06 - Inference on the Mean Vector

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Multivariate vs Univariate Tests

The number of parameters in multivariate tests can be large: p - means; p - variances; $\binom{p}{2}$ covariances.

Arguments for multivariate tests:

- If we use p univariate tests - it will inflate the overall Type I error rate α, whereas multivariate test preserves exact α.
- For example, if we run $p = 10$ univariate tests on the mean at $\alpha = .05$. Let $H_0^i : \mu_i = \mu_{i0}$, $i = 1, \ldots, 10$.

$$
\begin{aligned}
&P(\text{overall Type 1 error}) \\
&\quad = P\{(\text{Reject } H_0^1) \cup \cdots \cup (\text{Reject } H_0^{10})\} \\
&\quad = 1 - P\{(\text{Fail to Reject } H_0^1) \cap \cdots \cap (\text{Fail to Reject } H_0^{10})\} \\
&\quad = 1 - 0.95^{10} = 0.40 \quad \{\text{if variables are independent - rare}\} \\
&\quad = \text{between .05 and .40} \quad \{\text{if variables are dependent}\}
\end{aligned}
$$

Multivariate vs Univariate Tests

- Univariate tests completely ignore correlations.
- Multivariate tests make direct use of the correlations.
- Multivariate tests are more powerful in many cases

  Power $=$ *probability reject* H_0 *when it is false*

- Many multivariate tests involving means have as a byproduct the construction of linear combinations of variables that reveal more about how the variables unite to reject the null hypothesis.

Univariate t-test on $H_0: \quad \mu = \mu_0$

- We will look only at the two-tailed test because 1 tailed tests do not readily generalize to the multivariate situation.
- $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$, where σ^2 is unknown, the appropriate test statistic is

$$t = \frac{(\bar{X} - \mu_0)}{s/\sqrt{n}}$$

  where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_i - \bar{X})^2$.
- The test statistic t has a student's t-distn with $n-1$ degrees of freedom.
- We reject H_0, that μ_0 is a plausible value of μ, if $|t|$ is exceeds a specified percentage point of a t-distn with $n-1$ df.

Multivariate Hotelling Test on $H_0: \quad \mu = \mu_0$

- The square of t (square distance) can be written as

$$t^2 = \frac{(\bar{X} - \mu_0)^2}{s^2/n} = n(\bar{X} - \mu_0)(s^2)^{-1}(\bar{X} - \mu_0)$$

- A multivariate generalization of the squared distance

$$\begin{aligned} T^2 &= (\bar{\mathbf{X}} - \mu_0)' \left(\frac{1}{n}\mathbf{S}\right)^{-1} (\bar{\mathbf{X}} - \mu_0) \\ &= n(\bar{\mathbf{X}} - \mu_0)'\mathbf{S}^{-1}(\bar{\mathbf{X}} - \mu_0) \end{aligned}$$

- The statistic T^2 is called Hotelling's T^2. If the observed T^2 is too large, the hypothesis $H_0 : \mu = \mu_0$ is rejected.

Hotelling Test for $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$}

- Under $H_0 : \mu = \mu_0$,

$$T^2 \text{ is distributed as } \frac{(n-1)p}{(n-p)} F_{p,n-p}$$

where $F_{p,n-p}$ denotes a r.v. with an F-distribution with p and $n-p$ df's.

- At the α level of significance, we reject H_0 in favor of H_1 if the observed

$$T^2 = n(\bar{\mathbf{X}} - \mu_0)'\mathbf{S}^{-1}(\bar{\mathbf{X}} - \mu_0) > \frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)$$

where $F_{p,n-p}(\alpha)$ is the upper $(100\alpha)th$ percentile of the $F_{p,n-p}$ distribution.

Properties of Hotelling T^2 test

- $n-1$ must be greater than p. If not, $\mathbf{S}$ is singular and no inverse exists.
- The distribution of Hotelling's T^2 when H_0 is true and $X_i \sim N_p(\mu, \Sigma)$ has two parameters - v and p.
- In the one sample case, $v = n-1$. In the two sample case, $v = n_1 + n_2 - 1$, where $n_1$and n_2 are the sample sizes of samples 1 and 2, respectively.
- Test is always 2-sided.
- In the multivariate case, $T^2_{p,n-1} = \frac{(n-1)p}{n-p} F_{p,n-p}$. Thus, if there are no T^2 table, we can use F tables.

Invariance of T^2 statistic

Suppose there are changes in the units of measurements for $\mathbf{X}$ of the form

$$\mathbf{Y}_{(p\times 1)} = \mathbf{C}_{(p\times p)}\mathbf{X}_{(p\times 1)} + \mathbf{d}_{(p\times 1)}, \;\; \mathbf{C} \text{ nonsingular}$$

This happens usually when variable X_i is transformed to $a_i(X_i - b_i)$, where $a_i > 0, b_i$ are constants.

Given observations $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, we have

$$\bar{\mathbf{y}} = \mathbf{C}\bar{\mathbf{x}} + \mathbf{d} \quad \text{and} \quad \mathbf{S}_y = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{y}_j - \bar{\mathbf{y}})(\mathbf{y_j} - \bar{\mathbf{y}})' = \mathbf{CSC}'$$

$$\mu_{\mathbf{Y}} = E(\mathbf{Y}) = E(\mathbf{CX} + \mathbf{d}) = E(\mathbf{CX}) + \mathbf{d} = \mathbf{C}\mu + \mathbf{d}$$

Invariance of T^2 statistic

Thus, $\mathbf{T}^2$ with the $\mathbf{y}'s$ and a hypothesis value $\mu_{\mathbf{Y},0} = \mathbf{C}\mu_{\mathbf{0}} + \mathbf{d}$ is

$$\begin{aligned} T^2 &= n(\bar{\mathbf{y}} - \mu_{\mathbf{Y},0})'\mathbf{S}^{-1}(\bar{\mathbf{y}} - \mu_{\mathbf{Y},0}) \\ &= n\left(\mathbf{C}(\bar{\mathbf{x}} - \mu_0)\right)'\left(\mathbf{CSC}'\right)^{-1}\mathbf{C}(\bar{\mathbf{x}} - \mu_0) \\ &= n(\bar{\mathbf{x}} - \mu_0)'\mathbf{C}'(\mathbf{CSC}')^{-1}\mathbf{C}(\bar{\mathbf{x}} - \mu_0) \\ &= n(\bar{\mathbf{x}} - \mu_0)'\mathbf{S}^{-1}(\bar{\mathbf{x}} - \mu_0) \end{aligned}$$

Unrestricted Maximum MVN Likelihood and MLE's

- The maximum of the multivariate normal likelihood with no restriction on the values of μ and Σ is

$$\max_{\mu,\Sigma} L(\mu, \Sigma) = \frac{1}{(2\pi)^{np/2}|\hat{\Sigma}|^{n/2}} \exp\left(-\frac{1}{2} \sum_{j=1}^{n} (\mathbf{x}_j - \hat{\mu})' \Sigma^{-1} (\mathbf{x}_j - \hat{\mu}) \right)$$

  where

$$\hat{\mu} = n^{-1} \sum_{j=1}^{n} \mathbf{x}_j, \quad \hat{\Sigma} = n^{-1} \sum_{j=1}^{n} (\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'$$

  are the maximum likelihood estimates (MLE) of Σ and μ, respectively.
- We can interpret $\hat{\mu}$ and $\hat{\Sigma}$ as choices for μ and Σ that best explain the observed values of the random variable.

Linear Algebra Review

(Result 4.9) Let $\mathbf{A}$ be a $k \times k$ symmetric matrix and $\mathbf{x}$ be a $k \times 1$ vector. Then

1. $\mathbf{x}'\mathbf{A}\mathbf{x} = tr(\mathbf{x}'\mathbf{A}\mathbf{x}) = tr(\mathbf{A}\mathbf{x}\mathbf{x}')$
2. $tr(\mathbf{A}) = \sum_{i=1}^{k} \lambda_i$, where the λ_i are the eigenvalue of $\mathbf{A}$.

(Result 4.10) Given a $p \times p$ symmetric positive definite matrix $\mathbf{B}$ and a scalar $b > 0$, it follows that

$$\frac{1}{|\Sigma|^b} e^{-tr(\Sigma^{-1}\mathbf{B})/2} \leq \frac{1}{|\mathbf{B}|^b} (2b)^{pb} e^{-bp}$$

for all positive definite $\Sigma_{(p \times p)}$, with equality holding only for $\Sigma = (1/2b)\mathbf{B}$.

Restricted MVN Likelihood under $H_0 : \mu = \mu_0$

- Under the hypothesis $H_0 : \mu = \mu_0$, the MVN likelihood is

$$L(\mu_0, \Sigma) = \frac{1}{(2\pi)^{np/2}|\Sigma|^{n/2}} \exp\left(-\frac{1}{2}\sum_{j=1}^{n}(\mathbf{x}_j - \mu_0)'\Sigma^{-1}(\mathbf{x}_j - \mu_0)\right)$$

- The mean μ_0 is now fixed, but Σ can be varied to find the value that is "most likely" to have led, with μ_0 fixed, to the observed sample.

- This value is obtained by maximizing $L(\mu_0, \Sigma)$ with respect to Σ. Applying Result 4.9 on the exponent, we have

$$
\begin{aligned}
-\frac{1}{2}\sum_{j=1}^{n}&(\mathbf{x}_j-\mu_0)'\Sigma^{-1}(\mathbf{x}_j-\mu_0)\\
&=-\frac{1}{2}\sum_{j=1}^{n} tr\left[\Sigma^{-1}(\mathbf{x}_j-\mu_0)(\mathbf{x}_j-\mu_0)'\right]\\
&=-\frac{1}{2}tr\left[\Sigma^{-1}\left(\sum_{j=1}^{n}(\mathbf{x}_j-\mu_0)(\mathbf{x}_j-\mu_0)'\right)\right]
\end{aligned}
$$

Restricted MVN Maximum Likelihood under $H_0 : \mu = \mu_0$

- Let $B = \sum_{j=1}^{n} (\mathbf{x}_j - \mu_0)(\mathbf{x}_j - \mu_0)'$ and $b = n/2$, and applying Result 4.10, we have

$$
\begin{aligned}
L(\mu_0, \Sigma) = \frac{1}{(2\pi)^{pb}|\Sigma|^{b}} e^{-\frac{1}{2}tr\left[\Sigma^{-1}\mathbf{B}\right]} &\leq \frac{1}{(2\pi)^{pb}|\mathbf{B}|^{b}} (2b)^{pb} e^{-bp} \\
&= \frac{1}{(2\pi)^{np/2}|\mathbf{B}|^{n/2}} (n)^{np/2} e^{-np/2} \\
&= \frac{1}{(2\pi)^{np/2}\left(n^{-p}|\mathbf{B}|\right)^{n/2}} e^{-np/2} \\
&= \frac{1}{(2\pi)^{np/2}|\hat{\Sigma}_0|^{n/2}} e^{-np/2} \\
&= \max_{\Sigma} L(\mu_0, \Sigma)
\end{aligned}
$$

where $\hat{\Sigma}_0 = n^{-1} \sum_{j=1}^{n} (\mathbf{x}_j - \mu_0)(\mathbf{x}_j - \mu_0)'$.

Likelihood Ratio Statistic

- There is a general principle in constructing test procedures called the **likelihood ratio method**. The T^2-statistic can be derived as the likelihood ratio test of $H_0 : \mu = \mu_0$.
- To determine whether μ_0 is a plausible value of μ, the maximum of $L(\mu_0, \Sigma)$ is compared with the unrestricted maximum of $L(\mu, \Sigma)$.

Likelihood Ratio Statistic

- The resulting ratio is called the likelihood ratio statistic

$$\text{Likelihood ratio} = \Lambda = \frac{\max_{\Sigma} L(\mu_0, \Sigma)}{\max_{\mu,\Sigma} L(\mu, \Sigma)}$$

$$= \frac{\frac{1}{(2\pi)^{np/2}|\hat{\Sigma}_0|^{n/2}} e^{-np/2}}{\frac{1}{(2\pi)^{np/2}|\hat{\Sigma}|^{n/2}} e^{-np/2}}$$

$$= \left(\frac{|\hat{\Sigma}|}{|\hat{\Sigma}_0|} \right)^{n/2}$$

Wilks Lambda Statistic

- The equivalent statistic $\Lambda^{2/n} = |\hat{\Sigma}|/|\hat{\Sigma}_0|$ is called the Wilk's lambda.
- The likelihood ratio test of $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ rejects H_0 if

$$\Lambda = \left(\frac{|\hat{\Sigma}|}{|\hat{\Sigma}_0|} \right)^{n/2} = \left(\frac{|\sum_{j=1}^{n} (\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'|}{|\sum_{j=1}^{n} (\mathbf{x}_j - \mu_0)(\mathbf{x}_j - \mu_0)'|} \right)^{n/2} < c_\alpha$$

where c_α is the lower $(100\alpha)th$ percentile of the distribution of Λ.

Wilks Lambda Statistic

- Determining the distribution of Λ is complicated, but, fortunately, we can write Λ in terms of the Hotelling's T^2 statistic.
- Under $H_0 : \mu = \mu_0$,

$$\Lambda^{2/n} = \left(1 + \frac{T^2}{(n-1)}\right)^{-1}, \text{ or}$$

$$T^2 = \frac{(n-1)|\hat{\Sigma}_0|}{|\hat{\Sigma}|} - (n-1)$$

Example: Reaven and Miller (1979) link

Reaven and Miller measured five variables in a comparison of normal patients and diabetics. We use partial data for normal patients only. The three variables of major interest were

- X_1= glucose intolerance,
- X_2 = insulin response to oral glucose,
- X_3= insulin resistance.

The two additional variables of minor interest were
* Y_1 = relative weight,
* Y_2 = fasting plasma glucose.

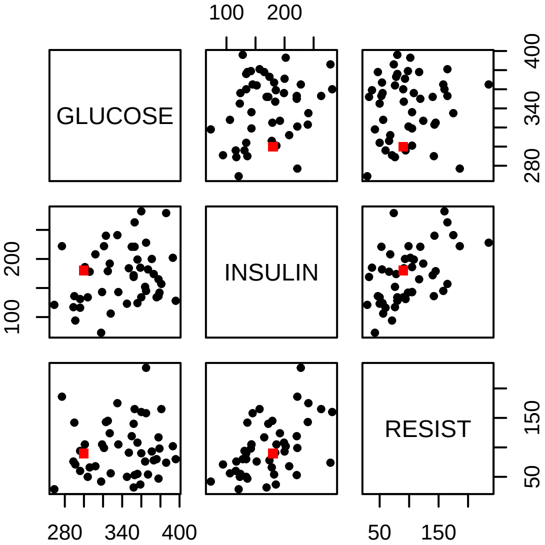
Reaven and Miller (1979): Descriptives Stats

```
patients <- read.csv("patients.csv", header = TRUE)
data.frame(Mean = colMeans(patients),
           Median = apply(patients, 2, median),
           Variance = apply(patients, 2, var))
```

```
#            Mean Median Variance
# WEIGHT     0.92   0.94  1.6e-02
# FASTING   90.41  90.00  7.1e+01
# GLUCOSE 340.83 351.00  1.1e+03
# INSULIN 171.37 170.50  2.4e+03
# RESIST   97.78  92.00  2.1e+03
```

```
X <- with(patients, cbind(GLUCOSE, INSULIN, RESIST))
```

$\mu_0 = (300, 180, 90)$ hypothesized mean, boundary levels



Test to see if the mean vector is different from specified vector

$$H_0 : \mu = [300, 180, 90]' \text{ vs } H_1 : \mu \neq [300, 180, 90]'$$

```
n <- nrow(X) # sample size
p <- ncol(X) # number of variables
(Xbar <- colMeans(X)) # sample mean
```

```
# GLUCOSE INSULIN  RESIST
#     341     171      98
```

```
(S <- cov(X))  # sample covariance matrix
```

```
#         GLUCOSE INSULIN RESIST
# GLUCOSE    1106     397    108
# INSULIN     397    2382   1143
# RESIST      108    1143   2136
```

```
Sinv <- matrixcalc::matrix.inverse(S)
```

Using raw code to compute T^2

```
# hypothesized mean
mu0 <- c(300, 180, 90)
# Hotelling T2
T2 <- n*t(Xbar - mu0)%*%Sinv%*%(Xbar - mu0)
# Critical Value at 5% level of significance
cval <- ((n-1)*p/(n-p))*qf(1 - 0.05, df1 = p, df2 = n-p)
# transformed T2 statistic,
# needed if compared to F crit value
T2.F <- T2/((n-1)*p/(n-p))
# Wilks Lambda
W <- (1 + T2/(n-1))^(-n/2)
```

```r
data.frame(T2 = T2, df1 = p, df2 = n-p,
           Fcrit = cval, T2.trans = T2.F, Wilks = W)
```

```
#   T2 df1 df2 Fcrit T2.trans   Wilks
# 1 87   3  43   8.9       28 1.7e-11
```

```r
# Since T2 = 87 > critical value = 8.9, then we reject H0
# at 5% level of significance.
```

Using HotellingsT2() from package ICSNP

`HotellingsT2()` statistic is the transformed T^2 so that it has the unscaled F-distribution.

```
library(ICSNP) # install ICSNP package
(patients.T2 <- HotellingsT2(X, mu = mu0))
```

```
#
#   Hotelling's one sample T2-test
#
# data:  X
# T.2 = 30, df1 = 3, df2 = 40, p-value = 4e-10
# alternative hypothesis: true location is not equal to c(3
```

```
patients.T2$statistic
```

```
# T.2
#  28
```