System Dynamics

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General Information

This is a living document created from Markdown and executable code. Feel free to reach out if you find typos or have suggestions. It's a passion project though, so my response time may vary.

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Course Description: An introductory course to the mathematical modeling of systems. Topics include mechanical elements and systems, electric circuits and analogous systems, fluid elements and systems, analysis of systems using transfer functions, state space equations, analog simulation and digital simulation. Also covered are block diagrams, Laplace transforms, and linear system analysis. Computer projects will be assigned that will use MATLAB/Python.

Course Instructor: Dirk M. Luchtenburg (dirk.luchtenburg@cooper.edu)

Patch Notes/Detritus

To-Do:

- Replace all figures with acceptable alternatives
- § 4.4
- Flesh out Chapter 6
- Finish Chapter 7

Log:

- 06.19.23 - Chapter 3 looks good
- 06.06.23 - Chapter 2 looks good
- 05.22.23 - Transplanted everything from the original LaTeX file here except for pictures. Some additional formatting changes needed, and a LOT of editing, but things look pretty good…
- 05.18.23 - Made the jump from a LaTeX document on Github to Quarto at Prof. Luchtenburg's recommendation. It's pretty cool…can embed inline code and whatnot!

1 Course Motivation and Overview

1.1 First Ramblings

The course is called Systems Engineering in the course catalog, but Prof. Luchtenburg calls it System Dynamics instead. It's a more apt name for what the course actually covers; if you look up Systems Engineering you'll find a completely different subject.

The course text is Ogata's *System Dynamics*, which doesn't fit particularly well for the course. Most feedback control textbooks, like Nise, start with a few chapters covering dynamic modeling and response, but they tend not to be very in depth. I can provide PDFs if you can't find them yourselves. Read the syllabus, it's pretty in depth.

Prof. Luchtenburg *should* put his notes up in the MS Teams Class Notebook and record his lectures (if he hasn't given up using Microsoft Teams yet). Your mileage may vary if you don't bother coming to class, though; the audio quality in the recordings isn't that great and the notes can be less detailed than the lectures themselves.

This document is a living document, which means I'll continue editing it as I see fit. I reserve the right to include material that may not be covered in the course, but better prepares you for concepts that may seem like they come out-of-the-blue otherwise.

I have this lousy habit of writing in the fourth person and cursing in my writing. Apologies in advance.

1.2 Why Model?

To start off, let's define a **model**. A mathematical model of a system is a set of differential equations that allows us to predict how a system behaves under different conditions. By creating a model, we can use equations and principles to describe a system's present behavior, identify key parameters that affect the system, and make educated guesses as to how the system will behave in the future.

You may have covered some rudimentary modeling in Ma111 or Ma240, where it was probably crammed in to satisfy those annoying kids that go "wHEN ARe we ever gOINg to usE tHIS iN ReAl LIfE". The methods we'll cover in this course will be more robust than simply going off a given formula.

Something we'll exploit heavily a lot in this course is the principle of **analogical models**, or generic representations of common physical phenomena. This turns out to be really useful because it provides a convenient and consistent way to represent and analyze complex systems that involve different types of "stuff". Say you're faced with a fluid flow system, and you haven't taken a fluid mechanics course yet. Using methods introduced in this course, we will be able to convert this system into more familiar, yet equivalent, like a mass-spring system or a series circuit. Using these analogies allows us to apply the same concepts and mathematical tools to different types of systems, which can greatly simplify the analysis and design process.

To conclude, modeling is a powerful tool that, if mastered, will allow you to understand, predict, and (as we'll see in ME351 next semester) **control** the behavior of physical systems, and it has many practical applications in engineering, physics, and other fields.

Don't suck at it.

2 The First Order

Preface

We'll grow more accustomed to the idea of analogical models after modeling a few simple systems. Let's start by throwing out some fundamental systems and developing intuition to dive into simplified models.

- Emptying a water tank
- Cooling of a lightbulb
- Discharge of an RC circuit

2.1 Gradients Make Stuff Flow

A cylindrical tank has a cross-sectional area of A, and an outflow rate of Q_{out}. The height of the fluid is defined by the function $h(t)$. The pressure outside the tank is P_∞, and the fluid has a density of ρ. Can we derive a governing equation for this system in terms of $\Delta P = P - P_\infty$, where $P(t)$ is the pressure at the bottom of the tank?

First off, because the tank is cylindrical, it's apparent that the volume $V(t)$ of water in the tank is proportional to the value of $h(t)$.

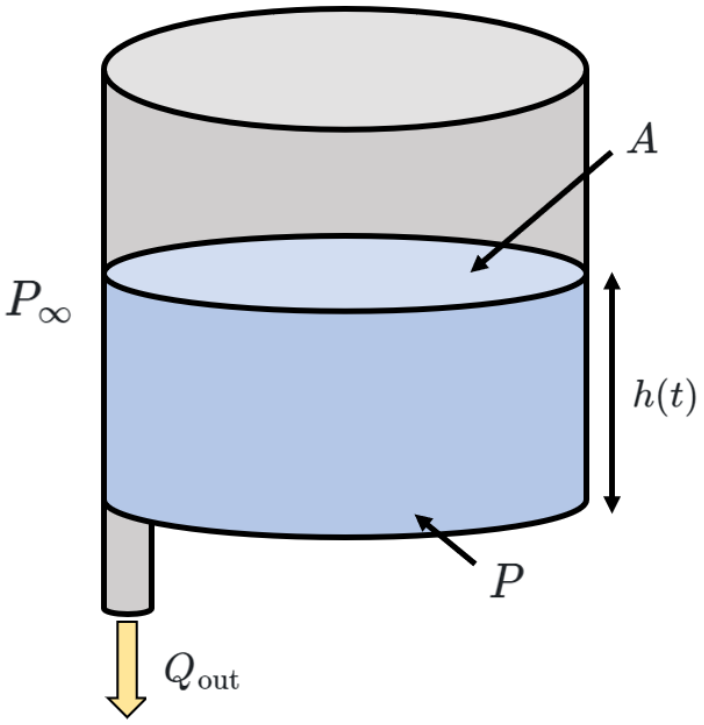
$$V(t) = Ah(t)$$

Because of conservation of mass, we know the rate at which the volume is changing is the difference between the inflow and the outflow.

$$\frac{d}{dt}V(t) = \dot{V} = 0 - Q_{\text{out}} = -Q_{\text{out}} \tag{1}$$

That's not very useful yet. Let's leverage some prior circuits knowledge here…charge moves because of a voltage difference ΔV, and comparably, fluid moves because of a pressure difference.[1] Ohm's law! We'll come back to that, but the main takeaway here is that the volumetric

[1] V refers to voltage here, and not volume as previously defined. This contingency also holds true for Ohm's law as defined below.


A
P_∞
$h(t)$
P
Q_{out}

outflow Q_{out} is related to the difference between the pressure inside the tank $P(t)$ and the atmospheric pressure P_∞ by a "resistance" value R.

That circuits analogy comes in handy really often, because it turns out Ohm's law translates directly into fluid flow.

$$\begin{aligned} \Delta V = V - V_0 &= IR \\ \Delta P = P(t) - P_\infty &= Q_{\text{out}} R \end{aligned} \tag{2}$$

These are called **constitutive equations**, or relationships between physical quantities that establish a connection between a material's internal response (like stress, strain, or deformation) and the external factors that influence it (like temperature, pressure, or applied loads).

We'll generalize a bit soon, but for now I understand if you don't get it. It's still very hand-wavey.

Let's leverage some hydrostatics now. Assuming the tank is open to the atmosphere at the top, we can derive an expression for ΔP using the hydrostatic pressure equation $\Delta P = \rho g \Delta h$.

$$\Delta P = P(t) - P_\infty = \rho g \left(h(t)\right) = \frac{\rho g V}{A} \tag{3}$$

Differentiating equation (3) yields the following.[2]

$$\Delta \dot{P} = \frac{\rho g \dot{V}}{A}$$

Using equations (1) and (2), we can manipulate this equation into a differential equation in terms of pressure differences.

$$\Delta \dot{P} = \frac{\rho g \left(-Q_{\text{out}}\right)}{A} = \frac{-\rho g \Delta P}{AR}$$

$$\Delta \dot{P} + \frac{\rho g}{AR} \Delta P = 0$$

$$R \left(\frac{A}{\rho g}\right) \Delta \dot{P} + \Delta P = 0$$

This is a neat little differential equation. It looks like the equation for an RC circuit if you've seen those before, with the voltage differences swapped out for pressure differences. To really

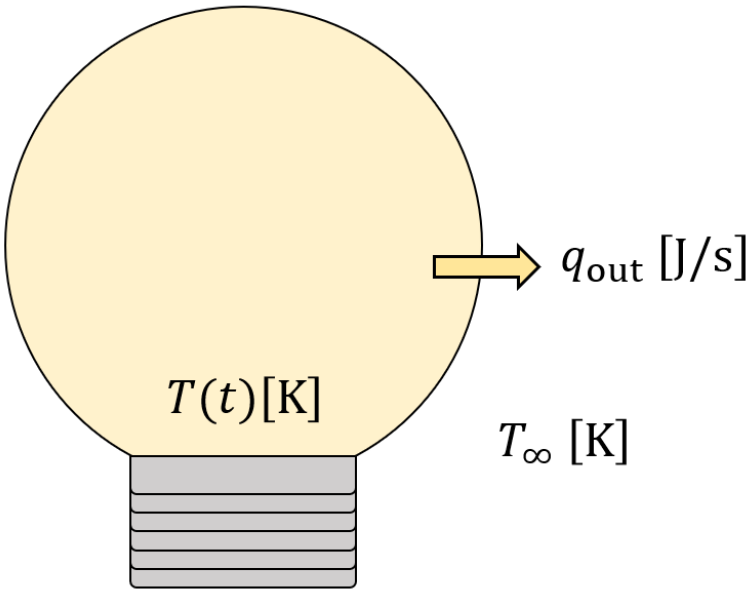
[2]Note that the derivative of ΔP is equal to the derivative of $P(t)$, because P_∞ is constant.

drive this comparison home, we define a "capacitance" $C = \frac{A}{\rho g}$ and slot it into our governing equation.

$$\boxed{RC\Delta\dot{P} + \Delta P = 0}$$

Perfect. Note that this isn't the only possible governing equation of the system - it's also possible to find a differential equation in terms of volume $V(t)$. Give it a go!

Let's move onto a second example: the cooling of a lightbulb. When we shut off power to the lightbulb, how can we measure its temperature as it cools to room temperature?



The bulb is initially very hot (with temperature T) compared to its environment (which has temperature T_∞). Heat is flowing outwards at q_{out}. This is seeming very familiar…a temperature difference is driving heat to leave through the resistance R of the bulb.

Let's go through the steps again. Energy's being conserved here, so we use the capacitive relationship relating accumulated heat Q and temperature T from ESC330:

$$Q = C\Delta T$$

Differentiate across the board…

$$\dot{Q} = -q_{\text{out}} = C\Delta\dot{T}$$

And now we're just chugging through the motions. Next is another constitutive relationship (which looks shudderingly close to Ohm's law!):

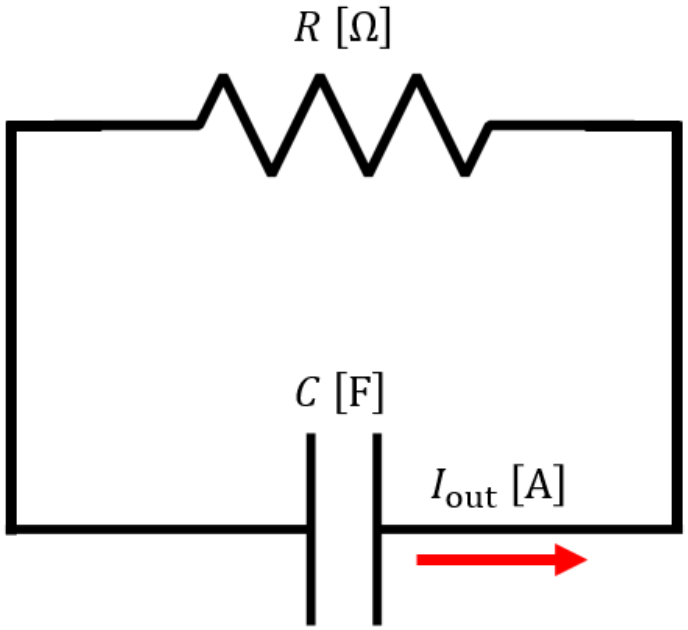
$$\Delta T = T - T_\infty = q_{\text{out}} R$$

We combine the last two equations and construct:

$$\boxed{RC\Delta\dot{T} + \Delta T = 0}$$

Which surprisingly, just looks like the equation from the first problem.

Ok, one more problem. Say we have an RC circuit - specifically, the variant where a resistor and a fully charged capacitor are connected in series without a attached voltage source.



As we've done twice before, we leverage physical principles to generate a governing equation for this system. Let's start with conservation of charge:

$$\dot{q} = -I_{\text{out}}$$

Next, Ohm's law:

$$\Delta V = I_{\text{out}} R = -\dot{q} R$$

$$\dot{q} = -\frac{\Delta V}{R}$$

Finally, we deal in the capacitive relationship:

$$q = C\Delta V$$

Manipulating these equations as we did before yields the following governing equation:

$$C\Delta\dot{V} = -\frac{\Delta V}{R}$$

$$\boxed{RC\Delta\dot{V} + \Delta V = 0}$$

All right, so all of these equations look really similar. What can we take away from this?

Gradients make stuff flow.

That's a really powerful statement…to those who understand it. Let's go through it piece by piece.

We define a **gradient** as a change in the magnitude of a potential observed at two different points. Some examples of potentials are:

- Temperature
- Voltage
- Displacement
- Species concentration

When a potential difference exists, "stuff" has a tendency to move. For example, when a temperature difference exists across a structure, heat will flow through it.

"Stuff" isn't the greatest word for something like this, (maybe "quantity" instead?) but to the best of my knowledge, a better word doesn't exist. We define "stuff" as something that can be stored, like charge in a capacitor, or fluid in a tank, or heat in a reservoir.

By generalizing these quantities, we can create widely applicable rules for modeling first order systems without having specific knowledge about the underlying physics.

$$\text{Stuff} = \text{Capacitance} \times \text{Gradient}$$

$$\text{Gradient} = \text{Flow of Stuff} \times \text{Resistance}$$

Also conservation. That's a big one.

$$\text{Rate of Change of Stuff} = \text{Inflow} - \text{Outflow}$$

2.2 The Time Constant

All of the governing equations we've derived thus far have been of the form $\tau\dot{y} + y = 0$. This form, also called the **canonical form**, is useful for understanding the behavior of the dynamic system.

To solve this differential equation, we'll guess a solution $y(t) = ce^{\alpha t}$, find its time derivative $\dot{y}(t) = \alpha ce^{\alpha t} = \alpha y$, and plug in.

$$\tau\dot{y} + y = 0$$
$$\tau\alpha e^{\alpha t} + e^{\alpha t} = 0$$
$$(\tau\alpha + 1)\, e^{\alpha t} = 0$$
$$\alpha = -\frac{1}{\tau}$$
$$y(t) = ce^{-\frac{t}{\tau}}$$

Let's toss in an initial condition $y(0) = y_0$ to get rid of that unknown constant.

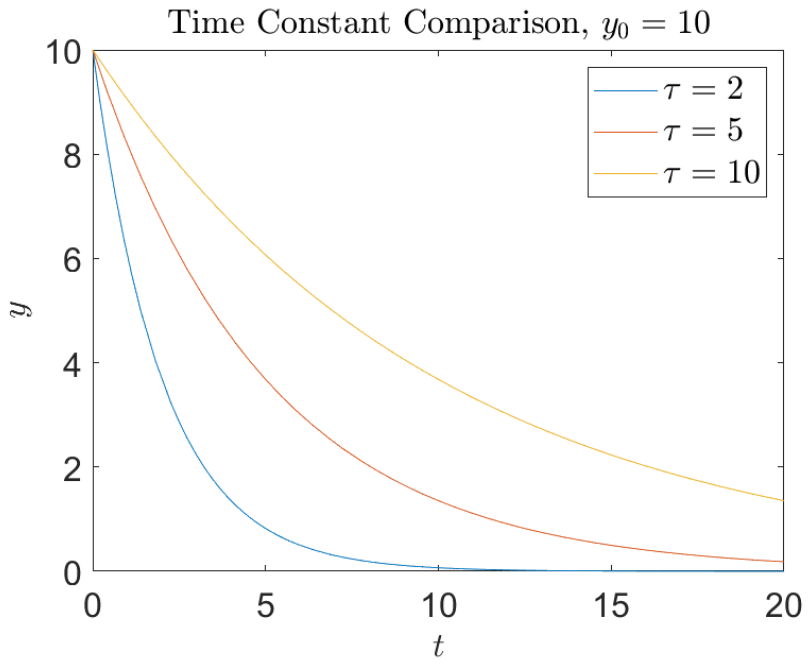
$$y(t) = y_0\, e^{-\frac{t}{\tau}}$$

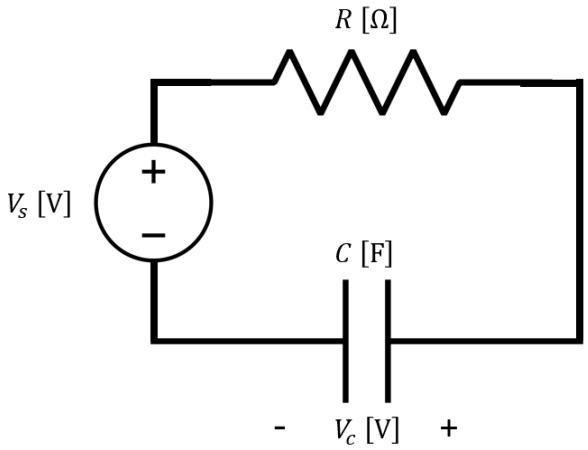
We call τ the **time constant** of the system.[3] The time constant of a system is used to describe how quickly the system response grows or decays. The smaller the time constant, the faster the decay.

So what happens when we set $t = \tau$?

$$y(\tau) = y_0 e^{-\frac{\tau}{\tau}} = y_0 e^{-1}$$

The time constant is the time at which the system response has decayed to $y_0 e^{-1}$, or approximately 37% of its initial value.[4]


Time Constant Comparison, $y_0 = 10$
$\tau = 2$
$\tau = 5$
$\tau = 10$
y
t
0
2
4
6
8
10
0
5
10
15
20


R [Ω]
V_s [V]
+
−
C [F]
-
V_c [V]
+

2.3 Let's Throw in an Input

We return to the RC circuit, but this time, let's say there *is* a voltage source with voltage V_s. The capacitor initially has a voltage of V_0, and as time passes, it will charge up through the resistor until it reaches the supply voltage of the source. (We define the voltage across the capacitor as $V_c(t)$.)

To expedite calculations, we'll use Kirchhoff's current law to find a governing equation for this scenario.

$$C\dot{V_c} + \frac{V_c - V_s}{R} = 0$$
$$RC\dot{V_c} + V_c = V_s$$

That's…different. This equation is no longer homogeneous, meaning we can't rely on our exponential function guess method anymore. We'll have to use more specialized methods from Ma240 instead of guessing and praying, like the method of undetermined coefficients or the Laplace transform if initial conditions are provided.

A more general form of this governing equation for one of these systems is:

$$\tau\dot{y} + y = u$$

where u represents a general input, constant or otherwise. For example, in the case that the voltage source is an alternator, V_s could be a sinusoidal function rather than a constant.

This feels like a good time to introduce the concept of the **unit step function**, which I'll denote $u_s(t)$. The unit step function is defined as follows:[5]

$$u_s(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

The unit step function is useful for physical modeling because it can be used to model situations where something changes rapidly from one value to another. (For example, the unit step function can be used to model the action of turning on a light switch at time $t = 0$, where 0 represents the light being off and 1 represents the light being on!).

A general form of the governing equation when the input is constant is:

[3] $\tau = RC$ always has units of time.

[4] Or alternatively, "the time constant is the time at which the system response has lost approximately 63% of its initial value".

[5] We don't care about what happens at $t = 0$. Stop it.

$$\tau\dot{y} + y = ku_s$$

where k is an arbitrary scale factor. Solving this equation yields:

$$y(t) = ce^{-\frac{t}{\tau}} + k$$

For the initial condition $y(0) = y_0$, the constant $c = y_0 - k$. Here's our updated solution:

$$y(t) = y_0 e^{-\frac{t}{\tau}} + k(1 - e^{-\frac{t}{\tau}})$$

When we graph this function for $y_0 = 0$, we see that it gradually grows towards $y = k$ as $t \to \infty$. Now we can analyze exponential growth - you see this behavior everywhere, like when you change a thermostat setting and the temperature slowly creeps towards your choice. The response of a system in the time domain when the input is switched from 0 to 1 very quickly is called a **step response**.

How could we find the time constant of this response? Let's take a look at what happens to $y(t)$ at $t = \tau$.

$$y(\tau) = y_0 e^{-\frac{\tau}{\tau}} + k(1 - e^{-\frac{\tau}{\tau}}) = y_0 e^{-1} + k(1 - e^{-1})$$

When we set $y_0 = 0$, this further simplifies to:

$$y(\tau) = k(1 - e^{-1}) \approx 0.63k$$

For this system, at $t = \tau$, the system response will have accumulated 63% of its steady state value.

The step input is just one of the test inputs we usually use; we'll look at a few more as the course progresses (such as sinusoidal waves, delta functions, etc.).

3 Our Second Order of Business

Preface

First order systems aren't that interesting. They exhibit a certain...simplicity in their response to a step input, characterized by pure exponential behavior. (However, it's important to appreciate the fundamental nature of this behavior, as it lays the foundation for understanding more complex dynamic systems.)

Nevertheless, first order systems lack the necessary components to sustain oscillatory behavior...which really stinks because oscillation represents a fundamental behavior observed in many natural and engineered systems.

Second order systems, on the other hand, *can* oscillate by themselves. [1]

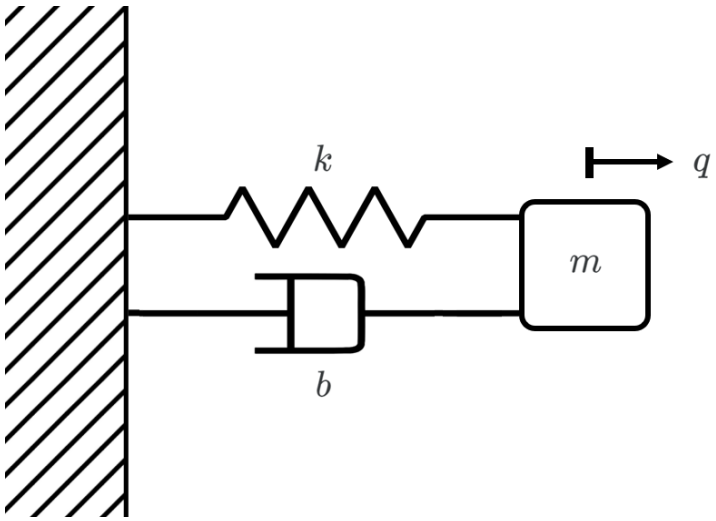
Mass-spring systems are pretty versatile models for a really wide range of phenomena (as long as you use enough mass-spring systems). They're really nice because having a good understanding of ONE mass-spring system provides us with the intuition for more complicated systems. While mass-spring systems may not capture *all* the intricacies and complexities of real-world phenomena, they serve as an invaluable tool for providing a framework for understanding a LOT of systems.

3.1 Mass and Spring

Say we have a mass-spring system where a mass m is attached to a wall with a spring with spring constant k and a dashpot with damping coefficient b, as pictured above. (Gravity isn't "turned on" in this scenario.) The equation of motion for a positive displacement q is:

$$ma = \sum F$$
$$m\ddot{q} = -kq - b\dot{q}$$
$$m\ddot{q} + b\dot{q} + kq = 0$$

[1] Try to convince yourself of this mathematically just based on what oscillation is.



This is a linear differential equation, so we're able to solve this pretty easily. We'll assume a solution of the form $q(t) = Ae^{st}$, where A is a constant to be determined.

$$m\left(s^2Ae^{st}\right) + b\left(sAe^{st}\right) + k\left(Ae^{st}\right) = 0$$

Cancelling out the common term Ae^{st}, we get:

$$ms^2 + bs + k = 0$$

This is known as the **characteristic (or auxiliary) equation**. We can solve this quadratic equation to find the values of s, which are also known as the **poles** of the system.

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

We can simplify this expression further by defining the **natural frequency** $\omega_n = \sqrt{\frac{k}{m}}$ and the **damping ratio** $\zeta = \frac{b}{2m\omega_n}$. With these definitions, the poles can be written as:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

In terms of the variables ζ and ω_n, the mass-spring equation can be rewritten as:

$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = 0$$

The damping ratio ζ and the natural frequency ω_n provide important insights into the behavior of the system. However, note that the value of ζ is crucial in determining the nature of the poles of the system; it directly affects whether they're real or imaginary.

Before we deal with that, though, I'll introduce some technology from complex analysis to make our lives a bit easier.

Complex analysis is the study of functions of a complex variable z, where z has a real component a and an imaginary component b. Complex numbers show up all the time in this course, whenever anything oscillates, really (like mass-spring systems or pendulums).

To take our first plunge into complex analysis, we need to define the **imaginary unit** i.[2] i is a number that satisfies the equation $i^2 = -1$. Since the square of any real number is always non-negative, there is no real number that safisties this equation. Thus, i is considered an imaginary number. (Ooh!)

Recall from algebra that every non-constant polynomial over $\mathbb{R}$ can be factored into linear and quadratic terms. Any real quadratic can be factored into linear terms over $\mathbb{C}$, the set of all complex numbers. As such, $\mathbb{C}$ has the roots of all polynomials over $\mathbb{R}$.

This is really special - in other words, every polynomial of degree n has n complex roots, counting multiplicity. This is known as the Fundamental Theorem of Algebra - it's a nice thing to stash in the back of your head, and it'll come back in a bit.

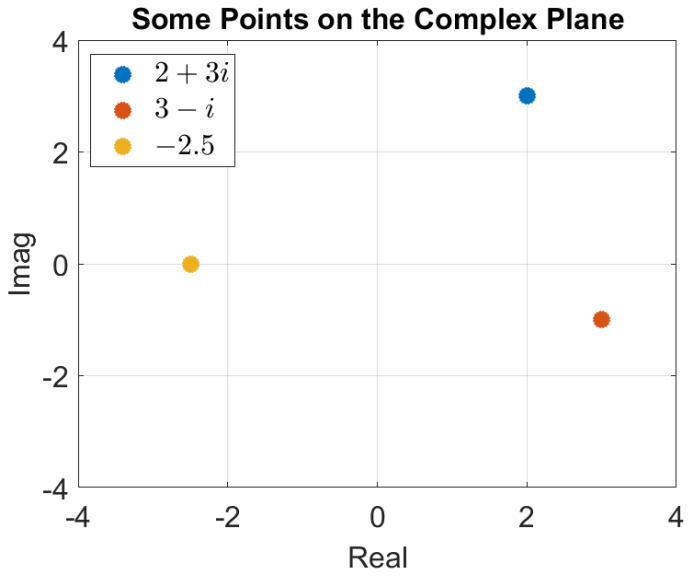
Let's conjure up a graphical representation of these complex numbers using Cartesian coordinates, where the real numbers are represented along the x-axis and the imaginary numbers along the y-axis. We'll call this the complex plane.

In this representation, a complex number $z = a + bi$ can be visualized as a point in the two-dimensional Cartesian plane, with the real part a as the x-coordinate and the imaginary part b as the y-coordinate.

By plotting these complex numbers on the plane, we can observe their geometric relationships. Additionally, algebraic operations can be understood in terms of vector addition, subtraction, and scalar multiplication.

Complex numbers can also be interpreted in the context of polar coordinates - in polar coordinates, a complex number $z = a + bi$ is described by two quantities: the magnitude r and the argument θ. The magnitude r represents the distance from the origin $(0, 0)$ to the point representing the complex number z in the complex plane. It's calculated using the Pythagorean theorem:

[2] Prof. Baglione likes to comment that mathematicians use i and engineers use j, which is pretty funny. Usually, j is used by electrical engineers for disambiguation with current, which is often represented with an i as well.



$$r = \sqrt{a^2 + b^2}$$

The argument, denoted as θ, represents the angle between the vector from the origin to z and the positive real axis, measured counterclockwise. The argument can be determined as follows (with additional precautions depending on the quadrant):

$$\theta = \arctan\left(\frac{b}{a}\right)$$

Converting back to Cartesian coordinates is trivial.

$$a = r\cos\theta \qquad\qquad b = r\sin\theta$$

It'd be criminal to not mention Euler's formula, which connects trigonometric functions to the complex exponential: [3]

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This result demonstrates the connection between exponentials and trigonometric functions, bringing together complex numbers, angles, and the unit circle. It also means that we can express complex numbers in the elegant form:

[3] You can prove this by substituting $x = i\theta$ into the Taylor series expansion of e^x. Do it, it's *very* rewarding.

$$z = a + bi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

This is *really* useful. It simplifies complex number arithmetic, enables efficient calculations of powers and roots, and provides a natural framework for solving differential equations involving complex variables.

Phew. Okay, back to springs.

We know from our study of differential equations that $q = e^{st}$ is a solution of the equation when s is a pole of the system. Let's break each possibility down case by case.

When $\zeta = 0$ (or when the system is **undamped**), the poles are:

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{\omega_n^2(\zeta^2 - 1)} = \pm\sqrt{-\omega_n^2} = \pm i\omega_n$$

And the solution is:

$$\begin{aligned} q(t) &= A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t} = A_1(\cos(\omega_n t) + i\sin(\omega_n t)) + A_2(\cos(\omega_n t) - i\sin(\omega_n t)) \\ &= (A_1 + A_2)\cos(\omega_n t) + i(A_1 - A_2)\sin(\omega_n t) = C_1\cos(\omega_n t) + C_2\sin(\omega_n t) \end{aligned}$$

Physically, when a mass-spring damper system is undamped, the mass will oscillate forever.

When we pick a damping ratio ζ between 0 and 1 (or the system is **underdamped**), the poles are:

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{\omega_n^2(\zeta^2 - 1)} = -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2}$$

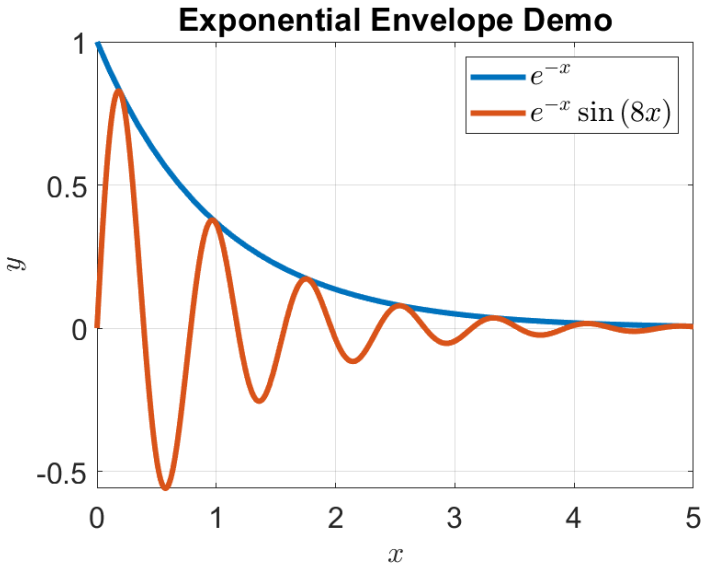
We further consolidate the definition of these poles using two additional definitions: we define the **damped frequency** $\omega_d = \omega_n\sqrt{1 - \zeta^2}$, and a "growth parameter" $\sigma = \zeta\omega_n$.[4] These result in a more concise representation of the poles:

$$s_{1,2} = -\sigma \pm i\omega_d$$

Our solution, given these poles, is:

$$q(t) = A_1 e^{(-\sigma + i\omega_d)t} + A_2 e^{(-\sigma - i\omega_d)t} = e^{-\sigma t}(C_1\cos(\omega_n t) + C_2\sin(\omega_n t))$$

Representing the solution as a product of a real exponential and a linear combination of sines and cosines allows us to characterize the exponential **envelope** by which the oscillation decays.



An envelope is a function that outlines the extremes of a function. The figure below shows a sinusoidal wave and its upper envelope.

Notably, the time constant τ of the envelope is equal to $1/\sigma$. Thus, we can eyeball the value of σ based on how we'd find the time constant (the value 63% less than the y-intercept of the envelope).

$$q(t) = e^{-\sigma t}\sin(\omega_d t + \varphi)$$

Most mechanical systems tend to have a pretty low damping ratio ($\zeta \simeq O(0.1)$),[5] and a good rule of thumb is that $\omega_n = \omega_d$.

When $\zeta = 1$, or the system is **critically damped**, the poles are:

$$s_{1,2} = -\zeta\omega_n = -\sigma$$

We use a trick from differential equations to fake another linearly independent solution, just chuck on an extra t.

$$q(t) = A_1 e^{-\zeta\omega_n t} + A_2 t e^{-\zeta\omega_n t} = A_1 e^{-\sigma t} + A_2 t e^{-\sigma t}$$

This doesn't really happen in the real world, but it's nice to cover all our bases - how about the case where $\zeta > 1$? The system is **overdamped** and our poles are:

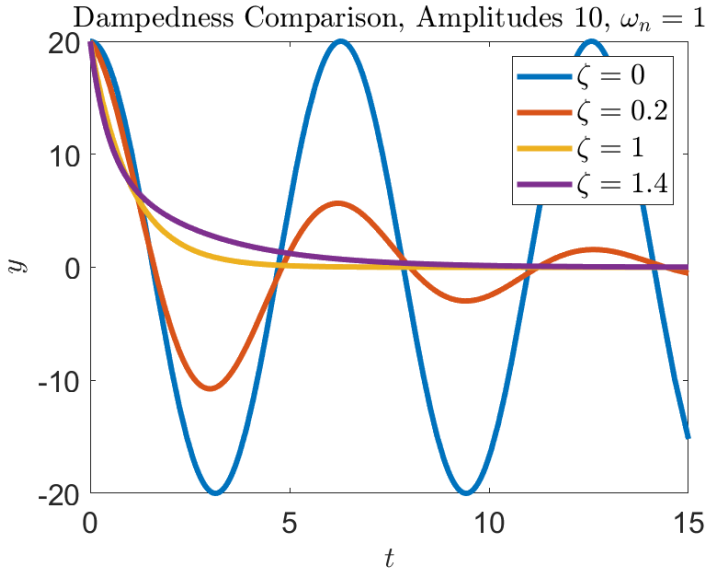
[4]Not sure if there's an agreed upon name for this value, but I think that term covers the necessary bases.
[5]This notation just means "on the order of".

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Both poles are negative here! Our solution is:

$$q(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Overdampedness implies that the mass can slowly return to equilibrium without ever overshooting as it would in the underdamped case.



3.2 Leveraging the Discriminant

I'm going to stray off from Prof. Luchtenburg for a second because *I* think this is really useful.

Suppose we want to identify the dampedness of a system immediately based on the equation of motion rather than solving for ζ. We can do exactly this using the **discriminant** of the characteristic equation of the system. Let's take a look at the canonical mass-spring system with a damper once again.

The equation of motion, assuming free motion, is:

$$m\ddot{q} + b\dot{q} + kq = 0$$

The characteristic equation is derived after plugging in $q = e^{st}$.

$$ms^2 + bs + k = 0$$

As stated in the section on damping, there are three forms of the general solution if there is damping present: both poles are real and distinct (the system is overdamped), both poles are real and equal (the system is critically damped), or both poles are complex conjugates (the system is underdamped). You may recall from algebra that the discriminant of a polynomial can reveal some properties of the roots without having to compute them. The discriminant of a quadratic is defined as follows:

$$\text{Disc}(ax^2 + bx + c) = b^2 - 4ac$$

Notably, this is the argument of the square root in the quadratic formula. If this expression is positive, the solutions to the quadratic are real and distinct. If this expression is 0, then there is only one real solution to the quadratic. If this expression is negative, the solutions to the quadratic are complex. So physically, finding the discriminant of the characteristic equation of the mass-spring system will tell us how damped it is.

$$\text{Disc}(ms^2 + bs + k) = b^2 - 4mk$$

$$\begin{aligned} b^2 - 4mk > 0 &\quad \rightarrow \quad \text{overdamped} \\ b^2 - 4mk = 0 &\quad \rightarrow \quad \text{critically damped} \\ b^2 - 4mk < 0 &\quad \rightarrow \quad \text{underdamped} \end{aligned}$$

And of course, if $b = 0$, then the system is undamped.

3.3 Pole Plots

We use a graphical tool called a **pole plot**, the plot of the roots of the characteristic equation on the complex plane, to determine the dampedness of a system from its poles. Let's go down the list:

- When both poles are on the imaginary axis, the system is undamped.
- When both poles are off the real axis, the system oscillates. If they're to the left of the imaginary axis, it'll decay exponentially (underdamped), and if they're to the right of the imaginary axis, it'll grow exponentially.
- If both poles are on the real axis to the left of the imaginary axis, it's overdamped.
- Rule of thumb: if there is ANY pole to the right of the imaginary axis, the response blows up.

Here's a nice chart from FPE. Complex conjugates are omitted for simplicity.

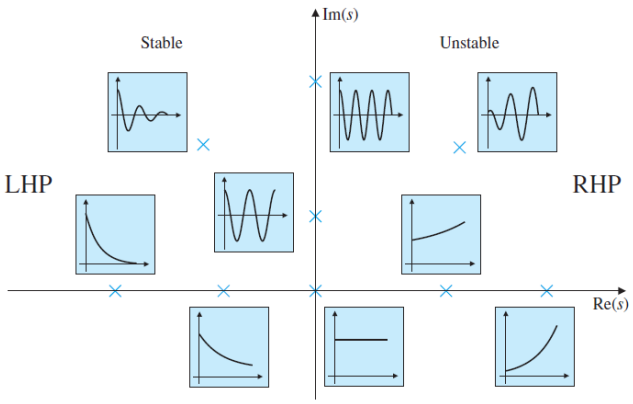


Figure 3.16
Time functions associated with points in the *s*-plane (LHP, left half-plane; RHP, right half-plane)

3.4 The Dominant Pole Approximation

The concept of pole "speed" is often useful when analyzing higher order systems. While we're able to gather valuable information from first and second order systems, doing this for systems of a higher order is more complicated.

A technique called the **dominant pole approximation** is applicable in these cases - namely, in the case that poles are substantially far apart, the slowest part of a system "dominates" the response and the faster parts are negligible for analysis.

It's better to explain by showing rather than telling here. Say we're provided with the following third order system (I don't know, like hurricane wind):

$$\frac{d^3x}{dt^3} + 110.1\frac{d^2x}{dt^2} + 1011\frac{dx}{dt} + 100x = u$$

The characteristic equation, expressed in terms of s, is:

$$s^3 + 110.1s^2 + 1011s + 100 = (s + 0.1)(s + 10)(s + 100) = 0$$

This equation has the roots -0.1, -10, and -100. By definition, these are the poles of our system.

Poles closer to the origin (in this case, $s = -0.1$) are considered **slower** than those farther away from the origin. This approximation poses that the slowest part of the system dominates the response, and that the effect of the faster poles can be ignored. (This is of course only the case if the gap between the slowest part of the system and the faster part(s) is large enough - this is, of course, just an approximation.)

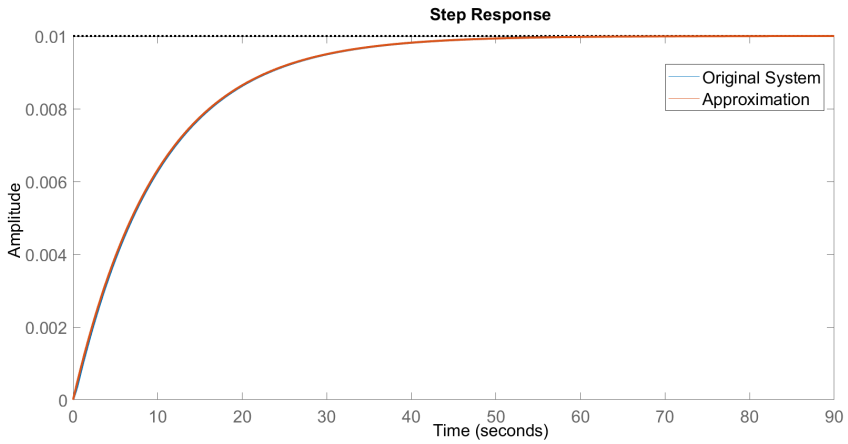
Let's continue with this problem. The slowest pole here is $s = -0.1$, so we assume that the response to:

$$\frac{d^3x}{dt^3} + 110.1\frac{d^2x}{dt^2} + 1011\frac{dx}{dt} + 100x = u$$

is similar to that of:

$$1000\left(\frac{dx}{dt} + 0.1x\right) = u$$

The scale factor of 1000 is necessary for the approximation - since we want the final value of the response to remain unchanged, we apply the additional condition that when all derivatives are zero, the original system and the approximation should be the same.[6]



This approximation can also work for systems with complex poles - we can quantify a pole's speed based on their real component.

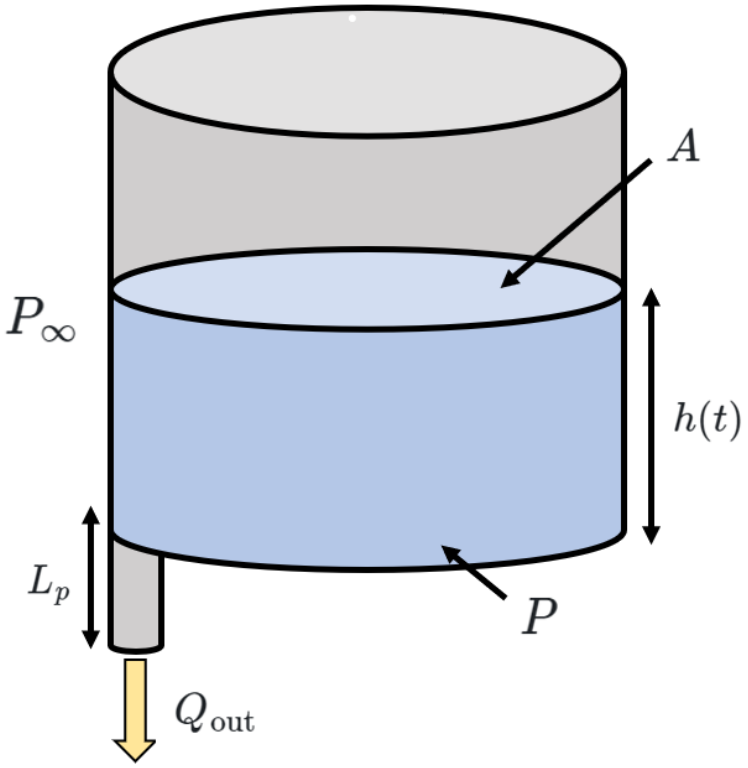
Remember that this is just an approximation. There are times when it's very cool and sleek to boot out a bunch of poles, but sometimes you're losing valuable information about a system's dynamic behavior. Prof. Fontaine says something that fits this situation pretty well in his Signal Processing class -

> Do no harm to the signal. Anything you do harms the signal. Sometimes the best option is to do nothing.

[6] I'm really dancing around poles, transfer functions, and the Final Value Theorem here so I might write a follow-up section to this in Chapter 6 when we've developed the technology for it.

3.5 The Tank, Revisited (Inertia)

Let's revisit the tank from our study of first order systems. However, we'll make one small change: the outflow pipe now has a defined length of L_p.



We'll model the same way we've been doing thus far. First, a conservation law:

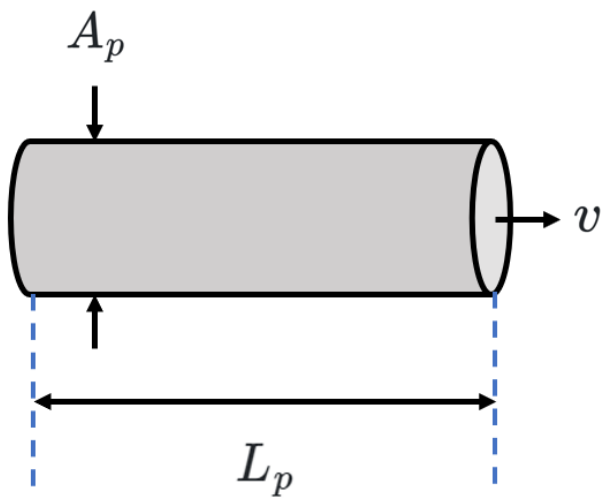
$$\dot{V} = -Q$$

Next, "Ohm's law":

$$Q = \frac{\Delta P}{R} = \frac{P - P_{\text{atm}}}{R}$$

Let's take a closer look at that pipe.

Assuming the cross-sectional area of the pipe A_p is constant, the pressure force $F_p = A_p \Delta P$ accelerates the fluid between the two ends of this pipe. Additionally, there is friction $F_f = -QRA_p$ on the liquid caused by the resistance of the pipe. By leveraging Newton's second law



of motion, we now have a relationship between the pressure difference ΔP and the velocity of the water v.

$$m\frac{dv}{dt} = F_p + F_f = A_p\Delta P - QRA_p$$

The mass m of the fluid between the two ends of this pipe is equal to the product of the density of the fluid ρ and the volume between the two ends V_p. (Notably, the volume $V_p = A_pL_p$.)

$$\rho V_p \frac{dv}{dt} = \rho A_p L_p \frac{dv}{dt} = A_p\Delta P - RQA_p$$

Because the product of the cross-sectional area A_p and the fluid velocity v is equal to the volumetric flow rate Q, we can rewrite this equation as follows:

$$\frac{\rho L_p}{A_p}\frac{dQ}{dt} = \Delta P - RQ$$

Ok, we can shed some light on what we're doing now. We define **inductance** (also referred to as **liquid-flow inertance** or **inertia**) as a term that describes the change in potential required for a unit rate of fluid flow. Inductance is the tendency of the fluid to move; it's created by the inertia of water flowing through the pipe. The mathematical definition of inductance is as follows:

$$L = \frac{\rho L_p}{A_p}$$

Note that this definition of inductance is only valid for flow systems, but analogous concepts occur in other fields (like inductors from circuit analysis)! Fluid components that have an inductance are analogous to these inductors, or mechanical components with inertia.

Let's wrap up this example. When we plug in the definition of L into our equation, a simple first order system rears its ugly head.[7]

$$L\frac{dQ}{dt} + RQ = \Delta P$$

Let's throw it into canonical form so we can see its time constant.

$$\left(\frac{L}{R}\right)\dot{Q} + Q = \frac{\Delta P}{R} \qquad\qquad \tau = \frac{L}{R}$$

To summarize, we've added a new tool to our arsenal: conservation of momentum (or Newton's second law).

$$\begin{gathered}\Delta P = L\dot{Q} + RQ \\ \dot{V} = -Q \\ C\Delta P = V\end{gathered}$$

By combining these three equations, we can use tools from our studies of mass-spring systems to analyze… well… any second order system.

$$\begin{aligned} L\Delta\ddot{P} + R\Delta\dot{P} + \frac{1}{C}\Delta P = 0 \qquad &\rightarrow \qquad \Delta\ddot{P} + \left(\frac{R}{L}\right)\Delta\dot{P} + \left(\frac{1}{LC}\right)\Delta P = 0 \\ m\ddot{q} + b\dot{q} + kq = 0 \qquad &\rightarrow \qquad \ddot{q} + \left(\frac{b}{m}\right)\dot{q} + \left(\frac{k}{m}\right)q = 0 \end{aligned}$$

We simply retrofit the definitions of the natural frequency ω_n and damping ratio ζ based on how we defined them for mass-spring systems to determine how the oscillations behave. Here's a quick example using the flow system scenario we've been tackling thus far:

$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = 0 \quad \longleftrightarrow \quad \Delta\ddot{P} + \left(\frac{R}{L}\right)\Delta\dot{P} + \left(\frac{1}{LC}\right)\Delta P = 0$$

$$\begin{aligned} 2\zeta\omega_n = \frac{R}{L} \qquad &\longrightarrow \qquad \zeta = \frac{R}{2L\omega_n} = \frac{R\sqrt{LC}}{2L} = \frac{R}{2}\sqrt{\frac{C}{L}} \\ \omega_n^2 = \frac{1}{LC} \qquad &\longrightarrow \qquad \omega_n = \frac{1}{\sqrt{LC}} = \frac{\sqrt{LC}}{LC} \end{aligned}$$

[7] The analog of this system in circuit analysis is called the RL circuit, which is often used as a passive filter.

4 Linear Algebra for Absolute Bozos

Preface

This chapter's going to be a smorgasbord of mathematical concepts I think are foundational to understanding this course from a theoretical perspective. A decent chunk of it will be review from Ma110 and Ma240, but it doesn't hurt to take a second look at these things (they keep coming back over and over). That being said, this chapter is skippable for those confident in their abilities.

What I'll try to do, instead of rehashing what you got out of Ma240, is reframe these concepts in a way that's more applicable to this course (and ME351). I'll also formalize a bunch of concepts from Ma326 that are really important to know for this class (and later on). But first, let's formalize a few more definitions that will come in handy later on.

A **linear combination** of elements of a set $x_1, x_2, ..., x_n$, is given by:

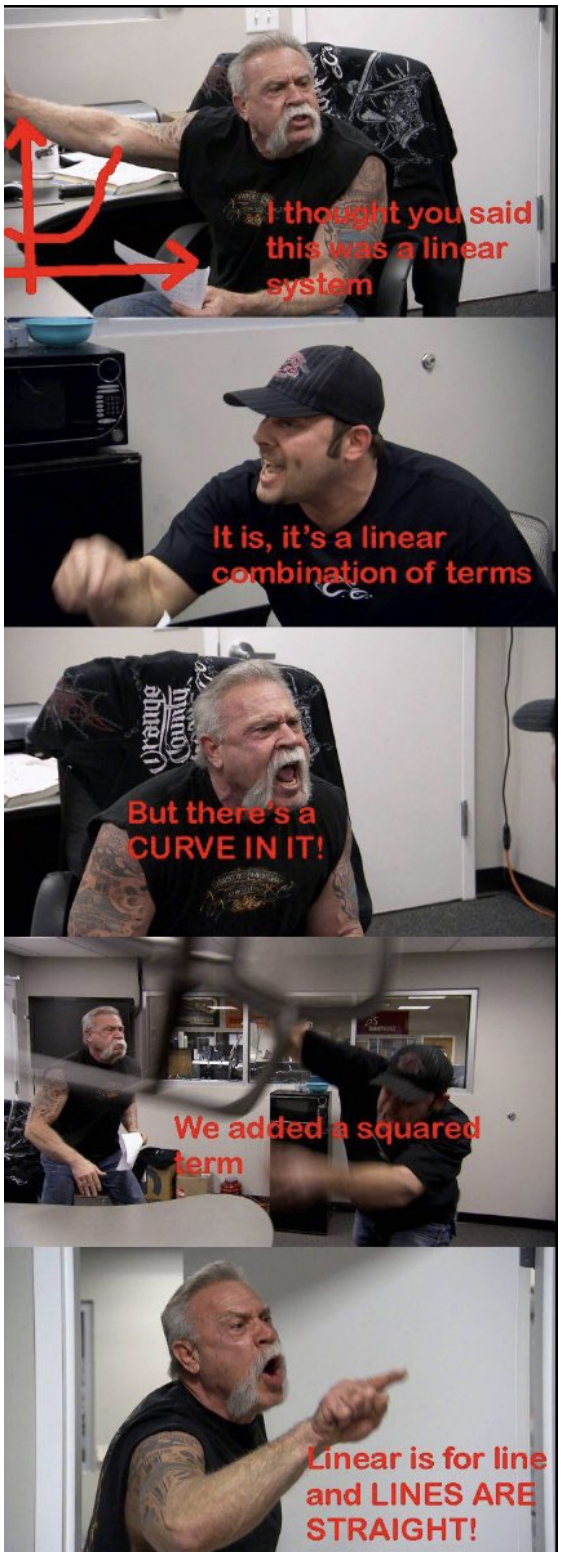
$$a_1x_1 + a_2x_2 + ... + a_nx_n$$

where each a_i is a constant. In layman's terms, a linear combination of variables is the sum of scaled versions of those variables, where the scaling factor is a scalar.

A set of variables are **linearly dependent** if one of the variables can be expressed as a linear combination of the others. More formally, a set of variables $x_1, x_2, ..., x_n$ is linearly dependent if there exist scalars $a_1, a_2, ...a_n$ (not all zero), such that:

$$a_1x_1 + a_2x_2 + ... + a_nx_n = 0$$

On the other hand, if no such scalars exist, the set of variables is said to be **linearly independent**.


I thought you said this was a linear system
It is, it's a linear combination of terms
But there's a CURVE IN IT!
We added a squared term
Linear is for line and LINES ARE STRAIGHT!

4.1 Linearity and Time-Invariance

A **linear time-invariant (LTI) system** is a mathematical model often used in control theory to describe the behavior of physical systems. It is characterized by two properties: linearity and time-invariance. I'll describe these separately in the context of differential equations. A differential equation of the form:

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)} + ... + a_1 \dot{y}(t) + a_0 y(t) = f(t)$$

where $y^{(i)}$ is the ith derivative of $y(t)$, is called **linear**. (The relationship between $y(t)$ and t is a linear mapping.) A system is linear if and only if it satisfies two properties: superposition and homogeneity:

- **Superposition** - if $x_1 \to y_1$ and $x_2 \to y_2$, then $x_1 + x_2 \to y_1 + y_2$
- **Homogeneity** - if k is a scalar and $x \to y$, then $kx \to ky$

If $\{a_i\}_0^n$, also called coefficients, are constants, the equation is also characterized as a constant coefficient differential equation. Physically, constant coefficients imply that the system behavior does not depend on time.

We can establish an equivalence between linear constant coefficient equations and linear time-invariant systems. Time-invariance is the principle that if we chug an input t_0 into a system that outputs $y(t_0)$, the input $t_0 + t_1$ will result in an output of $y(t_0 + t_1)$.[1]

Let's take a look at a few examples to make this more clear:

$$\dot{y} + \sin(t)y = 0$$

This system is not LTI, because the coefficient of y is not constant. (More explicitly, it's a function of t, so as $t \to \infty$, the behavior is affected.)

$$2\dot{y} + 3y = 0$$

This system *is* LTI, because the coefficients of each derivative of y are constant.

$$\dot{y} + \ln(y) = 0$$

This system isn't even linear, for obvious reasons.

Examples of LTI systems include first-order passive filters, second order systems such as springs and masses, and many other linear systems in control theory and signal processing.

[1] An equivalent definition in signal processing is that a system is time-invariant if it commutes with a "delay".

4.2 Matrices, at Lightspeed

I truly hope you know what a matrix is by now. If not, fasten your seatbelt.

A matrix is a rectangular array of numbers, of the form:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where a_{ij} is the entry at row i and column j. We say a matrix is of size $m \times n$, where m is the number of rows and n is the number of columns.

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{3\times 2}$$

Matrices are pretty neat. We can add numbers in the matrix elementwise and scale it by a scalar factor as follows:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

The **zero matrix 0** is an $m \times n$ matrix with all entries 0. The zero matrix plays an important role in linear algebra, as it is the additive identity for matrices. This means that adding a zero matrix to any matrix does not change the matrix, much like adding zero to any number does not change its value.

A **square matrix** is a matrix with the same number of rows and columns. Many concepts in linear algebra are designed with these in mind, such as determinants and eigenvalues.

Next, we'll tackle the **Kronecker delta**, which is is a useful and important symbol in mathematics, particularly in linear algebra and related fields. Its simple definition allows for the easy expression of many concepts and operations, making it a valuable tool for mathematicians and scientists. We define the Kronecker delta δ_{ij} as follows:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Following from this, the **identity matrix** $\mathbf{I}_n$ is defined as an $n \times n$ square matrix where $(\mathbf{I}_n)_{ij} = \delta_{ij}$, i.e.,

$$\mathbf{I}_1 = 1,\ \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ ...$$

The set of all matrices fixed at size $m \times n$ with scalar entries forms something we call a vector space. There's a lot of nuance attached to that name; if you care about it, take Ma326. Here are a few properties for now. (Bolded quantities are matrices, a, b, and 1 are scalars.)

- Matrix addition is commutative, i.e., $\mathbf{X} + \mathbf{Y} = \mathbf{Y} + \mathbf{X}$
- Matrix addition is associative, i.e., $(\mathbf{X} + \mathbf{Y}) + \mathbf{Z} = \mathbf{X} + (\mathbf{Y} + \mathbf{Z})$
- Each matrix has an additive identity, i.e., $\mathbf{X} + \mathbf{0} = \mathbf{X}$
- Each matrix has an additive inverse, i.e., $\mathbf{X} + \mathbf{Y} = \mathbf{0}$
- Each matrix has a scalar multiplicative identity, i.e., $1(\mathbf{X}) = \mathbf{X}$
- $(ab)\mathbf{X} = a(b\mathbf{X})$
- $a(\mathbf{X} + \mathbf{Y}) = a\mathbf{X} + a\mathbf{Y}$
- $(a + b)\mathbf{X} = a\mathbf{X} + b\mathbf{X}$

We can multiply two matrices of sizes $m \times n$ and $n \times p$, respectively, to produce another matrix of size $m \times p$. The operation, dubbed **matrix multiplication**, should not be confused with the scalar multiplication used before. It's defined as follows:

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{n} \mathbf{A}_{ik}\mathbf{B}_{kj}$$

Finding the product of two matrices may look intimidating based off that formula, but it's actually not that difficult once you understand the basic steps. First, you need to make sure that the matrices are compatible for multiplication. To do this, we need to ensure that the number of columns in the first matrix is the same as the number of rows in the second matrix. If they're not the same, you can't multiply them.[2]

Once you've determined that the matrices are compatible, you can start multiplying. To find each element in the product matrix, you need to multiply the corresponding row in the first matrix by the corresponding column in the second matrix. Specifically, for each element in the product matrix, you will:

- Take the row of the first matrix that corresponds to that element.
- Take the column of the second matrix that corresponds to that element.
- Multiply each corresponding pair of elements in the row and column.

[2]This is a really great gut check when you're finishing up a long calculation. If that matrix multiplication at the end of the problem is impossible, something must be up.

- Add up all of the products you got in the last step.

Keep doing this for every element in the product matrix until you've filled in all the entries. Here's a brief example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 3 + 3\cdot 5 & 1\cdot 2 + 2\cdot 4 + 3\cdot 6 \\ 4\cdot 1 + 5\cdot 3 + 6\cdot 5 & 4\cdot 2 + 5\cdot 4 + 6\cdot 6 \end{bmatrix} = \begin{bmatrix} 22 & 28 \\ 49 & 64 \end{bmatrix}$$

Matrix products show up everywhere. They're essential for fields like population modeling, network theory, signal processing, advanced dynamics, etc. Try to be as comfortable as possible with this operation before diving into the next chapter.

4.3 Transposition and Symmetry

The **transpose** of an $m \times n$ matrix $\mathbf{A}$ is the $n \times m$ matrix obtained by interchanging its rows and columns. It's often denoted as $\mathbf{A}^{\mathrm{T}}$ or $\mathbf{A}^{\mathrm{t}}$.[3] In other words, the rows of the original matrix become columns in the transposed matrix, and the columns become rows.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Let's follow up with a few more properties that involve transposition. (Bolded quantities are matrices, a, b are scalars.)

- $\left(\mathbf{X}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{X}$
- $(\mathbf{X} + \mathbf{Y})^{\mathrm{T}} = \mathbf{X}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$
- $a\mathbf{X}^{\mathrm{T}} = (a\mathbf{X})^{\mathrm{T}}$
- $(\mathbf{XY})^{\mathrm{T}} = \mathbf{Y}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}$

A **symmetric matrix** is a square matrix equal to its own transpose. (Another way to phrase this is: if $\mathbf{A}$ is an $n \times n$ matrix, then $\mathbf{A}$ is symmetric if and only if $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$.)

Trivially, the sum of two symmetric matrices is symmetric, and a scalar multiple of a symmetric matrix is symmetric.

There's other kinds of symmetry as well. For example, a **skew-symmetric matrix** is a square matrix equal to its negative. (Another way to phrase this is: if $\mathbf{A}$ is an $n \times n$ matrix, then $\mathbf{A}$ is skew-symmetric if and only if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$.) This turns out to be *much* more interesting than

[3] I prefer the former notation and will be using it henceforth.

its vanilla counterpart in dynamics, especially when dealing with concepts like inertial tensors and angular momentum.

Again, trivially, the sum of two skew-symmetric matrices is symmetric, and a scalar multiple of a skew-symmetric matrix is symmetric. Additionally, you may realize that every entry on the diagonal of a skew-symmetric matrix must be equal to 0. (Otherwise, how would the definition work?)

This factoid turns out to be crucial if we focus on the 3-space case, there are only three independent entries of a skew-symmetric matrix. We can define a skew-symmetric operator for vectors in 3-space skew() as follows:[4]

$$\text{skew}(x) = \text{skew}((x_1, x_2, x_3)) = \begin{bmatrix} 0 & -x_1 & x_2 \\ x_1 & 0 & -x_3 \\ -x_2 & x_3 & 0 \end{bmatrix}$$

This operator comes with a really cool property:

$$\text{skew}(x)^{\text{T}} = -\text{skew}(x)$$

which comes in handy a lot in advanced dynamics. Additionally, we can make an alternative definition of the vector cross product.

$$x \times y = \text{skew}(x)y$$

Prove it! It's kind of fun.

4.4 Invert It! Determine...It!

An $n \times n$ matrix $\mathbf{A}$ is called **invertible** if there exists a inverse $\mathbf{B}$ such that:

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

There are many algorithms for finding the inverse of a matrix; the list is too exhaustive to describe in detail here, so I'll just drop the Wikipedia link here and encourage you to explore on your own.

I'm going to dump some properties here, where $\mathbf{X}$ and $\{\mathbf{A}_i\}_{i=1}^{k}$ are invertible $n \times n$ matrices:

- $\left(\mathbf{X}^{-1}\right)^{-1} = \mathbf{X}$

[4] Yeah, you'll see awful notation everywhere for this thing. I'm going to use skew() because why not.

- $\left(\mathbf{X}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{X}^{-1}\right)^{\mathrm{T}}$
- $(\mathbf{A}_1\mathbf{A}_2)^{-1} = \mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$
- $(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{k-1}\mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1}\mathbf{A}_{k-1}^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$

But how can you tell *if* a matrix is invertible? We use a value called the **determinant** to help there.

When we have a matrix of size $n \times n$, its determinant tells us whether the matrix is invertible or not. If the determinant of a square matrix is non-zero, it means the matrix is invertible. In other words, it has an inverse matrix that can be found. On the other hand, if the determinant is zero, the matrix is not invertible, and it is called a **singular** matrix.

Finding the determinant of a matrix is an…annoying process to say the least if the matrix is of size larger than 2×2 - you'll have to use the method cofactor expansion or Laplace expansion for that, and…bleugh. For 2×2, though?

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Calculating the inverse and determinant by hand for matrices larger than size 2×2 shouldn't be necessary in this course - that's why we have tools like MATLAB and Python to help out.

While I did just get lazy and drop a Wikipedia link before, there is one method of finding the inverse of a matrix that I do want to highlight. It involves the **classical adjoint**, or **adjucate** of a matrix, a matrix that satisfies the following property:

$$\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = \det(\mathbf{A})\,\mathbf{I}$$

The classical adjoint of a 2×2 matrix $\mathbf{A}$ is:

$$\mathrm{adj} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and using the property that defines the classical adjoint, we can derive the following formula for the inverse of a matrix $\mathbf{A}$:

$$\mathbf{A}^{-1} = \frac{\mathrm{adj}\,\mathbf{A}}{\det \mathbf{A}}$$

This formula's still valid for matrices of size $n \times n$, but I find it useful when we're dealing with 2×2 matrices. Note that when a matrix $\mathbf{A}$ has a determinant of 0, the formula correctly lets you know that $\mathbf{A}^{-1}$ is undefined.

As I've done seemingly countless times now, here's another property dump, where k is a scalar, $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{B}$ is an $n \times n$ invertible matrix:

- $\det(k\mathbf{A})^{-1} = k^n \det(\mathbf{A}^{-1})$
- $\det(\mathbf{A}^{\mathrm{T}}) = \det(\mathbf{A})$
- $\det(\mathbf{B}^{-1}) = \det(\mathbf{B})^{-1}$

5 A Mishmash of Modeling Concepts

Preface - Time Domain Modeling

When we begin to analyze more complicated systems, it becomes less and less feasible to solve them analytically. As a result, we resort to setting up a system of differential equations rather than concatenating them into one as we've done in past chapters.

The state-space representation of a system describes the system's behavior over time in terms of a set of variables called **states**.[1] The state variables represent the current conditions of the system, and their evolution over time is described by a set of first order differential equations called **state equations**.

The state-space representation is a very powerful tool for modeling and analyzing physical systems, providing valuable insights into their behavior and enabling the development of control algorithms covered in ME351.

An important thing to note is that the state-space representation of a system is not unique. In fact, an infinite number of representations exist for a physical system.

5.1 The State-Space Approach

In the most general case, a state-state representation can be represented as the following:

$$\begin{cases} \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \\ \underline{y} = \underline{h}(\underline{x}, \underline{u}) \end{cases}$$

This might be a bit daunting at first, but it's just a lot of fancy notation for a concept that's pretty simple. $\underline{x}$ is the state, $\underline{u}$ is an input, and $\underline{y}$ is an output. Let's drive this concept home with an example.

Say we have a simple pendulum with length L and a point mass m at its end, as pictured below:

For this problem, the mass of the rod (and any potential friction in the hinge) is ignored. The equation of motion of the pendulum can be derived by summing moments about the point of

[1] The state space can be described as a Euclidean space where each state corresponds with an axis.

contact between the pendulum and the fixed surface.[2] Let's call that point of contact O for future bookkeeping purposes.

$$\sum M_O = J_O \ddot{\theta}$$

The moment arm for the weight mg is the horizontal displacement $L\sin(\theta)$, and $J_O = mL^2$ is the mass moment of inertia of the point mass m about point O. Let's crunch some numbers.

$$-mgL\sin(\theta) = mL^2\ddot{\theta}$$
$$mL^2\ddot{\theta} + mgL\sin(\theta) = 0$$
$$\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0$$

Now let's try putting this in state-space form. First, we select the state vector, which should adhere to the following points:

- Pick state variables that include all the relevant information about the system you're trying to model.
- The number of dimensions in the state vector should match the number of degrees of freedom of the system.
- The state vector should be **minimal**, meaning it should contain only the information necessary to describe the system, and not any redundant information. (Usually, the minimum number required is equal to the order of the differential equation that represents the system.)
- The components of the state vector must be linearly independent.

A good rule of thumb is that the state should correspond with the initial conditions provided. We define states θ (angle) and $\omega = \dot{\theta}$ (angular velocity), and start constructing our state equations.

$$\underline{x} = \begin{bmatrix} \theta \\ \omega \end{bmatrix} \qquad \dot{\underline{x}} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix}$$
$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$$

$$\dot{\underline{x}} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{g}{L}\sin(\theta) \end{bmatrix}$$

[2] Alternatively, you could sum forces in the parallel and perpendicular directions of motion to yield an equivalent result.

We've turned a second order differential equation into two first order differential equations. Let's move onto the second part of the representation: defining the output y. We're interested in the states' behavior over time, so our output is...just the state vector $\underline{x}$.

$$\underline{y} = \underline{h}(\underline{x}, \underline{u}) = \underline{x} = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

These two components make up the state-space representation of this pendulum system. Putting it in this form makes it easier to numerically solve using tools like Python or MATLAB.

A nonlinear solution can be unappealing, though perfectly valid. By using the small-angle approximation $\sin(\theta) \sim \theta$, we can refine this representation further.

$$\ddot{\theta} + \frac{g}{L}\sin(\theta) \sim \ddot{\theta} + \frac{g}{L}\theta = 0$$

Our system is now an LTI system. Linearity is very nice, because we can use matrix multiplication to make this representation pretty.

$$\dot{\underline{x}} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{g}{L}\theta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

Let's move onto the **output equation**, which is (in my opinion) less interesting than the state equation. What are we interested in analyzing here? Say we're interested in analyzing θ - or more succinctly, $y = \theta$.

$$\underline{y} = \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

Or, more interestingly, say we want to track both states over time (θ and ω). Our output is just the state, so we set $y = \underline{x}$. In these cases, the "coefficient" matrix is the $n \times n$ identity matrix $\mathbf{I}_n$, where n is the number of components in the state vector.

$$\underline{y} = \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \mathbf{I}_2 \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

We'll move on using this output equation. Let's make this more complicated by saying the pendulum has an input applied torque T, and the new equation of motion is:

$$\ddot{\theta} + \frac{g}{L}\theta = \frac{T}{mL^2}$$

Our revised state-space representation would be:

$$\underline{\dot{x}} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{g}{L}\theta + \frac{T}{mL^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} T = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} u$$

$$\underline{y} = \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

Using this example, we now extract the general form of the state-space representation of a linear system. A linear system is represented in state space by the following equations:

$$\underline{\dot{x}} = \mathbf{A}\underline{x} + \mathbf{B}\underline{u}$$

$$\underline{y} = \mathbf{C}\underline{x} + \mathbf{D}\underline{u}$$

for $t \geq t_0$, $\underline{x}(t_0)$, where:

- $\underline{x}$ is the state vector, of size $n \times 1$
- $\underline{y}$ is the output vector, of size $q \times 1$
- $\underline{u}$ is the input vector, of size $p \times 1$
- $\mathbf{A}$ is the system matrix, of size $n \times n$
- $\mathbf{B}$ is the input matrix, of size $n \times p$
- $\mathbf{C}$ is the output matrix, of size $q \times n$
- $\mathbf{D}$ is the feedforward matrix, of size $q \times p$

Let's try another example, this time a translational mechanical system. Block 1 of mass m is attached to a fixed wall by dashpot with damping coefficient b. Block 2, also of mass m, is attached to block 1 by a spring of spring constant k. Gravity is turned off.

First, we write the equations of motion of the network. (Just draw a free body diagram around each mass and don't fuck up your signs.)

> You want a hint? Newton guy.
>
> Prof. Luchtenburg

$$m\ddot{q_1} + b\dot{q_1} + kq_1 - kq_2 = 0$$
$$-kq_1 + m\ddot{q_2} + kq_2 = f(t)$$

This is a system of two second order differential equations, so we'll pick four states. We select our q_1, $\dot{q_1}$, q_2, and $\dot{q_2}$ to be our four state variables, because we're analyzing the kinematic

behavior of two masses obeying Newton's second law (N2L is a second order differential equation, which requires two initial conditions, and since there's two masses to analyze we have four).

Our first two state equations are easy: just define the derivatives. We get the other two by rearranging the equations of motion and isolating $\ddot{q_1}$ and $\ddot{q_2}$.

$$\underline{\dot{x}} = \begin{bmatrix} \dot{q_1} \\ \ddot{q_1} \\ \dot{q_2} \\ \ddot{q_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{b}{m} & \frac{k}{m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m} & 0 & -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q_1} \\ q_2 \\ \dot{q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} f(t)$$

Now, we didn't specify what output we wanted, but let's say we want to analyze the velocity of the second mass, or $\dot{q_2}$. We'd do the following:

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q_1} \\ q_2 \\ \dot{q_2} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} f(t)$$

which is equivalent to the expression $y = \dot{q_2}$. Even though it seems more complicated to put it in this form, it provides us with a **C** and **D** matrix, which is invaluable when analyzing systems computationally.

What makes this example more significant than the previous one is that there's now two different "entities" to analyze.[3] Rather than just analyzing multiple states (position, velocity) of a singular entity like we did with the pendulum, we're now taking a look at the position and velocity of *two* masses. That's kind of nifty, I think.

5.2 The Unit Impulse

THWACK!

You hear that? It's the sound of Prof. Baglione smacking something with an impact hammer, a tool used to simultaneously excite something and measure the impact force at the same time.

It's really common in mechanical (and electrical, ugh) scenarios to want to analyze a very large force over a very short period of time. Say we're interested in modeling a batter's hit, a car crash, or…I don't know…whacking a mass-spring system with an impact hammer.

Consider a model of a hammer strike, pictured as follows.

[3] You could try to put the two tanks problem from HW1 in state-space form for extra practice.

This strike is modeled as a rectangular "pulse", where the rule is:

$$F(t) = \begin{cases} 0 & \text{if } t \leq t_0 - \epsilon \\ \frac{F_0}{2\epsilon} & \text{if } t_0 - \epsilon < t < t_0 + \epsilon \\ 0 & \text{if } t \geq t_0 + \epsilon \end{cases}$$

When integrated over time, $F(t)$ yields the **impulse** over F.

$$\text{Imp} = \int_{0^-}^{\infty} F(t)\,dt = \int_{t_0-\epsilon}^{t_0+\epsilon} \frac{F_0}{2\epsilon}\,dt = \frac{F_0}{2\epsilon}\left((t_0+\epsilon) - (t_0-\epsilon)\right) = F_0$$

This trick of "ripping out the integrand" only works when $\epsilon \neq 0$. Nevertheless, we investigate the case where $\epsilon \to 0$.[4]

As ϵ gets smaller and smaller, the magnitude of force $\frac{F_0}{2\epsilon}$ gets larger and larger. Nevertheless, the impulse remains equal to F_0. We define the **impulse function** as a function $F(t)$ with the following two properties:

$$F(t - t_0) = 0, \quad t \neq t_0 \qquad\qquad \int_{0^-}^{\infty} F(t)\,dt = F_0$$

We call an impulse function where $F_0 = 1$ the **unit impulse function**, also called the **Dirac delta function**, denoted $\delta(t)$.[5] Let's rewrite the above principles for the unit impulse.

$$\delta(t - t_0) = 0, \quad t \neq t_0 \qquad\qquad \int_{0^-}^{\infty} \delta(t)\,dt = 1$$

It's important to realize that the unit impulse function…well…isn't a function. The idea of a "function" that is equal to 0 everywhere except $t = t_0$, where it's equal to ∞ is preposterous! Additionally, for this "function" to make sense, the integral would be equal to 0.

We don't care. We defined it that way.[6]

There exists an alternative definition of the unit impulse used in signal processing (that will also end up useful later on in this course).

$$\int_{0^-}^{\infty} \delta(t - t_0) f(t)\,dt = f(t_0)$$

[4] Figure from Zill, et al. (8)

[5] Not to be confused with the Kronecker delta!

[6] The less dismissive answer is that the unit impulse function is an example of something we call a generalized function. Go take a functional analysis course if you want to pursue this further.

where f is a continuous function. This is called the **sifting property** of the unit impulse function. The unit impulse function acts as a sampler; the juicy part of $\delta(t - t_0)$ is located at $t = t_0$, so integrating the product of the impulse $\delta(t - t_0)$ and a continuous function $f(t)$ yields $f(t_0)$.

Preface - Frequency Domain Prerequisites

While using the state-space representation of a system can be advantageous in situations where we have multiple inputs and multiple outputs, sometimes the linear algebra just gets too unwieldy, especially if you don't have Python or MATLAB sitting in front of you. (This isn't to say that transfer functions aren't easily implementable in Python or MATLAB, though.)

In lieu of modeling in the time domain, we can use **transfer functions** to mathematically model systems in the frequency domain. This ends up being really useful in ME351 when we talk about how to control physical systems rather than just analyzing them. Transfer functions are also much easier to translate into graphical interpretations of systems, like Bode plots.

5.3 The Laplace Transform

You've likely brushed upon Laplace transforms in Ma240, so I'll try to reintroduce them…err…less formally.

The Laplace transform is an essential tool for analyzing and designing control systems, making them a vital topic to cover in preparation for ME351. By learning about Laplace transforms, we can develop a deep understanding of the behavior of dynamic systems and how to control them.

Laplace transforms allow us to simplify differential equations that describe the behavior of a system in the time-domain, into simpler algebraic equations in the s-domain. These equations can be easily analyzed to determine important system characteristics such as stability, steady-state error, and transient response.

Furthermore, understanding Laplace transforms is crucial for designing controllers that can effectively control the behavior of a system. For example, by using Laplace transforms to analyze a system's frequency response, engineers can design controllers that attenuate unwanted frequencies, leading to more desirable system performance.

Let's cut to the chase. The (one-sided) Laplace transform of a function $f(t)$ is a new function $F(s)$, $s \in \mathbb{C}$, defined by:[7]

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} e^{-st} f(t) dt$$

As the Laplace transform is an integral transform, we know that it is linear. Thus, taking the Laplace transform of a linear combination of functions yields a linear combination of the Laplace transforms of the functions:

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \mathcal{L}\{\alpha f(t)\} + \mathcal{L}\{\beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s)$$

where α, β are constants. Here are a few common ones (easily verifiable with the integral definition).

$$\mathcal{L}\{\delta(t)\} = 1$$
$$\mathcal{L}\{1\} = \frac{1}{s}$$
$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Confirm them yourself! (Or find a table of Laplace transforms, I don't know…)

Our objective is to use the Laplace transform to model physical systems. Thus, it is imperative that we find a way to take the Laplace transform of a derivative. Provided f and its derivatives are continuous from $0 \leq t < \infty$, and are of exponential order,[8] we do so as follows:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty} e^{-st} \frac{df(t)}{dt} dt = [e^{-st} f(t)]_{0^-}^{\infty} - \int_{0^-}^{\infty} -se^{-st} f(t) dt = -f(0) + sF(s)$$

or:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$

By recursion, it is apparent that the Laplace transform of a second derivative is:

[7] The two-sided Laplace transform instead has integral limits of $-\infty$ and ∞, but for causal signal inputs, the one-sided Laplace transform is equivalent. (A causal signal is a signal that is 0 for all $t < 0$.)

[8] A function $f(t)$ is said to be of exponential order if there exist positive constants a and b such that $|f(t)| \leq ae^{bt}$ for all $t \geq 0$.

$$\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = sF'(s) - f'(0) = s(sF(s) - f(0)) - f'(0) = s^2F(s) - sf(0) - f'(0)$$

and an nth derivative is:

$$\mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

In the case that all initial conditions are zero, it's valid to write:

$$\mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s)$$

This will be handy very soon.

Finally, we'll introduce the notion of the **inverse Laplace transform**, which transforms an expression in the s-domain back into the time domain.[9] More concisely:

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

The inverse Laplace transform is also a linear transformation. Notably, it's not unique; there exist distinct f and g such that $\mathcal{L}\{f\} = \mathcal{L}\{g\}$.

Using the technology we've formalized[10] thus far, we can use the Laplace transform to solve initial value problems. Say we wanted to solve the IVP:

$$\ddot{y} - 4\dot{y} + 3y = e^{2t}$$

where y is a function of t, and $y(0) = \dot{y}(0) = 0$. First, we transform the whole equation into the s-domain.

$$s^2Y - 4sY + 3Y = \frac{1}{s-2}$$

Since our objective is to solve for $y(t)$, it's wise to isolate $Y(s)$ on one side, so when we take the inverse Laplace transform, we immediately have an explicit formula y.

[9] It might seem peculiar that I'm abstaining from providing a formula to directly calculate the inverse Laplace transform. You can look up Mellin's inverse formula for additional hardship, but rest assured that it's WAY more convenient to think of the inverse Laplace transform as just that - an inverse Laplace transform, instead of as its own distinct transform.

[10] Man, I said I was going to do this less formally!

$$Y(s^2 - 4s + 3) = \frac{1}{s-2}$$
$$Y = \frac{1}{(s-2)(s^2-4s+3)}$$

We then find the partial fraction decomposition of this rational function, so that transforming back to the time domain is easier. [11]

$$Y = -\left(\frac{1}{s-2}\right) + \frac{1}{2}\left(\frac{1}{s-1}\right) + \frac{1}{2}\left(\frac{1}{s-3}\right)$$

And finally, we transform back.

$$y = -e^{2t} + \frac{1}{2}e^{t} + \frac{1}{2}e^{3t}$$

which matches whatever we'd get if we tried using a time domain method like the method of undetermined coefficients. Of course, we did a problem with zero initial conditions, but rest assured, if your goal is just to solve differential equations, we can consider cases with nonzero initial conditions as well.

5.4 It's Convoluted!

But what if we wanted to find the Laplace transform of an *integral*?

Convolution is an operation that generates a function from two other piecewise continuous functions. We define the convolution $f * g$ of functions $f(t)$ and $g(t)$ as follows:

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

So what makes this operation special? The Laplace transform of the convolution $f(t) * g(t)$ is equal to the product of the Laplace transform of $f(t)$ and $g(t)$.

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\} = F(s)G(s)$$

Or alternatively:

[11] If you're not that comfortable with partial fraction decomposition, it might be a good idea to look up the Heaviside cover-up method. Thank me later.

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$$

So in the case that $g(t) = 1$, the Laplace transform of the integral of $f(t)$ is:

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

5.5 Transfer Functions and Block Diagrams

When possible, we try to think of all systems as black boxes with an input and an output. We've covered how to represent a system in the time domain using the state-space representation. Let's try something else.

Transfer functions are often preferred over state space representations in certain situations because they are simple and easy to work with, especially for linear systems with a single input and a single output. They can be easily manipulated using some algebra and can be used to compute a system's response to various input signals.

Transfer functions are also very useful for frequency-domain analysis, such as calculating the system's frequency response or designing filters to tweak the system's frequency characteristics.

Mathematically, a transfer function is the ratio of the Laplace transform of the output of a system $Y(s)$ to the Laplace transform of its input $U(s)$, with all initial conditions assumed to be zero. The transfer function is typically denoted as $H(s)$, where:

$$H(s) = \frac{Y(s)}{U(s)}$$

That's all there is to it. Let's try a variation of the system we tackled before.

$$\ddot{y} - 4\dot{y} + 3y = u$$

So instead of that exponential expression on the right side of the equation, we now have an arbitrary input $u(t)$. Because we're trying to find the transfer function, we assume zero initial conditions and take the Laplace transform of everything.

$$s^2Y - 4sY + 3Y = U$$

Then, we find H.

$$H = \frac{Y}{U} = \frac{1}{s^2 - 4s + 3}$$

Using this transfer function, we can find the system response to any input we want. Say we're interested in the response to a step input $u_s(t)$. We can retrofit the definition of a transfer function to solve for the system response.

$$Y = HU$$

In this scenario, U is the Laplace transform of $u_s(t)$, or $\frac{1}{s}$.

$$Y = \frac{1}{s^3 - 4s^2 + 3s}$$

which is left to complete as an exercise to the reader.[12]

Ah, one more thing! The roots of the numerator of the transfer function are called the *zeroes* of the system, and the roots of the denominator of the transfer function are called the *poles*. These definitions are consistent with the definitions we established earlier in the course.

This seems like as good a time as any to introduce the concept of a **block diagram**, or a visual representation of a system, using blocks and arrows to break down components. We literally represent components as black boxes, without regard for whatever's going on inside. All that matters is what comes in and what goes out. Seems perfect for a transfer function.

Here's a block diagram. U is our input signal, Y is the output signal, and H is the transfer function representation of the system. When a signal enters a system in a block diagram, the output signal will be the product of the signal and the system's transfer function.

We can use additional tools of block diagram algebra to simplify and manipulate these diagrams, which ends up being vital when we start introducing feedback control into the mix.

5.6 Electrical Networks and Impedance

In ESC221, you may have covered the series RLC circuit, pictured as follows:

where the source voltage V_s is the input of the system and the voltage across the capacitor V_c is the output. The governing equation of this system is a second order differential equation, but using transfer functions, we can turn this differential equation into an algebraic one.

[12] I'm too lazy to take the inverse Laplace transform of this. It's not difficult though, knock yourself out.

This is especially advantageous when conducting an alternating current (AC) analysis of the network, when V_s could be a sinusoidal function. However, the formulation described in this course is applicable to any input signal.

The **impedance** Z of a two-terminal passive component like a resistor, capacitor, or inductor, is defined as the ratio of the Laplace transform of the voltage $\tilde{V}$ to the Laplace transform of the current $\tilde{I}$.

Ohm's law states that $V = IR$. When we take the Laplace transform of the expression, we get:

$$\tilde{V} = \tilde{I}R$$

Thus, the impedance of a resistor is:

$$Z_R = \frac{\tilde{V}}{\tilde{I}} = R$$

That was...anticlimactic. Let's try a capacitor next. The capacitive relationship states that $q = \int I\,dt = CV$. When we take the Laplace transform of the expression, we get:

$$\frac{1}{s}\tilde{I} = C\tilde{V}$$

Thus, the impedance of a capacitor is:

$$Z_C = \frac{\tilde{V}}{\tilde{I}} = \frac{1}{sC}$$

That's more interesting. Finally, the inductive relationship states that $V = L\frac{dI}{dt}$. Again, Laplace transform:

$$\tilde{V} = sL\tilde{I}$$

and the impedance of an inductor is:

$$Z_L = \frac{\tilde{V}}{\tilde{I}} = sL$$

Here's a bit more context - think of impedance as the frequency domain "value" of a passive component. Complex resistance, if you will. Resistance is real, so there's no reason impedance is different in the Laplace'd analog of Ohm's law.

Relatedly, for a direct current (DC) circuit, $s = 0$.

Let's take a look at the circuit again. The governing equation, found using Kirchhoff's voltage law, is:

$$V_s - RI - L\frac{dI}{dt} - V_c = 0$$

Using these newfound impedances, we can immediately convert this system to a transfer function. Let's Laplace the shit out of this thing. (Keep in mind that a transfer function is output over input.)

$$\tilde{V_s} - \tilde{I}(Z_R + Z_L + Z_C) = 0$$
$$\tilde{V_s} - \tilde{I}(R + sL + \frac{1}{sC}) = 0$$
$$U = \tilde{V_s} = \frac{\tilde{I}}{R + sL + \frac{1}{sC}} \qquad Y = \tilde{V_c} = \frac{\tilde{I}}{sC}$$
$$\frac{Y}{U} = \frac{\frac{\tilde{I}}{sC}}{\frac{\tilde{I}}{R+sL+\frac{1}{sC}}} = \frac{s^2CL + sCR + 1}{s^2C^2}$$

We can expand this idea to more than just passive components.[13] Impedance is a reliable method for thinking about mechanical systems as well (such as mass-spring systems, systems involving gears, motors, etc.)

The pigeonhole here is that these systems are modeled as linear. The frequency domain does not translate well when we study nonlinear systems; instead, we tend to use more qualitative methods such as drawing phase portraits or bifurcation analysis.

Intermezzo - Discretization

Let's take a look at a first order continuous system.

$$\tau\dot{y} + y = Ku$$

We can discretize this system by using finite differences in lieu of derivatives. At time t_k, we can approximate the derivative $\dot{y}(t_k)$ as follows:

[13] In the electrical networks world, we can translate operational amplifiers (op amps) into impedances as well. Absolutely terrifying.

$$\dot{y}(t_k) \approx \frac{y(t_k) - y(t_{k-1})}{t_k - t_{k-1}}$$

Plugging in, we get:

$$\tau \frac{y(t_k) - y(t_{k-1})}{t_k - t_{k-1}} + y(t_{k-1}) = Ku(t_{k-1})$$

Let's define $\Delta t = t_k - t_{k-1}$ and $\alpha = \frac{\tau}{\Delta t}$ With a bit more algebra, we find that:

$$y(t_k) = \frac{\alpha - 1}{\alpha} y(t_{k-1}) + \frac{K}{\alpha} u(t_{k-1})$$

One more simplification: $\beta = \alpha^{-1}$:

$$y(t_k) = (1 - \beta) y(t_{k-1}) + K\beta u(t_{k-1})$$

Discretization makes it much easier to use computational tools like MATLAB and Python to manipulate a signal without preexisting toolboxes.

5.7 Impulse Response

Now that we've developed all this new technology, let's draw some conclusions.

First, let's circle back to our discussion of the unit impulse. We define the **impulse response** $h(t)$ as a function that describes the behavior of an LTI system when it's stimulated with a unit impulse. The impulse response is the output of the system resulting from this very short, narrow input signal.

Say we smack a damped mass-spring system with an impact hammer, exciting the mass with a force we'll model as the unit impulse $\delta(t)$.

The equation of motion of this system is:

$$m\ddot{x} + b\dot{x} + kx = \delta(t) \qquad \text{or} \qquad \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{1}{m}\delta(t)$$

That's extremely daunting to solve, but we'll use a nifty trick. We separate the equation of motion into two…"stages" - during the hammer thwack and after the hammer thwack. Assume the mass is at rest before we whack it.

For the hammer thwack itself, we integrate the equation of motion twice successively from 0 to time t, where t is immediately after the smack:

$$\int_{0^-}^{t} \left(m\ddot{x} + b\dot{x} + kx = \delta(\tau)\right)\, d\tau$$

$$m\dot{x} + bx + \int_{0^-}^{t} kx\, d\tau = 1 \qquad \longleftrightarrow \qquad m\dot{x} = 1 - bx - \int_{0^-}^{t} kx\, d\tau$$

$$\int_{0^-}^{t} \left(\int_{0^-}^{t} \left(m\ddot{x} + b\dot{x} + kx = \delta(\tau)\right)\, d\tau \right) d\tau$$

$$mx + \int_{0^-}^{t} bx\, d\tau + \int_{0^-}^{t} 1 - \left(\int_{0^-}^{t} kx\, d\tau \right) d\tau = 0 \qquad \longleftrightarrow \qquad mx = \int_{0^-}^{t} 1 - bx - \left(\int_{0^-}^{t} kx\, d\tau \right) d\tau$$

Evaluating these integrals at $t = 0^+$ yields the following two "initial conditions":

$$x(0^+) = 0 \qquad \dot{x}(0^+) = \frac{1}{m}$$

Since after the hammer crack there is no force exciting the system, solving the homogeneous version of the equation of motion with these initial conditions will yield the impulse response.

$$m\ddot{x} + b\dot{x} + kx = 0,\ x(0^+) = 0,\ \dot{x}(0^+) = \frac{1}{m}$$

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) = h(t)$$

You'll get much more familiar with the concept of mechanical impulse in ME301.

We can calculate the response of a system to an arbitrary force of varying magnitude using the impulse response. To calculate the response of a structure to an external force, we use the concept of superposition, which involves breaking down the force into small parts and adding up the response to each part to get the response.

For example, if we have a force that varies over time, we can divide it into small time intervals of length Δt, and at each interval t_k, we can calculate the response of the system to an impulse of force $\Delta F_k = F(t_k)\Delta t$.

To do this, we use the impulse response function $h(t)$, which represents the response of the system to a unit impulse of force applied at time $t = 0$. Then, the response of the system to

the impulse of force ΔF_k applied at time t_k is given by $h(t-t_k)\Delta F_k$, where $h(t-t_k)$ is the impulse response function shifted in time by t_k.

Thus, the response of the system to the external force $F(t)$ is given by the sum of the responses to all the impulses:

$$y(t) = \sum_{k=1}^{n} \Delta F_k h(t-t_k) = \sum_{k=1}^{n} F(t_k) h(t-t_k) \Delta t$$

By using smaller and smaller time intervals Δt, we can make this sum approach an integral:

$$y(t) = \int_0^t F(\tau) h(t-\tau)\, d\tau$$

which you may recognize as the convolution integral. While we used a mechanical framework to develop intuition for this concept, we can generalize further by saying that **if you have the impulse response $h(t)$ of the system, you can determine the response to an arbitrary input $u(t)$** as follows:

$$y(t) = \int_0^t u(\tau) h(t-\tau)\, d\tau$$

Let's close out with one more thought. What if we have an input signal $U(s) = 1$? Then the output signal is just equal to the transfer function $H(s)$. Thus, it's valid to say that **the impulse response of a system is equivalent to the inverse Laplace transform of the transfer function**.[14]

We've established that we have two main representations of a linear system: the state-space representation and a transfer function. What if we want to convert between them?

Well, it's pretty easy in Python/MATLAB - just use a command. What's

It's possible to come up with infinitely many state-space representations of a system, but the transfer function of a system is unique.

[14] This might be more apparent if you think about this outside the Laplace domain, where $y(t) = \delta(t)$.

6 Frequency Response

Preface

Just as the impulse response characterizes all systems in the time domain, the **frequency response** characterizes all systems in the frequency domain. The frequency response is a linear system's steady-state response to sinusoidal inputs.

More broadly, the frequency response describes how a system responds to different frequencies of input signals. Understanding the frequency response is crucial for designing and analyzing systems like filters and amplifiers. It also plays a critical role in understanding the behavior of natural systems such as the human ear, which responds differently to different frequencies of sound waves.

6.1 That Hertz!

So what exactly makes sinusoidal inputs so interesting?

Well, plugging a sinusoidal input into an LTI system yields a sinusoidal response of the same frequency (in the steady-state). However, the response *may* differ in amplitude and phase.

As such, it's common to use the phasor representation of a sinusoid for frequency response techniques, where the magnitude of the complex number is the amplitude of the wave, and the angle of the complex number is the phase. Say our input is $u(t) = A\sin(\omega t)$. We can express this in terms of complex exponentials as follows:

$$u(t) = A\sin(\omega t) = \frac{A}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

Using the convolution integral, let's solve for a potential $y(t)$.

$$
\begin{aligned}
y(t) &= \int_{0^-}^{t} h(\tau)u(t-\tau)\,d\tau \\
&= \frac{A}{2i}\int_{0^-}^{t} h(\tau)\left(e^{i\omega(t-\tau)} - e^{-i\omega(t-\tau)}\right)\,d\tau = \frac{A}{2i}\int_{0^-}^{t} h(\tau)\left(e^{i\omega t}e^{-i\omega\tau} - e^{-i\omega t}e^{i\omega\tau}\right)\,d\tau \\
&= \frac{A}{2i}\int_{0^-}^{t} h(\tau)\left(e^{i\omega t}e^{-i\omega\tau}\right)\,d\tau - \frac{A}{2i}\int_{0^-}^{t} h(\tau)\left(e^{-i\omega t}e^{i\omega\tau}\right)\,d\tau \\
&= \frac{A}{2i}e^{i\omega t}\int_{0^-}^{t} h(\tau)e^{-i\omega\tau}\,d\tau - \frac{A}{2i}e^{-i\omega t}\int_{0^-}^{t} h(\tau)e^{i\omega\tau}\,d\tau \\
&= \frac{A}{2i}\left(e^{i\omega t}H(s) - e^{-i\omega t}H(-s)\right)
\end{aligned}
$$

We define the **frequency response function** $H(i\omega)$ as the transfer function $H(s)$ evaluated at $s = i\omega$.

$$
\left[\frac{A}{2i}\left(e^{i\omega t}H(s) - e^{-i\omega t}H(-s)\right)\right]_{s=i\omega} = \frac{A}{2i}\left(H(i\omega)e^{i\omega t} - H(-i\omega)e^{-i\omega t}\right)
$$

Notice that since the Laplace transform of the impulse response is the transfer function of a system, and the frequency response function $H(i\omega)$ is complex, we can represent it as the product of a magnitude M and an exponential with phase φ:

$$
H(i\omega) = Me^{i\varphi}
$$

where $M = |H(i\omega)|$ and $\varphi = \measuredangle H(i\omega)$. We rewrite $y(t)$ as:

$$
y(t) = \frac{A}{2i}\left(Me^{i\omega t+\varphi} - Me^{-i\omega t-\varphi}\right) = AM\sin(\omega t + \varphi)
$$

Thus, if a system has a transfer function representation of $H(s)$, a sinusoidal input with amplitude A will "become" a sinusoidal output with magnitude AM, shifted by phase angle φ.

Let's be a tad less abstract now. The frequency response is characterized as the magnitude/gain $M(\omega)$ and the phase $\varphi(\omega)$. The magnitude is defined as the ratio of the output wave's amplitude to the input wave's amplitude, and the phase is defined as the offset of the output signal compared to the input signal.

A positive magnitude means that the output wave is scaled up/magnified from the input wave. A negative magnitude means that the output wave is scaled down/contracted from the input

wave. A positive phase means that the output wave leads the input wave, and a negative phase means that the output wave lags from the input wave.[1]

Intermezzo - Another Quick Dive into Complex Analysis

Thinking of complex numbers as vectors is great when trying to establish the notion of a "magnitude" (or more commonly in complex analysis textbooks, the modulus) of a complex number. The **magnitude** $|z|$ of a complex number $z = a + bi$ is defined as:

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

A useful result for this course is that the magnitude of the product/quotient of complex numbers is equal to the product/quotient of the modulus of the complex numbers. For example:

$$z = \frac{3+i}{4+2i} \qquad |z| = \left|\frac{3+i}{4+2i}\right| = \frac{\sqrt{3^2+1^2}}{\sqrt{4^2+2^2}} = \sqrt{\frac{10}{20}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

The "phase" φ (or more commonly in complex analysis textbooks, the argument) of a complex number is the polar angle from the positive real axis to the vector representation of z in the complex plane.

So say we're interested in finding the phase of $z = 3 + i$.

It's clear that the phase φ is equal to $\tan^{-1}\frac{1}{3}$.

(TALK ABOUT PRINCIPLE ARGUMENTS AND DOMAIN RESTRICTIONS)

Another useful result for this course is that the phase of the product of two complex numbers is equal to the sum of the phase of the complex numbers. (Relatedly, the phase of the quotient of two complex numbers is equal to the difference of the phase of the complex numbers.)

Say we're analyzing the system $\dot{y} + 3y = u$. What's the response to the sinusoidal input $u = A\cos(\omega t)$?

The transfer function (and thus, the frequency response function) of the system is trivial to find:

$$H(s) = \frac{1}{s+3} \qquad H(i\omega) = [H(s)]_{s=i\omega} = \frac{1}{i\omega + 3}$$

[1] There are opposite conventions for the sign of phase in controls and vibrations. This is because vibrations engineers usually get things wrong.

$$M = |H(i\omega)| = \left|\frac{1}{i\omega + 3}\right| = \frac{1}{\sqrt{\omega^2 + 9}}$$
$$\varphi = \measuredangle H(i\omega) = -\tan^{-1}\left(\frac{\omega}{3}\right)$$

Note that in the preface (and a few other times) that I specified that the frequency response only shows a system's **steady-state** response. The frequency response is only applicable to the steady-state response of a system, and it does not provide information about the system's transient behavior.

6.2 It's Pronounced Boh-dee

(TALK ABOUT BODE AT SOME POINT)

How about a damped mass-spring system? Here's the equation of motion of a hypothetical system:

$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = \omega_n^2 u$$

To find the transfer function, we take the Laplace transform of both sides (assuming zero initial conditions) and get:

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)Q = \omega_n^2 U$$
$$H = \frac{Q}{U} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(LOOK AT INITIAL RESPONSE AND GO TO PLOTTING WITH PIECEWISE LINEARS)

7 Dynamic Behavior and Linear Systems

Preface

The systems we've been encountering thus far have been of the form:

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x, u) \end{cases}$$

We've been skirting around the fact that the state-space representation of a system also allows for f and h to be nonlinear maps of x and u, since we've been focusing on analysis of linear systems, which uses the following form:

$$\begin{cases} \dot{x} = \mathbf{A}x + \mathbf{B}u \\ y = \mathbf{C}x + \mathbf{D}u \end{cases}$$

Well, we finally got here. First, we'll tackle the case where u is itself a function of x (say...$u = \Psi(x)$). In other words:

$$\begin{cases} \dot{x} = f(x, \Psi(x)) = f^*(x) \\ y = h(x, \Psi(x)) = h^*(x) \end{cases}$$

As usual, the dynamic behavior of the system only really hinges on the state equation, so we'll focus on that.

$$\dot{x} = f(x, \Psi(x)) = f^*(x)$$

Our objective in this chapter is to analyze the behavior of the solutions to this equation without solving it analytically. Understanding how to determine key traits of the system without having to solve the actual differential equation is crucial to understanding how to judge the stability of a system, which is a crucial topic in control theory.

We also brushed over what exactly makes the state-space representation so compelling over the transfer function approach, especially since we spent so much time pouring over the Laplace

transform and the frequency response. It's honestly a bit of a tossup - the transfer function representation of a system ends up being immensely powerful for methods of classical control, but being able to navigate the state space opens the door to useful tools from linear algebra.

We define the **phase space**, also colloquially called the **state space**, the abstract space where the system's states $x_1, x_2, ..., x_n$ are coordinates. For now, we'll focus on two-dimensional LTI systems so things are easier to visualize - the plane with axes x_1 and x_2 is known as the **phase plane**.

These two-dimensional LTI systems - also called **planar dynamical systems** - are sufficient to get a general idea of how qualitative analysis works. The ideas established here hold when we try to analyze dynamical behavior for systems with more state variables.

Steven Strogatz illustrates an excellent example to develop intuition for two-dimensional systems - Romeo and Juliet. I highly recommend reading the section titled "Love Affairs" in his book on nonlinear dynamics and chaos.

> "I'm not Dr. Phil, or Dr. Mark, or Dr. Dirk…so don't come to me with your relationship problems!"
>
> Prof. Luchtenburg

That being said, before we get to two dimensions, we have to touch upon the one-dimensional case. Let's first introduce some ideas in one dimension to beef up your intuition.

7.1 Panta Rei

Yeah, Heraclitus probably didn't say "Panta Rei". No matter, it's catchy, and it's the unofficial motto of this class.

When x only contains one state variable, the following equation is sufficient to describe the dynamic behavior of the system.[1] (with the assumptions established in the preface to this chapter).

$$\dot{x} = f^*(x)$$

Nonlinear system are finicky - they're usually pretty difficult to solve analytically. Alternatively, we analyze them qualitatively, using graphical methods, to map out their dynamic behavior. For the one-dimensional case, we've been doing this since Ma111.

Say we're interested in the following nonlinear system:

[1] Some of you might be wary about the usage of the word "system" to describe one equation. We'll adhere to the definitions established by Steven Strogatz, where the word "system" is used in the context of dynamical systems, not as a collection of more than one equation.

$$\dot{x} = \sin x$$

First, we'll solve this analytically by separating the variables:

$$\frac{dx}{\sin(x)} = dt$$
$$\int \csc(x)dx = \int dt$$
$$t = \ln|\csc(x) + \cot(x)| + c$$

While these results are analytically correct, they're pretty difficult to interpret. For example, provided initial conditions, how would the system behave as time elapses?

So rather than dealing with that…tommyrot…, we'll graphically portray this system as a vector field, graphing $\dot{x}$ against x.

There's a bit to break down here. First, notice the arrows on the x-axis. If $\dot{x} > 0$ (or the value of the sine wave is positive), we draw an arrow pointing in the $+x$ direction. Likewise, if $\dot{x} < 0$ (or the sine wave is negative), we draw an arrow pointing in the $-x$ direction.

At points where $\dot{x} = 0$, there is no flow at all. We call these points **equilibrium points**. When the point is book-ended by arrows pointing towards it, we call it a stable fixed point. When the point is book-ended by arrows pointing *away* from it, we denote it an unstable fixed point.

Looking back at our system, we can conclude what happens when we pick an arbitrary initial condition. For whatever x_0 we pick:

- if $\dot{x} > 0$, x will increase until it asymptotically approaches the next highest fixed point. (Say, $x_0 = 0.7\pi$. x will increase until it approaches the next fixed point - $x = \pi$.)
- if $\dot{x} < 0$, x will decrease until it asymptotically approaches the fixed point directly beneath it. (Say, $x_0 = 0.3\pi$. x will decrease until it approaches the fixed point - $x = 0$.)

We'll cover this idea of stability in more detail later in the course and in ME351, but this is really just meant to give you a head start in understanding the power of ignoring the math for just a second.

7.2 Eigen Eigen Eigen

Eigenvalue's a scary word, and it seemingly comes up everywhere. Let's break it down.

$$\mathbf{A}v = \lambda v$$

is what we call an **eigenproblem**. $\mathbf{A}$ is a square matrix, v is what we call an **eigenvector**, and λ is what we call an **eigenvalue**. All square matrices have eigenvalues and eigenvectors.

So why define them? Well, eigenvectors (often shortened to e-vects) are special. When you multiply a square matrix by one of its eigenvectors, it scales that eigenvector by some constant (the eigenvalue).

We can solve for the eigenvalues explicitly by rearranging the eigenproblem.

$$\begin{aligned}\mathbf{A}v &= \lambda v \\ \mathbf{A}v - \lambda v &= 0 \\ (\mathbf{A} - \lambda\mathbf{I})v &= 0\end{aligned}$$

The zero vector always works as a trivial eigenvector (check it out if you don't believe me) so we'll omit that possibility. We have to find a nonzero v that satisfies this relation. This means that just multiplying both sides of the equation by $(\mathbf{A} - \lambda\mathbf{I})^{-1}$ won't cut it.

Back to the eigenproblem. If we want to solve for a nonzero eigenvector, we want to find the case where the matrix $\mathbf{A} - \lambda\mathbf{I}$ is noninvertible.

Recall that if the determinant of a matrix is 0, the matrix is noninvertible. Given a square matrix $\mathbf{A}$, we define the **characteristic polynomial** of said matrix as $\det(\mathbf{A} - \lambda\mathbf{I})$. Following from this, we define the **characteristic equation** of a matrix $\mathbf{A}$ as the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

Suppose we wanted to find the eigenvalues of a 2×2 matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The eigenvalues of $\mathbf{A}$ can be solved for with the characteristic equation as follows.

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= \det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) \\ &= (a-\lambda)(d-\lambda) - bc = 0\end{aligned}$$

The characteristic equation can alternatively be defined in terms of the trace and determinant of the 2×2 square matrix:[2]

$$\lambda^2 - \text{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$$

and the eigenvalues of the matrix can be defined explicitly as the solutions of said equation.

$$\lambda_{1,2} = \frac{\text{tr}(\mathbf{A}) \pm \sqrt{(\text{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A})}}{2}$$

Great, we're halfway there. An eigenvector $(u_1, u_2)^T$ of the system associated with eigenvalue λ_k will satisfy the following equation:

$$[\det(\mathbf{A} - \lambda\mathbf{I})]_{\lambda=\lambda_k} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let's give this a shot with an example - what are the eigenvalues and eigenvectors of our favorite matrix?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

We plug $\mathbf{A}$ into the characteristic equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)(4-\lambda) - (2)(3) = 0$$

$$\lambda^2 - 5\lambda - 2 = 0$$

The solutions λ_1 and λ_2 are $\mathbf{A}$'s eigenvalues.

$$\lambda_1 = \frac{1}{2}\left(5 + \sqrt{33}\right) \qquad \lambda_2 = \frac{1}{2}\left(5 - \sqrt{33}\right)$$

Now let's find the eigenvectors associated with these eigenvalues.

I'm of course omitting a lot of nuance, but as always, if you want more background on eigenvalues and eigenvectors, take Ma326.

[2] Pretty easily verifiable, give it a go!

7.3 So What About Them Poles?

We hinted earlier that the poles of a system's transfer function are equivalent to the eigenvalues of a system.[3] Let's prove that by converting from the state-space representation to a transfer function. (We'll assume the system is SISO (single-input single-output) because transfer functions are only viable for SISO systems.)[4]

$$\dot{x} = \mathbf{A}x + \mathbf{B}u$$
$$y = \mathbf{C}x + \mathbf{D}u$$

First, we take the Laplace transform of each of these equations.

$$sX = \mathbf{A}X + \mathbf{B}U$$
$$Y = \mathbf{C}X + \mathbf{D}U$$

Given that we're interested in finding the transfer function, we're looking to isolate the expression equivalent to $H = \frac{Y}{U}$. To do this, we'll isolate X in one of these equations and substitute it into the other.

$$sX = \mathbf{A}X + \mathbf{B}U$$
$$(s\mathbf{I} - \mathbf{A})X = \mathbf{B}U$$
$$X = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U$$

Looks good so far. Now, we plug $(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}U$ in for X in the transformed output equation.

$$Y = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U + \mathbf{D}U = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D})U$$
$$H = \frac{Y}{U} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Since the quantity $(s\mathbf{I} - \mathbf{A})^{-1}$ is defined as the ratio of the classical adjoint of $(s\mathbf{I} - \mathbf{A})$ to the determinant of $(s\mathbf{I} - \mathbf{A})$. Thus, the poles of $H(s)$ are exactly the same as the solutions to the equation $\det(s\mathbf{I} - \mathbf{A}) = 0$.

With a simple adjustment to how we examine the eigenproblem, we find that this equation is mathematically equivalent to how we defined the characteristic equation of $\mathbf{A}$, when $s - \lambda$:

[3] Notably, this is NOT the case if the transfer function has pole-zero cancellation or the state-space representation isn't controllable and observable. This is a ME351 caveat though, we should be fine for now.

[4] Numerator is output and denominator is input. There's methods to consider MIMO (multiple-input multiple-output) systems as well using something called a **transfer function matrix**, but that's a story for another day.

$$\mathbf{A}v = \lambda v$$
$$0 = \lambda v - \mathbf{A}v$$
$$0 = (\lambda \mathbf{I} - \mathbf{A})v$$
$$\det(\lambda \mathbf{I} - \mathbf{A})v = 0$$

These definitions of the characteristic equation are functionally the same for our purposes; they differ by a factor of $(-1)^n$, (where n is the row/column length of square matrix A) which has no bearing on the roots of the equation.

This means that the poles of the system is solely dependent on $\mathbf{A}$. (More generally, the stability of the whole system solely hinges on $\mathbf{A}$.)

7.4 State Transition Matrices and the Matrix Exponential

7.5 Linearization

Linearization is an art. Sure, understanding the behavior of nonlinear systems qualitatively is cool and all, but nothing *really* compares to the beauty of an analytical solution.

That being said, as we've established earlier in the chapter, nonlinear systems are difficult to solve. However, if we simplify the problem so that a solution that applies *near* a point of interest suffices, our life becomes much easier.

A notable example of where we've encountered the idea of linearization before is the idea of a small-angle approximation, where:

$$\sin(\theta) \approx \theta \qquad \cos(\theta) \approx 1$$

for small θ's.

First, let's tackle the idea of a one-dimensional linearization (or as we called it in Ma111, a linear approximation). For an arbitrary function $y = f(x)$, if f is differentiable at a point x_0, the tangent line at $x = x_0$ passes through the point $x_0, f(x_0)$. This leads to the point-slope form of the tangent line at that point:

$$y = f'(x_0)(x - x_0) + f(x_0)$$

Observe that so long as $x_0 \approx x$, the linearization y will yield a close approximation to the true value of f at $x = x_0$. As our choice of x shifts further from x_0, the accuracy of our approximation gets lousier and lousier.

When we push forward from one dimension onwards, the path becomes murkier. If our objective is to develop control schemes for nonlinear dynamical systems based on linear models, it isn't really useful to linearize about *any* old point anymore. Rather, we select one of the nonlinear system's equilibrium points to linearize around.[5]

Note that a nonlinear system can have many equilibrium points, and that each can have a different linearization associated with it.

Let's take a look at a general SISO nonlinear system, defined as follows:

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x, u) \end{cases}$$

with an equilibrium point at $x = x_e$ and $u = u_e$. So long as $x - x_e$ and $u - u_e$ are very tiny, we can reliably study the local behavior of the system for (x, u) around the equilibrium points.

We define a new state variable $z = x - x_e$, a new input $v = u - u_e$, and a new output $w = y - h(x_e, u_e)$, and rewrite the nonlinear system in terms of these deviations.

$$\begin{cases} \dot{z} = f(x_e + z, u_e + v) \\ w = h(x_e + z, u_e + v) - h(x_e, u_e) \end{cases}$$

Now, we use the same linear approximations we used in the one-dimensional case:

$$f(x_e + z, u_e + v) \approx \left[\frac{\partial f}{\partial x}\right]_{(x_e, u_e)} z + \left[\frac{\partial f}{\partial u}\right]_{(x_e, u_e)} v + f(x_e, u_e)$$

$$h(x_e + z, u_e + v) \approx \left[\frac{\partial h}{\partial x}\right]_{(x_e, u_e)} z + \left[\frac{\partial h}{\partial u}\right]_{(x_e, u_e)} v + h(x_e, u_e)$$

Since $\dot{x} = f(x_e, u_e) = 0$ and $w = h(x_e + z, u_e + v) - h(x_e, u_e)$ at equilibrium, we can reorganize these equations into the following linear form.

$$\begin{cases} \dot{z} = \left[\frac{\partial f}{\partial x}\right]_{(x_e, u_e)} z + \left[\frac{\partial f}{\partial u}\right]_{(x_e, u_e)} v = Az + Bv \\ w = \left[\frac{\partial h}{\partial x}\right]_{(x_e, u_e)} z + \left[\frac{\partial h}{\partial u}\right]_{(x_e, u_e)} v = Cz + Dv \end{cases}$$

We define the **Jacobian linearization** of a system as:

[5] If this is a bit too hand-wavey for you, there's a theorem called the Hartman–Grobman theorem that posits that the local behavior of a dynamical system near an equilibrium point with $\mathrm{Re}(\lambda) \neq 0$ is functionally equivalent to that of a linearization about that equilibrium point.

$$\begin{cases} \dot{z} = Az + Bv \\ w = Cz + Dv \end{cases}$$

This definition is easily generalizable when there are multiple inputs and multiple outputs - just add vector dashes and bold your matrices![6]

$$\begin{cases} \dot{z} = \mathbf{A}z + \mathbf{B}v \\ w = \mathbf{C}z + \mathbf{D}v \end{cases}$$

7.6 Modal Decomposition

[6] Remind me to come back here and talk about trim at some point.

8 Auxiliary Topics

I don't like having an appendix at the end for these topics. What I'm probably going to do instead is spread them out throughout the book and put an asterisk on "optional sections". Ah well, content first!

8.1 A Quick and Dirty Introduction to Control

8.2 Too Many Blocks!

8.3 The Singular Value Decomposition