四. 线性常系数齐次递推关系

定义： $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0$

$a_0 = d_0,\ a_1 = d_1,\ \cdots\ a_{k-1} = d_{k-1}$

若 $c_1, c_2, \cdots c_k,\ d_0, d_1, \cdots d_{k-1}$ 都是常数

则"——"称为 k 阶的线性常系数齐次递推关系

Eg：① Fibonacci　$F_n = F_{n-1} + F_{n-2}$　$F_1 = F_2 = 1$

$\Downarrow$

二阶线性常系数齐次递推关系

② Hanoi　$a_n = 2a_{n-1} + 1$　$a_1 = 1$

$\Downarrow$

一阶线性常系数非齐次递推关系

与通式相对应的：

$$C(x) = x^k + c_1 x^{k-1} + \cdots + c_{k-1} x + c_k$$

称为"—"的特征多项式

重要环节：（以下简述证明过程，主要用结果）

根据递推关系得到母函数：

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

下面根据递推关系解 $G(x)$ [略]

得 $$(1 + c_1 x + c_2 x^2 + \cdots + c_k x^k)\, G(x) = \underbrace{\sum_{h=0}^{k-1} \left[c_h x^h \left(\sum_{j=0}^{k-1-h} a_j x^j \right) \right]}_{P(x)}$$

$$G(x) = \frac{P(x)}{\cdots} = \frac{P(x)}{\cdots}$$

$1+c_1x+c_2x^2+\cdots+c_kx^k \qquad R(x)$

（推导过程略）

$$G(x)=\frac{P(x)}{R(x)}=\frac{P(x)}{(1-a_1x)^{k_1}(1-a_2x)^{k_2}\cdots(1-a_tx)^{k_t}}$$

(1) 不同根的情况

Eg: Fibonacci 递推关系 → 序列的一般公式

（前面用母函数来推导，现在用特征多项式）

$$F_n=F_{n-1}+F_{n-2} \qquad F_1=F_2=1$$

特征方程：$x^2=x+1 \Rightarrow x^2-x-1=0$

$$G(x)=\frac{P(x)}{1-x-x^2}=\frac{P(x)}{(1-\frac{1+\sqrt5}{2}x)(1-\frac{1-\sqrt5}{2}x)}=\frac{A}{1-\alpha x}+\frac{B}{1-\beta x}$$

令 $P(x)=ax+b$ （$P(x)$的次方不超过 $k-1$）［这里好像没用］

$$G(x)=A(1+\alpha x+\alpha^2x^2+\cdots)+B(1+\beta x+\beta^2x^2+\cdots)$$

$$F_n=A\alpha^n+B\beta^n$$

$$\begin{cases}F_0=A+B=0\\ F_1=\frac{A}{1-\alpha}+\frac{B}{1-\beta}=1\end{cases} \Rightarrow \begin{cases}A=\frac{1}{\sqrt5}\\ B=-\frac{1}{\sqrt5}\end{cases}$$

$$G(x)=\frac{1}{\sqrt5}[(\alpha-\beta)x+(\alpha^2-\beta^2)x^2+\cdots]$$

$$F_n=\frac{\alpha^n-\beta^n}{\sqrt5}$$

Eg: $a_n - a_{n-1} - 12a_{n-2} = 0 \quad a_0 = 3 \quad a_1 = 26$

特征方程: $x^2 - x - 12 = 0 \Rightarrow (x-4)(x+3) = 0$

$$G(x) = \frac{P(x)}{1 - x - 12x^2}$$ 注意变换 $= \dfrac{P(x)}{(1-4x)(1+3x)}$

$$x^2\left(\frac{1}{x^2} - \frac{1}{x} - 12\right) = 1 - x - 12x^2$$

$P(x)$ 为一次多项式

$$G(x) = \frac{A}{1-4x} + \frac{B}{1+3x}$$

$$= A(\underbrace{1 + 4x + 4^2x^2 + \cdots}_{4^n x^n}) + B(\underbrace{1 - 3x + 3^2x^2 - 3^3x^3 + \cdots}_{(-3)^n x^n})$$

$$\begin{cases} a_0 = A + B = 3 \\ a_1 = 4A - 3B = 26 \end{cases}$$

$a_n = A \cdot 4^n + B(-3)^n$

$\Downarrow$

$$\begin{cases} A = 5 \\ B = -2 \end{cases}$$

$a_n = 5 \cdot 4^n - 2(-3)^n$

(2) 复根情况

Eg: $a_n - a_{n-1} + a_{n-2} = 0 \quad a_1 = 1 \quad a_2 = 0$ 补充 $a_0 = 1$

特征方程: $x^2 - x + 1 = 0 \quad x = \frac{1}{2} \pm \frac{\sqrt{-3}}{2}$

$$G(x) = \frac{P(x)}{\left(1 - \frac{1+\sqrt{-3}}{2}x\right)\left(1 - \frac{1-\sqrt{-3}}{2}x\right)}$$

$$= \frac{A}{1 - \frac{1+\sqrt{-3}}{2}x} + \frac{B}{1 - \frac{1-\sqrt{3}}{2}x}$$

$$a_n = A\left(\frac{1+\sqrt{3}i}{2}\right)^n + B\left(\frac{1-\sqrt{3}i}{2}\right)^n$$

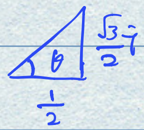
$$\begin{cases} a_0 = A + B \\ a_1 = A\left(\frac{1+\sqrt{3}i}{2}\right) + B\left(\frac{1-\sqrt{3}i}{2}\right) = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2}\left[1 - \frac{1}{\sqrt{3}}i\right] \\ B = \frac{1}{2}\left[1 + \frac{1}{\sqrt{3}}i\right] \end{cases}$$

$$a_n = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}i\right)\left(\frac{1+\sqrt{3}i}{2}\right)^n + \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}i\right)\left(\frac{1-\sqrt{3}i}{2}\right)^n$$

method 1：欧拉公式 $\frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{i\frac{\pi}{3}}$ （较复杂）

method 2： $a_n = A\rho^n \cos n\theta + B\rho^n \sin n\theta$

$\frac{1}{2} + \frac{\sqrt{3}}{2}i$



$\theta = \frac{\pi}{3}$ $\rho = 1$

用这种设法 $a_n = A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}$

$a_1 = 1$ $\begin{cases} \frac{A}{2} + \frac{\sqrt{3}}{2}B = 1 \\ -\frac{1}{2}A + \frac{\sqrt{3}}{2}B = 0 \end{cases} \Rightarrow A = 1 \quad B = \frac{1}{3}\sqrt{3}$

$a_2 = 0$

$$a_n = \cos\frac{n\pi}{3} + \frac{\sqrt{3}}{3}\sin\frac{n\pi}{3}$$

(3) 二重根情况

Eg： $a_n - 4a_{n-1} + 4a_{n-2} = 0$ $a_0 = 1$ $a_1 = 4$

特征方程： $x^2 - 4x + 4 = (x-2)^2 = 0$

$$R(x) = x^2\left(\frac{1}{x^2} - \frac{4}{x} + 4\right) = 1 - 4x + 4x^2 = (1-2x)^2$$

$$G(x) = \frac{P(x)}{(1-2x)^2}$$

重点是分式为两部分：

$$G(x) = \frac{A}{1-2x} + \frac{B}{(1-2x)^2}$$

$$= A(1+2x+2^2x^2+\cdots) + B(1+2x+2^2x^2+\cdots)^2$$

$$= A(1+2x+2^2x^2+\cdots) + B\underline{(1+2\cdot(2x)+3\cdot(2x)^2+4\cdot(2x)^3+\cdots)}$$

$$a_n = A\cdot 2^n + B\cdot(n+1)2^n = (\underbrace{A^*}_{A+B} + Bn)2^n$$

$$\begin{cases} a_0 = 1 = (1+B)\cdot 2 = 4 \\ A^* = 1 \end{cases} \Rightarrow a_n = (1+n)2^n$$

上述情况的一般情况

(1) 无重根

$$C(x) = (x-a_1)(x-a_2)\cdots(x-a_k)$$

$$G(x) = \frac{A_1}{1-a_1x} + \frac{A_2}{1-a_2x} + \cdots + \frac{A_k}{1-a_kx} = \sum_{h=1}^{k} \frac{A_h}{1-a_hx}$$

$$a_n = \sum_{h=1}^{k} A_h a_h^n$$

根据系数推出一个范德蒙德行列式 $\Rightarrow$ 解唯一

(2) 根是复数

$$\alpha_1 = a+bi = \rho e^{i\theta} \qquad e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$$\alpha_2 = a-bi = \rho e^{-i\theta}$$

$$\frac{A_1}{1-\alpha_1x} + \frac{A_2}{1-\alpha_2x} = A_1(1+\alpha_1x+\alpha_1^2x^2+\cdots)$$

$$+A_2(1+\alpha_2x+\alpha_2^2x^2+\cdots)$$

$$A_1\alpha_1^n + A_2\alpha_2^n = A_1\rho^n e^{in\theta} + A_2\rho^n e^{-in\theta}$$

$$= A_1 \rho^n (\cos n\theta + i n \sin\theta)$$

$$+ A_2 \rho^n (\cos n\theta - i n \sin\theta)$$

$$= (A_1 + A_2)\rho^n \cos n\theta + i(A_1 - A_2)\rho^n \sin n\theta = A\rho^n \cos n\theta + B\rho^n \sin n\theta$$

(3) 有重根

$$\frac{P(x)}{(1-\alpha x)^k} = \frac{A_1}{1-\alpha x} + \frac{A_2}{(1-\alpha x)^2} + \cdots + \frac{A_k}{(1-\alpha x)^k}$$

$$\Big\downarrow \text{二项式定理}$$

a_n 是 n 的 $k-1$ 次多项式与 α^n 之积：

$$(h_1 n^{k-1} + h_2 n^{k-2} + \cdots + h_k)\alpha^n$$