RECIPROCAL FUNCTIONS – INTRO

Recall: State the reciprocal of the following.

a) 5 b) $\frac{1}{35}$ c) -12 d) 4000 e) $\frac{-1}{750}$

Note the following properties:

1. The reciprocal of 0 is ____________________.
2. The reciprocal of undefined (ie _____) is _______________.
3. The reciprocal of a positive number is a ____________________ number.
4. The reciprocal of 1 is _____.
5. The reciprocal of positive number very close to 0 (ex. __________) is a __________ __________ ____________________ number (ie _____).
6. The reciprocal of a very big positive number (ie _____) is _____.
7. The reciprocal of a negative number is a ____________________ number.
8. The reciprocal of -1 is _____.
9. The reciprocal of a negative number very close to 0 (ex. __________) is a __________ __________ ____________________ number (ie _____).
10. The reciprocal of a very big negative number (ie _____) is _____.

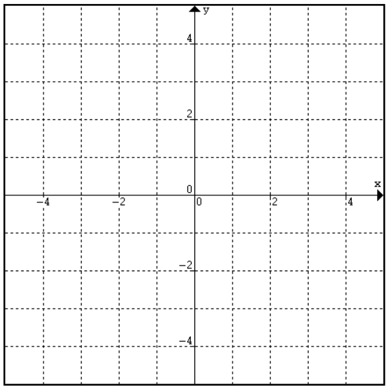
Ex. 1 Given the following functions, write the equations of the corresponding reciprocal functions.

a) $y = x$ c) $y = x^2$ e) $y = \sqrt{x+2}$

b) $y = 3x - 7$ d) $y = |x|$ f) $y = (x+1)^3$

Ex. 2

a) $y = 4$



The reciprocal graph of $y = 4$ is ___________.

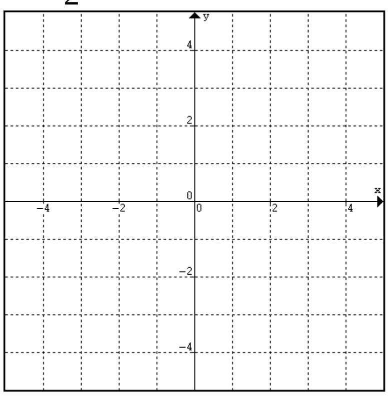
Similarly the reciprocal graph of $y = 100$ is

___________.

Note: The larger the y-value, the

________________ the reciprocal is to the x-axis.

b) $y = \frac{1}{2}$



The reciprocal graph of $y = \frac{1}{2}$ is ___________.

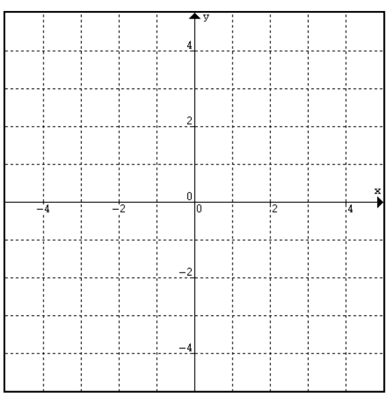
Similarly the reciprocal graph of $y = \frac{1}{50}$ is

___________.

Note: The smaller the y-value, the

________________ the reciprocal is to the x-axis.

c) $y = -8$



The reciprocal graph of $y = -8$ is ___________.

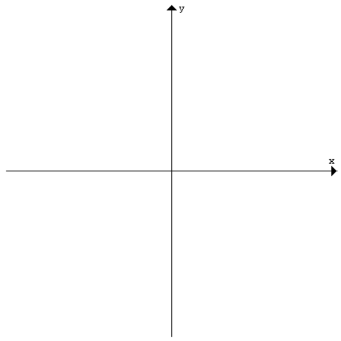
Similarly the reciprocal graph of $y = -75$ is

___________.

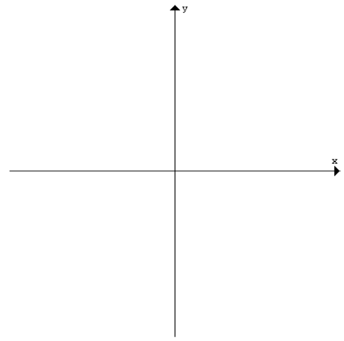
Note: The farther a y-value is from the x-axis, the ________________ the reciprocal is to the x-axis and vice versa.

Ex. 3 Sketch the following functions and their reciprocals.

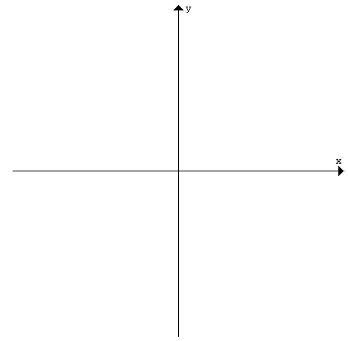
a) $y = x + 3$



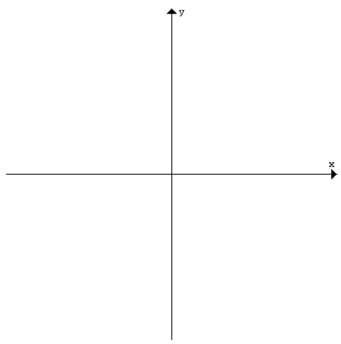
b) $y = -x - 3$



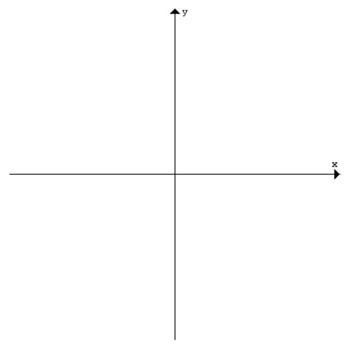
c) $y = 2x + 4$



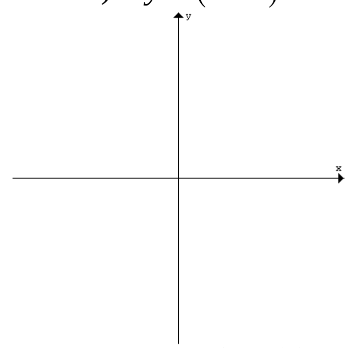
d) $y = -2x + 4$



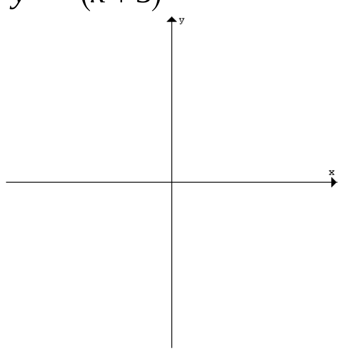
e) $y = x^2$



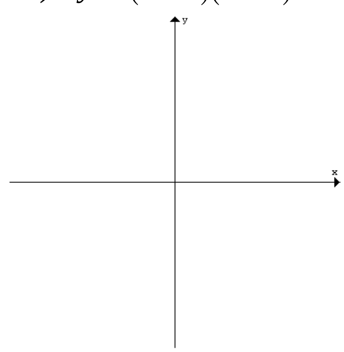
f) $y = (x-2)^2$



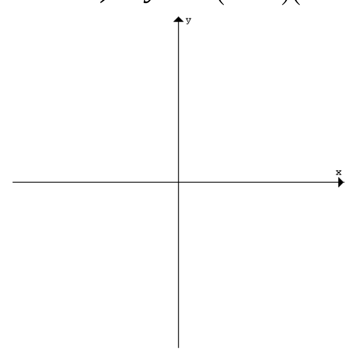
g) $y = -(x+3)^2$



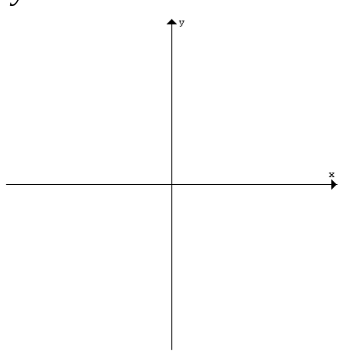
h) $y = (x-6)(x+2)$



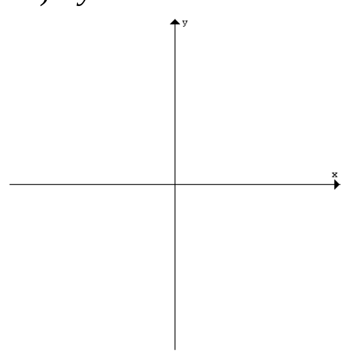
i) $y = -(x-1)(x-4)$



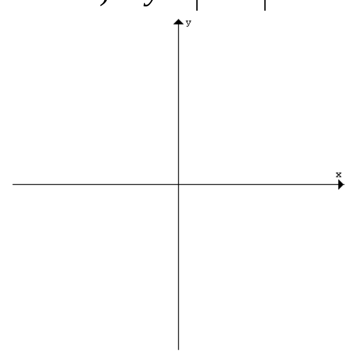
j) $y = 2x^2 + 4x + 4$



k) $y = -x^2 + 4x - 7$



l) $y = |x+2| - 2$



When the original function is above the x-axis, the reciprocal function is ________________ the x-axis.
When the original function is below the x-axis, the reciprocal function is ________________ the x-axis.

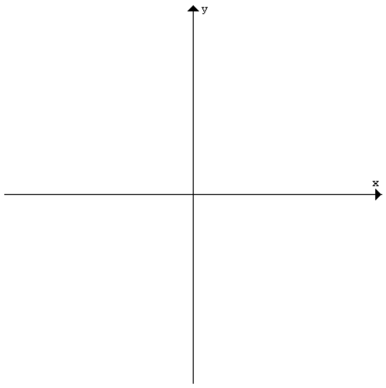
When the original function is increasing, the reciprocal function is ________________.
When the original function is decreasing, the reciprocal function is ________________.

The zeros of the original function become ______________________ ______________________ in the reciprocal function.

RECIPROCAL OF A LINEAR FUNCTION

Consider $f(x) = x - 3$ and $g(x) = \frac{1}{f(x)} = \frac{1}{x-3}$

a) Graph $f(x)$ and $g(x)$.



b) What is the x-intercept of $g(x)$?

c) What is the y-intercept of $g(x)$?

d) Determine the end behaviour of $g(x)$.

e) Determine the equation of the horizontal asymptote of $g(x)$.

f) Determine the equation of vertical asymptote of $g(x)$.

g) Determine the behaviour of $g(x)$ near the vertical asymptote.

h) Determine the positive/negative intervals.

i) Determine when the function is increasing and decreasing.

Ex. 2 Given $f(x) = \frac{3}{2x+4}$.

a) What is the x-intercept?

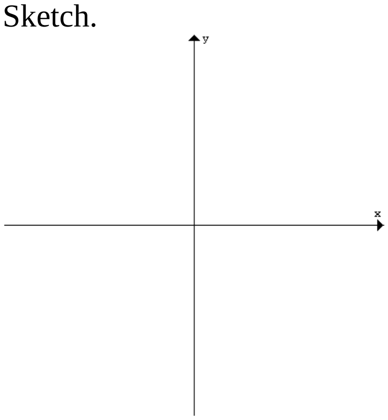
b) What is the y-intercept?

c) Determine the end behaviour.

d) Determine the equation(s) of all asymptote(s).

e) Determine the behaviour near the vertical asymptote.

f) Sketch.



g) Determine when the function is above and below the x-axis.

h) Determine the intervals during which the function is increasing and decreasing.

RECIPROCAL OF A QUADRATIC FUNCTION

Consider $f(x)=x^2+2x+1$ and $g(x)=\dfrac{1}{x^2+2x+1}$

a) What is the x-intercept of $g(x)$?

b) What is the y-intercept of $g(x)$?

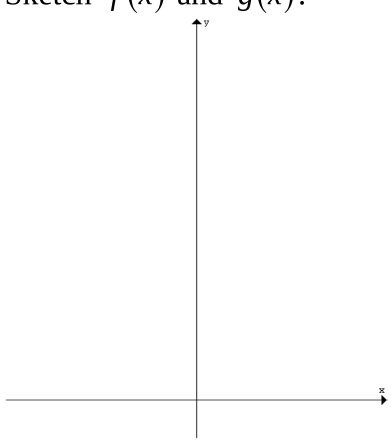
c) Find the equation of the vertical asymptote of $g(x)$.

d) Determine the behaviour of $g(x)$ near the vertical asymptote.

e) Determine the end behaviour of $g(x)$.

f) Find the equation of the horizontal asymptote of $g(x)$.

g) Sketch $f(x)$ and $g(x)$.



h) Determine the positive/negative intervals.

i) Determine when the function is increasing and decreasing.

j) What is the domain and range?

Consider $f(x) = x^2 + 4$ and $g(x) = \dfrac{1}{x^2 + 4}$

a) What is the x-intercept of $g(x)$?

b) What is the y-intercept of $g(x)$?

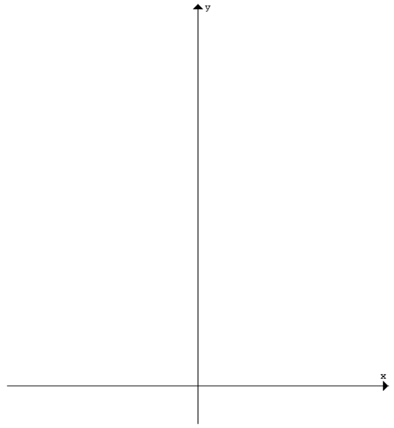
c) Find the equation of the vertical asymptote of $g(x)$.

d) Determine the behaviour of $g(x)$ near the vertical asymptote.

e) Determine the end behaviour of $g(x)$.

f) Find the equation of the horizontal asymptote of $g(x)$.

g) Sketch $f(x)$ and $g(x)$.



h) Determine the positive/negative intervals.

i) Determine when the function is increasing and decreasing.

j) What is the domain and range?

Consider $f(x) = x^2 - 4x - 5$ and $g(x) = \dfrac{1}{x^2 - 4x - 5}$

a) What is the x-intercept of $g(x)$?

b) What is the y-intercept of $g(x)$?

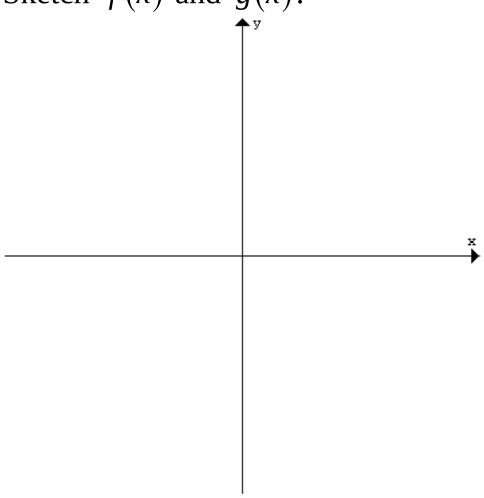
c) Find the equation of the vertical asymptote of $g(x)$.

d) Determine the behaviour of $g(x)$ near the vertical asymptote.

e) Determine the end behaviour of $g(x)$.

f) Find the equation of the horizontal asymptote of $g(x)$.

g) Sketch $f(x)$ and $g(x)$.



h) Determine the positive/negative intervals.

i) Determine when the function is increasing and decreasing.

j) What is the domain and range?

HOLES

A rational function has the form $h(x)=\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are both polynomials. To find the x-intercepts, set $h(x)=0$ and solve, which is the same as setting $f(x)=0$ and solving. To find the equation(s) of the vertical asymptote(s), set $g(x)=0$ and solve. What happens if some value of x causes both the numerator and denominator to equal zero (ie $h(x)$ is indeterminate)?

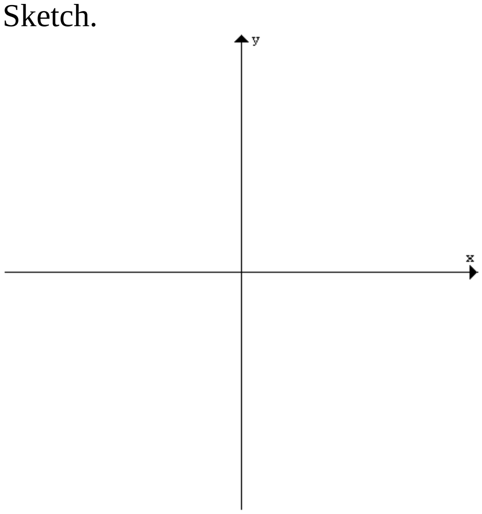
Consider $f(x)=\frac{x-2}{x^2-4}$

a) Simplify $f(x)$.

b) Determine the x-intercept(s).

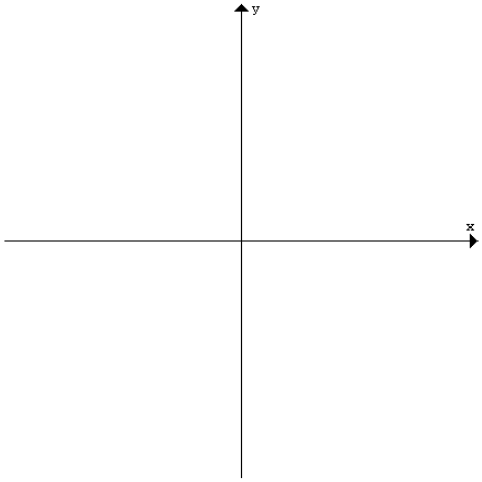
c) Determine the equation(s) of the vertical asymptote(s).

d) Sketch.



In general, if a value of x causes $h(x)$ to be indeterminate, the graph will have a hole at that value of x.

Ex. 1 Sketch: $f(x) = \dfrac{2x+2}{x^3+2x^2-x-2}$



RATIONAL FUNCTIONS IN THE FORM $f(x)=\dfrac{ax+b}{cx+d}$

Consider $f(x)=\dfrac{6x-11}{3x-6}$

a) Re-write in the form $f(x)=\dfrac{R}{D}+Q$.

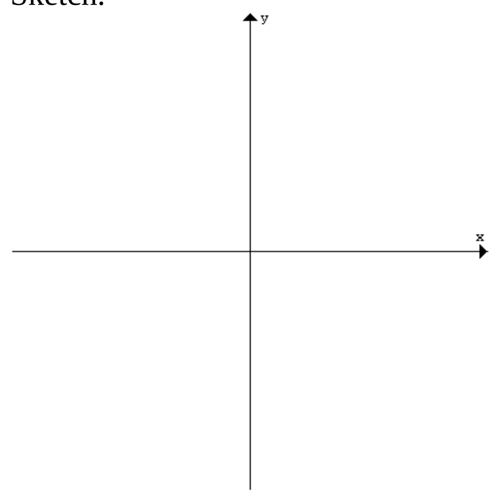
b) What is the x-intercept?

c) What is the y-intercept?

d) Determine the equation of the horizontal asymptote.

e) Determine the equation of the vertical asymptote.

f) Sketch.

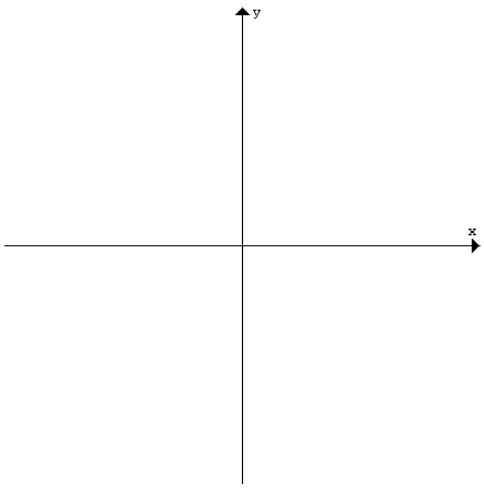


For rational functions in the form $f(x)=\dfrac{ax+b}{cx+d}$.

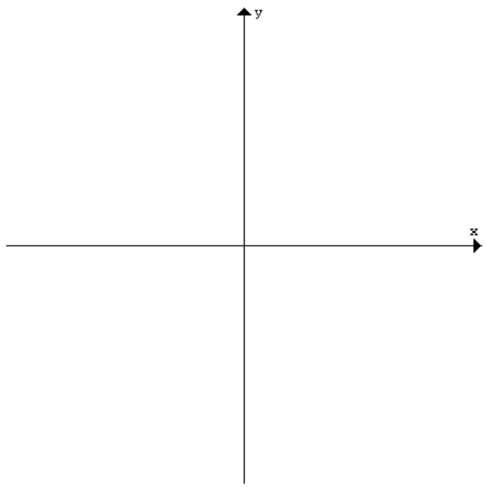
1. The x-intercept is
2. The y-intercept is
3. The vertical asymptote is
4. The horizontal asymptote is

Ex. 1 Sketch the following. Show all asymptotes and x and y intercepts.

a) $f(x)=-\dfrac{3x}{x+1}$



b) $f(x)=\dfrac{x}{2x+2}$



RATIONAL FUNCTIONS SUMMARY

A rational function has the form $h(x)=\dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are both polynomials.

Part A: Properties

1. Holes

   Factor the numerator, and factor the denominator. If there exists a common factor, $ax-b$, then a hole exists at ___________.

2. X-Intercept(s)

   Simplify the rational function (factor the numerator, factor the denominator, and reduce), set the ______________________ equal to ___________, and solve for _____.

3. Y-Intercept

   Set ___________, and solve for _____. Or, find ___________.

4. Vertical Asymptote(s)

   Simplify the rational function (factor the numerator, factor the denominator, and reduce), set the ______________________ equal to ___________, and solve for _____.

5. Horizontal Asymptote or Slant/Oblique Asymptote

   For functions in the form $h(x)=\dfrac{b}{cx+d}$, the equation of the horizontal asymptote is ______.

   For functions in the form $h(x)=\dfrac{ax+b}{cx+d}$, the equation of the horizontal asymptote is ______.

   Or, find the ___________________________. If $h(x)$ approaches a ________________, e, then the equation of the horizontal asymptote is ___________.

   For equations in the form $h(x)=\dfrac{f(x)}{g(x)}$, where the degree of $f(x)$ is _____ ___________ than the degree of $g(x)$, re-write $h(x)$ in the form $h(x)=\dfrac{a}{g(x)}+mx+b$, and the equation of the slant asymptote is ________________.

6. End Behaviour

   Find ________________ and ________________.

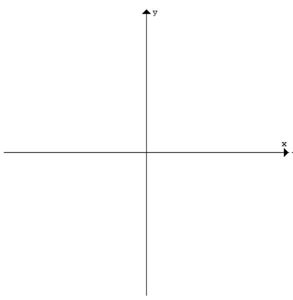
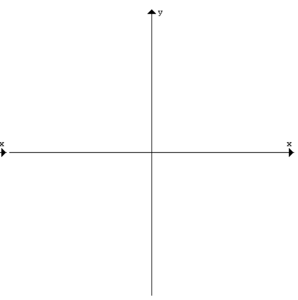
7. Behaviour near vertical asymptotes

   If there is a vertical asymptote at $x = a$, find ________________ and ________________.

Part B: Types

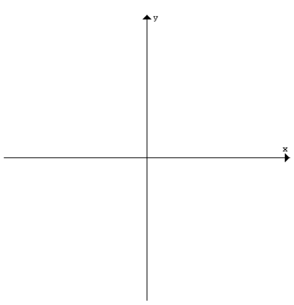
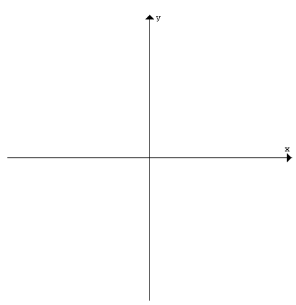
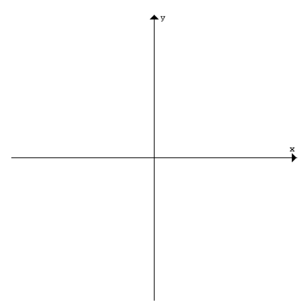
1. Rational functions in the form $y = \dfrac{a}{cx + d}$ and $y = \dfrac{ax + b}{cx + d}$

   These functions are transformations of the reciprocal function, $y = \dfrac{1}{x}$. These functions generally look like:

2. Rational functions in the form $y = \dfrac{a}{bx^2 + cx + d}$

   These functions are reciprocals of the ________________ function. There are three possible cases depending on the number of ___________ that the original quadratic function has.

0 zeros	1 zero	2 zeros
		

RATIONAL FUNCTIONS – APPLICATIONS

Ex. 1 The function $P(t) = \frac{30(7t+9)}{3t+2}$ models the population, in thousands, of a town t years since 1990.

a) What is the population in 1993?
b) What is the population in 1989? Is this function an appropriate model for the population at this time? Explain.
c) As time passes, what value will the population approach?
d) Sketch this function.

Ex. 2 Salt water is flowing into a large tank that contains pure water. The concentration of salt, c, in the tank at t minutes is given by $c(t)=\dfrac{10t}{25+t}$, where c is measured in grams per litre.

a) When does the salt concentration in the tank reach 3.75 g/L?

b) What happens to the concentration of salt in the tank during the 48 hour period after the salt water is added?

c) Sketch a graph of this relationship.

Notes on Solving Rational Equations

There is frequently more than one way to approach a problem. Here is an example of an approach to solving equations involving rational expressions and determining whether or not a solution might be extraneous.

Recall that the domain of an expression is the set of all real numbers that can be substituted in place of the variable. We can use this concept to our advantage when solving equations involving rational expressions. In fact, we find that a little effort at the beginning can save us quite a bit of time and effort at the end.

What we want to do is examine the expressions in the equation and determine the domain of each. Then we intersect the separate domains to arrive at the domain of the whole equation. If we have done this properly, when we get to the point that we have possible solutions to check, it is only necessary to check each solution to make sure it is in the domain of the equation. Anything in the domain of the equation is acceptable; everythin else is unacceptable.

Perhaps it would be a good idea to look at an example of how to do this.

Consider $\frac{1}{x}+\frac{1}{x-3}=\frac{x-2}{x-3}$.

For the first expression, $\frac{1}{x}$, the domain is $\{x \neq 0\}$.

For the second expression, $\frac{1}{x-3}$, the domain is $\{x \neq 3\}$.

For the third expression, $\frac{x-2}{x-3}$, the domain is $\{x \neq 3\}$

The intersection of these sets is $\{x \neq 0 \text{ and } x \neq 3\}$.

So what we know now is that neither **0** nor **3** can be a solution to this equation.

To solve the equation, our first step is to multiply both sides of the equation by the Lowest Common Denominator and then simplify the resulting expressions as much as possible.

$$\frac{1}{x}+\frac{1}{x-3}=\frac{x-2}{x-3}$$

$$x(x-3)\left(\frac{1}{x}+\frac{1}{x-3}\right)=x(x-3)\left(\frac{x-2}{x-3}\right)$$

$$x(x-3)\left(\frac{1}{x}\right)+x(x-3)\left(\frac{1}{x-3}\right)=x(x-3)\left(\frac{x-2}{x-3}\right)$$

$$(x-3)(1)+x(1)=x(x-2)$$

$$x-3+x=x^2-2x$$

$$2x-3=x^2-2x$$

At this point, we recognize that we have a quadratic equation to solve, so we get everything on one side of the equal sign, simplify everything, and factor the resulting polynomial.

$$2x-3=x^2-2x$$

$$2x-3-2x+3=x^2-2x-2x+3$$

$$0=x^2-4x+3$$

$$0=(x-3)(x-1)$$

Now, we can Factor to solve for **x**.

$$0=(x-3)(x-1)$$

$$x-3=0 \quad \text{or} \quad x-1=0$$

$$x-3+3=0+3 \quad \text{or} \quad x-1+1=0+1$$

$$x=3 \quad \text{or} \quad x=1$$

When we examine the domain for the equation, we find that it includes **1** but it does not include **3**. Therefore, **3** is extraneous and **1** is the only solution.

This type of problem allows us to see a definite difference between foresight and hindsight in solving problems. Foresight is better, but it requires more thought.

SOLVING RATIONAL INEQUALITIES

1. $\frac{12}{x-5} < 6$

2. $\dfrac{2x-10}{3x-3} > 2$

3. $\dfrac{4}{x+1} \le \dfrac{2}{x-1}$

4. $\dfrac{x^2 - x - 6}{x^2 + 2x + 1} \ge 0$