RT2 - Theory question

CS-341 : Introduction to Computer Graphics

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1 Reflection's iterative formula

1.1 Formula proof

Let's consider the formula we are trying to demonstrate:

$$c_b = \sum_{i=0}^{+\infty} (1 - \alpha_i)(\prod_{k=0}^{i-1} \alpha_k)c_i \tag{1}$$

Which can also be written as:

$$c_b = \sum_{i=0}^{n} (1 - \alpha_i)(\prod_{k=0}^{i-1} \alpha_k)c_i \tag{2}$$

With $n = +\infty$ (it is good to point out that what n represents concretely for us is the total number of reflections). We can demonstrate the validity of the formula via proof by induction. We first demonstrate that the formula is correct for a certain value of n. Let's consider $n = 1$ for instance. We know that:

$$c_b = (1 - \alpha_0)c_0 + \alpha_0 c^1 = (1 - \alpha_0)c_0 + \alpha_0((1 - \alpha_1)c_1 + \alpha_1 c^2) \tag{3}$$

And since we've established that we consider that if intersection number i doesn't occur, the corresponding color obtained without reflection at this intersection $c_i = 0$, then in our case $c^2 = (1 - \alpha_2)c_2 + \alpha_2 c^3 = \alpha_2 c^3$ since $c_2 = 0$ because $n = 1$. And since every subsequent c_i will also equal 0, we can intuitively deduce that $c^2 = 0$ (This could be proven more rigorously, but we felt like it was intuitive enough not to have to give a detailed proof). Therefore our c_b would end up being:

$$c_b = (1 - \alpha_0)c_0 + \alpha_0 c^1 = (1 - \alpha_0)c_0 + \alpha_0(1 - \alpha_1)c_1 \tag{4}$$

If we now consider the formula we are trying to prove for $n = 1$, we get:

$$\sum_{i=0}^{1} (1 - \alpha_i)(\prod_{k=0}^{i-1} \alpha_k)c_i = (1 - \alpha_0)c_0 + (1 - \alpha_1)\alpha_0 c_1 = c_b \tag{5}$$

We can thus confirm that the formula is valid for $n = 1$. We will now assume that it is valid for an arbitrary value of n and try proving its validity for a value $n + 1$. We have:

$$\sum_{i=0}^{n+1} (1 - \alpha_i)(\prod_{k=0}^{i-1} \alpha_k)c_i = \sum_{i=0}^{n} (1 - \alpha_i)(\prod_{k=0}^{i-1} \alpha_k)c_i + (1-\alpha_{n+1})(\prod_{k=0}^{n} \alpha_k)c_{n+1} = c_b^n + (1-\alpha_{n+1})(\prod_{k=0}^{n} \alpha_k)c_{n+1} = c_b^{n+1} \tag{6}$$

Where c_b^n is the value of c_b for n reflections. And since the formula is valid for $n + 1$, it is therefore valid for each and every value of $n \geq 1$ and is therefore valid for $n = +\infty$

1.2 Formula simplification for N reflections

Let us now assume that the N^{th} reflection is null. That is, that $\alpha_n = 0$, and let

$$c_b = lim_{n \to \infty} c_b^n = \sum_{i=0}^{\infty} (1-\alpha_i)[\prod_{k=0}^{i-1} \alpha_k] c_i \tag{7}$$

Then we have :

$$c_b = \sum_{i=0}^{N-1} (1-\alpha_i)[\prod_{k=0}^{i-1} \alpha_k] c_i + (1-\alpha_N)[\prod_{k=0}^{N-1} \alpha_k] c_i + \sum_{i=N+1}^{\infty} (1-\alpha_i)[\prod_{k=0}^{i-1} \alpha_k] c_i$$

$$c_b = \sum_{i=0}^{N-1} (1-\alpha_i)[\prod_{k=0}^{i-1} \alpha_k] c_i + [\prod_{k=0}^{N-1} \alpha_k] c_i + \underbrace{\alpha_n \sum_{i=N+1}^{\infty} (1-\alpha_i) \prod_{k=0, k \neq N}^{i} \alpha_k}_{=0}$$

$$c_b = \sum_{i=0}^{N} (1-\alpha_i)[\prod_{k=0}^{i-1} \alpha_k] c_i \tag{8}$$

1.3 Implementation

To take into account reflections in the final return color of a pixel, we need to iterate N times on every pixel, N being the number of reflections. Outside of the iteration loop, we keep track of the current multiplication of all α_i needed in the formula (10) product, initialized at 1 (here denoted α), and of the current computation of c_b, initialized at 0 (here denoted c).
At each iteration, we do the following :

- compute c_i, the color of the pixel without reflections (Blinn-Phong lighting)
- add to c the following : $(1-\alpha_i) * \alpha * c_i$, where α_i is the current collision material's reflection coefficient
- move the ray's origin for the next iteration to the current collision point and update the ray's direction accordingly
- multiply α_i to α