ECEN 743: Reinforcement Learning

RL with Function Approximation

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References

- [SB, Chapter 9-11]
- [BDP, Chapter 6]

Tabular Form to Function Approximation

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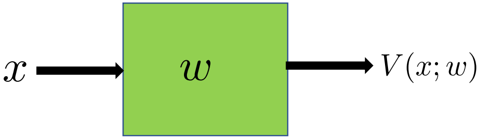

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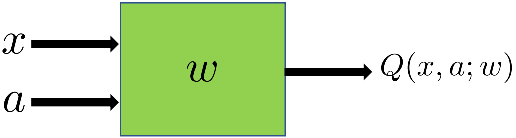

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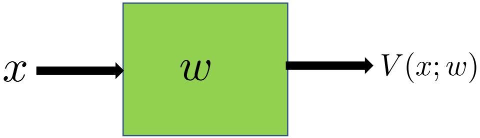

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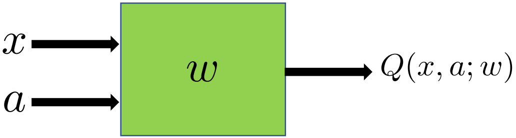

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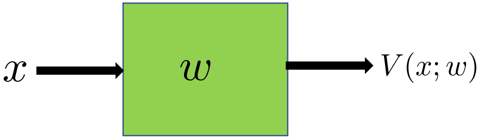

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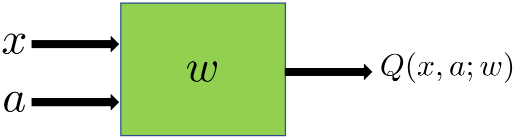

Function approximation will extrapolate/generalize the value/policy from seen states to unseen states

Linear Function Approximation

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$$\text{For each } s \in \mathcal{S},\ \phi(s) \in \mathbb{R}^d,\ \phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_d(s))^\top$$

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- $(\phi_i)_{i=1}^d$ are often assumed to be independent. So, they are also called basis functions

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- Polynomial basis functions

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  - Assume that $\mathcal{S} \subset \mathbb{R}^2$, so that we represent $s = (s_1, s_2)^\top$. We can consider feature vectors of many polynomial forms

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- Radial basis functions
  - Assume that $\mathcal{S} \subset \mathbb{R}^n$, and $\phi(s) \in \mathbb{R}^d$. Radial basis functions are defined as

$$\phi_i(s) = \exp\left(-\frac{\|s - \mu_i\|^2}{2\sigma_i^2}\right),$$

where μ_i, σ_i are fixed a priori

Matrix Notation

- We can write $V(w)(s) = (\Phi w)(s)$, where

$$\Phi = \begin{bmatrix} \phi_1(s_1) & \phi_2(s_1) & \cdots & \phi_d(s_1) \\ \phi_1(s_2) & \phi_2(s_2) & \cdots & \phi_d(s_2) \\ \vdots & & \ddots & \vdots \\ \phi_1(s_{|\mathcal{S}|}) & \phi_2(s_{|\mathcal{S}|}) & \cdots & \phi_d(s_{|\mathcal{S}|}) \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \cdots & \boldsymbol{\phi}_d \\ | & | & \cdots & | \end{bmatrix}$$

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- The actual value functions may be in an $|\mathcal{S}|$-dimensional space $\mathcal{V} \subset \mathbb{R}^{|\mathcal{S}|}$

Approximation as Projection

- How do we find the *best* approximation for $V \in \mathcal{V}$ in the set $\mathcal{V}_\phi = \{\Phi w, w \in \mathbb{R}^d\}$?

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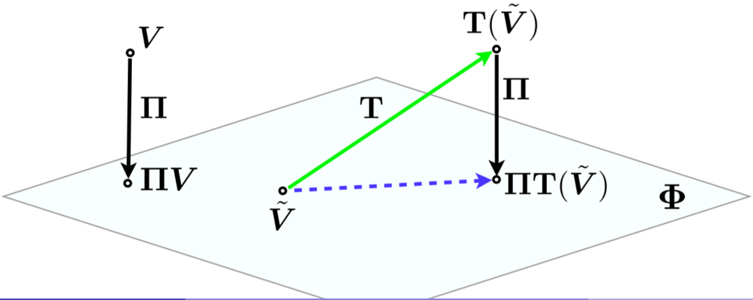

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- Pythagorean theorem: If V_1 and V_2 are orthogonal, $\|V_1 + V_2\|^2 = \|V_1\|^2 + \|V_2\|^2$

Approximate Dynamic Programming

Policy Evaluation with Linear Function Approximation

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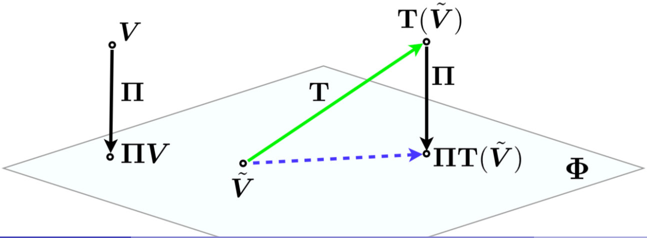

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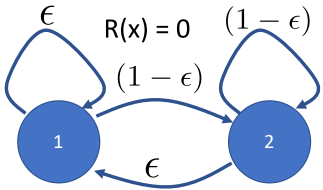

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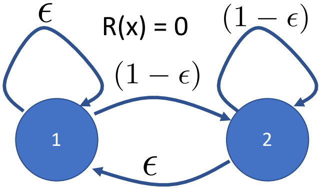

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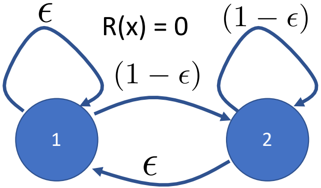

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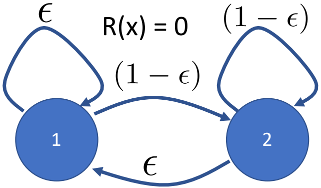

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- $V_k(1) = w_k, V_k(2) = 2w_k$

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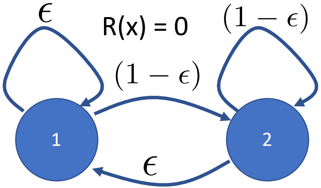
Policy Evaluation Iteration with Linear Function Approximation



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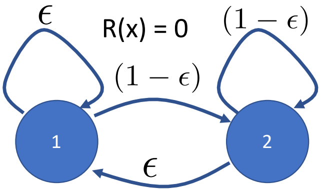

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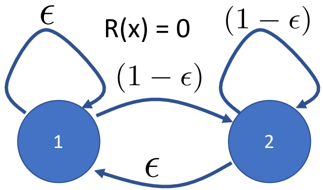

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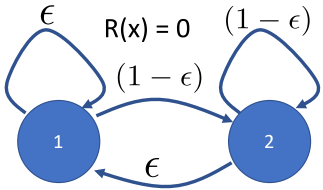

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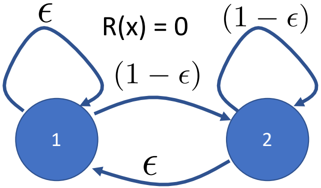

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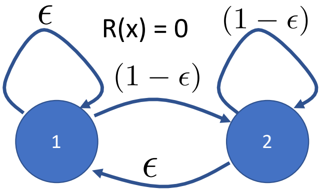

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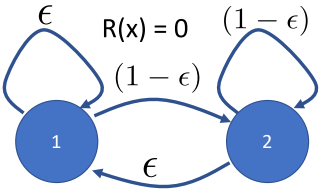

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Let $f : \mathbb{R}^n \to \mathbb{R}$ *be a convex function and let* X *be a random vector. Then,* $f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$.

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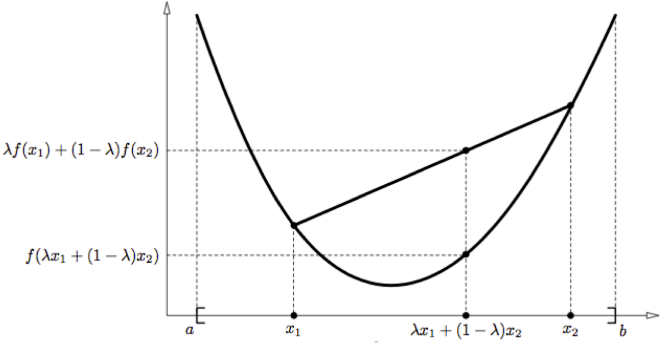

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Lemma (Non-expansive property of $\|\cdot\|_{2,\mu}$ projection)

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Proof: (Contraction property of ΠT_π w.r.t. $\|\cdot\|_{2,\mu}$)

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&= ||\Pi T_\pi \bar{V}_\pi - \Pi T_\pi V_\pi||_{2,\mu}^2 + ||\Pi V_\pi - V_\pi||_{2,\mu}^2
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Approximation Error

- ΠT_π is a contraction mapping. So, it has a unique fixed point $\bar{V}_\pi$. The iteration $V_{k+1} = \Pi T_\pi V_k$ will converge to $\bar{V}_\pi$
- What is the approximation error $\|V_\pi - \bar{V}_\pi\|_{2,\mu}$?

Proposition (Approximation Error)

$$\|\Pi V_\pi - V_\pi\|_{2,\mu} \leq \|\bar{V}_\pi - V_\pi\|_{2,\mu} \leq \frac{1}{\sqrt{1-\gamma^2}} \|\Pi V_\pi - V_\pi\|_{2,\mu}$$

Proof: Observe that $\bar{V}_\pi - \Pi V_\pi$ and $V_\pi - \Pi V_\pi$ are orthogonal vectors w.r.t. the inner product $\langle V_1, V_2 \rangle = \sum_s \mu(s) V_1(s) V_2(s)$. Now,

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&= ||\Pi T_\pi \bar{V}_\pi - \Pi T_\pi V_\pi||^2_{2,\mu} + ||\Pi V_\pi - V_\pi||^2_{2,\mu} \\
&\leq \gamma^2 ||\bar{V}_\pi - V_\pi||^2_{2,\mu} + ||\Pi V_\pi - V_\pi||^2_{2,\mu}
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This implies, $\|\bar{V}_\pi - V_\pi\|_{2,\mu} \leq \frac{1}{\sqrt{1-\gamma^2}} \|\Pi V_\pi - V_\pi\|_{2,\mu}$.

TD Learning with Function Approximation

Tabular TD to Function Approximation

- Tabular TD learning

$$V_{t+1}(s_t) = V_t(s_t) + \alpha_t(r_t + \gamma V_t(s_{t+1}) - V_t(s_t)), \text{ and, } V_{t+1}(s) = V_t(s) \text{ for all } s \neq s_t$$

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- Define the TD error $\delta_t(w) = r_t + \gamma V(w_t)(s_{t+1}) - V(w)(s_t)$
- Update the parameter in order to minimize the squared TD error;

$$w_{t+1} = w_t - \alpha_t \nabla_w \delta_t^2(w)|_{w=w_t}, \text{ where,}$$

$$\nabla_w \delta_t^2(w)|_{w=w_t} = -\delta_t(w_t) \nabla_w V(w)(s_t)|_{w=w_t}$$

TD Learning with Linear Function Approximation

- TD learning with function approximation: Define the TD error at time step t as $\delta_t(w) = r_t + \gamma V(w_t)(s_{t+1}) - V(w)(s_t)$. Update the parameter as

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Theorem (Convergence of TD learning with linear function approximation)

Let w^* be such that $V(w^*) = \Pi T_\pi V(w^*)$. Then $w_t \to w^*$ almost surely, where w_t is given by the TD learning update equation above.

Matrix Notation

- We can write $V(w)(s) = (\Phi w)(s)$, where

$$\Phi = \begin{bmatrix} \phi_1(s_1) & \phi_2(s_1) & \cdots & \phi_d(s_1) \\ \phi_1(s_2) & \phi_2(s_2) & \cdots & \phi_d(s_2) \\ \vdots & & \ddots & \vdots \\ \phi_1(s_{|\mathcal{S}|}) & \phi_2(s_{|\mathcal{S}|}) & \cdots & \phi_d(s_{|\mathcal{S}|}) \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \cdots & \boldsymbol{\phi}_d \\ | & | & \cdots & | \end{bmatrix}$$

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ODE Approximation

- TD learning update equation

$$w_{t+1} = w_t + \alpha_t \, (r_t + \gamma \, (\Phi w_t)(s_{t+1}) - (\Phi w_t)(s_t)) \, \phi(s_t)$$

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$$\dot{w} = \mathbb{E}\left[\left(r_t + \gamma \, (\Phi w)(s_{t+1}) - (\Phi w)(s_t)\right) \, \phi(s_t)\right]$$

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&= \sum_{s \in \mathcal{S}} \; \mu_\pi(s) \; (r_\pi(s) + \gamma \; (P_\pi \Phi w)(s) - (\Phi w)(s)) \; \phi(s)
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\end{aligned}$$

- This can be written as

$$\dot{w} = \Phi^\top D_\pi (r_\pi + \gamma \, (P_\pi \Phi w) - (\Phi w)),$$

where $D_\pi = \text{diag}(\mu(s_1), \ldots, \mu(s_{|\mathcal{S}|}))$

A Simple Result

- Define the inner product $\langle V_1, V_2 \rangle = V_1^\top D_\pi V_2$.

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Lemma

Let Π *be an orthogonal projection. Then,* $\langle \Pi V_1, V_2 \rangle = \langle V_1, \Pi V_2 \rangle$

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Lemma

Let Π be an orthogonal projection. Then, $\langle \Pi V_1, V_2 \rangle = \langle V_1, \Pi V_2 \rangle$

Proof: For any V_1, V_2, due to orthogonality, $\langle \Pi V_1, (V_2 - \Pi V_2) \rangle = \langle (V_1 - \Pi V_1), V_2 \rangle = 0$. We get the desired result from this. $\square$

Convergence of TD Learning with LFA

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$$\dot{U} = -2(w^* - w)^\top \dot{w}$$

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\end{aligned}$$

(we get (i) by the previous projection lemma)

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\end{aligned}$$

(we get (i) by the previous projection lemma)

$$\overset{(ii)}{\leq} -2\left[||\Phi w^* - \Phi w||_{2,\mu}^2 - ||(\Phi w^* - \Phi w)||_{2,\mu}\ ||(\Pi T_\pi \Phi w - \Phi w^*)||_{2,\mu}\right]$$

(we get (ii) by Cauchy–Schwarz inequality)

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&= -2(\Phi w^* - \Phi w)^\top\; D_\pi\; (T_\pi(\Phi w) - (\Phi w)) \\
&= -2(\Phi w^* - \Phi w)^\top\; D_\pi\; (T_\pi(\Phi w) - (\Phi w^*) + (\Phi w^*) - (\Phi w)) \\
&= -2\left[(\Phi w^* - \Phi w)^\top\; D_\pi\; (\Phi w^* - \Phi w) + (\Phi w^* - \Phi w)^\top\; D_\pi\; (T_\pi \Phi w - \Phi w^*)\right] \\
&= -2\left[||\Phi w^* - \Phi w||_{2,\mu}^2 + (\Phi w^* - \Phi w)^\top\; D_\pi\; (T_\pi \Phi w - \Phi w^*)\right] \\
&\overset{(i)}{=} -2\left[||\Phi w^* - \Phi w||_{2,\mu}^2 + (\Phi w^* - \Phi w)^\top\; D_\pi\; (\Pi T_\pi \Phi w - \Phi w^*)\right],
\end{aligned}
$$

(we get (i) by the previous projection lemma)

$$
\overset{(ii)}{\leq} -2\left[||\Phi w^* - \Phi w||_{2,\mu}^2 - ||(\Phi w^* - \Phi w)||_{2,\mu}\; ||(\Pi T_\pi \Phi w - \Phi w^*)||_{2,\mu}\right]
$$

(we get (ii) by Cauchy–Schwarz inequality)

$$
\leq -2\left[||\Phi w^* - \Phi w||_{2,\mu}^2 - \gamma||(\Phi w^* - \Phi w)||_{2,\mu}^2\right]
$$

Convergence of TD Learning with LFA

Proof: Let w^* be such that $\Pi T_\pi V(w^*) = V(w^*)$.

Let $U(t) = ||w^* - w(t)||_2^2$. We will show that $\dot{U} < 0$ for $w(t) \neq w^*$.

$$\begin{aligned}
\dot{U} &= -2(w^* - w)^\top \dot{w} \\
&= -2(w^* - w)^\top \Phi^\top D_\pi (r_\pi + \gamma\, (P_\pi \Phi w) - (\Phi w)) \\
&= -2(\Phi w^* - \Phi w)^\top\, D_\pi\, (T_\pi(\Phi w) - (\Phi w)) \\
&= -2(\Phi w^* - \Phi w)^\top\, D_\pi\, (T_\pi(\Phi w) - (\Phi w^*) + (\Phi w^*) - (\Phi w)) \\
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\end{aligned}$$

(we get (i) by the previous projection lemma)

$$\overset{(ii)}{\leq} -2\left[||\Phi w^* - \Phi w||_{2,\mu}^2 - ||(\Phi w^* - \Phi w)||_{2,\mu}\, ||(\Pi T_\pi \Phi w - \Phi w^*)||_{2,\mu}\right]$$

(we get (ii) by Cauchy–Schwarz inequality)

$$\begin{aligned}
&\leq -2\left[||\Phi w^* - \Phi w||_{2,\mu}^2 - \gamma ||(\Phi w^* - \Phi w)||_{2,\mu}^2\right] \\
&= -2(1-\gamma)||\Phi w^* - \Phi w||_{2,\mu}^2
\end{aligned}$$

Convergence of TD Learning

Proof: So, we have

$$\dot{U} \leq -2(1-\gamma)||\Phi w^* - \Phi w||_{2,\mu}^2$$

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Therefore, the ODE approximation of TD learning with linear function approximation will converge to the optimal parameter w^*.
Now, using the results from stochastic approximation theory, we can argue that TD learning with linear function approximation will converge to the optimal parameter w^*

$\square$

Convergence of RL Algorithms for Policy Evaluation

- on-policy policy evaluation
  - TD learning converges in the tabular setting
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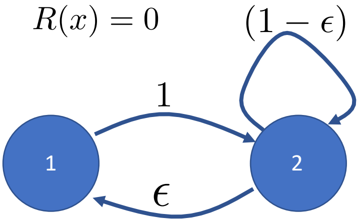
- off-policy policy evaluation algorithms are not guaranteed to converge

Example of Non-Convergence for Off-Policy Evaluation

- Consider the Markov chain induced by a policy π on an MDP. Clearly, $V_\pi(s) = 0$
- Use linear approximation with $\phi = \begin{bmatrix} 1 & 2 \end{bmatrix}^\top$
- We get $V(w)(s) = ws$. Optimal value of w is 0



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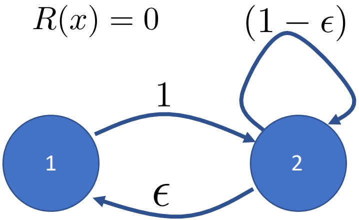

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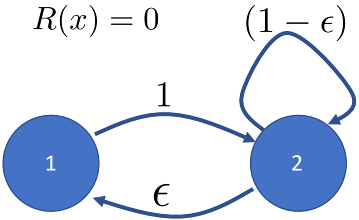

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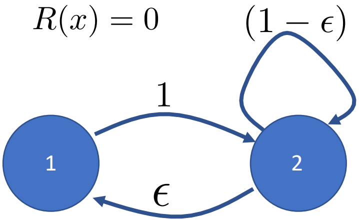

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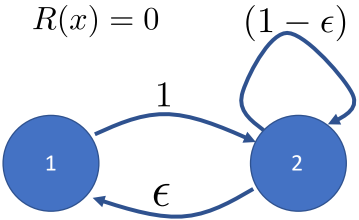

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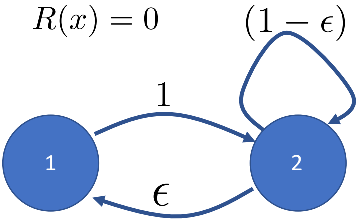

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- How do we perform TD learning using off-policy data obtained as above?

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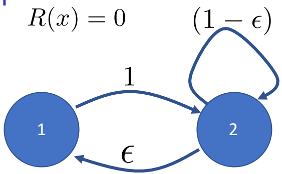

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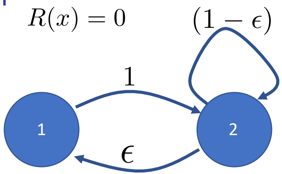

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$$w_{t+1} = w_t + \alpha_t w_t(\gamma\phi(s_{t+1}) - \phi(s_t))\ \phi(s_t)$$

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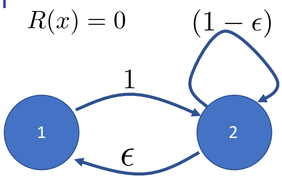

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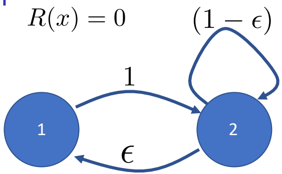
$$\begin{aligned}\mathbb{E}[w_{t+1}] &= \mathbb{E}[w_t] + \alpha_t \mathbb{E}[w_t]\left(\frac{1}{2}(\gamma\phi(2) - \phi(1))\phi(1)\right) \\ &\quad + \left(\frac{1}{2}(\gamma(\epsilon\phi(1) + (1-\epsilon)\phi(2)) - \phi(2))\phi(2)\right) \\ &= \mathbb{E}[w_t] + \alpha_t \mathbb{E}[w_t]\left(\frac{1}{2}(2\gamma - 1)\right) + \left(\frac{1}{2}(\gamma(4 - 2\epsilon) - 4)\right) \\ &= \mathbb{E}[w_t] + \alpha_t \mathbb{E}[w_t]\frac{1}{2}(6\gamma - 2\epsilon - 5)\end{aligned}$$

Example of Non-Convergence for Off-Policy Evaluation



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- This will diverge if $\gamma > 5/(6 - \epsilon)$

Q-Learning with Function Approximation

Tabular Q-Learning to Function Approximation

- Tabular Q-learning

$$Q_{t+1}(s_t, a_t) = Q_t(s_t, a_t) + \alpha_t \left(r_t + \gamma \max_b Q_t(s_{t+1}, b) - Q_t(s_t, a_t)\right)$$

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- Update the parameter in order to minimize the squared TD error;

$$w_{t+1} = w_t - \alpha_t \nabla_w \delta_t^2(w)|_{w=w_t}, \text{ where,}$$

$$\nabla_w \delta_t^2(w)|_{w=w_t} = -\delta_t(w_t) \nabla_w Q(w)(s_t, a_t)|_{w=w_t}$$

Q-Learning with Function Approximation

- Q-learning with function approximation: Define $\delta_t(w) = r_t + \gamma \max_b Q(w_t)(s_{t+1}, b) - Q(w)(s_t, a_t)$. Update the parameter as

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- Q-learning with **linear** function approximation: Approximate the value function as $Q(w)(s) = w^\top \phi(s, a)$, where $\phi(s) = (\boldsymbol{\phi}_1(s, a), \boldsymbol{\phi}_2(s, a), \ldots, \boldsymbol{\phi}_d(s, a))^\top$ is the feature vector. Then, update the parameter as

$$w_{t+1} = w_t + \alpha_t \left(r_t + \gamma \max_b Q(w_t)(s_{t+1}, b) - Q(w_t)(s_t, a_t) \right) \phi(s_t, a_t)$$

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- Q-learning
  - Q-learning converges in the tabular setting
  - Q-learning may not converge (even) in the linear function approximation setting
  - Convergence is not guaranteed for nonlinear setting

"Deadly Triad" [SB, Chapter 11]

- RL algorithms shows instability and divergence whenever we combine all of the following three elements:
- Function approximation: a scalable way of generalizing from a state space much larger than the memory and computational resources
- Bootstrapping: Update targets that include existing estimates rather than relying exclusively on actual rewards and complete returns
- Off-policy training: training on a distribution of transitions other than that produced by the target policy