Symmetry arguments play important role in simplifying the algebra involved in the problem. The use of symmetry arguments in writing the charges on different plates will be demonstrated later in the section of worked out examples.

31.4 FORCE BETWEEN THE PLATES OF A CAPACITOR

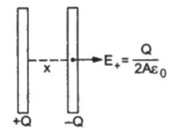

Figure 31.12

Consider a parallel-plate capacitor with plate area A. Suppose a positive charge $+Q$ is given to one plate and a negative charge $-Q$ to the other plate. The electric field due to only the positive plate is

$$E_+ = \frac{\sigma}{2\varepsilon_0} = \frac{Q}{2A\varepsilon_0}$$

at all points if the plate is large. The negative charge $-Q$ finds itself in the field of this positive charge. The force on $-Q$ is, therefore,

$$F = -QE_+$$

$$= (-Q)\frac{Q}{2A\varepsilon_0} = -\frac{Q^2}{2A\varepsilon_0}.$$

The magnitude of the force is

$$F = \frac{Q^2}{2A\varepsilon_0}.$$

This is the force with which the positive plate attracts the negative plate. This is also the force of attraction on the positive plate by the negative plate. Thus, the plates of a parallel-plate capacitor attract each other with a force

$$F = \frac{Q^2}{2A\varepsilon_0}. \qquad \text{... (31.7)}$$

31.5 ENERGY STORED IN A CAPACITOR AND ENERGY DENSITY IN ELECTRIC FIELD

Let us consider a parallel-plate capacitor of plate area A (figure 31.13). Suppose the plates of the capacitor are almost touching each other and a charge Q is given to the capacitor. One of the plates, say a, is kept fixed and the other, say b, is slowly pulled away from a to increase the separation from zero to d. The attractive force on the plate b at any instant due to the first plate is, from equation (31.7),

$$F = \frac{Q^2}{2A\varepsilon_0}.$$

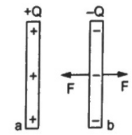

Figure 31.13

The person pulling the plate b must apply an equal force F in the opposite direction if the plate is only slowly moved.

The work done by the person during the displacement of the second plate is

$$W = Fd$$

$$= \frac{Q^2 d}{2A\varepsilon_0} = \frac{Q^2}{2\,C}$$

where C is the capacitance of the capacitor in the final position. The work done by the person must be equal to the increase in the energy of the system. Thus, a capacitor of capacitance C has a stored energy

$$U = \frac{Q^2}{2\,C} \qquad \text{... (31.8)}$$

where Q is the charge given to it. Using $Q = CV$, the above equation may also be written as

$$U = \frac{1}{2}CV^2 \qquad \text{... (31.9)}$$

or,

$$U = \frac{1}{2}QV. \qquad \text{... (31.10)}$$

Example 31.7

Find the energy stored in a capacitor of capacitance 100 μF *when it is charged to a potential difference of* 20 V.

Solution : The energy stored in the capacitor is

$$U = \frac{1}{2}CV^2 = \frac{1}{2}(100\ \mu\text{F})\,(20\ \text{V})^2 = 0{\cdot}02\ \text{J}.$$

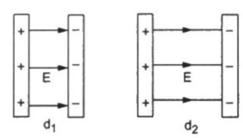

Figure 31.14

The energy stored in a capacitor is electrostatic potential energy. When we pull the plates of a capacitor apart, we have to do work against the electrostatic attraction between the plates. In which region of space is the energy stored ? When we increase the separation between the plates from d_1 to d_2, an amount $\frac{Q^2}{2A\varepsilon_0}(d_2 - d_1)$ of work is performed by us and this much energy goes into the capacitor. On the other hand, new electric field is created in a volume $A(d_2 - d_1)$ (figure 31.14). We conclude that the energy $\frac{Q^2}{2A\varepsilon_0}(d_2 - d_1)$ is stored in the volume $A(d_2 - d_1)$ which is now filled with the electric field. Thus, an electric field has energy associated with it. The energy stored per unit volume in the electric field is

$$u = \frac{\frac{Q^2(d_2 - d_1)}{2A\varepsilon_0}}{A(d_2 - d_1)} = \frac{Q^2}{2A^2\varepsilon_0}$$

$$= \frac{1}{2}\varepsilon_0\left(\frac{Q}{A\varepsilon_0}\right)^2 = \frac{1}{2}\varepsilon_0 E^2$$

where E is the intensity of the electric field.

Once it is established that a region containing electric field E has energy $\frac{1}{2}\varepsilon_0 E^2$ per unit volume, the result can be used for any electric field whether it is due to a capacitor or otherwise.

31.6 DIELECTRICS

In dielectric materials, effectively there are no free electrons. The monatomic materials are made of atoms. Each atom consists of a positively charged nucleus surrounded by electrons. In general, the centre of the negative charge coincides with the centre of the positive charge. Polyatomic materials, on the other hand, are made of molecules. The centre of the negative charge distribution in a molecule may or may not coincide with the centre of the positive charge distribution. If it does not coincide, each molecule has a permanent dipole moment $\vec{p}$. Such materials are known as *polar materials.* However, different molecules have different directions of the dipole moment because of the random thermal agitation in the material. In any volume containing a large number of molecules (say more than a thousand), the net dipole moment is zero. If such a material is placed in an electric field, the individual dipoles experience torque due to the field and they try to align along the field. On the other hand, thermal agitation tries to randomise the orientation and hence, there is a partial alignment. As a result, we get a net dipole moment in any volume of the material.

In nonpolar materials, the centre of the positive charge distribution in an atom or a molecule coincides with the centre of the negative charge distribution. The atoms or the molecules do not have any permanent dipole moment. If such a material is placed in an electric field, the electron charge distribution is slightly shifted opposite to the electric field. This induces dipole moment in each atom or molecule and thus, we get a dipole moment in any volume of the material.

Thus, when a dielectric material is placed in an electric field, dipole moment appears in any volume in it. This fact is known as *polarization* of the material. The *polarization vector* $\vec{P}$ is defined as the dipole moment per unit volume. Its magnitude P is often referred to as the polarization.

Consider a rectangular slab of a dielectric. The individual dipole moments are randomly oriented (figure 31.15a). In any volume containing a large number of molecules, the net charge is zero. When an electric field is applied, the dipoles get aligned along the field. Figure (31.15b) and (31.15c) show the effect of dipole alignment when a field is applied from left to right. We see that the interior is still charge free but the left surface of the slab gets negative charge and the right surface gets positive charge. The situation may be represented as in figure (31.15d). The charge appearing on the surface of a dielectric when placed in an electric field is called *induced charge.* As the induced charge appears due to a shift in the electrons bound to the nuclei, this charge is also called *bound charge.*

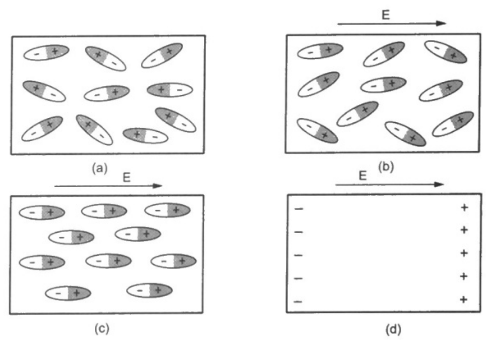

Figure 31.15

The surface charge density of the induced charge has a simple relationship with the polarization P. Suppose, the rectangular slab of figure (31.15) has a length l and area of cross-section A. Let σ_p be the magnitude of the induced charge per unit area on the faces. The dipole moment of the slab is then $(\sigma_p A)l = \sigma_p(Al)$. The polarization is dipole moment induced per unit volume. Thus,

$$P = \frac{\sigma_p\,(Al)}{Al} = \sigma_p. \qquad \ldots (31.11)$$

Although this result is deduced for a rectangular slab, it is true in general. The induced surface charge density is equal in magnitude to the polarization P.

Dielectric Constant

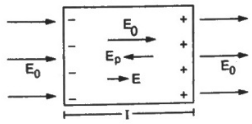

Figure 31.16

Because of the induced charges, an extra electric field is produced inside the material. Let $\vec{E}_0$ be the applied field due to external sources and $\vec{E}_p$ be the field due to polarization (figure 31.16). The resultant field is $\vec{E} = \vec{E}_0 + \vec{E}_p$. For homogeneous and isotropic dielectrics, the direction of $\vec{E}_p$ is opposite to the direction of $\vec{E}_0$. The resultant field $\vec{E}$ is in the same direction as the applied field $\vec{E}_0$ but its magnitude is reduced. We can write

$$\vec{E} = \frac{\vec{E}_0}{K}$$

where K is a constant for the given dielectric which has a value greater than one. This constant K is called the *dielectric constant* or *relative permittivity* of the dielectric. For vacuum, there is no polarization and hence $\vec{E} = \vec{E}_0$ and $K = 1$.

If a very high electric field is created in a dielectric, the outer electrons may get detached from their parent atoms. The dielectric then behaves like a conductor. This phenomenon is known as *dielectric breakdown.* The minimum field at which the breakdown occurs is called the *dielectric strength* of the material. Table (31.1) gives dielectric constants and dielectric strengths for some of the dielectrics.

Table 31.1 : *Dielectric constants and dielectric strengths*

Material	Dielectric constant	Dielectric strength ($kVmm^{-1}$)
Vacuum	1	∞
Pyrex Glass	5·6	≈ 14
Mica	3–6	12
Neoprene rubber	6·9	12
Bakelite	4·9	24
Plexiglas	3·40	40
Fused quartz	3·8	8
Paper	3·5	14
Polystyrene	2·6	25
Teflon	2·1	60
Strontium titanate	310	8
Titanium dioxide	100	6
Water	80	–
Glycerin	42·5	–
Benzene	2·3	–
Air (1 atm)	1·00059	3
Air (100 atm)	1·0548	–

31.7. PARALLEL-PLATE CAPACITOR WITH A DIELECTRIC

Consider a parallel-plate capacitor with plate area A and separation d between the plates (figure 31.17). A dielectric slab of dielectric constant K is inserted in the space between the plates. Suppose, the slab almost completely fills the space between the plates. A charge Q is given to the positive plate and $-Q$ to the negative plate of the capacitor. The electric field polarizes the dielectric so that induced charges $+Q_p$ and $-Q_p$ appear on the two faces of the slab.

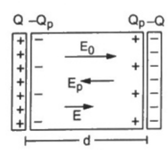

Figure 31.17

The electric field at a point between the plates due to the charges $+Q$, $-Q$ on the capacitor plates is

$$E_0 = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0} \qquad \dots \text{(i)}$$

in a direction left to right in the figure (31.17).

From the definition of dielectric constant, the resultant field is

$$E = \frac{E_0}{K} = \frac{Q}{\varepsilon_0 AK} \cdot \qquad \dots \text{(ii)}$$

The potential difference between the plates is

$$V = Ed = \frac{Qd}{\varepsilon_0 AK} \cdot$$

The capacitance is

$$C = \frac{Q}{V} = \frac{K\varepsilon_0 A}{d} = KC_0 \qquad \dots (31.12)$$

where $C_0 = \frac{\varepsilon_0 A}{d}$ is the capacitance without the dielectric. Thus,

The capacitance of a capacitor is increased by a factor of K when the space between the plates is filled with a dielectric of dielectric constant K.

This result is often taken as the definition of the dielectric constant.

Magnitude of the Induced Charge

From (i), the electric field at a point between the plates due to the charges $+Q$, $-Q$ is

$$E_0 = \frac{Q}{A\varepsilon_0} \cdot$$

The field due to the charges Q_p, $-Q_p$ is directed oppositely and has magnitude

$$E_p = \frac{\sigma_p}{\varepsilon_0} = \frac{Q_p}{A\varepsilon_0}.$$

The resultant field is

$$E = E_0 - E_p$$

$$= \frac{Q - Q_p}{A\varepsilon_0}. \quad \ldots \text{(iii)}$$

From equations (ii) and (iii),

$$\frac{Q - Q_p}{\varepsilon_0 A} = \frac{Q}{\varepsilon_0 AK}$$

or, $$Q - Q_p = \frac{Q}{K}$$

or, $$Q_p = Q\left(1 - \frac{1}{K}\right). \quad \ldots (31.13)$$

Example 31.8

Two parallel-plate capacitors, each of capacitance 40 μF, *are connected in series. The space between the plates of one capacitor is filled with a dielectric material of dielectric constant* $K = 4$. *Find the equivalent capacitance of the system.*

Solution : The capacitance of the capacitor with the dielectric is

$$C_1 = KC_0 = 4 \times 40\ \mu\text{F} = 160\ \mu\text{F}.$$

The other capacitor has capacitance $C_2 = 40\ \mu\text{F}$. As they are connected in series, the equivalent capacitance is

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{(160\ \mu\text{F})\,(40\ \mu\text{F})}{200\ \mu\text{F}} = 32\ \mu\text{F}.$$

Example 31.9

A parallel-plate capacitor has plate area A and plate separation d. The space between the plates is filled up to a thickness x (< *d*) *with a dielectric of dielectric constant K. Calculate the capacitance of the system.*

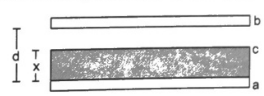

Figure 31.18

Solution :

The situation is shown in figure (31.18). The given system is equivalent to the series combination of two capacitors, one between *a* and *c* and the other between *c* and *b*. Here *c* represents the upper surface of the dielectric. This is because the potential at the upper surface of the dielectric is constant and we can imagine a thin metal plate being placed there.

The capacitance of the capacitor between *a* and *c* is

$$C_1 = \frac{K\varepsilon_0 A}{x}$$

and that between *c* and *b* is

$$C_2 = \frac{\varepsilon_0 A}{d - x}.$$

The equivalent capacitance is

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{K\varepsilon_0 A}{Kd - x(K-1)}.$$

31.8 AN ALTERNATIVE FORM OF GAUSS'S LAW

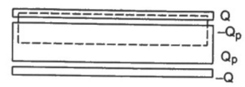

Figure 31.19

Let us again consider a parallel-plate capacitor with a charge Q. The space between the plates is filled with a dielectric slab of dielectric constant K. Let us consider a Gaussian surface as shown in figure (31.19). The charge enclosed by the surface is $Q - Q_p$. From Gauss's law,

$$\oint \vec{E}\cdot d\vec{S} = \frac{Q - Q_p}{\varepsilon_0} \quad \ldots \text{(i)}$$

$$= \frac{1}{\varepsilon_0}\left[Q - Q\left(1 - \frac{1}{K}\right)\right] = \frac{Q}{\varepsilon_0 K}$$

or, $$\oint K\vec{E}\cdot d\vec{S} = \frac{Q_{free}}{\varepsilon_0}. \quad \ldots (31.14)$$

Q_{free} is used in place of Q to emphasise that it is the free charge given to the plates and does not include the bound charge appearing due to polarization.

Equation (31.14) is taken as another form of Gauss's law. This form differs from the usual form of Gauss's law in two respects. Firstly, the charge Q_{free} appearing on the right-hand side is not the total charge inside the Gaussian surface. It is the free charge or external charge inside the Gaussian surface. The bound charge Q_p appearing due to polarization of the dielectric is left out. Secondly, an extra factor K appears on the left-hand side. The two differences compensate the effects of each other and the two forms of Gauss's law are identical. Either of the two may be used in any case.

Though we derived this result for a special case of parallel-plate capacitor, it is true in any situation where the dielectric used is homogeneous and isotropic. Let us now write Gauss's law in yet another form valid for any case.

Displacement Vector

The field due to the polarization is

$$E_p = \frac{\sigma_p}{\varepsilon_0} = \frac{P}{\varepsilon_0}$$

where P is the polarization (the dipole moment per unit volume). As the direction of E_p is opposite to the polarization vector $\vec{P}$, we write

$$\vec{E}_p = -\frac{\vec{P}}{\varepsilon_0}.$$

Now, $$\vec{E} = \vec{E}_0 + \vec{E}_p$$

or, $$\vec{E} = \vec{E}_0 - \frac{\vec{P}}{\varepsilon_0}$$

or, $$\varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E}_0 \quad \dots \text{(i)}$$

or, $$\oint (\varepsilon_0 \vec{E} + \vec{P}) \cdot d\vec{S} = \oint \varepsilon_0 \vec{E}_0 \cdot d\vec{S}$$

over any closed surface. As $\vec{E}_0$ is the field produced by the free charge Q_{free}, $\oint \varepsilon_0 \vec{E}_0 \cdot d\vec{S} = Q_{free}$ from Gauss's law. Thus,

$$\oint (\varepsilon_0 \vec{E} + \vec{P}) \cdot d\vec{S} = Q_{free}. \quad \dots \text{(ii)}$$

The quantity $\varepsilon_0 \vec{E} + \vec{P}$ is known as the *electric displacement vector* $\vec{D}$. Equation (ii) above may be written in terms of $\vec{D}$ as

$$\oint \vec{D} \cdot d\vec{S} = Q_{free} \quad \dots (31.15)$$

which is another form of Gauss's law.

If there is no polarization, $\vec{D} = \varepsilon_0 \vec{E}$ and Q_{free} is equal to the total charge inside the Gaussian surface. Equation (31.15) then reduces to the usual form of Gauss's law.

In case of homogeneous and isotropic dielectrics, $\vec{E}_0 = K\vec{E}$ so that equation (i) above gives $\vec{D} = \varepsilon_0 K\vec{E}$ and equation (31.15) reduces to (31.14).

31.9 ELECTRIC FIELD DUE TO A POINT CHARGE q PLACED IN AN INFINITE DIELECTRIC

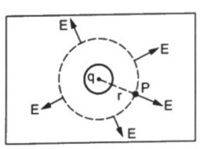

Figure 31.20

Suppose, a point charge q is placed inside an infinite dielectric and we wish to calculate the electric field at a point P at a distance r from the charge q (figure 31.20). We draw a spherical surface through P with the centre at q. From Gauss's law,

$$\oint K\, \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

or, $$KE\, 4\pi r^2 = \frac{q}{\varepsilon_0}$$

or, $$E = \frac{q}{4\pi\varepsilon_0 K r^2}. \quad \dots (31.16)$$

The field is radially away from the charge. Note that q is the total *free charge* inside the Gaussian surface.

It should be clear that the field $\frac{q}{4\pi\varepsilon_0 K r^2}$ is due to the free charge q and the polarization charges induced in the dielectric medium. Because of the radially outward field (assuming q to be positive), negative charges shift inward. This produces an induced charge $-q\left(1 - \frac{1}{K}\right)$ on the surface of the cavity in the dielectric in which the charge q is residing. The effective charge is, therefore, $q - q\left(1 - \frac{1}{K}\right) = q/K$ and hence the field is $\frac{q}{4\pi\varepsilon_0 K r^2}$.

31.10 ENERGY IN THE ELECTRIC FIELD IN A DIELECTRIC

Consider a parallel-plate capacitor filled with a dielectric of dielectric constant K. The energy stored in the capacitor is $U = \frac{1}{2} CV^2$. The energy density in the volume between the plates is

$$u = \frac{U}{Ad} = \frac{\frac{1}{2}\left(\frac{K\varepsilon_0 A}{d}\right)V^2}{Ad} = \frac{1}{2} K\varepsilon_0 \left(\frac{V}{d}\right)^2 = \frac{1}{2} K\varepsilon_0 E^2$$

where $E = V/d$ is the electric field between the plates.

We see that the energy density in dielectrics is greater than that in vacuum for the same electric field. The dipole moments interact with each other so as to give this additional energy.

31.11 CORONA DISCHARGE

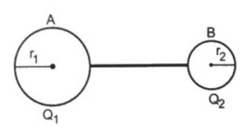

Figure 31.21

Let us consider two conducting spheres A and B connected to each other by a conducting wire. The radius of A is r_1 which is larger than the radius r_2 of B. A charge Q is given to this system. Suppose a part

Q_1 resides on the surface of A and the rest Q_2 on the surface of B. The potential of the sphere A is

$$V_1 = \frac{Q_1}{4\pi\varepsilon_0 r_1}$$

and that of the sphere B is

$$V_2 = \frac{Q_2}{4\pi\varepsilon_0 r_2}.$$

As the two spheres are connected by a conducting wire, their potentials must be the same. Thus,

$$\frac{Q_1}{4\pi\varepsilon_0 r_1} = \frac{Q_2}{4\pi\varepsilon_0 r_2}$$

or, $$\sigma_1 r_1 = \sigma_2 r_2$$

or, $$\frac{\sigma_1}{\sigma_2} = \frac{r_2}{r_1} \qquad \dots (31.17)$$

where σ_1 and σ_2 are charge densities on the two spheres. We see that the sphere with smaller radius has larger charge density to maintain the same potential.

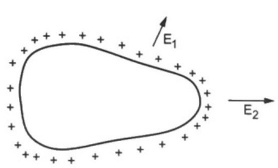

Figure 31.22

Now consider a single conductor with a nonspherical shape. If a charge is given to this conductor (figure 31.22), the charge density will not be uniform on the entire surface. A portion where the surface is more "flat" may be considered as part of a sphere of larger radius. The charge density at such a portion will be smaller from equation (31.17). At portions where the surface is more curved, the charge density will be larger. More precisely, the charge density will be larger where the radius of curvature is small.

The electric field just outside the surface of a conductor is σ/ε_0. Thus, the electric field near the portions of small radius of curvature (more curved part) is large as compared to the field near the portions of large radius of curvature (flatter part). If a conductor has a pointed shape like a needle and a charge is given to it, the charge density at the pointed end will be very high. Correspondingly, the electric field near these pointed ends will be very high which may cause dielectric breakdown in air. The charge may jump from the conductor to the air because of increased conductivity of the air. Often this discharge of air is accompanied by a visible glow surrounding the pointed end. This phenomenon is called *corona discharge.*

31.12 HIGH-VOLTAGE GENERATOR

In 1929, Robert J van de Graaff designed a machine which could produce large electrostatic potential difference, of the order of 10^7 volts. This machine, known as *van de Graaff generator,* is now described.

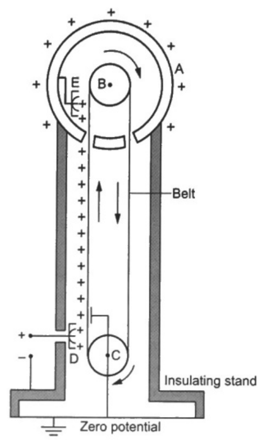

Figure 31.23

A hollow, metallic sphere A is mounted on an insulating stand. A pulley B is mounted at the centre of the sphere and another pulley C is mounted near the bottom. A belt of insulating material (such as silk) goes over the pulleys. The pulley C is continuously driven by an electric motor, or by hand for a smaller machine used for demonstration. The belt, therefore, continuously moves. Two comb-shaped conductors D and E, having a number of metallic needles, are mounted near the pulleys. The needles point towards the belt. The lower comb D is maintained at a positive potential of the order of 10^4 volts by a power supply system. The upper comb E is connected to the metallic sphere A.

Because of the high electric field near the needles of D, the air becomes conducting (corona discharge). The negative charges in the air move towards the needles and the positive charges towards the belt. This positive charge sticks to the belt. The negative charge neutralises some of the positive charge on the comb D. The power supply maintains the positive potential of the needles by supplying more positive charge to it. Effectively, positive charge is transferred from the power supply to the belt. As the belt moves, this positive charge is physically carried upwards. When it reaches near the upper comb E, corona discharge takes place and the air becomes conducting. The negative charges of the air move towards the belt and the positive charges towards the needles of the comb. The negative charges neutralise the positive charge on the belt. The positive charges of the air which have moved to the comb are transferred to the sphere. Effectively, the positive charge on the belt is transferred to the sphere. This positive charge quickly goes to the outer surface of the sphere.

The machine, thus, continuously transfers positive charge to the sphere. The potential of the sphere keeps on increasing. The main limiting factor on the value of this high potential is the radius of the sphere. If the electric field just outside the sphere is sufficient for dielectric breakdown of air, no more charge can be transferred to it. The dielectric strength of air is $3 \times 10^{6}\ \mathrm{V\ m^{-1}}$. For a conducting sphere, the electric field just outside the sphere is $E = \frac{Q}{4\pi\varepsilon_0 R^2}$ and the potential of the sphere is $V = \frac{Q}{4\pi\varepsilon_0 R}$. Thus, $V = ER$. To have a field of $3 \times 10^{6}\ \mathrm{V\ m^{-1}}$ with a sphere of radius 1 m, its potential should be 3×10^{6} V. Thus, the potential of a sphere of radius 1 m can be raised to 3×10^{6} V by this method. The potential can be increased by enclosing the sphere in a highly evacuated chamber.

Worked Out Examples

1. *A parallel-plate capacitor has plates of area* 200 cm^2 *and separation between the plates* 1·00 mm. *What potential difference will be developed if a charge of* 1·00 nC (*i.e.,* $1{\cdot}00 \times 10^{-9}$ C) *is given to the capacitor? If the plate separation is now increased to* 2·00 mm, *what will be the new potential difference?*

Solution : The capacitance of the capacitor is $C = \frac{\varepsilon_0 A}{d}$

$$= 8{\cdot}85 \times 10^{-12}\ \mathrm{Fm^{-1}} \times \frac{200 \times 10^{-4}\ \mathrm{m}^2}{1 \times 10^{-3}\ \mathrm{m}}$$

$$= 0{\cdot}177 \times 10^{-9}\ \mathrm{F} = 0{\cdot}177\ \mathrm{nF}.$$

The potential difference between the plates is

$$V = \frac{Q}{C} = \frac{1\ \mathrm{nC}}{0{\cdot}177\ \mathrm{nF}} = 5{\cdot}65\ \text{volts}.$$

If the separation is increased from 1·00 mm to 2·00 mm, the capacitance is decreased by a factor of 2. If the charge remains the same, the potential difference will increase by a factor of 2. Thus, the new potential difference will be

$$5{\cdot}65\ \text{volts} \times 2 = 11{\cdot}3\ \text{volts}.$$

2. *An isolated sphere has a capacitance of* 50 pF. *(a) Calculate its radius. (b) How much charge should be placed on it to raise its potential to* 10^4 V?

Solution : (a) The capacitance of an isolated sphere is $C = 4\pi\varepsilon_0 R$. Thus,

$$50 \times 10^{-12}\ \mathrm{F} = \frac{R}{9 \times 10^{9}\ \mathrm{mF^{-1}}}$$

or, $R = 50 \times 10^{-12} \times 9 \times 10^{9}\ \mathrm{m} = 45\ \mathrm{cm}.$

(b) $Q = CV$

$$= 50 \times 10^{-12}\ \mathrm{F} \times 10^{4}\ \mathrm{V} = 0{\cdot}5\ \mu\mathrm{C}.$$

3. *Consider the connections shown in figure (31-W1). (a) Find the capacitance between the points A and B. (b) Find the charges on the three capacitors. (c) Taking the potential at the point B to be zero, find the potential at the point D.*

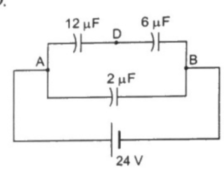

Figure 31-W1

Solution : (a) The 12 μF and 6 μF capacitors are joined in series. The equivalent of these two will have a capacitance given by

$$\frac{1}{C} = \frac{1}{12\ \mu\mathrm{F}} + \frac{1}{6\ \mu\mathrm{F}},$$

or, $C = 4\ \mu\mathrm{F}.$

The combination of these two capacitors is joined in parallel with the 2 μF capacitor. Thus, the equivalent capacitance between A and B is

$$4\ \mu\mathrm{F} + 2\ \mu\mathrm{F} = 6\ \mu\mathrm{F}.$$

(b) The charge supplied by the battery is

$$Q = CV = 6\ \mu\mathrm{F} \times 24\ \mathrm{V} = 144\ \mu\mathrm{C}.$$

The potential difference across the 2 μF capacitor is 24 V. The charge on this capacitor is, therefore,

$$2\ \mu\mathrm{F} \times 24\ \mathrm{V} = 48\ \mu\mathrm{C}.$$

The charge on the 12 μF and 6 μF capacitor is, therefore,

$$144\ \mu\mathrm{C} - 48\ \mu\mathrm{C} = 96\ \mu\mathrm{C}.$$

(c) The potential difference across the 6 μF capacitor is

$$\frac{96\ \mu\mathrm{C}}{6\ \mu\mathrm{F}} = 16\ \mathrm{V}.$$

As the potential at the point B is taken to be zero, the potential at the point D is 16 V.

4. *If 100 volts of potential difference is applied between a and b in the circuit of figure (31-W2a), find the potential difference between c and d.*

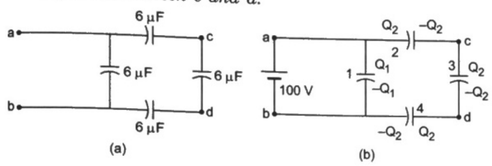

Figure 31-W2

Solution : The charge distribution on different plates is shown in figure (31-W2b). Suppose charge $Q_1 + Q_2$ is given by the positive terminal of the battery, out of which Q_1 resides on the positive plate of capacitor (1) and Q_2 on that of (2). The remaining plates will have charges as shown in the figure.

Take the potential at the point b to be zero. The potential at a will be 100 V. Let the potentials at points c and d be V_c and V_d respectively. Writing the equation $Q = CV$ for the four capacitors, we get,

$$Q_1 = 6\ \mu\text{F} \times 100\ \text{V} = 600\ \mu\text{C} \quad \dots \text{(i)}$$

$$Q_2 = 6\ \mu\text{F} \times (100\ \text{V} - V_c) \quad \dots \text{(ii)}$$

$$Q_2 = 6\ \mu\text{F} \times (V_c - V_d) \quad \dots \text{(iii)}$$

$$Q_2 = 6\ \mu\text{F} \times V_d. \quad \dots \text{(iv)}$$

From (ii) and (iii),

$$100\ \text{V} - V_c = V_c - V_d$$

or, $$2V_c - V_d = 100\ \text{V} \quad \dots \text{(v)}$$

and from (iii) and (iv),

$$V_c - V_d = V_d$$

or, $$V_c = 2V_d. \quad \dots \text{(vi)}$$

From (v) and (vi),

$$V_d = \frac{100}{3}\ \text{V} \quad \text{and} \quad V_c = \frac{200}{3}\ \text{V}$$

so that $V_c - V_d = \dfrac{100}{3}$ V.

5. *Find the charges on the three capacitors shown in figure (31-W3a).*

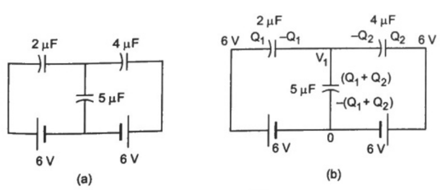

Figure 31-W3

Solution : Take the potential at the junction of the batteries to be zero. Let the left battery supply a charge Q_1 and the right battery a charge Q_2. The charge on the 5 μF capacitor will be $Q_1 + Q_2$. Let the potential at the junction of the capacitors be V_1. The charges at different plates and potentials at different points are shown in figure (31-W3b).

Note that the charges on the three plates which are in contact add to zero. It should be so, because, these plates taken together form an isolated system which cannot receive charges from the batteries. Applying the equation $Q = CV$ to the three capacitors, we get,

$$Q_1 = 2\ \mu\text{F}(6\ \text{V} - V_1) \quad \dots \text{(i)}$$

$$Q_2 = 4\ \mu\text{F}(6\ \text{V} - V_1) \quad \dots \text{(ii)}$$

and $$Q_1 + Q_2 = 5\ \mu\text{F}(V_1 - 0). \quad \dots \text{(iii)}$$

From (i) and (ii),

$$2Q_1 - Q_2 = 0 \ \text{ or, } \ Q_2 = 2Q_1.$$

From (ii) and (iii),

$$5Q_2 + 4(Q_1 + Q_2) = 20\ \mu\text{F} \times 6\ \text{V}$$

or, $$4Q_1 + 9Q_2 = 120\ \mu\text{C}$$

or, $$4Q_1 + 18Q_1 = 120\ \mu\text{C}$$

or, $Q_1 = 5{\cdot}45\ \mu$C and $Q_2 = 10{\cdot}9\ \mu$C.

Thus, the charges on the 2 μF, 4 μF and 5 μF capacitors are 5·45 μC, 10·9 μC and 16·35 μC respectively.

6. *Find the equivalent capacitance of the system shown in figure (31-W4a) between the points a and b.*

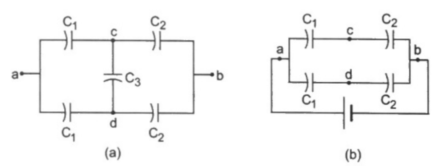

Figure 31-W4

Solution : Suppose, the capacitor C_3 is removed from the given system and a battery is connected between a and b. The remaining system is shown in figure (31-W4b). From the symmetry of the figure, the potential at c will be the same as the potential at d. Thus, if the capacitor C_3 is connected between c and d, it will have no charge. The charges of all the remaining four capacitors will remain unchanged. Thus, the system of capacitors in figure (31-W4a) is equivalent to that in the figure (31-W4b). The equivalent capacitance of the system in figure (31-W4b) can be calculated by applying the formulae for series and parallel combinations. C_1 and C_2 are connected in series. Their equivalent capacitance is

$$C = \frac{C_1 C_2}{C_1 + C_2}.$$

Two such capacitors are joined in parallel. So the equivalent capacitance of the given system is

$$2C = \frac{2\,C_1C_2}{C_1 + C_2}\cdot$$

7. ***Find the equivalent capacitance between the point A and B in figure (31-W5a).***

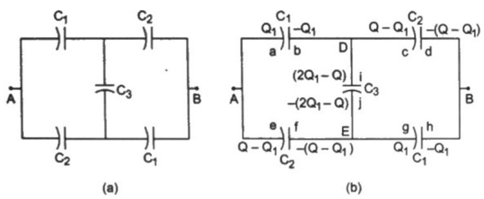

Figure 31-W5

Solution : Let us connect a battery between the points *A* and *B*. The charge distribution is shown in figure (31-W5b). Suppose the positive terminal of the battery supplies a charge $+Q$ and the negative terminal a charge $-Q$. The charge Q is divided between plates a and e. A charge Q_1 goes to the plate a and the rest $Q - Q_1$ goes to the plate e. The charge $-Q$ supplied by the negative terminal is divided between plates d and h. Using the symmetry of the figure, charge $-Q_1$ goes to the plate h and $-(Q - Q_1)$ to the plate d. This is because if you look into the circuit from A or from B, the circuit looks identical. The division of charge at A and at B should, therefore, be similar. The charges on the other plates may be written easily. The charge on the plate i is $2\,Q_1 - Q$ which ensures that the total charge on plates b, c and i remains zero as these three plates form an isolated system.

We have,

$$V_A - V_B = (V_A - V_D) + (V_D - V_B)$$

$$= \frac{Q_1}{C_1} + \frac{Q - Q_1}{C_2} \qquad \dots \text{(i)}$$

Also,
$$V_A - V_B = (V_A - V_D) + (V_D - V_E) + (V_E - V_B)$$

$$= \frac{Q_1}{C_1} + \frac{2\,Q_1 - Q}{C_3} + \frac{Q_1}{C_1}\cdot \qquad \dots \text{(ii)}$$

We have to eliminate Q_1 from these equations to get the equivalent capacitance $Q/(V_A - V_B)$.

The first equation may be written as

$$V_A - V_B = Q_1\left(\frac{1}{C_1} - \frac{1}{C_2}\right) + \frac{Q}{C_2}$$

or, $$\frac{C_1C_2}{C_2 - C_1}(V_A - V_B) = Q_1 + \frac{C_1}{C_2 - C_1}Q. \qquad \dots \text{(iii)}$$

The second equation may be written as

$$V_A - V_B = 2\,Q_1\left(\frac{1}{C_1} + \frac{1}{C_3}\right) - \frac{Q}{C_3}$$

or, $$\frac{C_1C_3}{2(C_1 + C_3)}(V_A - V_B) = Q_1 - \frac{C_1}{2(C_1 + C_3)}Q. \qquad \dots \text{(iv)}$$

Subtracting (iv) from (iii),

$$(V_A - V_B)\left[\frac{C_1C_2}{C_2 - C_1} - \frac{C_1C_3}{2(C_1 + C_3)}\right]$$

$$= \left[\frac{C_1}{C_2 - C_1} + \frac{C_1}{2(C_1 + C_3)}\right]Q$$

or, $$(V_A - V_B)\,[2\,C_1C_2(C_1 + C_3) - C_1C_3(C_2 - C_1)]$$

$$= C_1[2\,(C_1 + C_3) + (C_2 - C_1)]\,Q$$

or, $$C = \frac{Q}{V_A - V_B} = \frac{2\,C_1C_2 + C_2C_3 + C_3C_1}{C_1 + C_2 + 2\,C_3}.$$

8. ***Twelve capacitors, each having a capacitance C, are connected to form a cube (figure 31-W6a). Find the equivalent capacitance between the diagonally opposite corners such as A and B.***

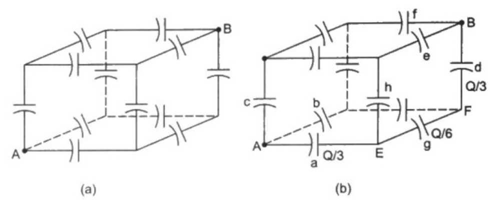

Figure 31-W6

Solution : Suppose the points A and B are connected to a battery. The charges appearing on some of the capacitors are shown in figure (31-W6b). Suppose the positive terminal of the battery supplies a charge $+\,Q$ through the point A. This charge is divided on the three plates connected to A. Looking from A, the three sides of the cube have identical properties and hence, the charge will be equally distributed on the three plates. Each of the capacitors a, b and c will receive a charge $Q/3$.

The negative terminal of the battery supplies a charge $-\,Q$ through the point B. This is again divided equally on the three plates connected to B. Each of the capacitors d, e and f gets equal charge $Q/3$.

Now consider the capacitors g and h. As the three plates connected to the point E form an isolated system, their total charge must be zero. The negative plate of the capacitor a has a charge $-\,Q/3$. The two plates of g and h connected to E should have a total charge $Q/3$. By symmetry, these two plates should have equal charges and hence each of these has a charge $Q/6$.

The capacitors a, g and d have charges $Q/3$, $Q/6$ and $Q/3$ respectively.

We have,

$$V_A - V_B = (V_A - V_E) + (V_E - V_F) + (V_F - V_B)$$

$$= \frac{Q/3}{C} + \frac{Q/6}{C} + \frac{Q/3}{C} = \frac{5Q}{6C}$$

or, $$C_{eq} = \frac{Q}{V_A - V_B} = \frac{6}{5} C.$$

9. *The negative plate of a parallel plate capacitor is given a charge of -20×10^{-8} C. Find the charges appearing on the four surfaces of the capacitor plates.*

Solution **:**

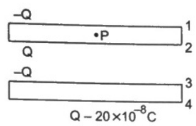

Figure 31-W7

Let the charge appearing on the inner surface of the negative plate be $-Q$. The charge on its outer surface will be $Q - 20 \times 10^{-8}$ C.

The charge on the inner surface of the positive plate will be $+Q$ from Gauss's law and that on the outer surface will be $-Q$ as the positive plate is electrically neutral. The distribution is shown in figure (31-W7).

To obtain the value of Q, consider the electric field at a point P inside the upper plate.

Field due to surface (1) $= \frac{Q}{2\varepsilon_0 A}$ upward,

due to surface (2) $= \frac{Q}{2\varepsilon_0 A}$ upward,

due to surface (3) $= \frac{Q}{2\varepsilon_0 A}$ downward

and due to surface (4) $= \frac{Q - 20 \times 10^{-8}\text{ C}}{2\varepsilon_0 A}$ upward.

As P is a point inside the conductor, the field here must be zero. Thus,

$$Q = -Q + 20 \times 10^{-8}\text{ C}$$

or, $$Q = 10 \times 10^{-8}\text{ C}.$$

The charges on the four surfaces may be written immediately from figure (31-W7).

10. *Three capacitors of capacitances 2 μF, 3 μF and 6 μF are connected in series with a 12 V battery. All the connecting wires are disconnected, the three positive plates are connected together and the three negative plates are connected together. Find the charges on the three capacitors after the reconnection.*

Solution **:** The equivalent capacitance of the three capacitors joined in series is given by

$$\frac{1}{C} = \frac{1}{2\,\mu\text{F}} + \frac{1}{3\,\mu\text{F}} + \frac{1}{6\,\mu\text{F}}$$

or, $$C = 1\,\mu\text{F}.$$

The charge supplied by the battery $= 1\,\mu\text{F} \times 12\text{ V}$
$= 12\,\mu\text{C}$.

As the capacitors are connected in series, 12 μC charge appears on each of the positive plates and −12 μC on each of the negative plates. The charged capacitors are now connected as shown in figure (31-W8).

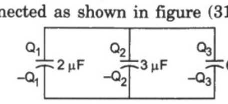

Figure 31-W8

The 36 μC charge on the three positive plates now redistribute as Q_1, Q_2 and Q_3 on the three connected positive plates. Similarly, −36 μC redistributes as $-Q_1$, $-Q_2$ and $-Q_3$. The three positive plates are now at a common potential and the three negative plates are also at a common potential. Let the potential difference across each capacitor be V. Then

$$Q_1 = (2\,\mu\text{F})\,V,$$
$$Q_2 = (3\,\mu\text{F})\,V,$$

and $$Q_3 = (6\,\mu\text{F})\,V.$$

Also, $$Q_1 + Q_2 + Q_3 = 36\,\mu\text{C}.$$

Solving these equations,

$$Q_1 = \frac{72}{11}\,\mu\text{C},\ Q_2 = \frac{108}{11}\,\mu\text{C} \text{ and } Q_3 = \frac{216}{11}\,\mu\text{C}.$$

11. *The connections shown in figure (31-W9a) are established with the switch S open. How much charge will flow through the switch if it is closed ?*

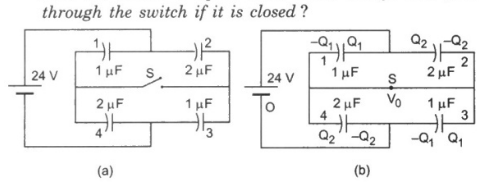

Figure 31-W9

Solution **:** When the switch is open, capacitors (2) and (3) are in series. Their equivalent capacitance is $\frac{2}{3}$ μF. The charge appearing on each of these capacitors is, therefore, $24\text{ V} \times \frac{2}{3}\,\mu\text{F} = 16\,\mu\text{C}$.

The equivalent capacitance of (1) and (4), which are also connected in series, is also $\frac{2}{3}$ μF and the charge on each of these capacitors is also 16 μC. The total charge on the two plates of (1) and (4) connected to the switch is, therefore, zero.

The situation when the switch S is closed is shown in figure (31-W9b). Let the charges be distributed as shown in the figure. Q_1 and Q_2 are arbitrarily chosen for the positive plates of (1) and (2). The same magnitude of charges will appear at the negative plates of (3) and (4).

Take the potential at the negative terminal to be zero and at the switch to be V_0.

Writing equations for the capacitors (1), (2), (3) and (4),

$$Q_1 = (24\text{ V} - V_0) \times 1\ \mu\text{F} \quad \ldots \text{(i)}$$

$$Q_2 = (24\text{ V} - V_0) \times 2\ \mu\text{F} \quad \ldots \text{(ii)}$$

$$Q_1 = V_0 \times 1\ \mu\text{F} \quad \ldots \text{(iii)}$$

$$Q_2 = V_0 \times 2\ \mu\text{F}. \quad \ldots \text{(iv)}$$

From (i) and (iii), $V_0 = 12$ V.

Thus, from (iii) and (iv),

$$Q_1 = 12\ \mu\text{C} \text{ and } Q_2 = 24\ \mu\text{C}.$$

The charge on the two plates of (1) and (4) which are connected to the switch is, therefore, $Q_2 - Q_1 = 12\ \mu$C.

When the switch was open, this charge was zero. Thus, 12 μC of charge has passed through the switch after it was closed.

12. *Each of the three plates shown in figure (31-W10a) has an area of 200 cm² on one side and the gap between the adjacent plates is 0·2 mm. The emf of the battery is 20 V. Find the distribution of charge on various surfaces of the plates. What is the equivalent capacitance of the system between the terminal points?*

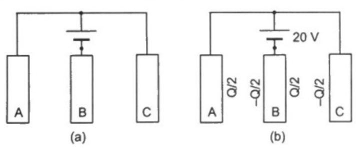

Figure 31-W10

Solution : Suppose the negative terminal of the battery gives a charge $-Q$ to the plate B. As the situation is symmetric on the two sides of B, the two faces of the plate B will share equal charges $-Q/2$ each. From Gauss's law, the facing surfaces will have charges $Q/2$ each. As the positive terminal of the battery has supplied just this much charge $(+Q)$ to A and C, the outer surfaces of A and C will have no charge. The distribution will be as shown in figure (31-W10b).

The capacitance between the plates A and B is

$$8{\cdot}85 \times 10^{-12}\text{ F m}^{-1} \times \frac{200 \times 10^{-4}\text{ m}^2}{2 \times 10^{-4}\text{ m}}$$

$$= 8{\cdot}85 \times 10^{-10}\text{ F} = 0{\cdot}885\text{ nF}.$$

Thus, $\quad \frac{Q}{2} = 0{\cdot}885\text{ nF} \times 20\text{ V} = 17{\cdot}7\text{ nC}.$

The distribution of charge on various surfaces may be written from figure (31-W10b).

The equivalent capacitance is

$$\frac{Q}{20\text{ V}} = 1{\cdot}77\text{ nF}.$$

13. *Find the capacitance of the infinite ladder shown in figure (31-W11).*

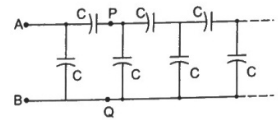

Figure 31-W11

Solution : As the ladder is infinitely long, the capacitance of the ladder to the right of the points P, Q is the same as that of the ladder to the right of the points A, B. If the equivalent capacitance of the ladder is C_1, the given ladder may be replaced by the connections shown in figure (31-W12).

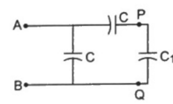

Figure 31-W12

The equivalent capacitance between A and B is easily found to be $C + \frac{CC_1}{C + C_1}\cdot$ But being equivalent to the original ladder, the equivalent capacitance is also C_1.

Thus, $\quad C_1 = C + \frac{CC_1}{C + C_1}$

or, $\quad C_1C + C_1^2 = C^2 + 2\,CC_1$

or, $\quad C_1^2 - CC_1 - C^2 = 0$

giving $C_1 = \frac{C + \sqrt{C^2 + 4\,C^2}}{2} = \frac{1 + \sqrt{5}}{2}\,C.$

Negative value of C_1 is rejected.

14. *Find the energy stored in the electric field produced by a metal sphere of radius R containing a charge Q.*

Solution : Consider a thin spherical shell of radius $x\ (> R)$ and thickness dx concentric with the given metal sphere. The energy density in the shell is

$$u = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0 x^2}\right)^2.$$

The volume of the shell is $4\pi x^2 dx$. The energy contained in the shell is, therefore,

$$dU = \frac{1}{2}\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0 x^2}\right)^2 \times 4\pi x^2 dx = \frac{Q^2 dx}{8\pi\varepsilon_0 x^2}\cdot$$

The energy contained in the whole space outside the sphere is

$$U = \int_R^\infty \frac{Q^2 dx}{8\pi\varepsilon_0 x^2} = \frac{Q^2}{8\pi\varepsilon_0 R}\cdot$$

As the field inside the sphere is zero, this is also the total energy stored in the field.