System Dynamics

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Spring 2023

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Chapter 1

Course Overview and Motivation

Preface

The course is called Systems Engineering in the course catalog, but Prof. Luchtenburg calls it System Dynamics instead. It's a more apt name for what the course actually covers; if you look up Systems Engineering you'll find a completely different subject.

The course text is Ogata's *System Dynamics*, which doesn't fit particularly well for the course. Most feedback control textbooks, like Nise, start with a few chapters covering dynamic modeling and response, but they tend not to be very in depth. I can provide PDFs if you can't find them yourselves. Read the syllabus, it's pretty in depth.

Prof. Luchtenburg *should* put his notes up in the MS Teams Class Notebook and record his lectures. Your mileage may vary if you don't bother coming to class, though; the audio quality in the recordings isn't that great and the notes can be less detailed than the lectures themselves.

Most figures here are taken from Prof. Luchtenburg's Class Notebook. When I take figures from another source, I will mention it inline. This document is a living document, which means I'll continue editing it as I see fit. I reserve the right to include material that may not be covered in the course, but better prepares you for concepts that may seem like they come out-of-the-blue otherwise.

I have this lousy habit of writing in the fourth person and cursing in my writing. Apologies in advance.

1.1 Why Model?

To start off, let's define a **model**. A mathematical model of a system is a set of differential equations that allows us to predict how a system behaves under different conditions. By creating a model, we can use equations and principles to describe a system's present behavior, identify key parameters that affect the system, and make educated guesses as to how the system will behave in the future.

You may have covered some rudimentary modeling in Ma111 or Ma240, where it was probably crammed in to satisfy those annoying kids that go "wHEN ARe we ever gOINg to usE tHIS iN ReAl LIfE". The methods we'll cover in this course will be more robust than simply going off a given formula.

Something we'll exploit heavily a lot in this course is the principle of **analogical models**, or generic representations of common physical phenomena. This turns out to be really useful because it provides a convenient and consistent way to represent and analyze complex systems that involve different types of "stuff". Say you're faced with a fluid flow system, and you haven't taken a fluid mechanics course yet. Using methods introduced in this course, we will be able to convert this system into something equivalent, yet more familiar, like a mass-spring system or a series circuit. Using these analogies allows us to apply the same concepts and mathematical tools to different types of systems, which can greatly simplify the analysis and design process.

To conclude, modeling is a powerful tool that allows us to understand, predict, and (as we'll see in ME351 next semester) **control** the behavior of physical systems, and it has many practical applications in engineering, physics, and other fields.

Don't suck at it.

Chapter 2

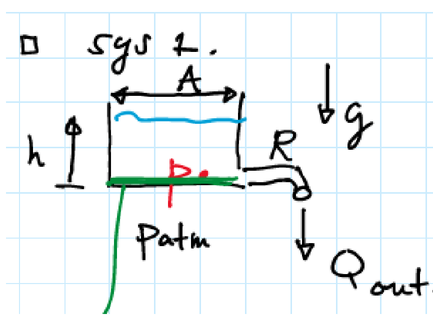
The First Order

Preface

We'll grow more accustomed to the idea of analogous models after modeling a few simpler systems. Let's start by throwing out some fundamental systems and developing intuition to dive into simplified models.

- Emptying a water tank
- Cooling of a lightbulb
- Discharge of an RC circuit

2.1 Gradients Make Stuff Flow



A cylindrical tank is filled to a level h, has a cross-sectional area of A, and an outflow rate of Q_{out}. The pressure outside the tank is P_∞. Can we derive a governing equation for this system? Well, we can try with a few physical principles.

Let's start with a **conservation law**. We know there's a volume $\mathcal{V}$ of water proportional to the value of h. Or...

$$\Delta \mathcal{V} = A\Delta h$$

We'll take the time derivative of that equation to get some more familiar variables. (You might recognize the math here from related rates in Ma111.)

$$\frac{d}{dt}[\Delta \mathcal{V} = A\Delta h] = -Q_{\text{out}}$$

That's not very useful yet. Let's leverage some prior circuits knowledge here...charge moves because of a **voltage difference** ΔV, and comparably, fluid moves because of a **pressure difference**. Ohm's law! We'll come back to that, but the main takeaway here is that the outflow Q_{out} is related to the difference between the pressure inside the tank P and the atmospheric pressure P_∞.

That circuits analogy comes in handy really often, because it turns out Ohm's law translates directly into fluid flow.

$$\Delta V = V - V_0 = IR$$

$$\Delta P = P - P_\infty = Q_{\text{out}} R$$

These are called **constitutive equations**, or relationships between physical quantities. (The flow is proportional to a level difference, or gradient.)

We'll generalize a bit soon, but for now I understand if you don't get it. It's still very hand-wavey.

Let's leverage some hydrostatics now. The tank is open to the atmosphere at the top, so we can actually derive an expression for P, the pressure at the base of the tank. We'll also define ρ, the density of the fluid, and C, or capacitance, as $\frac{A}{\rho g}$, because it ends up being useful in the circuit analogy.[1]

$$\Delta P = \rho g \Delta h = \rho g \frac{\Delta \cancel{V}}{A} = \frac{1}{C} \Delta \cancel{V}$$

$$C = \frac{A}{\rho g}$$

OK, I think we're all set. I've been pretty lax with the "delta's", but it should still be readable. Let me know if things need clarification.

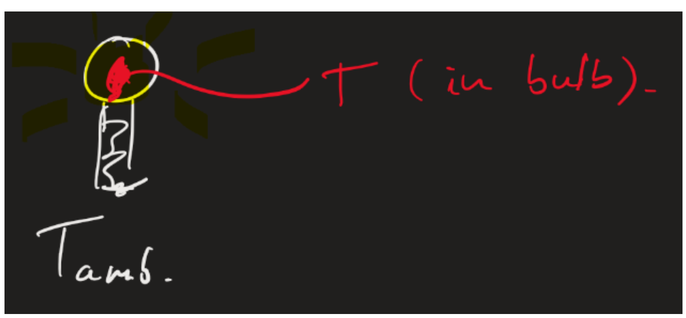
$$\dot{\cancel{V}} = -Q_{\text{out}}$$

$$C \dot{\Delta P} = -\frac{\Delta P}{R}$$

$$\boxed{RC \dot{\Delta P} + \Delta P = 0}$$

This is a really nice differential equation. It looks like the equation for an RC circuit if you've seen those before, with the voltage differentials swapped out for pressure differentials.

Let's move onto a second example: the cooling of a lightbulb. When we shut off power to the lightbulb, how can we measure its temperature as it cools to room temperature?



The bulb is initially very hot (with temperature T) compared to its environment (which has temperature T_∞). Heat is flowing outwards at $\dot{q}_{\text{out}}$.[2] This is seeming very familiar...a temperature difference is driving heat to leave through the resistance R of the bulb.

Let's go through the steps again. What's being conserved here?[3] Internal energy! (Or heat, since there's no work in this system.) It might be a bit early in the semester to have seen the capacitive relationship relating heat q and temperature T in ESC330, but here it is:

[1] You'll see this technique leveraged again in ME342.
[2] I'm not a fan of the usual notation here, so I'm using $\dot{q}$ for the flow of heat and q for heat.
[3] Someone said kinetic energy. What a statistical mechanics-esque answer.

$$\Delta q = C\Delta T$$

Differentiate across the board...

$$\frac{d}{dt}(C\Delta T) = C\dot{T} = -\dot{q}_{\text{out}}$$

And now we're just chugging through the motions. Next is another constitutive relationship (which looks shudderingly close to Ohm's law!):

$$\Delta T = T - T_\infty = \dot{q}_{\text{out}} R$$

Using this and the conservation equation, we construct:

$$\boxed{RC\dot{\Delta T} + \Delta T = 0}$$

Again. Familiar. Very familiar. Maybe there's some unifying theory in the background here. We'll generalize that equation to what we call its **canonical form** $\tau\dot{y} + y = 0$, a first order differential equation. Let's throw in an initial condition $y(0) = y_0$ just so we don't have any undetermined constants at the end.

To solve this differential equation, we'll guess a solution $y(t) = ce^{\alpha t}$, find its time derivative $\dot{y}(t) = \alpha ce^{\alpha t} = \alpha y$, and plug in.

$$\tau\dot{y} + y = 0$$

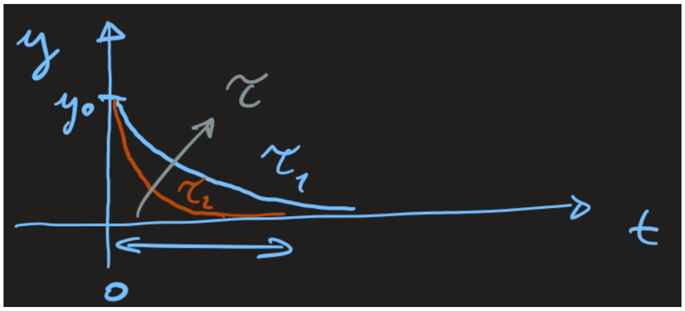
$$\tau\alpha e^{\alpha t} + e^{\alpha t} = 0$$

$$(\tau\alpha + 1)\, e^{\alpha t} = 0$$

$$\alpha = -\frac{1}{\tau}$$

$$y(t) = ce^{-\frac{t}{\tau}} = y_0\, e^{-\frac{t}{\tau}}$$

If you look at the graph below, it's just exponential decay from $(0, y_0)$. We call τ the **time constant** of the system, and it's commonly used to describe how quickly an exponential decays or grows. Different systems have different time constants. (Notably, RC always has units of time). The smaller the time constant, the faster the decay. (Assume for the figure below that $\tau_1 = 10$ and $\tau_2 = 5$.)



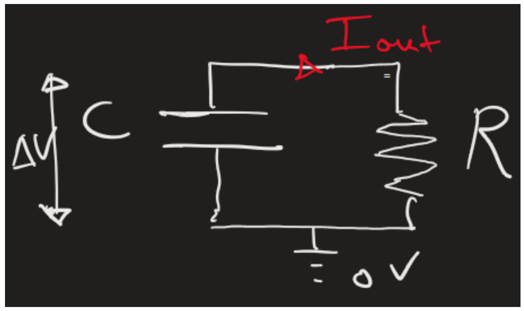
So what happens when we set $t = \tau$? Let's plug it in and find out.

$$y(\tau) = y_0 e^{-\frac{\tau}{\tau}} = y_0 e^{-1}$$

So the time constant is the time at which the system response has decayed to $y_0 e^{-1}$, or approximately 37% of its initial value. We can also reframe this definition as, "the time constant is the time at which the system response has lost approximately 63% of its initial value". There is a different definition of the time constant for increasing systems that we will explore in a later section.

2.2 The RC Circuit and Final Generalization

Say we have an RC circuit with a full capacitor.



The outflow of charge from the capacitor is represented as a negative current:

$$\dot{q} = -I_{\text{out}}$$

Here's Ohm's law:

$$V = I_{\text{out}}R = \dot{q}R$$

Finally, we deal in the capacitive relationship (from Ph213):

$$q = C\Delta V$$

Chug everything together and we get:

$$C\Delta\dot{V} = -\frac{\Delta V}{R}$$

$$\boxed{RC\Delta\dot{V} + \Delta V = 0}$$

which is the same equation we've gotten before. (Notably, we don't have to have this equation in terms of the voltage difference; as you'll see in ESC221 this semester, there's a form of the equation in terms of current as well.)

Final takeaways:

- Most first order systems we'll analyze in this class are the same mathematically!
- **Level differences (gradients) make stuff flow.**

"Stuff" isn't the greatest word for something like this, (maybe quantity or 'energy' instead?) but that's the best we have. Stuff can be stored, like charge in a capacitor, or fluid in a tank, or heat in a reservoir. However, by generalizing these quantities, we can create widely applicable rules for modeling first order systems.

$$\text{Stuff} = \text{Capacitance} \times \text{Level Difference}$$

$$\text{Level Difference} = \text{Flow of Stuff} \times \text{Resistance}$$

Also conservation. That's a biggie.

$$\text{Rate of Change of Stuff} = \text{Inflow} - \text{Outflow}$$

We've only discussed scenarios where there isn't anything flowing in thus far. In these cases, to solve nonhomogeneous differential equations, we'll have to use more specialized methods from Ma240 instead of guessing and praying, like the Laplace transform or the method of undetermined coefficients (or as I affectionately call it, MUC).

2.3 Let's Throw in an Input

A more simple form of the governing equation for one of these first order systems is:

$$\tau \dot{y} + y = ku$$

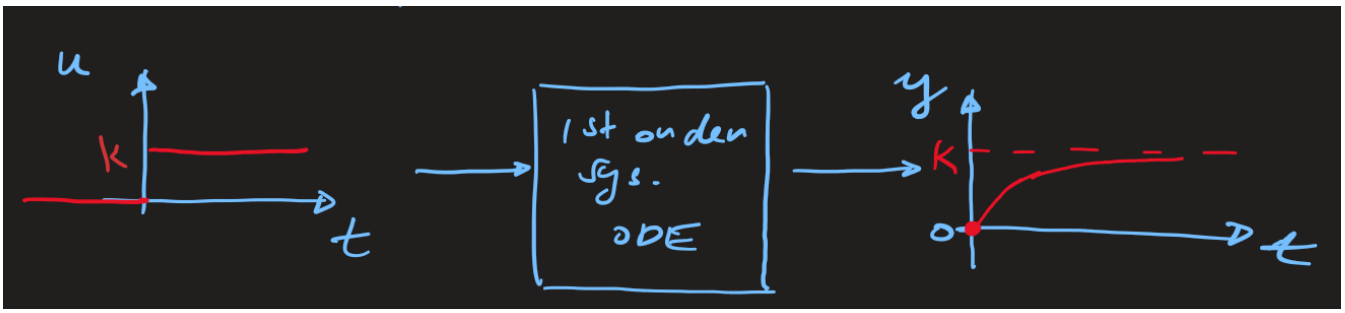
when we have a constant input. Think of it as turning on a light switch at time $t = 0$. k is just a scale factor, and $u(t)$ is the unit-step function, which is just 0 when $t < 0$ and 1 when $t > 0$.[4]

$$y(t) = ce^{-\frac{t}{\tau}} + k$$

For the initial condition $y(0) = y_0$, the undetermined coefficient $c = y_0 - k$. Here's our updated solution:

$$y(t) = y_0 e^{-\frac{t}{\tau}} + k(1 - e^{-\frac{t}{\tau}})$$

When we graph this function for $y_0 = 0$, we see that it gradually grows towards $y = k$ as $t \to \infty$. Now we can analyze exponential growth. You see this behavior everywhere, like when you change a thermostat setting and the temperature slowly creeps towards your choice. This is what we call a **step response**.



How could we find the time constant of this response? Let's take a look at what happens to $y(t)$ at $t = \tau$.

$$y(\tau) = y_0 e^{-\frac{\tau}{\tau}} + k(1 - e^{-\frac{\tau}{\tau}}) = y_0 e^{-1} + k(1 - e^{-1})$$

When we set $y_0 = 0$, this further simplifies to:

$$y(\tau) = k(1 - e^{-1}) \approx 0.63k$$

For this system, at $t = \tau$, the system response will have accumulated 63% of its steady state value.

The step input is just one of the test inputs we usually use; we'll look at a few more as the course progresses (such as sinusoidal waves, delta functions, etc.).

[4] We don't care about what happens at $t = 0$. Stop it.

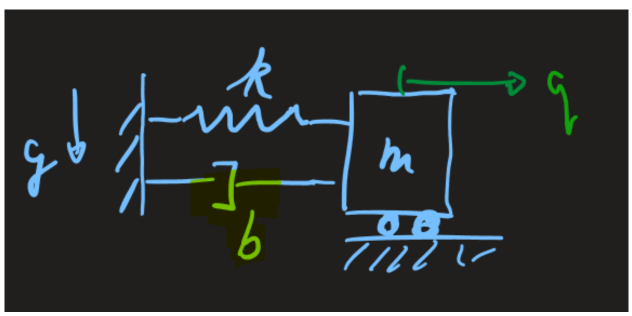
Chapter 3

Our Second Order of Business

Preface

First order systems are honestly pretty boring. When we put in a step input, we just get a pure exponential. We won't be able to get more interesting behavior, like oscillation, because that's just mathematically impossible.[1]

Second order systems, on the other hand, *can* oscillate by themselves. Try to convince yourself of this mathematically just based on what oscillation is.



Mass-spring systems are pretty good models of everything in the world (as long as you use enough mass-spring systems). They're really nice because having a good understanding of ONE mass-spring system provides us with the intuition for more complicated systems.

Say we have a mass-spring system where a mass m is attached to a wall with a spring k and a damper b. Gravity isn't "turned on", so if you want to visualize the system, that mass is floating. The equation of motion for a positive displacement q is:[2]

$$m\ddot{q} = -kq - b\dot{q}$$

Or in its more familiar form:

$$m\ddot{q} + b\dot{q} + kq = 0$$

This is the famed mass-spring equation. Say we have an input - a force u acting on the mass in the positive direction. Now our equation of motion is:

$$m\ddot{q} + b\dot{q} + kq = u$$

[1]Prove it!

[2]Prof. Luchtenburg went off on a tangent about Hooke being a genius for realizing that spring motion is linear near the origin here. That was pretty funny.

This is a linear differential equation, so we'll solve this by plugging in an educated guess. Let's try $q(t) = Ae^{st}$, because differentiating this function n times just multiplies it by s^n.

$$ms^2Ae^{st} + bsAe^{st} + kAe^{st} = u$$

$$ms^2 + bs + k = u$$

First, we'll solve the homogeneous equation, or the case where $u = 0$.[3]

$$ms^2 + bs + k = 0$$

This is known as the characteristic (or auxiliary) equation. We can now use algebra to solve for the roots of the equation, or by proxy, the solution of the differential equation.

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

These are also called the **poles** of the system, but we're getting ahead of ourselves. Let's simplify further.

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

We define the **natural frequency** $\omega_n = \sqrt{\frac{k}{m}}$ and the **damping ratio** $\zeta = \frac{b}{2m\omega_n}$. Using these definitions, we can reorganize the mass-spring equation in terms of these variables.

$$m\ddot{q} + b\dot{q} + kq = 0 \qquad \rightarrow \qquad \ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = 0$$

And the poles of this equation are:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

I just realized this is my second time having this lecture today, so I'm just going to copy-paste my ME301 notes here.

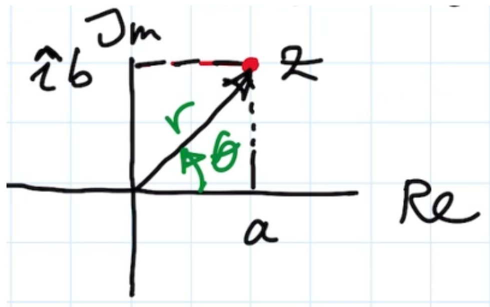
3.1 A Quick Dive into Complex Analysis

Complex analysis is the study of functions of a complex variable z, where z has a real component a and an imaginary component b. Complex numbers show up all the time in this course, whenever anything oscillates, really (like mass-spring systems or pendulums).

To take our first plunge into complex analysis, we need to define the **imaginary unit** i.[4] For now, let's define i as one of the two solutions to the quadratic equation $x^2 = -1$. (The other is $-i$, of course.) This is really special, because now we can describe the solutions of ALL polynomials[5] as the sum of a real number a and another real number b multiplied by the imaginary unit i.

Let's conjure up a graphical representation of these numbers using Cartesian coordinates, where we define one axis as "real" and the other as "imaginary". We'll call this the complex plane. An arbitrary point $z = a + ib$ is plotted below.

[3] If you're curious why we do this, you can read up on it in a linear algebra textbook. Think it's theorem 3.9 in Friedberg's Linear Algebra.
[4] You'll see people, especially electrical engineers, use j instead, because i is commonly used for current. We're better than them.
[5] Regardless if its coefficients are real or complex!



We can also interpret these numbers in the context of polar coordinates, where θ is the angle between the vector from the origin to z and the real axis, and r is the magnitude of the aforementioned vector. It's not difficult to translate between Cartesian coordinates and polar coordinates, but I'll dump the formulas here anyway.

$$a = r\cos\theta \qquad b = r\sin\theta \qquad r = \sqrt{a^2+b^2} \qquad \theta = \arctan\left(\frac{b}{a}\right)$$

It'd be criminal to not mention Euler's formula[6], which posits that:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

A lot of the nuance of complex numbers is best understood by analyzing this **complex exponential**, especially because:

$$z = a + ib = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

3.2 Damping!

When $\zeta = 0$ (or the system is **undamped**), we have the poles $s_{1,2} = \pm i\omega_n^2$. This implies that our solution is a linear combination of sines and cosines, endlessly oscillating, forever and ever. (That's kind of depressing to be honest.)

"*What next? Euler guy.*"

-Prof. Luchtenburg

$$\begin{aligned} x(t) &= A_1e^{i\omega_n t} + A_2e^{-i\omega_n t}\ {}^{7} = A_1(\cos(\omega_n t) + i\sin(\omega_n t)) + A_2(\cos(\omega_n t) - i\sin(\omega_n t)) \\ &= (A_1 + A_2)\cos(\omega_n t) + i(A_1 - A_2)\sin(\omega_n t) = C_1\cos(\omega_n t) + C_2\sin(\omega_n t) \end{aligned}$$

This reasoning carries over if we pick a damping ratio ζ between 0 and 1 (or the system is **underdamped**). The poles are:

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{\omega_n^2(\zeta^2-1)} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} = \sigma \pm i\omega_d$$

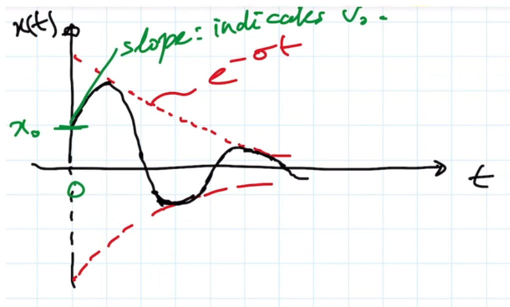
We define the **damped frequency** $\omega_d = \omega_n\sqrt{1-\zeta^2}$. (In practice, $\omega_d \approx \omega_n$, because $\zeta \ll 1$.) Additionally, we define $\sigma = i\omega_n$ (for some reason). After substituting these new variables in, our solution becomes:

$$x(t) = A_1e^{(-\sigma+i\omega_d)t} + A_2e^{(-\sigma-i\omega_d)t} = e^{-\sigma t}(C_1\cos(\omega_n t) + C_2\sin(\omega_n t))$$

In the lattermost form, it's obvious that σ in fact *does* have a use other than bookkeeping; it defines the exponential **envelope** by which the oscillation decays. An envelope is a function that outlines how a function grows/decays (it's the red dashed line in the figure below).

[6] You can prove this using the Taylor series representation of e^x. You should do it, it's *very* rewarding

[7] Apparently A_1 and A_2 are complex conjugates. Don't quote me on that.



Notably, the time constant τ of the envelope is equal to $1/\sigma$. Thus, we can eyeball the value of σ based on how we'd find the time constant (the value 63% less than the y-intercept of the envelope).

$$x(t) = e^{-\sigma t} \sin(\omega_d t + \varphi)$$

Most mechanical systems tend to have a very low damping ratio ($\zeta \simeq O(0.1)$[8]), and as mentioned before, a good rule of thumb is that $\omega_n = \omega_d$.

When we apply some initial conditions (like $q(0) = q_0$ and $\dot{q}(0) = v_0$), our solution becomes:

$$q(t) = e^{\sigma t}\left(q_0 \cos(\omega_d t) + \frac{\sigma q_0 + v_0}{\omega_d} \sin(\omega_d t)\right)$$

When $\zeta = 1$, or the system is **critically damped**, the poles are:

$$s_{1,2} = -\zeta\omega_n$$

We use a trick from differential equations to fake another linearly independent solution, just chuck on an extra t.

$$q(t) = A_1 e^{-\zeta\omega_n t} + A_2 t e^{-\zeta\omega_n t}$$

This doesn't really happen in the real world, but it's nice to cover all our bases. How about when $\zeta > 1$? We call this case **overdamped**, because our poles are:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Both poles are negative here! Our solution is:

$$q(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

3.3 Pole Plots

I've been emphasizing poles a lot in the last few pages, but why? What's the importance of these seemingly arbitrary values? In fact, we can infer the dynamics of a system based on its poles.[9]

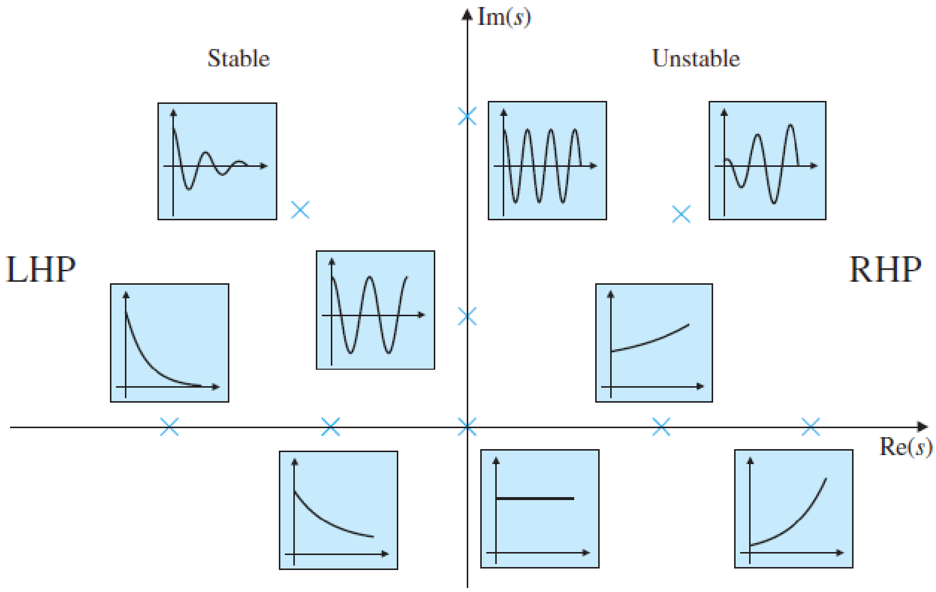
To analyze poles, we use a graphical tool called a **pole plot**, the plot of the roots of the characteristic equation on the complex plane. Let's go down the list:

- When both poles are on the imaginary axis, the system is undamped.
- When both poles are off the real axis, the system oscillates. If they're to the left of the imaginary axis, it'll decay exponentially (underdamped), and if they're to the right of the imaginary axis, it'll grow exponentially.
- If both poles are on the real axis to the left of the imaginary axis, it's overdamped.

[8] Related rates of growth from Ma111, or big-O notation if you've taken ECE264.
[9] This is the crux of a lot we do in ME351. If any of you remember what an eigenvalue is from Ma110, that'll come in handy in a bit.

- Rule of thumb: if there is ANY pole to the right of the imaginary axis, the response blows up.

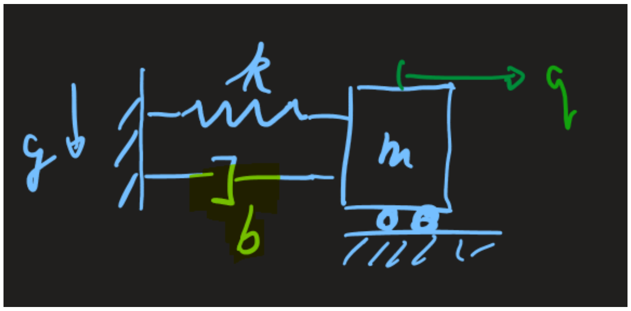
Here's a nice chart from FPE. Complex conjugates are omitted for simplicity.



3.4 Leveraging the Discriminant

I'm going to stray off from Prof. Luchtenburg for a second because *I* think this is useful.

Suppose we want to identify the dampedness of a system based on the equation of motion rather than solving for ζ. We can do exactly this using the **discriminant** of the characteristic equation of the system. Let's take a look at the canonical mass-spring system with a damper once again.



The equation of motion, assuming free motion, is:

$$m\ddot{q} + b\dot{q} + kq = 0$$

The characteristic equation is derived after plugging in $q = e^{st}$.

$$ms^2 + bs + k = 0$$

As stated in the section on damping, there are three forms of the general solution if there is damping present: both poles are real and distinct (the system is overdamped), both poles are real and equal (the system is critically damped), or both poles are complex conjugates (the system is underdamped). You may recall from Algebra that the discriminant of a polynomial can reveal some properties of the roots without actually computing them. The discriminant of a quadratic is defined as follows:

$$\text{Disc}(ax^2 + bx + c) = b^2 - 4ac$$

This is the argument of the square root in the quadratic formula. If this expression is positive, the solutions to the quadratic are real and distinct. If this expression is 0, then there is only one real solution to the quadratic. If this expression is negative, the solutions to the quadratic are complex. So physically, finding the discriminant of the characteristic equation of the mass-spring system will tell us how damped it is.

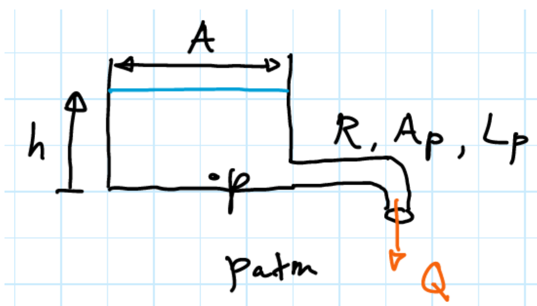
$$\text{Disc}(ms^2 + bs + k) = b^2 - 4mk$$

$$\begin{aligned} b^2 - 4mk > 0 \quad &\rightarrow \quad \text{overdamped} \\ b^2 - 4mk = 0 \quad &\rightarrow \quad \text{critically damped} \\ b^2 - 4mk < 0 \quad &\rightarrow \quad \text{underdamped} \end{aligned}$$

And of course, if $b = 0$, then the system is undamped.

3.5 The Tank, Revisited (Inertia)

Let's revisit the tank from our study of first order systems. However, we'll make one small change: the outflow pipe now has a defined length of L_p.



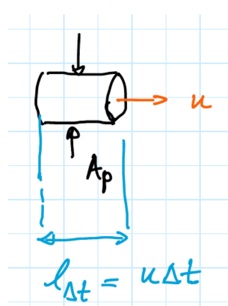
Let's model the same way we've been doing thus far. First, a conservation law:

$$\dot{V} = -Q$$

Next, "Ohm's law":

$$Q = \frac{\Delta P}{R} = \frac{P - P_{\text{atm}}}{R}$$

If we isolate a piece of the pipe (with length L_p) as shown below, we can demystify this system a bit.



Assuming the cross-sectional area of the pipe A_p is constant, the pressure force $F_p = A_p \Delta P$ accelerates the fluid between the two ends of this pipe. Additionally, there is friction $F_f = -QRA_p$ on the liquid caused by the resistance of the pipe. By leveraging Newton's second law of motion, we now have a relationship between the pressure difference ΔP and the velocity of the water v.

$$m\frac{dv}{dt} = F_p + F_f = A_p \Delta P - QRA_p$$

The mass m of the fluid between the two ends of this pipe is equal to the product of the density of the fluid ρ and the volume between the two ends $\forall_p$. (Notably, the volume $\forall_p = A_p L_p$.)

$$\rho \forall_p \frac{dv}{dt} = \rho A_p L_p \frac{dv}{dt} = A_p \Delta P - RQA_p$$

Because the product of the cross-sectional area A_p and the fluid velocity v is equal to the volumetric flow rate Q, we can rewrite this equation as follows:

$$\frac{\rho L_p}{A_p}\frac{dQ}{dt} = \Delta P - RQ$$

Ok, we can shed some light on what we're doing now. We define **inductance** (also referred to as **liquid-flow inertance** or **inertia**) as a term that describes the change in potential required for a unit rate of fluid flow. Inductance is the tendency of the fluid to move; it's created by the inertia of water flowing through the pipe. The mathematical definition of inductance is as follows:

$$L = \frac{\rho L_p}{A_p}$$

Note that this definition of inductance is only valid for flow systems, but analogous concepts occur in other fields (like inductors from circuit analysis)! Fluid components that have an inductance are analogous to these inductors, or mechanical components with inertia.

Let's wrap up this example. When we plug in the definition of L into our equation, a simple first order system rears its head.[10]

$$L\frac{dQ}{dt} + RQ = \Delta P$$

Let's throw it into canonical form so we can see its time constant.

$$\left(\frac{L}{R}\right)\dot{Q} + Q = \frac{\Delta P}{R} \qquad\qquad \tau = \frac{L}{R}$$

To summarize, we've added a new tool to our arsenal: conservation of momentum (or Newton's second law).

$$\begin{aligned} \Delta P &= L\dot{Q} + RQ \\ \dot{\forall} &= -Q \\ C\Delta P &= \forall \end{aligned}$$

By combining these three equations, we can use tools from our studies of mass-spring systems to analyze... well ... any second order system.

$$L\Delta\ddot{P} + R\Delta\dot{P} + \frac{1}{C}\Delta P = 0 \qquad \rightarrow \qquad \Delta\ddot{P} + \left(\frac{R}{L}\right)\Delta\dot{P} + \left(\frac{1}{LC}\right)\Delta P = 0$$

$$m\ddot{q} + b\dot{q} + kq = 0 \qquad \rightarrow \qquad \ddot{q} + \left(\frac{b}{m}\right)\dot{q} + \left(\frac{k}{m}\right)q = 0$$

We simply retrofit the definitions of the natural frequency ω_n and damping ratio ζ based on how we defined them for mass-spring systems to determine how the oscillations behave. Here's a quick example using the flow system analogy we've been using thus far:

[10] The analog of this system in circuit analysis is called the RL circuit, which is often used as a passive filter.

$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = 0 \quad \longleftrightarrow \quad \Delta\ddot{P} + \left(\frac{R}{L}\right)\Delta\dot{P} + \left(\frac{1}{LC}\right)\Delta P = 0$$

$$2\zeta\omega_n = \frac{R}{L} \qquad \longrightarrow \qquad \zeta = \frac{R}{2L\omega_n} = \frac{R\sqrt{LC}}{2L} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$\omega_n^2 = \frac{1}{LC} \qquad \longrightarrow \qquad \omega_n = \frac{1}{\sqrt{LC}} = \frac{\sqrt{LC}}{LC}$$

Chapter 4

An Engineering Student's Butchering of Ma326

Preface

This chapter's going to be a smorgasbord of mathematical concepts I think are foundational to understanding this course from a theoretical perspective. A decent chunk of it will be review from Ma110 and Ma240, but it doesn't hurt to take a second look at these things (they keep coming back over and over).

What I'll try to do, instead of rehashing what you got out of Ma240, is reframe these concepts in a way that's more applicable to this course (and ME351). I'll also formalize a bunch of concepts from Ma326 that are really important to know for this class (and later on). But first, let's formalize a few more definitions that will come in handy later on.

A **linear combination** of elements of a set $x_1, x_2, \ldots, x_n$, is given by:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

where each a_i is a constant. In layman's terms, a linear combination of variables is the sum of scaled versions of those variables, where the scaling factor is a scalar.

A set of variables are **linearly dependent** if one of the variables can be expressed as a linear combination of the others. More formally, a set of variables $x_1, x_2, \ldots, x_n$ is linearly dependent if there exist scalars $a_1, a_2, \ldots a_n$ (not all zero), such that:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

On the other hand, if no such scalars exist, the set of variables is said to be **linearly independent**.

4.1 Linearity and Time-Invariance

A **linear time-invariant (LTI) system** is a mathematical model often used in control theory to describe the behavior of physical systems. It is characterized by two properties: linearity and time-invariance. I'll describe these separately in the context of differential equations. A differential equation of the form:

$$a_ny^{(n)}(t) + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y}(t) + a_0y(t) = f(t)$$

where $y^{(i)}$ is the ith derivative of $y(t)$, is called **linear**. (The relationship between $y(t)$ and t is a linear mapping.) A system is linear if and only if it satisfies two properties: superposition and homogeneity:

- **Superposition** - if $x_1 \to y_1$ and $x_2 \to y_2$, then $x_1 + x_2 \to y_1 + y_2$
- **Homogeneity** - if k is a scalar and $x \to y$, then $kx \to ky$

If ${a_i}_0^n$, also called coefficients, are constants, the equation is also characterized as a **constant coefficient** differential equation. Physically, constant coefficients imply that the system behavior does not depend on time.

We can establish an equivalence between linear constant coefficient equations and linear time-invariant systems.
Time-invariance is the principle that if we chug an input t_0 into a system that outputs $y(t_0)$, the input $t_0 + t_1$ will result in an output of $y(t_0 + t_1)$.[1]

Let's take a look at a few examples to make this more clear:

$$\dot{y} + \sin(t)y = 0$$

This system is not LTI, because the coefficient of y is not constant. (More explicitly, it's a function of t, so as $t \to \infty$, the behavior is affected.)

$$2\dot{y} + 3y = 0$$

This system *is* LTI, because the coefficients of each derivative of y are constant.

$$\dot{y} + \ln(y) = 0$$

This system isn't even linear, for obvious reasons.

Examples of LTI systems include first-order passive filters, second order systems such as springs and masses, and many other linear systems in control theory and signal processing.

4.2 Matrices, at Lightspeed

I truly hope you know what a matrix is by now. If not, fasten your seatbelt.

A matrix is a rectangular array of numbers, of the form:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where a_{ij} is the entry at row i and column j. We say a matrix is of size $m \times n$, where m is the number of rows and n is the number of columns.

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{3\times 2}$$

Matrices are pretty neat. We can add numbers in the matrix elementwise and scale it by a scalar factor as follows:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

[1] An equivalent definition in signal processing is that a system is time-invariant if it commutes with a "delay".

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

The **zero matrix $\mathbf{0}$** is an $m \times n$ matrix with all entries 0. The zero matrix plays an important role in linear algebra, as it is the additive identity for matrices. This means that adding a zero matrix to any matrix does not change the matrix, much like adding zero to any number does not change its value.

A **square matrix** is a matrix with the same number of rows and columns. Many concepts in linear algebra are designed with these in mind, such as determinants and eigenvalues.

Next, we'll tackle the **Kronecker delta**, which is is a useful and important symbol in mathematics, particularly in linear algebra and related fields. Its simple definition allows for the easy expression of many concepts and operations, making it a valuable tool for mathematicians and scientists. We define the Kronecker delta δ_{ij} as follows:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Following from this, the **identity matrix $\mathbf{I}_n$** is defined as an $n \times n$ square matrix where $(\mathbf{I}_n)_{ij} = \delta_{ij}$, i.e.,

$$\mathbf{I}_1 = 1,\ \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \ldots$$

The set of all matrices fixed at size $m \times n$ with scalar entries forms something we call a vector space. There's a lot of nuance attached to that name; if you care about it, take Ma326. Here are a few properties for now. (Bolded quantities are matrices, a, b, and 1 are scalars.)

- Matrix addition is commutative, i.e., $\mathbf{X} + \mathbf{Y} = \mathbf{Y} + \mathbf{X}$
- Matrix addition is associative, i.e., $(\mathbf{X} + \mathbf{Y}) + \mathbf{Z} = \mathbf{X} + (\mathbf{Y} + \mathbf{Z})$
- Each matrix has an additive identity, i.e., $\mathbf{X} + \mathbf{0} = \mathbf{X}$
- Each matrix has an additive inverse, i.e., $\mathbf{X} + \mathbf{Y} = \mathbf{0}$.
- Each matrix has a scalar multiplicative identity, i.e., $1(\mathbf{X}) = \mathbf{X}$
- $(ab)\mathbf{X} = a(b\mathbf{X})$
- $a(\mathbf{X} + \mathbf{Y}) = a\mathbf{X} + a\mathbf{Y}$
- $(a + b)\mathbf{X} = a\mathbf{X} + b\mathbf{X}$

We can multiply two matrices of sizes $m \times n$ and $n \times p$, respectively, to produce another matrix of size $m \times p$. The operation, dubbed **matrix multiplication**, should not be confused with the scalar multiplication used before. It's defined as follows:

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{n} \mathbf{A}_{ik}\mathbf{B}_{kj}$$

Finding the product of two matrices may look intimidating based off that formula, but it's actually not that difficult once you understand the basic steps. First, you need to make sure that the matrices are compatible for multiplication. To do this, we need to ensure that the number of columns in the first matrix is the same as the number of rows in the second matrix. If they're not the same, you can't multiply them.[2]

Once you've determined that the matrices are compatible, you can start multiplying. To find each element in the product matrix, you need to multiply the corresponding row in the first matrix by the corresponding column in the second matrix. Specifically, for each element in the product matrix, you will:

[2] This is a really great gut check when you're finishing up a long calculation. If that matrix multiplication at the end of the problem is impossible, something must be up.

- Take the row of the first matrix that corresponds to that element.
- Take the column of the second matrix that corresponds to that element.
- Multiply each corresponding pair of elements in the row and column.
- Add up all of the products you got in the last step.

Keep doing this for every element in the product matrix until you've filled in all the entries. Here's a brief example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 3 + 3\cdot 5 & 1\cdot 2 + 2\cdot 4 + 3\cdot 6 \\ 4\cdot 1 + 5\cdot 3 + 6\cdot 5 & 4\cdot 2 + 5\cdot 4 + 6\cdot 6 \end{bmatrix} = \begin{bmatrix} 22 & 28 \\ 49 & 64 \end{bmatrix}$$

Matrix products show up everywhere. They're essential for fields like population modeling, network theory, signal processing, advanced dynamics, etc. Try to be as comfortable as possible with this operation before diving into the next chapter.

4.3 Transposition and Symmetry

The **transpose** of an $m \times n$ matrix $\mathbf{A}$ is the $n \times m$ matrix obtained by interchanging its rows and columns. It's often denoted as $\mathbf{A}^{\mathrm{T}}$ or $\mathbf{A}^{\mathbf{t}}$.[3] In other words, the rows of the original matrix become columns in the transposed matrix, and the columns become rows.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Let's follow up with a few more properties that involve transposition. (Bolded quantities are matrices, a, b are scalars.)

- $\left(\mathbf{X}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{X}$
- $(\mathbf{X}+\mathbf{Y})^{\mathrm{T}} = \mathbf{X}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$
- $a\mathbf{X}^{\mathrm{T}} = (a\mathbf{X})^{\mathrm{T}}$
- $(\mathbf{X}\mathbf{Y})^{\mathrm{T}} = \mathbf{Y}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}$

A **symmetric** matrix is a square matrix equal to its own transpose. (Another way to phrase this is: if $\mathbf{A}$ is an $n \times n$ matrix, then $\mathbf{A}$ is symmetric if and only if $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$.)

Trivially, the sum of two symmetric matrices is symmetric, and a scalar multiple of a symmetric matrix is symmetric.

There's other kinds of symmetry as well. For example, a **skew-symmetric** matrix is a square matrix equal to its negative. (Another way to phrase this is: if $\mathbf{A}$ is an $n \times n$ matrix, then $\mathbf{A}$ is skew-symmetric if and only if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$.) This turns out to be *much* more interesting than its vanilla counterpart in dynamics, especially when dealing with things like inertial tensors and angular momentum.

Again, trivially, the sum of two skew-symmetric matrices is symmetric, and a scalar multiple of a skew-symmetric matrix is symmetric. Additionally, you may realize that every entry on the diagonal of a skew-symmetric matrix must be equal to 0. (Otherwise, how would the definition work?)

This factoid turns out to be crucial if we focus on the 3-space case, there are only three independent entries of a skew-symmetric matrix. We can define a skew-symmetric operator for vectors in 3-space skew() as follows:[4]

[3] I prefer the former notation and will be using it henceforth.
[4] Yeah, you'll see awful notation everywhere for this thing. I'm going to use skew() because why not.

$$\text{skew}(\underline{x}) = \text{skew}(x_1, x_2, x_3) = \begin{bmatrix} 0 & -x_1 & x_2 \\ x_1 & 0 & -x_3 \\ -x_2 & x_3 & 0 \end{bmatrix}$$

This operator comes with a really cool property:

$$\text{skew}(\underline{x})^{\mathrm{T}} = -\text{skew}(\underline{x})$$

which comes in handy a lot in advanced dynamics. Additionally, we can make an alternative definition of the vector cross product.

$$\underline{x} \times \underline{y} = \text{skew}(\underline{x})\underline{\mathrm{y}}$$

Prove it! It's kind of fun.

4.4 Eigen Eigen Eigen

Chapter 5

Modeling in Different Domains

Preface - Time Domain Modeling

When we begin to analyze more complicated systems, it becomes less and less feasible to solve them analytically. As a result, we resort to setting up a system of differential equations rather than concatenating them into one as we've done in past chapters.

The state-space representation of a system describes the system's behavior over time in terms of a set of variables called **states**.[1] The state variables represent the current conditions of the system, and their evolution over time is described by a set of first order differential equations called **state equations**.

The state-space representation is a very powerful tool for modeling and analyzing physical systems, providing valuable insights into their behavior and enabling the development of control algorithms covered in ME351.

An important thing to note is that the state-space representation of a system is not unique. In fact, an infinite number of representations exist for a physical system.

5.1 The State-Space Approach

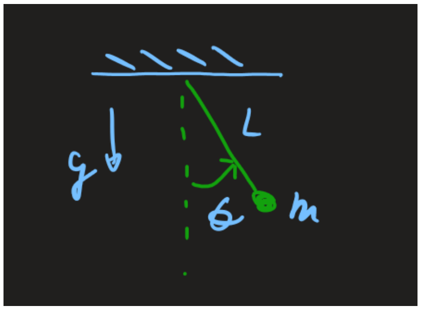
In the most general case, a state-state representation can be represented as the following:

$$\begin{cases} \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \\ \underline{y} = \underline{h}(\underline{x}, \underline{u}) \end{cases}$$

This might be a bit daunting at first, but it's just a lot of fancy notation for a concept that's pretty simple. $\underline{x}$ is the state, $\underline{u}$ is an input, and $\underline{y}$ is an output. Let's drive this concept home with an example.

Say we have a simple pendulum with length L and a point mass m at its end, as pictured below:

[1]The state space can be described as a Euclidean space where each state corresponds with an axis.



For this problem, the mass of the rod (and any potential friction in the hinge) is ignored. The equation of motion of the pendulum can be derived by summing moments about the point of contact between the pendulum and the fixed surface.[2] Let's call that point of contact O for future bookkeeping purposes.

$$\sum M_O = J_O \ddot{\theta}$$

The moment arm for the weight mg is the horizontal displacement $L\sin(\theta)$, and $J_O = mL^2$ is the mass moment of inertia of the point mass m about point O. Let's crunch some numbers.

$$-mgL\sin(\theta) = mL^2\ddot{\theta}$$

$$mL^2\ddot{\theta} + mgL\sin(\theta) = 0$$

$$\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0$$

Now let's try putting this in state-space form. First, we select the state vector, which should adhere to the following points:

- Pick state variables that include all the relevant information about the system you're trying to model.
- The number of dimensions in the state vector should match the number of degrees of freedom of the system.
- The state vector should be **minimal**, meaning it should contain only the information necessary to describe the system, and not any redundant information. (Usually, the minimum number required is equal to the order of the differential equation that represents the system.)
- The components of the state vector must be linearly independent.

A good rule of thumb is that the state should correspond with the initial conditions provided. We define states θ (angle) and $\omega = \dot{\theta}$ (angular velocity), and start constructing our state equations.

$$\underline{x} = \begin{bmatrix} \theta \\ \omega \end{bmatrix} \qquad \dot{\underline{x}} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix}$$

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$$

$$\dot{\underline{x}} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{g}{L}\sin(\theta) \end{bmatrix}$$

We've turned a second order differential equation into two first order differential equations. Let's move onto the second part of the representation: defining the output y. We're interested in the states' behavior over time, so our output is...just the state vector $\underline{x}$.

$$\underline{y} = \underline{h}(\underline{x}, \underline{u}) = \underline{x} = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

[2] Alternatively, you could sum forces in the parallel and perpendicular directions of motion to yield an equivalent result.

These two components make up the state-space representation of this pendulum system. Putting it in this form makes it easier to numerically solve using tools like Python or MATLAB.

A nonlinear solution can be unappealing, though perfectly valid. By using the small-angle approximation $\sin(\theta) \sim \theta$, we can refine this representation further.

$$\ddot{\theta} + \frac{g}{L}\sin(\theta) \sim \ddot{\theta} + \frac{g}{L}\theta = 0$$

Our system is now an LTI system. Linearity is very nice, because we can use matrix multiplication to make this representation pretty.

$$\underline{\dot{x}} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{g}{L}\theta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

Let's move onto the **output equation**, which is (in my opinion) less interesting than the state equation. What are we interested in analyzing here? Say we're interested in analyzing θ - or more succinctly, $y = \theta$.

$$\underline{y} = \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

Or, more interestingly, say we want to track both states over time (θ and ω). Our output is just the state, so we set $y = \underline{x}$. In these cases, the "coefficient" matrix is the $n \times n$ identity matrix $\mathbf{I}_n$, where n is the number of components in the state vector.

$$\underline{y} = \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \mathbf{I}_2 \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

We'll move on using this output equation. Let's make this more complicated by saying the pendulum has an input applied torque T, and the new equation of motion is:

$$\ddot{\theta} + \frac{g}{L}\theta = \frac{T}{mL^2}$$

Our revised state-space representation would be:

$$\underline{\dot{x}} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{g}{L}\theta + \frac{T}{mL^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} T = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} u$$

$$\underline{y} = \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

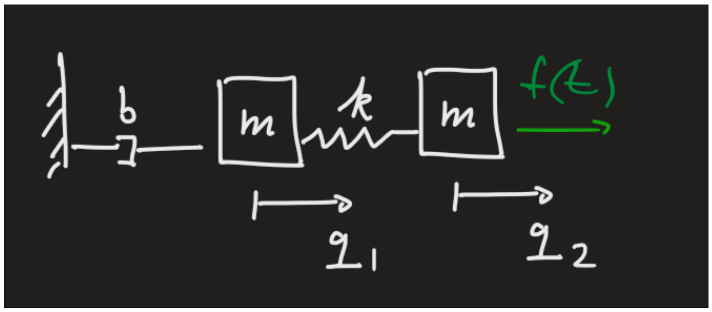
Using this example, we now extract the general form of the state-space representation of a linear system. A linear system is represented in state space by the following equations:

$$\underline{\dot{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u}$$
$$\underline{y} = \mathbf{C}\underline{x} + \mathbf{D}\underline{u}$$

for $t \geq t_0$, $\underline{x}(t_0)$, where:
$\underline{x}$ is the state vector, of size $n \times 1$
$\underline{y}$ is the output vector, of size $q \times 1$
$\underline{u}$ is the input vector, of size $p \times 1$
A is the system matrix, of size $n \times n$
B is the input matrix, of size $n \times p$
C is the output matrix, of size $q \times n$
D is the feedforward matrix, of size $q \times p$

5.2 Another Mass? Seriously?

Let's try another example, this time a translational mechanical system. Block 1 of mass m is attached to a fixed wall by dashpot with damping coefficient b. Block 2, also of mass m, is attached to block 1 by a spring of spring constant k. Gravity is turned off.



First, we write the equations of motion of the network. (Just draw a free body diagram around each mass and don't fuck up your signs.)

"*You want a hint? Newton guy.*"

-Prof. Luchtenburg

$$m\ddot{q}_1 + b\dot{q}_1 + kq_1 - kq_2 = 0$$

$$-kx_1 + m\ddot{q}_2 + kq_2 = f(t)$$

This is a system of two second order differential equations, so we'll pick four states. We select our q_1, $\dot{q}_1$, q_2, and $\dot{q}_2$ to be our four state variables, because we're analyzing the kinematic behavior of two masses obeying Newton's second law (N2L is a second order differential equation, which requires two initial conditions, and since there's two masses to analyze we have four).

Our first two state equations are easy: just define the derivatives. We get the other two by rearranging the equations of motion and isolating $\ddot{q}_1$ and $\ddot{q}_2$.

$$\dot{\underline{x}} = \begin{bmatrix} \dot{q}_1 \\ \ddot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{b}{m} & \frac{k}{m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m} & 0 & -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} f(t)$$

Now, we didn't specify what output we wanted, but let's say we want to analyze the acceleration of the second mass, or $\ddot{q}_2$. We'd do the following:

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} f(t)$$

which is equivalent to the expression $y = \dot{q}_2$. Even though it seems more complicated to put it in this form, it provides us with a **C** and **D** matrix, which is invaluable when analyzing systems computationally.

What makes this example more significant than the previous one is that there's now two different "entities" to analyze.[3] Rather than just analyzing multiple states (position, velocity) of a singular entity like we did with the pendulum, we're now taking a look at the position and velocity of TWO masses. That's kind of nifty, I think.

[3] You could try to put the two tanks problem from HW1 in state-space form for extra practice.

Preface - Frequency Domain Modeling

While using the state-space representation of a system can be advantageous in situations where we have multiple inputs and multiple outputs, sometimes the linear algebra just gets too unwieldy, especially if you don't have Python or MATLAB sitting in front of you. (This isn't to say that transfer functions aren't easily implementable in Python or MATLAB, though.)

In lieu of modeling in the time domain, we can use **transfer functions** to mathematically model systems in the frequency domain. This ends up being really useful in ME351 when we talk about how to control physical systems rather than just analyzing them. Transfer functions are also much easier to translate into graphical interpretations of systems, like Bode plots.

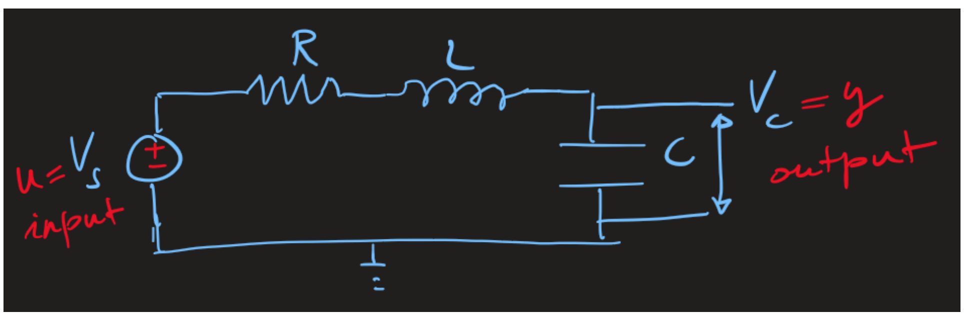
5.3 The Laplace Transform

5.4 Block Diagrams

When possible, we try to think of all systems as black boxes with an input and an output. As such, that's how we

5.5 Electrical Networks and Impedance

In ESC221, you may have covered the series RLC circuit, pictured as follows:



where the source voltage V_s is the input of the system and the voltage across the capacitor V_c is the output. The governing equation of this system is a second order differential equation, but using transfer functions, we can turn this differential equation into an algebraic one.

This is especially advantageous when conducting an alternating current (AC) analysis of the network, when V_s could be a sinusoidal function. However, the formulation described in this course is applicable to any input signal.

The **impedance** Z of a two-terminal passive component like a resistor, capacitor, or inductor, is defined as the ratio of the Laplace transform of the voltage $\tilde{V}$ to the Laplace transform of the current $\tilde{I}$.

Ohm's law states that $V = IR$. When we take the Laplace transform of the expression, we get:

$$\tilde{V} = \tilde{I}R$$

Thus, the impedance of a resistor is:

$$Z_R = \frac{\tilde{V}}{\tilde{I}} = R$$

That was...anticlimactic. Let's try a capacitor next. The capacitive relationship states that $q = \int I\,dt = CV$. When we take the Laplace transform of the expression, we get:

$$\frac{1}{s}\tilde{I} = C\tilde{V}$$

Thus, the impedance of a capacitor is:

$$Z_C = \frac{\tilde{V}}{\tilde{I}} = \frac{1}{sC}$$

That's more interesting. Finally, the inductive relationship states that $V = L\frac{dI}{dt}$. Again, Laplace transform:

$$\tilde{V} = sL\tilde{I}$$

and the impedance of an inductor is:

$$Z_L = \frac{\tilde{V}}{\tilde{I}} = sL$$

Here's a bit more context - think of impedance as the frequency domain "value" of a passive component. Complex resistance, if you will. Resistance is real, so there's no reason impedance is different in the Laplace'd analog of Ohm's law.

Relatedly, for a direct current (DC) circuit, $s = 0$.

Let's take a look at the circuit again. The governing equation, found using Kirchhoff's voltage law, is:

$$V_s - RI - L\frac{dI}{dt} - V_c = 0$$

Using these newfound impedances, we can immediately convert this system to a transfer function. Let's Laplace the shit out of this thing. (Keep in mind that a transfer function is output over input.)

$$\tilde{V_s} - \tilde{I}(Z_R + Z_L + Z_C) = 0$$

$$\tilde{V_s} - \tilde{I}(R + sL + \frac{1}{sC}) = 0$$

$$U = \tilde{V_s} = \frac{\tilde{I}}{R + sL + \frac{1}{sC}} \qquad Y = \tilde{V_c} = \frac{\tilde{I}}{sC}$$

$$\frac{Y}{U} = \frac{\frac{\tilde{I}}{sC}}{\frac{\tilde{I}}{R+sL+\frac{1}{sC}}} = \frac{s^2CL + sCR + 1}{s^2C^2}$$

We can expand this idea to more than just passive components.[4] Impedance is a reliable method for thinking about mechanical systems as well (such as mass-spring systems, systems involving gears, motors, etc.)

The pigeonhole here is that these systems are modeled as linear. The frequency domain does not translate well when we study nonlinear systems; instead, we tend to use more qualitative methods such as drawing phase portraits or bifurcation analysis.

[4]In the electrical networks world, we can translate operational amplifiers (op amps) into impedances as well. Absolutely terrifying.

Chapter 6

Call and Response!

Preface

It's important to note that the transfer function is just a description of the system, not its inputs and outputs. From the section on block diagrams, we know that the Laplace transform of the output signal is equal to the product of the Laplace transform of the input signal and the transfer function of the system.

As such, it's advantageous to think about how different systems react to common input signals to quantify the stability of a system. For example, how does your heating system react when you input a temperature setting on your thermostat?

6.1 It's About Time

First, let's revisit the **step response**.

6.2 The Squiggly One

talk about mag, phase, some basic filters