Lab 6 - EKF - pages 8-11

Sensor Orientation – LAB 2
Circular motion of a vehicle
(background for Lab 3 and Lab 6)

Abstract

The labs 2, 3 and 6 deal with a simplified example of a Kalman filter for the GPS/INS integration. It assumes the uniform clockwise motion of a virtual vehicle on a circular track and is based on simulated measurements (Sect. 1). The sensor errors are modeled in a realistic style involving a random constant and Gauss-Markov processes. The example becomes more complicated when replacing the circular track by a spiral and allowing changes of the angular rate of the trajectory (Sect. 4).

The Labs 2 and 3 concern the simulation of the inertial measurements (nominal and realistic) along the circular track and their processing by strapdown inertial navigation.

1 Circular motion with constant angular rate

1.1 Assumptions

The motion is described in the two-dimensional mapping frame (m-frame). For the sake of simplicity, it is assumed that the m-frame is an inertial frame, i.e., it is nonaccelerated and nonrotating. The $\mathbf{x}_1^m$-axis points towards the north, the $\mathbf{x}_2^m$ points towards the east. Furthermore, the effect of gravitational acceleration is neglected as the problem is assuemd to be strictly two-dimensional.

When modeling a circular motion, polar coordinates are most suitable. These are defined by the radius r and the position angle ψ. The transformation between the Cartesian and polar coordinates in the m-frame is given by

$$\mathbf{x}^m(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = r(t) \begin{bmatrix} \cos\psi(t) \\ \sin\psi(t) \end{bmatrix} . \tag{1}$$

For the latter vector, the shorthand notation $\mathbf{e}^r(t)$ will be used. Applying the assumtion of constant radius (circular motion) and constant angular rate (uniform velocity), the differential equatios

$$\dot{r} = 0 \,, \tag{2}$$
$$\ddot{\psi} = 0 \tag{3}$$

control the motion of the vehicle. Solving these differential equations yields

$$r = r_0 \,, \tag{4}$$
$$\psi = \omega_0 t + \psi_0 \,, \tag{5}$$

where r_0, ω_0 $(= \dot{\psi})$, and ψ_0 are constants and the latter is assumed to be zero (i.e., the initial vehicle position at t_0 is on the $\mathbf{x}_1^m$-axis).

Note that in the following, the time dependence of the various quantities is usually not explicitly indicated which is done for simplicity.

1.2 Nominal sensor measurements

It is assumed that the vehicle is equipped with a 2D strapdown IMU, comprising two accelerometers with horizontal input axes and a single gyro with vertical input axis. What do these sensors measure?

To solve this question, the body (b-) frame of the IMU is introduced. Thereby, the $\mathbf{x}_1^b$-axis defines the along axis of the vehicle and the $\mathbf{x}_2^b$-axis represents its across axis, pointing towards the right.

In the simplified case of a perfect circular motion, the axes of the b-frame when expressed in the m-frame are given by

$$(\mathbf{x}_1^b)^m = \begin{bmatrix} \cos(\psi + \pi/2) \\ \sin(\psi + \pi/2) \end{bmatrix} = \begin{bmatrix} -\sin\psi \\ \cos\psi \end{bmatrix}, \tag{6}$$

$$(\mathbf{x}_2^b)^m = \begin{bmatrix} \cos(\psi + \pi) \\ \sin(\psi + \pi) \end{bmatrix} = \begin{bmatrix} -\cos\psi \\ -\sin\psi \end{bmatrix}. \tag{7}$$

Using now the unity vector $\mathbf{e}^r$ defined further above, one finds that

$$(\mathbf{x}_2^b)^m = -\mathbf{e}^r, \qquad (\mathbf{x}_1^b)^m = \frac{d\,\mathbf{e}^r}{d\,\psi} = \mathbf{e}^\psi. \tag{8}$$

(The notation $(\mathbf{x}_j^b)^m$ indicates the j^{th} axis of the b-frame expressed in the m-frame.) Note that the coordinate frame spanned by the vectors $\mathbf{e}^r$, $\mathbf{e}^\psi$ – the $r\psi$-frame – has the same origin as the m-frame but rotates continuously. On the other hand, the axes of the $r\psi$- and b-frames are closely related. These relations are: $\mathbf{x}_1^b = \mathbf{e}^\psi = \mathbf{x}_2^{r\psi}$, $\mathbf{x}_2^b = -\mathbf{e}^r = -\mathbf{x}_1^{r\psi}$ (see Fig. 1).

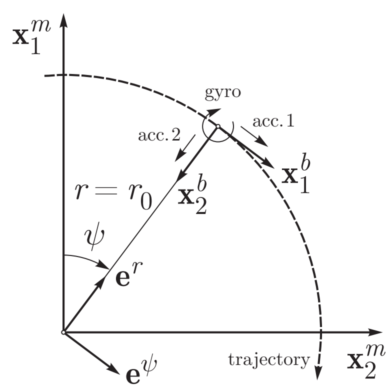

Figure 1: Simulated circular motion and coordinate frames

After these preparations, the nominal observations of the accelerometers may be determined by deriving Eq. (1) twice after time. The first derivative yields

$$\begin{aligned} \dot{\mathbf{x}}^m &= \dot{r}\,\mathbf{e}^r + r\,\dot{\mathbf{e}}^r \\ &= \dot{r}\,\mathbf{e}^r + r\,\dot{\psi}\,\mathbf{e}^\psi, \end{aligned} \tag{9}$$

where the relation $\dot{\mathbf{e}}^r = \dot{\psi}\,\mathbf{e}^\psi$ has been used. Similiarly, the second derivative is found to be

$$
\begin{aligned}
\ddot{\mathbf{x}}^m &= \ddot{r}\,\mathbf{e}^r + \dot{r}\,\dot{\mathbf{e}}^r + \dot{r}\,\dot{\psi}\,\mathbf{e}^\psi + r\,\ddot{\psi}\,\mathbf{e}^\psi + r\,\dot{\psi}\,\dot{\mathbf{e}}^\psi \\
&= (\ddot{r} - r\,\dot{\psi}^2)\,\mathbf{e}^r + (2\dot{r}\,\dot{\psi} + r\,\ddot{\psi})\,\mathbf{e}^\psi\,, \qquad (10)
\end{aligned}
$$

where now the relation $\dot{\mathbf{e}}^\psi = -\dot{\psi}\,\mathbf{e}^r$ has been used. Thereby, one can easily identify the apparent accelerations that are due to the rotation of the $r\psi$-frame with respect to the inertial space: centrifugal acceleration $(-r\,\dot{\psi}^2)$, Coriolis acceleration $(2\dot{r}\,\dot{\psi})$, and tangential acceleration $(r\,\ddot{\psi})$.

In case of a circular motion with constant angular rate, some simplifications can be made that are in accordance with Eqs. (2) and (3). Thus, Eqs. (9) and (10) convert to

$$
\begin{aligned}
\dot{\mathbf{x}}^m &= r\,\dot{\psi}\,\mathbf{e}^\psi\,, \qquad (11)\\
\ddot{\mathbf{x}}^m &= -r\,\dot{\psi}^2\,\mathbf{e}^r\,. \qquad (12)
\end{aligned}
$$

Thus, there is only an along-track velocity but no across-track component (the radius remains unchanged); in contrast, there is only an across-track acceleration (i.e., the apparent centrifugal acceleration) but there is no along-track component (the angular rate is constant).

Consequently, the nominal measurements are found to be

$$
\mathbf{f}^b = \begin{bmatrix} f_1^b \\ f_2^b \end{bmatrix} = \begin{bmatrix} 0 \\ r\,\omega_0^2 \end{bmatrix} \qquad (13)
$$

for the accelerometers, where ω_0 has been used instead of $\dot{\psi}$; and

$$
\boldsymbol{\omega}_{mb}^b = \begin{bmatrix} \omega_{mb}^b \end{bmatrix} = \begin{bmatrix} \omega_0 \end{bmatrix} \qquad (14)
$$

in case of the gyro. As the m-frame is assumed to be inertial and since there is only one gyro, ω_{mb}^b is used rather than the conventional quantity $\boldsymbol{\omega}_{ib}^b$.

Note that Eq. (13) clearly shows the effect of an apparent acceleration, because there is always a nonzero across-track acceleration although the radius remains constant. Hence, the observed accleration in the b-frame is only due to its rotation with respect to the m-frame.

1.3 Strapdown inertial navigation

As usual, it is necessary to define the initial conditions and to integrate the sensor measurements to obtain the current state vector of the vehicle.

Initial conditions

These are defined by five quantities, i.e., the initial position $\mathbf{x}_0^m$, the initial velocity $\mathbf{v}_0^m$, and the initial heading (or yaw) angle α. For the sake of simplicity, it will be assumed that there is no accleration phase – in other words: the measurements are only started when the vehicle is already in a "steady state" of motion. Furthermore, it is supposed that the along axis of the vehicle is always aligned with its velocity vector, i.e., there is no drift.

In this case, the initial conditions are given by $\mathbf{x}_0^m = [n_0, 0]^{\mathrm{T}}$, $\mathbf{v}_0^m = [0, v_0]^{\mathrm{T}}$, and $\alpha_0 = \psi_0 + \pi/2 = \pi/2$. From the latter, the initial attitude matrix is found by evaluating its general form at $\alpha = \alpha_0$:

$$\mathbf{R}_b^m = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{R}_b^m(\alpha_0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} . \tag{15}$$

Note that this definition holds as $\mathbf{R}_m^b = \mathbf{R}_b^{m\mathrm{T}} = \mathbf{R}(\alpha)$. Further, note that it is not necessary in this example to introduce the local-level frame since it would always be parallel to the m-frame due to the 2D (planar) modeling.

Dead reckoning

Attitude computation
As a first step, the current attitude matrix must be computed. This is achieved by numerical integration of the corresponding differential equation which is obtained by differencing the first part of Eq. (15) and rearranging the result:

$$\dot{\mathbf{R}}_b^m = \begin{bmatrix} -\sin\alpha & -\cos\alpha \\ \cos\alpha & -\sin\alpha \end{bmatrix} \dot{\alpha} = \mathbf{R}_b^m \begin{bmatrix} 0 & -\dot{\alpha} \\ \dot{\alpha} & 0 \end{bmatrix} = \mathbf{R}_b^m \, \mathbf{\Omega}_{mb}^b \, , \tag{16}$$

where $\dot{\alpha} = \omega_{mb}^b$ is the gyro measurement.

When using a sufficiently small integration interval, it may be assumed that the angular rate remains constant during that interval. Thus, the numerical integration of Eq. (16) is given by

$$\mathbf{R}_b^m(t_k) = \mathbf{R}_b^m(t_{k-1}) \, \exp\left(\mathbf{\Omega}_{mb}^b(t_k) \cdot (t_k - t_{k-1}) \right) . \tag{17}$$

Alternatively, the attitude angle itself may be computed from the first- or second-order approximations

$$\alpha(t_k) = \alpha(t_{k-1}) + \dot{\alpha}(t_k) \cdot (t_k - t_{k-1}) \, , \tag{18}$$

$$\alpha(t_k) = \alpha(t_{k-1}) + \frac{1}{2}\big(\dot{\alpha}(t_k) + \dot{\alpha}(t_{k-1})\big) \cdot (t_k - t_{k-1}) \, . \tag{19}$$

Navigation computation
As the m-frame is supposed to be inertial and the gravitational field is neglected, the navigation equations are given by

$$\dot{\mathbf{v}}^m = \mathbf{f}^m \, , \qquad \dot{\mathbf{x}}^m = \mathbf{v}^m \, . \tag{20}$$

Using the integrated attitude matrix, the specific-force measurements of the accelerometers need to be resolved in the m-frame, i.e., $\mathbf{f}^m = \mathbf{R}_b^m \, \mathbf{f}^b$. Afterwards, the transformed measurement can be integrated to obtain the current velocity vector. When using a first-order approximation, the velocity vector follows from

$$\mathbf{v}^m(t_k) = \mathbf{v}^m(t_{k-1}) + \mathbf{R}_b^m(t_k) \, \mathbf{f}^b(t_k) \cdot (t_k - t_{k-1}) \, . \tag{21}$$

Again, a second-order approximation is obtained from

$$\begin{aligned} \mathbf{v}^m(t_k) &= \mathbf{v}^m(t_{k-1}) + \ldots \\ &+ \frac{1}{2}\left(\mathbf{R}_b^m(t_k) \, \mathbf{f}^b(t_k) + \mathbf{R}_b^m(t_{k-1}) \, \mathbf{f}^b(t_{k-1}) \right) \cdot (t_k - t_{k-1}) \, . \end{aligned} \tag{22}$$

Finally, the current position is found in analogy to the velocity vector by using either a first- or second-order approximation:

$$
\begin{aligned}
\mathbf{x}^m(t_k) &= \mathbf{x}^m(t_{k-1}) + \mathbf{v}^m(t_k) \cdot (t_k - t_{k-1})\,, &(23)\\
\mathbf{x}^m(t_k) &= \mathbf{x}^m(t_{k-1}) + \frac{1}{2}\left(\mathbf{v}^m(t_k) + \mathbf{v}^m(t_{k-1})\right) \cdot (t_k - t_{k-1})\,. &(24)
\end{aligned}
$$

2 Realistic sensor measurements (LAB 3)

2.1 Continuous modeling

In contrast to the nominal observations, the real sensor data are corrupted by both random and systematic error terms. It is assumed that the time-correlated biases can be modeled as Gauss-Markov processes of first order (GM1). In general terms, the autocorrelation function (ACF) of such a bias-variation process $B(t)$ is given by

$$\mathrm{acov}(\tau)_B = \sigma^2\, \mathrm{e}^{-\beta\tau} \tag{25}$$

where τ is a time lag, σ^2 is the value of the ACF for $\tau = 0$, and $\beta = 1/T$ is the inverse of the process correlation time T. Without proof, the differential equation used to model a GM1 process is given by

$$\dot{b}(t) = -\beta\, b(t) + \sqrt{2\,\sigma^2\,\beta} \cdot u(t) \tag{26}$$

where $u(t)$ is a unity white-noise process that drives the evolution of the Gauss-Markov time series. Unity white noise means that the amplitude of its power-spectral density (PSD) is 1. Note that the presence of nonwhite (i.e., time-correlated) error processes will require an augmentation of the state vector in the Kalman filter.

Accelerometers
It is expected that each accelerometer is affected by white noise and by a time-variable bias, i.e., errors in the accelerometer measurements are defined by $\delta f_j^b(t) = w_{A_j}(t) + b_{A_j}(t)$, where $j = 1, 2$. According to Eq. (26), the time-variable biases are described by the differential equation

$$\dot{b}_{A_j}(t) = -\beta_A\, b_{A_j}(t) + \sqrt{2\,\sigma_A^2\,\beta_A} \cdot u_{A_j}(t)\,. \tag{27}$$

While the terms with superscript A_j are different for each sensor, those with superscript A alone (β_A, σ_A) are assumed to be equal for both accelerometers.

Gyroscope
In case of the gyro, three types of errors are considered: white noise, a random constant bias, and a time-variable bias (or bias variation). Thus, errors in the gyro measurement are defined by $\delta\omega_{mb}^b(t) = w_G(t) + b_C + b_G(t)$. The latter two error sources are modeled by the following differential equations:

$$\dot{b}_C \;=\; 0\,, \tag{28}$$

$$\dot{b}_G(t) \;=\; -\beta_G\, b_G(t) + \sqrt{2\,\sigma_G^2\,\beta_G} \cdot u_G(t)\,. \tag{29}$$

While Eq. (28) reflects the lack of a time dependence of the random constant, the time-correlated bias variation is again modeled as a GM1 process.

2.2 Simulation of errors at discrete epochs

The discrete versions of the above error processes are obvious for the white-noise and random-constant processes.

In case of the white noise, the error is simulated by scaling a Gaussian random variable with unity variance by the desired standard deviation (either σ_{w_A} or σ_{w_G}). The unity random variable is obtained, e.g., in Matlab by using the standard function "rand". Note that this function must be evaluated at every update epoch of the IMU, yielding a sequence of white-noise realizations.

In case of the random constant, a single value is chosen at the beginning of the simulation, either by deterministic assignment or by a random-number generator (which must be stored for later comparison with the Kalman-filter estimation). The value of the random constant is supposed to remain unchanged during the whole simulation.

In case of the GM1 process, the situation is a little more complicated. Generally, the time-discrete model of such a process, denoted here simply by x_k, reads

$$x_k = \mathrm{e}^{-\beta \Delta t} \, x_{k-1} + w_k \, , \tag{30}$$

where w_k is a driving white sequence (the discrete analogon of continuous white noise) and $\Delta t = t_k - t_{k-1}$ is the update interval of the IMU. As GM processes are known to be stationary, each element of the discrete process must have the same variance σ^2 (also known as steady-state variance of the process). By variance propagation of Eq. (30), one finds that

$$\sigma^2 = \mathrm{e}^{-2\beta \Delta t} \, \sigma^2 + q_k \, , \tag{31}$$

where q_k is the variance (the squared standard deviation) of the driving white sequence. (Note that Eq. (31) relies on the fact that white noise is completely uncorrelated over time as is reflected by the characteristic "white".) With the help of Eq. (31), one finds the required variance of the driving white sequence to generate the desired GM process:

$$q_k = \mathrm{var}(w_k) = (1 - \mathrm{e}^{-2\beta \Delta t}) \, \sigma^2 \, . \tag{32}$$

Note that the square root of q_k (i.e., the standard deviation of the white sequence) is used to generate the sequence of process elements in the simulation. It is used to scale the random number obtained by the random generator and added as a "real" w_k in Eq. (30), i.e., $w_k = \mathrm{randn} \cdot \sqrt{q_k}$. The process may either start with a deterministic value or with an initial random number.

3 Kalman filter (LAB 6)

It is common practice when integrating an INS with some other navigation system to use the errors of the INS output as the elements of the state vector of the filter. The dynamics of these error states are obtained by perturbing the differential equations that control the INS. For the given example, this procedure is shown in the following.

3.1 Continuous system dynamics

The perturbation is effected by forming the total differentials of the navigation equations for the attitude-, velocity-, and position terms (this is in fact a linear approximation). The perturbations are indicated by the symbol δ:

$$\begin{array}{lclclcl} \dot{\alpha} & = & \omega_{mb}^{b} & \Rightarrow & \delta\dot{\alpha} & = & \delta\omega_{mb}^{b}\,, \\ \dot{\mathbf{v}}^{m} & = & \mathbf{R}_{b}^{m}\,\mathbf{f}^{b} & \Rightarrow & \delta\dot{\mathbf{v}}^{m} & = & \delta\mathbf{R}_{b}^{m}\,\mathbf{f}^{b} + \mathbf{R}_{b}^{m}\,\delta\mathbf{f}^{b}\,, \\ \dot{\mathbf{x}}^{m} & = & \mathbf{v}^{m} & \Rightarrow & \delta\dot{\mathbf{x}}^{m} & = & \delta\mathbf{v}^{m}\,. \end{array} \tag{33}$$

In the second line of Eq. (33), the perturbation of the rotation matrix may be replaced by

$$\delta\mathbf{R}_{b}^{m} = \mathbf{R}_{b}^{m} \begin{bmatrix} 0 & -\delta\alpha \\ \delta\alpha & 0 \end{bmatrix}, \tag{34}$$

which is similar to Eq. (15). Using this result, one finds that

$$\begin{bmatrix} 0 & -\delta\alpha \\ \delta\alpha & 0 \end{bmatrix} \begin{bmatrix} f_1^b \\ f_2^b \end{bmatrix} = \delta\alpha \begin{bmatrix} -f_2^b \\ f_1^b \end{bmatrix}. \tag{35}$$

Left-multiplication by $\mathbf{R}_{b}^{m}$ and neglecting for the moment $\delta\alpha$ yields

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} -f_2^b \\ f_1^b \end{bmatrix} = \begin{bmatrix} -f_1^b \sin\alpha - f_2^b\cos\alpha \\ f_1^b\cos\alpha - f_2^b \sin\alpha \end{bmatrix} = \begin{bmatrix} -f_2^m \\ f_1^m \end{bmatrix}. \tag{36}$$

Substituting Eqs. (34) and (36) into Eq. (33) yields the following system:

$$\begin{bmatrix} \delta\dot{\alpha} \\ \delta\dot{v}_n \\ \delta\dot{v}_e \\ \delta\dot{x}_n \\ \delta\dot{x}_e \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -f_2^m & 0 & 0 & 0 & 0 \\ f_1^m & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\alpha \\ \delta v_n \\ \delta v_e \\ \delta x_n \\ \delta x_e \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\omega_{mb}^{b} \\ \delta f_1^b \\ \delta f_2^b \end{bmatrix}, \tag{37}$$

where $v_n = v_1^m$, $v_e = v_2^m$ denote the north and east velocity, and $x_n = x_1^m$, $x_e = x_2^m$ denote the north and east coordinate, respectively. The meaning of this system of equations may be understood easily by analyzing its rows:

- Temporal changes of an error in the attitude angle α (i.e., a misalignment of the system about the vertical axis) arise from errors in the gyro data (row 1).
- Depending on the magnitude of the transformed specific forces f_1^m, f_2^m, the misalignment propagates into temporal changes of the velocity errors. These are superimposed by errors of the accelerometer measurements (rows 2 and 3).

- Errors in the velocity vector propagate as temporal changes of the position errors (rows 4 and 5). Since the IMU sensors do not directly measure temporal changes of position, there are no direct contributions of the sensor errors.

3.2 State-vector augmentation

The shortcoming of Eq. (37) is that the time-correlated errors of the sensors are not considered yet. Therefore, the state vector must be augmented by the terms b_C, b_G, b_{A_1}, and b_{A_2}, referring to the random-constant bias and bias variation of the gyro, as well as to the bias variation of accelerometer 1 and 2, respectively. The differential equations controlling these terms were defined in Sect. 2. Thus, the extended model includes nine elements in the state vector:

$$\delta \mathbf{x} = \begin{bmatrix} \delta\alpha & \delta v_n & \delta v_e & \delta x_n & \delta x_e & b_C & b_G & b_{A_1} & b_{A_2} \end{bmatrix}^{\mathrm{T}} . \tag{38}$$

The augmented system dynamics are given by

$$\delta \dot{\mathbf{x}} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix} \delta \mathbf{x} + \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \mathbf{w} , \tag{39}$$

where the vector $\mathbf{w}$ identifies the driving (non-unity) white-noise terms.

The individual sub-matrices of $\mathbf{F}$ are defined as follows: $\mathbf{F}_{11}$ (5×5) is the same as the first matrix on the right-hand side of Eq. (37); $\mathbf{F}_{12}$ (5×4) is an extension of the second matrix on the right-hand side of Eq. (37) that accounts for two (instead of one) gyro impacts:

$$\mathbf{F}_{12} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; \tag{40}$$

$\mathbf{F}_{21}$ is a zero matrix with dimension 4×5; and $\mathbf{F}_{22}$ is a 4×4 diagonal matrix that models the temporal changes of the bias terms:

$$\mathbf{F}_{22} = \mathrm{diag} \begin{pmatrix} 0 & -\beta_G & -\beta_A & -\beta_A \end{pmatrix} . \tag{41}$$

The structure of Eq. (41) reflects that the random-constant gyro bias should not change with time and that the temporal change of the bias variations modeled as GM1 processes depends on the respective correlation times ($T = 1/\beta$).

The individual sub-matrices of $\mathbf{G}$ are defined as follows: $\mathbf{G}_{11}$ (5×3) is the same as the second matrix on the right-hand side of Eq. (37); it models the influence of white noise onto the misalignment and velocity-error states. $\mathbf{G}_{12}$ and $\mathbf{G}_{21}$ are zero matrices of dimensions 5×3 and 4×3, respectively. $\mathbf{G}_{22}$, finally, is given by

$$\mathbf{G}_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{42}$$

According to these definitions, the white-noise vector $\mathbf{w}$ has has six elements. The PSD amplitudes of these elements are summarized in the diagonal matrix

$$\mathbf{W} = \mathrm{diag}\left(\; \sigma^2_{G_w} \quad \sigma^2_{A_w} \quad \sigma^2_{A_w} \quad 2\,\sigma^2_G\,\beta_G \quad 2\,\sigma^2_A\,\beta_A \quad 2\,\sigma^2_A\,\beta_A \; \right) , \tag{43}$$

where the first three elements relate to the white noise sources influcening the gyro and accelerometer measurements, respectively, and the other elements are known from Sect. 2.

3.3 Discrete system dynamics

The discrete version of Eq. (39) is found by the following procedure. First of all, the matrix $\mathbf{A}$ is formed by

$$\mathbf{A} = \begin{bmatrix} -\mathbf{F} & \mathbf{G}\mathbf{W}\mathbf{G}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{F}^{\mathrm{T}} \end{bmatrix} \Delta t \,, \tag{44}$$

where Δt is the update interval of the Kalman filter and $\mathbf{0}$ is a 9×9 zero-matrix (thus, $\mathbf{A}$ is a 18×18 matrix). Subsequently, the exponential of $\mathbf{A}$ is formed:

$$\mathbf{B} = \mathrm{e}^{\mathbf{A}} \,. \tag{45}$$

From mathematics, it is known that the matrix exponential is computed in complete analogy to a scalar exponential:

$$\mathrm{e}^{\mathbf{A}} = \mathbf{I} + \frac{1}{1!}\mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + ... \tag{46}$$

Without proof, the individual 9×9 sub-matrices of $\mathbf{B}$ are found to be

$$\begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} ... & \mathbf{\Phi}^{-1}\mathbf{Q} \\ \mathbf{0} & \mathbf{\Phi}^{\mathrm{T}} \end{bmatrix} , \tag{47}$$

where $\mathbf{\Phi}$ is the discrete state-transition matrix, $\mathbf{Q}$ is the discrete driving noise matrix that includes white noise accumulated during the update interval Δt, and "..." indicates that the sub-matrix $\mathbf{B}_{11}$ is not needed any further. Note that $\mathbf{\Phi}$ and $\mathbf{Q}$ follow from

$$\mathbf{\Phi} = \mathbf{B}_{22}^{\mathrm{T}} \,, \qquad \mathbf{Q} = \mathbf{B}_{22}^{\mathrm{T}}\,\mathbf{B}_{12} \,. \tag{48}$$

3.4 Measurement update

It is assumed that the position of the vehicle is updated by GPS. The corresponding observation equation is given by

$$\delta\mathbf{z} = \mathbf{H}\,\delta\mathbf{x} + \mathbf{v} \,, \tag{49}$$

where $\delta\mathbf{z}$ is the difference of the GPS minus the INS position, i.e.,

$$\delta\mathbf{z} = \begin{bmatrix} x_n^{\mathrm{GPS}} - x_n^{\mathrm{INS}} \\ x_e^{\mathrm{GPS}} - x_e^{\mathrm{INS}} \end{bmatrix} . \tag{50}$$

The design matrix $\mathbf{H}$ is given by

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} , \tag{51}$$

where the nonzero terms indicate a direct observation of the INS position error states (state-vector elements 4 and 5, respectively). The measurement covariance matrix $\mathbf{R}^{\mathrm{GPS}}$ is finally given by

$$\mathbf{R}^{\mathrm{GPS}} = \begin{bmatrix} \sigma_n^2 & \sigma_{n,e} \\ \sigma_{n,e} & \sigma_e^2 \end{bmatrix}^{\mathrm{GPS}} . \tag{52}$$

Now, all preparations are made to set up the extended Kalman filter for the GPS/INS integration. Typically, the errors in the attitude, velocity, and position are fed back to the INS to ensure the validity of the linerity assumption made for the continuous version of the system dynamics.