1 Conditional Heteroscedasticity Models

1.1 Model Introduction

We here first define the return or growth rate of a series. For example, if x_t is the value of a stock at time t, then the return or relative gain, y_t, of the stock at time t is

$$y_t \equiv \frac{x_t - x_{t-1}}{x_{t-1}}$$

The GARCH$(1,1)$ is:

$$\begin{aligned} y_t &= \sigma_t \epsilon_t \\ \sigma_t^2 &= \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 \end{aligned}$$

1.2 Estimation and Prediction

The standard method of estimating parameters in GARCH(m,r) is conditional maximum likelihood (we can consider ARCH(m) as special case of GARCH$(m,0)$), given by

$$L(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid y_1, ...y_m) = \prod_{t=m+1}^{n} f_{\boldsymbol{\alpha},\boldsymbol{\beta}}(y_t | y_{t-1}, ...y_{t-m})$$

and the conditional density is given by (assume first r terms of $\sigma_1^2 = ... = \sigma_r^2 = 0$)

$$y_t \mid y_{t-1}, ...y_{t-m} \sim N(0, \omega + \sum_{j=1}^{m} \alpha_j y_{t-j}^2 + \sum_{j=1}^{r} \beta_j \sigma_{t-j}^2)$$

We here only focused on the simplest Garch model GARCH$(1,1)$, the log likelihood of GARCH$(1,1)$ is defined as:

$$\begin{aligned} l(\omega, \alpha, \beta) &= -\ln L(\omega, \alpha, \beta | y) \\ &= \frac{n}{2} \ln(2\pi) + \frac{1}{2} \sum_{t=2}^{n} (\ln(\omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2) + \frac{y_t^2}{\omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2}) \end{aligned}$$

And we define the estimated $\tilde{\theta}$ on a specific interval I as

$$\tilde{\boldsymbol{\theta}}_I = \text{argmax } l_I(\boldsymbol{\theta}) = \text{argmax } l_I(\omega, \alpha, \beta)$$

1.3 Test of homogeneity

Denote the possible change point set as $\mathcal{T}(I)$ within the interval I, every point $\tau \in \mathcal{T}(I)$ splits the interval I into two subintervals $J = [t_0, \tau]$ and $J^c = I \setminus J = [\tau + 1, t_1]$, a change point alternative with the location τ means that there two parameter values satisfies $\boldsymbol{\theta}(t) = \boldsymbol{\theta}_1$ when $t \in J$ and $\boldsymbol{\theta}(t) = \boldsymbol{\theta}_2$ when $t \in J^c$ ($\boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2$).

The LR test statistics can be defined as:

$$
\begin{aligned}
T_{I,\tau} &= 2[\max_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}\{l_J(\boldsymbol{\theta}_1) + l_{J^c}(\boldsymbol{\theta}_2)\} - \max_{\boldsymbol{\theta}} l_I(\boldsymbol{\theta})] \\
&= 2[l_J(\tilde{\boldsymbol{\theta}}_1) + l_{J^c}(\tilde{\boldsymbol{\theta}}_2) - l_I(\tilde{\boldsymbol{\theta}}_I)]
\end{aligned}
$$

We should define the final test statistics as the maximum of all possible points over $\tau \in \mathcal{T}(I)$.

$$
T_I = \max_{\tau \in \mathcal{T}(I)} T_{I,\tau}
$$

A change point is detected within I if the test statistics T_I exceeds a critical value ξ.

2. Adaptive Nonparametric Estimation

2.1 Choice of critical value

Since we have no information on the distribution of our test statistics, here we use paramtric bootstrap to simulate the critical value.

The algorithms:

1. Given $\boldsymbol{\theta}_I = (\alpha, \beta, \omega)$ (assumed known) and $n = |I|$(the length of the interval), generate GARCH$(1,1)$ series.
2. Split the interval into 10 subintervals with points $\tau \in \mathcal{T}(I) = [n/10, 2n/10, ..., 9n/10]$, estimate corresponding parameters $\tilde{\boldsymbol{\theta}}_1, \tilde{\boldsymbol{\theta}}_2$ using every possible τ, calculate the test statistics $T_{I,\tau} = 2[l_J(\tilde{\boldsymbol{\theta}}_1) + l_{J^c}(\tilde{\boldsymbol{\theta}}_2) - l_I(\hat{\boldsymbol{\theta}}_I)]$. Thus, we can have the maximal value of test statistics as T_I
3. Repeat Step1~2 B times (Take $B = 200$), obtain the 10th(95% percentile) largest value as the critical value
4. Repeat Step1~3 with different values of $n = |I|$ and $\boldsymbol{\theta} = (\alpha, \beta, \omega)$, obtain critical values under different conditions

With $n = 100$, $\omega = 0.1$ and different values of α and β, the following are points and smoothed estimated curves of $T_{I,\tau}$ without step3.

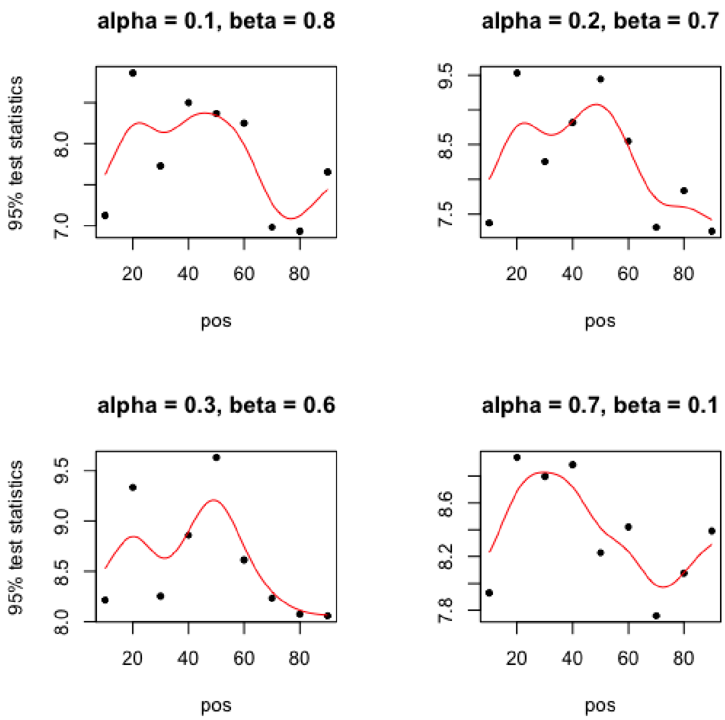

Figure 1: Different critical values in terms of position, alpha and beta (n=100)

Table 1: Test statistics with different alpha, beta and interval length

	n = 50	n = 100	n = 200	n = 400	n = 800
alpha = 0.1, beta = 0.8	8.803	8.862	9.644	8.846	10.36
alpha = 0.2, beta = 0.7	9.179	9.53	10.25	9.474	10.98
alpha = 0.3, beta = 0.6	9.388	9.631	10.28	9.706	11.26
alpha = 0.7, beta = 0.1	9.19	8.938	9.486	9.015	11.12

We here simply fit a linear regression model to estimate critical value. The following is the model summary:

$$\text{crit value} = \underset{(.1504)}{-10.31} + \underset{(2.52*10^{-5})}{0.0021n} + \underset{(0.012)}{24.48\alpha} + \underset{(0.012)}{20.73\beta} \quad (R^2 = 0.6731)$$

(Lower parenthesis show the p-values for corresponding covariate, and the total adjusted R^2 is 0.6731)

2.2 Choice of homogeneous interval

The procedures are as followed:

1. Set $k = 2$
2. Select interval $I_k = [t_k, T]$ and possible change points set $\mathcal{T}(I_k) = I_{k-1} \setminus I_{k-2}$

3. Based on the test of homogeneity within I_k against change-point alternative mentioned above and corresponding critical value ξ_k obtained by bootstrap, acquire test statistics T_{I_k} and determine whether or not reject the null hypothese

4. If a change point is detected for I_k, namely in $\mathcal{T}(I_k) = I_{k-1} \setminus I_{k-2}$, then set $\hat{I}_k = I_{k-2} = [t_{k-2}, T]$ as our final invertal for estimation and terminate the loop. Otherwise, increase $k = k + 1$ and do the detection again, continue with the step until $k > K$

2.3 Prediction and Error

The prediction is given by

$$\hat{\sigma}^2_{t+1} = \hat{\omega} + \hat{\alpha} y_t^2 + \hat{\beta} \hat{\sigma}_t^2$$

Therefore, following the above theory and algorithms, here is how we obtain the prediction.

1. We always assume the log returns of the original data follows GARCH model. Transfer the data x_t into the form of log returns: $y_t = \nabla \log(x_t)$

2. Parametric GARCH$(1,1)$: Estimate a GARCH$(1,1)$ model for the entire data sequence y_t.

3. Stepwise parametric GARCH$(1,1)$: We make estimation at every datapoint only use the former data.

4. Adaptive nonparametric GARCH$(1,1)$: Similar as 2, we do estimation at every timepoint T, but using the algorithm we have discussed above: determine the time homogeneous interval first, then do estimation and prediction.

5. In order to decrease the randomness effect between predictions $\hat{\sigma}_t^2$ and log returns y_t, we here use the average the prediction errors over one month throughout the time span to better display the outcome.

2.4 Alternative Models

2.4.1 Local Linear Regression

Given data $\{(x_i, Y_i)\ i = 1...n\}$ with x_i's fixed, we can use local linear regression to estimate $\hat{Y}$ of given x. The purpose of local linear regression is to minimize the weighted sum of squares

$$\hat{\alpha}, \hat{\beta} = \operatorname{argmin} \sum_{i=1}^{n} w_i(x)(Y_i - (\alpha + \beta x))^2$$

in which the weight function $w_i(x) = K((x_i - x)/h)$ is a kernel function with bandwidth h.

2.4.2 Wavelets

The potential problem of local linear regression is that, though categorized as nonparametric method, it has a fixed parameter which is the bandwidth h. We can use leave-one-out cross-validation or generalized cross-validation to choose the optimal h, but still in some situations the local linear method is not performing well.

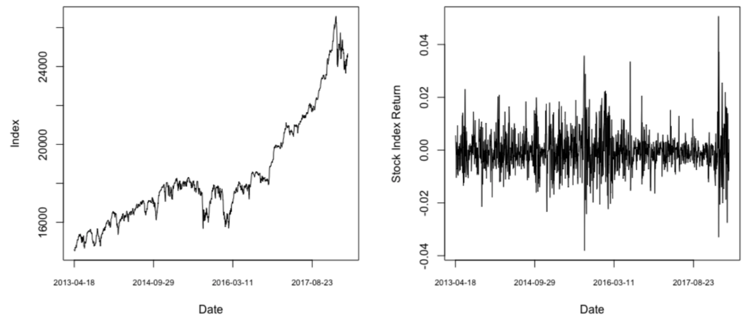

Figure 2: Stock index and returns

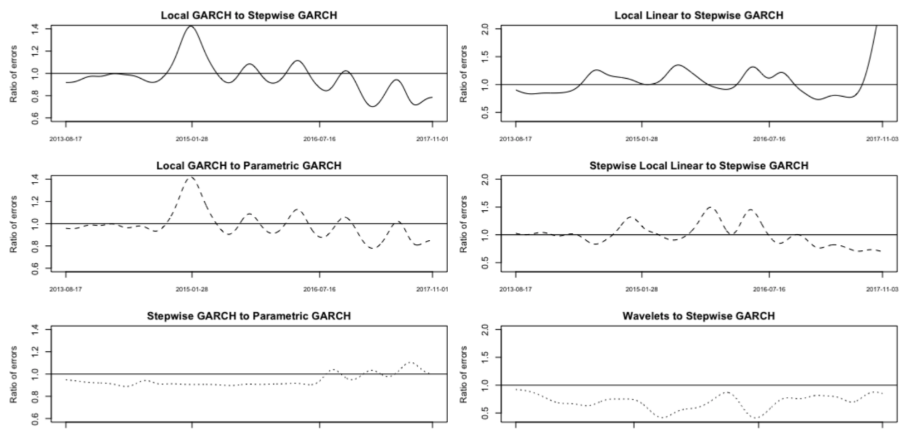

Figure 3: The ratio of squared errors among all methods

3 Real-time Application: Dow Jones Stock Index

We here apply the real time series consisting of log-returns of the Dow Jones Industrial Average in recent 5 years.

The data ranges from 04/18/2013 to 04/17/2018, with 1259 time points in total. Shown in “Figure 2”, left is the stock index, right is the corresponding returns.

Left part of “Figure 3” shows the errors ratio among three methods: parametric GARCH (global paramters), stepwise GARCH (only use data of former time-points) and our suggested adaptive GARCH. Right part shows the error ratios of local linear, stepwise local linear, wavelets regression with the stepwise GARCH.