第二章 递推关系与母函数

一、母函数

$$eg:\ C_{m+n}^{k} = C_m^0 C_n^k + C_m^1 C_n^{k-1} + \cdots + C_m^k C_n^0 .$$

上一章给了组合意义，这次正式证明

巧妙构造

$(1+x)^n (1+\frac{1}{x})^m$

$= \{C_n^0 + C_n^1 x^1 + C_n^2 x^2 + \cdots + C_n^n x^n\} \times \{C_m^0 + C_m^1 (\frac{1}{x})^1 + C_m^2 (\frac{1}{x})^2 + \cdots\}$

※ $(1+x)^n (1+\frac{1}{x})^m = (1+x)^n (1+x)^m \cdot (1+\frac{1}{x})^m \cdot \frac{1}{(1+x)^m}$

$= (1+x)^{m+n} \cdot \left(\frac{1+\frac{1}{x}}{1+x}\right)^m = (1+x)^{m+n} \cdot \frac{1}{x^m}$

$= \frac{1}{x^m} \{C_{m+n}^0 + C_{m+n}^1 x^1 + \cdots + C_{m+n}^m x^m + \cdots + C_{m+n}^{m+n} x^{m+n}\}$

比较常数项可证

对于序列 $c_0, c_1, c_2 \cdots$ 构造一函数（一一对应）

$$G(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

$G(x)$ 为序列 $c_0, c_1, c_2 \cdots$ 的母函数

如 $(1+x)^n$ 称为序列 $C_n^0, C_n^1 \cdots C_n^n$ 的母函数（长度可有限可无限）

1. 由递推关系求母函数

汉诺塔： $H(n) = 2H(n-1) + 1 \quad H(1) = 1$

补充 $H_0 = 0$

令 $\{H_n\}$ 的母函数为 $G(x) = H_0 + H_1 x + H_2 x^2 + \cdots$

根据递推式化简：

$$x : H_1 = 2H_0 + 1$$

$$x^2 : H_2 = 2H_1 + 1$$

$$\vdots$$

故
$$\begin{aligned} G(x) &= H_0 + H_1 x + H_2 x^2 + \cdots \\ &= 2x[H_0 + H_1 x + H_2 x^2 + \cdots] + [x + x^2 + x^3 + \cdots] \\ &= 2x \sum_{k=0}^{\infty} H_k x^k + \sum_{k=1}^{\infty} x^k \end{aligned}$$

☆☆☆核心：$1 + x + x^2 + \cdots = \dfrac{1}{1-x}$ (后面的变换都是以这个为基础)

question：是等比数列求和：$\frac{1-x^n}{1-x}$，但只有 $|x|<1$ 时 $n\to\infty$ 才可以推出，不知道为啥默认 $|x|<1$

$$G(x) = 2xG(x) + \frac{x}{1-x} \Rightarrow G(x) = \frac{x}{(1-x)(1-2x)}$$

2. 由母函数求序列（待定系数）

$$\frac{x}{(1-x)(1-2x)} = \frac{A}{1-2x} + \frac{B}{1-x} \Rightarrow A=1 \quad B=-1$$

$$\begin{aligned} G(x) &= \frac{1}{1-2x} - \frac{1}{1-x} \\ &= (1 + 2x + 2^2x^2 + \cdots) - (1 + x + x^2 + \cdots) \end{aligned}$$

$$H_n = 2^n - 1 \quad n = 1, 2 \cdots$$

二、Fibonacci 序列

基本公式 $\begin{cases} F_n = F_{n-1} + F_{n-2} \\ F_1 = F_2 = 1 \end{cases}$

1. 递推关系求母函数

令 $G(x) = F_1 x + F_2 x^2 + F_3 x^3 + \cdots$

$x^3: F_3 = F_2 + F_1$

$x^4: F_4 = F_3 + F_2$

$\vdots$

$G(x) = F_1 x + F_2 x^2 + (F_1 + F_2) x^3 + (F_2 + F_3) x^4 \cdots$

$$\begin{aligned} &= F_1 x + F_2 x^2 \\ &\quad + F_1 x^3 + F_2 x^4 + F_3 x^5 + \cdots \\ &\quad + F_2 x^3 + F_3 x^4 + F_4 x^5 + \cdots \end{aligned}$$

故 $G(x) = x + x^2 + x[G(x) - x] + x^2 G(x)$

$$G(x) = \frac{x}{1 - x - x^2}$$

2. 母函数求序列

$1 - x - x^2 = \left(1 - \frac{1-\sqrt{5}}{2}x\right)\left(1 - \frac{1+\sqrt{5}}{2}x\right)$ [离谱的基本公式]

$$G(x) = \frac{x}{\left(1 - \frac{1+\sqrt{5}}{2}x\right)\left(1 - \frac{1-\sqrt{5}}{2}x\right)} = \frac{A}{1 - \frac{1+\sqrt{5}}{2}x} + \frac{B}{1 - \frac{1-\sqrt{5}}{2}x}$$

经过各种令 $x = \cdots$ 得 $A = \frac{1}{\sqrt{5}}$ $B = -\frac{1}{\sqrt{5}}$

$$G(x) = \frac{1}{\sqrt{5}}\left[\frac{1}{1 - \frac{1+\sqrt{5}}{2}x} - \frac{1}{1 - \frac{1-\sqrt{5}}{2}x}\right]$$

令 $\alpha = \frac{1+\sqrt{5}}{2}$ $\beta = \frac{1-\sqrt{5}}{2}$

令 $\alpha = \frac{1+\sqrt{5}}{2}$ $\quad \beta = \frac{1-\sqrt{5}}{2}$

$$G(x) = \frac{1}{\sqrt{5}}\left[\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right]$$ → ps：观察这个形式和前面相似

$$= \frac{1}{\sqrt{5}}\left[(\alpha-\beta)x + (\alpha^2-\beta^2)x + \cdots\right]$$

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad (\text{且 } \beta^n \to 0)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

3. 若干等式证明

(1) $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$

$$F_3 = F_1 + F_2$$

$$\Downarrow$$

$$\begin{aligned}
F_1 &= \cancel{F_3} - F_2 \\
F_2 &= \cancel{F_4} - \cancel{F_3} \\
&\ \ \vdots \\
F_{n-1} &= \cancel{F_{n+1}} - \cancel{F_n} \\
F_n &= F_{n+2} - \cancel{F_{n+1}}
\end{aligned}$$

竖着加一下

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - F_2 = F_{n+2} - 1$$

(2) $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$

$$\begin{aligned}
F_1 &= F_2 \\
F_3 &= F_4 - F_2 \\
F_5 &= F_6 - F_4
\end{aligned}$$

$$\boxed{F_{2n-1}} = \boxed{F_{2n} - F_{2n-2}}$$

同理

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$$

(3) $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$

$$F_1^2 = \cancel{F_2 F_1}$$

$$F_2^2 = F_2(F_3 - F_1) = \cancel{F_2 F_3} - \cancel{F_2 F_1}$$

$$F_3^2 = F_3(F_4 - F_2) = F_3 F_4 - \cancel{F_3 F_2}$$

$$\vdots$$

$$F_n^2 = F_n(F_{n+1} - F_{n-1}) = F_n F_{n+1} - \cancel{F_n F_{n-1}}$$

wow! >o●

三. 母函数的性质（基本掌握）

设 $\{a_k\}$ $\{b_k\}$ 两个序列对应的母函数为 $A(x)$ $B(x)$

性质1：若 $b_k = \begin{cases} 0 & k < l \\ a_{k-l} & k \geq l \end{cases}$ 则 $B(x) = x^l A(x)$

证：$B(x) = 0 + 0 + \cdots + 0 + b_l x^l + b_{l+1} x^{l+1} + \cdots$

$= a_0 x^l + a_1 x^{l+1} + \cdots$

$= x^l A(x)$

性质2：若 $b_k = a_{k+l}$

则 $B(x) = \left[A(x) - \sum_{k=0}^{l-1} a_k x^k\right] / x^l$

$$\sum_{k=0}$$

证： $B(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

$= a_l + a_{l+1} x + a_{l+2} x^2 + \cdots$

$= \frac{1}{x^l}(a_l x^l + a_{l+1} x^{l+1} + \cdots)$

$= \frac{1}{x^l}[A(x) - a_0 - a_1 x - a_2 x^2 - \cdots - a_{l-1} x^{l-1}]$

$= [A(x) - \sum_{k=0}^{l-1} a_k x^k] / x^l$

性质3： 若 $b_k = \sum_{l=0}^{k} a_l$，则 $B(x) = \frac{A(x)}{1-x}$

证： $b_0 = a_0$

$b_1 = a_0 + a_1$

$b_2 = a_0 + a_1 + a_2$

$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

$= a_0(1 + x + x^2 + \cdots) + a_1 x(1 + x + x^2 + \cdots)$

$+ a_2 x^2 (1 + x + x^2 + \cdots)$

$= a_0 \frac{1}{1-x} + a_1 x \cdot \frac{1}{1-x} + a_2 x^2 \cdot \frac{1}{1-x}$

$= \frac{A(x)}{1-x}$

性质4： 若 $\sum_{k=0}^{\infty} a_k$ 收敛 $b_k = \sum_{h=k}^{\infty} a_h$

$$B(x) = \frac{A(1) - xA(x)}{1-x}$$

证： $b_0 = a_0 + a_1 + \cdots = A(1)$

$$b_1 = a_1 + a_2 + \cdots = A(1) - a_0$$

$$b_2 = a_2 + a_3 + \cdots = A(1) - a_0 - a_1$$

$$B(x) = A(1)(1+x+\cdots) - a_0 x(1+x+\cdots)$$

$$- a_1 x^2 (1+x+\cdots) + \cdots$$

$$= A(1)\frac{1}{1-x} - [a_0 + a_1 x + \cdots]\frac{x}{1-x}$$

$$= \frac{A(1) - xA(x)}{1-x}$$

性质5：若 $b_k = k a_k$ 则 $B(x) = xA'(x)$ $\quad A'(x) = \frac{d}{dx}A(x)$

证：$B(x) = 0a_0 + a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots$

$$= x(a_1 + 2a_2 x + 3a_3 x^2 + \cdots)$$

$$= xA'(x)$$

性质6：若 $C_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0 = \sum_{h=0}^{k} a_h b_{k-h}$

$$C(x) = A(x)B(x)$$

证：1: $C_0 = a_0 b_0$

x: $C_1 = a_0 b_1 + a_1 b_0$

x^2: $C_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$

$\vdots$

$$C(x) = a_0(b_0 + b_1 x + \cdots) + a_1 x(b_0 + b_1 x + \cdots)$$

$$+ a_2 x^2 (b_0 + b_1 x + \cdots)$$

$$= a_0 B(x) + a_1 x B(x) + a_2 x^2 B(x)$$

$$= A(x)B(x)$$

性质7： 若 $b_k = \frac{a_k}{1+k}$ 则 $B(x) = \frac{1}{x}\int_0^x A(x)\,dx$

证： $1: b_0 = \frac{a_0}{1} = a_0$

$x: b_1 = \frac{1}{2}a_1$

$x^2: b_2 = \frac{1}{3}a_2$

$B(x) = a_0 + \frac{1}{2}a_1x + \frac{1}{3}a_2x^2 + \cdots$

$\because \int_0^x A(x)\,dx = \int_0^x (a_0 + a_1x^2 + a_2x^3 + \cdots)\,dx$

$= a_0x + \frac{1}{2}a_1x^2 + \frac{1}{3}a_1x^3 + \cdots$

$\therefore B(x) = \frac{1}{x}\int_0^x A(x)\,dx$

（简单过一下，证明都是比较好想的）