Quantum Algorithms

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CWI

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QuSoft
Research Center for Quantum Software

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- Goal: Get an understanding of how quantum algorithms work <u>and</u> how they do not work.
- We will focus on the techniques and insight.
  (The not so famous, but really cool stuff)

Overview

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- A quick quiz!

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- For a qubit:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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- X changes $|0\rangle$ into $|1\rangle$ and vice versa.
- Z adds a -1 in front of $|1\rangle$.
- H changes $|0\rangle$ and $|1\rangle$ into $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$.

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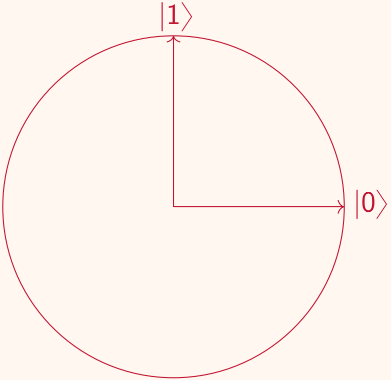

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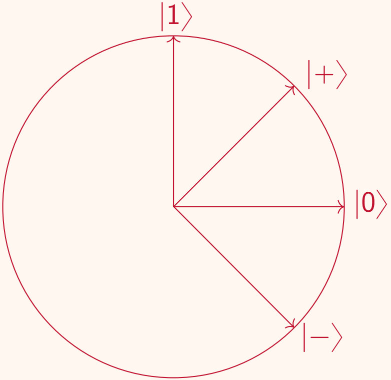

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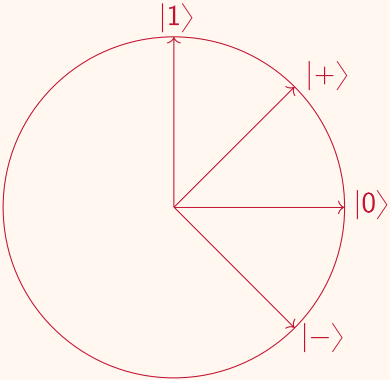
This is a X gate! Z is just X in the $\{|+\rangle, |-\rangle\}$ basis (and vice versa).

Reflections

We can also see this in our image.



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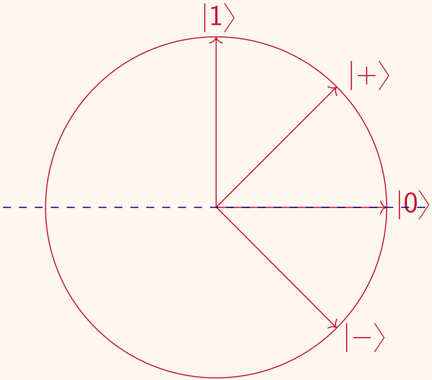

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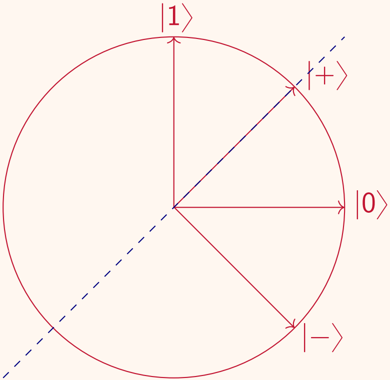

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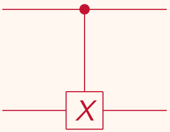

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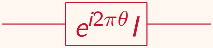

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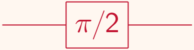

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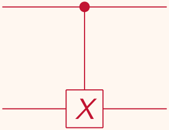

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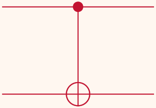

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- We often want a controlled phase oracle:

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How would you convert between the two types of oracle?

Amplitude amplification

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$$\mathcal{O}\left(\frac{1}{p}\right)$$

  times.

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$\Rightarrow \mathcal{O}(n)$ repetitions suffice.

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- $|G\rangle |1\rangle$ is the "Good" part of the state, $|B\rangle |0\rangle$ is the "Bad" part.

Quantum algorithms with small success probability

Back to quantum:

- Quantum algorithms are inherently random.
- A Hadamard gate can be used to flip a coin:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

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- Is we just measure then the success probability is $p = |\alpha_G|^2$.

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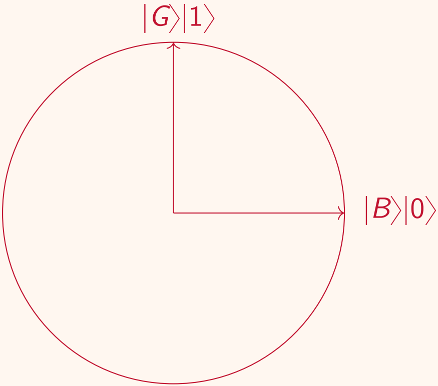
Three states - a picture



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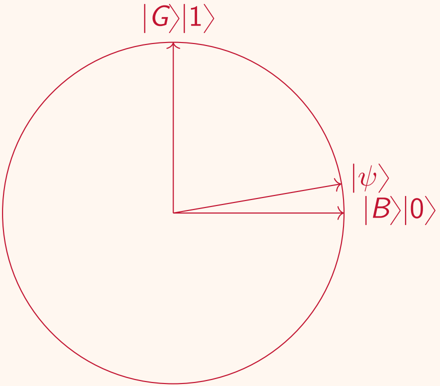

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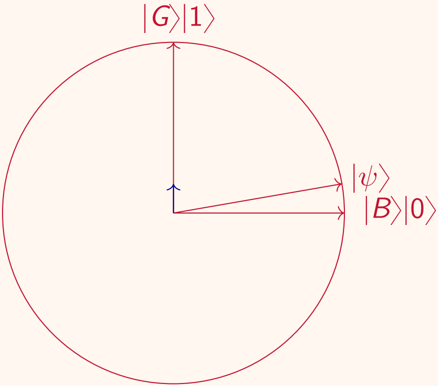

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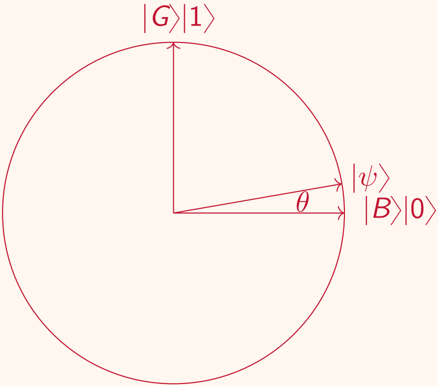

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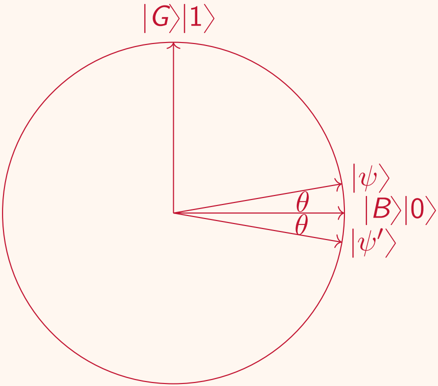

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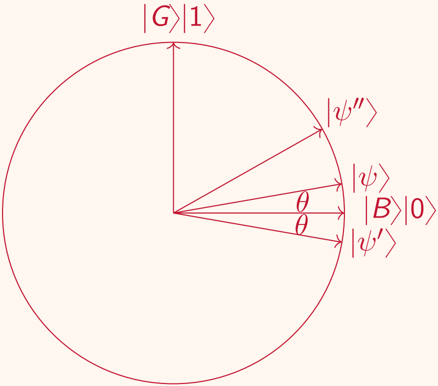

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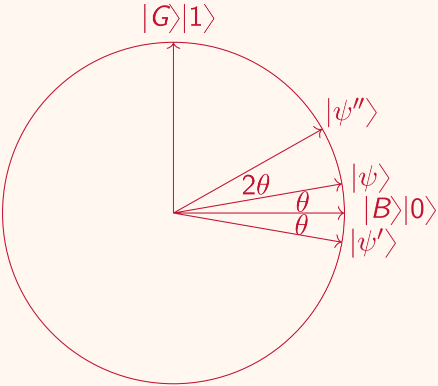

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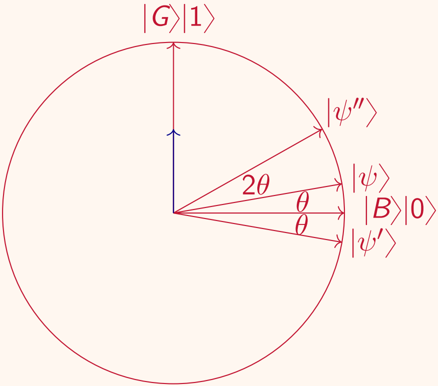

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- Nice, but can we actually implement these reflections?

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Use that $|\psi\rangle = U|0\rangle$:

1. Apply U^{-1} to map $|\psi\rangle$ to $|0\rangle$.
2. Reflect through $|0\rangle$.
3. Apply U to map $|0\rangle$ to back to $|\psi\rangle$.

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Before amplitude amplification there was Grover ('96).

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- Randomly pick an i and repeat: $\mathcal{O}(n/k)$ queries.

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Let U be:

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With $\mathcal{O}\left(\sqrt{\frac{n}{k}}\right)$ iterations we are done!

Phase estimation

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Eigenvectors and eigenvalues

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In general for a unitary U and eigenvector $|\psi\rangle$ of U:

$$U|\psi\rangle = e^{i2\pi\theta}|\psi\rangle$$

Remember: $e^{i\theta}$ is called a phase.

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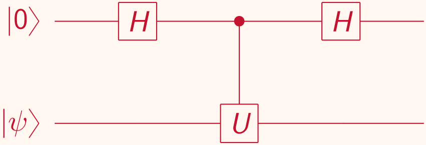

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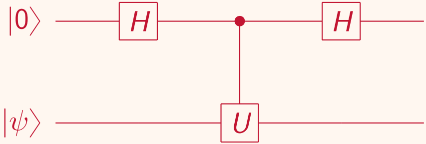

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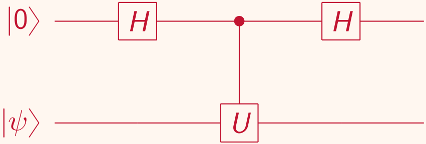

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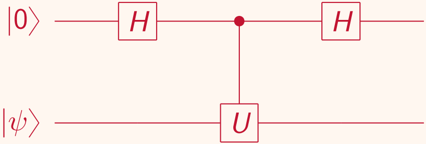

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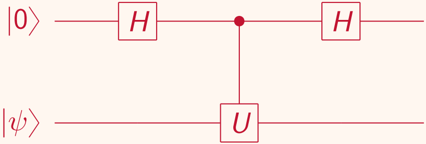

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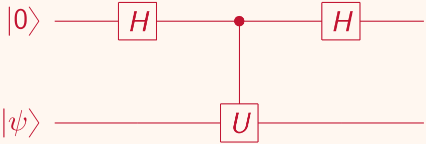

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- If this is only one bit ($\theta = 0.0$ or $\theta = 0.1 = \frac{1}{2}$) then we know what to do.

Last bit phase estimation

Let $U|\psi\rangle = e^{i2\pi\theta}|\psi\rangle$ for $\theta = 0.b_3b_2b_1$, then we can calculate b_1!

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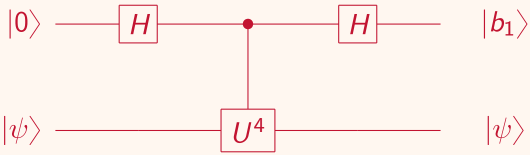
- Simply use our previous circuit to get the last bit:



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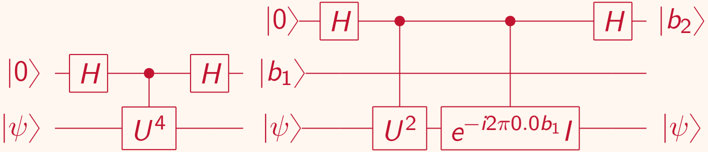

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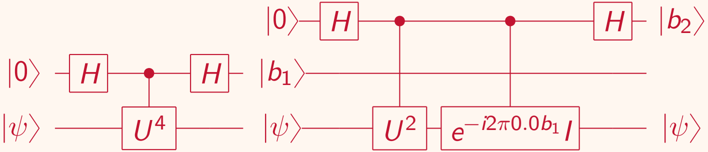

- This works because $U^2 e^{-i2\pi 0.b_1}$ adds a phase with $0.b_2b_1 - 0.0b_1 = 0.b_2$.

Three bit phase estimation

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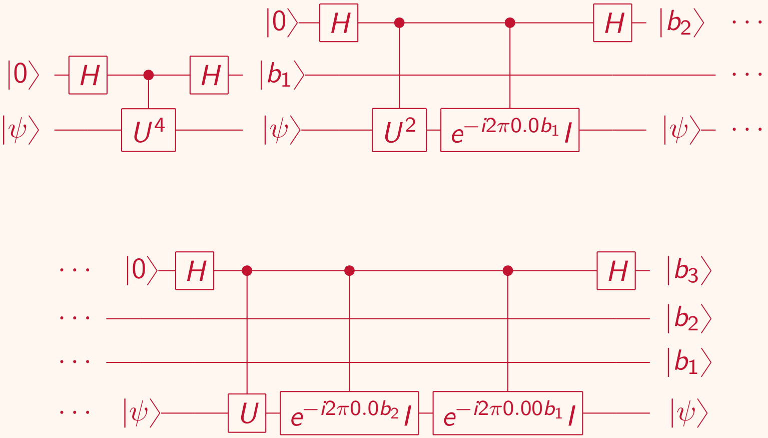
Let $U|\psi\rangle = e^{i2\pi\theta}|\psi\rangle$ for $\theta = 0.b_3b_2b_1$, can we calculate all the bits? Yes! just keep going:



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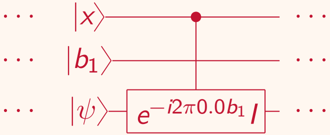

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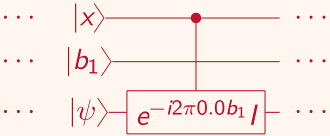

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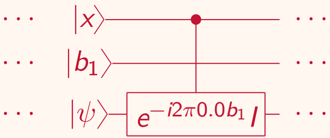

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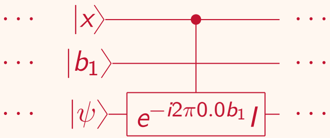

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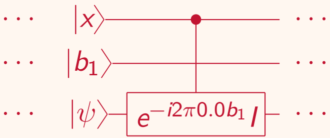

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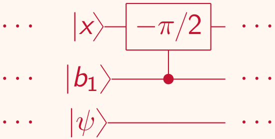

The inverse quantum Fourier transform

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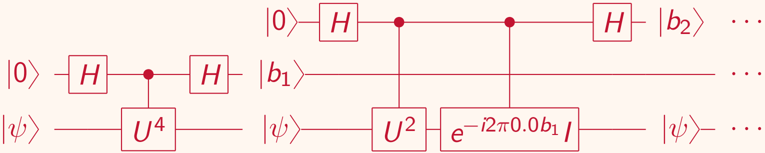

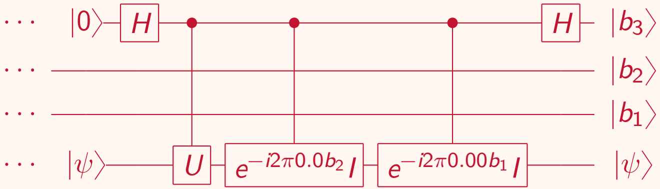

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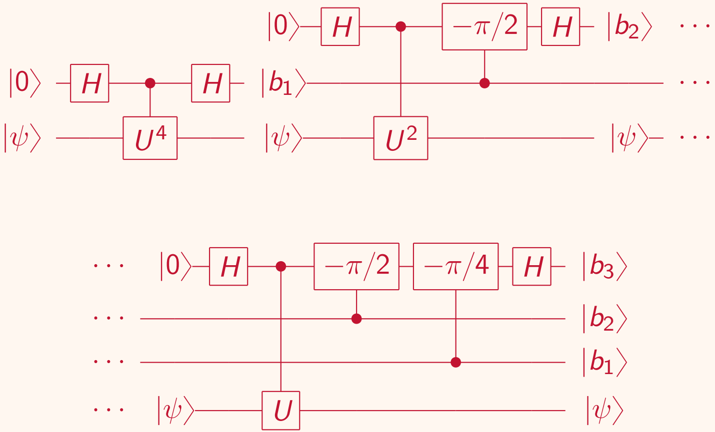

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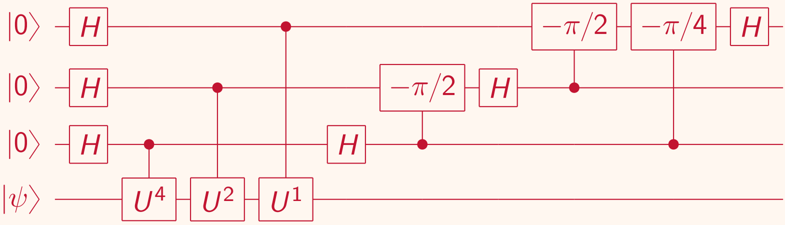

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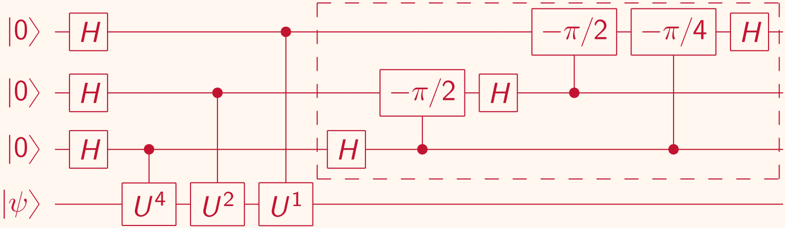

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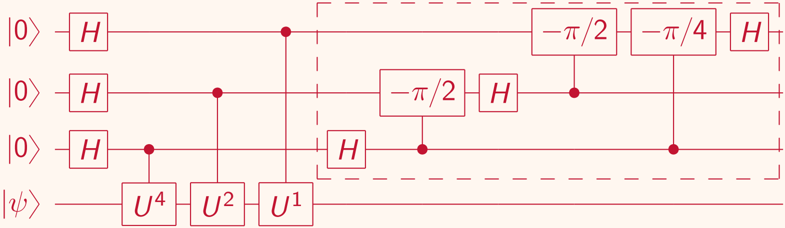

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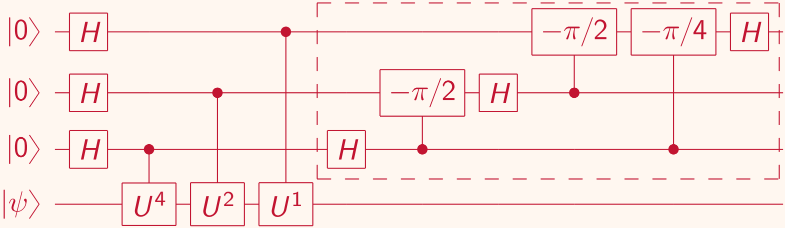

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In total we need to apply U^k for $k \sim \frac{1}{\varepsilon}$

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- Measuring the first register gives us a random θ_i, what happens to the second register?

Amplitude estimation

A problem with amplitude amplification

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- For example for search: how many solutions are there?
- Too many iterations decreases the success probability again!

A closer look at the amplitude amplification iterate

In the basis $\{|G\rangle|1\rangle, |B\rangle|0\rangle\}$ the iterate is a rotation:

$$\begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

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The eigenvectors also lie in the 2-dimensional subspace.

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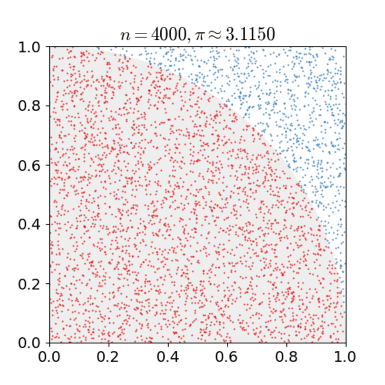
- Both tell us θ and hence the probability up to error ε in

$$\mathcal{O}\left(\frac{1}{\varepsilon}\right) \text{ uses of } U$$

Example: Speeding up Monte Carlo methods

In classical computing a Monte Carlo method samples a lot of times to estimate a probability.



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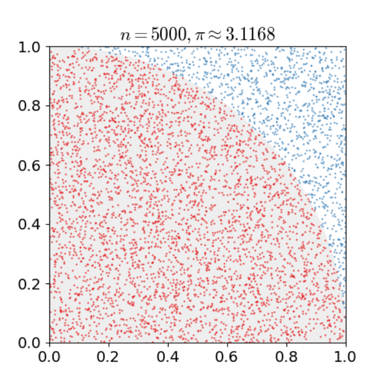

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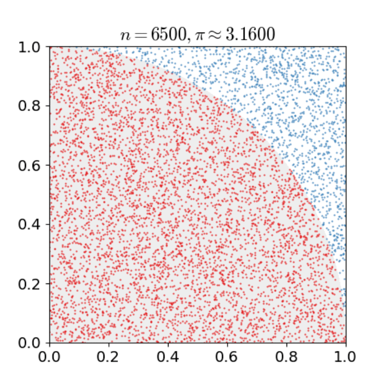

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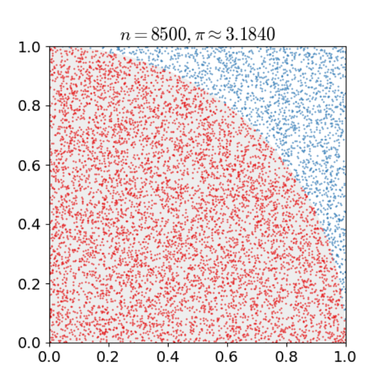

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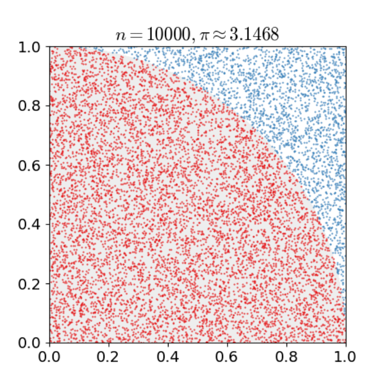

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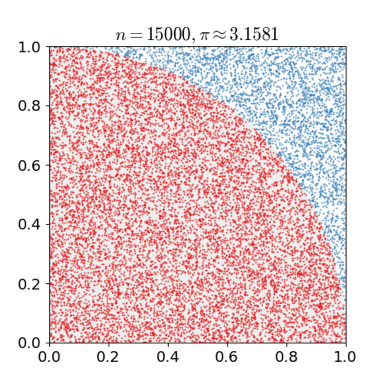

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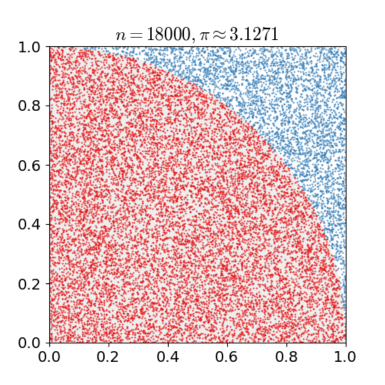

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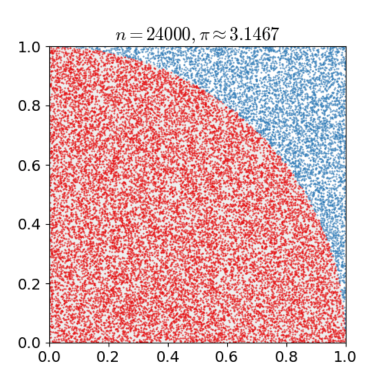

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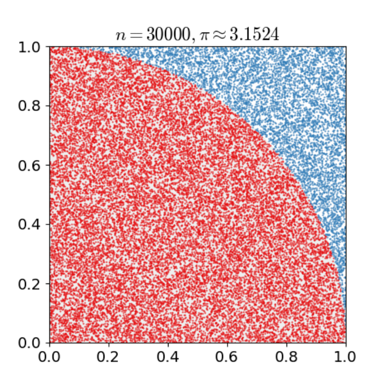

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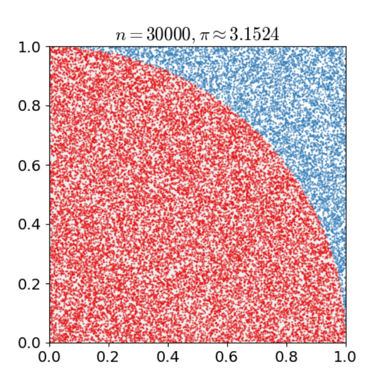

A classical algorithm requires $\sim \frac{1}{\varepsilon^2}$ samples to estimate, quantum $\frac{1}{\varepsilon}$.

Overview of some other algorithms

Some other algorithms

There are many more quantum algorithms, often using the discoed techniques. We will quickly discuss some other important algorithms:

- Shor's algorithm & factoring integers
- HHL for "solving" linear systems
- Faster gradient computation

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- Quantum computers can break this in <u>polynomial time</u>.

Shor's factoring algorithm ('94)

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- Let U be a unitary such that $U|f(a)\rangle = |f(a)\rangle$.
- Now $U^k|f(a)\rangle = |f(a+k)\rangle = |f(a)\rangle$ so $|f(a)\rangle$ has eigenvalue 1 under U^k.

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- Finding this period is believed to be hard classically.
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Shor's factoring algorithm ('94)

A rough sketch:

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- Small detail: we can implement U^k faster then k repetition, why?

HHL algorithm for systems of linear equations

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- Use phase estimation on all n coordinates at the same time!

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5. Quantum computers can solve linear systems of equations faster.
6. Quantum computers are wierd but cool.

That was it!