System Dynamics

Benjamin Aziel

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Table of contents

Patch Notes/Detritus **3**

1 Course Motivation and Overview **4**

1.1 Preface 4

1.2 Why Model? 4

2 The First Order **6**

Preface 6

2.1 Gradients Make Stuff Flow 6

2.2 The RC Circuit and Final Generalization 9

2.3 Let's Throw in an Input 10

3 Our Second Order of Business **12**

Preface 12

3.1 Mass and Spring 12

3.2 A Quick Dive into Complex Analysis 14

3.3 Damping! 15

3.4 Pole Plots 16

3.5 Leveraging the Discriminant 17

3.6 The Tank, Revisited (Inertia) 18

Patch Notes/Detritus

05.18.23 - Made the jump from a LaTeX document on Github to Quarto at Prof. Luchtenburg's recommendation. It's pretty cool...can embed inline code and whatnot!

1 Course Motivation and Overview

1.1 Preface

The course is called Systems Engineering in the course catalog, but Prof. Luchtenburg calls it System Dynamics instead. It's a more apt name for what the course actually covers; if you look up Systems Engineering you'll find a completely different subject.

The course text is Ogata's *System Dynamics*, which doesn't fit particularly well for the course. Most feedback control textbooks, like Nise, start with a few chapters covering dynamic modeling and response, but they tend not to be very in depth. I can provide PDFs if you can't find them yourselves. Read the syllabus, it's pretty in depth.

Prof. Luchtenburg *should* put his notes up in the MS Teams Class Notebook and record his lectures. Your mileage may vary if you don't bother coming to class, though; the audio quality in the recordings isn't that great and the notes can be less detailed than the lectures themselves.

Most figures here are taken from Prof. Luchtenburg's Class Notebook. When I take figures from another source, I will mention it inline. This document is a living document, which means I'll continue editing it as I see fit. I reserve the right to include material that may not be covered in the course, but better prepares you for concepts that may seem like they come out-of-the-blue otherwise.

I have this lousy habit of writing in the fourth person and cursing in my writing. Apologies in advance.

1.2 Why Model?

To start off, let's define a **model**. A mathematical model of a system is a set of differential equations that allows us to predict how a system behaves under different conditions. By creating a model, we can use equations and principles to describe a system's present behavior, identify key parameters that affect the system, and make educated guesses as to how the system will behave in the future.

You may have covered some rudimentary modeling in Ma111 or Ma240, where it was probably crammed in to satisfy those annoying kids that go "wHEN ARe we ever gOINg to usE tHIS iN ReAl LIfE". The methods we'll cover in this course will be more robust than simply going off a given formula.

Something we'll exploit heavily a lot in this course is the principle of **analogical models**, or generic representations of common physical phenomena. This turns out to be really useful because it provides a convenient and consistent way to represent and analyze complex systems that involve different types of "stuff". Say you're faced with a fluid flow system, and you haven't taken a fluid mechanics course yet. Using methods introduced in this course, we will be able to convert this system into something equivalent, yet more familiar, like a mass-spring system or a series circuit. Using these analogies allows us to apply the same concepts and mathematical tools to different types of systems, which can greatly simplify the analysis and design process.

To conclude, modeling is a powerful tool that allows us to understand, predict, and (as we'll see in ME351 next semester) **control** the behavior of physical systems, and it has many practical applications in engineering, physics, and other fields.

Don't suck at it.

2 The First Order

Preface

We'll grow more accustomed to the idea of analogous models after modeling a few simpler systems. Let's start by throwing out some fundamental systems and developing intuition to dive into simplified models.

- Emptying a water tank
- Cooling of a lightbulb
- Discharge of an RC circuit

2.1 Gradients Make Stuff Flow

A cylindrical tank is filled to a level h, has a cross-sectional area of A, and an outflow rate of Q_{out}. The pressure outside the tank is P_∞. Can we derive a governing equation for this system? Well, we can try with a few physical principles.

Let's start with a **conservation law**. We know there's a volume V of water proportional to the value of h. Or…

$$\Delta V = A\Delta h$$

We'll take the time derivative of that equation to get some more familiar variables. (You might recognize the math here from related rates in Ma111.)

$$\frac{d}{dt}\left(\Delta V = A\Delta h\right) = -Q_{\text{out}}$$

That's not very useful yet. Let's leverage some prior circuits knowledge here…charge moves because of a **voltage difference** ΔV, and comparably, fluid moves because of a **pressure difference**. Ohm's law! We'll come back to that, but the main takeaway here is that the outflow Q_{out} is related to the difference between the pressure inside the tank P and the atmospheric pressure P_∞.

That circuits analogy comes in handy really often, because it turns out Ohm's law translates directly into fluid flow.

$$\Delta V = V - V_0 = IR$$
$$\Delta P = P - P_\infty = Q_{\text{out}} R$$

These are called **constitutive equations**, or relationships between physical quantities. (The flow is proportional to a level difference, or gradient.)

We'll generalize a bit soon, but for now I understand if you don't get it. It's still very hand-wavey.

Let's leverage some hydrostatics now. The tank is open to the atmosphere at the top, so we can actually derive an expression for P, the pressure at the base of the tank. We'll also define ρ, the density of the fluid, and C, or capacitance, as $\frac{A}{\rho g}$, because it ends up being useful in the circuit analogy.[1]

$$\Delta P = \rho g \Delta h = \rho g \frac{\Delta V}{A} = \frac{1}{C} \Delta V$$
$$C = \frac{A}{\rho g}$$

OK, I think we're all set. I've been pretty lax with the "delta's", but it should still be readable. Let me know if things need clarification.

$$\dot{V} = -Q_{\text{out}}$$
$$C\dot{P} = -\frac{\Delta P}{R}$$
$$\boxed{RC\dot{P} + \Delta P = 0}$$

This is a really nice differential equation. It looks like the equation for an RC circuit if you've seen those before, with the voltage differentials swapped out for pressure differentials.

Let's move onto a second example: the cooling of a lightbulb. When we shut off power to the lightbulb, how can we measure its temperature as it cools to room temperature?

The bulb is initially very hot (with temperature T) compared to its environment (which has temperature T_∞). Heat is flowing outwards at $\dot{q}_{\text{out}}$. [2] This is seeming very familiar…a temperature difference is driving heat to leave through the resistance R of the bulb.

[1] You'll see this technique leveraged again in ME342.

[2] I'm not a fan of the usual notation here, so I'm using $\dot{q}$ for the flow of heat and q for heat accumulated.

Let's go through the steps again. What's being conserved here?[3] Internal energy! (Or heat, since there's no work in this system.) It might be a bit early in the semester to have seen the capacitive relationship relating heat q and temperature T in ESC330, but here it is:

$$\Delta q = C\Delta T$$

Differentiate across the board…

$$\frac{d}{dt}(C\Delta T) = C\dot{T} = -\dot{q}_{\text{out}}$$

And now we're just chugging through the motions. Next is another constitutive relationship (which looks shudderingly close to Ohm's law!):

$$\Delta T = T - T_\infty = \dot{q}_{\text{out}}R$$

Using this and the conservation equation, we construct:

$$\boxed{RC\dot{T} + \Delta T = 0}$$

Again. Familiar. Very familiar. Maybe there's some unifying theory in the background here. We'll generalize that equation to what we call its **canonical form** $\tau\dot{y} + y = 0$, a first order differential equation. Let's throw in an initial condition $y(0) = y_0$ just so we don't have any undetermined constants at the end.

To solve this differential equation, we'll guess a solution $y(t) = ce^{\alpha t}$, find its time derivative $\dot{y}(t) = \alpha ce^{\alpha t} = \alpha y$, and plug in.

$$\begin{aligned}
\tau\dot{y} + y &= 0 \\
\tau\alpha e^{\alpha t} + e^{\alpha t} &= 0 \\
(\tau\alpha + 1)\, e^{\alpha t} &= 0 \\
\alpha &= -\frac{1}{\tau} \\
y(t) = ce^{-\frac{t}{\tau}} &= y_0\, e^{-\frac{t}{\tau}}
\end{aligned}$$

If you look at the graph below, it's just exponential decay from $(0, y_0)$. We call τ the **time constant** of the system, and it's commonly used to describe how quickly an exponential decays or grows. Different systems have different time constants. (Notably, RC always has units of

[3] Someone said kinetic energy. What a statistical mechanics-esque answer.

time). The smaller the time constant, the faster the decay. (Assume for the figure below that $\tau_1 = 10$ and $\tau_2 = 5$.)

So what happens when we set $t = \tau$? Let's plug it in and find out.

$$y(\tau) = y_0 e^{-\frac{\tau}{\tau}} = y_0 e^{-1}$$

So the time constant is the time at which the system response has decayed to $y_0 e^{-1}$, or approximately 37% of its initial value. We can also reframe this definition as, "the time constant is the time at which the system response has lost approximately 63% of its initial value".

2.2 The RC Circuit and Final Generalization

Say we have an RC circuit with a full capacitor.

The outflow of charge from the capacitor is represented as a negative current:

$$\dot{q} = -I_{\text{out}}$$

Here's Ohm's law:

$$\Delta V = I_{\text{out}} R = -\dot{q} R$$
$$\dot{q} = -\frac{\Delta V}{R}$$

Finally, we deal in the capacitive relationship (from Ph213):

$$q = C \Delta V$$

We differentiate the capacitive relationship in order to set these equations equal to each other and get:

$$C\dot{V} = -\frac{\Delta V}{R}$$
$$\boxed{RC\dot{V} + \Delta V = 0}$$

which is the same equation we've gotten before. (Notably, we don't have to have this equation in terms of the voltage difference; as you'll see in ESC221 this semester, there's a form of the equation in terms of current as well.)

Final takeaways: * Most first order systems we'll analyze in this class are the same mathematically! * **Level differences (gradients) make stuff flow.**

"Stuff" isn't the greatest word for something like this, (maybe quantity or 'energy' instead?) but that's the best we have. Stuff can be stored, like charge in a capacitor, or fluid in a tank, or heat in a reservoir. However, by generalizing these quantities, we can create widely applicable rules for modeling first order systems.

$$\text{Stuff} = \text{Capacitance} \times \text{Level Difference}$$

$$\text{Level Difference} = \text{Flow of Stuff} \times \text{Resistance}$$

Also conservation. That's a biggie.

$$\text{Rate of Change of Stuff} = \text{Inflow} - \text{Outflow}$$

We've only discussed scenarios where there isn't anything flowing in thus far. In these cases, to solve nonhomogeneous differential equations, we'll have to use more specialized methods from Ma240 instead of guessing and praying, like the Laplace transform or the method of undetermined coefficients (or as I affectionately call it, MUC).

2.3 Let's Throw in an Input

A more simple form of the governing equation for one of these first order systems is:

$$\tau \dot{y} + y = ku$$

when we have a constant input. Think of it as turning on a light switch at time $t = 0$. k is just a scale factor, and $u = u_s(t)$, where u_s is the unit step function, which is just 0 when $t < 0$ and 1 when $t > 0$.[4]

$$y(t) = ce^{-\frac{t}{\tau}} + k$$

For the initial condition $y(0) = y_0$, the undetermined coefficient $c = y_0 - k$. Here's our updated solution:

$$y(t) = y_0 e^{-\frac{t}{\tau}} + k(1 - e^{-\frac{t}{\tau}})$$

When we graph this function for $y_0 = 0$, we see that it gradually grows towards $y = k$ as $t \to \infty$. Now we can analyze exponential growth. You see this behavior everywhere, like when you change a thermostat setting and the temperature slowly creeps towards your choice. This is what we call a **step response**.

[4] We don't care about what happens at $t = 0$. Stop it.

How could we find the time constant of this response? Let's take a look at what happens to $y(t)$ at $t = \tau$.

$$y(\tau) = y_0 e^{-\frac{\tau}{\tau}} + k(1 - e^{-\frac{\tau}{\tau}}) = y_0 e^{-1} + k(1 - e^{-1})$$

When we set $y_0 = 0$, this further simplifies to:

$$y(\tau) = k(1 - e^{-1}) \approx 0.63k$$

For this system, at $t = \tau$, the system response will have accumulated 63% of its steady state value.

The step input is just one of the test inputs we usually use; we'll look at a few more as the course progresses (such as sinusoidal waves, delta functions, etc.).

3 Our Second Order of Business

Preface

First order systems are honestly pretty boring. When we put in a step input, we just get a pure exponential. We won't be able to get more interesting behavior, like oscillation, because that's just mathematically impossible.[1]

Second order systems, on the other hand, *can* oscillate by themselves. Try to convince yourself of this mathematically just based on what oscillation is.

Mass-spring systems are pretty good models of everything in the world (as long as you use enough mass-spring systems). They're really nice because having a good understanding of ONE mass-spring system provides us with the intuition for more complicated systems.

3.1 Mass and Spring

Say we have a mass-spring system where a mass m is attached to a wall with a spring k and a damper b. Gravity isn't "turned on", so if you want to visualize the system, that mass is floating. The equation of motion for a positive displacement q is:[2]

$$m\ddot{q} = -kq - b\dot{q}$$

Or in its more familiar form:

$$m\ddot{q} + b\dot{q} + kq = 0$$

This is the famed mass-spring equation. Say we have an input - a force u acting on the mass in the positive direction. Now our equation of motion is:

$$m\ddot{q} + b\dot{q} + kq = u$$

[1]Prove it!

[2]Prof. Luchtenburg went off on a tangent about Hooke being a genius for realizing that spring motion is linear near the origin here. That was pretty funny.

This is a linear differential equation, so we'll solve this by plugging in an educated guess. Let's try $q(t) = Ae^{st}$, because differentiating this function n times just multiplies it by s^n.

First, we'll solve the homogeneous equation, or the case where $u = 0$.[3]

$$\begin{gathered} m\ddot{q} + b\dot{q} + kq = 0 \\ ms^2Ae^{st} + bsAe^{st} + kAe^{st} = 0 \\ ms^2 + bs + k = 0 \end{gathered}$$

This is known as the characteristic (or auxiliary) equation. We can now use algebra to solve for the roots of the equation, or by proxy, the solution of the differential equation.

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

These are also called the **poles** of the system, but we're getting ahead of ourselves. Let's simplify further.

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

We define the **natural frequency** $\omega_n = \sqrt{\frac{k}{m}}$ and the **damping ratio** $\zeta = \frac{b}{2m\omega_n}$. Using these definitions, we can reorganize the mass-spring equation in terms of these variables.

$$m\ddot{q} + b\dot{q} + kq = 0 \qquad \rightarrow \qquad \ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = 0$$

And the poles of this equation are:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

[3]If you're curious why we do this, you can read up on it in a linear algebra textbook. Think it's theorem 3.9 in Friedberg's Linear Algebra.

3.2 A Quick Dive into Complex Analysis

Complex analysis is the study of functions of a complex variable z, where z has a real component a and an imaginary component b. Complex numbers show up all the time in this course, whenever anything oscillates, really (like mass-spring systems or pendulums).

To take our first plunge into complex analysis, we need to define the **imaginary unit** i.[4] For now, let's define i as one of the two solutions to the quadratic equation $x^2 = -1$. (The other is $-i$, of course.) This is really special, because now we can describe the solutions of ALL polynomials[5] as the sum of a real number a and another real number b multiplied by the imaginary unit i.

Let's conjure up a graphical representation of these numbers using Cartesian coordinates, where we define one axis as "real" and the other as "imaginary". We'll call this the complex plane. An arbitrary point $z = a + ib$ is plotted below.

We can also interpret these numbers in the context of polar coordinates, where θ is the angle between the vector from the origin to z and the real axis, and r is the magnitude of the aforementioned vector. It's not difficult to translate between Cartesian coordinates and polar coordinates, but I'll dump the formulas here anyway.

$$a = r\cos\theta \qquad b = r\sin\theta \qquad r = \sqrt{a^2 + b^2} \qquad \theta = \arctan\left(\frac{b}{a}\right)$$

It'd be criminal to not mention Euler's formula, which posits that: [6]

$$e^{i\theta} = \cos\theta + i\sin\theta$$

A lot of the nuance of complex numbers is best understood by analyzing this **complex exponential**, especially because:

$$z = a + ib = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

[4]You'll see people, especially electrical engineers, use j instead, because i is commonly used for current. We're better than them.

[5]Regardless if its coefficients are real or complex!

[6]You can prove this using the Taylor series representation of e^x. You should do it, it's *very* rewarding

3.3 Damping!

When $\zeta = 0$ (or the system is **undamped**), we have the poles $s_{1,2} = \pm i\omega_n$. This implies that our solution is a linear combination of sines and cosines, endlessly oscillating, forever and ever. (That's kind of depressing to be honest.)

> What next? Euler guy.
>
> Prof. Luchtenburg

$$\begin{aligned} x(t) = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t} &= A_1(\cos(\omega_n t) + i\sin(\omega_n t)) + A_2(\cos(\omega_n t) - i\sin(\omega_n t)) \\ &= (A_1 + A_2)\cos(\omega_n t) + i(A_1 - A_2)\sin(\omega_n t) = C_1\cos(\omega_n t) + C_2\sin(\omega_n t) \end{aligned}$$

This reasoning carries over if we pick a damping ratio ζ between 0 and 1 (or the system is **underdamped**). The poles are:

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{\omega_n^2(\zeta^2 - 1)} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} = \sigma \pm i\omega_d$$

We define the **damped frequency** $\omega_d = \omega_n\sqrt{1-\zeta^2}$. (In practice, $\omega_d \approx \omega_n$, because $\zeta \ll 1$.) Additionally, we define $\sigma = \zeta\omega_n$ (for some reason). After substituting these new variables in, our solution becomes:

$$x(t) = A_1 e^{(-\sigma + i\omega_d)t} + A_2 e^{(-\sigma - i\omega_d)t} = e^{-\sigma t}(C_1\cos(\omega_n t) + C_2\sin(\omega_n t))$$

In the lattermost form, it's obvious that σ in fact *does* have a use other than bookkeeping; it defines the exponential **envelope** by which the oscillation decays. An envelope is a function that outlines how a function grows/decays (it's the red dashed line in the figure below).

Notably, the time constant τ of the envelope is equal to $1/\sigma$. Thus, we can eyeball the value of σ based on how we'd find the time constant (the value 63% less than the y-intercept of the envelope).

$$x(t) = e^{-\sigma t}\sin(\omega_d t + \varphi)$$

Most mechanical systems tend to have a very low damping ratio ($\zeta \simeq O(0.1)$)[7], and as mentioned before, a good rule of thumb is that $\omega_n = \omega_d$.

When we apply some initial conditions (like $q(0) = q_0$ and $\dot{q}(0) = v_0$), our solution becomes:

$$q(t) = e^{\sigma t}\left(q_0\cos(\omega_d t) + \frac{\sigma q_0 + v_0}{\omega_d}\sin(\omega_d t)\right)$$

[7] Related rates of growth from Ma111, or big-O notation if you've taken ECE264.

When $\zeta = 1$, or the system is **critically damped**, the poles are:

$$s_{1,2} = -\zeta\omega_n$$

We use a trick from differential equations to fake another linearly independent solution, just chuck on an extra t.

$$q(t) = A_1 e^{-\zeta\omega_n t} + A_2 t e^{-\zeta\omega_n t}$$

This doesn't really happen in the real world, but it's nice to cover all our bases. How about when $\zeta > 1$? We call this case **overdamped**, because our poles are:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Both poles are negative here! Our solution is:

$$q(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

3.4 Pole Plots

I've been emphasizing poles a lot in the last few pages, but why? What's the importance of these seemingly arbitrary values? In fact, we can infer the dynamics of a system based on its poles.[8]

To analyze poles, we use a graphical tool called a **pole plot**, the plot of the roots of the characteristic equation on the complex plane. Let's go down the list:

- When both poles are on the imaginary axis, the system is undamped.
- When both poles are off the real axis, the system oscillates. If they're to the left of the imaginary axis, it'll decay exponentially (underdamped), and if they're to the right of the imaginary axis, it'll grow exponentially.
- If both poles are on the real axis to the left of the imaginary axis, it's overdamped.
- Rule of thumb: if there is ANY pole to the right of the imaginary axis, the response blows up.

Here's a nice chart from FPE. Complex conjugates are omitted for simplicity.

[8] This is the crux of a lot we do in ME351. If any of you remember what an eigenvalue is from Ma110, that'll come in handy in a bit.

3.5 Leveraging the Discriminant

I'm going to stray off from Prof. Luchtenburg for a second because *I* think this is useful.

Suppose we want to identify the dampedness of a system based on the equation of motion rather than solving for ζ. We can do exactly this using the **discriminant** of the characteristic equation of the system. Let's take a look at the canonical mass-spring system with a damper once again.

The equation of motion, assuming free motion, is:

$$m\ddot{q} + b\dot{q} + kq = 0$$

The characteristic equation is derived after plugging in $q = e^{st}$.

$$ms^2 + bs + k = 0$$

As stated in the section on damping, there are three forms of the general solution if there is damping present: both poles are real and distinct (the system is overdamped), both poles are real and equal (the system is critically damped), or both poles are complex conjugates (the system is underdamped). You may recall from Algebra that the discriminant of a polynomial can reveal some properties of the roots without actually computing them. The discriminant of a quadratic is defined as follows:

$$\text{Disc}(ax^2 + bx + c) = b^2 - 4ac$$

This is the argument of the square root in the quadratic formula. If this expression is positive, the solutions to the quadratic are real and distinct. If this expression is 0, then there is only one real solution to the quadratic. If this expression is negative, the solutions to the quadratic are complex. So physically, finding the discriminant of the characteristic equation of the mass-spring system will tell us how damped it is.

$$\text{Disc}(ms^2 + bs + k) = b^2 - 4mk$$

$$\begin{aligned}
b^2 - 4mk > 0 \quad &\rightarrow \quad \text{overdamped} \\
b^2 - 4mk = 0 \quad &\rightarrow \quad \text{critically damped} \\
b^2 - 4mk < 0 \quad &\rightarrow \quad \text{underdamped}
\end{aligned}$$

And of course, if $b = 0$, then the system is undamped.

3.6 The Tank, Revisited (Inertia)

Let's revisit the tank from our study of first order systems. However, we'll make one small change: the outflow pipe now has a defined length of L_p.

Let's model the same way we've been doing thus far. First, a conservation law:

$$\dot{V} = -Q$$

Next, "Ohm's law":

$$Q = \frac{\Delta P}{R} = \frac{P - P_{\text{atm}}}{R}$$

If we isolate a piece of the pipe (with length L_p) as shown below, we can demystify this system a bit.

Assuming the cross-sectional area of the pipe A_p is constant, the pressure force $F_p = A_p \Delta P$ accelerates the fluid between the two ends of this pipe. Additionally, there is friction $F_f = -QRA_p$ on the liquid caused by the resistance of the pipe. By leveraging Newton's second law of motion, we now have a relationship between the pressure difference ΔP and the velocity of the water v.

$$m\frac{dv}{dt} = F_p + F_f = A_p \Delta P - QRA_p$$

The mass m of the fluid between the two ends of this pipe is equal to the product of the density of the fluid ρ and the volume between the two ends V_p. (Notably, the volume $V_p = A_p L_p$.)

$$\rho V_p \frac{dv}{dt} = \rho A_p L_p \frac{dv}{dt} = A_p \Delta P - RQA_p$$

Because the product of the cross-sectional area A_p and the fluid velocity v is equal to the volumetric flow rate Q, we can rewrite this equation as follows:

$$\frac{\rho L_p}{A_p} \frac{dQ}{dt} = \Delta P - RQ$$

Ok, we can shed some light on what we're doing now. We define **inductance** (also referred to as **liquid-flow inertance** or **inertia**) as a term that describes the change in potential required for a unit rate of fluid flow. Inductance is the tendency of the fluid to move; it's created by the inertia of water flowing through the pipe. The mathematical definition of inductance is as follows:

$$L = \frac{\rho L_p}{A_p}$$

Note that this definition of inductance is only valid for flow systems, but analogous concepts occur in other fields (like inductors from circuit analysis)! Fluid components that have an inductance are analogous to these inductors, or mechanical components with inertia.

Let's wrap up this example. When we plug in the definition of L into our equation, a simple first order system rears its head.[9]

$$L \frac{dQ}{dt} + RQ = \Delta P$$

Let's throw it into canonical form so we can see its time constant.

$$\left(\frac{L}{R}\right)\dot{Q} + Q = \frac{\Delta P}{R} \qquad \tau = \frac{L}{R}$$

To summarize, we've added a new tool to our arsenal: conservation of momentum (or Newton's second law).

$$\begin{aligned} \Delta P &= L\dot{Q} + RQ \\ \dot{V} &= -Q \\ C\Delta P &= V \end{aligned}$$

By combining these three equations, we can use tools from our studies of mass-spring systems to analyze... well... any second order system.

$$\begin{aligned} L\Delta\ddot{P} + R\Delta\dot{P} + \frac{1}{C}\Delta P = 0 \qquad &\rightarrow \qquad \Delta\ddot{P} + \left(\frac{R}{L}\right)\Delta\dot{P} + \left(\frac{1}{LC}\right)\Delta P = 0 \\ m\ddot{q} + b\dot{q} + kq = 0 \qquad &\rightarrow \qquad \ddot{q} + \left(\frac{b}{m}\right)\dot{q} + \left(\frac{k}{m}\right)q = 0 \end{aligned}$$

We simply retrofit the definitions of the natural frequency ω_n and damping ratio ζ based on how we defined them for mass-spring systems to determine how the oscillations behave. Here's a quick example using the flow system analogy we've been using thus far:

$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = 0 \quad \longleftrightarrow \quad \Delta\ddot{P} + \left(\frac{R}{L}\right)\Delta\dot{P} + \left(\frac{1}{LC}\right)\Delta P = 0$$

[9]The analog of this system in circuit analysis is called the RL circuit, which is often used as a passive filter.

$$
\begin{aligned}
2\zeta\omega_n &= \frac{R}{L} \qquad \longrightarrow \qquad \zeta = \frac{R}{2L\omega_n} = \frac{R\sqrt{LC}}{2L} = \frac{R}{2}\sqrt{\frac{C}{L}} \\
\omega_n^2 &= \frac{1}{LC} \qquad \longrightarrow \qquad \omega_n = \frac{1}{\sqrt{LC}} = \frac{\sqrt{LC}}{LC}
\end{aligned}
$$