

# System Dynamics

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# Chapter 1

## Course Overview and Motivation

### Preface

The course is called Systems Engineering in the course catalog, but Prof. Luchtenburg informally calls it System Dynamics. It's a more apt name for what the course actually covers; if you look up Systems Engineering you'll find a completely different subject.

The course text is Ogata's *System Dynamics*, which is a pretty lousy book, I'd recommend reading the first few chapters of Nise or FPE instead. I can provide PDFs if you can't find them yourselves. Read the syllabus, it's pretty in depth.

Prof. Luchtenburg *should* put his notes up in the MS Teams Class Notebook and record his lectures. Your mileage may vary if you don't bother coming to class.

Most figures here are taken from Prof. Luchtenburg's Class Notebook. When I take figures from another source, I will mention it there. This document is a living document, which means I'll continue editing it as I see fit.

I have this lousy habit of writing in the fourth person and cursing in my writing. Apologies in advance.

### 1.1 Why Model?

Good question.

# Chapter 2

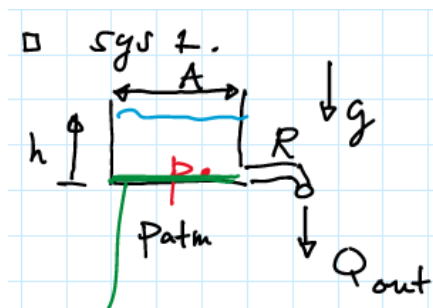
## The First Order

### Preface

Let's start by throwing out some systems and testing familiarity with first-order systems and simplified models.

- Emptying a water tank
- Cooling of a lightbulb
- Discharge of an RC circuit

### 2.1 Gradients Make Stuff Flow (Very Hand-Wavey Edition)



A cylindrical tank is filled to a level  $h$ , has a cross-sectional area of  $A$ , and an outflow rate of  $Q_{\text{out}}$ . The pressure outside the tank is  $P_{\infty}$ . Can we derive a governing equation for this system? Well, we can try with a few physical principles.

Let's start with a **conservation law**. We know there's a volume  $V$  of water proportional to the value of  $h$ . Or...

$$\Delta V = A\Delta h$$

We'll take the time derivative of that equation to get some more familiar variables. (You might recognize the math here from related rates in Ma111.)

$$\frac{d}{dt}[\Delta V = A\Delta h] = -Q_{\text{out}}$$

That's not very useful yet. Let's leverage some prior circuits knowledge here...charge moves because of a **voltage difference**  $\Delta V$ , and comparably, fluid moves because of a **pressure difference**. Ohm's law! We'll come back to that, but the main takeaway here is that the outflow  $Q_{\text{out}}$  is related to the difference between the pressure inside the tank  $P$  and the atmospheric pressure  $P_{\infty}$ .

That circuits analogy comes in handy really often, because it turns out Ohm's law translates directly into fluid flow.

$$\Delta V = V - V_0 = IR$$

$$\Delta P = P - P_\infty = Q_{\text{out}} R$$

These are called **constitutive equations**, or relationships between physical quantities. (The flow is proportional to a level difference, or gradient.)

We'll generalize a bit soon, but for now I understand if you don't get it. It's still very hand-wavey.

Let's leverage some hydrostatics now. The tank is open to the atmosphere at the top, so we can actually derive an expression for  $P$ , the pressure at the base of the tank. We'll also define  $\rho$ , the density of the fluid, and  $C$ , or capacitance, as  $\frac{A}{\rho g}$ , because it ends up being useful in the circuit analogy.<sup>1</sup>

$$\Delta P = \rho g \Delta h = \rho g \frac{\Delta V}{A} = \frac{1}{C} \Delta V$$

$$C = \frac{A}{\rho g}$$

OK, I think we're all set. I've been pretty lax with the "delta's", but it should still be readable. Let me know if things need clarification.

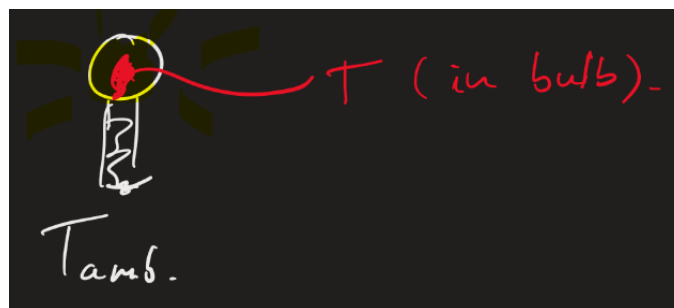
$$\dot{V} = -Q_{\text{out}}$$

$$C \dot{\Delta P} = -\frac{\Delta P}{R}$$

$$RC \dot{\Delta P} + \Delta P = 0$$

This is a really nice differential equation. It looks like the equation for an RC circuit if you've seen those before, with the voltage differentials swapped out for pressure differentials.

Let's move onto a second example: the cooling of a lightbulb. When we shut off power to the lightbulb, how can we measure its temperature as it cools to room temperature?



The bulb is initially very hot (with temperature  $T$ ) compared to its environment (which has temperature  $T_\infty$ ). Heat is flowing outwards at  $\dot{q}_{\text{out}}$ .<sup>2</sup> This is seeming very familiar...a temperature difference is driving heat to leave through the resistance  $R$  of the bulb.

Let's go through the steps again. What's being conserved here?<sup>3</sup> Internal energy! (Or heat, since there's no work in this system.) It might be a bit early in the semester to have seen the capacitive relationship relating heat  $q$  and temperature  $T$  in ESC330, but here it is:

<sup>1</sup>You'll see this technique leveraged again in ME342.

<sup>2</sup>I'm not a fan of the usual notation here, so I'm using  $\dot{q}$  for the flow of heat and  $q$  for heat.

<sup>3</sup>Someone said kinetic energy. What a statistical mechanics-esque answer.

$$\Delta q = C \Delta T$$

Differentiate across the board...

$$\frac{d}{dt}(C \Delta T) = C \dot{T} = -\dot{q}_{\text{out}}$$

And now we're just chugging through the motions. Next is another constitutive relationship (which looks shudderingly close to Ohm's law!):

$$\Delta T = T - T_{\infty} = \dot{q}_{\text{out}} R$$

Using this and the conservation equation, we construct:

$$RC \dot{\Delta T} + \Delta T = 0$$

Again. Familiar. Very familiar. Maybe there's some unifying theory in the background here.

## 2.2 Solving that Damn Equation (and Time Constants)

We'll generalize that equation to what we call its **canonical form**  $\tau \dot{y} + y = 0$ , a first order differential equation. Let's throw in an initial condition  $y(0) = y_0$  just so we don't have any undetermined constants at the end.

To solve this differential equation, we'll guess a solution  $y(t) = ce^{\alpha t}$ , find its time derivative  $\dot{y}(t) = \alpha ce^{\alpha t} = \alpha y$ , and plug in.

$$\tau \dot{y} + y = 0$$

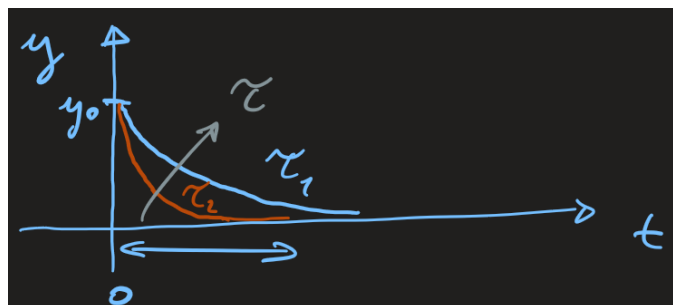
$$\tau \alpha e^{\alpha t} + e^{\alpha t} = 0$$

$$(\tau \alpha + 1) e^{\alpha t} = 0$$

$$\alpha = -\frac{1}{\tau}$$

$$y(t) = ce^{-\frac{t}{\tau}} = y_0 e^{-\frac{t}{\tau}}$$

If you look at the graph below, it's just exponential decay from  $(0, y_0)$ . We call  $\tau$  the **time constant** of the system, and it's commonly used to describe how quickly an exponential decays or grows. Different systems have different time constants. (Notably,  $RC$  always has units of time). The smaller the time constant, the faster the decay. (Assume for the figure below that  $\tau_1 = 10$  and  $\tau_2 = 5$ .)



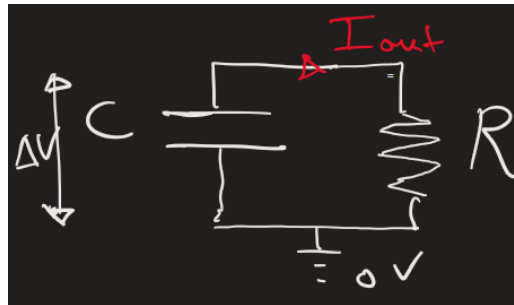
So what happens when we set  $t = \tau$ ? Let's plug it in and find out.

$$y(\tau) = y_0 e^{-\frac{\tau}{\tau}} = y_0 e^{-1}$$

So the time constant is the time at which the system response has decayed to  $y_0 e^{-1}$ , or approximately 37% of its initial value. We can also reframe this definition as, “the time constant is the time at which the system response has lost approximately 63% of its initial value”. There is a different definition of the time constant for increasing systems that we will explore in the next section.

## 2.3 The RC Circuit and Final Generalization

Say we have an RC circuit with a full capacitor.



The outflow of charge from the capacitor is represented as a negative current:

$$\dot{q} = -I_{\text{out}}$$

Here's Ohm's law:

$$V = I_{\text{out}} R = \dot{q} R$$

Finally, we deal in the capacitive relationship (from Ph213):

$$q = C \Delta V$$

Chug everything together and we get:

$$C \Delta \dot{V} = -\frac{\Delta V}{R}$$

$$\boxed{RC \Delta \dot{V} + \Delta V = 0}$$

which is the same equation we've gotten before. (Notably, we don't have to have this equation in terms of the voltage difference; as you'll see in ESC221 this semester, there's a form of the equation in terms of current as well.)

Final takeaways:

- Most first order systems we'll analyze in this class are the same mathematically!
- **Level differences make stuff flow.**

“Stuff” isn't the greatest word for something like this, (maybe quantity instead?) but that's the best we have. Stuff can be stored, like charge in a capacitor, or fluid in a tank, or heat in a reservoir. However, by generalizing these quantities, we can create widely applicable rules for modeling first order systems.

$$\text{Stuff} = \text{Capacitance} \times \text{Level Difference}$$

$$\text{Level Difference} = \text{Flow of Stuff} \times \text{Resistance}$$

Also conservation. That's a biggie.

$$\text{Rate of Change of Stuff} = \text{Inflow} - \text{Outflow}$$

We've only discussed scenarios where there isn't anything flowing in thus far. In these cases, to solve nonhomogeneous differential equations, we'll have to use more specialized methods from Ma240 instead of guessing and praying, like the Laplace transform or the method of undetermined coefficients (or as I affectionately call it, MUC).

## 2.4 Let's Throw in an Input

A more simple form of the governing equation for one of these first order systems is:

$$\tau \dot{y} + y = ku$$

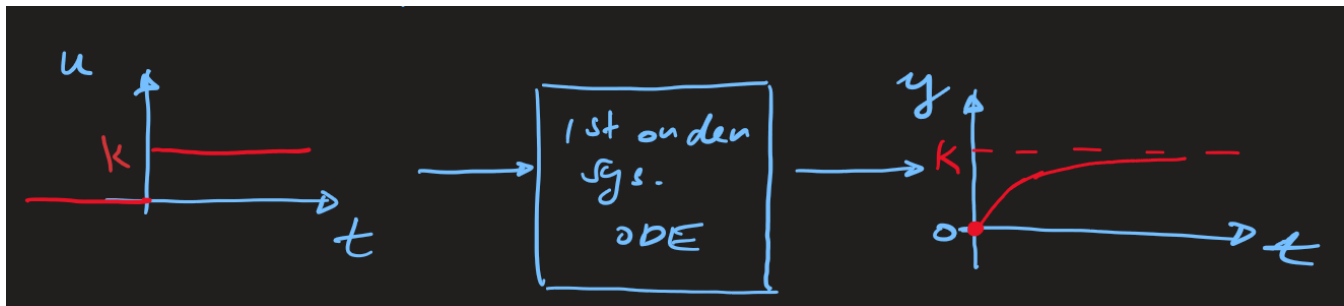
when we have a constant input. Think of it as turning on a light switch at time  $t = 0$ .  $k$  is just a scale factor, and  $u(t)$  is the unit-step function, which is just 0 when  $t < 0$  and 1 when  $t > 0$ .<sup>4</sup>

$$y(t) = ce^{-\frac{t}{\tau}} + k$$

For the initial condition  $y(0) = y_0$ , the undetermined coefficient  $c = y_0 - k$ . Here's our updated solution:

$$y(t) = y_0 e^{-\frac{t}{\tau}} + k(1 - e^{-\frac{t}{\tau}})$$

When we graph this function for  $y_0 = 0$ , we see that it gradually grows towards  $y = k$  as  $t \rightarrow \infty$ . Now we can analyze exponential growth. You see this behavior everywhere, like when you change a thermostat setting and the temperature slowly creeps towards your choice. This is what we call a **step response**.



How could we find the time constant of this response? Let's take a look at what happens to  $y(t)$  at  $t = \tau$ .

$$y(\tau) = y_0 e^{-\frac{\tau}{\tau}} + k(1 - e^{-\frac{\tau}{\tau}}) = y_0 e^{-1} + k(1 - e^{-1})$$

When we set  $y_0 = 0$ , this further simplifies to:

$$y(\tau) = k(1 - e^{-1}) \approx 0.63k$$

For this system, at  $t = \tau$ , the system response will have accumulated 63% of its steady state value.

The step input is just one of the test inputs we usually use; we'll look at a few more as the course progresses (such as sinusoidal waves, delta functions, etc.).

<sup>4</sup>We don't care about what happens at  $t = 0$ . Stop it.



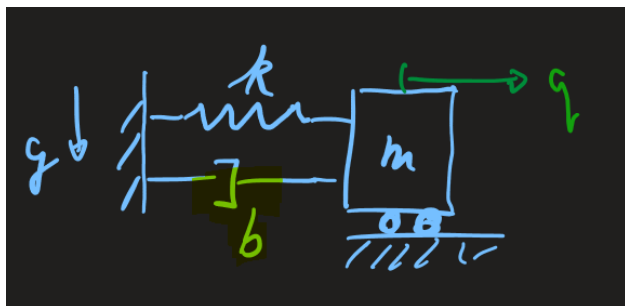
## Chapter 3

# Our Second Order of Business

### Preface

First order systems are honestly pretty boring. When we put in a step input, we just get a pure exponential. We won't be able to get more interesting behavior, like oscillation, because that's just mathematically impossible.<sup>1</sup>

Second order systems, on the other hand, *can* oscillate by themselves. Try to convince yourself of this mathematically just based on what oscillation is.



Mass-spring systems are pretty good models of everything in the world (as long as you use enough mass-spring systems). They're really nice because having a good understanding of ONE mass-spring system provides us with the intuition for more complicated systems.

Say we have a mass-spring system where a mass  $m$  is attached to a wall with a spring  $k$  and a damper  $b$ . Gravity isn't "turned on", so if you want to visualize the system, that mass is floating. The equation of motion for a positive displacement  $q$  is:<sup>2</sup>

$$m\ddot{q} = -kq - b\dot{q}$$

Or in its more familiar form:

$$m\ddot{q} + b\dot{q} + kq = 0$$

This is the famed mass-spring equation. Say we have an input - a force  $u$  acting on the mass in the positive direction. Now our equation of motion is:

$$m\ddot{q} + b\dot{q} + kq = u$$

---

<sup>1</sup>Prove it!

<sup>2</sup>Prof. Luchtenburg went off on a tangent about Hooke being a genius for realizing that spring motion is linear near the origin here. That was pretty funny.

This is a linear differential equation, so we'll solve this by plugging in an educated guess. Let's try  $q(t) = Ae^{st}$ , because differentiating this function  $n$  times just multiplies it by  $s^n$ .

$$ms^2 Ae^{st} + bsAe^{st} + kAe^{st} = u$$

$$ms^2 + bs + k = u$$

First, we'll solve the homogeneous equation, or the case where  $u = 0$ .<sup>3</sup>

$$ms^2 + bs + k = 0$$

This is known as the characteristic (or auxiliary) equation. We can now use algebra to solve for the roots of the equation, or by proxy, the solution of the differential equation.

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

These are also called the **poles** of the system, but we're getting ahead of ourselves. Let's simplify further.

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

We define the **natural frequency**  $\omega_n = \sqrt{\frac{k}{m}}$  and the **damping ratio**  $\zeta = \frac{b}{2m\omega_n}$ . Using these definitions, we can reorganize the mass-spring equation in terms of these variables.

$$m\ddot{q} + b\dot{q} + kq = 0 \quad \rightarrow \quad \ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = 0$$

And the poles of this equation are:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

I just realized this is my second time having this lecture today, so I'm just going to copy-paste my ME301 notes here.

## 3.1 Intermezzo - Complex Analysis

Complex analysis is the study of functions of a complex variable  $z$ , where  $z$  has a real component  $a$  and an imaginary component  $b$ . Complex numbers show up all the time in this course, whenever anything oscillates, really (like mass-spring systems and pendula).

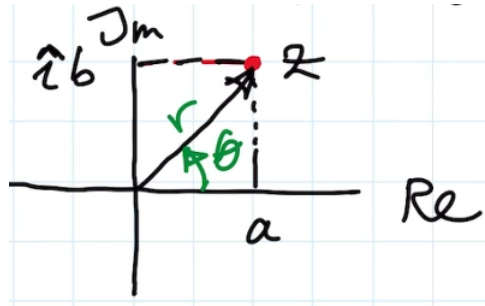
To take our first plunge into complex analysis, we need to define the **imaginary unit**  $i$ .<sup>4</sup> For now, let's define  $i$  as one of the two solutions to the quadratic equation  $x^2 = -1$ . (The other is  $-i$ , of course.) This is really special, because now we can describe the solutions of ALL polynomials<sup>5</sup> as the sum of a real number  $a$  and another real number  $b$  multiplied by the imaginary unit  $i$ .

Let's conjure up a graphical representation of these numbers using Cartesian coordinates, where we define one axis as "real" and the other as "imaginary". We'll call this the complex plane. An arbitrary point  $z = a + ib$  is plotted below.

<sup>3</sup>If you're curious why we do this, you can read up on it in a linear algebra textbook. Think it's theorem 3.9 in Friedberg's Linear Algebra.

<sup>4</sup>You'll see people, especially electrical engineers, use  $j$  instead, because  $i$  is commonly used for current. We're better than them.

<sup>5</sup>Regardless if its coefficients are real or complex!



We can also interpret these numbers in the context of polar coordinates, where  $\theta$  is the angle between the vector from the origin to  $z$  and the real axis, and  $r$  is the magnitude of the aforementioned vector. It's not difficult to translate between Cartesian coordinates and polar coordinates, but I'll dump the formulas here anyway.

$$a = r \cos \theta \quad b = r \sin \theta \quad r = \sqrt{a^2 + b^2} \quad \theta = \arctan\left(\frac{b}{a}\right)$$

It'd be criminal to not mention Euler's formula<sup>6</sup>, which posits that:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

A lot of the nuance of complex numbers is best understood by analyzing this **complex exponential**, especially because:

$$z = a + ib = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

## 3.2 Damping (You Guys are Going to See This Over and Over and Over)

When  $\zeta = 0$  (or the system is **undamped**), we have the poles  $s_{1,2} = \pm i\omega_n$ . This implies that our solution is a linear combination of sines and cosines, endlessly oscillating, forever and ever. (That's kind of depressing to be honest.)

"What next? Euler guy."

-Prof. Luchtenburg

$$\begin{aligned} x(t) &= A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t} \text{ } ^7 = A_1 (\cos(\omega_n t) + i \sin(\omega_n t)) + A_2 (\cos(\omega_n t) - i \sin(\omega_n t)) \\ &= (A_1 + A_2) \cos(\omega_n t) + i(A_1 - A_2) \sin(\omega_n t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) \end{aligned}$$

This reasoning carries over if we pick a damping ratio  $\zeta$  between 0 and 1 (or the system is **underdamped**). The poles are:

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{\omega_n^2(\zeta^2 - 1)} = -\zeta\omega_n \pm i\omega_n \sqrt{1 - \zeta^2} = \sigma \pm i\omega_d$$

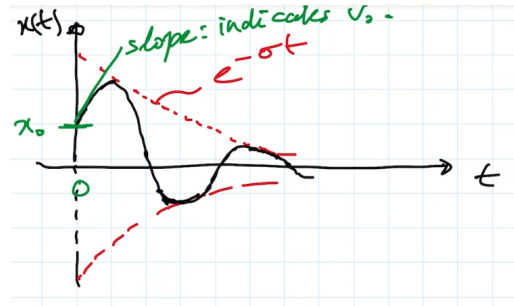
We define the **damped frequency**  $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ . (In practice,  $\omega_d \approx \omega_n$ , because  $\zeta \ll 1$ .) Additionally, we define  $\sigma = i\omega_n$  (for some reason). After substituting these new variables in, our solution becomes:

$$x(t) = A_1 e^{(-\sigma + i\omega_d)t} + A_2 e^{(-\sigma - i\omega_d)t} = e^{-\sigma t} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))$$

In the lattermost form, it's obvious that  $\sigma$  in fact *does* have a use other than bookkeeping; it defines the exponential **envelope** by which the oscillation decays. An envelope is a function that outlines how a function grows/decays (it's the red dashed line in the figure below).

<sup>6</sup>You can prove this using the Taylor series representation of  $e^x$ . You should do it, it's *very* rewarding

<sup>7</sup>Apparently  $A_1$  and  $A_2$  are complex conjugates. Don't quote me on that.



Notably, the time constant  $\tau$  of the envelope is equal to  $1/\sigma$ . Thus, we can eyeball the value of  $\sigma$  based on how we'd find the time constant (the value 63% less than the  $y$ -intercept of the envelope).

$$x(t) = e^{-\sigma t} \sin(\omega_d t + \varphi)$$

Most mechanical systems tend to have a very low damping ratio ( $\zeta \simeq O(0.1)$ <sup>8</sup>), and as mentioned before, a good rule of thumb is that  $\omega_n = \omega_d$ .

When we apply some initial conditions (like  $q(0) = q_0$  and  $\dot{q}(0) = v_0$ ), our solution becomes:

$$q(t) = e^{\sigma t} \left( q_0 \cos(\omega_d t) + \frac{\sigma q_0 + v_0}{\omega_d} \sin(\omega_d t) \right)$$

When  $\zeta = 1$ , or the system is **critically damped**, the poles are:

$$s_{1,2} = -\zeta\omega_n$$

We use a trick from differential equations to fake another linearly independent solution, just chuck on an extra  $t$ .

$$q(t) = A_1 e^{-\zeta\omega_n t} + A_2 t e^{-\zeta\omega_n t}$$

This doesn't really happen in the real world, but it's nice to cover all our bases. How about when  $\zeta > 1$ ? We call this case **overdamped**, because our poles are:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

Both poles are negative here! Our solution is:

$$q(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

### 3.3 Pole Plots

I've been emphasizing poles a lot in the last few pages, but why? What's the importance of these seemingly arbitrary values? In fact, we can infer the dynamics of a system based on its poles.<sup>9</sup>

To analyze poles, we use a graphical tool called a **pole plot**, the plot of the roots of the characteristic equation on the complex plane. Let's go down the list:

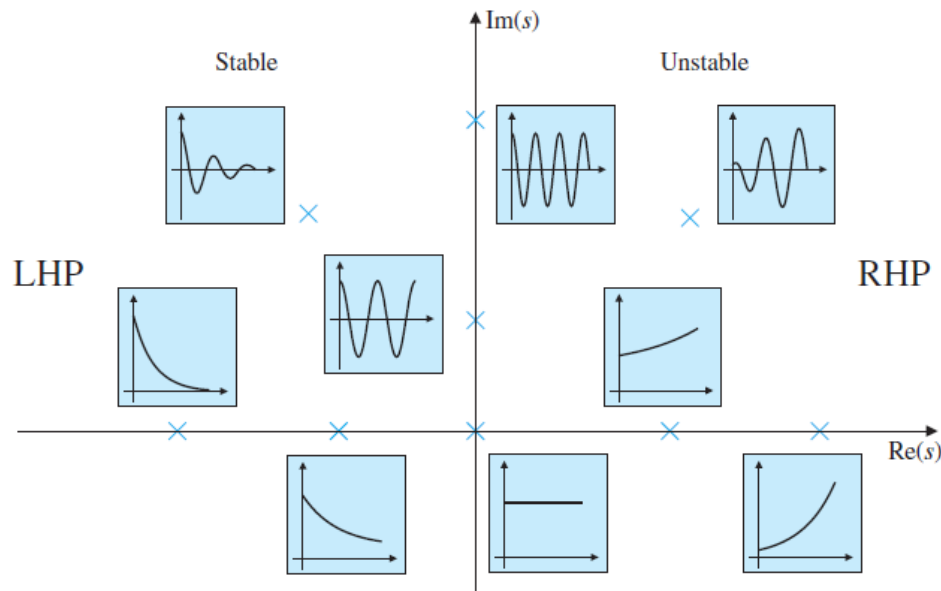
- When both poles are on the imaginary axis, the system is undamped.
- When both poles are off the real axis, the system oscillates. If they're to the left of the imaginary axis, it'll decay exponentially (underdamped), and if they're to the right of the imaginary axis, it'll grow exponentially.
- If both poles are on the real axis to the left of the imaginary axis, it's overdamped.

<sup>8</sup>Related rates of growth from Ma111, or big-O notation if you've taken ECE264.

<sup>9</sup>This is the crux of a lot we do in ME351. If any of you remember what an eigenvalue is from Ma110, that'll come in handy in a bit.

- Rule of thumb: if there is ANY pole to the right of the imaginary axis, the response blows up.

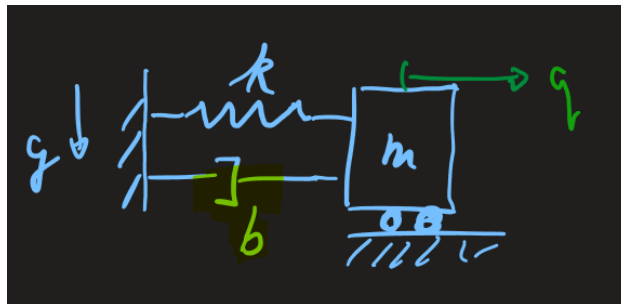
Here's a nice chart from FPE. Complex conjugates are omitted for simplicity.



### 3.4 Leveraging the Discriminant

I'm going to stray off from Prof. Luchtenburg for a second because *I* think this is useful.

Suppose we want to identify the dampedness of a system based on the equation of motion rather than solving for  $\zeta$ . We can do exactly this using the **discriminant** of the characteristic equation of the system. Let's take a look at the canonical mass-spring system with a damper once again.



The equation of motion, assuming free motion, is:

$$m\ddot{q} + b\dot{q} + kq = 0$$

The characteristic equation is derived after plugging in  $q = e^{st}$ .

$$ms^2 + bs + k = 0$$

As stated in the damping section, there are three forms of the general solution if there is damping present: both poles are real and distinct (the system is overdamped), both poles are real and equal (the system is critically damped), or both poles are complex conjugates (the system is underdamped). You may recall from Algebra that the discriminant of a polynomial can reveal some properties of the roots without actually computing them. The discriminant of a quadratic is defined as follows:

$$\text{Disc}(ax^2 + bx + c) = b^2 - 4ac$$

This is the argument of the square root in the quadratic formula. If this expression is positive, the solutions to the quadratic are real and distinct. If this expression is 0, then there is only one real solution to the quadratic. If this expression is negative, the solutions to the quadratic are complex. So physically, finding the discriminant of the characteristic equation of the mass-spring system will tell us how damped it is.

$$\text{Disc}(ms^2 + bs + k) = b^2 - 4mk$$

$$b^2 - 4mk > 0 \rightarrow \text{overdamped}$$

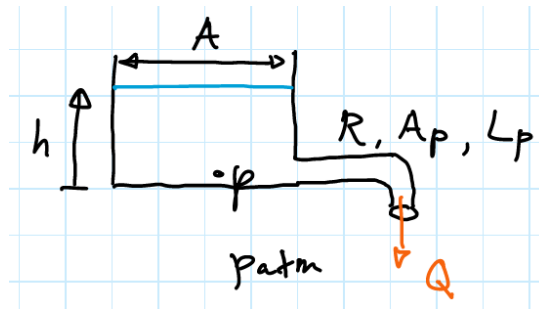
$$b^2 - 4mk = 0 \rightarrow \text{critically damped}$$

$$b^2 - 4mk < 0 \rightarrow \text{underdamped}$$

And of course, if  $b = 0$ , then the system is undamped.

### 3.5 The Tank, Revisited (Inertia)

Let's revisit the tank from our study of first order systems. However, we'll make one small change: the outflow pipe now has a defined length of  $L_p$ .



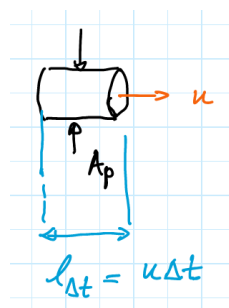
Let's model the same way we've been doing thus far. First, a conservation law:

$$\dot{V} = -Q$$

Next, "Ohm's law":

$$Q = \frac{\Delta P}{R} = \frac{P - P_{\text{atm}}}{R}$$

If we isolate a piece of the pipe (with length  $L_p$ ) as shown below, we can demystify this system a bit.



Assuming the cross-sectional area of the pipe  $A_p$  is constant, the pressure force  $F_p = A_p \Delta P$  accelerates the fluid between the two ends of this pipe. Additionally, there is friction  $F_f = -QRA_p$  on the liquid caused by the resistance of the pipe. By

leveraging Newton's second law of motion, we now have a relationship between the pressure difference  $\Delta P$  and the velocity of the water  $v$ .

$$m \frac{dv}{dt} = F_p + F_f = A_p \Delta P - Q R A_p$$

The mass  $m$  of the fluid between the two ends of this pipe is equal to the product of the density of the fluid  $\rho$  and the volume between the two ends  $V_p$ . (Notably, the volume  $V_p = A_p L_p$ .)

$$\rho V_p \frac{dv}{dt} = \rho A_p L_p \frac{dv}{dt} = A_p \Delta P - R Q A_p$$

Because the product of the cross-sectional area  $A_p$  and the fluid velocity  $v$  is equal to the volumetric flow rate  $Q$ , we can rewrite this equation as follows:

$$\frac{\rho L_p}{A_p} \frac{dQ}{dt} = \Delta P - R Q$$

Ok, we can shed some light on what we're doing now. We define **inductance** (also referred to as **liquid-flow inertance** or **inertia**) as a term that describes the change in potential required for a unit rate of fluid flow. Inductance is the tendency of the fluid to move; it's created by the inertia of water flowing through the pipe. The mathematical definition of inductance is as follows:

$$L = \frac{\rho L_p}{A_p}$$

Note that this definition of inductance is only valid for flow systems, but analogous concepts occur in other fields (like inductors from circuit analysis)! Fluid components that have an inductance are analogous to these inductors, or mechanical components with inertia.

Let's wrap up this example. When we plug in the definition of  $L$  into our equation, a simple first order system rears its head.<sup>10</sup>

$$L \frac{dQ}{dt} + R Q = \Delta P$$

Let's throw it into canonical form so we can see its time constant.

$$\left(\frac{L}{R}\right) \dot{Q} + Q = \frac{\Delta P}{R} \quad \tau = \frac{L}{R}$$

To summarize, we've added a new tool to our arsenal: conservation of momentum (or Newton's second law).

$$\Delta P = L \dot{Q} + R Q$$

$$\dot{V} = -Q$$

$$C \Delta P = V$$

By combining these three equations, we can use tools from our studies of mass-spring systems to analyze... well ... any second order system.

$$L \Delta \ddot{P} + R \Delta \dot{P} + \frac{1}{C} \Delta P = 0 \quad \rightarrow \quad \Delta \ddot{P} + \left(\frac{R}{L}\right) \Delta \dot{P} + \left(\frac{1}{LC}\right) \Delta P = 0$$

$$m \ddot{q} + b \dot{q} + k q = 0 \quad \rightarrow \quad \ddot{q} + \left(\frac{b}{m}\right) \dot{q} + \left(\frac{k}{m}\right) q = 0$$

We simply retrofit the definitions of the natural frequency  $\omega_n$  and damping ratio  $\zeta$  based on how we defined them for mass-spring systems to determine how the oscillations behave. Here's a quick example using the flow system analogy we've been using thus far:

<sup>10</sup>The analog of this system in circuit analysis is called the RL circuit, which is often used as a passive filter.

$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2q = 0 \quad \longleftrightarrow \quad \Delta\ddot{P} + \left(\frac{R}{L}\right)\Delta\dot{P} + \left(\frac{1}{LC}\right)\Delta P = 0$$

$$2\zeta\omega_n = \frac{R}{L} \quad \longrightarrow \quad \zeta = \frac{R}{2L\omega_n} = \frac{R\sqrt{LC}}{2L} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$\omega_n^2 = \frac{1}{LC} \quad \longrightarrow \quad \omega_n = \frac{1}{\sqrt{LC}} = \frac{\sqrt{LC}}{LC}$$



# Chapter 4

## Cool Math Games

### Preface

This chapter's going to be a smorgasbord of mathematical concepts I think are foundational to understanding this course from a theoretical perspective. A decent chunk of it will be review from Ma240, but it doesn't hurt to take a second look at these things (they keep coming back over and over).

What I'll try to do, instead of rehashing what you got out of Ma240, is reframe these concepts in a way that's more applicable to this course (and ME351). But first, let's formalize a few more definitions that will come in handy later on.

A **linear combination** of elements of a set  $x_1, x_2, \dots, x_n$ , is given by:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n$$

where each  $a_i$  is a constant. In layman's terms, a linear combination of variables is the sum of scaled versions of those variables, where the scaling factor is a scalar.

A set of variables are **linearly dependent** if one of the variables can be expressed as a linear combination of the others. More formally, a set of variables  $x_1, x_2, \dots, x_n$  is linearly dependent if there exist scalars  $a_1, a_2, \dots, a_n$  (not all zero), such that:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$$

On the other hand, if no such scalars exist, the set of variables is said to be **linearly independent**.

### 4.1 Linearity and Time-Invariance

A **linear time-invariant (LTI) system** is a mathematical model often used in control theory to describe the behavior of physical systems. It is characterized by two properties: linearity and time-invariance. I'll describe these separately in the context of differential equations. A differential equation of the form:

$$a_ny^{(n)}(t) + a_{n-1}y^{(n-1)} + \dots + a_1\dot{y}(t) + a_0y(t) = f(t)$$

where  $y^{(i)}$  is the  $i$ th derivative of  $y(t)$ , is called **linear**. (The relationship between  $y(t)$  and  $t$  is a linear mapping.) A system is linear if and only if it satisfies two properties: superposition and homogeneity:

- **Superposition** - if  $x_1 \rightarrow y_1$  and  $x_2 \rightarrow y_2$ , then  $x_1 + x_2 \rightarrow y_1 + y_2$
- **Homogeneity** - if  $k$  is a scalar and  $x \rightarrow y$ , then  $kx \rightarrow ky$

<sup>2</sup>If you don't remember, it's by plugging in a guess solution  $e^{st}$ .

## Chapter 5

# The State-Space Approach

### Preface

When we begin to analyze more complicated systems, it becomes less and less feasible to solve them analytically. As a result, we resort to setting up a system of differential equations rather than concatenating them into one as we've done in past chapters.

The state-space representation of a system describes the system's behavior over time in terms of a set of variables called **states**.<sup>1</sup> The state variables represent the current conditions of the system, and their evolution over time is described by a set of first order differential equations called **state equations**.

The state-space representation is a very powerful tool for modeling and analyzing physical systems, providing valuable insights into their behavior and enabling the development of control algorithms covered in ME351.

An important thing to note is that the state-space representation of a system is not unique. In fact, an infinite number of representations exist for a physical system.

### 5.1 General Formulation

In the most general case, a state-state representation can be represented as the following:

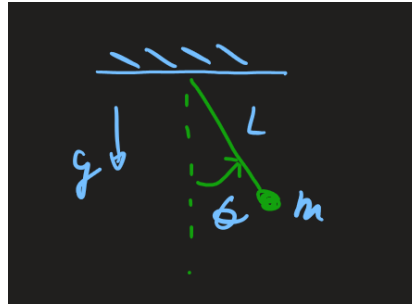
$$\begin{cases} \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \\ \underline{y} = \underline{h}(\underline{x}, \underline{u}) \end{cases}$$

This might be a bit daunting at first, but it's just a lot of fancy notation for a concept that's pretty simple.  $\underline{x}$  is the state,  $\underline{u}$  is an input, and  $\underline{y}$  is an output. Let's drive this concept home with an example.

Say we have a simple pendulum with length  $L$  and a point mass  $m$  at its end, as pictured below:

---

<sup>1</sup>The state space can be described as a Euclidean space where each state corresponds with an axis.



For this problem, the mass of the rod (and any potential friction in the hinge) is ignored. The equation of motion of the pendulum can be derived by summing moments about the point of contact between the pendulum and the fixed surface.<sup>2</sup> Let's call that point of contact  $O$  for future bookkeeping purposes.

$$\sum M_O = J_O \ddot{\theta}$$

The moment arm for the weight  $mg$  is the horizontal displacement  $L \sin(\theta)$ , and  $J_O = mL^2$  is the mass moment of inertia of the point mass  $m$  about point  $O$ . Let's crunch some numbers.

$$-mgL \sin(\theta) = mL^2 \ddot{\theta}$$

$$mL^2 \ddot{\theta} + mgL \sin(\theta) = 0$$

$$\ddot{\theta} + \frac{g}{L} \sin(\theta) = 0$$

Now let's try putting this in state-space form. First, we select the state vector, which should adhere to the following points:

- Pick state variables that include all the relevant information about the system you're trying to model.
- The number of dimensions in the state vector should match the number of degrees of freedom of the system.
- The state vector should be **minimal**, meaning it should contain only the information necessary to describe the system, and not any redundant information. (Usually, the minimum number required is equal to the order of the differential equation that represents the system.)
- The components of the state vector must be linearly independent.

We define states  $\theta$  (angle) and  $\omega = \dot{\theta}$  (angular velocity), and start constructing our state equations.

$$\underline{x} = \begin{bmatrix} \theta \\ \omega \end{bmatrix} \quad \dot{\underline{x}} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix}$$

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$$

$$\dot{\underline{x}} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{g}{L} \sin(\theta) \end{bmatrix}$$

We've turned a second order differential equation into two first order differential equations. Let's move onto the second part of the representation: defining the output  $y$ . We're interested in the states' behavior over time, so our output is...just the state vector  $\underline{x}$ .

$$\underline{y} = \underline{h}(\underline{x}, \underline{u}) = \underline{x} = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

These two components make up the state-space representation of this pendulum system. Putting it in this form makes it easier to numerically solve using tools like Python or MATLAB.

<sup>2</sup>Alternatively, you could sum forces in the parallel and perpendicular directions of motion to yield an equivalent result.

A nonlinear solution can be unappealing, though perfectly valid. By using the small-angle approximation  $\sin(\theta) \sim \theta$ , we can refine this representation further.

$$\ddot{\theta} + \frac{g}{L} \sin(\theta) \sim \ddot{\theta} + \frac{g}{L} \theta = 0$$

Our system is now an LTI system. Linearity is very nice, because we can use matrix multiplication to make this representation pretty.

$$\begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{g}{L} \theta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \\ \underline{y} &= \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \end{aligned}$$

Let's say the pendulum had an input applied torque  $T$ , and the new equation of motion was:

$$\ddot{\theta} + \frac{g}{L} \theta = \frac{T}{mL^2}$$

Our revised state-space representation would be:

$$\begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{g}{L} \theta + \frac{T}{mL^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} T = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} u \\ \underline{y} &= \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \end{aligned}$$

Using this example, we now extract the general form of the state-space representation of a linear system. A linear system is represented in state space by the following equations:

$$\begin{aligned} \dot{\underline{x}} &= \mathbf{A}\underline{x} + \mathbf{B}\underline{u} \\ \underline{y} &= \mathbf{C}\underline{x} + \mathbf{D}\underline{u} \end{aligned}$$

for  $t \geq t_0$ ,  $\underline{x}(t_0)$ , where:

$\underline{x}$  is the state vector, of size  $n \times 1$

$\underline{y}$  is the output vector, of size  $q \times 1$

$\underline{u}$  is the input vector, of size  $p \times 1$

$\mathbf{A}$  is the system matrix, of size  $n \times n$

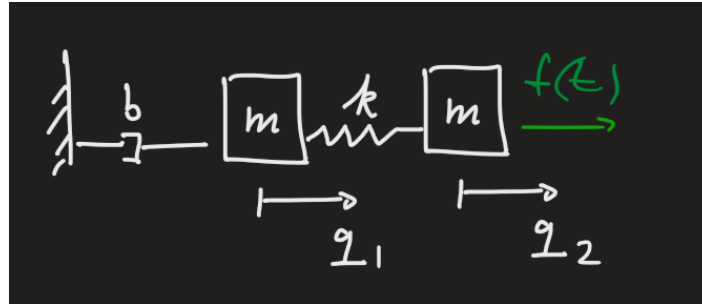
$\mathbf{B}$  is the input matrix, of size  $n \times p$

$\mathbf{C}$  is the output matrix, of size  $q \times n$

$\mathbf{D}$  is the feedforward matrix, of size  $q \times p$

## 5.2 Another Mass? Seriously?

Let's try another example, this time a translational mechanical system. Block 1 of mass  $m$  is attached to a fixed wall by dashpot with damping coefficient  $b$ . Block 2, also of mass  $m$ , is attached to block 1 by a spring of spring constant  $k$ . Gravity is turned off.



First, we write the equations of motion of the network. (Just draw a free body diagram around each mass and don't fuck up your signs.)

$$m\ddot{q}_1 + b\dot{q}_1 + kq_1 - kq_2 = 0$$

$$-kx_1 + m\ddot{q}_2 + kq_2 = f(t)$$

This is a system of two second order differential equations, so we'll pick four states. We select our  $q_1$ ,  $\dot{q}_1$ ,  $q_2$ , and  $\dot{q}_2$  to be our state variables.

Our first two state equations are easy: just define the derivatives. We get the other two by rearranging the equations of motion and isolating  $\ddot{q}_1$  and  $\ddot{q}_2$ .

$$\dot{\underline{x}} = \begin{bmatrix} \dot{q}_1 \\ \ddot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{b}{m} & \frac{k}{m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m} & 0 & -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} f(t)$$

Now, we didn't specify what output we wanted, but let's say we want to analyze the acceleration of the second mass, or  $\ddot{q}_2$ . We'd do the following:

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} f(t)$$

which is equivalent to the expression  $y = \ddot{q}_2$ . Even though it seems more complicated to put it in this form, it provides us with a **C** and **D** matrix, which is invaluable when analyzing systems computationally.