Maximal Connectivity of $\overrightarrow{S_{n,k}}$

Dan Steffy

Advisor: Eddie Cheng

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Abstract

Star graphs are a popular model considered as a topology for interconnection networks. In this paper we will consider the directed (n, k)-star graph, $\overrightarrow{S_{n,k}}$, presented by Cheng and Lipman in [8], and show that it is maximally connected.[1]

[1]This work is part of a larger research project in collaboration with Eddie Cheng and William A. Lindsey.

1 Background

1.1 Motivation

An important research topic today is the study of interconnection networks, or interconnection topologies. One application which has generated a large amount of attention in the fields of computer science and computer engineering is the use of interconnection networks as a way to link processors for multiprocessor computing. Researchers have found that linking many processors and running them in parallel can prove much more cost effective and powerful in many applications than using a single processor supercomputer. In any such multiprocessor arrangement, the method of connecting the processors is of crucial importance to the cost, speed and reliability of the system.

1.2 Terminology

Interconnection network topologies can be best described in the terms of graphs, so before we go into further detail, we will define some of the commonly used terminology. The following terminology is all standard and can be found in any introductory text on graph theory, such as [17].

We will define a **graph** G to be a set of vertices V, along with a set of edges E, where edges are pairs of vertices from V and can be thought to represent connections between two vertices. We will denote elements of the edge set as single letters such as u, v, w and we will represent edges with pairs of edges such as uv. In addition to considering edges as undirected connections between vertices, directions can be assigned to edges converting them into directed edges between vertices. A graph whose edges are directed is called a **directed graph** or a **digraph**, and an assignment of direction to the edges of an undirected graph is called an **orientation**. Edges in a directed graph are often referred to as **arcs** rather than edges.

For each vertex in a graph G, we will define its **degree** to be the number of edges incident to it. In a directed graph each vertex has an **in degree** and **out degree**, which are the number of directed edges entering and leaving each vertex, respectively. In a graph G a **path** is a set of non-repeating vertices $u_0, u_1, \ldots, u_n$ for which an edge connects each pair of successive vertices, and the length of a path is the number of edges it contains. Given two vertices the **shortest path** between them is the shortest of all paths connecting them. A graph is said to be **connected** if every pair of vertices is connected by a path. For example, the graph in Figure 1 has six vertices, each with degree two, and is connected. Similarly, in a directed graph a **directed path** is the same as a path in a standard graph with the requirement that all the edges point in the same direction. A directed graph is **strongly connected** if a directed path connects each vertex with every other vertex in both directions.

The **diameter** of a graph is the length of the longest of all the shortest paths between every pair of vertices. A graph with at least $k+1$ vertices is said to be k**-connected** if the removal of $k-1$ vertices will not disconnect the graph. A **subgraph** of a graph G is a graph whose vertex set is a subset of the vertex set of G and the edges induced by these vertices are the same as in G. Two graphs are said to be **isomorphic** if there exists a one to one correspondence between their vertices, and if two vertices have an edge between them in one graph, the corresponding vertices have an edge between them in the other graph, or equivalently, their structures are identical. We assume the reader is familiar with standard results and terminology on permutations.

1.3 Past Approaches

The terms from graph theory are useful in describing interconnection network topologies precisely, by thinking of processors as vertices and connections as edges. Some desirable properties of an interconnection network topology are low vertex degrees, low diameter, and high connectivity. These properties in graphs would correspond to less required physical connections, less distance between processors and high resilience of the network when applied to interconnection networks. Now that we have defined some basic terms we can look at the description of some commonly used models. One of the standard and earliest and most throughly studied models for this application is Q_n, the n-cube which is also known as the hypercube. For any positive integral value of n, Q_n has vertex set consisting of all bit strings of length n (strings made up of ones and zeros of length n). Edges are places between vertices whose bit strings differ in exactly one bit position. It is easy to see that Q_n has 2^n vertices of degree n, and diameter of n.

1.4 Star Graphs

In [16], Akers, Harel and Krishnamurthy proposed the star graph as an alternative to the n-cube. Some recent papers on star graphs or variants of star graphs include [16, 8, 14, 17, 7, 11, 15, 1, 2, 10, 9, 4, 6, 5, 13, 12, 3]. The star graph, S_n, is defined to have vertex set of all permutations, or arrangements of n symbols. So for example S_3 would have vertex set of $\{123,132,213,231,312,321\}$.

Edges are placed between two vertices if one permutation can be obtained from the other by a transposition of the first element, with any other element

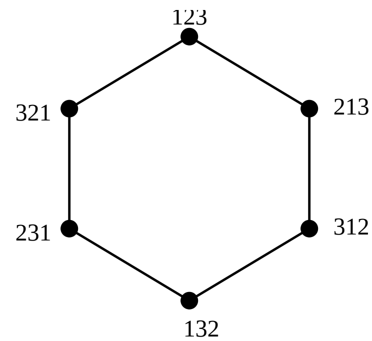

Figure 1: Example of S_3

in the list of symbols. So for example, in S_4 an edge would exist between vertices 1234 and 2134, since 2134 can be obtained by interchanging the first and second elements of 1234. We call an edge between two vertices that interchange their first and i-th position an ***i*-edge**. A picture of S_3 can be seen in Figure 1, and a picture of S_4 can be seen in Figure 2.

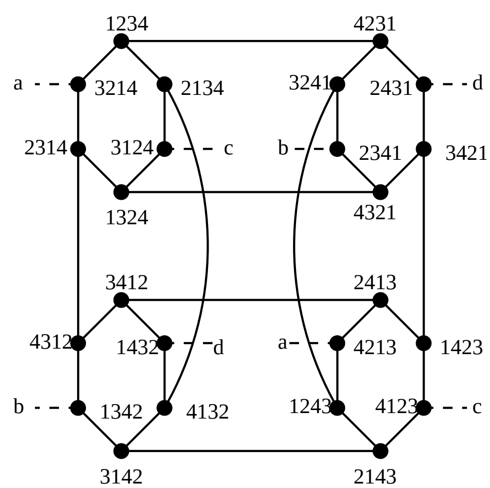

Figure 2: Example of S_4

We state some basic properties of star graphs as a theorem.

Theorem 1 *The star graph S_n has $n!$ vertices, all of degree $(n-1)$ and diameter of $\left\lfloor \frac{3}{2}(n-1) \right\rfloor$.*

In [14], Day and Tripathi introduced an orientation for S_n. The newly defined directed star graph, denoted US_n, can be useful as a topology for connecting processors as well. In some applications it may be useful to build a multiprocessor system where connections between processors transmit data *one way*, and in such a case understanding the structure of US_n and other directed interconnection topologies is important. In Figure 3 an example of US_4, the directed version of S_4, is given.

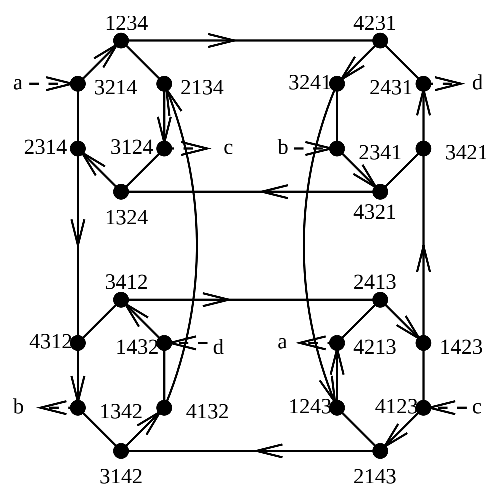

Figure 3: Example of US_4

Although S_n has proven to be an attractive alternative to Q_n, one drawback it has is the restriction on the number of vertices. Since S_n has $n!$ vertices, anyone wanting to build a multiprocessor network using this topology is forced to build one with $n!$ vertices for some value of n. This leaves them with few choices, since for example S_6 has 720 vertices while S_7 has 5040. This led in part to the introduction in [11] of (n, k)-star graphs, a generalization of star graphs. The (n, k)-star graph, denoted $S_{n,k}$, is defined similarly to the star graph. The vertex set of $S_{n,k}$ is the set of arrangements of k out of n symbols, where $k < n$ and thus has $\binom{n}{k}$ vertices. Edges are placed between vertices similarly to S_n, they are placed between vertices if one can be changed into the other by swapping the first element with any of the other $n-1$ available letters. An edge in $S_{n,k}$ is called a **star edge** if it is an i-edge where $i \leq k$, all edges that are not star edges are called **residual edges**. A picture of $S_{4,2}$ is depicted in Figure 4.

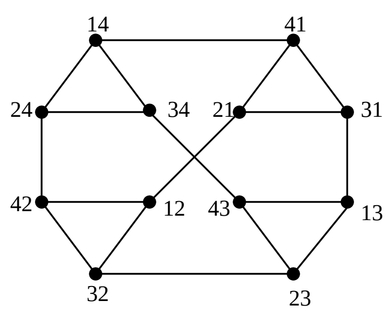

Figure 4: Example of $S_{4,2}$

2 Problem

2.1 Connectivity

In [8] Cheng and Lipman introduced an orientation for $S_{n,k}$ and studied some of its properties. The goal of this project is to extend the results of Cheng and Lipman and further explore the connectivity properties of $\overrightarrow{S_{n,k}}$, the directed (n,k)-star graph. We will show that $\overrightarrow{S_{n,k}}$ is **maximally connected**. A directed graph is said to be **maximally connected** if it is k-connected, and k is the minimum in or out degree of all the vertices. This result was proven to be correct for US_n in [7] by Cheng and Lipman, but still remains to be shown for $\overrightarrow{S_{n,k}}$.

In order to prove these results, we will rely on established results and the internal structure of $\overrightarrow{S_{n,k}}$. Many of the previous connectivity results proven about various star graphs rely heavily on their structure. For example, in the standard star graph S_n, the subgraph formed by choosing a set of vertices sharing a common symbol in any position is isomorphic to S_{n-1}. Notice that in Figure 2, we can see that S_4 has four copies of S_3.

Similar results occur in $S_{n,k}$. Choosing vertices with a common choice of their k symbols forms a subgraph isomorphic to S_k. Also, if one chooses vertices sharing a symbol in any position other than the first, they form a graph isomorphic to $S_{n-1,k-1}$, we will call such graphs **substars**. If one looks at vertices sharing identical choices of all chosen k letters *except* the first, the resulting subgraph will be isomorphic to a **complete graph** on $n-k+1$ vertices, a graph where edges are present between every pair of vertices. We will call such subgraphs **fundamental cliques**. These interesting properties of star graphs allow many results to be proven inductively.

2.2 Orientation of $S_{n,k}$

Before we prove any results about the directed (n,k)-star graph, $\overrightarrow{S_{n,k}}$, we will give the definition of the direction as defined by Cheng and Lipman in [8]. Define arcs within $\overrightarrow{S_{n,k}}$ that are between vertices within a substar as **star arcs**. Define all other arcs as **residual arcs**. Now associate each vertex $[a_1, a_2, \ldots, a_k]$ in $\overrightarrow{S_{n,k}}$, with the following permutation on n letters $[a_1, a_2, \ldots, a_k, x_1, \ldots, x_{n-k}]$ where $x_1 < x_2 < \ldots < x_{n-k}$. We will call vertices in $\overrightarrow{S_{n,k}}$ **even** or **odd** if their associated permutations are even or odd respectively.

It can be locally determined if an edge is a star edge or a residual edge. For the star edges it is easy to see that the vertices connected by the star edge are not of the same parity. Suppose $\pi_a\pi_b$ is a star edge, where π_a and π_b are vertices, without loss of generality we may assume π_a is even and π_b

is odd. To assign order to such edges we will use the standard Day-Tripathi rule for US_n to assign order, if $\pi_a\pi_b$ is an i-edge then the edge is oriented from π_a to π_b if i is even and from π_b to π_a if i is odd.

To assign order on the edges within the fundamental cliques we do the following. For any residual edge it is easy to see which fundamental clique it belongs to. The vertices will be of the form $\pi_i = [x_i, a_2, \ldots, a_k]$ and $\pi_j = [x_j, a_2, \ldots, a_k]$, and will lie within the fundamental clique with the vertices $[x_1, a_2, \ldots, a_k]$, $[x_2, a_2, \ldots, a_k]$, $\ldots$, $[x_{n-k+1}, a_2, \ldots, a_k]$ where $x_1 < x_2 < \ldots < x_{n-k+1}$. It is clear that if we consider the vertices in this order, indexed by their leading variable, that the parity of their associated permutations alternates. The first permutation in this list, π_1, may have an associated permutation that is either even or odd. If its associated permutation is odd, or if there are an odd number of elements in the clique, orient the edges as follows: given two vertices π_i and π_j where $x_i < x_j$, we will map from π_i to π_j if π_i and π_j have different parity, and from π_j to π_i if π_i and π_j have the same parity. If the associated permutation of the first element, π_1, is even, and there are an even number of elements in the fundamental clique, assign the exact opposite orientation.

Additional properties about this orientation and justification behind it can be found in [8]. Now we are ready to show that under this orientation

$\overrightarrow{S_{n,k}}$ becomes maximally connected.

3 Maximal Connectivity

For positive integers n, as mentioned previously, define the **complete graph** on n vertices, K_n, to be the graph with n vertices and single edges between every pair of vertices. Define $\overrightarrow{K_n}$ by assigning the following order on K_n: Label all vertices with elements from $\{1, 2, \ldots, n\}$ and for vertices u and v with $u < v$, direct the arc from u to v if u and v are of different parity, and from v to u if they are of the same parity.

Lemma 1 *Let $2p \geq 4$, then $\overrightarrow{K_{2p}}$ is $(p-1)$-connected.*

Proof: We will show this by induction. For a base case we consider $\overrightarrow{K_4}$ which is clearly strongly connected. Now let us partition the vertices of $\overrightarrow{K_{2p}}$ as follows: Place the vertices labeled $2p-1$ and $2p$ in their own sets. Let A be all the odd indexed vertices besides $2p-1$ and let B be all the even indexed vertices less than $2p$, as pictured in Figure 5.

Consider deleting S from $\overrightarrow{K_{2p}}$ where $|S| = p-2$.

Case 1: $2p-1$, $2p \notin S$.

Since $|B| = p-1$ and $|S| = p-2$, the digraph C induced by $2p-1$, $2p$ and $B - S$ is strongly connected. Now, any element in $A - S$ clearly

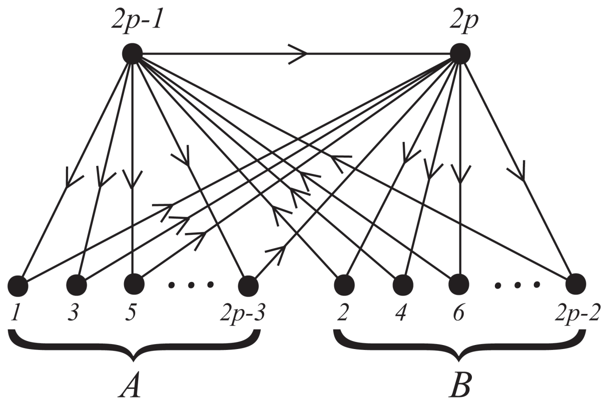

Figure 5: Graph from Lemma 1

has an arc from it to C and an arc from C back to it, therefore $\overrightarrow{K_n} - S$ is strongly connected.

Case 2: $2p-1$, $2p \in S$.

In this case $\overrightarrow{K_n} - \{2p-1, 2p\}$ is just $\overrightarrow{K_{n-2}}$ which we know from induction is $(p-2)$-connected, therefore $\overrightarrow{K_n} - S$ is strongly connected.

Case 3: $2p-1 \in S$, $2p \notin S$.

Again by induction we see that $\overrightarrow{K_{n-2}} - \{S - \{2p-1\}\}$ is strongly connected and since $|S - \{2p-1\}| = p-3$, $A - S$ and $B - S$ are nonempty. Now we can see there is an arc from $2p$ to $\overrightarrow{K_{n-2}} - \{S - \{2p-1\}\}$, and an arc entering $2p$ from $\overrightarrow{K_{n-2}} - \{S - \{2p-1\}\}$. Therefore $\overrightarrow{K_n} - S$ is strongly connected.

Case 4: $2p-1 \notin S$, $2p \in S$.

Again by induction we see that $\overrightarrow{K_{n-2}} - \{S - \{2p\}\}$ is strongly connected and since $|S - \{2p\}| = p - 3$, $A - S$ and $B - S$ are nonempty. Now we can see there is an arc from $2p-1$ to $\overrightarrow{K_{n-2}} - \{S - \{2p\}\}$, and an arc entering $2p$ from $\overrightarrow{K_{n-2}} - \{S - \{2p\}\}$. Therefore $\overrightarrow{K_2p} - S$ is strongly connected.

Lemma 2 $\overrightarrow{K_{2p+1}}$ *is p-connected.*

Proof: We will show this by induction. Clearly $\overrightarrow{K_5}$ is 2-connected. Partition the vertices of $\overrightarrow{K_{2p+1}}$ as follows: Place $2p$ and $2p+1$ in their own sets and let A be all the odd indexed vertices besides $2p+1$, let B be all the even indexed vertices besides $2p$ as pictured in Figure 6.

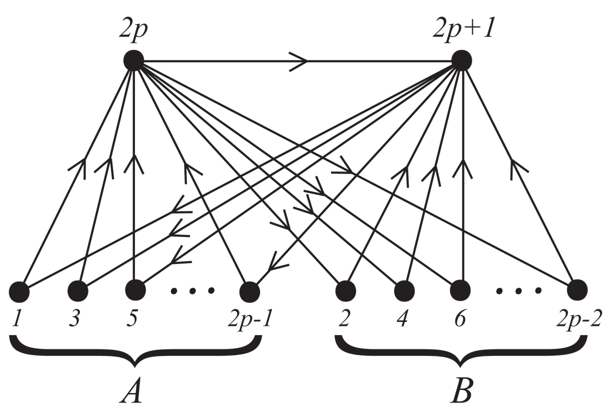

Figure 6: Graph from Lemma 2

Consider deleting S from $\overrightarrow{K_{2p+1}}$ where $|S| = p - 1$.

Case 1: $2p$, $2p+1 \notin S$

Since $|S| = p-1$ and $|A| = p$, the digraph induced by $2p$, $2p+1$, and $A - \{S\}$ is strongly connected, as there is a directed arc from $2p$ to $2p+1$, and every vertex in A has an arc leaving it directed into $2p$ and an arc entering it from $2p+1$. Now note that the subgraph induced by $2p$, $2p+1$, and $A - \{S\}$ is strongly connected, and since every element of B has an arc from it entering and leaving this subgraph, $\overrightarrow{K_{2p+1}} - S$ is strongly connected.

Case 2: $2p$, $2p+1 \in S$

Since $\overrightarrow{K_{2p+1}} - \{2p, 2p+1\}$ is isomorphic to $\overrightarrow{K_{2p-1}}$ which is $(p-1)$-connected by induction, deleting $S - \{2p, 2p+1\}$ from it will not disconnect it since $|S - \{2p, 2p+1\}| = p-3$. Therefore $\overrightarrow{K_{2p+1}} - S$ is strongly connected.

Case 3: $2p \in S$, $2p+1 \notin S$

By our induction step, $\overrightarrow{K_{2p-1}}$ is $(p-1)$-connected. Since $|S - \{2p\}| = p-2$ we know that $\overrightarrow{K_{2p-1}} - (S - \{2p\})$ must be strongly connected. We must also show that $\{2p+1\}$ is connected to the digraph C induced by $\{1, 2, \ldots, 2p\}$. Since $|A| = p$ and $|B| = p-1$ and $|S - \{2p\}| = p-2$, we know that there must be points from both A and B that are not in

S, and since all points from A have arcs to them from $\{2p+1\}$ and all points in B have arcs leaving them to $\{2p+1\}$, we know that $\{2p+1\}$ is connected to C.

Case 4: $2p \notin S$, $2p+1 \in S$

Similar to case 3, we may use our induction step to see that $\overrightarrow{K_{2p-1}}$ is $(p-1)$-connected. Since $|S - \{2p+1\}| = p-2$ we know that $\overrightarrow{K_{2p-1}} - (S - \{2p+1\})$ must be strongly connected. We must also show that $\{2p\}$ is connected to this larger component. Since $|A| = p$ and $|B| = p-1$ and $|S - \{2p+1\}| = p-2$, we know that there must be points from both A and B that are not in S. Since all points from A have arcs from them to $\{2p\}$ and all points in B have arcs to them from $\{2p\}$, $\{2p\}$ is connected to the larger component.

Therefore we may conclude that $\overrightarrow{K_{2p+1}}$ is p-connected.

Lemma 3 *Let H be the graph obtained from $S_{n,2}$ by contracting each fundamental clique to a vertex. Then the resulting graph is isomorphic to K_n.*

Proof: Since there is exactly one vertex between every fundamental clique, the result follows directly.

Lemma 4 *Let $\overrightarrow{H}$ be the directed graph obtained from $\overrightarrow{S_{n,2}}$ by contracting each fundamental clique to a vertex, then the resulting graph is an orientation of K_n with the same connectivity properties as $\overrightarrow{K_n}$, it is $\lfloor \frac{n-1}{2} \rfloor$- connected.*

Proof: Each fundamental clique is uniquely identified by the symbol in its second position. Let F_i represent the fundamental clique where i is located in the second position. That is, vertices in F_i are in the form $[x, i]$ where $x \in \{1, 2, \ldots, n\} - \{i\}$. Now for each distinct F_i and F_j there is exactly one edge connecting them, which is between vertices $[j, i]$ and $[i, j]$. Also, define the parity of each F_i to be that of its index, namely F_i is odd if and only if i is odd. We will show that if we the F_i's in order then their orientation is exactly opposite to that of $\overrightarrow{K_n}$, which gives it the same connectivity properties.

Claim: If the vertices of F_i are arranged in lexicographical order, the parities of their associated permutations will alternate, with the parity of the leading element being the same as i.

Justification: For $i = 1$ it is clear that the first vertex, $[2, 1]$, will have odd degree. For $i \neq 1$, the first vertex, $[1, i]$ shares the same parity as i because to move from its associated permutation to the identity, one need do nothing if $i = 2$, or make $i - 2$ moves otherwise. Now to see that the parities will alternate, clearly the parity of the first $i - 1$ elements will alternate. The $(i-1)$-th permutation and the i-th permutation will have alternating parities because a single simple swap will go from one to the other. It is also clear that the remaining vertices will alternate in parity.

Claim: If i is odd, then for the first $i-1$ elements of the clique, the parity of their leading element will be the same as the parity of their associated permutation, and for the remaining elements of the clique, the parity of their leading coefficient will be opposite the parity of their associated permutation.

Justification: If $i = 1$ then the first element clearly has an odd associated permutation. For other odd values of i, as we saw previously, it will take an odd number of swaps to get from the associated permutation of $[1, i]$ to the identity. Therefore the rest of our claim follows directly from the alternation of the parity of the associated permutations of the elements of the clique when placed in lexicographical order.

Claim: If i is even then the first $i - 1$ elements of the clique will have opposite parities for their associated permutations, and leading elements. The remaining vertices will have a leading element with the same parity as that of their associated permutation.

Justification: For even values of i, as we saw previously, it will take an even number of swaps to get from the associated permutation of $[1, i]$ to the identity. Therefore the rest of our claim follows directly from the alternation of the parity of the associated permutations of the elements of the clique when placed in lexicographical order.

Now consider different cases regarding the parity of i and j. We will consider the behavior of the edge between F_i and F_j. Without loss of generality, assume $i < j$.

Case 1: i is odd and j is even.

$[j, i]$ has the opposite parity as j and $[i, j]$ has the opposite parity as i, so $[j, i]$ is an odd vertex and $[i, j]$ is an even vertex. Therefore the edge will be directed from $[i, j]$ to $[j, i]$. So the edge goes from F_j to F_i as required. (This is the orientation of a 2-edge following the Day-Tripathi orientation.)

Case 2: i is even and j is odd.

In this case, since $i < j$, $[j, i]$ will be even and $[i, j]$ will be even. So the path will be directed from $[j, i]$ to $[i, j]$. So the edge goes from F_j to F_i as required.

Case 3: Both i and j are even.

In this case, $[j, i]$ will have the same parity as j and therefore even. Vertex $[i, j]$ will have opposite parity as i and will be odd. Therefore the arc between them will be directed from $[j, i]$ to $[i, j]$. So the edge goes from F_i to F_j as required.

Case 4: Both i and j are odd.

In this case, $[j, i]$ will have the opposite parity as j and is therefore even. Vertex $[i, j]$ will have the same parity as i and is therefore odd. This will mean the edge from $[j, i]$ will be directed to $[i, j]$. So the edge goes from F_i to F_j as required.

We may now conclude that this orientation is the exact opposite of $\overrightarrow{K_n}$, and therefore is $\lfloor \frac{n-1}{2} \rfloor$- connected..

Theorem 2 $\overrightarrow{S_{2p+1,2}}$ *is* p*-connected when* $2p + 1 \geq 5$.

Proof: Let us delete a subset of the vertices S from $\overrightarrow{S_{2p+1,2}}$ with $|S| = p-1$. We will call a fundamental clique damaged if it contains elements from S, otherwise we will say it is undamaged. Let d be the number of damaged fundamental cliques. Since $|S| = p-1$ we know that $d \leq p-1$. Let G be the graph induced by the vertices of the undamaged fundamental cliques. Since each fundamental clique is strongly connected it follows from Lemma 4 that G is strongly connected.

Case 1: Every damaged fundamental clique has at most $p-2$ vertices from S.

Choose any damaged fundamental clique with α deleted vertices, $\alpha \leq p-2$. Let C_1 be the vertices of this damaged fundamental clique with arcs leaving the fundamental clique, and let C_2 be the set of vertices with edges entering the clique. Clearly $|C_1| = p$ and $|C_2| = p$. Since $\alpha+d-1 \leq p-1$ there is at least one edge entering and one edge leaving the damaged fundamental clique connecting it to the rest of the graph.

Case 2: There exists one damaged fundamental clique, A, with $p-1$ deleted vertices.

In this case there is only one damaged fundamental clique. Since there are $2p$ vertices in the fundamental clique and exactly half have edges from them leaving the clique, and half have edges entering them, at least one edge will be directed into the clique from the rest of the graph and at least one edge will leave the clique into the rest of the graph. Let X denote the digraph induced by the undamaged fundamental clique, and let Y denote the strong component containing X. Suppose u is a vertex in A that is not deleted. Now suppose the $2p$ vertices in A, when indexed by their leading element are in the order *even*, *odd*, $\ldots$, *even*, *odd*. Now consider two cases.

Case 2a: u is even.

Since u is even there is an edge leaving u to a vertex in X. In order to show that u is part of Y, we need only show that there exists a vertex in X with a directed path from it leading to u.

Construct sets C_i, with $1 \leq i \leq p$, by assigning each vertex with an odd index that is less than u, and each consecutive pair of vertices with index larger than u in the order *odd-even*, to one of the C_i's, as pictured in Figure 7.

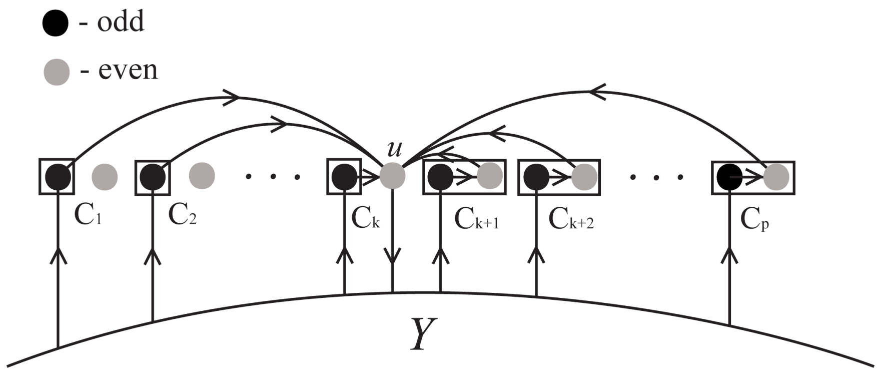

Figure 7: Picture from Case 2a

Since $|A| = 2p$, it is clear that there there are in fact p such sets. Now, for each of the single vertex C_i's, where the vertices are less than u, by the definition of the orientation, directed edges go directly into these vertices from X, followed by directed edges directly to u. For the C_i's with two vertices, there is a directed edge from X going into the odd vertex, then a directed edge going from the odd to the even vertex, then another directed edge going from the even vertex directly to u. This gives us a total of exactly p mutually disjoint directed paths from a vertex in X to a vertex in a vertex into one of the C_i's to u. Since we only delete $p-1$ vertices in total, at least one of these paths remains intact, and thus u is part of Y.

Case 2b: u is odd.

Since u is odd there is an edge entering u and connecting it with X. In order to show that u is part of Y, we need only show that there exists a vertex in X with a directed path leading to it from u. Construct sets C_i, with $1 \leq i \leq p$, by assigning each vertex with an even index that is greater than u, and each consecutive pair of vertices with lower index than u in order *odd-even*, to one of the C_i's, as pictured in Figure 8.

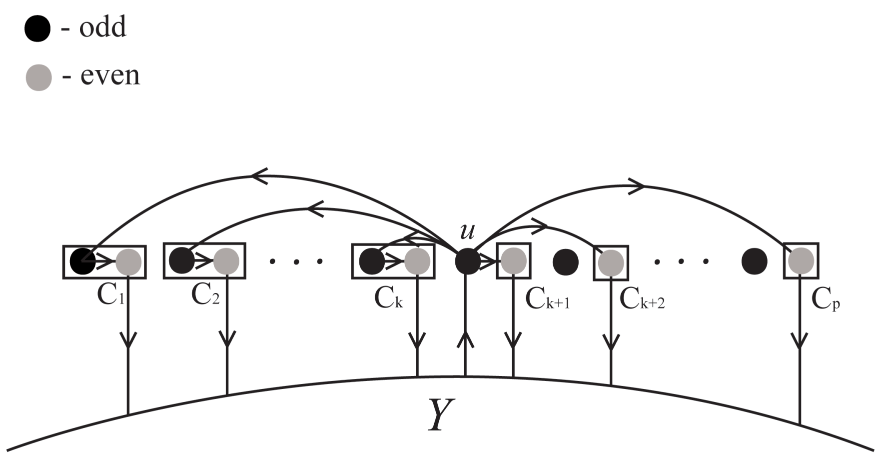

Figure 8: Picture from Case 2b

Since $|A| = 2p$, it is clear that there there are in fact p such sets. Now, for each of the single vertex C_i's, where the vertices are greater than u, by the definition of the orientation, directed edges go directly from u to each of these vertices, and another directed edge goes from each of these vertices into X. For the C_i's with two vertices, there is a directed edge from u going into the odd vertex, then a directed edge going from the odd to the even vertex, then another directed edge going from the even vertex directly to X. This gives us a total of exactly p mutually disjoint directed paths from u to a vertex in one of the C_i's to a vertex in X. Since we only delete $p-1$ vertices in total, at least one of these paths remains intact, and thus u is part of Y.

Now if the $2p$ edges in A, when indexed by their leading element, are in the order *odd*, *even*, $\ldots$, *odd*, *even*, we may achieve the same result by reindexing the vertices in the opposite order, and reversing the direction of the arcs mentioned in the previous two sub cases.

Theorem 3 $\overrightarrow{S_{2p,2}}$ *is* $(p-1)$*-connected when* $2p \geq 6$.

Proof: In $\overrightarrow{S_{2p,2}}$ there are $2p$ fundamental cliques, each of size $2p-1$, each of which is $(p-1)$-connected by Lemma 2. Let us consider deleting S from $\overrightarrow{S_{2p,2}}$

where $|S| = p-2$. Define a damaged fundamental clique as a fundamental clique containing elements of S, and an undamaged fundamental clique as one that is not damaged. Let d be the number of damaged fundamental cliques. Since $|S| = p-2$ we know that $d \leq p-2$. Let G be the digraph induced by the vertices in the undamaged fundamental cliques. Since each fundamental clique is strongly connected it follows from Lemma 2 and Lemma 4 that G is strongly connected. Since $|S| = p-2$, every damaged fundamental clique has at most $p-2$ deleted vertices, and is therefore strongly connected. Now, suppose H is a damaged fundamental clique with α deleted vertices. Since $\alpha + d - 1 \leq p - 2$ and the set of even vertices and the set of odd vertices are at least of size $p-1$, arcs exist in both directions between H and the rest of the graph. Now, since if we consider each fundamental clique as being contracted to a single vertex, since at most $p-2$ of the fundamental cliques have any vertices deleted from them, we may use Lemmas 4 to see that each undamaged component is connected to every other undamaged component, and therefore the graph is connected.

Corollary 4 $\overrightarrow{S_{n,2}}$ *is* $\lfloor \frac{n-1}{2} \rfloor$*-connected when* $n \geq 5$.

Proof: This follows directly from the previous two theorems.

Theorem 5 $\overrightarrow{S_{n,k}}$ *is* $\left\lfloor \frac{n-1}{2} \right\rfloor$*-connected when* $n \geq 5$*, and* $k \geq 2$.

Proof: We will show this by induction on k. By the previously proven lemmas and theorems we know that $\overrightarrow{S_{n,2}}$ is $\lfloor \frac{n-1}{2} \rfloor$-connected. Define H_i to be a subgraph of $\overrightarrow{S_{n,k}}$ isomorphic to $\overrightarrow{S_{n-1,k-1}}$ induced by selecting vertices with i in the k-th position. For any distinct i and j there are $\frac{(n-2)!}{(n-k)!}$ independent arcs between H_i and H_j, and they will be between edges of the form $[j, a_1, a_2, \ldots, a_{k-2}, i]$, $[i, a_1, a_2, \ldots, a_{k-2}, j]$. Also it is easy to see that there are exactly $\frac{(n-2)!}{2(n-k)!}$ arcs directed from H_i to H_j, and the same number of arcs directed from H_j to H_i since $n \geq 5$. Now consider the deletion of a subset T of the vertices where $|T| = \lfloor \frac{n-1}{2} \rfloor - 1$. Let T_i be the vertices in both T and H_i for each i.

Suppose n is even, then $\lfloor \frac{n-1}{2} \rfloor - 1 = \frac{n-2}{2} - 1$, We already know that each H_i is $\lfloor \frac{n-2}{2} \rfloor = (\frac{n-2}{2})$-connected, therefore $H_i - T_i$ is strongly connected. We are done if we can show that there is at least one arc from $H_i - T_i$ to $H_j - T_j$ and vice versa. In order to show this, we need only show that, the number of arcs going one way from H_i into H_j is greater than $\left(\frac{n-1}{2} - 1\right)$. Since $k \geq 3$,

$$\frac{(n-2)!}{2(n-k)!} - \left(\frac{n-2}{2} - 1\right) \geq \frac{(n-2)!}{2(n-3)!} - \left(\frac{n-2}{2} - 1\right)$$
$$\geq \frac{(n-2)}{2} - \left(\frac{n-2}{2} - 1\right) = 1$$

Therefore, when n is even, each H_i is strongly connected to each H_j and we are done.

Now we will examine the case when n is odd. In this case $n = 2p+1$ for some p and $\lfloor \frac{n-1}{2} \rfloor - 1 = p - 1$ and H_i is $\lfloor \frac{n-2}{2} \rfloor = \lfloor \frac{2p-1}{2} \rfloor = (p-1)$-connected. Therefore $H_i - T_i$ is strongly connected if $|T_i| \leq p - 2$.

Case 1: $|T_i| \leq p - 2$ for all i.

We are done if $\frac{(n-2)!}{2(n-k)!} - 1 \geq (p-1)$. If $k \geq 4$ then

$$\left\lfloor \frac{(n-2)!}{2(n-k)!} \right\rfloor - (p-1) \geq \left\lfloor \frac{(n-2)!}{2(n-4)!} \right\rfloor - (p-1) \geq \left\lfloor \frac{(2p-1)(2p-2)}{2} \right\rfloor - (p-1)$$

$$\geq \frac{(2p-1)(2p-2)}{2} - (p-1) \geq 1$$

Hence there is an arc from $H_i - T_i$ and $H_j - T_j$ and vice versa and each $H_i - T_i$ is strongly connected so $\overrightarrow{S_{n,k}} - T$ is strongly connected. However, if $k = 3$ then

$$\left\lfloor \frac{(n-2)!}{2(n-k)!} \right\rfloor - (p-1) \geq \left\lfloor \frac{(n-2)!}{2(n-2)!} \right\rfloor - (p-1)$$

$$\geq \left\lfloor \frac{(2p-1)}{2} \right\rfloor - (p-1) \geq (p-1) - (p-1) = 0$$

Which gives us the possibility that there are no arcs between subgraphs $H_i - T_i$ and $H_j - T_j$. This situation will only occur when $T = T_i \bigcup T_j$. Since $n \geq 5$ there are at least three other H_i's that remain undamaged and are thus strongly connected to each other. Since $|T_i| \leq p-2$ for all i, each of $H_i - T_i$ and $H_j - T_j$ must be strongly connected to one of these undamaged H_i's, and hence the entire graph is strongly connected.

Case 2: $|T_1| = p - 1$ and $|T_i| = 0$ for all $i \neq 1$.

Clearly the directed graph induced by the union of the vertices in $H_2, H_3, \ldots, H_n$ is strongly connected, let us denote this graph by X and the strong component containing X by Y. Let C be a strongly connected component of $H_1 - T_1$.

Case 2a: C is not a single isolated vertex.

If C contains a star arc then we are done because if uv is a star arc one of the vertices has an odd associated permutation and one has an even associated permutation. Then the two arcs between $\{u, v\}$ and X must be in opposite directions. Suppose C contains no star arc. Then C must be a subgraph of a directed fundamental clique of H_1. Since H_1 is isomorphic to $\overrightarrow{S_{n-1,k-1}}$, a fundamental clique is of size $(n-1)-(k-1)+1 = n-k+1$. At this point it is enough to show that C contains an odd vertex and an even vertex, because they will give arcs between C and X in both directions through star arcs. Since C is a subgraph of a directed fundamental clique, if it were to contain all odd or all even vertices then it would not be strongly connected by the definition of the orientation. So it must contain at least one vertex of each parity.

Case 2b: C is an isolated vertex, π_1.

We will show that there is a directed path containing π_1 that starts and ends in X. Assume that the arc connecting π_1 to Y is directed from π_1 to a vertex x in X. Within H_1, π_1 has p arcs directed into it and $p-1$ arcs leaving it. Therefore there must exist an arc entering π_1 from another vertex π_2 in $H_1 - T_1$. If there is an arc directed from a vertex in X to π_2 then we are done because we have a path from Y to π_2 to π_1 to X. If not, we may show by the same reasoning that π_2 must also have an arc entering it, so since at least one arc is entering π_2 we can find π_3 in H_1 where there is an arc from π_3 to π_2. Again, if there is an arc from a vertex in X directed to π_3 we are done. If not then we may repeat this process and eventually either find a vertex π_m with an arc from X leading into it in which case we will be done, or we will continue to grow this path and it will eventually generate a directed cycle $\pi_k \pi_{k-1} \ldots \pi_m$ since H_1 has finitely many vertices. If we find such a cycle, this gives us a strongly connected component in $H_1 - T_1$, which we have previously shown must be part of Y. Thus there exists a path from Y into this cycle, from which there exists a directed path to π_1 by construction, and hence π_1 is strongly connected to X and in Y. Above we assumed that the arc connecting π_1 to X was directed from π_1 to a vertex y in Y, if it is the other way around and the arc connecting π_1 to X is directed from a vertex x in X to π_1, then the exact same argument with all the directions reversed can be used to show π_1 is part of Y.

We have now shown that $\overrightarrow{S_{n,k}}$ is $\lfloor \frac{n-1}{2} \rfloor$-connected when $n \geq 5$, and $k \geq 2$.

4 Conclusion

In this paper we have shown that $\overrightarrow{S_{n,k}}$ is $\lfloor \frac{n-1}{2} \rfloor$-connected and hence maximally connected. This is an interesting nontrivial result. Showing this connectivity property helps to further show the strength and resilience of $\overrightarrow{S_{n,k}}$. This reinforces the knowledge that the $\overrightarrow{S_{n,k}}$, and the directions defined on it in [8] are well defined. Now that we have shown maximal connectivity, there are other connectivity properties that can also be looked to such as loose and tight super connectivity, which are known to hold for US_n, as shown by Cheng and Lipman in [7]. The now established property of maximal connectivity in $\overrightarrow{S_{n,k}}$ can be used to help show these stronger properties.

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