Introductory Game Theory, Lecture 3

Xiaoye Liao

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		B	
		L	R
A	T	$2, -1$	$1, 1$
	B	$0, 1$	$1, -1$

For player A:

- If B plays the pure strategy L, then the best response in pure strategy for A is T.
- If B plays the pure strategy R, then the best response in pure strategy for A is also T.
- So the set of dominant strategies for A is $\triangle\{T\}$, that is, T.

For player B:

- If A plays the pure strategy T, then the best response in pure strategy for B is R.
- If A plays the pure strategy B, then the best response in pure strategy for B is also L.
- So B does not have any dominant strategy.

Dominant strategy and rationality

- The existence of a dominant strategy will make a player's choice easy, as different beliefs about others' strategies will not affect the optimality of a dominant strategy.
- A dominant strategy is clearly a "best" strategy. Thus, if a player has any dominant strategies, then playing any of them is rational and will be what we can predict based on the principle of rationality.
- If a player even has a *strictly* dominant strategy, then playing that strategy is the *only* choice that is consistent with the principle of rationality 1 (and also the whole principle of rationality because a strictly dominant strategy is unquestionably the best choice).
- **Question:** Can we say that playing a dominant strategy, if feasible, is the *only* choice that is consistent with the principle of rationality 1?

- **Definition.** For a player i, a strategy σ_i is said to *strictly dominate* a strategy σ_i' if

$$U_i(\sigma_i, \sigma_{-i}) > U_i(\sigma_i', \sigma_{-i}) \text{ for all feasible strategies } \sigma_{-i} \text{ of other players'.}$$

- In words, σ_i strictly dominates σ_i' if σ_i always outperforms σ_i' no matter what the other players will play.
- **Definition.** For a player i, a strategy σ_i is said to *weakly dominate* a strategy σ_i' if

$$U_i(\sigma_i, \sigma_{-i}) \geqslant U_i(\sigma_i', \sigma_{-i}) \text{ for } \textit{all} \text{ feasible strategies } \sigma_{-i} \text{ of other players', and}$$

$$U_i(\sigma_i, \sigma_{-i}) > U_i(\sigma_i', \sigma_{-i}) \text{ for } \textit{some} \text{ feasible strategies } \sigma_{-i} \text{ of other players'.}$$

- In words, σ_i strictly dominates σ_i' if σ_i is always no worse than σ_i' no matter what the other players will play, and is better than σ_i' for some strategies of others.
- **Definition.** A strategy σ_i is said to be *weakly dominated* if there is a strategy σ_i weakly dominating it; a strategy σ_i is said to be *strictly dominated* if there is a strategy σ_i strictly dominating it.

Example

Consider the following normal-form game

			B	
		L	M	R
A	T	$4, 3$	$3, 1$	$3, 0$
	B	$2, 0$	$1, 1$	$3, 3$

For player A

- T is not strictly/weakly dominated as it is the *unique* best response to L and M;
- B is weakly, but not strictly, dominated by T.
- Every other strictly mixed strategy of A is weakly but not strictly dominated by T.

For player B

- L and R are not strictly/weakly dominated. Why?
- But M is strictly dominated (and hence weakly dominated) by $\sigma_B(0.5) = \frac{1}{2}L \oplus \frac{1}{2}R$: for every mixed strategy of player A, $\sigma_A(p) = pT \oplus (1-p)B$,

$$U_B(\sigma_A(p), M) = 1,$$

$$U_B(\sigma_A(p), \sigma_B(0.5)) = \frac{1}{2} \cdot p \cdot 3 + \frac{1}{2}(1-p) \cdot 0 + \frac{1}{2} \cdot p \cdot 0 + \frac{1}{2}(1-p) \cdot 3 = \frac{3}{2}.$$

There is, unfortunately, no simple systematic algorithm to identify all dominated strategies, and so in practice some “attention” is needed. But there are still some useful facts that can save our work.

- When seeking for dominated strategies for player i, we only need to consider all possible *pure strategies* for other players.
- If a pure strategy is dominated, then every mixed strategy which has the pure strategy in its support is also dominated. Thus, it is often helpful to identify dominated pure strategies first.
- If a pure strategy is a best response to some strategies of others', then it is not strictly dominated; if a pure strategy is the unique best response to some strategies of others', then it is not weakly dominated.

There is another side of the principle of rationality out of dominance.

- We have argued that dominant strategies are rational choices, and strictly dominant strategies are the only rational choices. That is, dominant strategies are “good” strategies in strategic situations.
- We can also attack the problem from the opposite side, that is, what strategies will NOT be played by rational players? That is, what strategies are “bad” strategies?
- It should be clear that if a strategy is *strictly* dominated, then it fails to maximize the player’s expected payoff in all possible cases, because there is another feasible strategy that always works better. Thus, a rational player will never play a strictly dominated strategy.
- **Question:** Is playing weakly dominated strategies necessarily irrational?

Dominance and rationality: a summary

The principle of rationality implies that in any strategic situation,

- a rational player will play a strictly dominant strategy if such a strategy exists;
- a rational player will never play a strictly dominated strategy;
- a rational player will play a best response to his conjecture/belief about what others will play.

- **Definition.** A strategy σ_i is called a *never-best response* to player i if $\sigma_i \notin BR_i(\sigma_{-i})$ for all possible strategies of the other players σ_{-i}.
- It is easy to see that a strictly dominated strategy is a never-best response.
- **Theorem.** For a *two*-player *finite* game, a strategy is a never-best response if and only if it is strictly dominated.
- Thus, in any two-player finite game, a never-best strategy will never be played by a rational player.
- For finite games with more than two players, the conclusion is still true if we allow players to randomize in a *correlated* way. If we only allow agents to randomize independently, then there are games in which a never-best response is not strictly dominated.

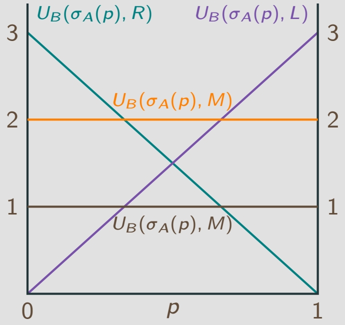
Let us revisit the example we have seen before and show in a direct way that M is a never-best response.

		B: L	M	R
A	T	$4, 3$	$3, 1$	$3, 0$
	B	$2, 0$	$1, 1$	$3, 3$

- Let A's strategy be $\sigma_A(p) = pT \oplus (1-p)B$.
- Then B's expected payoffs from playing L, M, and R are, respectively,

$$U_B(\sigma_A(p), L) = 3p, \quad U_B(\sigma_A(p), M) = 1, \quad U_B(\sigma_A(p), R) = 3(1-p).$$

- We can plot the three functions about p in the same coordinate system to reach the desired conclusion.
- If M yields to B a payoff of 2 rather than 1, then the line for $U_B(\sigma_A(p), M)$ will be shifted up to the orange level. This makes the point that a strategy that is not a best response to any pure strategies can still be a best response to a strictly mixed strategy.



Solution Concept

Definition

- In general, a *solution concept* is a proposed approach or method that can be utilized to solve various problems.
- In game theory, “solving a game” means *predicting how a game will be played*, namely, how players will behave in a strategic situation. These predictions are called “solutions”.
- A solution concept in game theory is a *formal rule for predicting agents’ behaviors in games*.
- A solution concept is a general criterion that is used to select strategy profiles in a strategic situation that are “reasonable” or “plausible”.
- Under the principle of rationality, an outcome is reasonable or plausible if it is consistent with the requirements of rational decision making, namely, everyone chooses an “optimal” strategy in a properly defined sense.

Dominant-strategy Equilibrium

- **Definition.** A *strategy profile* $\sigma^* = (\sigma_i^*)_{i \in N}$ is said to be a *dominant-strategy equilibrium* if σ_i^* is a dominant strategy for each player i.
- That is, a dominant-strategy equilibrium is a situation in which every player plays a dominant strategy.
- You should note that a dominant-strategy equilibrium is a strategy profile, not an outcome or a payoff vector. This principle applies to other solution concepts, too.
- Dominant-strategy equilibrium is often considered a quite convincing solution, because a dominant strategy (if exists) is, in any sense, an optimal/rational choice.
- In general, a dominant-strategy equilibrium may not exist (like in matching pennies or rock-paper-scissors).

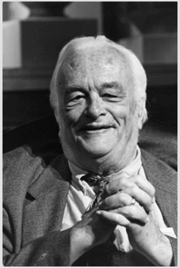
Example 1: Prisoners' Dilemma

Prisoners' dilemma refers to a class of strategic situations with the feature that every player has a strictly dominant strategy, but the resulting dominant-strategy equilibrium is not "socially desirable".

- The original version was proposed by *Flood* and *Dresher* in 1950, as given by the following payoff matrix:

		Prisoner B	
		D	C
Prisoner A	D	$-0.5, -0.5$	$-10, 0$
	C	$0, -10$	$-5, -5$

- It is easy to see that for each prisoner, C is a strictly dominant strategy, and so (C, C) is a (strictly) dominant-strategy equilibrium, which yields a payoff of -5 to each prisoner.
- From the "social" point of view, this outcome is not desirable, because the outcome (D, D) *Pareto dominates* (C, C). So everybody doing the best results in an undesirable social outcome.
- Typical real world examples: Nuclear arms race, trade war, doping in sports, etc..
- Key message: Rationality at individual level may contradict efficiency/optimality at social level.



Second-price sealed-bid auction is an auction form (or trading mechanism) initially proposed by Canadian economist *William Vickrey* (1914-1996), who won the Nobel Prize largely because of this contribution.

Protocol:

- An object is to be sold through an auction.
- There are $n \geqslant 2$ bidders, indexed by $i = 1, 2, \ldots, n$. Bidder i's WTP is v_i, which is perfectly known by himself/herself, but not others (i.e., *private value auction*).
- Each bidder i submits a bid b_i in a sealed envelope so that no one sees others' bids (this is where the term "sealed-bid" comes from).
- The auctioneer then opens all the n envelopes and identifies the highest bid.
- The object is given to a bidder who submits the highest bid (if $k \geqslant 2$ bidders submit the highest bid, then each of them gets the object with probability $1/k$).
- The winner of the object pays the highest bid submitted by the losers (this is where the name "second-price" comes from).

Example 2: Second-price sealed-bid auction (cont.)

Analysis: Let us first represent this strategic situation as a normal-form game.

- A simultaneous move game;
- Set of players $N = \{1, 2, \ldots, n\}$, $n \geqslant 2$;
- Set of pure strategies for each player i: $[0, \infty)$;
- Payoff: If bidder i wins the object and pays p for it, he/she receives a payoff of $v_i - p$; otherwise his/her payoff is 0.

  Let player i's bid be b_i and denote $\overline{b} = \max\{b_1, b_2, \ldots, b_n\}$, $\overline{b}_{-i} = \max\{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n\}$ (i.e., the highest bid made by people other than i). According to the rules of auction, we can write i's expected payoff at each bid profile $(b_1, b_2, \cdots, b_n)$ as

$$u_i(b_1, b_2, \ldots, b_n) = \begin{cases} 0, & \text{if } b_i < \overline{b} \\ \dfrac{v_i - \overline{b}_{-i}}{k}, & \text{if } b_i = \overline{b} \end{cases},$$

where $k = \left|\{b_j \mid b_j = \overline{b}, j = 1, 2, \ldots, n\}\right|$ = the number of bidders who submit the highest bid.

Claim: Bidding one's WTP, namely, $b_i(v_i) = v_i$, is a dominant strategy for each player i.

- Fix an arbitrary pure strategy profile for other bidders.
- Note that $\overline{b}_{-i}$ is a random variable from i's point of view, but below we will fix an arbitrary realization of this random variable.

	$\overline{b}_{-i} > v_i$	$\overline{b}_{-i} = v_i$	$\overline{b}_{-i} < v_i$
$b_i > v_i$	$\leqslant 0$	0	$v_i - \overline{b}_{-i}$
$b_i = v_i$	0	0	$v_i - \overline{b}_{-i}$
$b_i < v_i$	0	0	$\leqslant v_i - \overline{b}_{-i}$

- Observe that for each of the three possibilities about $\overline{b}_{-i}$, bidding the WTP truthfully is a best response, and is hence a dominant strategy.
- Indeed, bidding one's WTP is a dominant strategy even if bidder i can bid after knowing others' bids.

Conclusion: In a second-price sealed bid auction (with private value), there is a dominant-strategy equilibrium, in which everyone bids his/her WTP.

Rationalizability

Big idea of rationalizability

- *Rationalizability* is a solution concept based on exploring the implications of players' knowledge/belief about others' rationality. It is the implication of *common knowledge of rationality*.
- If a player knows that other players are rational, then he/she must know that others will not play strictly dominated strategies.
- As a result, the player should not form a conjecture/belief in which some of the other players will play strictly dominated strategies, by which a restriction has been imposed on the player's belief/conjecture.
- If all players are rational, then every player should reason in this way, and so no player should ever conjecture that others will play strictly dominated strategies.
- Such a restriction on players' beliefs/conjectures may help us narrow down the set of strategies players can reasonably choose.
- **Key point:** The above conclusion is everything we can deduce based on the knowledge that *every player knows that every player is rational*.

Consider the following normal-form game:

		Bob	
		L	R
Andrew	T	$2, 0$	$-1, 1$
	M	$0, 1$	$0, 0$
	B	$-1, -1$	$2, 0$

- We observe that M is strictly dominated by $\frac{1}{2}T \oplus \frac{1}{2}B$ (try proving this by yourself), but Bob does not have any dominant strategy.
- If Bob knows that Andrew is rational, then Bob knows that Andrew will not play M.
- Given this knowledge, Bob will find R the unique best response to each remaining strategy for Andrew, which then implies that Bob will play R for sure.
- It does seem that we can go further in this way. We are kind of inclined to reach the following conclusion:

  Since Bob will play R for sure, Andrew should only play B.

- But this is not an implication of “players knowing that everyone is rational”. It is the consequence of “Andrew knows that Bob knows that Andrew is rational”.

- By assuming higher order knowledge about players' rationality, we can proceed in a similar manner to eliminate as many (pure) strategies as possible.
- If it is common knowledge that every player is rational, then we have the following **algorithm**:
  - First eliminate strictly dominated *pure* strategies of all players (*every player is rational*);
  - In the reduced game, eliminate strictly dominated *pure* strategies of all players (*every player knows that every player is rational*);
  - Repeat the above step till no more (pure) strategies can be eliminated (*every player knows that ... that every player is rational*).
- **Terminology:** The process to analyze a game, as defined by the algorithm above, is called *Iterative Elimination of Strictly Dominated Strategies* (*IESDS*).
- **Definition:** A strategy that can survive IESDS is called a *rationalizable strategy*. The set of rationalizable *pure* strategies for a player is called the *rationalizable set* of the player.

IESDS and rationalizable strategies (cont.)

Common knowledge of rationality will lead us to conclude that *players will only play rationalizable strategies*.

Several notable features of IESDS

- For a finite game, IESDS will always stop after finitely many rounds of elimination. When some players have infinitely many pure strategies, there might be infinitely many rounds of elimination, and we will focus on the *limit* of IESDS.
- In a finite game, every player must have at least one rationalizable strategy, that is, IESDS will not eliminate all strategies for a player.
- In general, IESDS may not lead us to a sharp prediction, that is, players may have multiple rationalizable strategies. If IESDS does yield a unique outcome in a game, then the game is called a *dominance-solvable game*.

Example 1

Consider the following normal-form game:

		Player B		
		ℓ	m	r
Player A	U	$1,1$	$0,4$	$2,2$
	C	$2,4$	$2,1$	$1,2$
	D	$1,0$	$0,1$	$0,2$

- **Round 1:** For player A, D is strictly dominated by C and is hence eliminated; for player B, no pure strategy can be eliminated. (*Player A is rational*)
- **Round 2:** For player A, no pure strategy can be eliminated; for player B, r is strictly dominated by $0.5\ell \oplus 0.5m$ and is hence eliminated. (*Player B knows that player A is rational and player B is rational*)
- **Round 3:** For player A, U is strictly dominated by C and is hence eliminated; for player B, no pure strategy can be eliminated. (*Player A knows that player B knows that player A is rational, and player B is rational, and player A is rational*)
- **Round 4:** For player A, no more pure strategy can be eliminated (as there is only one left); for player B, m is strictly dominated by ℓ and is hence eliminated. (*Based on what knowledge/assumption?*)
- **Conclusion:** Common knowledge of rationality leads us to the unique prediction (C,ℓ).