Sensor Orientation – LAB 4
Circular motion of a vehicle
(background for Labs ~~5,7,8,9,10~~ **2, 3, 5, 6**)

Abstract

The labs ~~4, 5, 7, 8, 9 and 10~~ 2, 3 and 5, 6 deal with a simplified example of a Kalman filter for the GPS/INS integration. It assumes the uniform clockwise motion of a virtual vehicle on a circular track and is based on simulated measurements (Sect. 1). The sensor errors are modeled in a realistic style involving a random constant and Gauss-Markov processes. The example becomes more complicated when replacing the circular track by a spiral and allowing changes of the angular rate of the trajectory (Sect. 4).

The Labs 4 and 5 concern the simulation of the inertial measurements (nominal and realistic) along the circular track and their processing by strapdown inertial navigation.

1 Circular motion with constant angular rate

1.1 Assumptions

The motion is described in the two-dimensional mapping frame (m-frame). For the sake of simplicity, it is assumed that the m-frame is an inertial frame, i.e., it is nonaccelerated and nonrotating. The $\mathbf{x}_1^m$-axis points towards the north, the $\mathbf{x}_2^m$ points towards the east. Furthermore, the effect of gravitational acceleration is neglected as the problem is assuemd to be strictly two-dimensional.

When modeling a circular motion, polar coordinates are most suitable. These are defined by the radius r and the position angle ψ. The transformation between the Cartesian and polar coordinates in the m-frame is given by

$$\mathbf{x}^m(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = r(t) \begin{bmatrix} \cos\psi(t) \\ \sin\psi(t) \end{bmatrix} . \tag{1}$$

For the latter vector, the shorthand notation $\mathbf{e}^r(t)$ will be used. Applying the assumtion of constant radius (circular motion) and constant angular rate (uniform velocity), the differential equatios

$$\dot{r} = 0\,, \tag{2}$$

$$\ddot{\psi} = 0 \tag{3}$$

control the motion of the vehicle. Solving these differential equations yields

$$r = r_0\,, \tag{4}$$

$$\psi = \omega_0 t + \psi_0\,, \tag{5}$$

where r_0, ω_0 $(= \dot{\psi})$, and ψ_0 are constants and the latter is assumed to be zero (i.e., the initial vehicle position at t_0 is on the $\mathbf{x}_1^m$-axis).

Note that in the following, the time dependence of the various quantities is usually not explicitly indicated which is done for simplicity.

1.2 Nominal sensor measurements

It is assumed that the vehicle is equipped with a 2D strapdown IMU, comprising two accelerometers with horizontal input axes and a single gyro with vertical input axis. What do these sensors measure?

To solve this question, the body (b-) frame of the IMU is introduced. Thereby, the $\mathbf{x}_1^b$-axis defines the along axis of the vehicle and the $\mathbf{x}_2^b$-axis represents its across axis, pointing towards the right.

In the simplified case of a perfect circular motion, the axes of the b-frame when expressed in the m-frame are given by

$$(\mathbf{x}_1^b)^m = \begin{bmatrix} \cos(\psi + \pi/2) \\ \sin(\psi + \pi/2) \end{bmatrix} = \begin{bmatrix} -\sin\psi \\ \cos\psi \end{bmatrix}, \tag{6}$$

$$(\mathbf{x}_2^b)^m = \begin{bmatrix} \cos(\psi + \pi) \\ \sin(\psi + \pi) \end{bmatrix} = \begin{bmatrix} -\cos\psi \\ -\sin\psi \end{bmatrix}. \tag{7}$$

Using now the unity vector $\mathbf{e}^r$ defined further above, one finds that

$$(\mathbf{x}_2^b)^m = -\mathbf{e}^r, \quad (\mathbf{x}_1^b)^m = \frac{d\,\mathbf{e}^r}{d\,\psi} = \mathbf{e}^\psi. \tag{8}$$

(The notation $(\mathbf{x}_j^b)^m$ indicates the j^{th} axis of the b-frame expressed in the m-frame.) Note that the coordinate frame spanned by the vectors $\mathbf{e}^r$, $\mathbf{e}^\psi$ – the $r\psi$-frame – has the same origin as the m-frame but rotates continuously. On the other hand, the axes of the $r\psi$- and b-frames are closely related. These relations are: $\mathbf{x}_1^b = \mathbf{e}^\psi = \mathbf{x}_2^{r\psi}$, $\mathbf{x}_2^b = -\mathbf{e}^r = -\mathbf{x}_1^{r\psi}$ (see Fig. 1).

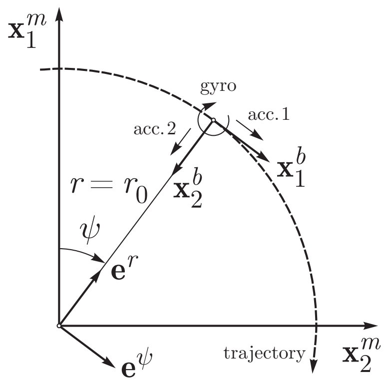

Figure 1: Simulated circular motion and coordinate frames

After these preparations, the nominal observations of the accelerometers may be determined by deriving Eq. (1) twice after time. The first derivative yields

$$\begin{aligned} \dot{\mathbf{x}}^m &= \dot{r}\,\mathbf{e}^r + r\,\dot{\mathbf{e}}^r \\ &= \dot{r}\,\mathbf{e}^r + r\,\dot{\psi}\,\mathbf{e}^\psi, \end{aligned} \tag{9}$$

where the relation $\dot{\mathbf{e}}^r = \dot{\psi}\, \mathbf{e}^\psi$ has been used. Similiarly, the second derivative is found to be

$$\begin{aligned}
\ddot{\mathbf{x}}^m &= \ddot{r}\, \mathbf{e}^r + \dot{r}\, \dot{\mathbf{e}}^r + \dot{r}\, \dot{\psi}\, \mathbf{e}^\psi + r\, \ddot{\psi}\, \mathbf{e}^\psi + r\, \dot{\psi}\, \dot{\mathbf{e}}^\psi \\
&= (\ddot{r} - r\, \dot{\psi}^2)\, \mathbf{e}^r + (2\dot{r}\, \dot{\psi} + r\, \ddot{\psi})\, \mathbf{e}^\psi\,, \qquad (10)
\end{aligned}$$

where now the relation $\dot{\mathbf{e}}^\psi = -\dot{\psi}\, \mathbf{e}^r$ has been used. Thereby, one can easily identify the apparent accelerations that are due to the rotation of the $r\psi$-frame with respect to the inertial space: centrifugal acceleration $(-r\, \dot{\psi}^2)$, Coriolis acceleration $(2\dot{r}\, \dot{\psi})$, and tangential acceleration $(r\, \ddot{\psi})$.

In case of a circular motion with constant angular rate, some simplifications can be made that are in accordance with Eqs. (2) and (3). Thus, Eqs. (9) and (10) convert to

$$\dot{\mathbf{x}}^m = r\, \dot{\psi}\, \mathbf{e}^\psi\,, \qquad (11)$$

$$\ddot{\mathbf{x}}^m = -r\, \dot{\psi}^2\, \mathbf{e}^r\,. \qquad (12)$$

Thus, there is only an along-track velocity but no across-track component (the radius remains unchanged); in contrast, there is only an across-track acceleration (i.e., the apparent centrifugal acceleration) but there is no along-track component (the angular rate is constant).

Consequently, the nominal measurements are found to be

$$\mathbf{f}^b = \begin{bmatrix} f_1^b \\ f_2^b \end{bmatrix} = \begin{bmatrix} 0 \\ r\, \omega_0^2 \end{bmatrix} \qquad (13)$$

for the accelerometers, where ω_0 has been used instead of $\dot{\psi}$; and

$$\boldsymbol{\omega}_{mb}^b = \begin{bmatrix} \omega_{mb}^b \end{bmatrix} = \begin{bmatrix} \omega_0 \end{bmatrix} \qquad (14)$$

in case of the gyro. As the m-frame is assumed to be inertial and since there is only one gyro, ω_{mb}^b is used rather than the conventional quantity $\boldsymbol{\omega}_{ib}^b$.

Note that Eq. (13) clearly shows the effect of an apparent acceleration, because there is always a nonzero across-track acceleration although the radius remains constant. Hence, the observed accleration in the b-frame is only due to its rotation with respect to the m-frame.

1.3 Strapdown inertial navigation

As usual, it is necessary to define the initial conditions and to integrate the sensor measurements to obtain the current state vector of the vehicle.

Initial conditions

These are defined by five quantities, i.e., the initial position $\mathbf{x}_0^m$, the initial velocity $\mathbf{v}_0^m$, and the initial heading (or yaw) angle α. For the sake of simplicity, it will be assumed that there is no accleration phase – in other words: the measurements are only started when the vehicle is already in a "steady state" of motion. Furthermore, it is supposed that the along axis of the vehicle is always aligned with its velocity vector, i.e., there is no drift.