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Sensitivity of the linear programming condition number to right-hand side scaling

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Abstract The Renegar condition number for linear programs quantifies how sensitive the optimal solution status of problem instances are to perturbations in the problem data. Although condition numbers are an important characteristic of linear programs, little work has been done to understand how they change when changes are made to the underlying LP. In this paper we present results in this direction, studying how the LP condition number is affected by scaling the right-hand side of a linear program. We prove that, as a function of the right-hand side scaling factor, the condition number is a convex function. Additionally, we provide a simple closed form for the minimizer of this function, and deduce analogous results for scaling of the objective function.

Keywords Linear programming · Condition number · finite-precision arithmetic

1 Introduction

Linear Programming is among the most widely applied tools in optimization. Software implementations of algorithms for solving Linear Programs (LPs) are typically based on finite-precision floating-point arithmetic. Because of the limitations of this arithmetic, problem instances may not even be representable exactly using floating-point numbers, and solution algorithms can suffer from

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numerical inaccuracy. In practice, these implementations often provide good approximations to true optimal solutions, but it is important to understand their limitations and develop bounds on these possible inaccuracies. Theoretical analysis of the bit-complexity of polynomial-time linear programming algorithms also often rely on finding approximate solutions of a prescribed accuracy which are later transformed to exact rational solutions (see [15]).

Renegar [13] first defined the notion of a condition number of an LP, which is based on the size of the smallest perturbations needed to change the feasibility status of an LP. LP condition numbers provide an measure of the magnitude of potential computational inaccuracies that could arise in a numerical solution process, and other authors have since introduced additional condition measures for linear programming [3,6]. Linear programming condition numbers arise in the complexity analysis of interior-point methods under finite-precision arithmetic [5,13,14,17] and in the smoothed analysis of linear programming complexity [16]. It has been observed that ill-conditioned and ill-posed LPs are not necessarily difficult to solve in practice [10], as the behavior of solution algorithms depends on many factors; for example, the behavior of the simplex algorithm depends on the LP bases encountered during the solution process.

It is well understood how to compute the LP condition number, but this calculation can be prohibitively expensive for practical purposes [9,10,12]. Little is known about how preconditioning techniques for LPs (e.g., [8,11]) affect the condition number. Some recent research has looked into bounding how small perturbations affect the condition measure (see [1,2,4,7]), but these have mainly focused on random perturbations.. In this paper we study how the condition number of a problem instance changes as certain types of modifications are made; in particular the scaling of the right-hand side or objective function. Even these relatively simple modifications can lead to interesting behavior of the condition number, and the ideas developed here may serve as a starting point for evaluating other types of problem modifications.

The paper is structured as follows. In Section 2 we introduce some necessary definitions. In Section 3 we study some helpful properties of the condition number. In Section 4 we develop our main results regarding the effect of right-hand side scaling on the condition number of a linear program. Section 5 concludes and discusses possible future directions.

2 Definitions

Following Renegar [13] we will represent an LP

$$z = \min_{x} \{c^T x : Ax \leq b\},$$

as a data triple (A, b, c). Let $\mathcal{F}$ be the set of linear programs whose feasible regions have open interiors and finite optima, $\bar{\mathcal{F}}$ be its closure, and $\mathcal{F}^C$ be its complement. We say that $d = (A, b, c)$ is *ill-posed* if $d \in \bar{\mathcal{F}} \cap \mathcal{F}^C$, and we define the *distance to ill-posedness* as $\rho_d = \inf\{\|(\alpha,\beta,\gamma)\|_\infty : d+(\alpha,\beta,\gamma) \in \bar{\mathcal{F}} \cap \mathcal{F}^C\}$, where $\|(A,b,c)\|_\infty = \max\{\|A\|_1, \|b\|_\infty, \|c\|_\infty\}$. A vector $\delta = (\alpha,\beta,\gamma)$ such that $d+\delta$ is ill-posed is is called a *perturbation to ill-posedness* and such a vector with $\|(\alpha,\beta,\gamma)\|_\infty = \rho_d$ is called a *minimal perturbation to ill-posedness.* Throughout the paper, unless otherwise noted, we use $\|\cdot\|$ to denote the ∞-norm for vectors, and the 1-norm for matrices. The *condition number* of d is defined as

$$C(d) = \frac{\|d\|_\infty}{\rho_d}.$$

If the distance to ill-posedness is zero, the problem is said to be *ill-posed.*

For an LP $d = (A,b,c)$ and scalar λ we use d_λ to denote the LP $(A, \lambda b, c)$; throughout the paper we will restrict our attention to the case when $\lambda > 0$. Similarly, if $\delta = (\alpha,\beta,\gamma)$ is a perturbation to ill-posedness for d then we use δ_λ to denote $(\alpha, \lambda\beta, \gamma)$. As we often study the condition number of d_λ as a function of λ we define $C_d(\lambda) := C(d_\lambda)$.

There may be multiple minimal perturbations to ill-posedness, and in some cases it is helpful for us to consider minimal perturbations that are further minimal with respect to additional criteria. We say that a minimal perturbation to ill-posedness δ is a *strongly minimal* perturbation to ill-posedness if it satisfies the following condition: There does not exist a minimal perturbation $\hat{\delta}$ such that $|\hat{\delta}_i| \le |\delta_i|$ for all components i and $|\hat{\delta}_i| \ne |\delta_i|$ for some component i. In other words, there does not exist a minimal perturbation to ill-posedness that is strictly less than δ in component-wise absolute value.

3 Properties of the Condition Number

In this section, we prove some results about the types of perturbations to ill-posedness that may be minimal, and how the form of the minimal perturbation of d_λ can change as λ is varied. Recall that a linear program $d = (A,b,c)$ is ill-posed if $d \in \bar{\mathcal{F}} \cap \mathcal{F}^C$, i.e., the feasible region of at least one of the primal or dual problem does not have an open interior. Thus a perturbation to ill-posedness is a perturbation which either eliminates the interior of either the primal or dual LP.

Proposition 1 *Let $d = (A,b,c) \in \mathcal{F}$ be any well-posed linear program and $\delta = (\alpha,\beta,\gamma)$ be a perturbation to ill-posedness.Then at least one of $(\alpha,\beta,0)$ or $(\alpha,0,\gamma)$ is also a perturbation to ill-posedness.*

Proof Suppose that d is a well-posed linear program, and $\delta = (\alpha,\beta,\gamma)$ is a perturbation to ill-posedness. Since $d+\delta$ is ill-posed, the feasible region of at least one of the primal or the dual problems does not have an open interior. In the case that the feasible region of $d+\delta$ does not have an open interior, $d+(\alpha,\beta,0)$ also yields a program whose primal feasible region has no open interior since γ has no effect on the feasible region, and $(\alpha,\beta,0)$ is therefore a perturbation to ill-posedness. Similarly, if the feasible region of the dual of

$d + \delta$ does not have an open interior, then $(\alpha, 0, \gamma)$ is a perturbation to ill-posedness. □

We immediately arrive at the following:

Corollary 1 *All strongly minimal perturbations to ill-posedness take one of the following forms:* $(\alpha, 0, 0)$, $(0, \beta, 0)$, $(0, 0, \gamma)$, $(\alpha, \beta, 0)$, *or* $(\alpha, 0, \gamma)$.

Thus, throughout the paper, when discussing perturbations to primal infeasibility we may assume that $\gamma = 0$ and when discussing perturbations to dual infeasibility we may assume that $\beta = 0$.

Proposition 2 *Suppose that* $d = (A, b, c) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}^n$ *is linear program and* $\lambda \in \mathbb{R}^+$. *Then* d *is well posed if and only if* d_λ *is well posed. Moreover,* δ *is a perturbation to ill-posedness for* d *if and only if* δ_λ *is a perturbation to ill-posedness for* d_λ.

Proof Since $\lambda > 0$ we observe that x is feasible for d if and only if λx is feasible for d_λ and the costs of these solution vectors for their respective problems differs only by the factor λ. It follows that d is has an open interior, or finite optimum precisely when d_λ shares the same property. Thus d is well posed if and only if d_λ is well posed. To prove the second claim, we simply note that $d_\lambda + \delta_\lambda = (d + \delta)_\lambda$, so the first result implies that $d_\lambda + \delta_\lambda$ is ill-posed if and only if $d + \delta$ is ill-posed. □

We also present the following basic result about the condition number as a function of a scalar:

Proposition 3 *Let* $d \in \mathcal{F}$ *be a well-posed linear program. Then* $C_d(\lambda)$ *is a continuous function from* $\mathbb{R}^+ \to \mathbb{R}$.

Proof We first note that $C_d(\lambda)$ is a ratio of $\|d_\lambda\|_\infty$ and ρ_{d_λ}, as defined in Section 2. It is clear that $\|d_\lambda\|_\infty$ is a continuous function of λ so it is sufficient to show that ρ_{d_λ} is continuous function of λ over the domain $\mathbb{R}^+$. Since d is well posed, Proposition 2 tells us that d_λ is well posed for any positive value of λ, therefore ρ_{d_λ} is strictly positive on $\mathbb{R}^+$.

To show continuity of ρ_{d_λ} we suppose, for contradiction that it is discontinuous; without loss of generality we may assume that the discontinuity occurs at the value $\lambda = 1$. Then there exists $\epsilon > 0$ such that for any $\gamma > 0$, there is a $\rho \in (-\gamma, \gamma)$, a minimal perturbation to ill-posedness δ for d, and a perturbation to ill-posedness δ' for d such that $\|\delta - \delta'_{1+\rho}\| > \epsilon$; this is because by Proposition 2, if $\delta'_{1+\rho}$ is a minimal perturbation to ill-posedness for $d_{1+\rho}$, then δ' is a (possibly non-minimal) perturbation to ill-posedness for d. Thus we see that $\|\delta'\| \geq \|\delta\|$. In the above, we may choose γ such that $0 < \gamma < \min\{\frac{\epsilon}{2\|d_{1.5}\|}, \frac{1}{2}\}$.

Then we have that

$$\begin{aligned}\epsilon &< \|\delta_1\| - \|\delta'_{1+\rho}\| \\ &\le \|\delta'_1\| - \|\delta'_{1+\rho}\| \\ &\le \|\delta'_1 - \delta'_{1+\rho}\| \\ &\le |\rho|\|\beta'\| \\ &< \gamma\|\beta'\|\end{aligned}$$

from which we obtain $2\|d_{1.5}\| < \frac{\epsilon}{\gamma} < \|\beta'\|$. But $(1+\rho)\|\beta'\| \le \|\delta'_{1+\rho}\| \le \|d_{1+\rho}\|$, and so $1+\rho < \frac{\|d_{1+\rho}\|}{2\|d_{1.5}\|} \le \frac{1}{2}$, because $\|d_{1+\rho}\| < \|d_{1.5}\|$. But this implies that $|\rho| > \frac{1}{2}$, which is a contradiction because $|\rho| < \gamma < \frac{1}{2}$, and so $C_d(\lambda)$ must be continuous. □

4 Main Results

As we attempt to build a complete picture of $C_d(\lambda)$, we begin with a few basic properties that hold. As noted in Corollary 1, all strongly minimal perturbations must take one of five forms. We begin by demonstrating how these forms are related. In particular, we show that if δ is a perturbation to ill-posedness such that δ_μ is a strongly minimal perturbation for d_μ then there is an interval, I, of strictly positive length with $\mu \in I$ such that for all $\lambda \in I, \delta_\lambda$ is minimal perturbation for d_λ. We then characterize the form of strongly minimal perturbations that are in adjacent such intervals. As a first result, we show that $C_d(\lambda)$ has a minimizer.

Proposition 4 *For any well-posed linear program d with $b \neq 0$, $C_d(\lambda)$ is non-increasing on the interval $(0, \|A\|/\|b\|]$ and non-decreasing on the interval $[\|A\|/\|b\|, \infty)$.*

Proof First, we show that $C_d(\lambda)$ is non-increasing on $(0, \|A\|/\|b\|]$. Let $0 < \lambda_2 < \lambda_1 \le \|A\|/\|b\|$, and suppose $C_d(\lambda_2) < C_d(\lambda_1)$, with minimal perturbations to ill-posedness given by δ_{λ_2} and δ'_{λ_1}, respectively (where δ, δ' are perturbations to ill-posedness for d). Then $\|d_{\lambda_1}\| = \|A\| = \|d_{\lambda_2}\|$, and so $\|\delta'_{\lambda_1}\| < \|\delta_{\lambda_1}\|$. But then $\|\delta'_{\lambda_2}\| < \|\delta_{\lambda_2}\|$, which contradicts the minimality of δ_{λ_2}, so $C_d(\lambda)$ is non-increasing on $(0, \|A\|/\|b\|]$.

Next we show that $C_d(\lambda)$ is non-decreasing on $[\|A\|/\|b\|, \infty)$. Let $\frac{\|A\|}{\|b\|} \le x < y$, and suppose $C_d(x) > C_d(y)$, with minimal perturbations given by δ_x and δ'_y, respectively. Since $x, y \ge \frac{\|A\|}{\|b\|}$, we have that $\frac{x\|b\|}{\|\delta_x\|} > \frac{y\|b\|}{\|\delta'_y\|}$, which is equivalent to $\frac{x}{y}\|\delta'_y\| < \|\delta_x\|$. But $\|\delta'_x\| \le \frac{x}{y}\|\delta'_y\|$, which contradicts the minimality of δ_x, so $C_d(\lambda)$ is non-decreasing on $[\|A\|/\|b\|, \infty)$. □

This leads us to the useful corollary:

Corollary 2 *For any well-posed linear program d with $b \neq 0$, $C_d(\lambda)$ has at least one global minimizer. One such minimizer is given by $\lambda = \|A\|/\|b\|$.*

In the following we sometimes assume that $C_d(\lambda)$ is minimized at $\lambda = 1$ in order to simplify the notation. This can be done without loss of generality because for other cases, we may consider the program $d' = d_{1/\lambda}$ whose minimal perturbation at 1 is precisely the minimal perturbation for d_λ, and by Lemma 2 results derived for d' can be translated back to related results for d.

Proposition 5 *Let d be a well-posed linear program with $b \neq 0$ with minimal perturbation to ill-posedness $\delta = (\alpha, \beta, 0)$ with $\|\alpha\| > \|\beta\|$. Then δ_λ is a minimal perturbation to ill-posedness for d_λ for all λ in the interval $[1, \frac{\|\alpha\|}{\|\beta\|}]$.*

Proof Suppose that there is a perturbation to ill-posedness for d, δ', and a number $\lambda \in [1, \frac{\|\alpha\|}{\|\beta\|}]$ such that $\|\delta'_\lambda\| < \|\delta_\lambda\|$. Since $1 \leq \lambda$, $\|\delta'\| \leq \|\delta'_\lambda\|$. Also, since $\lambda \leq \frac{\|\alpha\|}{\|\beta\|}$, we have that $\|\delta_\lambda\| = \|\delta\| = \|\alpha\|$. Combining these, along with our supposition, we conclude that $\|\delta'\| < \|\delta\|$, which contradicts the minimality of δ, and so the result is proven. $\square$

Proposition 6 *Let d be a well-posed linear program with $b \neq 0$ with minimal perturbation to ill-posedness $\delta = (\alpha, \beta, 0)$ such that $\|\alpha\| < \|\beta\|$. Then for all $\lambda \in [\frac{\|\alpha\|}{\|\beta\|}, 1]$, δ_λ is a minimal perturbation.*

Proof Proceeding as before, suppose there exists a perturbation to ill-posedness for d, δ', and number $\lambda \in [\frac{\|\alpha\|}{\|\beta\|}, 1]$ such that $\|\delta'_\lambda\| < \|\delta_\lambda\|$. Then $\|\lambda\beta'\| \leq \|\delta'_\lambda\| < \|\delta_\lambda\| = \|\lambda\beta\|$, where equality is true because $\lambda \geq \frac{\|\alpha\|}{\|\beta\|}$. But then $\|\beta'\| < \|\beta\|$, so $\|\delta'\| = \max\{\|\alpha'\|, \|\gamma'\|\}$. From this we see that $\|\delta'_\lambda\| = \|\delta'\|$, and so $\|\delta'\| < \|\delta_\lambda\| < \|\delta\| \leq \|\delta'\|$, which is a contradiction, and so the result is proven. $\square$

We now work toward the proof of our main result, namely that if d is a well-posed linear program, then the function $C_d(\lambda)$ is convex on $(0, \infty)$ We begin by showing that there are no local maxima except in neighborhoods where $C_d(\lambda)$ is constant. We use this to show that C_d is convex to the right of the all global minima. After this, we show that to the left of the global minima, $C_d(\lambda)$ behaves as $f(x) = \frac{k}{\lambda}$ for some constant k. Since C_d is convex and decreasing to the left of the global minima, convex and increasing to the right of the global minima, and otherwise constant, this will establish our main result.

Lemma 1 *Let d be a well-posed linear program with $b \neq 0$ such that $C_d(\lambda)$ is nondecreasing on the interval $[1, 1+\epsilon]$, with $C_d(1) < C_d(1+\epsilon)$, for some $\epsilon > 0$. Then for every $1 \leq \lambda < \lambda'$ we have that $C_d(\lambda) \leq C_d(\lambda')$.*

Proof Suppose that d is as in the assumptions, $\delta = (\alpha, \beta, \gamma)$ is a minimal perturbation to ill-posedness for d, and $\delta' = (\alpha', \beta', \gamma')$ is a minimal perturbation to ill-posedness for $d_{1+\epsilon}$. Then since $C_d(1) < C_d(1+\epsilon)$ we have

$$\frac{\|d\|}{\|\delta\|} < \frac{\|d_{1+\epsilon}\|}{\|\delta'\|}.$$

First we consider the case when $\|d\| = \|d_{1+\epsilon}\|$. In this case we have $\|\delta'\| < \|\delta\|$ and thus $\delta'_{1/(1+\epsilon)}$ is a perturbation to ill-posedness for d satisfying $\|\delta'_{1/(1+\epsilon)}\| \leq \|\delta'\| < \|\delta\|$, but this contradicts the minimality of δ.

We may restrict our attention to the remaining case where $\|d\| < \|d_{1+\epsilon}\|$. In this case we have that $\|d_{1+\epsilon}\| = (1+\epsilon)\|b\|_\infty$ and thus for all $\lambda \geq 1+\epsilon$, $\|d_\lambda\| = \lambda\|b\|_\infty$. Suppose, for contradiction, that there is some $\lambda > 1+\epsilon$ where $C_d(\lambda) < C_d(1+\epsilon)$. Let $\hat{\delta}$ be a minimal perturbation to ill-posedness for d_λ. Since $\lambda > 1+\epsilon$ we have that

$$\frac{\lambda\|b\|_\infty}{\|\hat{\delta}\|} < \frac{(1+\epsilon)\|b\|_\infty}{\|\delta'\|},$$

or, equivalently

$$\frac{\lambda}{1+\epsilon}\|\delta'\| < \|\hat{\delta}\|.$$

But $\delta'_{\lambda/(1+\epsilon)}$ is a perturbation to ill-posedness for d_λ, and

$$\|\delta'_{\lambda/(1+\epsilon)}\| \leq \frac{\lambda}{1+\epsilon}\|\delta'\| < \|\hat{\delta}\|,$$

which contradicts the minimality of $\hat{\delta}$. Therefore, $C_d(\lambda) \geq C_d(1+\epsilon)$ for all $\lambda \geq 1+\epsilon$. Finally, by assumption we had that $C_d(\lambda)$ is nondecreasing on $[1, 1+\epsilon]$, which establishes the result. □

As a consequence of this, we immediately have the following:

Corollary 3 *If x is a local extremum of C_d and not a global minimum, and $[a,b]$ is a maximal interval containing x on which C_d is constant, then for any $\epsilon > 0$, x is not a global extremum for C_d restricted to $[a-\epsilon, b+\epsilon]$.*

Lemma 2 *Let $d \in \mathcal{F}$ be a well-posed linear program with $b \neq 0$, with $0 < y < z$ such that $\min_{x\in\mathbb{R}^+} C_d(x) \leq C_d(y) < C_d(z)$, and for any $\epsilon > 0$, $C_d(y+\epsilon) > C_d(y)$. Then for any $t \in (0,1)$, $tC_d(y) + (1-t)C_d(z) \geq C_d(ty+(1-t)z)$.*

Proof From the assumptions it follows that $C_d(ty+(1-t)z) > C_d(y)$. Following from the same ideas used in the proof of Lemma 1, taking the limit as ϵ goes to zero, for all $\lambda \geq y$ we have $\|d_\lambda\|_\infty = \lambda\|b\|_\infty$. By way of contradiction, suppose that $tC_d(y) + (1-t)C_d(z) < C_d(ty+(1-t)z)$, with minimal perturbations δ, δ', and $\hat{\delta}$. We then have that

$$t\frac{\|d_y\|}{\|\delta\|} + (1-t)\frac{\|d_z\|}{\|\delta'\|} < \frac{\|d_{ty+(1-t)z}\|}{\|\hat{\delta}\|},$$

and because $\|d_\lambda\| \geq \lambda\|b\|$, we obtain that

$$t\frac{y\|b\|}{\|\delta\|} + (1-t)\frac{z\|b\|}{\|\delta'\|} < (ty+(1-t)z)\frac{\|b\|_\infty}{\|\hat{\delta}\|}.$$

Since $\|\delta\| \geq \|\delta'\|$ we obtain

$$(ty + (1-t)z)\frac{\|b\|}{\|\delta\|} < (ty + (1-t)z)\frac{\|b\|}{\|\hat{\delta}\|},$$

which is equivalent to

$$\|\hat{\delta}\| < \|\delta\|.$$

But then $\hat{\delta}_{y/(ty+(1-t)z)}$ is a perturbation to ill-posedness for d_y, and

$$\|\hat{\delta}_{y/(ty+(1-t)z)}\| \leq \|\hat{\delta}\| < \|\delta\|,$$

which contradicts the minimality of δ. □

Since C_d is increasing immediately to the right of the largest global minimizer, there is at least one place where the assumptions of the previous lemma are satisfied, we thus have the following.

Corollary 4 *If k is the largest global minimizer of C_d, then C_d is convex on $[k, \infty)$.*

Lemma 3 *Let k be the smallest number that minimizes $C_d(\lambda)$, then $C_d(\lambda)$ is decreasing on the interval $(0, k]$ and a minimal perturbation for $C_d(\lambda)$ is given by $(0, \lambda\beta, 0)$, where $(0, \beta, 0)$ is a minimal perturbation for d_k.*

Proof We begin by showing that for all $\lambda \in (0, k], \|d_\lambda\| = \|d_k\|$, then by showing a minimal perturbation for d_λ is given by δ_λ where δ_k is a minimal perturbation at $\lambda = k$, after which we will show that δ must be of the form $(0, \beta, 0)$.

Without loss of generality, suppose $k = 1$ is the smallest value that is a global minimizer for C_d with minimal perturbation δ, and suppose that there exists $\kappa < 1$ such that $\|d_\kappa\| \neq \|d\|$. Because the only difference between d_κ and d is the scalar applied to the b component, this means that $\|d_\kappa\| < \|d\| = \|b\|_\infty$, since if $\|d\| = \|b\|_\infty$ then scaling b down would not affect $\|d\|$. Further, there is an $\epsilon > 0$ such that for all $\lambda \in (1-\epsilon, 1], \|d_\lambda\| = \lambda\|b\|_\infty$. Choose $\lambda \in (1-\epsilon, 1)$, with minimal perturbation δ'. Since 1 is the smallest value for which C_d is minimized, $C_d(\lambda) > C_d(1)$ and $\|d_\lambda\| = \lambda\|b\|_\infty < \|b\|_\infty = \|d\|$. But this yields

$$\lambda\frac{\|b\|_\infty}{\|\delta'\|} > \frac{\|b\|_\infty}{\|\delta\|},$$

which is equivalent to $\frac{1}{\lambda}\|\delta'\| < \|\delta\|$. But $\delta'_{1/\lambda}$ is a perturbation to ill-posedness for d, and $\|\delta'_{1/\lambda}\| \leq \frac{1}{\lambda}\|\delta'\| < \|\delta\|$, which contradicts the minimality of δ, and so for all $\lambda < 1$ we may now assume that $\|d_\lambda\| = \|d\|$.

Suppose there is some $\lambda < 1$ such that δ_λ is not a minimal perturbation for d_λ. Then its minimal perturbation is given by $\delta' \neq \delta_\lambda$. Since C_d is decreasing, and $\|d_\lambda\| = \|d\|$, we have that $\|\delta'\| < \|\delta_\lambda\| \leq \lambda\|\delta\|$, or $\frac{1}{\lambda}\|\delta'\| < \|\delta\|$. But $\delta'_{1/\lambda}$ is a perturbation to ill-posedness for d, and $\|\delta'_{1/\lambda}\| \leq \frac{1}{\lambda}\|\delta'\| < \|\delta\|$, which contradicts the minimality of δ, so for all $\lambda \leq 1$, a minimal perturbation for d_λ is given by δ_λ.

Now, suppose that $\delta = (\alpha, \beta, 0)$, with $\alpha \neq \mathbf{0}$, and let $(0, \beta', 0)$ be any perturbation to ill-posedness for d, and pick $\lambda < \frac{\|\alpha\|}{\|\beta'\|}$. Then $(0, \lambda\beta', 0)$ is a perturbation to ill-posedness for d_λ and

$$\|(0, \lambda\beta', 0)\| < \frac{\|\alpha\|}{\|\beta'\|}\|\beta'\| = \|\alpha\| \leq \|\delta_\lambda\|,$$

which contradicts the minimality of δ_λ for d_λ. Therefore, α must be the zero matrix, and the result is shown. □

Corollary 5 *Suppose that the l is the smallest minimizer of $C_d(\cdot)$. Then for all $\lambda \in (0, l]$, $C_d(\lambda) = \frac{1}{\lambda}C_d(l)$.*

We now collect our results together into the main theorem of this paper. Since C_d is continuous, and $(0, \infty)$ can be split into three intervals, $(0, l), [l, u]$, (possibly with $l = u$) and (u, ∞) such that C_d is decreasing and convex on $(0, l)$, constant on $[l, u]$, and increasing and convex on (u, ∞), we have the following.

Theorem 1 *For any well-posed linear program d with $b \neq 0$, the function $C_d(\cdot)$ is convex on the interval $(0, \infty)$. Moreover, $C_d(\lambda)$ is minimized at $\lambda = \|A\|/\|b\|$.*

The preceding results have given us a good understanding of how the condition number of an LP changes as we scale the right-hand side. However, we observe that for a well-posed linear program $d = (A, b, c)$, had we instead considered the condition number of $(A, b, \gamma c)$ as a function of γ, the arguments made could have been applied in a near identical fashion. This observation leads us to the following two results.

Theorem 2 *Let $d = (A, b, c)$ be a well posed linear program with $c \neq 0$, then the function $f(\gamma) := C(A, b, \gamma c)$ is convex on $\mathbb{R}^+$. Moreover, $f(\gamma)$ is minimized when $\gamma = \|A\|/\|c\|$.*

Combining the results regarding right-hand side and objective scaling we may also conclude the following.

Corollary 6 *For a well-posed linear program $d = (A, b, c)$ with $b \neq 0$ and $c \neq 0$ the condition number of $d_{\lambda,\gamma} := (A, \lambda b, \gamma c)$ for $\lambda, \gamma \in \mathbb{R}^+$ is minimized when $\|A\| = \|\lambda b\| = \|\gamma c\|$.*

5 Conclusion and Future Work

In this paper we have explored how the linear programming condition number is affected by right-hand side scaling of the underlying LP. Although this work provides first steps in understanding how the LP condition number is affected by some specific modifications we think it will be interesting to study how the condition number is affected by other problem modifications such as translation of the feasible region or LP preprocessing techniques.

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