{E}_{s_0 \sim \mu_0} \mathbb{E}_{a_0 \sim \pi_\theta(s_0, \cdot)}[\nabla (r(s_0, a_0) + \gamma \mathbb{E}_{s_1 \sim P(\cdot | s_0, a_0)}[V_{\pi_\theta}(s_1)])] \\
&= \mathbb{E}_{s_0 \sim \mu_0} \mathbb{E}_{a_0 \sim \pi_\theta(s_0, \cdot)}[\nabla \log \pi_\theta(s_0, a_0)\ Q_{\pi_\theta}(s_0, a_0)] + \\
&\qquad \gamma \mathbb{E}_{s_0 \sim \mu_0} \mathbb{E}_{a_0 \sim \pi_\theta(s_0, \cdot)} \mathbb{E}_{s_1 \sim P(\cdot | s_0, a_0)}[\nabla V_{\pi_\theta}(s_1)] \\
&= \sum_{(s,a)} \mathbb{P}(s_0 = s, a_0 = a)[\nabla \log \pi_\theta(s, a)\ Q_{\pi_\theta}(s, a)] + \gamma \sum_s \mathbb{P}(s_1 = s)\ \nabla V_{\pi_\theta}(s)
\end{aligned}
$$

# Proof of Policy Gradient Theorem: Form 2

Proof:

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$$= \sum_{(s,a)} \mathbb{P}(s_0 = s, a_0 = a)[\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(s,a)]$$

$$+ \gamma \sum_{(s,a)} \mathbb{P}(s_1 = s, a_1 = a)[\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(s,a)] + \gamma^2 \sum_{s} \mathbb{P}(s_2 = s)\ \nabla V_{\pi_\theta}(s)$$

# Proof of Policy Gradient Theorem: Form 2

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$$
\begin{aligned}
&= \sum_{(s,a)} \mathbb{P}(s_0 = s, a_0 = a)[\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(s,a)] \\
&+ \gamma \sum_{(s,a)} \mathbb{P}(s_1 = s, a_1 = a)[\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(s,a)] + \gamma^2 \sum_{s} \mathbb{P}(s_2 = s)\ \nabla V_{\pi_\theta}(s) \\
&\quad \vdots
\end{aligned}
$$

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&= \sum_{(s,a)} \mathbb{P}(s_0 = s, a_0 = a)[\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(s,a)] \\
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&\quad \vdots \\
&= \sum_{t=0}^{\infty} \gamma^t \sum_{(s,a)} \mathbb{P}(s_t = s, a_t = a)[\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(s,a)]
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&\qquad \vdots \\
&= \sum_{t=0}^{\infty} \gamma^t \sum_{(s,a)} \mathbb{P}(s_t = s, a_t = a)[\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(s,a)] \\
&= \sum_{(s,a)} [\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(x,a)] \sum_{t=0}^{\infty} \gamma^t \mathbb{P}(s_t = s, a_t = a)
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$$

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&\quad \vdots \\
&= \sum_{t=0}^{\infty} \gamma^t \sum_{(s,a)} \mathbb{P}(s_t = s, a_t = a)[\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(s,a)] \\
&= \sum_{(s,a)} [\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(x,a)] \sum_{t=0}^{\infty} \gamma^t \mathbb{P}(s_t = s, a_t = a) \\
&= \frac{1}{(1-\gamma)} \sum_{(s,a)} [\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(s,a)] \rho_{\pi_\theta}(s,a)
\end{aligned}
$$

# Proof of Policy Gradient Theorem: Form 2

Proof:

$$
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&= \sum_{(s,a)} \mathbb{P}(s_0 = s, a_0 = a)[\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(s,a)] \\
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&\quad \vdots \\
&= \sum_{t=0}^{\infty} \gamma^t \sum_{(s,a)} \mathbb{P}(s_t = s, a_t = a)[\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(s,a)] \\
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&= \frac{1}{(1-\gamma)} \sum_{(s,a)} [\nabla \log \pi_\theta(s,a)\ Q_{\pi_\theta}(s,a)] \rho_{\pi_\theta}(s,a) \\
&= \frac{1}{(1-\gamma)} \mathbb{E}_{(s,a)\sim \rho_{\pi_\theta}(\cdot)} [Q_{\pi_\theta}(s,a)\ \nabla \log \pi_\theta(s,a)]
\end{aligned}
$$

$\square$

# Estimating Policy Gradient

- We have $\nabla V_{\pi_\theta} = (1/(1-\gamma))\ \mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[Q_{\pi_\theta}(s,a)\ \nabla \log \pi_\theta(s,a)\right]$
- How do we estimate $\widehat{\nabla V_{\pi_\theta}}$?

# Estimating Policy Gradient

- We have $\nabla V_{\pi_\theta} = (1/(1-\gamma))\ \mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[Q_{\pi_\theta}(s,a)\ \nabla\log\pi_\theta(s,a)\right]$
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# Estimating Policy Gradient

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## Proposition

$$\begin{aligned}\nabla V_{\pi_\theta} &= (1/(1-\gamma))\ \mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[Q_{\pi_\theta}(s,a)\ \nabla \log \pi_\theta(s,a)\right] \\ &= \mathbb{E}_{\tau\sim P_{\mathrm{traj},\theta}(\tau)}\left[\sum_{t=0}^{\infty}\gamma^t\ Q_{\pi_\theta}(s_t,a_t)\ \nabla \log \pi_\theta(s_t,a_t)\right]\end{aligned}$$

# Estimating Policy Gradient

Proof:

$\square$

# Estimating Policy Gradient

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$$\mathbb{E}_{\tau\sim P_{\mathrm{traj},\theta}(\tau)}[\sum_{t=0}^{\infty}\gamma^t\ Q_{\pi_\theta}(s_t,a_t)\ \nabla\log\pi_\theta(s_t,a_t)]$$

$\square$

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Proof:

$$\mathbb{E}_{\tau\sim P_{\text{traj},\theta}(\tau)}[\sum_{t=0}^{\infty}\gamma^t\ Q_{\pi_\theta}(s_t,a_t)\ \nabla\log\pi_\theta(s_t,a_t)]$$

$$=\sum_x\sum_a\sum_{t=0}^{\infty}\mathbb{P}(s_t=s,a_t=a)\ \gamma^t\ Q_{\pi_\theta}(s,a)\ \nabla\log\pi_\theta(s,a)$$

$\square$

# Estimating Policy Gradient

Proof:

$$\mathbb{E}_{\tau \sim P_{\text{traj},\theta}(\tau)}[\sum_{t=0}^{\infty} \gamma^t \; Q_{\pi_\theta}(s_t, a_t) \; \nabla \log \pi_\theta(s_t, a_t)]$$

$$= \sum_x \sum_a \sum_{t=0}^{\infty} \mathbb{P}(s_t = s, a_t = a) \; \gamma^t \; Q_{\pi_\theta}(s, a) \; \nabla \log \pi_\theta(s, a)$$

$$= \sum_s \sum_a Q_{\pi_\theta}(s, a) \; \nabla \log \pi_\theta(s, a) \; (\sum_{t=0}^{\infty} \mathbb{P}(s_t = s, a_t = a)\gamma^t)$$

$\square$

# Estimating Policy Gradient

Proof:

$$\mathbb{E}_{\tau \sim P_{\text{traj},\theta}(\tau)}[\sum_{t=0}^{\infty} \gamma^t \; Q_{\pi_\theta}(s_t, a_t) \; \nabla \log \pi_\theta(s_t, a_t)]$$

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$$= (1/(1-\gamma)) \sum_x \sum_a Q_{\pi_\theta}(s, a) \; \nabla \log \pi_\theta(s, a) \rho_{\pi_\theta}(s, a)$$

$\square$

# Estimating Policy Gradient

Proof:

$$\mathbb{E}_{\tau \sim P_{\text{traj},\theta}(\tau)}[\sum_{t=0}^{\infty} \gamma^t \; Q_{\pi_\theta}(s_t, a_t) \; \nabla \log \pi_\theta(s_t, a_t)]$$

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$$= (1/(1-\gamma)) \sum_x \sum_a Q_{\pi_\theta}(s, a) \; \nabla \log \pi_\theta(s, a) \rho_{\pi_\theta}(s, a)$$

$$= (1/(1-\gamma)) \mathbb{E}_{(s,a) \sim \rho_{\pi_\theta}(\cdot)} [Q_{\pi_\theta}(s, a) \; \nabla \log \pi_\theta(s, a)]$$

□

# Estimating Policy Gradient

Proof:

$$
\begin{aligned}
&\mathbb{E}_{\tau\sim P_{\text{traj},\theta}(\tau)}[\sum_{t=0}^{\infty}\gamma^t\ Q_{\pi_\theta}(s_t,a_t)\ \nabla\log\pi_\theta(s_t,a_t)]\\
&=\sum_{x}\sum_{a}\sum_{t=0}^{\infty}\mathbb{P}(s_t=s,a_t=a)\ \gamma^t\ Q_{\pi_\theta}(s,a)\ \nabla\log\pi_\theta(s,a)\\
&=\sum_{s}\sum_{a}Q_{\pi_\theta}(s,a)\ \nabla\log\pi_\theta(s,a)\ (\sum_{t=0}^{\infty}\mathbb{P}(s_t=s,a_t=a)\gamma^t)\\
&=(1/(1-\gamma))\sum_{x}\sum_{a}Q_{\pi_\theta}(s,a)\ \nabla\log\pi_\theta(s,a)\rho_{\pi_\theta}(s,a)\\
&=(1/(1-\gamma))\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot)}\left[Q_{\pi_\theta}(s,a)\ \nabla\log\pi_\theta(s,a)\right]\\
&=\nabla V_{\pi_\theta}
\end{aligned}
$$

□

# Estimating Policy Gradient

- We have $\nabla V_{\pi_\theta} = (1/(1-\gamma))\ \mathbb{E}_{(x,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[Q_{\pi_\theta}(x,a)\ \nabla \log \pi_\theta(x,a)\right]$
- How do we estimate $\widehat{\nabla V_{\pi_\theta}}$?
- We will use an equivalent form

## Proposition

$$\begin{aligned}\nabla V_{\pi_\theta} &= (1/(1-\gamma))\ \mathbb{E}_{(x,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[Q_{\pi_\theta}(x,a)\ \nabla \log \pi_\theta(x,a)\right] \\ &= \mathbb{E}_{\tau\sim P_{\mathrm{traj},\theta}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t\ Q_{\pi_\theta}(s_t,a_t)\ \nabla \log \pi_\theta(s_t,a_t)\right]\end{aligned}$$

# Estimating Policy Gradient

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- For a trajectory $\tau$, define:

$$\widehat{Q_{\pi_\theta}}(s_t,a_t) = \sum_{m\geq t}^{\infty} \gamma^{(m-t)} r(s_m,a_m)$$

$$\widehat{\nabla V_{\pi_\theta}} = \sum_{t=0}^{\infty} \gamma^t\ \widehat{Q_{\pi_\theta}}(s_t,a_t)\ \nabla \log \pi_\theta(s_t,a_t)$$

# Variance Reduction using Baseline

- Advantage function of a policy $\pi$ is defined as
$$A_\pi(s,a) = Q_\pi(s,a) - V_\pi(s)$$
- Note that $\mathbb{E}_{a\sim\pi(s,\cdot)}[A_\pi(s,a)] = 0$

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- Note that $\mathbb{E}_{a\sim\pi(s,\cdot)}[A_\pi(s,a)] = 0$

## Proposition

$$\nabla V_{\pi_\theta} = \frac{1}{(1-\gamma)}\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[A_{\pi_\theta}(s,a)\ \nabla\log\pi_\theta(s,a)\right]$$

# Variance Reduction using Baseline

**Proof:**

Let $b_{\pi_\theta}(s)$ be an action independent baseline.

We will first show that $\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[b_{\pi_\theta}(s)\ \nabla \log \pi_\theta(s,a)\right] = 0.$

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$$\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[b_{\pi_\theta}(s)\ \nabla \log \pi_\theta(s,a)\right]$$

# Variance Reduction using Baseline

## Proof:

Let $b_{\pi_\theta}(s)$ be an action independent baseline.

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$$
\begin{aligned}
&\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[b_{\pi_\theta}(s)\ \nabla \log \pi_\theta(s,a)\right] \\
&\qquad = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(s)\ \mathbb{E}_{a\sim\pi_\theta(s,\cdot)}[\nabla \log \pi_\theta(s,a)]]
\end{aligned}
$$

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Let $b_{\pi_\theta}(s)$ be an action independent baseline.

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&\qquad = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(s)\ \mathbb{E}_{a\sim\pi_\theta(s,\cdot)}[\nabla \log \pi_\theta(s,a)]] \\
&\qquad = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(x)\ \sum_a \pi_\theta(s,a)\nabla \log \pi_\theta(s,a)]
\end{aligned}$$

# Variance Reduction using Baseline

## Proof:

Let $b_{\pi_\theta}(s)$ be an action independent baseline.

We will first show that $\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[b_{\pi_\theta}(s)\ \nabla \log \pi_\theta(s,a)\right] = 0$.

$$
\begin{aligned}
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&\qquad = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(x)\ \sum_a \nabla\pi_\theta(s,a)] = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(s)\ \nabla \sum_a \pi_\theta(s,a)]
\end{aligned}
$$

# Variance Reduction using Baseline

## Proof:

Let $b_{\pi_\theta}(s)$ be an action independent baseline.

We will first show that $\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[b_{\pi_\theta}(s)\ \nabla \log \pi_\theta(s,a)\right] = 0$.

$$
\begin{aligned}
&\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[b_{\pi_\theta}(s)\ \nabla \log \pi_\theta(s,a)\right] \\
&\qquad = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(s)\ \mathbb{E}_{a\sim\pi_\theta(s,\cdot)}[\nabla \log \pi_\theta(s,a)]] \\
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&\qquad = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(x)\ \sum_a \nabla\pi_\theta(s,a)] = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(s)\ \nabla\sum_a \pi_\theta(s,a)] \\
&\qquad = \mathbb{E}_{x\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[b_{\pi_\theta}(s)\ \nabla(1)\right] = 0
\end{aligned}
$$

# Variance Reduction using Baseline

Proof:

Let $b_{\pi_\theta}(s)$ be an action independent baseline.

We will first show that $\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[b_{\pi_\theta}(s)\ \nabla \log \pi_\theta(s,a)\right] = 0$.

$$
\begin{aligned}
&\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[b_{\pi_\theta}(s)\ \nabla \log \pi_\theta(s,a)\right] \\
&\quad = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(s)\ \mathbb{E}_{a\sim\pi_\theta(s,\cdot)}[\nabla \log \pi_\theta(s,a)]] \\
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&\quad = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(x)\ \sum_a \nabla\pi_\theta(s,a)] = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(s)\ \nabla\sum_a \pi_\theta(s,a)] \\
&\quad = \mathbb{E}_{x\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[b_{\pi_\theta}(s)\ \nabla(1)\right] = 0
\end{aligned}
$$

So, for any such baseline $b_\pi(s)$ , we get

$$
\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot)}\left[(Q_{\pi_\theta}(s,a) - b_{\pi_\theta}(s))\ \nabla \log \pi_\theta(s,a)\right] = \nabla V_{\pi_\theta}
$$

# Variance Reduction using Baseline

## Proof:

Let $b_{\pi_\theta}(s)$ be an action independent baseline.

We will first show that $\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[b_{\pi_\theta}(s)\ \nabla\log\pi_\theta(s,a)\right] = 0$.

$$
\begin{aligned}
&\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[b_{\pi_\theta}(s)\ \nabla\log\pi_\theta(s,a)\right] \\
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&\qquad = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(x)\ \sum_a \pi_\theta(s,a)\nabla\log\pi_\theta(s,a)] \\
&\qquad = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(x)\ \sum_a \nabla\pi_\theta(s,a)] = \mathbb{E}_{s\sim\rho_{\pi_\theta}(\cdot)}[b_{\pi_\theta}(s)\ \nabla\sum_a \pi_\theta(s,a)] \\
&\qquad = \mathbb{E}_{x\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[b_{\pi_\theta}(s)\ \nabla(1)\right] = 0
\end{aligned}
$$

So, for any such baseline $b_\pi(s)$ , we get

$$
\mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot)}\left[(Q_{\pi_\theta}(s,a) - b_{\pi_\theta}(s))\ \nabla\log\pi_\theta(s,a)\right] = \nabla V_{\pi_\theta}
$$

This is in particular true for the baseline $b_{\pi_\theta}(x) = V_{\pi_\theta}(x)$

# Policy Gradient Theorem(s)

## Theorem

*The following as expressions for* $\nabla V_{\pi_\theta}$*:*

$$\nabla V_{\pi_\theta} = \mathbb{E}_{\tau \sim P_{\mathrm{traj},\theta}(\cdot)} \left[ R(\tau) \sum_{t=0}^{\infty} \nabla \log \pi_\theta(s_t, a_t) \right] \tag{1}$$

$$\nabla V_{\pi_\theta} = \frac{1}{(1-\gamma)} \mathbb{E}_{(s,a) \sim \rho_{\pi_\theta}(\cdot,\cdot)} \left[ Q_{\pi_\theta}(s,a) \ \nabla \log \pi_\theta(s,a) \right] \tag{2}$$

$$\nabla V_{\pi_\theta} = \mathbb{E}_{\tau \sim P_{\mathrm{traj},\theta}(\tau)} \left[ \sum_{t=0}^{\infty} \gamma^t \ Q_{\pi_\theta}(s_t, a_t) \ \nabla \log \pi_\theta(s_t, a_t) \right] \tag{3}$$

$$\nabla V_{\pi_\theta} = \frac{1}{(1-\gamma)} \mathbb{E}_{(s,a) \sim \rho_{\pi_\theta}(\cdot,\cdot)} \left[ A_{\pi_\theta}(s,a) \ \nabla \log \pi_\theta(s,a) \right] \tag{4}$$

# Actor-Critic Algorithm

- We have, $\nabla V_{\pi_\theta} = \mathbb{E}_{(s,a)\sim\rho_{\pi_\theta}(\cdot,\cdot)}\left[Q_{\pi_\theta}(s,a)\ \nabla \log \pi_\theta(s,a)\right]$

# Actor-Critic Algorithm

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- This is an example of two timescale stochastic approximation algorithm

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- We will use Kullback–Leibler divergence as a measure of distance between two policies

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# Taylor Series Expansion of KL Divergence

- Taylor series expansion of a function $f(y)$ around $y_0$

$$f(y) \approx f(y_0) + (y - y_0)^\top \nabla_y f(y)|_{y_0} + (y - y_0)^\top \nabla_y^2 f(y)|_{y_0} (y - y_0) + \cdots$$

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D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta) = D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_{\theta_0}) &+ (\theta - \theta_0)^\top \nabla_\theta D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0} \\
&+ (\theta - \theta_0)^\top \nabla_\theta^2 D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0} (\theta - \theta_0) + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\theta D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0} &= \nabla_\theta \mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s,\cdot)} \left[ \log \frac{\pi_{\theta_0}(s,a)}{\pi_\theta(s,a)} \right] |_{\theta_0} \\
&= -\mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s,\cdot)} \left[ \frac{\nabla \pi_\theta(s,a)|_{\theta_0}}{\pi_{\theta_0}(s,a)} \right] \\
&= -\mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} [\sum_a \nabla \pi_\theta(s,a)|_{\theta_0}]
\end{aligned}
$$

# Taylor Series Expansion of KL Divergence

- Taylor series expansion of $D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)$

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D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta) = D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_{\theta_0}) &+ (\theta - \theta_0)^\top \nabla_\theta D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0} \\
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&= -\mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s,\cdot)} \left[ \frac{\nabla \pi_\theta(s,a)|_{\theta_0}}{\pi_{\theta_0}(s,a)} \right] \\
&= -\mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} [\sum_a \nabla \pi_\theta(s,a)|_{\theta_0}] \\
&= -\mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} [\nabla \sum_a \pi_\theta(s,a)|_{\theta_0}] = 0
\end{aligned}
$$

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$$
\nabla_\theta^2 D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0} = \nabla_\theta^2 \mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s, \cdot)} \left[ \log \frac{\pi_{\theta_0}(s, a)}{\pi_\theta(s, a)} \right] |_{\theta_0}
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&= -\mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s, \cdot)} \nabla \left[ \frac{\nabla \pi_\theta(s,a)}{\pi_\theta(s,a)} \right] |_{\theta_0}
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&= -\mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s,\cdot)} \nabla \left[\frac{\nabla \pi_\theta(s,a)}{\pi_\theta(s,a)}\right] |_{\theta_0} \\
&= -\mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s,\cdot)} \left[\frac{\nabla^2 \pi_\theta(s,a)}{\pi_\theta(s,a)} - \frac{\nabla \pi_\theta(s,a)\ \nabla \pi_\theta(s,a)^\top}{(\pi_\theta(s,a))^2}\right] |_{\theta_0}
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&+ (\theta - \theta_0)^\top \nabla_\theta^2 D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0}(\theta - \theta_0) + \cdots
\end{aligned}
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\nabla_\theta^2 D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0} &= \nabla_\theta^2 \mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s,\cdot)} \left[ \log \frac{\pi_{\theta_0}(s,a)}{\pi_\theta(s,a)} \right] |_{\theta_0} \\
&= -\mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s,\cdot)} \nabla \left[ \frac{\nabla \pi_\theta(s,a)}{\pi_\theta(s,a)} \right] |_{\theta_0} \\
&= -\mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s,\cdot)} \left[ \frac{\nabla^2 \pi_\theta(s,a)}{\pi_\theta(s,a)} - \frac{\nabla \pi_\theta(s,a)\ \nabla \pi_\theta(s,a)^\top}{(\pi_\theta(s,a))^2} \right] |_{\theta_0} \\
&= \mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s,\cdot)} \left[ \frac{\nabla \pi_\theta(s,a)\ \nabla \pi_\theta(s,a)^\top}{(\pi_\theta(s,a))^2} \right] |_{\theta_0}
\end{aligned}
$$

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\nabla_\theta^2 D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0} &= \nabla_\theta^2 \mathbb{E}_{s\sim\rho_{\pi_{\theta_0}}} \mathbb{E}_{a\sim\pi_{\theta_0}(s,\cdot)} \left[\log \frac{\pi_{\theta_0}(s,a)}{\pi_\theta(s,a)}\right] |_{\theta_0} \\
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&= \mathbb{E}_{s\sim\rho_{\pi_{\theta_0}}} \mathbb{E}_{a\sim\pi_{\theta_0}(s,\cdot)} \left[\frac{\nabla \pi_\theta(s,a)\ \nabla \pi_\theta(s,a)^\top}{(\pi_\theta(s,a))^2}\right] |_{\theta_0} \\
&= \mathbb{E}_{s\sim\rho_{\pi_{\theta_0}}} \mathbb{E}_{a\sim\pi_{\theta_0}(s,\cdot)} \left[\nabla \log \pi_\theta(s,a)\ \nabla \log \pi_\theta(s,a)^\top\right] |_{\theta_0}
\end{aligned}
$$

# Taylor Series Expansion of KL Divergence

- Taylor series expansion of $D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)$

$$
\begin{aligned}
D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta) = D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_{\theta_0}) &+ (\theta - \theta_0)^\top \nabla_\theta D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0} \\
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\end{aligned}
$$

$$
\begin{aligned}
\nabla_\theta^2 D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0} &= \nabla_\theta^2 \mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s,\cdot)} \left[ \log \frac{\pi_{\theta_0}(s,a)}{\pi_\theta(s,a)} \right] |_{\theta_0} \\
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&= -\mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s,\cdot)} \left[ \frac{\nabla^2 \pi_\theta(s,a)}{\pi_\theta(s,a)} - \frac{\nabla \pi_\theta(s,a)\ \nabla \pi_\theta(s,a)^\top}{(\pi_\theta(s,a))^2} \right] |_{\theta_0} \\
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&= \mathbb{E}_{s \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(s,\cdot)} \left[ \nabla \log \pi_\theta(s,a)\ \nabla \log \pi_\theta(s,a)^\top \right] |_{\theta_0}
\end{aligned}
$$

- Fisher information matrix

$$
F(\theta_0) = \mathbb{E}_{x \sim \rho_{\pi_{\theta_0}}} \mathbb{E}_{a \sim \pi_{\theta_0}(x,\cdot)} \left[ \nabla \log \pi_\theta(x,a)\ \nabla \log \pi_\theta(x,a)^\top \right] |_{\theta_0}
$$

# Taylor Series Expansion of KL Divergence

- Taylor series expansion of $D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)$

$$\begin{aligned} D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta) = D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_{\theta_0}) + (\theta - \theta_0)^\top \nabla_\theta D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0} \\ + (\theta - \theta_0)^\top \nabla_\theta^2 D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0}(\theta - \theta_0) + \cdots \end{aligned}$$

$$\begin{aligned} \nabla_\theta^2 D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0} &= \nabla_\theta^2 \mathbb{E}_{s\sim\rho_{\pi_{\theta_0}}} \mathbb{E}_{a\sim\pi_{\theta_0}(s,\cdot)} \left[\log \frac{\pi_{\theta_0}(s,a)}{\pi_\theta(s,a)}\right]|_{\theta_0} \\ &= -\mathbb{E}_{s\sim\rho_{\pi_{\theta_0}}} \mathbb{E}_{a\sim\pi_{\theta_0}(s,\cdot)} \nabla \left[\frac{\nabla \pi_\theta(s,a)}{\pi_\theta(s,a)}\right]|_{\theta_0} \\ &= -\mathbb{E}_{s\sim\rho_{\pi_{\theta_0}}} \mathbb{E}_{a\sim\pi_{\theta_0}(s,\cdot)} \left[\frac{\nabla^2 \pi_\theta(s,a)}{\pi_\theta(s,a)} - \frac{\nabla \pi_\theta(s,a)\ \nabla \pi_\theta(s,a)^\top}{(\pi_\theta(s,a))^2}\right]|_{\theta_0} \\ &= \mathbb{E}_{s\sim\rho_{\pi_{\theta_0}}} \mathbb{E}_{a\sim\pi_{\theta_0}(s,\cdot)} \left[\frac{\nabla \pi_\theta(s,a)\ \nabla \pi_\theta(s,a)^\top}{(\pi_\theta(s,a))^2}\right]|_{\theta_0} \\ &= \mathbb{E}_{s\sim\rho_{\pi_{\theta_0}}} \mathbb{E}_{a\sim\pi_{\theta_0}(s,\cdot)} \left[\nabla \log \pi_\theta(s,a)\ \nabla \log \pi_\theta(s,a)^\top\right]|_{\theta_0} \end{aligned}$$

- Fisher information matrix

$$F(\theta_0) = \mathbb{E}_{x\sim\rho_{\pi_{\theta_0}}} \mathbb{E}_{a\sim\pi_{\theta_0}(x,\cdot)} \left[\nabla \log \pi_\theta(x,a)\ \nabla \log \pi_\theta(x,a)^\top\right]|_{\theta_0}$$

- $\nabla_\theta^2 D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)|_{\theta_0} = F(\theta_0)$

# Natural Policy Gradient

- Using Taylor series expansion of $D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta)$

$$D_{KL}^{\pi_{\theta_0}}(\pi_{\theta_0}, \pi_\theta) \approx (\theta - \theta_0)^\top F(\theta_0)(\theta - \theta_0)$$

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- We we want to find a direction $\delta\theta$, where

$$\arg\max_{\delta} \ \delta^\top(\nabla V_{\pi_\theta}), \quad \text{s.t. } D_{KL}^{\pi_\theta}\big(\pi_\theta, \pi_{(\theta+\delta\theta)}\big) \leq \epsilon$$

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$$\theta \leftarrow \theta + \delta\theta$$

$$\delta\theta = \sqrt{\frac{\epsilon}{(\nabla V)^\top F(\theta)^{-1}(\nabla V)}} \ F(\theta)^\dagger(\nabla V)$$

# Example: Policy Gradient for LQG

- Consider a scalar Linear Quadratic Gaussian (LQG) problem

$$x_{t+1} = Ax_t + Bu_t + w_t,\ w_t \sim \mathcal{N}(0, \sigma^2),\ R_t = -Qx_t^2 - Pu_t^2$$

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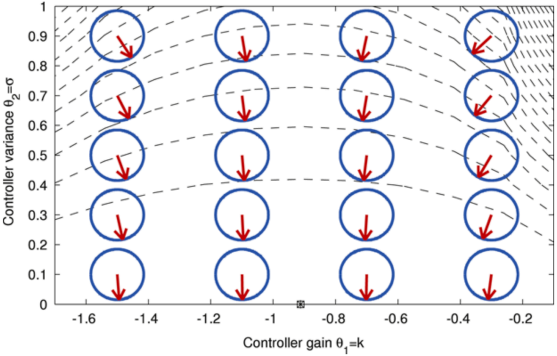
- Perform vanilla PG: $\theta \leftarrow \theta + \alpha \widehat{\nabla V_{\pi_\theta}}$
- How will the vanilla PG perform?

# Example: Vanilla Policy Gradient for LQG




- Control policy: $u_t \sim \mathcal{N}(\theta_1 x_t, \theta_2)$, $\theta \leftarrow \theta + \alpha \widehat{\nabla V_{\pi_\theta}}$

Figure from Peters, Jan, and Stefan Schaal. "Reinforcement learning of motor skills with policy gradients", *Neural*

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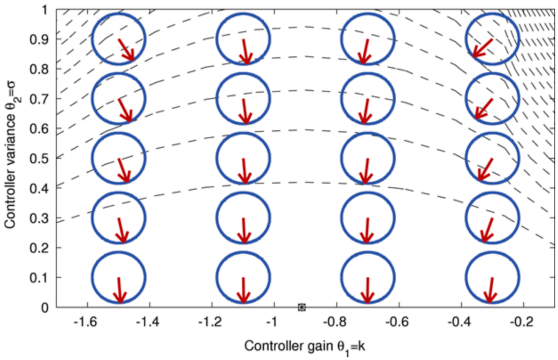


- Control policy: $u_t \sim \mathcal{N}(\theta_1 x_t, \theta_2)$, $\theta \leftarrow \theta + \alpha \widehat{\nabla V_{\pi_\theta}}$
- Decrease of exploration has a stronger immediate effect on the expected return. So, the gradient mainly points in that direction.

Figure from Peters, Jan, and Stefan Schaal. "Reinforcement learning of motor skills with policy gradients", *Neural*

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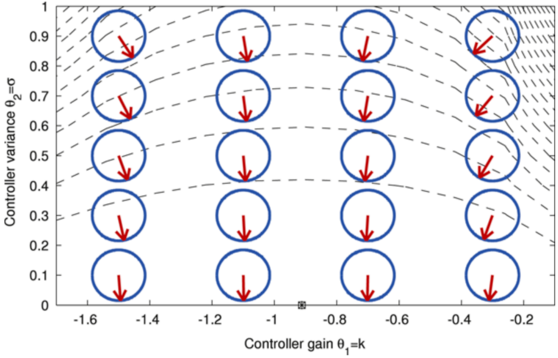
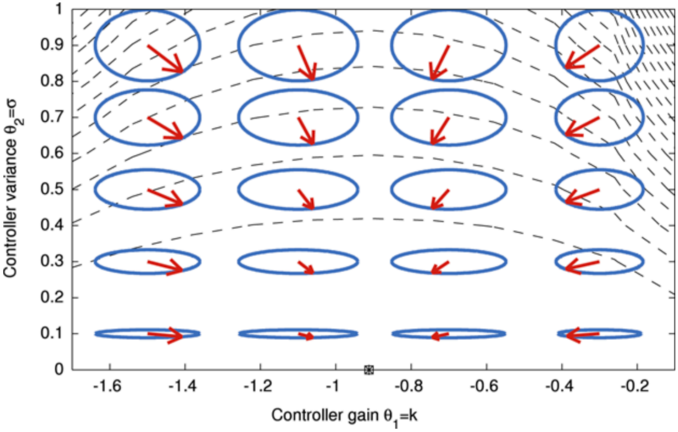



- Control policy: $u_t \sim \mathcal{N}(\theta_1 x_t, \theta_2)$, $\theta \leftarrow \theta + \alpha \widehat{\nabla V_{\pi_\theta}}$
- Decrease of exploration has a stronger immediate effect on the expected return. So, the gradient mainly points in that direction.
- This will quickly reaches the 'plateau of zero exploration' and results in a very slow convergence

Figure from Peters, Jan, and Stefan Schaal. "Reinforcement learning of motor skills with policy gradients", *Neural*

# Example: Natural Policy Gradient for LQG




- Control policy: $u_t \sim \mathcal{N}(\theta_1 x_t, \theta_2)$, $\theta \leftarrow \theta + \alpha \widehat{F}(\theta)^{-1} \widehat{\nabla V_{\pi_\theta}}$

Figure from Peters, Jan, and Stefan Schaal. "Reinforcement learning of motor skills with policy gradients", *Neural networks*, 2008.