

## 07 - Confidence Regions for the Mean

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## Confidence Regions for $\mu$

- The set of all  $\mu$  satisfying this inequality form an ellipsoid.
- For  $p > 3$ , this is hard to visualize and so this equality is of more mathematical interest than of practical use.
- The hypothesized mean value  $\mu_0$  lies within the confidence region if the computed the generalized square distance satisfies

$$n(\bar{\mathbf{X}} - \mu_0)' \mathbf{S}^{-1} (\bar{\mathbf{X}} - \mu_0) \leq \frac{(n-1)p}{(n-p)} F_{p, n-p}(\alpha)$$

- This approach is analogous to testing

$$H_0 : \mu = \mu_0 \quad \text{vs} \quad H_0 : \mu \neq \mu_0$$

where  $T^2$ -test would not reject  $H_0$  when

$$T^2 \leq \frac{(n-1)p}{(n-p)} F_{p, n-p}(\alpha).$$

## Simultaneous Confidence Statements for $\mathbf{a}'\mu$

- A  $100(1 - \alpha)\%$  simultaneous confidence intervals involving the  $(p \times 1)$  vector  $\mathbf{a}$  for  $\mathbf{a}'\mu$  is

$$\mathbf{a}'\bar{\mathbf{X}} - \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{\mathbf{a}'\mathbf{S}\mathbf{a}}{n}} \leq \mathbf{a}'\mu \leq \mathbf{a}'\bar{\mathbf{X}} + \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{\mathbf{a}'\mathbf{S}\mathbf{a}}{n}}$$

- The Simultaneous  $T^2$  confidence intervals for all  $\mathbf{a}'\mu$  are just the shadows (projection), of the confidence ellipsoid on the component axes.

In particular, if we let  $\mathbf{a}' = [0, \dots, 0, 1, 0, \dots, 0]$  where 1 is on the  $i$ th row of  $\mathbf{a}$ , then a  $100(1 - \alpha)\%$  confidence interval for  $\mathbf{a}'\mu = \mu_i$  ( $p = 1$ ) is

$$\bar{X}_i - \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{s_{ii}}{n}} \leq \mu_i \leq \bar{X}_i + \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{s_{ii}}{n}}$$

$$\bar{X}_i - \sqrt{F_{1,n-1}(\alpha)} \sqrt{\frac{s_{ii}}{n}} \leq \mu_i \leq \bar{X}_i + \sqrt{F_{1,n-1}(\alpha)} \sqrt{\frac{s_{ii}}{n}}$$

$$\bar{X}_i - t_{n-1}(\alpha/2) \sqrt{\frac{s_{ii}}{n}} \leq \mu_i \leq \bar{X}_i + t_{n-1}(\alpha/2) \sqrt{\frac{s_{ii}}{n}}$$

## Simultaneous Confidence Statements for $\mathbf{a}'\mu$

- We can also make statements about the differences  $\mu_i - \mu_k$  corresponding to  $\mathbf{a}' = [0, \dots, 0, a_i, 0, \dots, a_k, \dots, 0]$ , where  $a_i = 1$  and  $a_k = -1$ . In this case ( $p = 2$ ),  $\mathbf{a}'\mathbf{S}\mathbf{a} = s_{ii} - 2s_{ik} + s_{kk}$ , we have the interval

$$(\bar{X}_i - \bar{X}_k) - \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{s_{ii} - 2s_{ik} + s_{kk}}{n}} \leq \mu_i - \mu_k \leq (\bar{X}_i - \bar{X}_k) + \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{s_{ii} - 2s_{ik} + s_{kk}}{n}}$$

- Which set of intervals is better (smaller) depends on the relative sizes of  $n$  and  $p$ , and even the number of means compared, say  $m$   $\mu_i$ 's.
- Perhaps the best way is to calculate both sets (critical values) and use the set yielding the narrower intervals.

## Patients Example : Simultaneous Statentents

Test to see if  $\mu_2 = \mu_5$ .

$$H_0 : \mu_2 = \mu_5 \text{ vs } H_1 : \mu_2 \neq \mu_5$$

```
patients <- read.csv("patients.csv", header=TRUE)
(Xbar <- colMeans(patients))
```

```
# WEIGHT FASTING GLUCOSE INSULIN RESIST
#      0.92    90.41   340.83   171.37    97.78
```

```
S <- cov(patients) # cov matrix
n <- nrow(patients)
```

We want to test whether there is enough evidence against  $\mu_2 = \mu_5$ .

$$H_0 : \mu_2 = \mu_5 \text{ vs } H_1 : \mu_2 \neq \mu_5$$

```
p <- 2 # comparing two means
(cval <- ((n-1)*p/(n-p))*qf(0.95,
                           df1 = p, df2 = n-p)) # F critical value
```

```
# [1] 6.6
```

```
a <- c(0,1,0,0,-1)
Ybar <- t(a)%*%Xbar
SY <- t(a)%*%S%*%a
LL <- Ybar - sqrt(cval)*sqrt(SY/n)
UL <- Ybar + sqrt(cval)*sqrt(SY/n)
data.frame(Mean.D = Ybar, Lower.lim = LL, Upper.lim = UL)
```

```
#   Mean.D Lower.lim Upper.lim
# 1    -7.4      -25       11
```

The 95% simultaneous confidence interval for  $\mu_2 - \mu_5$  is (-25, 11). Since 0 is inside the confidence interval, then it is plausible that  $H_0$  holds.

## Bonferroni Method of Multiple Comparisons

- Suppose prior to the collection of data, confidence statements about  $m$  linear combinations  $\mathbf{a}'_1\mu, \mathbf{a}'_2\mu, \dots, \mathbf{a}'_m\mu$  are required.
- Let  $C_i$  denote the confidence statement about the value  $\mathbf{a}'_i\mu$  with  $P(C_i \text{ true}) = 1 - \alpha_i, i = 1, 2, \dots, m$ .

$$P[\text{all } C_i \text{ true}] \geq 1 - \sum_{i=1}^m \alpha_i$$

- A special case of the Bonferroni allows the investigator to control the overall error rate  $\sum_{i=1}^m \alpha_i$ , regardless of the correlation structure.
- We consider the individual t-intervals

$$\bar{X}_i \pm t_{n-1} \left( \frac{\alpha_i}{2} \right) \sqrt{\frac{s_{ii}}{n}}, \quad i = 1, 2, \dots, m,$$

where  $\alpha_i = \alpha/m$ .



```
p <- 5; alpha <- 0.05
tci <- qt(1 - alpha/2, df=n-1)
bc.tci <- qt(1 - alpha/(2*p), df=n-1)
T2ci <- sqrt(((n-1)*p/(n-p))*qf(0.95, df1=p, df2=n-p))
CI <- function(cval, Xbar, S, n){
  cbind( Xbar - cval * sqrt(diag(S/n)) ,
        Xbar + cval * sqrt(diag(S/n))) }

# t-intervals
t <- CI(tci, Xbar, S, n)

# bonferroni corrected t-intervals
bct <- CI(bc.tci, Xbar, S, n)

# Hotelling T2-intervals
T2 <- CI(T2ci, Xbar, S, n)
```

```

# Confidence Intervals for the mean vector with
# alpha = 0.05
# One-at-a-time t, bonferroni corrected t,
# and T2 intervals
colnames(t) <- colnames(bct) <-
  colnames(T2) <- c("LL", "UL")
data.frame(t = t, Bonf.t = bct, T2 = T2)

```

#	t.LL	t.UL	Bonf.t.LL	Bonf.t.UL	T2.LL	T2.UL
# WEIGHT	0.88	0.96	0.87	0.97	0.85	0.99
# FASTING	87.92	92.91	87.08	93.74	85.88	94.95
# GLUCOSE	330.95	350.70	327.64	354.02	322.87	358.78
# INSULIN	156.88	185.86	152.02	190.72	145.02	197.72
# RESIST	84.06	111.51	79.45	116.11	72.83	122.74

## Comparison of Interval Widths

```
# One-at-a-time t, bonferroni corrected t,  
# and T2 intervals  
data.frame(t.width = t[,2] - t[,1],  
            Bonf.t.width = bct[,2] - bct[,1],  
            T2.width = T2[,2] - T2[,1])
```

#	t.width	Bonf.t.width	T2.width
# WEIGHT	0.076	0.1	0.14
# FASTING	4.989	6.7	9.07
# GLUCOSE	19.756	26.4	35.92
# INSULIN	28.986	38.7	52.70
# RESIST	27.452	36.7	49.91

## Observations

- In general, the width of  $T^2$ -intervals, relative to  $t$  and bonferroni corrected  $t$  intervals, increases as  $p$  increases (for fixed  $n$ ) and decreases as  $n$  increases (for fixed  $p$ ).
- The confidence level associated with any collection of  $T^2$ -intervals, for fixed  $n$  and  $p$ , is  $1 - \alpha$ , and the overall confidence associated with a collection of  $t$  intervals, for the same  $n$ , can be much less.
- The bonferroni correction guarantees that overall confidence level is greater than or equal to 0.95.
- Because Bonferroni correction is easy to apply and provide relatively short confidence intervals; it often used in practice.