

07 - Confidence Regions for the Mean

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Confidence Regions for μ

- The set of all μ satisfying this inequality form an ellipsoid.
- For $p > 3$, this is hard to visualize and so this equality is of more mathematical interest than of practical use.
- The hypothesized mean value μ_0 lies within the confidence region if the computed the generalized square distance satisfies

$$n(\bar{\mathbf{X}} - \mu_0)' \mathbf{S}^{-1} (\bar{\mathbf{X}} - \mu_0) \leq \frac{(n-1)p}{(n-p)} F_{p, n-p}(\alpha)$$

- This approach is analogous to testing

$$H_0 : \mu = \mu_0 \quad \text{vs} \quad H_0 : \mu \neq \mu_0$$

where T^2 -test would not reject H_0 when

$$T^2 \leq \frac{(n-1)p}{(n-p)} F_{p, n-p}(\alpha).$$

Simultaneous Confidence Statements for $\mathbf{a}'\mu$

- A $100(1 - \alpha)\%$ simultaneous confidence intervals involving the $(p \times 1)$ vector \mathbf{a} for $\mathbf{a}'\mu$ is

$$\mathbf{a}'\bar{\mathbf{X}} - \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{\mathbf{a}'\mathbf{S}\mathbf{a}}{n}} \leq \mathbf{a}'\mu \leq \mathbf{a}'\bar{\mathbf{X}} + \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{\mathbf{a}'\mathbf{S}\mathbf{a}}{n}}$$

- The Simultaneous T^2 confidence intervals for all $\mathbf{a}'\mu$ are just the shadows (projection), of the confidence ellipsoid on the component axes.

In particular, if we let $\mathbf{a}' = [0, \dots, 0, 1, 0, \dots, 0]$ where 1 is on the i th row of \mathbf{a} , then a $100(1 - \alpha)\%$ confidence interval for $\mathbf{a}'\mu = \mu_i$ ($p = 1$) is

$$\bar{X}_i - \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{s_{ii}}{n}} \leq \mu_i \leq \bar{X}_i + \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{s_{ii}}{n}}$$

$$\bar{X}_i - \sqrt{F_{1,n-1}(\alpha)} \sqrt{\frac{s_{ii}}{n}} \leq \mu_i \leq \bar{X}_i + \sqrt{F_{1,n-1}(\alpha)} \sqrt{\frac{s_{ii}}{n}}$$

$$\bar{X}_i - t_{n-1}(\alpha/2) \sqrt{\frac{s_{ii}}{n}} \leq \mu_i \leq \bar{X}_i + t_{n-1}(\alpha/2) \sqrt{\frac{s_{ii}}{n}}$$

Simultaneous Confidence Statements for $\mathbf{a}'\mu$

- We can also make statements about the differences $\mu_i - \mu_k$ corresponding to $\mathbf{a}' = [0, \dots, 0, a_i, 0, \dots, a_k, \dots, 0]$, where $a_i = 1$ and $a_k = -1$. In this case ($p = 2$), $\mathbf{a}'\mathbf{S}\mathbf{a} = s_{ii} - 2s_{ik} + s_{kk}$, we have the interval

$$\begin{aligned} (\bar{X}_i - \bar{X}_k) - \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{s_{ii} - 2s_{ik} + s_{kk}}{n}} \leq \mu_i \leq \\ (\bar{X}_i - \bar{X}_k) + \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{s_{ii} - 2s_{ik} + s_{kk}}{n}} \end{aligned}$$

- Which set of intervals is better (smaller) depends on the relative sizes of n and p , and even the number of means compared, say m μ_i 's.
- Perhaps the best way is to calculate both sets (critical values) and use the set yielding the narrower intervals.

Patients Example : Simultaneous Statentents

Test to see if $\mu_2 = \mu_5$.

$$H_0 : \mu_2 = \mu_5 \quad \text{vs} \quad H_1 : \mu_2 \neq \mu_5$$

```
patients <- read.csv("patients.csv", header=TRUE)
(Xbar <- colMeans(patients))
```

```
# WEIGHT FASTING GLUCOSE INSULIN RESIST
#      0.92    90.41   340.83   171.37    97.78
```

```
S <- cov(patients) # cov matrix
n <- nrow(patients)
```

We want to see if $\mu_2 = \mu_5$.

$$H_0 : \mu_2 = \mu_5 \quad \text{vs} \quad H_1 : \mu_2 \neq \mu_5$$

```
p <- 2 # comparing two means
(cval <- ((n-1)*p/(n-p))*qf(0.95,
                             df1 = p, df2 = n-p)) # F critical value
```

```
# [1] 6.6
```

```
a <- c(0,1,0,0,-1)
Ybar <- t(a)%*%Xbar
SY <- t(a)%*%S%*%a
LL <- Ybar - sqrt(cval)*sqrt(SY/n)
UL <- Ybar + sqrt(cval)*sqrt(SY/n)
data.frame(Mean.D = Ybar, Lower.lim = LL, Upper.lim = UL)
```

```
#   Mean.D Lower.lim Upper.lim
# 1   -7.4      -25      11
```

The 95% simultaneous confidence interval for $\mu_2 - \mu_5$ is (-25, 11). Since 0 is inside the confidence interval, then it is plausible that H_0 holds.

Bonferroni Method of Multiple Comparisons

- Suppose prior to the collection of data, confidence statements about m linear combinations $\mathbf{a}'_1\mu, \mathbf{a}'_2\mu, \dots, \mathbf{a}'_m\mu$ are required.
- Let C_i denote the confidence statement about the value $\mathbf{a}'_i\mu$ with $P(C_i \text{ true}) = 1 - \alpha_i, i = 1, 2, \dots, m$.

$$P[\text{all } C_i \text{ true}] \geq 1 - \sum_{i=1}^m \alpha_i$$

- A special case of the Bonferroni allows the investigator to control the overall error rate $\sum_{i=1}^m \alpha_i$, regardless of the correlation structure.
- We consider the individual t-intervals

$$\bar{X}_i \pm t_{n-1} \left(\frac{\alpha_i}{2} \right) \sqrt{\frac{s_{ii}}{n}}, \quad i = 1, 2, \dots, m,$$

where $\alpha_i = \alpha/m$.


```
p <- 5; alpha <- 0.05
tci <- qt(1 - alpha/2, df=n-1)
bc.tci <- qt(1 - alpha/(2*p), df=n-1)
T2ci <- sqrt(((n-1)*p/(n-p))*qf(0.95, df1=p, df2=n-p))
CI <- function(cval, Xbar, S, n){
  cbind( Xbar - cval * sqrt(diag(S/n)) ,
         Xbar + cval * sqrt(diag(S/n))) }

# t-intervals
t <- CI(tci, Xbar, S, n)

# bonferroni corrected t-intervals
bct <- CI(bc.tci, Xbar, S, n)

# Hotelling T2-intervals
T2 <- CI(T2ci, Xbar, S, n)
```

```

# Confidence Intervals for the mean vector with
# alpha = 0.05
# One-at-a-time t, bonferroni corrected t,
# and T2 intervals
colnames(t) <- colnames(bct) <-
      colnames(T2) <- c("LL", "UL")
data.frame(t = t, Bonf.t = bct, T2 = T2)

```

#	t.LL	t.UL	Bonf.t.LL	Bonf.t.UL	T2.LL	T2.UL
# WEIGHT	0.88	0.96	0.87	0.97	0.85	0.99
# FASTING	87.92	92.91	87.08	93.74	85.88	94.95
# GLUCOSE	330.95	350.70	327.64	354.02	322.87	358.78
# INSULIN	156.88	185.86	152.02	190.72	145.02	197.72
# RESIST	84.06	111.51	79.45	116.11	72.83	122.74

Comparison of Interval Widths

```
# One-at-a-time t, bonferroni corrected t,  
# and T2 intervals  
data.frame(t.width = t[,2] - t[,1],  
            Bonf.t.width = bct[,2] - bct[,1],  
            T2.width = T2[,2] - T2[,1])
```

#	t.width	Bonf.t.width	T2.width
# WEIGHT	0.076	0.1	0.14
# FASTING	4.989	6.7	9.07
# GLUCOSE	19.756	26.4	35.92
# INSULIN	28.986	38.7	52.70
# RESIST	27.452	36.7	49.91

Observations

- In general, the width of T^2 -intervals, relative to t and bonferroni corrected t intervals, increases as p increases (for fixed n) and decreases as n increases (for fixed p).
- The confidence level associated with any collection of T^2 -intervals, for fixed n and p , is $1 - \alpha$, and the overall confidence associated with a collection of t intervals, for the same n , can be much less.
- The bonferroni correction guarantees that overall confidence level is greater than or equal to 0.95.
- Because Bonferroni correction is easy to apply and provide relatively short confidence intervals; it often used in practice.