ECEN 743: Reinforcement Learning

Exploration in RL

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References

- [UCB Algorithm] Chapter 7, Tor Lattimore and Csaba Szepesvári, "Bandit algorithms". Cambridge University Press, 2020.
- [Posterior Sampling Algorithm] Chapter 36, Tor Lattimore and Csaba Szepesvári, "Bandit algorithms". Cambridge University Press, 2020.

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  - This is one of the fundamental (research) problems of RL

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- How to select arms if the mean values are unknown a priori?

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- Objective: select actions to maximize the expected cumulative reward

$$\max_{(a_t)_{t=1}^T} \quad \mathbb{E}[\sum_{t=1}^{T} r(a_t)]$$

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- What is the fundamental lower bound? Is there any algorithm that achieves the lower bound?

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Upper Confidence Bound (UCB) Algorithm

UCB algorithm

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Performance of UCB Algorithm

Theorem (Auer et al, 2002)

Under UCB algorithm,

$$\mathrm{Reg}(T) \leq 8 \sum_{k=2}^{K} \frac{\log T}{(\mu_1 - \mu_k)} + 4K$$

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Lemma (Hoeffding's inequality)

Let $\hat{\mu}(t) = \frac{1}{t}\sum_{\tau=1}^{t} X_\tau$, where X_τs are i.i.d. random variables with $\mathbb{E}[X_\tau] = \mu$. Also, let $b_l \leq X_\tau \leq b_h$. Then,

$$\mathbb{P}(|\hat{\mu}(t) - \mu| \geq \epsilon) \leq 2 \exp\left(-\frac{2t\epsilon^2}{(b_h - b_l)^2}\right)$$

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- We will choose $m_k = C \log T$ and show that under this choice, $T\mathbb{P}(G^c)$ is a constant

Proof of the UCB Algorithm - 2

- We will bound the probability of $G_k^c = \{\mu_1 \leq \min_t g_1(t)\}^c \cup \left\{g_{k,m_k} \leq \mu_1\right\}^c$

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- So, we can bound the probability of the fist set as

$$\mathbb{P}(\{\mu_1 \ge \min_t g_1(t)\}) \le 2T\delta$$

Proof of the UCB Algorithm - 3

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- Note that this can be achieved by selecting $m_k \ge \frac{2\log(1/\delta)}{\Delta_k^2}$. Choose this m_k.

Proof of the UCB Algorithm - 3

- We will bound the probability of $G_k^c = \{\mu_1 \leq \min_t g_1(t)\}^c \cup \{g_{k,m_k} \leq \mu_1\}^c$
- We got a bound for the first set. Now, we will consider the second set.
- We first assume that m_k is large enough so that $\Delta_k - \sqrt{\frac{\log(1/\delta)}{2m_k}} \geq \frac{1}{2}\Delta_k$
- Note that this can be achieved by selecting $m_k \geq \frac{2\log(1/\delta)}{\Delta_k^2}$. Choose this m_k.
- Now, we can bound the probability of the second set as

$$\mathbb{P}(\{g_{k,m_k} \geq \mu_1\}) = \mathbb{P}(\{\hat{\mu}_{k,m_k} + \sqrt{\frac{\log(1/\delta)}{2m_k}} \geq \mu_1\}) = \mathbb{P}(\{\hat{\mu}_{k,m_k} - \mu_k + \sqrt{\frac{\log(1/\delta)}{2m_k}} \geq \mu_1 - \mu_k\})$$

$$= \mathbb{P}(\{\hat{\mu}_{k,m_k} - \mu_k \geq \Delta_k - \sqrt{\frac{\log(1/\delta)}{2m_k}}\}) \leq \mathbb{P}(\{\hat{\mu}_{k,m_k} - \mu_k \geq \frac{\Delta_k}{2}\})$$

$$\leq 2\text{exp}(-2m_k\frac{\Delta_k^2}{4}) = 2\text{exp}(-\log(1/\delta)) = 2\delta$$

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- We will bound $G_k^c = \{\mu_1 \leq \min_t g_1(t)\}^c \cup \left\{g_{k,m_k} \leq \mu_1\right\}^c$

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- We will bound $G_k^c = \{\mu_1 \le \min_t g_1(t)\}^c \cup \left\{g_{k,m_k} \le \mu_1\right\}^c$
- Combining the probability bound for the first and second term, we get

$$\mathbb{P}(G_k^c) \le 2T\delta + 2\delta = 2(T+1)\delta$$

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- Now, using the definition of regret

$$\text{Reg}(T) = \sum_{k=1}^{K} \Delta_k \mathbb{E}[n_k(T)] \leq \sum_{k=1}^{K} \frac{4}{\Delta_k} \log(T) + 4K$$

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- For UCB Algorithm we first proved that, $\mathbb{E}[n_k(T)] \leq \frac{4\log(T)}{\Delta_k^2} + 4$. Using, this, we concluded that $\text{Reg}(T) = \sum_{k=1}^{K} \Delta_k \mathbb{E}[n_k(T)] \leq \sum_{k=1}^{K} \frac{4}{\Delta_k} \log(T) + 4K$.

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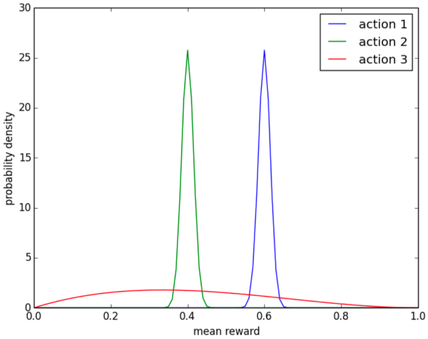
- We can actually write

$$\text{Reg}(T) = \min\{\frac{4K}{\Delta_{\min}} \log T,\ 5\sqrt{KT}\}$$

Posterior Sampling Algorithm

Bandit Algorithm using Posterior Sampling

- Which arm will you select?



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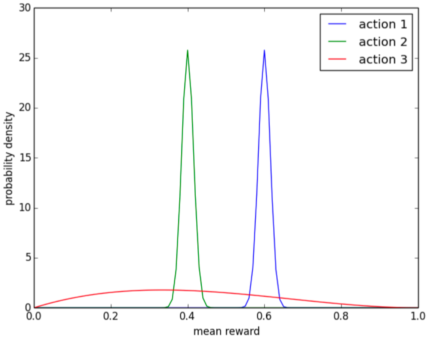

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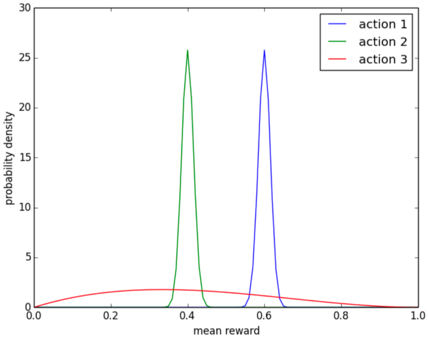

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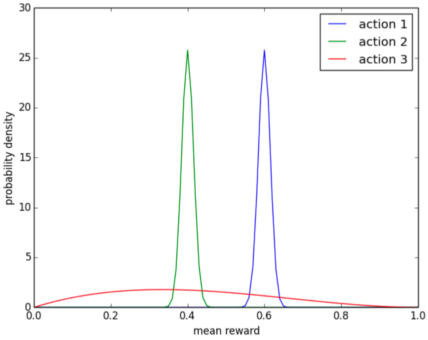

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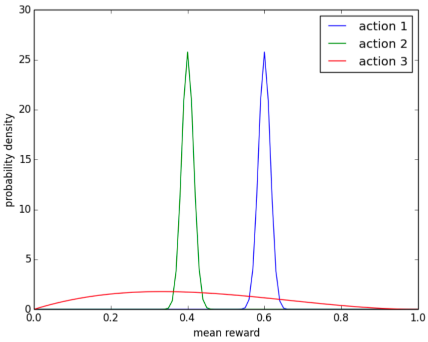

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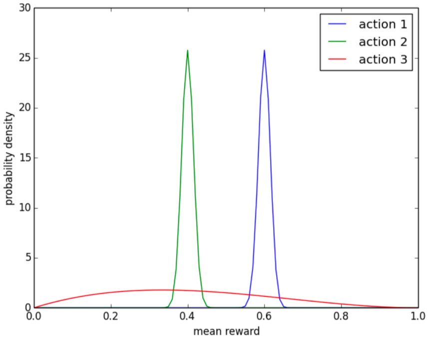

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- Posterior Sampling (also known as Thompson Sampling) overcomes this difficulty by sampling

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- For example, exponential families have conjugate priors

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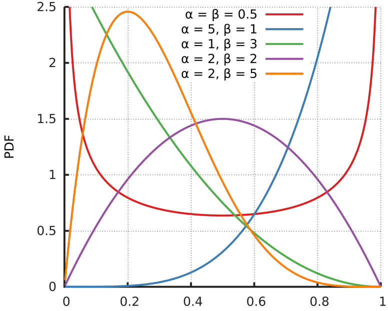

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- So, Bernoulli and Beta are conjugate priors

Posterior Sampling Algorithm for Bernoulli MAB

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Algorithm 6 Posterior Sampling Algorithm

for $t = 1, 2, \ldots$ **do**
  # sample the parameter
  for $k = 1, \ldots, K$ **do**
    $\theta_k \sim \text{Beta}(\alpha_k, \beta_k)$
  end for
  # select action
  $a_t = \arg\max_k \ \theta_k$
  Observe the reward R_t
  # update the posterior
  $(\alpha_{a_t}, \beta_{a_t}) \leftarrow (\alpha_{a_t} + R_t, \beta_{a_t} + R_t)$
end for

Bayesian Regret

- Recall the regret definition

$$\text{Reg}(T) = \mu_1 T - \mathbb{E}[\sum_{t=1}^{T} r(a_t)] = \mu_1 T - \mathbb{E}[\sum_{t=1}^{T} \sum_{k=1}^{K} \mathbb{1}\{a(t) = k\} X_k(t)]$$

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- We will make it more precise as follows:
- Let $\theta = (\theta_1, \theta_2, \ldots, \theta_K)$ be the parameter of the reward distribution, i.e., $X_k \sim p(\cdot|\theta_k)$. Denote $\mu^*(\theta) = \max_k\{\mu_k(\theta_k)\}$ where $\mu_k(\theta_k)$ is the mean reward from arm k.

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- Bayesian regret is defined as

$$\text{Bayes-Reg}(T) = \mathbb{E}_{\theta\sim p_{\text{prior}}}[\text{Reg}(T;\theta)]$$

Performance of Posterior Sampling Algorithm

Theorem

Under posterior sampling algorithm,

$$\textit{Reg}(T;\theta) = O\left(\frac{K}{\Delta_{\min}}\log T\right)$$

$$\textit{Bayes-Reg}(T) = \mathbb{E}_{\theta\sim p_{\text{prior}}}[\textit{Reg}(T;\theta)] = O(\sqrt{KT\log T})$$

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Performance of Posterior Sampling Algorithm

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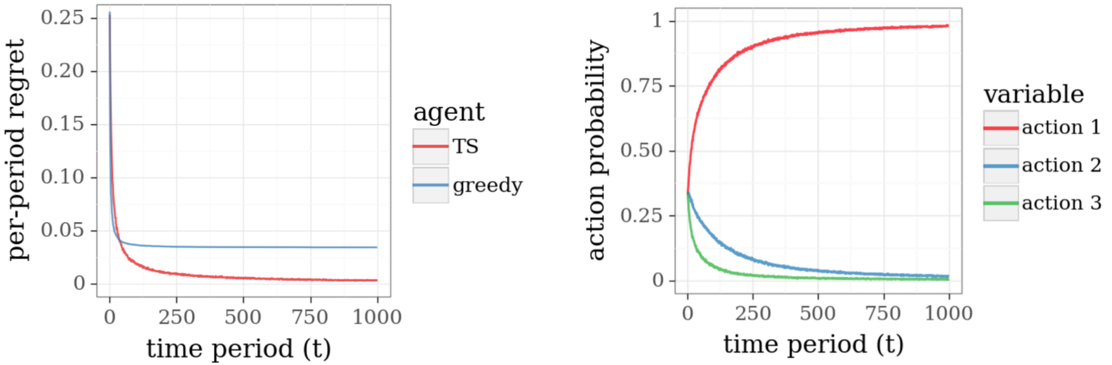

Exploration in Deep RL

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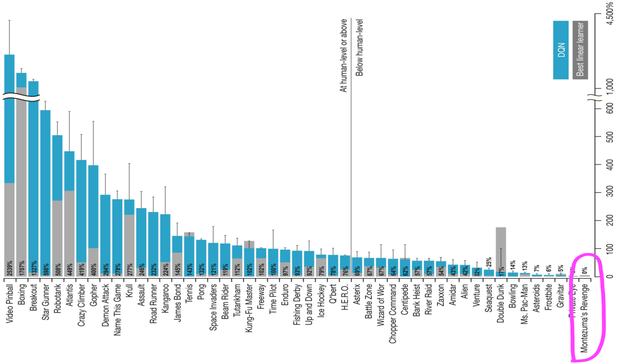
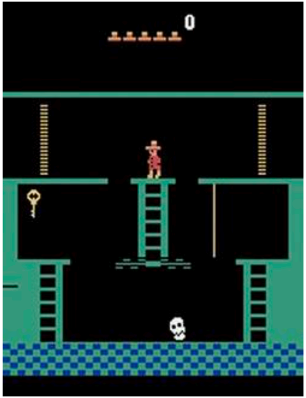


Exploration in Montezuma's Revenge

- There is reward for getting key and opening door
- No reward for getting killed by the skull (is it good? bad?)



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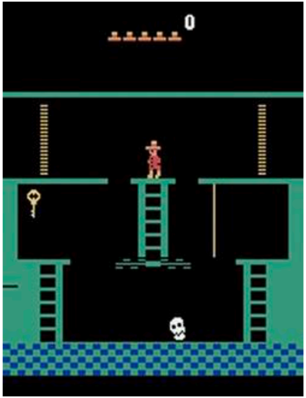

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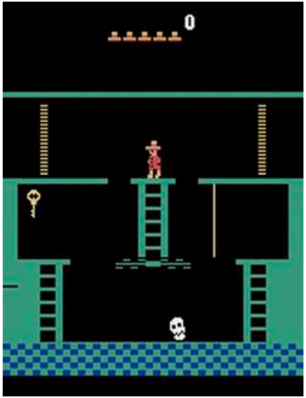

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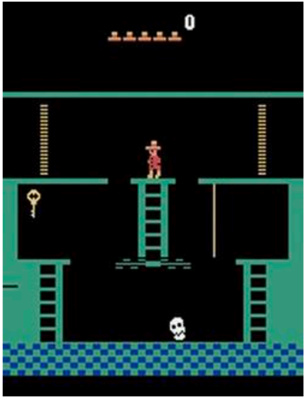

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- Probability of getting a reward exponentially decreases with the length of the sequence
- Random exploration is unlikely to get a reward signal in such problems

Some References

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  - [Strehl and Littman, 2008, Bellemare et al., 2016, Tang et al., 2017]
- Prediction-based exploration
  - [Schmidhuber, 1991, Pathak et al., 2017, Burda et al., 2018a, Houthooft et al., 2016, Burda et al., 2018b]
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  - [Badia et al., 2019, 2020]

- For a good survey of exploration in deep RL, see [Weng, 2020]

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