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Distributional Modelling

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Executive Summary

This paper is intended to model the unconditional return distribution of Berkshire Hathaway Inc. (BRK) and International Business Machines Corporation (IBM) for different time horizons (daily return, 5-day return, 10-day return and 30-day return) using different distribution candidates (normal distribution, t-distribution and variance gamma distribution). Knowledge of the distribution can help fund managers to manage their fund risk.

Key Findings

Full Period (Task 2 & Task 3)

Table 1 below shows the summary of distribution parameters by fitting the bivariate time series data of BRK and IBM return of the **full period** (from 17 March 1980 to 17 February 2017) into the three distribution models.

Table 1: Summary of Fitting Results – Parameters and Log-likelihood (Full Period)

	Daily			5-day			10-day			30-day		
	(1)	**(2)**	**(3)**	**(1)**	**(2)**	**(3)**	**(1)**	**(2)**	**(3)**	**(1)**	**(2)**	**(3)**
nu		2.844			3.537			3.680			7.478	
λ			0.614			1.377			1.419			2.776
μ_{brk}	0.0008	0.0003	-1.117e-18	0.0042	0.0029	0.0027	0.0083	0.0053	0.0051	0.0246	0.0217	0.0199
μ_{ibm}	0.0005	0.0003	3.979e-17	0.0025	0.0022	0.0018	0.0050	0.0050	0.0042	0.0150	0.0140	0.0126
σ_{brk}	2.122e-04	2.368e-04	1.909e-04	0.0010	0.0011	0.0009	0.0020	0.0021	0.0018	0.0055	0.0059	0.0058
σ_{ibm}	2.848e-04	3.962e-04	3.410e-04	0.0013	0.0015	0.0013	0.0026	0.0030	0.0027	0.0082	0.0080	0.0079
Covariance	5.652e-05	6.168e-05	5.079e-05	0.0003	0.0003	0.0002	0.0007	0.0007	0.0006	0.0014	0.0015	0.0015
Log-likelihood	51230.11	53869.21	$\boxed{54314.36}$	7324.56	$\boxed{7677.84}$	7655.26	3058.99	$\boxed{3198.83}$	3194.56	681.29	690.20	$\boxed{691.03}$

Note: (1): Gaussian Distribution; (2): Student-t Distribution; (3) Variance Gamma Distribution

- Based on the maximum log-likelihood, **Variance Gamma** distribution performed best on the **daily** and **30-day** return datasets, while **Student-t** distribution performed best on the **5-day** and **10-day** return datasets.

Out-of-Sample Test (Task 4)

To perform out-of-sample test, we segregated the full period into half, and took the **first half** (from 17 March 1980 to 17 August 1998) as the **training set** to fit the distributions.

Parameters estimated from the **training set** are summarized in Table 2 as follows:

Table 2: Summary of Fitting Results – Parameters and Log-likelihood (Training Set – First Half Period)

	Daily			5-day			10-day			30-day		
	(1)	**(2)**	**(3)**	**(1)**	**(2)**	**(3)**	**(1)**	**(2)**	**(3)**	**(1)**	**(2)**	**(3)**
nu		3.338			4.505			5.261			18.418	
λ			0.485			1.791			2.110			8.598
μ_{brk}	0.0013	0.0008	1.437e-17	0.0065	0.0051	0.0048	0.0130	0.0098	0.0094	0.0387	0.0380	0.0379
μ_{ibm}	0.0006	0.0002	2.075e-17	0.0028	0.0023	0.0019	0.0058	0.0053	0.0048	0.0177	0.0182	0.0179
σ_{brk}	2.007e-04	1.899e-04	1.772e-04	0.0011	0.0010	0.0010	0.0023	0.0022	0.0021	0.0062	0.0063	0.0063
σ_{ibm}	2.673e-04	3.333e-04	4.085e-04	0.0012	0.0013	0.0012	0.0025	0.0026	0.0025	0.0083	0.0081	0.0081
Covariance	3.929e-05	2.632e-05	2.503e-05	0.00022	0.0002	0.0002	0.0005	0.0005	0.0005	0.0010	0.0010	0.0010
Log-likelihood	25832.82	26861.91	$\boxed{27577.71}$	3669.31	$\boxed{3797.29}$	3784.91	1504.61	$\boxed{1545.13}$	1541.82	330.23	330.99	$\boxed{331.00}$

Note: (1): Gaussian Distribution; (2): Student-t Distribution; (3) Variance Gamma Distribution

- The best-performed distributions are **consistent** with the result under the full period fitting.

In-Sample vs Out-of-Sample VaR (Task 5)

In-sample VaR is calculated based on the respective quantile (0.1%, 1% and 5%) of the **test set**, i.e. second half (from 18 August 1998 to 17 February 2017) of data points.

Out-of-sample VaR is calculated based on the respective quantile (0.1%, 1% and 5%) of **10,000 random data points** simulated from the multivariate distribution derived from the training set (i.e parameters in **Table 2**).

Results are summarized in Table 3 as follows and **best-performed** out-of-sample VaR estimation is highlighted. The best-performed out-of-sample VaR estimation is determined by which estimation is closest to the in-sample VaR.

Table 3: In-Sample vs Out-of-Sample VaR

		99.9%				99%				95%			
		BRK		IBM		BRK		IBM		BRK		IBM	
		In	Out	In	Out	In	Out	In	Out	In	Out	In	Out
Daily	**(1)**		-4.08%		-4.92%		-3.09%		-3.75%		-2.18%		-2.69%
	(2)	-6.52%	-8.98%	-8.29%	-9.78%	-4.04%	-3.55%	-4.93%	**-4.85%**	-2.02%	-1.88%	-2.61%	**-2.58%**
	(3)		**-6.78%**		**-9.28%**		**-4.11%**		-5.96%		**-2.12%**		-3.32%
5-day	**(1)**		-9.54%		-10.24%		-7.11%		**-7.75%**		-4.84%		-5.41%
	(2)	-12.29%	-14.08%	-16.83%	**-15.56%**	-8.40%	-7.61%	-10.62%	-8.93%	-4.50%	**-4.44%**	-5.75%	-5.17%
	(3)		**-11.83%**		-12.53%		**-7.68%**		-8.73%		-4.70%		**-5.59%**
10-day	**(1)**		-12.74%		-14.32%		**-9.71%**		-11.15%		-6.66%		-7.51%
	(2)	-17.12%	-19.14%	-19.73%	**-21.61%**	-9.47%	-11.68%	-13.27%	**-13.51%**	-5.57%	**-6.56%**	-8.04%	-7.55%
	(3)		**-16.56%**		-17.53%		-11.10%		-12.25%		-6.59%		**-7.72%**
30-day	**(1)**		**-21.87%**		-25.08%		-14.61%		-18.76%		-9.07%		-12.88%
	(2)	-22.04%	-20.49%	-27.40%	-30.84%	-16.85%	-14.95%	-21.25%	-19.42%	-8.69%	**-9.03%**	-13.80%	-12.99%
	(3)		-21.41%		**-26.66%**		**-15.04%**		**-19.82%**		-9.37%		**-13.19%**

Note: (1): Gaussian Distribution; (2): Student-t Distribution; (3) Variance Gamma Distribution

Performance Metrics and Recommendation

The performance metrics of Table 2 (relating to MLE) and Table 3 (relating to VaR) are summarized in Table 4 as follows.

Table 4: Performance Metrics

	Daily	5-day	10-day	30-day
MLE	Variance Gamma	Student-T	Student-T	Variance Gamma
VaR – 99.9%	Variance Gamma	Variance Gamma/Student-T	Variance Gamma/Student-T	Variance Gamma/Gaussian
VaR – 99%	Variance Gamma/Student-T	Variance Gamma/Gaussian	Gaussian/Student-T	Variance Gamma
VaR – 95%	Variance Gamma/Student-T	Variance Gamma/Student-T	Variance Gamma/Student-T	Variance Gamma/Student-T

- Based on the mean squared error, **Variance Gamma** distribution dominates in the estimation of VaR with the highest count among the different time horizons.
- However, based on highest MLE, Variance Gamma distribution is tied with Student-t distribution.
- Since each multivariate distribution MLE is similar, we could get a **better estimate by averaging** the VAR output of each distribution under the same time horizon.

Robustness testing

For robustness testing of Task 3, we divided the dataset into 2 parts; **1st half** (from 17 March 1980 to 17 August 1998) and **2nd half** (from 18 August 1998 to 17 February 2017). For the same time horizon, the multivariate distribution MLE values do not differ much for the 1st half and 2nd half datasets. The best multivariate distribution is almost consistent across different time horizons, with the exception of daily and 10-day time horizons. The summary of the MLE values are found in Table 5.1.

Table 5.1: Summary of Fitting Results – Log-likelihood on Full Period, 1st Half and 2nd Half												
	Daily			5-day			10-day			30-day		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
Full period	51230.11	53869.21	**54314.36**	7324.56	**7677.84**	7655.26	3058.99	**3198.83**	3194.56	681.29	690.20	**691.03**
1st half	25832.82	26861.91	**27577.71**	3669.31	**3797.29**	3784.91	1504.61	**1545.13**	1541.82	330.23	330.99	**331.00**
2nd half	25430.65	**27089.58**	26985.07	3676.35	**3897.90**	3886.98	1584.924	1653.58	**1657.01**	365.57	387.74	**389.83**

We further **compared** the **optimal multivariate distribution parameters** for the 1st half and 2nd half datasets as follows in Table 5.2. The multivariate distribution MLE values do not differ much for the 1st half and 2nd half datasets.

Table 5.2: Summary of Fitting Results – Parameters Estimated for the optimal multivariate distribution												
	Daily – Variance Gamma			5-day – Student-T			10-day – Student-T			30-day – Variance Gamma		
	Full	1st Half	2nd Half	Full	1st Half	2nd Half	Full	1st Half	2nd Half	Full	1st Half	2nd Half
nu				3.537	4.505	2.71481	3.680	5.261	3.352			
λ	0.614	0.485	0.9593							2.776	8.598	1.151
μ_{brk}	-1.12e-18	1.44e-17	-0.0001	0.0029	0.0051	0.0010	0.0053	0.0098	0.0027	0.0199	0.0379	0.0075
μ_{ibm}	3.98e-17	2.08e-17	0.0002	0.0022	0.0023	0.0015	0.0050	0.0053	0.0037	0.0126	0.0179	0.0100
σ_{brk}	1.91e-04	1.77e-04	1.87e-04	0.0011	0.0010	0.0014	0.0021	0.0022	0.0018	0.0058	0.0063	0.0040
σ_{ibm}	3.41e-04	4.09e-04	2.82e-04	0.0015	0.0013	0.0021	0.0030	0.0026	0.0032	0.0079	0.0081	0.0084
Covariance	5.08e-05	2.50e-05	6.99e-05	0.0003	0.0002	0.0005	0.0007	0.0005	0.0007	0.0015	0.0010	0.0016
Log-likelihood	54314.36	27577.71	26985.07	7677.84	3797.29	3897.897	3198.83	1545.13	1653.58	691.03	331.00	389.83

For robustness testing of Tasks 4 and 5, we used the 2nd half dataset to fit the distribution, simulated 10,000 data points and compared this to the 1st half dataset. The results are shown in Table 5.3 below. The VaRs under the "2nd" columns are calculated from random data points simulated from the multivariate distribution derived from the 2nd half dataset.

Table 5.3: Out-of-Sample VaR from multivariate distributions derived from 1st Half and 2nd Half													
		99.9%				99%				95%			
		BRK		IBM		BRK		IBM		BRK		IBM	
		1ST	2ND	1ST	2ND	1ST	2ND	1ST	2ND	1ST	2ND	1ST	2ND
Daily	**(1)**	-4.08%	-4.43%	-4.92%	-5.17%	-3.09%	-3.32%	-3.75%	-3.96%	-2.18%	-2.39%	-2.69%	-2.88%
	(2)	-8.98%	-12.67%	-9.78%	-18.46%	-3.55%	-4.62%	-4.85%	-5.61%	-1.88%	-2.03%	-2.58%	-2.57%
	(3)	-6.78%	-6.51%	-9.28%	-6.93%	-4.11%	-3.89%	-5.96%	-4.56%	-2.12%	-2.25%	-3.32%	-2.83%
5-day	**(1)**	-9.54%	-9.56%	-10.24%	-12.03%	-7.11%	-6.89%	-7.75%	-8.83%	-4.84%	-4.93%	-5.41%	-6.11%
	(2)	-14.08%	-22.38%	-15.56%	-27.93%	-7.61%	-9.31%	-8.93%	-11.68%	-4.44%	-4.62%	-5.17%	-5.68%
	(3)	-11.83%	-12.31%	-12.53%	-16.40%	-7.68%	-8.07%	-8.73%	-9.95%	-4.70%	-5.01%	-5.59%	-5.93%
10-day	**(1)**	-12.74%	-11.33%	-14.32%	-15.03%	-9.71%	-8.72%	-11.15%	-11.84%	-6.66%	-6.14%	-7.51%	-7.98%
	(2)	-19.14%	-21.81%	-21.61%	-31.59%	-11.68%	-10.80%	-13.51%	-13.62%	-6.56%	-5.73%	-7.55%	-7.43%
	(3)	-16.56%	-16.01%	-17.53%	-23.83%	-11.10%	-10.34%	-12.25%	-13.81%	-6.59%	-6.03%	-7.72%	-8.19%
30-day	**(1)**	-21.87%	-20.18%	-25.08%	-25.94%	-14.61%	-15.07%	-18.76%	-18.84%	-9.07%	-10.05%	-12.88%	-13.03%
	(2)	-20.49%	-42.09%	-30.84%	-61.98%	-14.95%	-17.33%	-19.42%	-25.34%	-9.03%	-8.94%	-12.99%	-13.03%
	(3)	-21.41%	-29.32%	-26.66%	-36.54%	-15.04%	-17.19%	-19.82%	-24.27%	-9.37%	-9.81%	-13.19%	-14.29%

Note: (1): Gaussian Distribution; (2): Student-t Distribution; (3) Variance Gamma Distribution

- Across different data set and among same distribution, the 99% and 95% VaR values do not differ much.
- 99.9% VAR is highly dependent on outliers. As such, among all distributions, there is a larger spread of values for 99.9% VaR, with the largest spread being multivariate Gaussian distribution. This is probably due to the data being simulated by the joint covariance multivariate distribution function which tend to give more extreme values, as compared to a univariate distribution function.

- Knowledge of these distributions will enable fund managers to choose the appropriate multivariate distribution based on their market outlook. For example, if there is a lot of uncertainty in the market, a multivariate Gaussian distribution would be more appropriate as it generates more extreme values.

Methodology

Stationarity Tests

Before fitting the distributions, it is important to check whether the data is stationary or not. Fitting of distributions can only be allowed on stationary data. Appendix I shows the time-series of BRK and IBM returns on different time horizons. The following tests were performed to ensure that the data was stationary.

1. Augmented Dickey-Fuller Test (Unit Root Test)

   We check the p-value in this test. If the p-value <= 0.01, then the data is stationary. In our case, the daily returns for both BRK and IBM had p-values less than or equal to 0.01. Please refer to Appendix II for the results.

2. Checking Auto-correlation plots

   From the auto correlation plots, one can determine if the data is correlated with its lagged values. We found that the daily returns for both BRK and IBM were not correlated with any lag values. Please refer to Appendix II for the ACF plots.

Out-of-sample testing

For out-of-sample tests, we used one half of the dataset to fit a multivariate distribution. Then, we simulated 10,000 random data points from the multivariate distribution derived from one half of the dataset. Lastly, we compared the respective quantiles (0.1%, 1% and 5%) of the out-of-sample and in-sample VaR.

Visualization of data

As part of visualization of the data, we plotted the Bivariate Daily Return Data. For comparison, we plotted the actual distributions and the multivariate fitted distributions. We also created heat maps and threejs Javascript plots. Appendix III shows the daily return plots. Similar plots were done for 5-day, 10-day and 30-day return data.

Assumptions and Model Risks

- The time series data is assumed to be stationary. Stationarity tests were performed with satisfactory results. However, there is a small chance it might be a false positive.

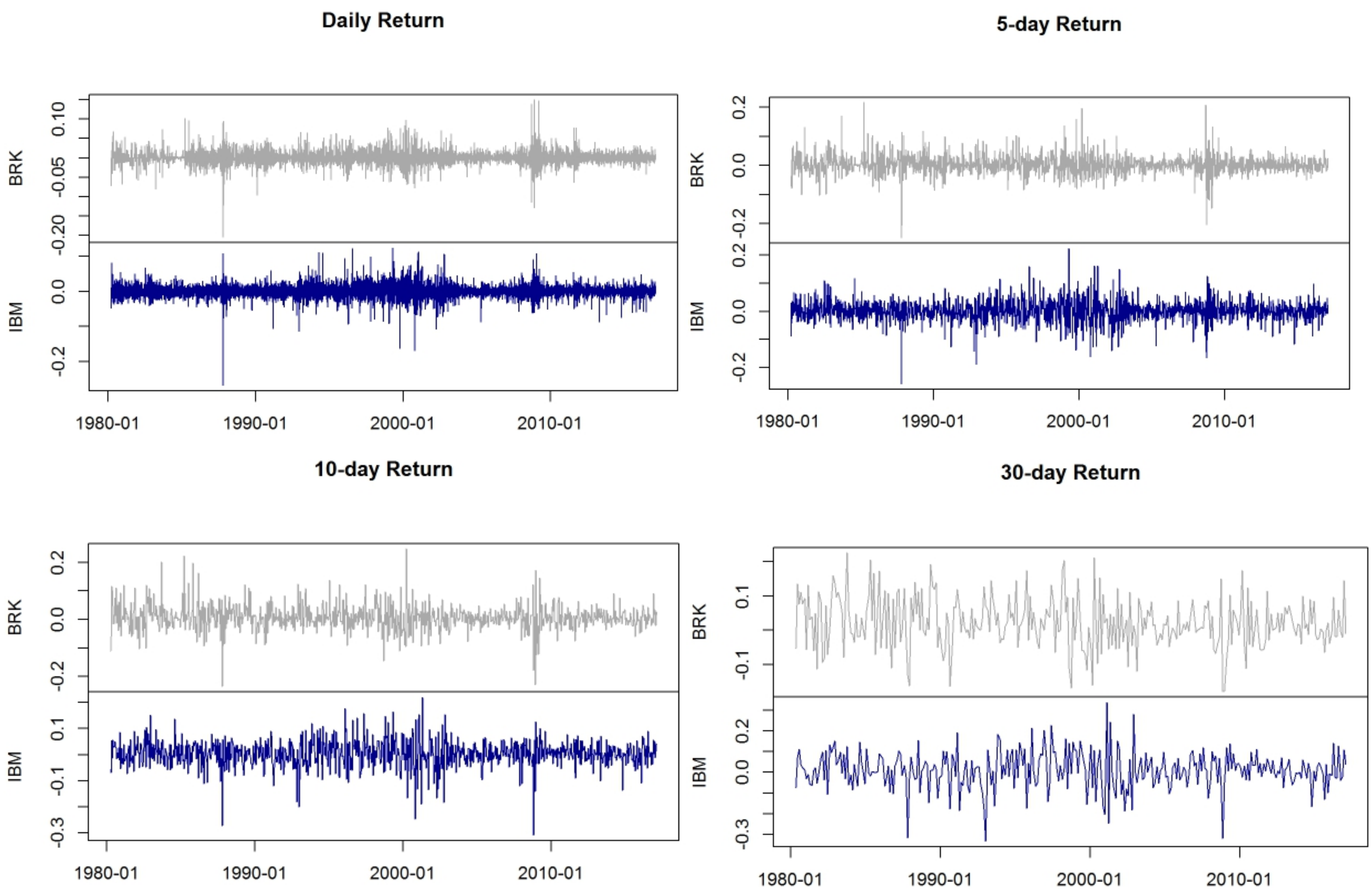
- We assumed that the stocks pay no dividend and that returns are calculated based on capital gains.

- When we split up the data into 2 different time frames, the data set might be too small to capture extreme events in the financial market. As such, the fitted multivariate distribution might not be a good distribution to reflect the returns. Furthermore, we assumed that the data set between these 2 time frames are comparable.

- VaR at high confidence level (i.e. 99.9%) will be highly dependent on the outlier of the fitted distributions. As such this is highly dependent on the number of samples generated from the multivariate distribution. If we generate too little data points, we might not be able to capture the extreme values, leading to poor estimation.

Appendix I

Time-series of BRK and IBM returns on different time horizons:



Appendix II – Stationarity Checks

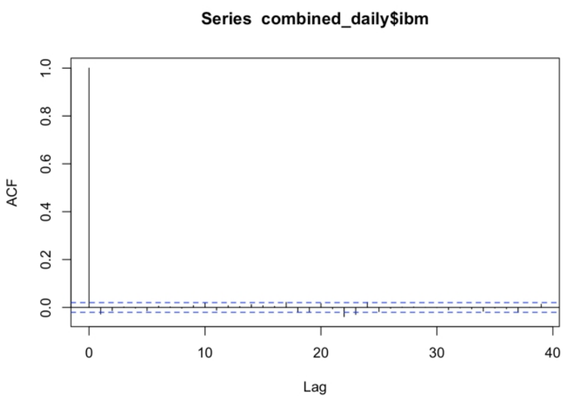
Augmented Dickey-Fuller Test Results

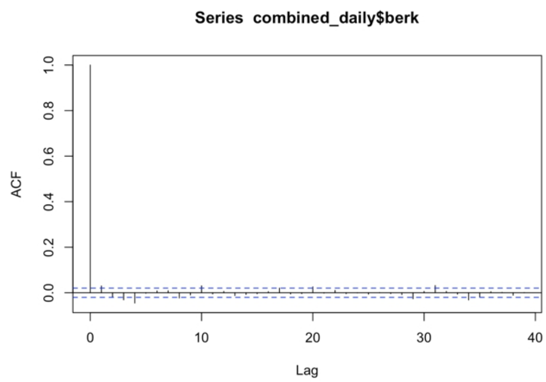
```
##
##  Augmented Dickey-Fuller Test
##
## data:  combined_daily$ibm
## Dickey-Fuller = -99.202, Lag order = 0, p-value = 0.01
## alternative hypothesis: stationary
```

```
##
##  Augmented Dickey-Fuller Test
```

```
##
## data:  combined_daily$berk
## Dickey-Fuller = -93.712, Lag order = 0, p-value = 0.01
## alternative hypothesis: stationary
```

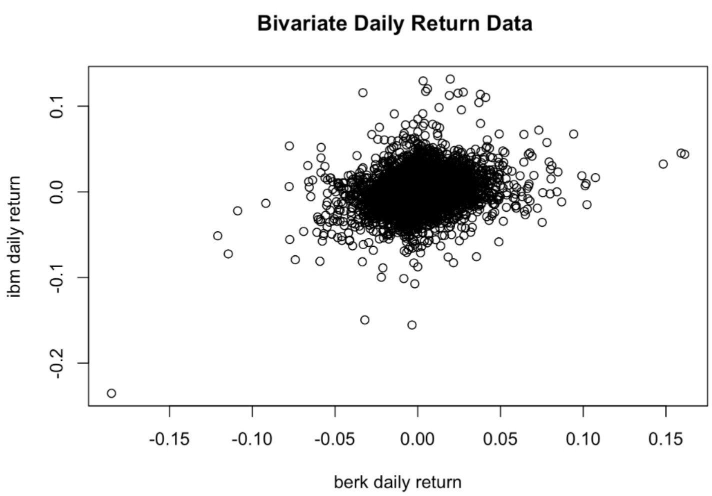
ACF plots



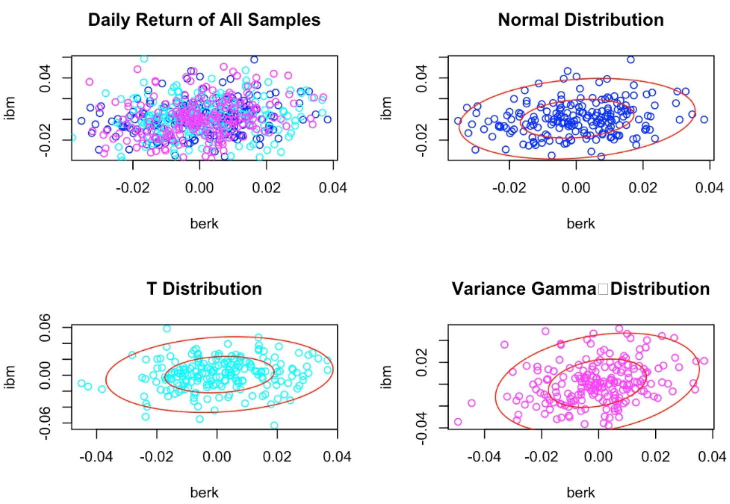


Appendix III

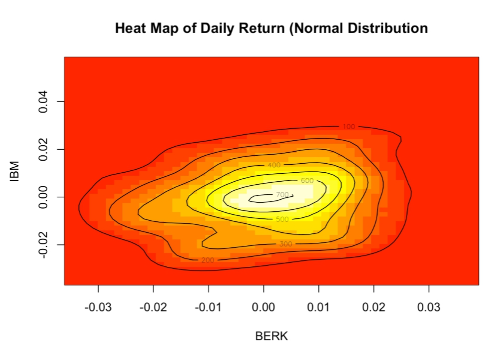
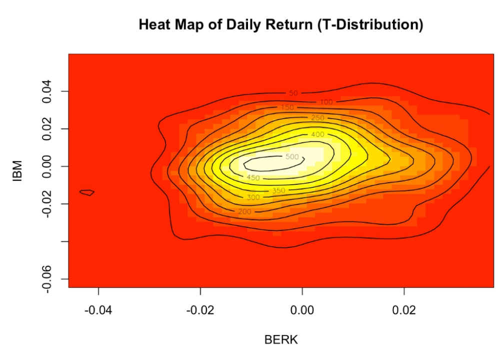
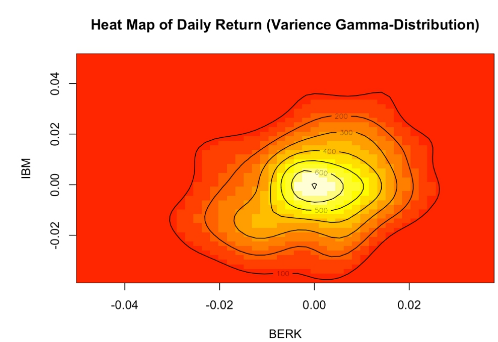
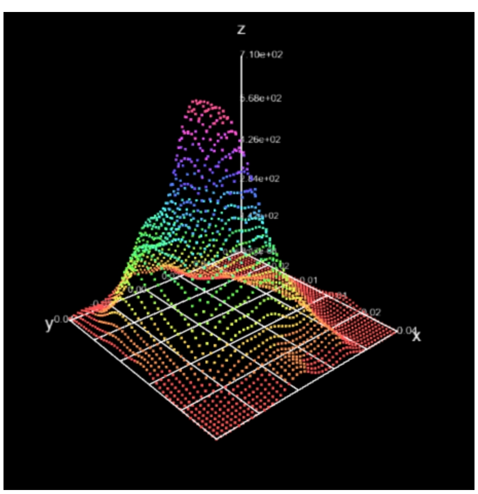
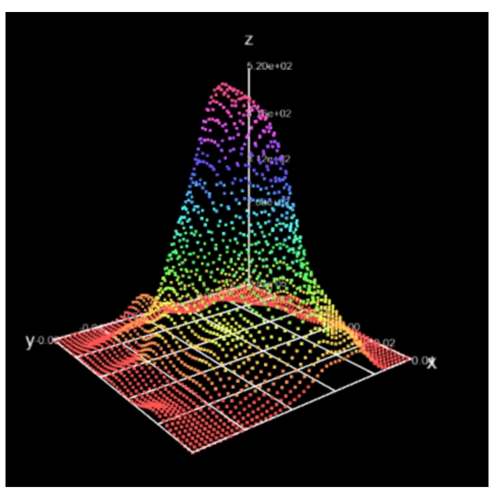
Bivariate Daily Return Data. Similar charts were plotted for 5-day, 10-day and 30-day return data.



This chart shows the actual distribution and the fitted distributions of daily returns. Similar charts were plotted for 5-day, 10-day and 30-day return data.



The charts below show the heat maps of daily return distributions and the threejs Javascript plots. Similar charts were plotted for 5-day, 10-day and 30-day return data.

		
Normal distribution 	Student-t distribution 	Variance Gamma distribution