ECEN 628 TAKE HOME EXAM AND ASSIGNMENT
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13.4)

$$\ddot{y}(t) = u(t)$$

Performance Index:

$$I = \int_0^\infty [y^2(t) + u^2(t)]dt$$

Reducing system to its coefficient matrices:

$$A = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$$

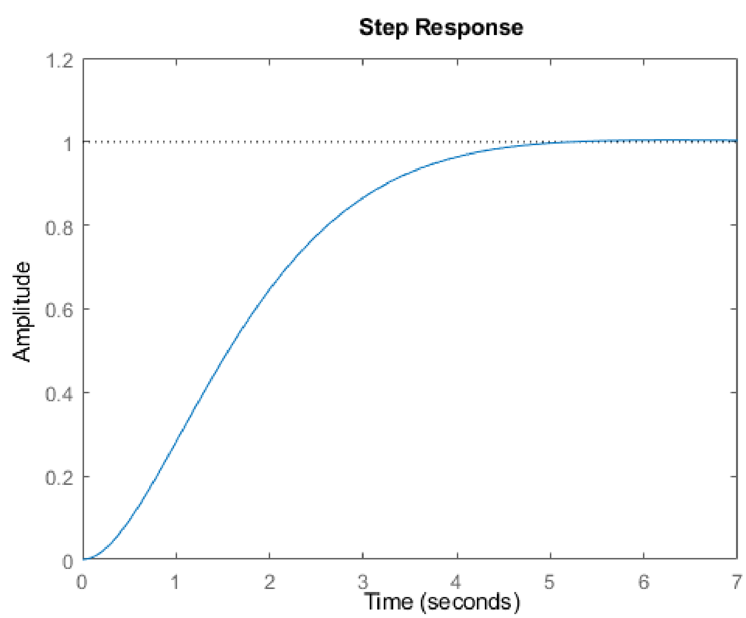
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [1 \quad 0]$$

$$D = 0$$

Q = eye(2) (Identity matrix of order 2)
R = 1
Solving ARE for K:

$$K = 1.00 \quad 1.7321$$



13.5) Characteristic matrices of the system are:

$$A = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

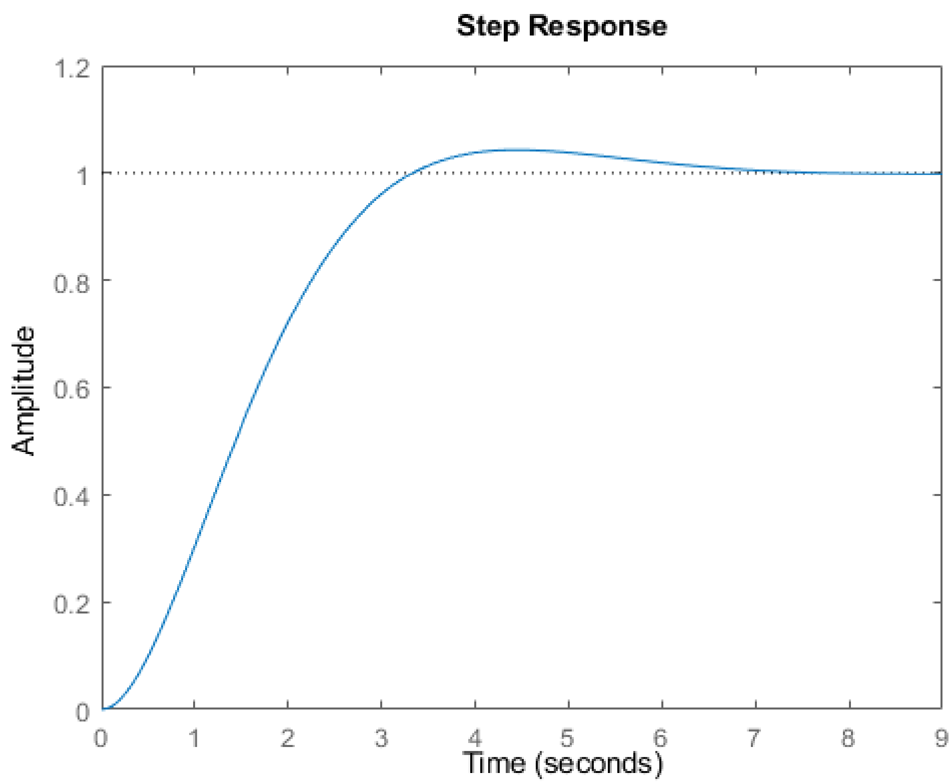
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D = 0$$

The Q matrix and the R matrix are being modified to include a new weight term called "s". As a result, the updated Q matrix will have a different form.

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

R = 1
Solving ARE for K:



13.7) I have solved this in MATLAB and SIMULINK.

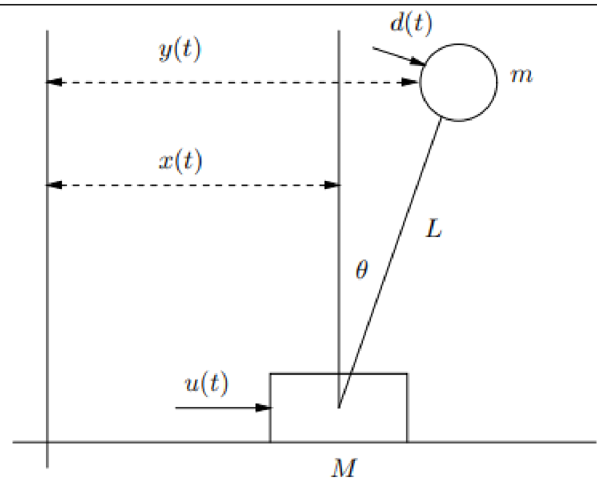
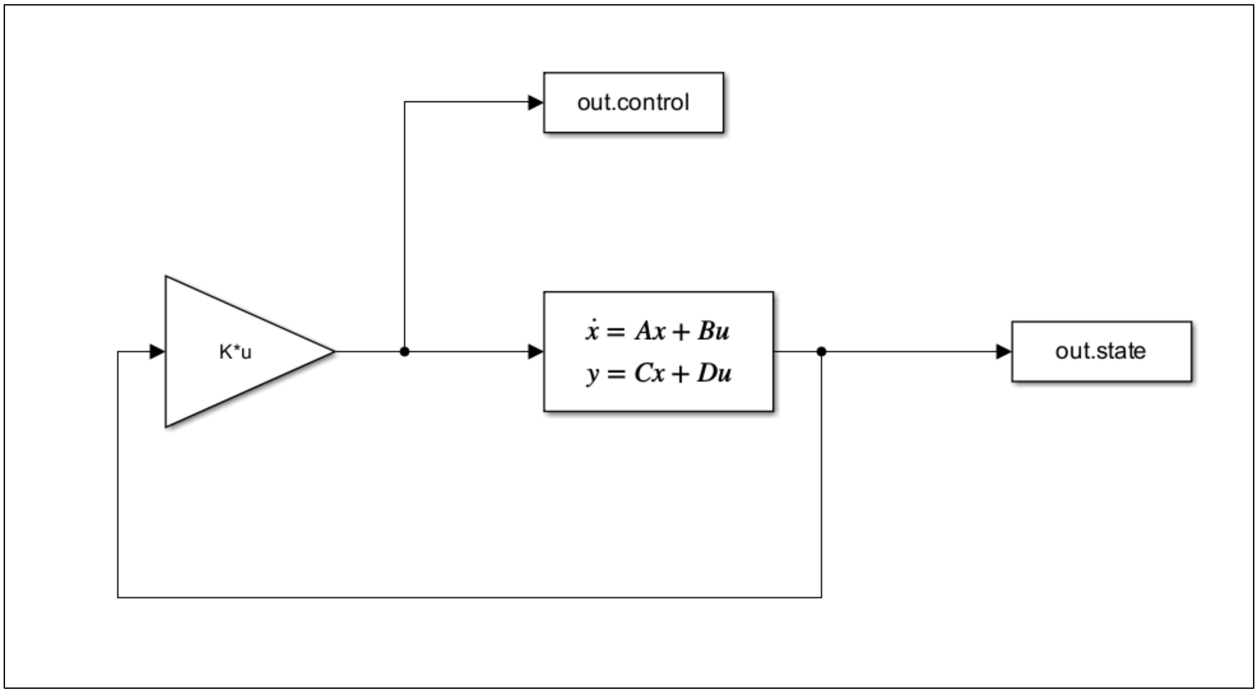


Figure 13.7
The inverted pendulum.

$$\dot{x}(t)=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -\dfrac{m}{M}g & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & \dfrac{M+m}{ML}g & 0\end{bmatrix}x(t)+\begin{bmatrix}0\\ \dfrac{1}{M}\\ 0\\ -\dfrac{1}{ML}\end{bmatrix}u(t)+\begin{bmatrix}0\\ -\dfrac{1}{M}\\ 0\\ \dfrac{M+m}{mML}\end{bmatrix}d(t)$$

Assume that

$$d(t)=0,\quad M=2\text{ kg},\quad m=1\text{ kg},\quad L=0.5\text{ m},\quad g=9.18\text{ m/s}^2$$



(GRAPHS ARE ATTACHED AT THE END WITH THE CODE AND SIMULINK MODEL)

I have just written a simple code which is passed to LQR command. The models generated are plotted after the Simulink model runs which created the output vectors.

13.8) I have solved this in MATLAB and SIMULINK

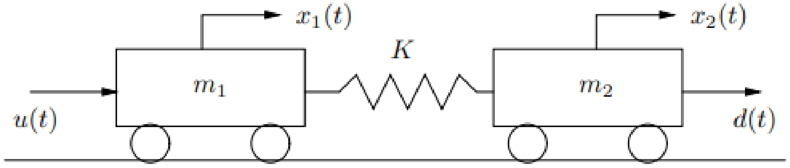


Figure 13.8
Two-mass spring system.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k}{m_1} & \dfrac{k}{m_1} & 0 & 0 \\ \dfrac{k}{m_2} & -\dfrac{k}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{m_1} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{m_2} \end{bmatrix} d(t)$$

where

$x_1(t)$ is the position of body 1 (m)

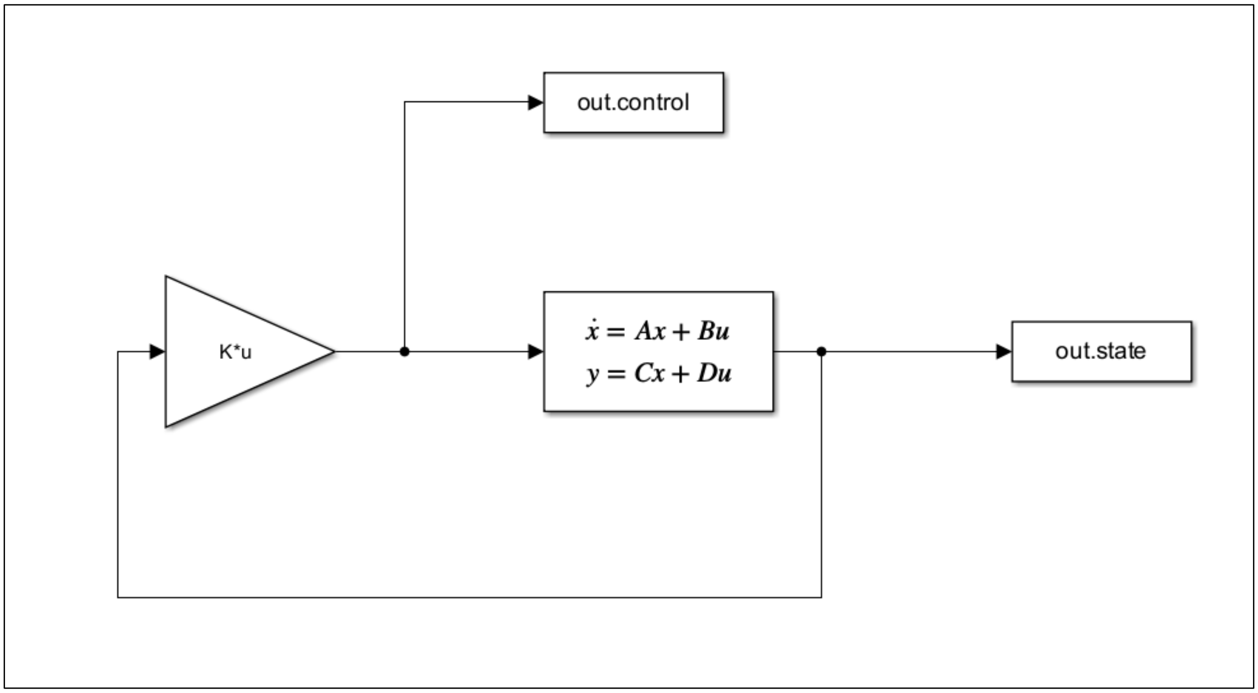
$x_2(t)$ is the position of body 2 (m)

$x_3(t)$ is the velocity of body 1 (m/s)

$x_4(t)$ is the velocity of body 2 (m/s)

$u(t)$ is the control-force input (N)

$d(t)$ is a disturbance-force input (N)



(GRAPHS ARE ATTACHED AT THE END WITH THE CODE AND SIMULINK MODEL)

I have just written a simple code which is passed to LQR command. The models generated are plotted after the Simulink model runs which created the output vectors.

13.9)

$$A = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D = 0$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$$

$$R = 1$$

Initializing $\rho = 1$, and solving ARE for K

$$K = 1.00\ 1.7321$$

- When r is equal to 1, the eigenvalues of the matrix have a complex conjugate form, specifically -0.8660 +/- i0.500. These values can be represented mathematically as a function.
- The complex conjugates always appear in pairs, where the only difference is the sign of the imaginary component. In this case, the eigenvalues are -0.8660 + 0.5i and -0.8660 - 0.5i.
- These eigenvalues are important because they can provide information about the behavior and stability of the system represented by the matrix.
- The plot shows that when the value of r approaches 0, the eigenvalues of the system matrix move towards the origin. This behavior indicates that the system becomes less stable as more oscillations are allowed.
- However, this decrease in stability is accompanied by a significant reduction in the cost of control since the energy of the system is lowered.
- On the other hand, as r approaches infinity, the eigenvalues of the system matrix also approach infinity, which results in a more stable system.
- However, this stability comes at a cost, as the energy of the system increases, and more oscillations are damped out.
- Consequently, the cost of control also increases significantly. Overall, the behavior of the eigenvalues in response to changes in "rho" provides valuable insight into the stability and control of the system.

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Q13.10)

Same matrix as in the previous question

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 2r \end{bmatrix}$$

Solving K in the ARE in MATLAB using icare command

The gain

$$K = [1.00\ 2.00]$$

(GRAPHS ARE ATTACHED AT THE END WITH THE CODE)

- The behavior of a system can be analyzed using its frequency response, which shows how the system responds to different frequencies. In this case, the frequency response is characterized by a curve and its associated eigenvalues.
- However, looking at the gain and phase margins, which are measures of how much the system's gain and phase can be varied without causing instability, provides more insight. The gain margin, which indicates the amount by which the gain of the system can be increased before it becomes unstable, appears to be infinite for any value of a parameter called rho. This means that variations in gain have no effect on stability.
- On the other hand, the phase margin, which indicates the amount by which the phase of the system can be varied before it becomes unstable, is always -180 degrees. This means that even small variations in phase can cause the system to become unstable, which is a sign of marginal stability.
- In conclusion, although the system may seem stable based on its frequency response curve and eigenvalues, it is not robust because small variations in phase can cause instability.

13.11)

We can define the system as state-space representation: x'(t) = a x(t) + b u(t)

For LQ control, the cost function of the form

$$\text{Iq} = \int_{o}^{inf} (y^T Q y + u^T R u) dt$$

Comparing this with given cost function I, we get Q = q and R = ρ.

Solving Algebraic Ricatti Equation (ARE),

$$A^{\mathrm{T}}\mathrm{P} + \mathrm{PA} - \mathrm{P\,B\,R^{-1}B^{T}P} + \mathrm{Q} = 0$$

In MATLAB we get gain as

$$K = \frac{\sqrt{(a^2 + \frac{q\rho}{\mathrm{b}^2}} + a}{b}$$

$$p = \frac{q}{a}$$

Substituting the value of K in the optimal control law, we get:

$$u^*(t) = -Kx(t) = \frac{\sqrt{(a^2 + \frac{q\rho}{\mathrm{b}^2}} + a}{b} x(t)$$

Thus, the CLTF becomes

$$G(s) = \frac{-K}{(s-A+BK)}.$$

Substituting the values of A, B, and K and finding the closed-loop eigenvalues of the system from det|s – A + BK|.
Solving this MATLAB, gives

$$\lambda = -\sqrt{(\boldsymbol{a^2} + \frac{q\mathbf{b^2}}{\rho})}.$$

- When the value of the parameter r approaches 0 in the system, the closed-loop eigenvalue lambda approaches a, where a is a constant in the system. This means that as r gets smaller, the system becomes more and more unstable, and for sufficiently small values of r, it becomes completely unstable and cannot be controlled.
- On the other hand, when the value of r approaches infinity, the closed-loop eigenvalue lambda approaches k, where k is another constant in the system. This means that as r gets larger, the system becomes more stable, but at the cost of a slower response.
- In conclusion, the behavior of the system is highly dependent on the value of the parameter r. As r approaches 0, the system becomes more unstable, while as r approaches infinity, the system becomes more stable but with a slower response. It is crucial to find the right balance between stability and responsiveness to ensure the optimal performance of the system.

$$\sqrt{\frac{qb^2}{a^2}}$$

Hence, the system is underdamped.

13.12)

State-space representation:

$$x'(t) = Ax(t) + B\,u(t)$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Cost Function:

$$\text{Iq} = \int_{o}^{inf} (y^T Q y + u^T R u)dt$$

Hence after comparing to Iq, we get

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}$$

$$R = [\,\rho\,].$$

ARE:

$$A^{\mathrm{T}}\mathrm{P} + \mathrm{PA} - \mathrm{P\,B\,R^{-1}B^{T}P} + \mathrm{Q} = 0$$

Solving for P:

$$\frac{dP}{dt} = -A^T P - PA + PBR^{-1}B^T P + Q$$

Solving ARE in MATLAB using icare command:

$$P = \begin{bmatrix} \sqrt{\rho} & 0 \\ 0 & \sqrt{\dfrac{\boldsymbol{q} + \mathbf{2}\sqrt{\boldsymbol{\rho}}}{\boldsymbol{\rho}}} \end{bmatrix}$$

Substituting the values of A, B, Q, R, and P into ARE, we get:

$$\boldsymbol{K} = \begin{bmatrix} \dfrac{\mathbf{1}}{\sqrt{\boldsymbol{\rho}}} & \sqrt{\dfrac{\boldsymbol{q} + \mathbf{2}\sqrt{\boldsymbol{\rho}}}{\boldsymbol{\rho}}} \end{bmatrix}$$

Thus, the CLTF becomes:

$$\mathrm{G(s)} = \frac{-\mathrm{K}}{(\mathrm{s}-\mathrm{A}+\mathrm{BK})}.$$

Substituting the values of A, B, and K and finding the CL eigenvalues of the system from

$$det|s - A + BK|.$$

$$\boldsymbol{det}|\ \boldsymbol{s}^{\mathbf{2}} + \left(\boldsymbol{q} + \mathbf{2}\sqrt{\rho}\right)\boldsymbol{s} + \rho$$

Solving this MATLAB, gives

$$\lambda = \begin{bmatrix} \dfrac{-\sqrt{\boldsymbol{q} + \mathbf{2}\sqrt{\boldsymbol{\rho}}} + \sqrt{\boldsymbol{q} - \mathbf{2}\sqrt{\boldsymbol{\rho}}}}{2\sqrt{\boldsymbol{\rho}}} \\ \dfrac{\sqrt{\boldsymbol{q} + \mathbf{2}\sqrt{\boldsymbol{\rho}}} - \sqrt{\boldsymbol{q} - \mathbf{2}\sqrt{\boldsymbol{\rho}}}}{2\sqrt{\boldsymbol{\rho}}} \end{bmatrix}$$

- As the r approaches 0 in the system, the system becomes very sensitive to any changes in the control input u. This means that even small changes in the control input can result in large oscillations in the system, making it unstable. This sensitivity is because when r is small, the feedback gain of the system is large, amplifying the effect of any changes in the control input.

- On the other hand, as r approaches infinity, the control input becomes more and more constrained, resulting in a slow response from the system. This is because when r is large, the feedback gain of the system is small, reducing the effect of any changes in the control input.
- In conclusion, the system's sensitivity to changes in the control input depends on the value of the parameter r. When r is small, the system becomes very sensitive and can become unstable, while when r is large, the system has a slow response to changes in the control input. A balance between sensitivity and responsiveness must be found to ensure the stability and optimal performance of the system.

13.15)

State space representation:

$$x^{\cdot}(t) = A_p x(t) + B_p u(t) + E_p d(t)$$

$$y(t) = C_p x(t)$$

$$x(t) = [x1(t)\,;\; x2(t);\; x3(t);\; x4(t)]$$

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- To ensure that the system can track arbitrary step and ramp inputs accurately, we need to minimize the error between the output of the system and the desired reference signal. This can be achieved by using a weighting matrix, which assigns penalties to the different components of the error based on their relative importance.
- The choice of weighting matrices can have a significant impact on the performance of the system. In this case, we have chosen a diagonal weighting matrix Q with positive constants q1, q2, q3, and q4, and a diagonal weighting matrix R with positive constants r1 and r2. These values reflect the relative importance of the different components of the error.
- To compute the controller gains K in MATLAB, we can use the chosen values for Q and R. In this case, the values are Q = diag([10 10 1 1]) and R = diag([1 1]). By using these

values, the controller gains K can be calculated using the linear quadratic regulator (LQR) algorithm, which minimizes the weighted sum of the error and the control effort.

- In conclusion, by choosing appropriate weighting matrices, we can minimize the error between the system output and the desired reference signal and ensure accurate tracking of arbitrary step and ramp inputs. The specific values chosen for the weighting matrices can be used to reflect the relative importance of different components of the error, and the controller gains K can be calculated using MATLAB's LQR algorithm.

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