System Model

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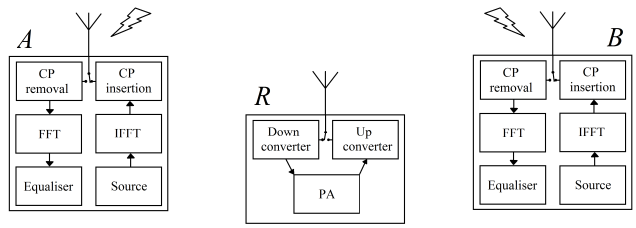

Fig. 1. Nodes A and B wish to communicate with each other via the relay, node R. The figure represents the state of the half-duplex system in its first time slot.

***Abstract*—This document describes the system model used in the programs** Po_numeric **and** Po_analytic**.** Po_numeric **calculates the numerical outage probability of this system, while** Po_analytic **calculates the theoretical outage probability. Here, we also explain why the analytical analysis will fail for fixed-gain relaying if the per-hop channels are not sufficiently dispersive. This failure can be observed in** Po_numeric **and** Po_analytic**.**

I. System Model

Consider a two-hop, two-way, time-division duplexing (TDD) AF OFDM relay network operating over a total of n subcarriers (Fig. 1). We consider an l tap quasi-static linear time-invariant channel impulse response for links A-R and B-R, respectively. After time-domain sampling, the impulse response for hop $\beta \in \{A, B\}$ to the relay can be represented as the time-domain vector

$$\tilde{H}_\beta = \sqrt{\frac{n}{l}} \frac{\left[\tilde{h}_{\beta 0} \quad \tilde{h}_{\beta 1} \quad \cdots \quad \tilde{h}_{\beta,l-1}\right]^T}{\left|\left[\tilde{h}_{\beta 0} \quad \tilde{h}_{\beta 1} \quad \cdots \quad \tilde{h}_{\beta,l-1}\right]^T\right|}, \tag{1}$$

where the ith entry of $\tilde{H}_\beta$ corresponds to ith tap of the channel, and $\tilde{h}_{\beta i}$, $i \in \{0, \ldots, l-1\}$, is an i.i.d. ZMCG random variable with total variance μ_β. The prefactor $\sqrt{n/l}$ is a normalization term. After taking the unitary FFT of the channel's l tap impulse response, the frequency response for the kth subcarrier between links A-R and B-R are given by $h_{Ak} \sim \mathcal{CN}(0, \mu_A)$ and $h_{Bk} \sim \mathcal{CN}(0, \mu_B)$, $k \in \{1, \ldots, n\}$, respectively. Note, within our system model, individual subcarriers of the same hop are necessarily correlated with each other.

We will now describe the relaying protocol from transmission to reception. This takes place over two time slots and is detailed in the following subsections.

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A. First Time Slot

Nodes A and B construct OFDM symbol vectors comprised of n symbols, which we denote by the frequency-domain vectors $X_A = [x_{A1}, \ldots, x_{An}]^T$ and $X_B = [x_{B1}, \ldots, x_{Bn}]^T$, where $\sigma_A^2 := \mathbb{E}[|x_{Ak}|^2]$ and $\sigma_B^2 := \mathbb{E}[|x_{Bk}|^2]$. It is assumed that the symbols $\{x_{Ak}\}$ and $\{x_{Bk}\}$ are chosen uniformly and independently from a quadrature phase-shift keying (QPSK) constellation. Before being passed through an SEL, the time-domain transmit vector at node β is given by

$$\tilde{X}_\beta = \left[\tilde{x}_{\beta,n-2l-1} \quad \tilde{x}_{\beta,n-2l} \quad \cdots \quad \tilde{x}_{\beta,n-1}\right]^T, \tag{2}$$

where $\tilde{x}_{\beta,i}$ is the ith entry of $\tilde{X}_\beta := \mathbf{F}^{-1} X_\beta$.

Source Transmission: The sources are subject to maximum transmit power constraints, $p_{max,\beta}$, $\beta \in \{A, B\}$. To ensure that these constraints are not exceeded, nodes A and B, respectively, pass their time-domain waveforms through soft envelope limiters (SEL) (to be described next). The output of the SEL in the time-domain given that the input is the kth element of (2) is

$$\tilde{y}_{\beta k} = \min\left\{\sqrt{p_{max\beta}}, |\tilde{x}_{\beta k}|\right\} \exp\left(\mathbf{i} \arg \tilde{x}_{\beta k}\right), \qquad \beta \in \{A, B\}. \tag{3}$$

By considering the theory presented in [1] (Bussgang's theorem), and provided the number of subcarriers is sufficiently large, the frequency domain output of the SEL on each subcarrier can be written as

$$y_{\beta k} = \zeta_\beta x_{\beta k} + \varrho_{\beta k}, \qquad \beta \in \{A, B\}, \tag{4}$$

where ζ_β is given by

$$\zeta_\beta = 1 - e^{-\frac{p_{max\beta}}{\sigma_\beta^2}} + \sqrt{\frac{\pi p_{max\beta}}{4\sigma_\beta^2}} \operatorname{erfc}\left(\frac{p_{max\beta}}{\sigma_\beta^2}\right) \tag{5}$$

and $\varrho_{\beta k}$ is uncorrelated with $x_{\beta k}$ and well approximated by a ZMCG random variable with total variance η_β given by

$$\eta_\beta = \sigma_\beta^2 \left(1 - e^{-\frac{p_{max\beta}}{\sigma_\beta^2}}\right) - \zeta_\beta^2 \sigma_\beta^2. \tag{6}$$

The average transmit power, p_β, on each subcarrier at node β is given by

$$p_\beta = \sigma_\beta^2 \left(1 - e^{-\frac{p_{max\beta}}{\sigma_\beta^2}}\right). \tag{7}$$

The received signal at the relay on the kth subcarrier is then given by

$$y_{Rk} = h_{Ak}\zeta_A x_{Ak} + h_{Bk}\zeta_B x_{Bk} + v_{Rk} + h_{Ak} d_{Ak} + h_{Bk} d_{Bk}, \tag{8}$$

where $v_{Rk} \sim \mathcal{CN}(0, n_0)$ is the noise term on the kth subcarrier at the relay.

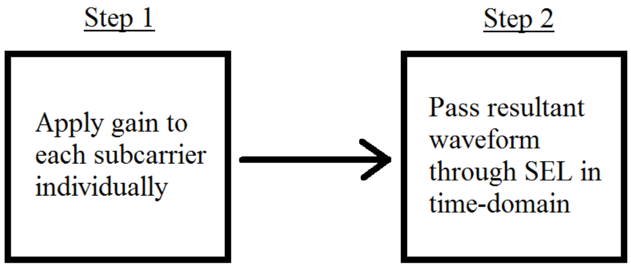

Fig. 2. Figure illustrating the two step process used to model the amplification at the relay before transmission.

B. Relay Amplification Model

Once y_{Rk}, (8), has been received at the relay, the relay performs the amplification process and then transmits the resultant signal. This process takes place over two distinct steps, see Fig. 2.

a) Step One: The relay applies the amplification factor $g_{\alpha k}$ to its received signal, where $\alpha \in \{FG, VG\}$ denotes whether FG or VG has been considered. For the FG scenario, the amplification factor for the kth subcarrier is given by

$$g_{FGk} \quad = \quad \sqrt{\frac{\sigma_R^2}{p_A\mu_A + p_B\mu_B + n_0}}; \tag{9}$$

where σ_R^2 is the average input power to the SEL at the relay. For the VG scenario, we assume the gain for the kth subcarrier takes the form

$$g_{VGk} = \sqrt{\frac{\sigma_R^2}{p_A|h_{Ak}|^2 + p_B|h_{Bk}|^2 + n_0}}. \tag{10}$$

Note, (9) is independent of k. However, to aid exposition, we refrain from removing the subscript k.

b) Step Two: Before transmission, the relay passes the amplified time-domain waveform through an SEL (3) to limit its maximum transmit power to p_{maxR}. As an immediate consequence of the central limit theorem, this time-domain signal converges in distribution to a stationary ZMCG variable with variance σ_R^2 as the number of subcarriers grows large. This allows us to apply Bussgang's theorem, as we did in (4), so that we can write the frequency-domain output of the relay's SEL as

$$x_{Rk} = \zeta_R g_{\alpha k} y_{Rk} + \varrho_{Rk}, \tag{11}$$

where ζ_R is given by (5) and ϱ_{Rk} is uncorrelated with y_{Rk} and well approximated by a ZMCG random variable with variance η_R given by (6). The average transmit power on each subcarrier at the relay is given by p_R (see (7)).

C. Second Time Slot

By assuming channel reciprocity, which follows from the TDD nature of the channel, and that the entire relaying process has taken place within the coherence time of the channel, the received signal on the kth subcarrier at node $\beta \in \{A, B\}$ is

$$y_{\beta k} = h_{\beta k} x_{Rk} + v_{\beta k}, \tag{12}$$

where $v_{\beta k} \sim \mathcal{CN}(0, n_0)$ is the additive noise term on the kth carrier at node β. Note, to obtain (12), node β must first remove the CP from the received time-domain block and then perform an FFT on this block. The kth element of the output vector of the FFT will then be given by (12).

To perfectly remove all self-interference from the network, A and B should calculate

$$\underline{y}_{Ak} = y_{Ak} - \zeta_A\zeta_R g_{\alpha k} h_{Ak}^2 x_{Ak}, \ \underline{y}_{Bk} = y_{Bk} - \zeta_B\zeta_R g_{\alpha k} h_{Bk}^2 x_{Bk}. \tag{13}$$

However, in our model the destination nodes is unaware of the relay's amplifier distortion characteristics. In this scenario, they calculate

$$\underline{y}_{Ak} = y_{Ak} - \zeta_A g_{\alpha k} h_{Ak}^2 x_{Ak}, \ \underline{y}_{Bk} = y_{Bk} - \zeta_B g_{\alpha k} h_{Bk}^2 x_{Bk}. \tag{14}$$

Thus, there will be self-interference at A and B given, respectively, by

$$g_{\alpha k}(\zeta_R - 1) h_{Ak}^2 x_{Ak} \qquad \text{and} \qquad g_{\alpha k}(\zeta_R - 1) h_{Bk}^2 x_{Bk}. \tag{15}$$

II. A Subtlety for FG Relaying

It is important that we discuss a particular subtlety that occurs when applying Bussgang's theorem to FG relaying. For the Bussgang parameters at the relay to be completely determined by the average input power to the SEL[1] (see σ_R^2 in (9) and (10)), the input to the relay's SEL should be stationary. For FG, this stationarity is contingent on there being a sufficient number of *significant* taps within the channel. This is *not* the case for VG.

To understand why a sufficient number of taps is required for FG, consider the extreme scenario in which the channels are flat across all the subcarriers; i.e., the channels have a single tap impulse response. Furthermore, note that a quasi-static fading model has been considered, so the channel coefficients are considered to be fixed for a single OFDM time-domain block. For this, the subcarrier responses will be independent of their indices, i.e., $h_{\beta i} = h_{\beta j} \ \forall \ i, j$. With $h_{\beta i} = h_\beta \sim \mathcal{CN}(0, \mu_\beta)$ for all i, this allows us to write the time-domain waveform at the relay's SEL as

$$\tilde{Y}_R(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e^{\frac{i2\pi kt}{n}} g_{FGk} \tag{16}$$

$$\times \left(h_A\zeta_A x_{Ak} + h_B\zeta_B x_{Bk} + v_{Rk} + h_A d_{Ak} + h_B d_{Bk}\right). \tag{17}$$

Because the gains are fixed and independent of k, and the channel coefficients are assumed to be quasi-static, $\tilde{Y}_R(t)$ will approach a ZMCG variable with conditional variance

$$\mathbb{V}\left[\tilde{Y}_R(t) \,|\{h_A, h_B\}\right] = \left(\frac{\sigma_R^2}{p_A\mu_A + p_B\mu_B + n_0}\right) \tag{18}$$

$$\times \left(p_A |h_A|^2 + p_B |h_B|^2 + n_0\right).$$

Note conditioning on the channel coefficients is performed to account for the quasi-static nature of the channel. Thus, we will not have the requirement for the Bussgang parameters that the input to the SEL be a stationary ZMCG variable with

[1]When this is the case, analytic calculations become tractable.

variance σ_R^2. In particular, this will be a function of the quasi-static random variables[2] h_A and h_B. The Bussgang parameters will then become functions of these instantaneous parameters, varying from one fading block to next. However, as the number of channel taps grows, h_{1i} and h_{1j}, $i \neq j$, will become increasingly decorrelated and the averaging performed by the inverse FFT in (17) will tend to remove the dependence of $\tilde{Y}_R(t)$'s conditional variance on the instantaneous realizations of h_{Ak} and h_{Bk}. Concretely, $\tilde{Y}_R(t)$ will approach a stationary ZMCG random variable with conditional variance σ_R^2 as the number of channel taps grows large. To see this, observe the following:

$$\begin{aligned}
&\mathbb{V}\left[\tilde{Y}_R\left(t\right) \mid \{h_{A1},\ldots,h_{An},h_{B1},\ldots,h_{Bn}\}\right] \\
=&\frac{1}{n}\sum_{k=1}^{n}\mathbb{V}\left[e^{\frac{i2\pi kt}{n}} g_{FGk}\left(h_{Ak}\zeta_A x_{Ak} + h_{Bk}\zeta_B x_{Bk} + v_{Rk}\right.\right. \\
&\qquad\qquad\qquad\qquad\qquad \left.+h_{1k}d_{Sk}\right) \left| h_{1k} \right] \\
=&\left(\frac{\sigma_R^2}{p_A\mu_A + p_B\mu_B + n_0}\right) && (19) \\
&\times\frac{1}{n}\sum_{k=1}^{n}\left(p_A\left|h_{Ak}\right|^2 + p_B\left|h_{Bk}\right|^2 + n_0\right). && (20)
\end{aligned}$$

In the limit as the number of channel taps and subcarriers grow large, $\left|h_{\beta i}\right|^2$ becomes independent of $\left|h_{\beta j}\right|^2$ for all $i \neq j$ and $\beta \in \{A, B\}$, and from (20) and the law of large numbers

$$\mathbb{V}\left[\tilde{Y}_R\left(t\right) \mid \{h_{A1},\ldots,h_{An},h_{B1},\ldots,h_{Bn}\}\right]] \longrightarrow \sigma_R^2.$$

This shows us that $\tilde{Y}_R\left(t\right)$ approaches a stationary random variable as the number of channel taps grows large. Of course, in practice the number of channel taps will be finite. However, from heuristic observations, we find that 16 or more channel taps allows for very accurate analytical modeling of FG systems using Bussgang's theorem. This can be observed from the program.

REFERENCES

[1] D. Dardari, V. Tralli, and A. Vaccari, "A Theoretical Characterization of Nonlinear Distortion Effects in OFDM Systems," *IEEE Transactions on Communications*, vol. 48, no. 10, pp. 1755–1764, 2000.

[2] Equation (19) also illustrates why a sufficient number of taps is not required for VG: for VG, the denominator of the first bracketed term will be identical to the second bracketed term, and therefore cancel with it.