Quick Template

***Abstract*—**

I. Introduction

II. Derivation Q-ary Tree with SIC

The probability generating function of a tree can be written in a recursive way, considering all possible split probabilities,

$$Q_n(z) = \sum_{n_1,n_2,\dots,n_d}^{n} \binom{n}{n_1, n_2, \dots, n_d} \prod_{j=1}^{d} p_j^{n_j} Q_{n_j}(z). \tag{1}$$

In p_j given as a splitting probability to branch j. Due to the T-ary channel model, we have singletons up to and including T arrivals,

$$Q_0(z) = Q_1(z) = \dots = Q_T(z) = z. \tag{2}$$

We assume Poisson arrivals for the number of collided users with a mean x.

$$Q(x,z) := \sum_{n=0}^{\infty} Q_n(z) e^{-z} \frac{x^n}{n!}. \tag{3}$$

We can re-adjust this equation to use the Poisson effect on most parts as,

First getting rid off the region where we have to different $Q_n(z)$ functions through the following modification,

$$\begin{gathered} Q(x,z) = \sum_{n=T+1}^{\infty} \sum_{n_1,n_2,\dots,n_d}^{n} \binom{n}{n_1, n_2, \dots, n_d} \\ \prod_{j=1}^{d} p_j^{n_j} Q_{n_j}(z) e^{-z} \frac{x^n}{n!} + z \left(\sum_{j=0}^{T} \frac{x^j}{j!} \right) e^{-x}. \end{gathered} \tag{4}$$

Then, we put back the same part that enables a single sum from 0,

$$\begin{gathered} Q(x,z) = \sum_{n=0}^{\infty} \sum_{n_1,n_2,\dots,n_d}^{n} \binom{n}{n_1, n_2, \dots, n_d} \\ \prod_{j=1}^{d} p_j^{n_j} Q_{n_j}(z) e^{-z} \frac{x^n}{n!} \\ + z \left(\sum_{j=0}^{T} \frac{x^j}{j!} \right) e^{-x} - z^d \left(\sum_{j=0}^{T} \frac{x^j}{j!} \right) e^{-x}. \end{gathered} \tag{5}$$

Merging these terms we get,

$$\begin{gathered} Q(x,z) = \sum_{n=0}^{\infty} \sum_{n_1,n_2,\dots,n_d}^{n} \binom{n}{n_1, n_2, \dots, n_d} \\ \prod_{j=1}^{d} p_j^{n_j} Q_{n_j}(z) e^{-z} \frac{x^n}{n!} + \left(z - z^d\right) \left(\sum_{j=0}^{T} \frac{x^j}{j!} \right) e^{-x}. \end{gathered} \tag{6}$$

Right hand side of equation can be re-written as,

$$\begin{gathered} \sum_{n_1=0}^{\infty} Q_{n_1}(z) e^{-p_1 x} \frac{(p_1 x)^{n_1}}{n_1!} \times \cdots \\ \times \sum_{n_{d-1}=0}^{\infty} Q_{n_{d-1}}(z) e^{-p_{d-1} x} \frac{(p_{d-1} x)^{n_{d-1}}}{n_{d-1}!} \times \\ \times \sum_{n_d = n - \sum_{j=1}^{d-1} n_j}^{\infty} Q_{n_d}(z) e^{-p_d x} \frac{(p_d x)^{n_d}}{n_d!} \end{gathered} \tag{7}$$

Thus the whole equation can be re-written as,

$$Q(x,z) = \prod_{j=1}^{d} Q(p_j x, z) + \left(z - z^d\right) \left(\sum_{j=0}^{T} \frac{x^j}{j!} \right) e^{-x}. \tag{8}$$

Derivative with respect to z and inputting one yields,

$$L(x) = \sum_{j=1}^{d} L(p_j x) - (d-1) \left(\sum_{j=0}^{T-1} \frac{x^j}{j!} \right) e^{-x}. \tag{9}$$

We treat $L(x)$ as a power series,

$$L(x) = \sum_{n=0}^{\infty} \alpha_n x^n. \tag{10}$$

As L is the average tree length we can write it as the derivative of the probability generating function,

$$L(x) = \left. \frac{\partial Q(x,z)}{\partial z} \right|_{z=1} = e^{-x} \sum_{n=0}^{\infty} L_n \frac{x^n}{n!}. \tag{11}$$

Writing, exponential in power series we can then break down the problem to equating coefficients for

$$\frac{L_n}{n!} = \sum_{i=0}^{n} \frac{\alpha_i}{(n-i)!}. \tag{12}$$

From Eq. (2) we now that,

$$L_0(z) = L_1(z) = \dots = L_T(z) = 1. \tag{13}$$

This gives us $\alpha_0 = 1$ and $\alpha_1 = \alpha_2 = \dots = \alpha_T = 0$. We then, use Eq. 25 with the exponential function of $L(x)$ to find the coefficients,

$$\sum_{n=0}^{\infty} \alpha_n x^n = \sum_{j=1}^{d} \sum_{n=0}^{\infty} \alpha_n p_j^n x^n - (d-1) \sum_{n=0}^{\infty} \left(\sum_{j=0}^{T} \frac{x^j}{j!} \right) (-1)^n \frac{x^n}{n!}. \tag{14}$$

This gives us,

$$\alpha_n = \sum_{j=0}^{T} \binom{n}{j} \frac{(-1)^{n-j+1}}{n!} \cdot \frac{d-1}{1 - \sum_{j=1}^{d} p_j^n}. \tag{15}$$

Finally we can write the average tree length as in,

$$L_n = 1 + \sum_{i=T+1}^{n} \binom{n}{i} \sum_{j=0}^{T} \binom{i}{j} \frac{(-1)^{i-j+1} (d-1)}{1 - \sum_{j=1}^{d} p_j^n}, \, n \geq T. \tag{16}$$

III. Derivation Q-ary Tree without SIC

The probability generating function of a tree can be written in a recursive way, considering all possible split probabilities,

$$Q_n(z) = z \sum_{n_1, n_2, \dots, n_d}^{n} \binom{n}{n_1, n_2, \dots, n_d} \prod_{j=1}^{d} p_j^{n_j} Q_{n_j}(z). \tag{17}$$

In p_j given as a splitting probability to branch j. Due to the T-ary channel model, we have singletons up to and including T arrivals,

$$Q_0(z) = Q_1(z) = \dots = Q_T(z) = z. \tag{18}$$

We assume Poisson arrivals for the number of collided users with a mean x.

$$Q(x, z) := \sum_{n=0}^{\infty} Q_n(z) e^{-z} \frac{x^n}{n!}. \tag{19}$$

We can re-adjust this equation to use the Poisson effect on most parts as,

$$Q(x,z) = z \sum_{n=T}^{\infty} \sum_{n_1, n_2, \dots, n_d}^{n} \binom{n}{n_1, n_2, \dots, n_d} \prod_{j=1}^{d} p_j^{n_j} Q_{n_j}(z) e^{-z} \frac{x^n}{n!} + z \left(\sum_{j=0}^{T} \frac{x^j}{j!} \right) e^{-x}. \tag{20}$$

$$Q(x,z) = z \sum_{n=0}^{\infty} \sum_{n_1, n_2, \dots, n_d}^{n} \binom{n}{n_1, n_2, \dots, n_d} \prod_{j=1}^{d} p_j^{n_j} Q_{n_j}(z) e^{-z} \frac{x^n}{n!} \tag{21}$$
$$+ z \left(\sum_{j=0}^{T} \frac{x^j}{j!} \right) e^{-x} - z^{d+1} \left(\sum_{j=0}^{T} \frac{x^j}{j!} \right) e^{-x}.$$

$$Q(x,z) = \sum_{n=0}^{\infty} \sum_{n_1, n_2, \dots, n_d}^{n} \binom{n}{n_1, n_2, \dots, n_d} \prod_{j=1}^{d} p_j^{n_j} Q_{n_j}(z) e^{-z} \frac{x^n}{n!} + \left(z - z^{d+1} \right) \left(\sum_{j=0}^{T} \frac{x^j}{j!} \right) e^{-x}. \tag{22}$$

Right hand side of equation can be re-written as,

$$\sum_{n_1=0}^{\infty} Q_{n_1}(z) e^{-p_1 x} \frac{(p_1 x)^{n_1}}{n_1!} \times \cdots \times \sum_{n_{d-1}=0}^{\infty} Q_{n_{d-1}}(z) e^{-p_{d-1} x} \frac{(p_{d-1} x)^{n_{d-1}}}{n_{d-1}!} \times \times \sum_{n_d = n - \sum_{j=1}^{d-1} n_j}^{\infty} Q_{n_d}(z) e^{-p_d x} \frac{(p_d x)^{n_d}}{n_d!} \tag{23}$$

Thus the whole equation can be re-written as,

$$Q(x,z) = \prod_{j=1}^{d} Q(p_j x, z) + \left(z - z^{d+1} \right) \left(\sum_{j=0}^{T} \frac{x^j}{j!} \right) e^{-x}. \tag{24}$$

Derivative with respect to z and inputting one yields,

$$L(x) = \sum_{j=1}^{d} L(p_j x) - d \left(\sum_{j=0}^{T} \frac{x^j}{j!} \right) e^{-x}. \tag{25}$$

We treat $L(x)$ as a power series,

$$L(x) = \sum_{n=0}^{\infty} \alpha_n x^n. \tag{26}$$

As L is the average tree length we can write it as the derivative of the probability generating function,

$$L(x) = \left. \frac{\partial Q(x,z)}{\partial z} \right|_{z=1} = e^{-x} \sum_{n=0}^{\infty} L_n \frac{x^n}{n!}. \tag{27}$$

Writing, exponential in power series we can then break down the problem to equating coefficients for

$$\frac{L_n}{n!} = \sum_{i=0}^{n} \frac{\alpha_i}{(n-i)!}. \tag{28}$$

From Eq. (2) we now that,

$$L_0(z) = L_1(z) = \dots = L_T(z) = 1. \tag{29}$$

This gives us $\alpha_0 = 1$ and $\alpha_1 = \alpha_2 = \dots = \alpha_T = 0$. We then, use Eq. 25 with the exponential function of $L(x)$ to find the coefficients,

$$\sum_{n=0}^{\infty}\alpha_n x^n = \sum_{j=1}^{d}\sum_{n=0}^{\infty}\alpha_n p_j^n x^n \\ -d\sum_{n=0}^{\infty}\left(\sum_{j=0}^{T-1}\frac{x^j}{j!}\right)(-1)^n\frac{x^n}{n!}. \tag{30}$$

This gives us,

$$\alpha_n = \sum_{j=0}^{T}\binom{n}{j}\frac{(-1)^{n-j+1}}{n!}\cdot\frac{d}{1-\sum_{j=1}^{d}p_j^n}. \tag{31}$$

Finally we can write the average tree length as in,

$$L_n = 1 + \sum_{i=T+1}^{n}\binom{n}{i}\sum_{j=0}^{T}\binom{i}{j}\frac{d(-1)^{i-j+1}}{1-\sum_{j=1}^{d}p_j^n},\ n>T. \tag{32}$$

IV. RECURSIVE Q-ARY WITH SIC

The probability generating function of a tree can be written in a recursive way, considering all possible split probabilities,

$$Q_n(z) = \sum_{n_1,n_2,...,n_d}^{n}\binom{n}{n_1,n_2,...,n_d}\prod_{j=1}^{d}p_j^{n_j}Q_{n_j}(z). \tag{33}$$

derivative of that with respect to z and setting $z=1$ we get,

$$L_n = \sum_{n_1,n_2,...,n_d}^{n}\binom{n}{n_1,n_2,...,n_d}\sum_{k=1}^{d}L_{n_k}\prod_{j=1}^{d}p_j^{n_j}. \tag{34}$$

We can get rid of the last product by assuming that we set the first probability p_1 and we fix the others as $p_j=\frac{1-p_1}{d-1}$ with $j\neq 1$. With this we have,

$$L_n = \sum_{n_1,n_2,...,n_d}^{n}\binom{n}{n_1,n_2,...,n_d}\sum_{k=1}^{d}L_{n_k}p_1^{n_1}\left(\frac{1-p_1}{d-1}\right)^{n-n_1}. \tag{35}$$

We can also write the multiple n choose n_1,n_2,n_3 as,

$$\sum_{n_1,n_2,...,n_d}^{n}\binom{n}{n_1,n_2,...,n_d} = \sum_{n_1}^{n}\sum_{n2}^{n-n1}\cdots \\ \sum_{n_d}^{n-\sum_{d-1}^{j=1}n_j}\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-\sum_{d-1}^{j=1}n_j}{n_d} \tag{36}$$

Also if we write the sum on the RHS in an open way we get,

$$L_n = \sum_{n_1}^{n}\sum_{n_2}^{n-n_1}\cdots\sum_{n_d}^{n-\sum_{d-1}^{j=1}n_j}\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots \\ \binom{n-\sum_{d-1}^{j=1}n_j}{n_d}L_{n_1}p_1^{n_1}\left(\frac{1-p_1}{d-1}\right)^{n-n_1}+ \\ +\sum_{n_1}^{n}\sum_{n_2}^{n-n_1}\cdots\sum_{n_d}^{n-\sum_{d-1}^{j=1}n_j}\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots \\ \binom{n-\sum_{d-1}^{j=1}n_j}{n_d}L_{n_2}p_1^{n_1}\left(\frac{1-p_1}{d-1}\right)^{n-n_1}+ \\ \cdots \\ +\sum_{n_1}^{n}\sum_{n_2}^{n-n_1}\cdots\sum_{n_d}^{n-\sum_{d-1}^{j=1}n_j}\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots \\ \binom{n-\sum_{d-1}^{j=1}n_j}{n_d}L_{n_d}p_1^{n_1}\left(\frac{1-p_1}{d-1}\right)^{n-n_1}. \tag{37}$$

As the top part only depends on n_1 we can simplify all the inner sums as,

$$\sum_{n_1}^{n}\binom{n}{n_1}(d-1)^{n-n_1}L_{n_1}p_1^{n_1}\left(\frac{1-p_1}{d-1}\right)^{n-n_1} = \\ = \sum_{n_1}^{n}\binom{n}{n_1}L_{n_1}p_1^{n_1}(1-p_1)^{n-n_1}. \tag{38}$$

For the other parts, the projection is, if the partitioning constraint that is $n=\sum_{i=1}^{d}n_i$ hols, we can also re-write them as the above expression i.e.,

$$= \sum_{n_2}^{n}\binom{n}{n_2}L_{n_2}p_1^{n_1}(1-p_1)^{n-n_1}. \tag{39}$$

Thus we can re-group all the expressions back as,

$$L_n = \sum_{j=1}^{d}\sum_{n_j}^{n}\binom{n}{n_j}L_{n_j}p_1^{n_1}(1-p_1)^{n-n_1}. \tag{40}$$

Lastly, we can re-write the recursive expression as

$$L_n = \frac{\sum_{j=1}^{d}\sum_{n_j}^{n-1}\binom{n}{n_j}p_1^{n_1}(1-p_1)^{n-n_1}L_{n_j}}{1-p_1^n-(d-1)\left(\frac{1-p_1}{d-1}\right)^n}. \tag{41}$$

beware that such a derivation is possible given the probabilities p_j have a constraint such that $p_j\ \forall j\neq 1$ we have $p_j=\frac{1-p_1}{d-1}$. This is a rule also used in the work of Flajolet to investigate a multi-dimensional problem with a single dimension. And we can use p_1 as a design parameter to adjust the system. Eq. (32) can be used with the same constraint to show that in such a setting biased coins are optimal with derivating with respect to p_1. However, please note that in case different constraints are applied such that $p_1=p_2$ and $p_j=\frac{1-2p_1}{d-2}$. From derivation it can be shown that optimal p_1 becomes dependent on the number of users

n. This can be used to optimized the tree with low number of users. Remember that the Eq. (41), obeys the constraint $\sum_{j=1}^{d} n_j = n$.

V. Recursive Q-ary without SIC

The probability generating function of a tree can be written in a recursive way, considering all possible split probabilities,

$$Q_n(z) = z \sum_{n_1,n_2,...,n_d}^{n} \binom{n}{n_1, n_2, ..., n_d} \prod_{j=1}^{d} p_j^{n_j} Q_{n_j}(z). \tag{42}$$

derivative of that with respect to z and setting $z = 1$ we get,

$$L_n = 1+ \sum_{n_1,n_2,...,n_d}^{n} \binom{n}{n_1, n_2, ..., n_d} \sum_{k=1}^{d} L_{n_k} \prod_{j=1}^{d} p_j^{n_j}. \tag{43}$$

we can shortcut and write the result from the with SIC as in Eq. (40),

$$L_n = 1 + \sum_{j=1}^{d} \sum_{n_j}^{n} \binom{n}{n_j} L_{n_j} p_1^{n_1} \left(1 - p_1\right)^{n-n_1}. \tag{44}$$

and converting it to a recursive form we get,

$$L_n = \frac{1 + \sum_{j=1}^{d} \sum_{n_j}^{n-1} \binom{n}{n_j} p_1^{n_1} \left(1 - p_1\right)^{n-n_1} L_{n_j}}{1 - p_1^n - (d-1)\left(\frac{1-p_1}{d-1}\right)^n}. \tag{45}$$

VI. Conclusions