02 - Review of Basic Matrix Results

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Review of Basic Matrix Results

Given matrices $\mathbf{A}$ and $\mathbf{B}$ of appropriate dimensions

1. Transposition: $(\mathbf{A}+\mathbf{B})' = \mathbf{A}' + \mathbf{B}' \quad (\mathbf{AB})' = (\mathbf{B}'\mathbf{A}')$
2. Trace: Given a square matrix $\mathbf{A}$
   - $tr(\mathbf{A}) = \sum diag(\mathbf{A})$
   - $tr(k\mathbf{A}) = k \cdot tr(\mathbf{A})$, $tr(\mathbf{A}') = tr(\mathbf{A})$
   - $tr(\mathbf{A}+\mathbf{B}) = tr(\mathbf{A}) + tr(\mathbf{B}), \quad tr(\mathbf{AB}) = tr(\mathbf{BA})$
3. Determinant
   - $|\mathbf{A}'| = |\mathbf{A}|, \quad |k\mathbf{A}| = k^n|\mathbf{A}|, \quad |A^{-1}| = 1/|\mathbf{A}|$
   - $\begin{vmatrix} \mathbf{T} & \mathbf{U} \\ \mathbf{V} & \mathbf{W} \end{vmatrix} = |\mathbf{T}| \cdot |\mathbf{W} - \mathbf{V}\mathbf{T}^{-1}\mathbf{U}|$
   - $\begin{vmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{W} \end{vmatrix} = |\mathbf{T}| \cdot |\mathbf{W}|$

Inverse Matrices

1. Inverse of a sum of Matrices:

$$(\mathbf{R}+\mathbf{STU})^{-1} = \mathbf{R}^{-1} - \mathbf{R}^{-1}\mathbf{S}(\mathbf{T}^{-1}+\mathbf{UR}^{-1}\mathbf{S})^{-1}\mathbf{UR}^{-1}$$

2. Inverse of a partitioned matrix:

$$\begin{pmatrix} \mathbf{T} & \mathbf{U} \\ \mathbf{V} & \mathbf{W} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{T}^{-1}+\mathbf{T}^{-1}\mathbf{UQ}^{-1}\mathbf{VT}^{-1} & -\mathbf{T}^{-1}\mathbf{UQ}^{-1} \\ -\mathbf{Q}^{-1}\mathbf{VT}^{-1} & \mathbf{Q}^{-1} \end{pmatrix}$$

where $\mathbf{Q} = \mathbf{W} - \mathbf{VT}^{-1}\mathbf{U}$.

Linear Independence

1. A list of $v_1, \ldots, v_m$ vectors in $\mathbb{R}^p$ is called **linearly independent** if the only choice of scalars $c_1, \ldots, c_m \in \mathbb{R}$ that makes $c_1 v_1 + \cdots + c_k v_k$ equal 0 vector is $c_1 = \cdots = c_m = 0$.

2. The (subset) columns $v_1, \ldots, v_j$ of matrix $\mathbf{A}$ is **linearly independent** if the only choice of scalars $c_1, \ldots, c_j \in \mathbb{R}$ that makes $c_1 v_1 + \cdots + c_j v_j$ equal 0 vector is $c_1 = \cdots = c_j = 0$.

3. The rank of matrix $\mathbf{A}$ is the maximum number of linearly independent columns of $\mathbf{A}$.

Eigenvalues and Eigenvectors

1. A scalar λ is said to be an **eigenvalue** of $p \times p$ matrix $\mathbf{A}$ if there exists an $p \times 1$ vector x such that

$$\mathbf{Ax} = \lambda \mathbf{x}$$

then x is an **eigenvector** of $\mathbf{A}$. By Cramer's Rule, the eigenvalues λ of $\mathbf{A}$ satisfies the characteristic equation

$$|\mathbf{A} - \lambda I| = 0 \quad \text{(Why?)}$$

- There is a correspondence between square matrices $\mathbf{A}$ and linear transformations T defined by $T(x) = \mathbf{A}x$.
- Geometrically an eigenvector of $\mathbf{A}$ points in a direction that is stretched by the linear transformation and the eigenvalue is the factor by which it is stretched. If the eigenvalue is negative, the direction is reversed.

2. Given $\mathbf{A}$ with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, with multiplicities $t_1, t_2, \ldots, t_k$.

- $Rank(A)$ equals the number of nonzero eigenvalues
- $tr(A) = \sum_{i=1}^{k} t_i \lambda_i$
- $|A| = \prod_{i=1}^{k} \lambda_i^{t_i}$

```
(A <- matrix(c(13, -4, 2, -4, 11, -2, 2, -2, 8), 3, 3,
             byrow=TRUE))
```

```
#      [,1] [,2] [,3]
# [1,]   13   -4    2
# [2,]   -4   11   -2
# [3,]    2   -2    8
```

```
# Use `eigen()` to get the eigenvalues/eigenvectors
# pairs. Returns a named list.
ev <- eigen(A)
```

```r
(values <- ev$values) # extract eigenvalues
```

```
# [1] 17  8  7
```

```r
(vectors <- ev$vectors) # extract eigenvectors
```

```
#            [,1]       [,2]      [,3]
# [1,]  0.7453560  0.6666667 0.0000000
# [2,] -0.5962848  0.6666667 0.4472136
# [3,]  0.2981424 -0.3333333 0.8944272
```

The eigenvalues are always returned in decreasing order, and each column of `vectors` corresponds to the elements in `values`.

Visualizing Eigenvectors and Eigenvalues[1]

[1] `http://setosa.io/ev/eigenvectors-and-eigenvalues/`

Special Matrices

$\mathbf{A}$ is a real symmetric matrix if $\mathbf{A}^T = \mathbf{A}$ and all entries in $\mathbf{A}$ are real numbers.

Properties of a a real symmetric matrix $\mathbf{A}$
* all eigenvalues are real
* eigenvectors corresponding to a distinct eigenvalues are orthogonal, $x_i^T x_j = 0$ if $\lambda_i \neq \lambda_j$

$$\begin{aligned} \text{positive definite} &\iff \mathbf{x}'\mathbf{A}\mathbf{x} > 0 \ \text{ for all vector } \mathbf{x} \\ &\iff \text{all eigenvalues of } \mathbf{A} \text{ are } \ > 0 \\ &\iff \text{is non-singular} \end{aligned}$$

$$\begin{aligned} \text{positive semi-definite} &\iff \mathbf{x}'\mathbf{A}\mathbf{x} \geq 0 \ \text{ for all vector } \mathbf{x} \\ &\iff \text{all eigenvalues of } \mathbf{A} \text{ are } \ \geq 0 \end{aligned}$$

Verify properties of eigenvalues/eigenvectors.

Eigenvectors are orthogonal, $V'V = I$

```r
t(vectors)%*%vectors # same as `crossprod(vectors)`
```

```
#              [,1]         [,2]         [,3]
# [1,] 1.000000e+00 3.053113e-16 5.551115e-17
# [2,] 3.053113e-16 1.000000e+00 0.000000e+00
# [3,] 5.551115e-17 0.000000e+00 1.000000e+00
```

```r
zapsmall(crossprod(vectors)) # rounding small #'s
```

```
#      [,1] [,2] [,3]
# [1,]    1    0    0
# [2,]    0    1    0
# [3,]    0    0    1
```

Trace and Determinant of a Matrix in R

```r
library(matrixcalc) # load `matrixcalc` package
matrix.trace(A)  # trace
```

```
# [1] 32
```

```r
sum(values) # verify with sum of eigenvalues
```

```
# [1] 32
```

```r
det(A) # determinant of a matrix
```

```
# [1] 952
```

```r
prod(values) # verify with product of eigenvalues
```

```
# [1] 952
```

Rank and Inverse of a Matrix

```r
matrix.rank(A) # rank of a matrix, need `matlib`
```

```
# [1] 3
```

```r
sum(values != 0) # number of non-zero eigenvalues
```

```
# [1] 3
```

```r
(A.inv <- matrix.inverse(A)) # inverse
```

```
#             [,1]       [,2]        [,3]
# [1,]  0.08823529 0.02941176 -0.01470588
# [2,]  0.02941176 0.10504202  0.01890756
# [3,] -0.01470588 0.01890756  0.13340336
```

Inverse of A: $A^{-1}A = I$

```
zapsmall(A.inv %*% A) # check
```

```
#      [,1] [,2] [,3]
# [1,]    1    0    0
# [2,]    0    1    0
# [3,]    0    0    1
```

Check whether A Positive Definite and Positive Semi-Definite

```
c(is.positive.definite(A), is.positive.semi.definite(A))
```

```
# [1] TRUE TRUE
```

Singular, Idempotent, Orthogonal Matrix

Singular Matrix: $\mathbf{A}$ $k \times k$ matrix is singular if $Rank(\mathbf{A}) < k$

Idempotent matrix: Let $\mathbf{A}$ be a $k \times k$ matrix, $\mathbf{A}$ is idempotent if

$$\mathbf{A} \cdot \mathbf{A} = \mathbf{A}$$

Orthogonal matrix: A square matrix A is orthogonal if

$$\mathbf{A}'\mathbf{A} = \mathbf{A}\mathbf{A}' = \mathbf{I_k}$$

if $\mathbf{A}$ is non-singular $\mathbf{A}' = \mathbf{A}^{-1}$

New Matrix Verification

```
B <- matrix( c( 2, -1, 2, -1, 2, -1, 2, -1, 2 ),
             nrow=3, byrow=TRUE )
c(is.positive.definite(B), is.positive.semi.definite(B))
```

```
# [1] FALSE  TRUE
```

```
eigen(B)$values # one eigenvalue is zero
```

```
# [1] 4.732051e+00 1.267949e+00 8.881784e-16
```

```
c(is.singular.matrix(B), is.idempotent.matrix(B))
```

```
# [1]  TRUE FALSE
```

Review other matrix results.

1. Our Textbook
2. The Matrix Cookbook by Petersen and Pedersen (2012)[2]
3. Linear Algebra Abridged by Sheldon Axler (2016)[3]

[2] http://www2.imm.dtu.dk/pubdb/views/publication_details.php?id=3274

[3] http://linear.axler.net/LinearAbridged.pdf