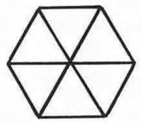
Euler's Theorem

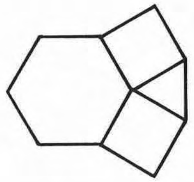
Euler's theorem for tilings demonstrates that there is always a consistent relationship among the components of a tessellation: the number of polygons + the number of vertices = the number of edges + 1.

$$P + V = E + 1$$

$$6 + 7 = 12 + 1$$
$$13 = 13$$



$$4 + 10 = 13 + 1$$
$$14 = 14$$

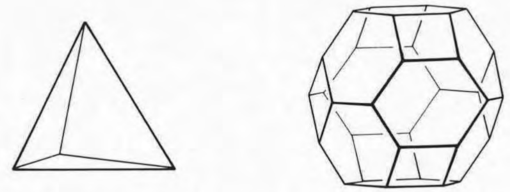


3

Polyhedra

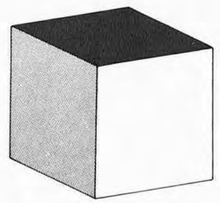
Polyhedron

A polyhedron is formed by enclosing a portion of three-dimensional space with four or more plane polygons. (Plural: polyhedra).



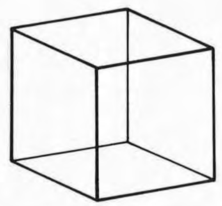
Faces

The faces of a polyhedron are polygons.



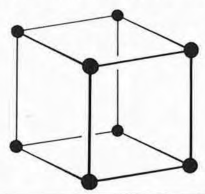
Edges

The edges of a polyhedron are formed where common sides of neighboring polygons meet.



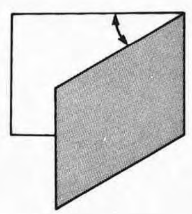
Vertex

A vertex of a polyhedron is a point where edges intersect. Three or more polygons must meet at each vertex. The sum of the face angles of the polygons meeting at a vertex must always equal less than 360°



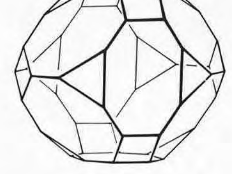
Dihedral Angle

A dihedral angle is the angle formed by two polygons joined along a common edge.



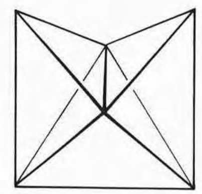
Convex Polyhedron

A polyhedron is convex if every dihedral angle is less than 180°.



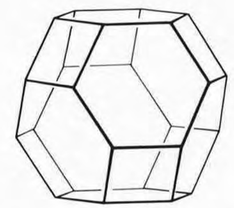
Concave Polyhedron

A polygon is concave if at least one of its dihedral angles is more than 180°.



Uniform

A polyhedron is uniform if all of its vertices are the same or congruent.

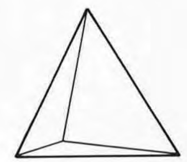


Euler's Theorem for Polyhedra

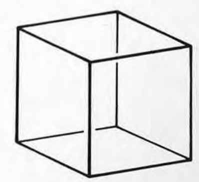
Euler's theorem for polyhedra demonstrates the constant relationship among the components of a given polyhedron: the number of polygons + the number of vertices = the number of edges + 2.

$$P + V = E + 2$$

$$4 + 4 = 6 + 2$$
$$8 = 8$$



$$6 + 8 = 12 + 2$$
$$14 = 14$$

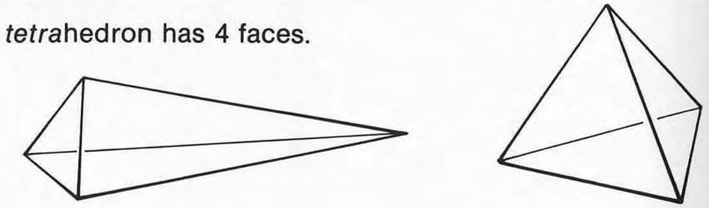


Naming Polyhedra

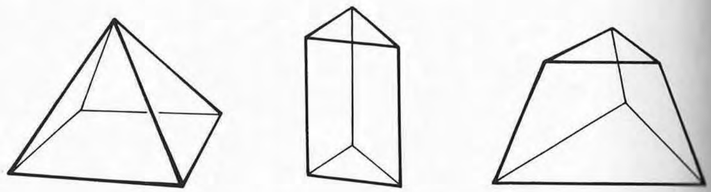
n-hedron

Polyhedra are usually named by the number of faces they have. An n-hedron is a polyhedron with an unspecified number of faces.

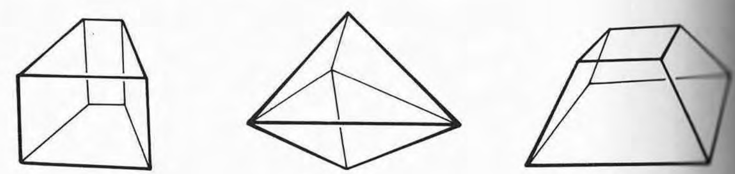
A *tetra*hedron has 4 faces.



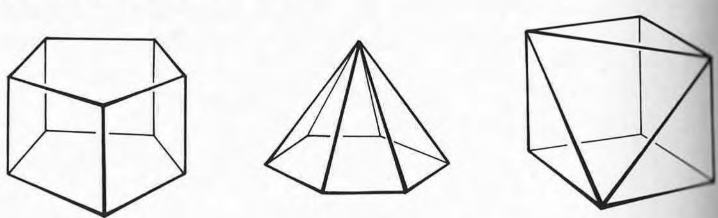
A *penta*hedron has 5 faces.



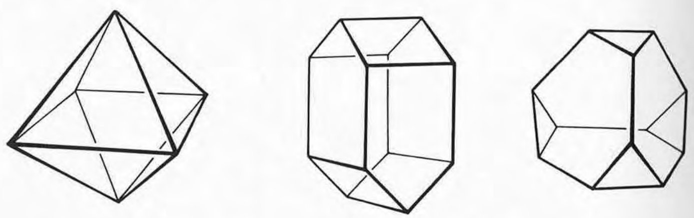
A *hexa*hedron has 6 faces.



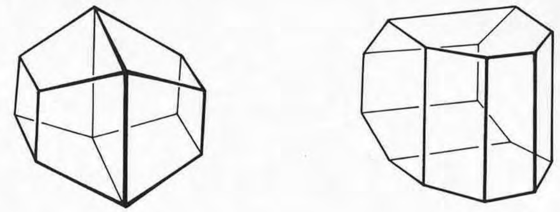
A *septa*hedron has 7 faces.



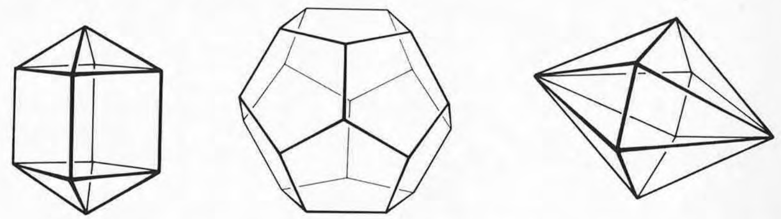
An *octa*hedron has 8 faces.



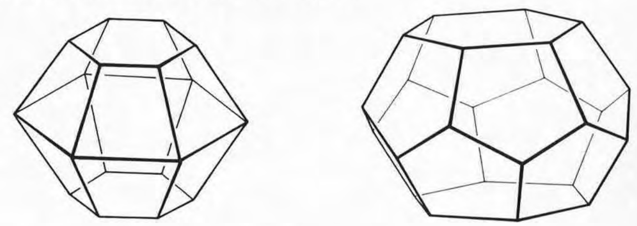
A *deca*hedron has 10 faces.



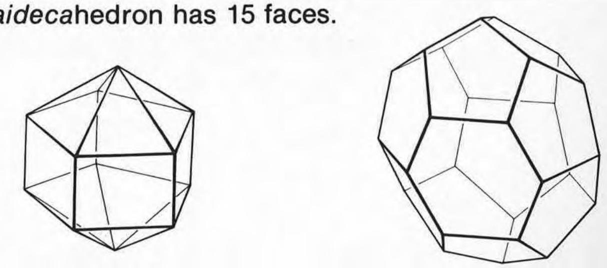
A *dodeca*hedron has 12 faces.



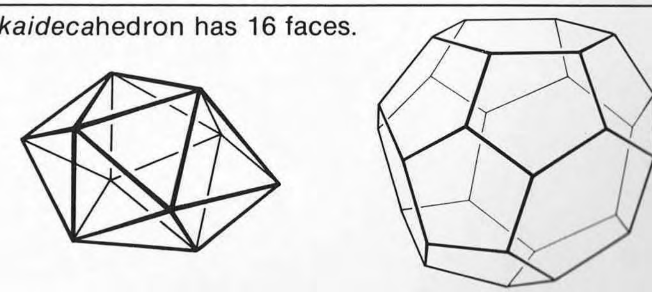
A *tetrakaideca*hedron has 14 faces.



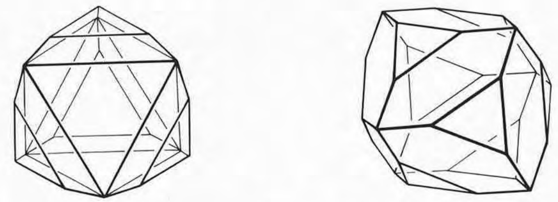
A *pentakaideca*hedron has 15 faces.



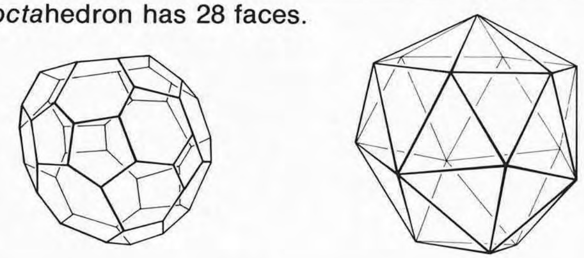
A *hexakaideca*hedron has 16 faces.



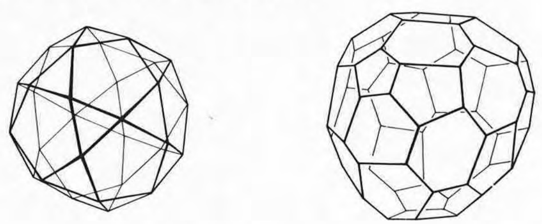
An *icosa*hedron has 20 faces.



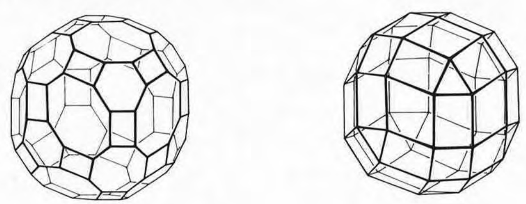
An *icosiocta*hedron has 28 faces.



An *icosidodeca*hedron has 32 faces.

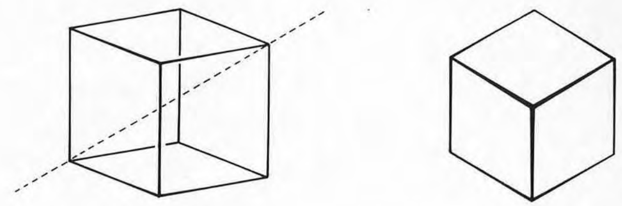


A *pentaconta*hedron has 50 faces.

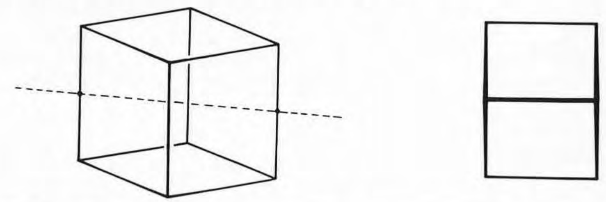


Polyhedra and Symmetry

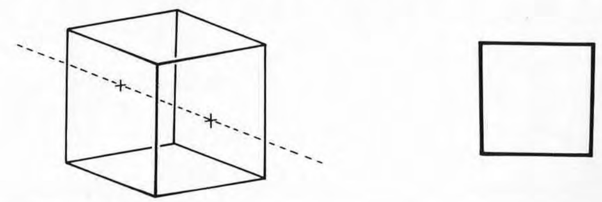
The symmetry properties of a polyhedron are determined by viewing it from different orientations. The number of orientations that produce views of symmetry may vary for different polyhedra. Views of symmetry are characterized as rotational and/or mirror. A particular view of symmetry may occur more than once in a given polyhedron. One possible view of symmetry may be determined by looking at a polyhedron toward a centered vertex.



Another view of symmetry may be determined by looking at a polyhedron toward a centered edge.



A third view of symmetry may be determined by looking at a polyhedron toward a centered face.



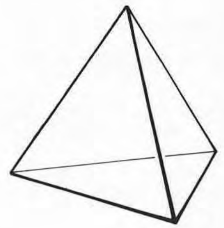
Regular Polyhedra

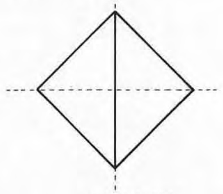
Regular polyhedra are uniform and have faces of all of one kind of congruent regular polygon. There are five regular polyhedra. The regular polyhedra were an important part of Plato's natural philosophy, and thus have come to be called the Platonic Solids.

In the views of symmetry shown below, each type of rotational symmetry and the number of times it occurs is indicated by: n-fold(x). The mirror planes are represented by dotted lines.

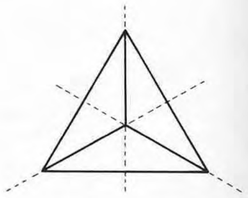
Tetrahedron

Faces:
4 triangles
Vertices:
4, each with 3 edges meeting
Edges:
6
Dihedral angle:
70° 32′
Views of symmetry:





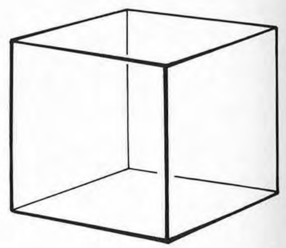
2-fold (3)

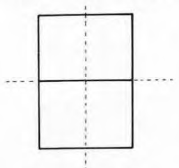


3-fold (4)

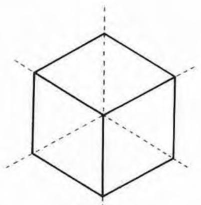
Cube (hexahedron)

Faces:
6 squares
Vertices:
8, each with 3 edges meeting
Edges:
12
Dihedral angle:
90°
Views of symmetry:

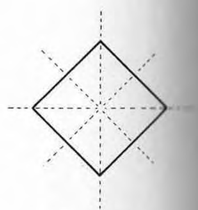




2-fold (6)



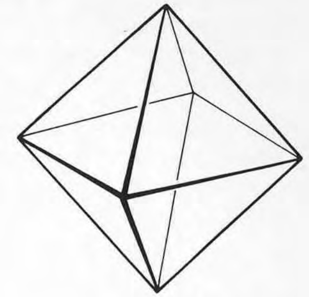
3-fold (4)

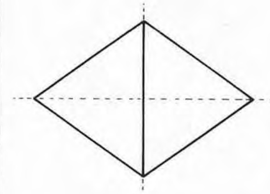


4-fold (3)

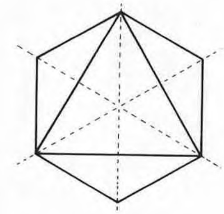
Octahedron

Faces:
8 triangles
Vertices:
6, each with 4 edges meeting
Edges:
12
Dihedral angle:
109° 28′
Views of symmetry:

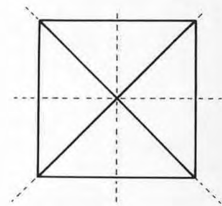




2-fold (6)



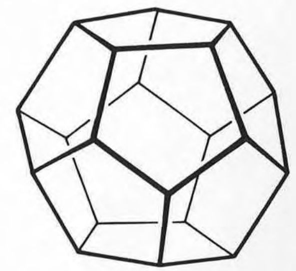
3-fold (4)

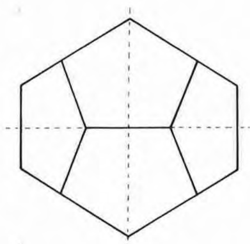


4-fold (3)

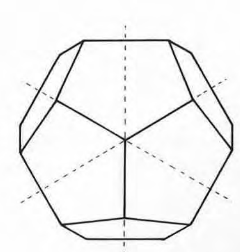
Dodecahedron

Faces:
12 pentagons
Vertices:
20, each with 3 edges meeting
Edges:
30
Dihedral angle:
116° 34′
Views of symmetry:

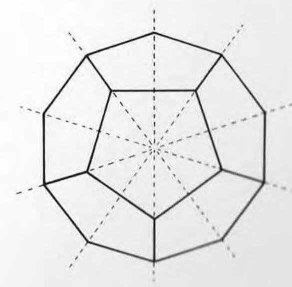




2-fold (15)



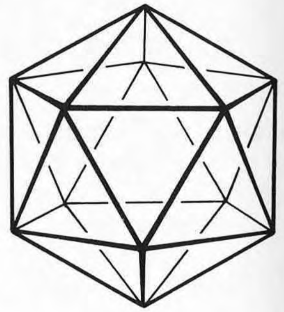
3-fold (10)

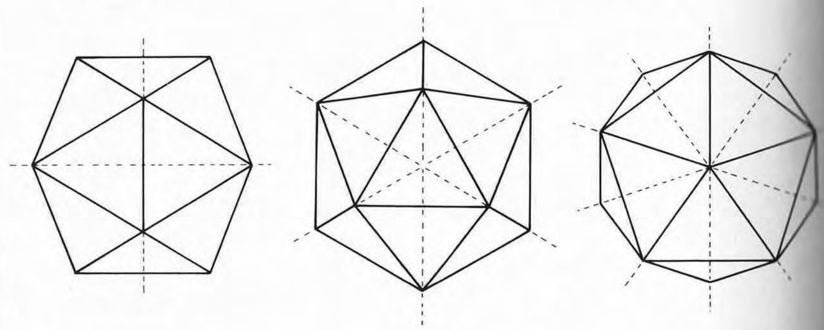


5-fold (6)

Icosahedron

Faces:
20 triangles
Vertices:
12, each with 5 edges meeting
Edges:
30
Dihedral angle:
138° 11′
Views of symmetry:





2-fold (15) 3-fold (10) 5-fold (6)

Semiregular Polyhedra

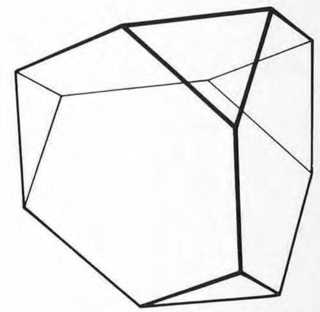
A semiregular polyhedron has regular polygons as faces, but the faces are not all of the same kind. As in regular polyhedra, the vertices are congruent. There are thirteen semiregular polyhedra. It is generally believed that they were described by Archimedes, and thus are called the Archimedean Polyhedra.

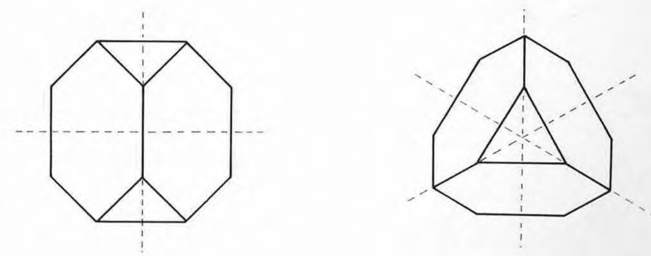
As in the regular polyhedra, in the views of symmetry shown below, each type of rotational symmetry and the number of times it occurs is indicated by: n-fold(x), and the mirror planes are represented by dotted lines.

Five semiregular polyhedra are derived by truncating the five regular polyhedra. Truncation is done so that all new faces are regular polygons. The polyhedra formed are the truncated tetrahedron, truncated cube, truncated octahedron, truncated dodecahedron, and truncated icosahedron.

Truncated Tetrahedron

Faces:
4 hexagons ⎤ 8 total
4 triangles ⎦
Vertices:
12, each with 3 edges meeting
Edges:
18
Dihedral angles:
70° 32′ (hexagon-hexagon)
109° 28′ (triangle-hexagon)
Views of symmetry:





2-fold (3) 3-fold (4)

Truncated cube

Faces:
- 8 triangles
- 6 octagons

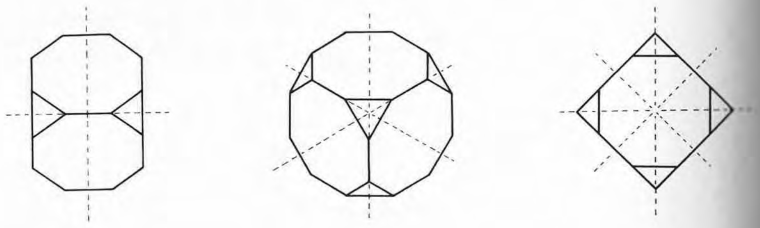
] 14 total

Vertices:
- 24, each with 3 edges meeting

Edges:
- 36

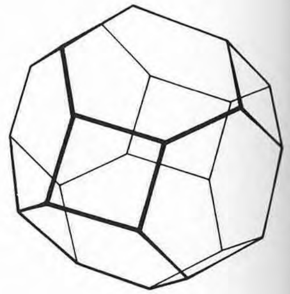
Dihedral angles:
- 125° 16′ (octagon-triangle)
- 90° (octagon-octagon)

Views of symmetry:



2-fold (6) 3-fold (4) 4-fold (3)

Truncated octahedron



Faces:
- 6 squares
- 8 hexagons

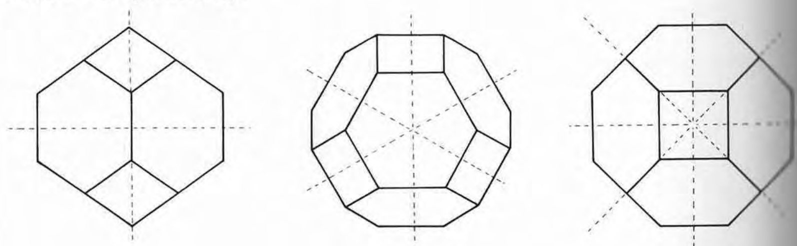
] 14 total

Vertices:
- 24, each with 3 edges meeting

Edges:
- 36

Dihedral angles:
- 125° 16′ (square-hexagon)
- 109° 28′ (hexagon-hexagon)

Views of symmetry:



2-fold (6) 3-fold (4) 4-fold (3)

Truncated dodecahedron

Faces:
- 20 triangles
- 12 decagons

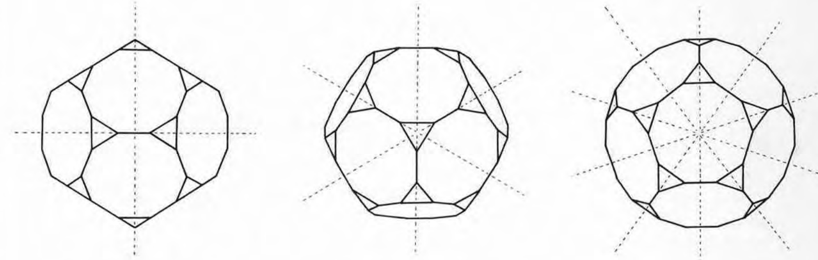
] 32 total

Vertices:
- 60, each with 3 edges meeting

Edges:
- 90

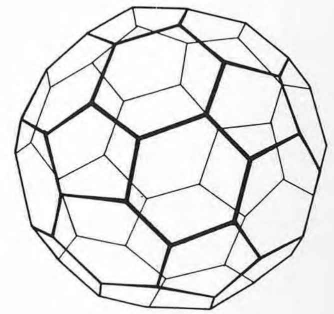
Dihedral angles:
- 116° 34′ (decagon-decagon)
- 142° 37′ (decagon-triangle)

Views of symmetry:



2-fold (15) 3-fold (10) 5-fold (6)

Truncated icosahedron



Faces:
- 12 pentagons
- 20 hexagons

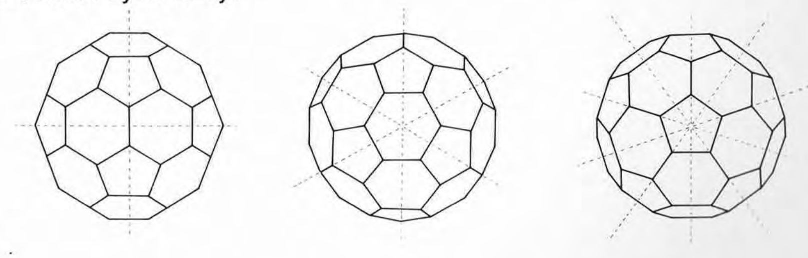
] 32 total

Vertices:
- 60, each with 3 edges meeting

Edges:
- 90

Dihedral angles:
- 138° 11′ (hexagon-hexagon)
- 142° 37′ (hexagon-pentagon)

Views of symmetry:



2-fold (15) 3-fold (10) 5-fold (6)

Quasiregular Polyhedra

A quasiregular polyhedron has two kinds of faces, with each face of one kind being entirely surrounded by the face of the other kind. Two semiregular polyhedra are quasiregular: the cuboctahedron and the icosidodecahedron.

Cuboctahedron

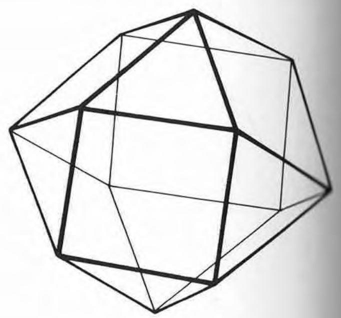
Faces:
- 8 triangles, 6 squares — 14 total

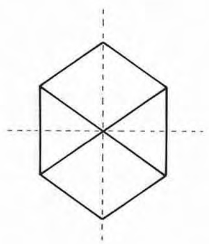
Vertices:
- 12, each with 4 edges meeting

Edges:
- 24

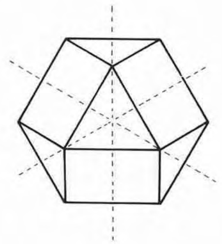
Dihedral angle:
- 125° 16′

Views of symmetry:

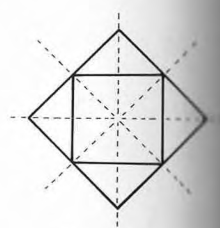




2-fold (6)



3-fold (4)



4-fold (3)

Icosidodecahedron

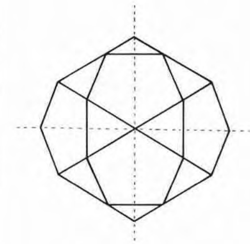
Faces:
- 20 triangles, 12 pentagons — 32 total

Vertices:
- 30, each with 4 edges meeting

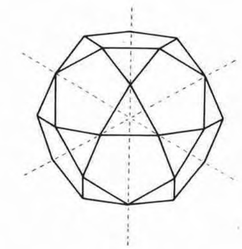
Edges:
- 60

Dihedral angle:
- 142° 37′

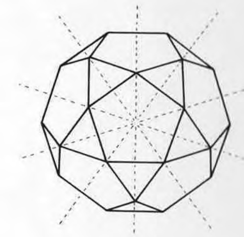
Views of symmetry:



2-fold (15)



3-fold (10)



5-fold (6)

The snub cuboctahedron (also known as the snub cube) and the snub icosidodecahedron (also known as the snub dodecahedron) are derivations of the cuboctahedron and icosidodecahedron respectively that are formed by adding extra equilateral triangles to the original polyhedron.

Snub cuboctahedron

Faces:
- 32 triangles
- 6 squares

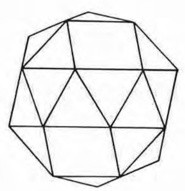
38 total

Vertices:
24, each with 5 edges meeting

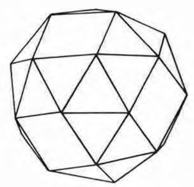
Edges:
60

Dihedral angles:
- 142° 59′ (square-triangle)
- 153° 14′ (triangle-triangle)

Views of symmetry:



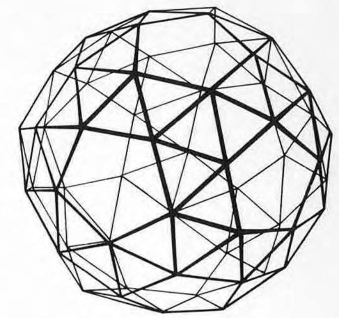
2-fold (6)



3-fold (4)

4-fold (3)

Snub icosidodecahedron



Faces:
- 80 triangles
- 12 pentagons

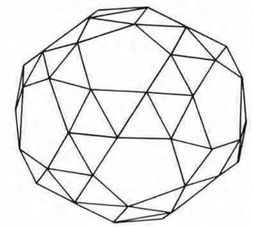
92 total

Vertices:
60, each with 5 edges meeting

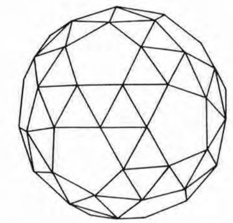
Edges:
150

Dihedral angles:
- 152° 16′ (pentagon-triangle)
- 164° 11′ (triangle-triangle)

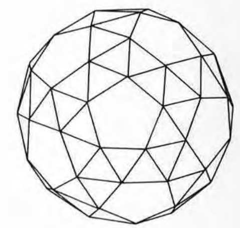
Views of symmetry:



2-fold (15)



3-fold (10)



5-fold (6)

Truncating the cuboctahedron in two different ways gives rise to the truncated cuboctahedron (also known as the greater rhombicuboctahedron) and the rhombicuboctahedron.

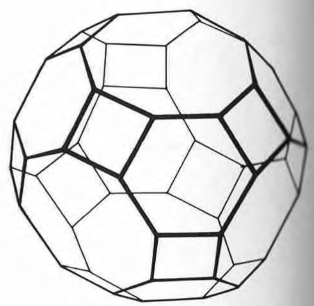
Truncated cuboctahedron
(Greater rhombicuboctahedron)

Faces:
12 squares
8 hexagons
6 octagons
26 total

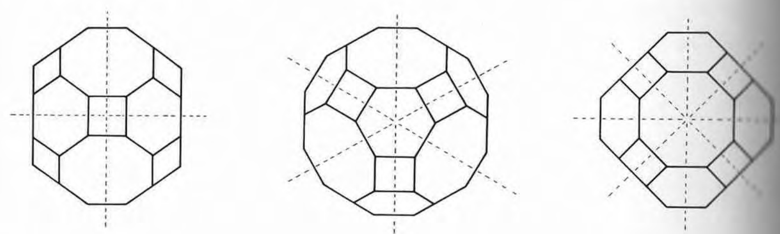
Vertices:
48, each with 3 edges meeting

Edges:
72

Dihedral angles:
135° (octagon-square)
125°16′ (octagon-hexagon)
144°44′ (hexagon-square)



Views of symmetry:



2-fold (6) 3-fold (4) 4-fold (3)

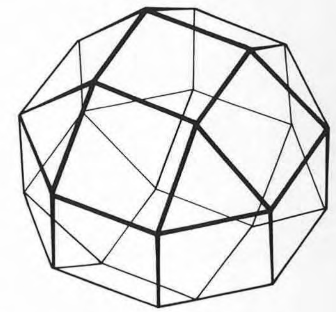
Rhombicuboctahedron

Faces:
8 triangles
18 squares
26 total

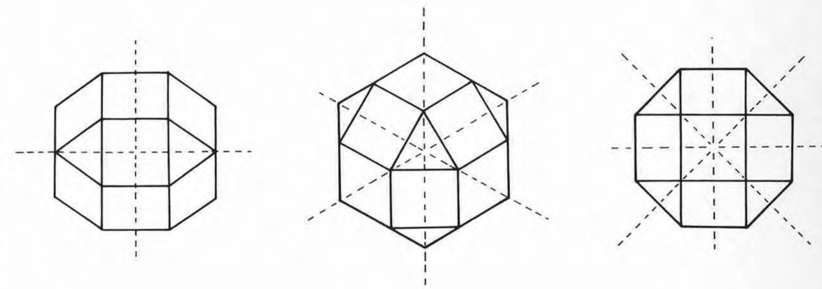
Vertices:
24, each with 4 edges meeting

Edges:
48

Dihedral angles:
135° (square-square)
144°44′ (square-triangle)



Views of symmetry:



2-fold (6) 3-fold (4) 4-fold (3)

Truncating the icosidodecahedron in two different ways gives rise to the truncated icosidodecahedron (also known as the greater rhombicosidodecahedron), and the rhombicosidodecahedron.

Truncated icosidodecahedron
(Greater rhombicosidodecahedron)

Faces:
- 30 squares
- 20 hexagons
- 12 decagons

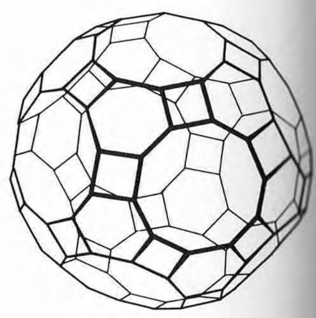
62 total

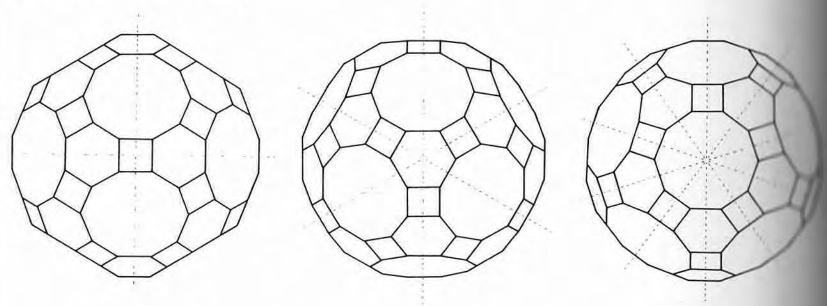
Vertices:
- 120, each with 3 edges meeting

Edges:
- 180

Dihedral angles:
- 148° 17′ (decagon-square)
- 142° 37′ (decagon-hexagon)
- 159° 6′ (hexagon-square)

Views of symmetry:





2-fold (15) 3-fold (10) 5-fold (6)

Rhombicosidodecahedron

Faces:
- 20 triangles
- 30 squares
- 12 pentagons

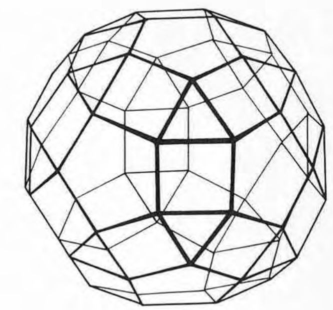
62 total

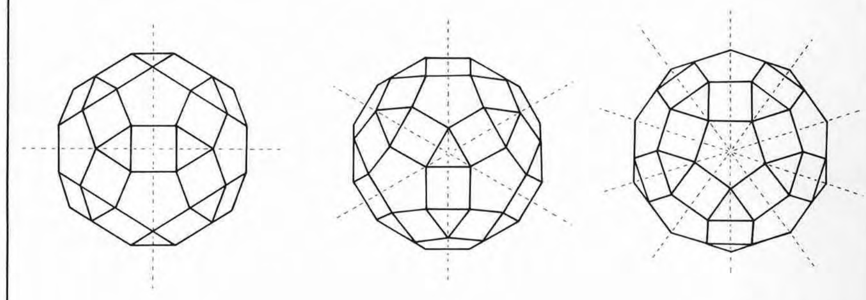
Vertices:
- 60, each with 4 edges meeting

Edges:
- 120

Dihedral angles:
- 148° 17′ (pentagon-square)
- 159° 6′ (triangle-square)

Views of symmetry:

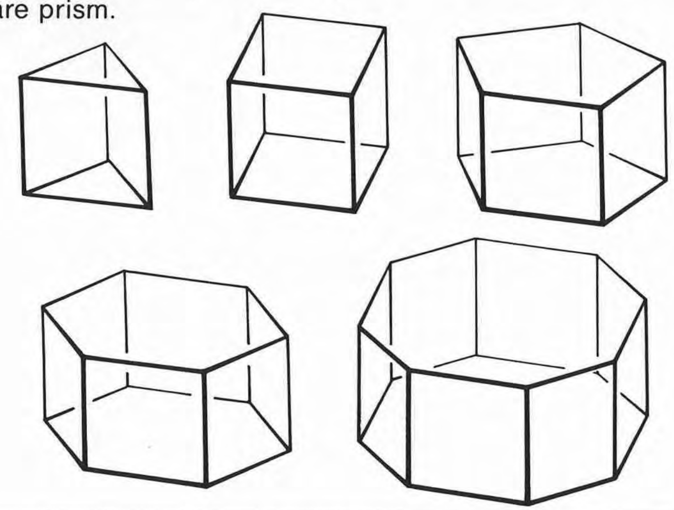




2-fold (15) 3-fold (10) 5-fold (6)

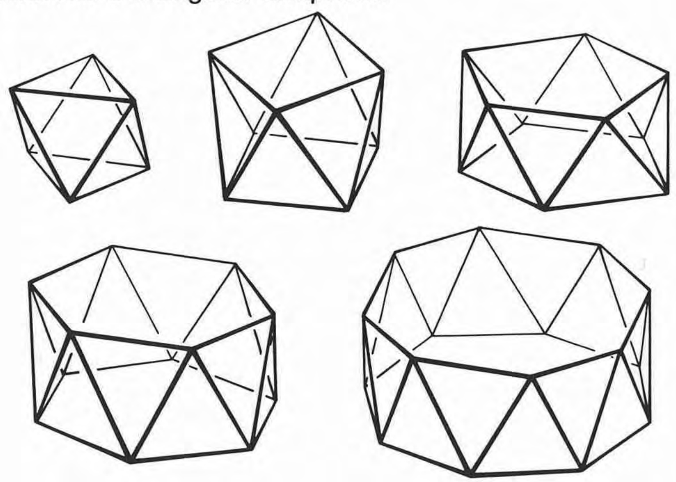
Prism

A prism is a polyhedron with two congruent and parallel faces that are joined by a set of parallelograms. The prism is semiregular if all the polygons are regular. A cube can be considered a square prism.



Antiprism

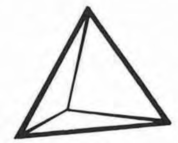
An antiprism is a polyhedron with two congruent and parallel faces that are joined by a set of triangles. The antiprism is semiregular if all the polygons are regular. The octahedron can be considered a triangular antiprism.



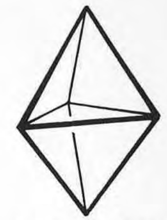
Prisms and antiprisms correspond to the infinite number of possible polygons.

Deltahedra

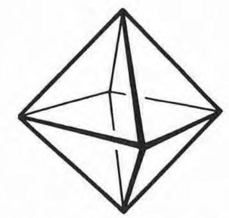
Deltahedra are convex polyhedra whose faces consist entirely of equilateral triangles. The deltahedra are the only polyhedra, beside the regular polyhedra, that have faces consisting of only one kind of regular polygon. Three of the eight deltahedra are regular polyhedra. The remaining five do not have congruent vertices.



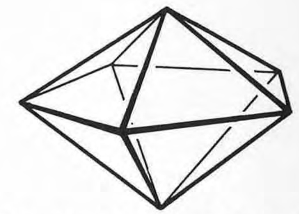
Tetrahedron (4-hedron)



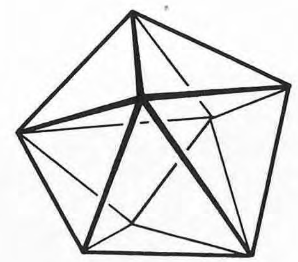
Triangular Dipyramid (6-hedron)



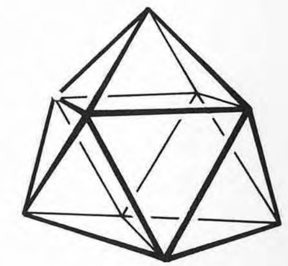
Octahedron (8-hedron)



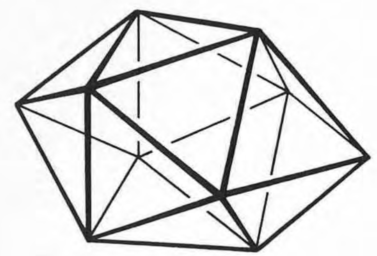
Pentagonal Dipyramid (10-hedron)



12-hedron



14-hedron

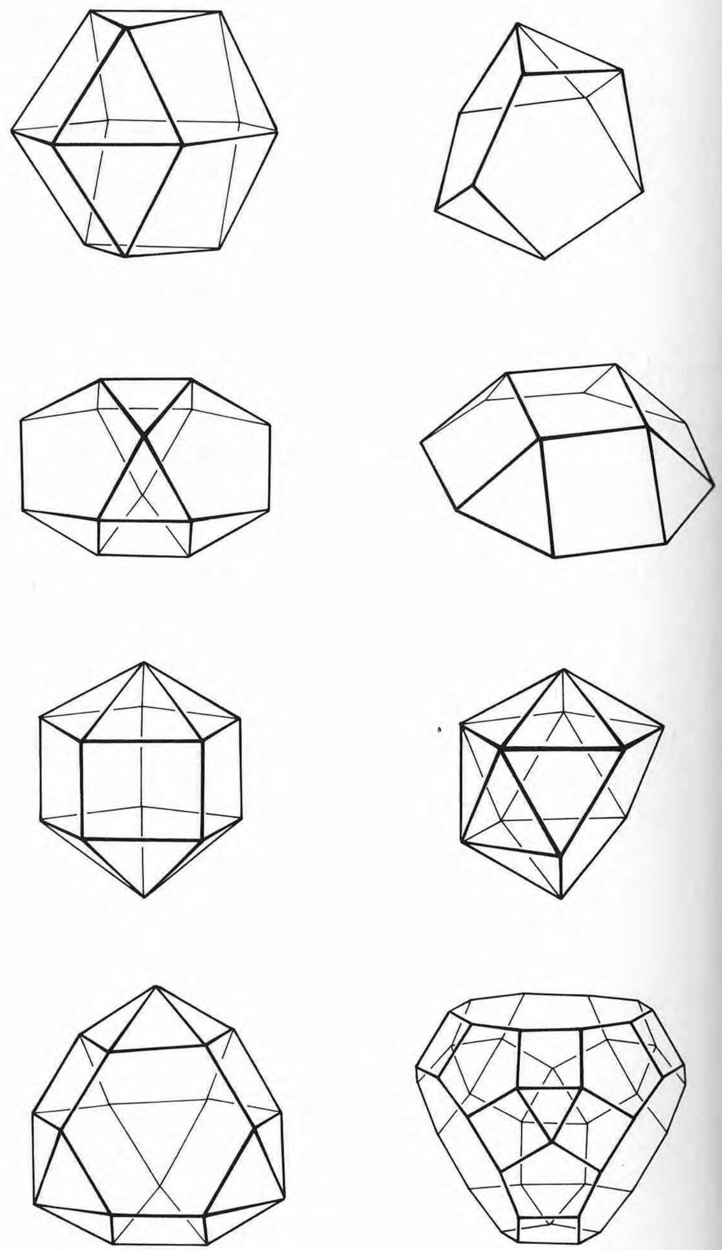


16-hedron

Icosahedron (20-hedron)

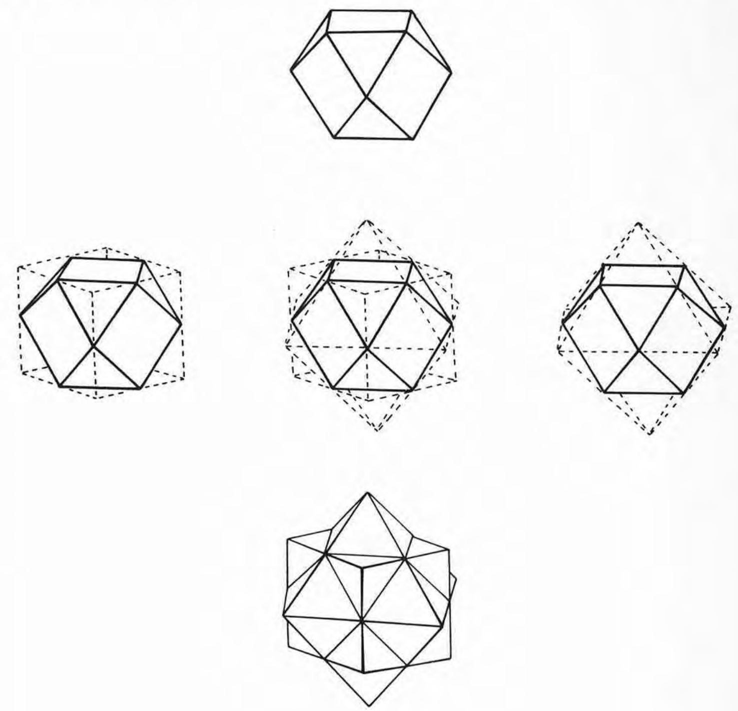
Other Convex Polyhedra with Regular Faces

In addition to the regular and semiregular polyhedra, the deltahedra, and the infinite class of prisms and antiprisms, there are 87 more convex polyhedra whose faces are entirely regular polygons. These polyhedra do not have congruent vertices. Only regular triangles, squares, pentagons, hexagons, octagons, and decagons may be used for the formation of polyhedra with regular faces. Some of these additional polyhedra are shown below.



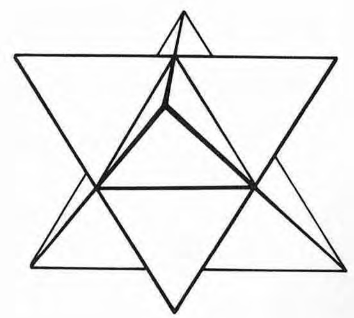
Stellated Polyhedra

A stellated polyhedron is formed by extending in the same plane each face of a convex polyhedron until the faces intersect to form a new enclosing shape. Usually stellations are performed on polyhedra whose faces are all alike, but the operation can be done on other polyhedra.



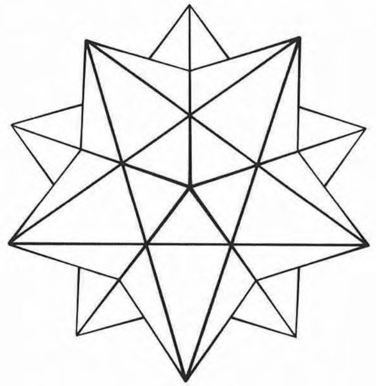
Stella Octangula

The stella octangula is the polyhedron formed by stellating an octahedron. The stella octangula can also be thought of as an intersection of two tetrahedra.

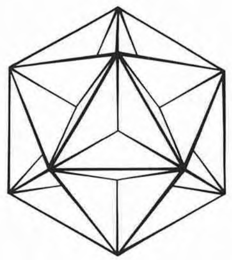


Kepler-Poinsot Solids

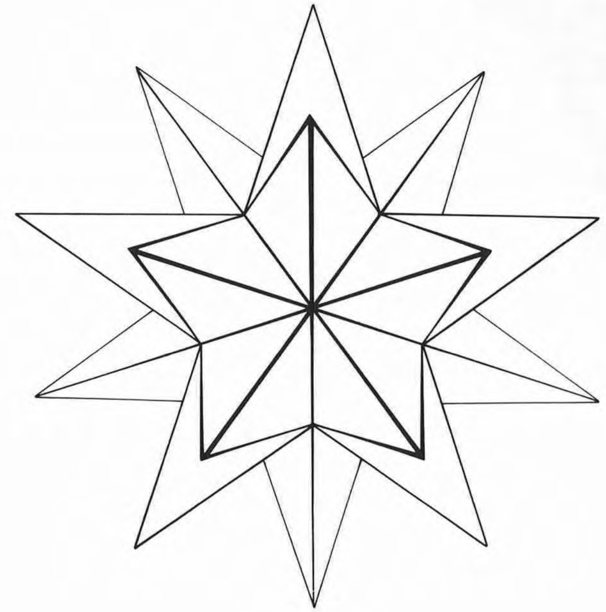
The stellations of the dodecahedron and icosahedron are called the Kepler-Poinsot Solids. The regular dodecahedron can be stellated to form the small stellated dodecahedron.



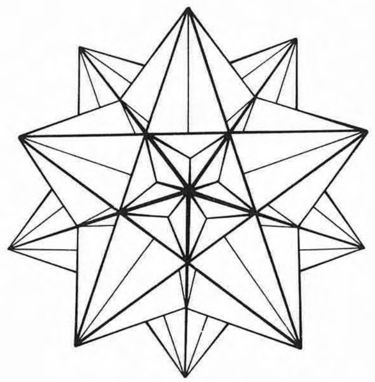
The regular dodecahedron can be stellated in a different manner to form the great dodecahedron.



The dodecahedron can again be stellated to form the great stellated dodecahedron.



The icosahedron can be stellated to form the great icosahedron.



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Dual Polyhedra