

Introductory Game Theory, Lecture 1

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Game Theory: What Is It about?

- A strategic situation is a state of affairs in which the *outcome* depends on the *decisions of all parties* (individuals, organizations, etc.).
- This implies that each party's *payoff* (i.e., the subjective evaluation of the outcome) depends on other parties' decisions.
- So strategic situations are those that feature *payoff interdependence*.
- Such payoff interdependence typically implies *strategic interdependence*, namely, how each party will behave relies on how other parties will behave.

Strategic situations are abundant in our real world.

Examples of strategic situations

- Wars
- Bidding in auctions
- Cellphone pricing
- Penalty kick in football matches

Examples of non-strategic situations

- Deciding what to eat for dinner today
- Choosing your consumption bundle on competitive markets

A brief history about thoughts:

- The investigation of strategic situations can be traced back to ancient times. The most stark example is the military treaties *The Art of War* by Sun Tzu (5th century BC), which is the first systematic theory about one of the most important strategic situations—war.
- The modern idea of strategic situation originated from *games* (by its literal meaning) like nim, chess, and hex, which are obvious examples that feature payoff interdependence.
- People later realized that all strategic situations could be analyzed under a uniform framework. Thus, the term "game" has been extended to mean all mathematical models of strategic situations.

- Under the uniform framework, people have developed a modern theory which investigates *rational decision making* in mathematical models of strategic situations (i.e., games).
- Such a theory is thus called *game theory*.
- Uniform framework:
 - (i) The uniform representation of key factors of a strategic situation, which transforms the original situation into a game;
 - (ii) Exploring the implication of *rationality* in games, which generates a general principle of "reasonable outcomes", called *solution concepts*.

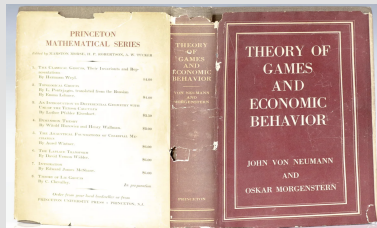
- Early development of game theory was not systematic and was scattered in the discussion of concrete cases.
 - Military books: *The Art of War*, *Sun Bin's Art of War*, *Thirty-six Stratagems*, etc.;
 - J. Waldegrave (1713) proposed a *minimax mixed strategy solution* to a two-person card game;
 - E. Zermelo (1913) proved that for games like chess there exists a strategy that guarantees that some player will never lose;
 - E. Borel (1938) proved a *minimax theorem* for a special two-person zero-sum game;
 - A. Cournot (1838) and J. Bertrand (1883) each proposed a model of *oligopolistic competition* and a solution.

A BRIEF HISTORY OF GAME THEORY (CONT.)

- The birth of modern game theory: *On the Theory of Games of Strategy* (1928) by J. von Neumann (1903-1957), and the publication of the book *Theory of Games and Economic Behavior* (1944) by J. von Neumann and O. Morgenstern (1902-1977).



Morgenstern and von Neumann

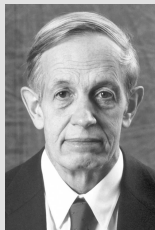


Theory of Games and Economic Behavior

- Laid down the foundations for modern game theory (unified framework, basic notions, and paradigm of analysis);
- Focusing on a particular class of games: *zero-sum games*.

A BRIEF HISTORY OF GAME THEORY (CONT.)

- A landmark in the development of game theory: *Equilibrium Points in n -person Games* (PNAS, 1950) by J. Nash (1928-2015).



John F. Nash

EQUILIBRIUM POINTS IN N -PERSON GAMES

By JOHN F. NASH, JR.*

PRINCETON UNIVERSITY

Communicated by S. Lefschetz, November 16, 1949

One may define a concept of an n -person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n -tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability

Title of the paper *Equilibrium Points in n -person Games*

- Proposed the fundamental solution concept of *Nash equilibrium*, which is widely applicable to strategic analysis (not just zero-sum games);
- Laid down the principle of rationality in strategic situations;
- Most modern development in solution concept is based on Nash equilibrium.

Formal Ingredients of Strategic Situations

- There are four key factors for describing a strategic situation:
 1. The set of involved parties, called *players* or sometimes *agents*
 2. The set of feasible *actions/choices* for each player
 3. The *payoffs* associated with each possible outcome
 4. The *information/knowledge* players have about the three factors above, others' decisions, and others' knowledge/information
- The four ingredients form the *structure* of a strategic situation.

SOME CLARIFICATION

- A player is not necessarily an individual, but a decision-making entity that is appropriately defined when setting up the model.
- A player's payoff from an outcome is his/her evaluation (satisfaction/desirability/utility) of the outcome. We often assume that every player evaluates an uncertain outcome by its expected payoff.
- A player's knowledge needs to specify not only what he/she knows about external parameters (like payoffs), but also what he/she knows about the other players' knowledge and beliefs about these parameters, as well as what he/she knows about the other players' knowledge of his/her own belief, and so on.
- Unless declared otherwise, we will assume that the set of players, players' feasible actions, and players' payoffs are *common knowledge*:
 - Every player knows it;
 - Every player knows that every player knows it;
 - Every player knows that every player knows that every player knows it;
 - ... (ad infinitum)

Consider the situation that you (A) and your friend (B) play rock-paper-scissors.

- Set of players: $\{A, B\}$;
- Feasible set of actions for each player: $a_A, a_B \in \{R, P, S\}$, a_i being the action for player i , $i \in \{A, B\}$;
- Payoffs: $u_A(a_A, a_B)$ and $u_B(a_A, a_B)$, where

$$u_A(R, S) = u_A(S, P) = u_A(P, R) = 1$$

$$u_A(R, R) = u_A(P, P) = u_A(S, S) = 0$$

$$u_A(S, R) = u_A(P, S) = u_A(R, P) = -1$$

and $u_B(a_A, a_B) = -u_A(a_A, a_B)$ for every (a_A, a_B) .

- The above factors are common knowledge.

Representation of Strategic Situations

So far, people have invented two convenient ways to represent important factors of a strategic situation:

- *Extensive-form game*

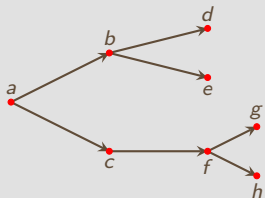
- initially proposed in *von Neumann* (1928) and its current form is due to *Kuhn* (1953);
- a representation that uses *game trees* and *information sets* as its building blocks;

- *Normal-form (or strategic form) game*

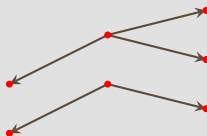
- initially proposed in *von Neumann* (1928) and became popular since *Nash* (1950);
- a representation that uses *strategies* and *payoff matrix* as its building blocks.

Extensive-form Representation

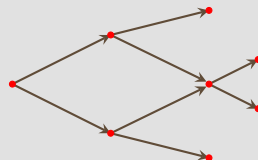
- A *tree* is a set of *nodes* and *directed edges* connecting these nodes such that
 - there is a unique *initial node* (i.e., a node that has no incoming edges), sometimes called the *origin* or the *root*;
 - for every non-initial node there is exactly one incoming edge.



A tree



Not a tree

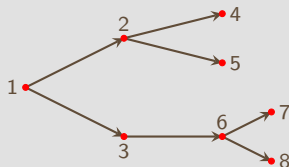


Not a tree

- Notice that in a tree, every node is connected with the origin with a unique *path* (i.e., a set of consecutive edges of the same direction).

Example. In the tree above, the unique path connecting the initial node *a* and node *e* is $a \rightarrow b \rightarrow e$, the unique path connecting node *c* and *h* is $c \rightarrow f \rightarrow h$.

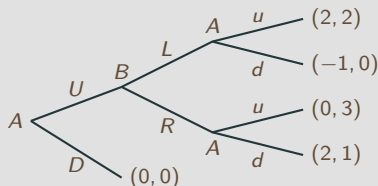
- If two nodes are connected by a path, then the starting node is called a *predecessor* of the ending node, and the ending node is called a *successor* of the starting node.



For example, in this tree, the initial node 1 is a predecessor of each of the other nodes; node 5 is a successor of node 2; node 3 is a predecessor of node 7, etc..

- A node that has no successors (i.e., a node from which no edges start) is called a *terminal node*.
In the tree above, the set of terminal nodes is $Z = \{4, 5, 7, 8\}$.
- For simplicity, we often skip the arrows of edges and postulate that the tree grows either rightward or downward, which often suffices to determine the root and the direction of each edge (i.e., departing from the node closer to the root).

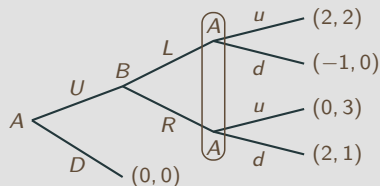
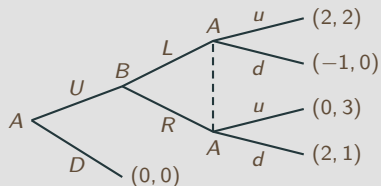
- A tree that represents a strategic situation is called a *game tree*.
- In a game tree, the first three formal ingredients of a strategic situation are depicted in the following way:
 - each non-terminal node, called a *decision node*, represents a position/stage/step in a game, at which exactly one player is to make a decision, the name of the player being labelled explicitly;
 - each edge represents a feasible action for the player who decides/moves at the starting node of the edge;
 - Players' payoffs are specified at each terminal node, which captures a specific completion of the game.



- In a game tree, a path is also called a *history*. A path that connects the origin and a terminal node is called a *terminal history*.

GAME TREE (CONT.)

- To capture each player's knowledge about the history (i.e., what have already happened) at his/her decision-making stage, we need another building block: *information sets*.
- An information set is a collection of node(s) such that
 - the same player is to move at each of the node(s);
 - the same actions are available at each of the node(s).
- In a game tree, all nodes in the same information set are joined together by dashed lines or are boxed together. Nothing needs to be done if an information set is a singleton.

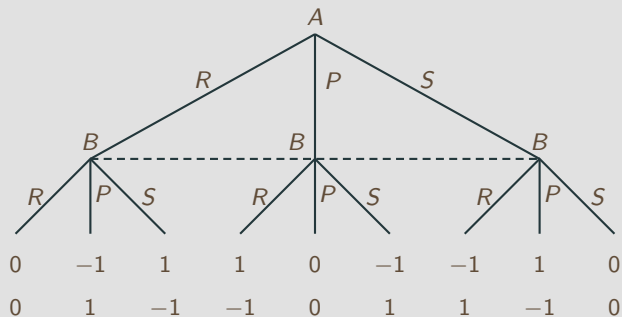


- Information sets are used to capture “*unobservability*”: two histories are not distinguishable if they end with nodes in the same information set.

- It might look at the first glance that extensive-form games can only describe strategic situations in which parties are moving *sequentially*. How about games in which some players move *simultaneously*?
- In game theory, however, the objective timing does not really matter too much. Timing is more relevant to players' knowledge about the history of play.
- For example, consider rock-paper-scissors again.
 - **Version 1:** A simultaneous-move game as what it is often considered to be;
 - **Version 2:** You first covertly write down a letter from $\{R, P, S\}$ on paper, after which your opponent does the same thing. We finally check what you two have put down and determine the outcome.
- While in version 2 people move sequentially, you can easily see that it is not different from version 1 in terms of players' *strategic consideration*.
- Thus, "simultaneous moves" basically means "moves made without seeing others' moves".

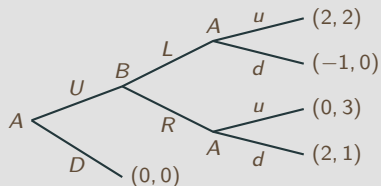
UNOBSERVABILITY AND TIMING OF MOVES

The extensive-form representation for rock-paper-scissors:

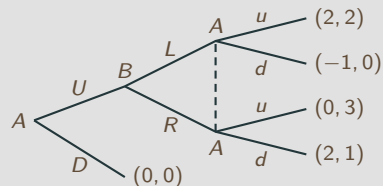


At each terminal node, the top number is the payoff to *A* and the bottom to *B*

- An extensive-form game is said to be a *game with perfect information* (or *perfect observation*) if every information set is a singleton.

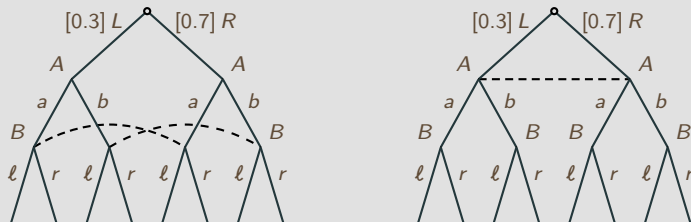


A game with perfect information



A game with imperfect information

- In many games, players are often faced with uncertainty about some relevant fact, in which case we say that there is *incomplete information*.
- In extensive-form games, uncertainty/chance is modeled as the knowledge about the action of *Nature*, an imaginary player.
- Nature has no payoff. At each decision node for Nature, a probability distribution over the departing edges is specified, which captures players' belief about the uncertainty.
- A decision node of Nature is denoted as a hollow node.



- In this way, we have transformed games with incomplete information to games with imperfect information.

Strategies in Extensive-form Games

- In an extensive-form game, a (*behavior*) *strategy* of a player is a *feasible*, *complete*, and *contingent* plan that specifies how the player will behave at each of his/her information sets.
 - *Feasible*: The behavior as specified by a strategy at each information set must be a probability distribution over all actions that are available at the information set;
 - *Complete*: The way to behave has to be specified at *every* information set, not just those that will be reached in playing the game;
 - *Contingent*: The way to behave is made dependent on the player's knowledge about the history of play.
- Since a player cannot distinguish decision nodes in the same information set, his/her strategy has to prescribe the same behavior at each node of the same information set.
- A behavior strategy is called a *pure strategy* if it prescribes a deterministic action (i.e., assigns a probability of 1 to an action) at each information set.

Verbal description

- A strategy for player A in the game, σ_A , can be

$$\sigma_A(\emptyset) = 0.5L \oplus 0.5R,$$

$$\sigma_A(La) = \sigma_A(Lb) = \ell, \quad \sigma_A(Ra) = \sigma_A(Rb) = r$$

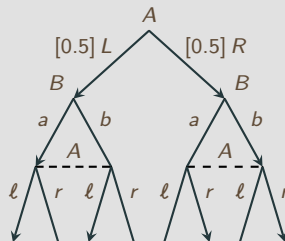
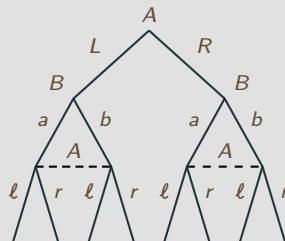
- A strategy for player B in the game, σ_B , can be

$$\sigma_B(L) = a, \quad \sigma_B(R) = b.$$

- Note that σ_A is a behavior strategy but not a pure strategy; σ_B , however, is a pure strategy.

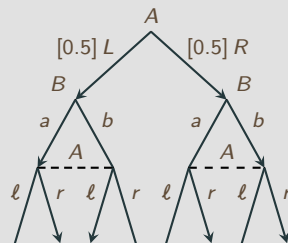
Graphical description

- We can also describe the *strategy profile* (σ_A, σ_B) using arrows in the game tree.



EXAMPLES (CONT.)

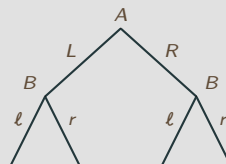
- The strategy profile (σ_A, σ_B) as illustrated in the game tree on the right is not valid.
- While σ_A is a valid strategy for player A , σ_B prescribes different behaviors at $L a$ and $L b$, which are not distinguishable to B .



- A *mixed strategy* for a player is a probability distribution over the set of pure strategies for the player.
- Typically, a mixed strategy cannot be well represented in the game tree. It is often described verbally.
- For example, consider the game tree illustrated in the right figure.

Here player B has four pure strategies $\sigma_B^1, \sigma_B^2, \sigma_B^3, \sigma_B^4$, where

$$\begin{aligned} \sigma_B^1(L) = \sigma_B^1(R) = \ell; & \quad \sigma_B^2(L) = \sigma_B^2(R) = r; \\ \sigma_B^3(L) = \ell, \sigma_B^3(R) = r; & \quad \sigma_B^4(L) = r, \sigma_B^4(R) = \ell. \end{aligned}$$



- A mixed strategy of B is $p_1\sigma_B^1 \oplus p_2\sigma_B^2 \oplus p_3\sigma_B^3 \oplus p_4\sigma_B^4$, with $p_i \geq 0$ for each $i = 1, 2, 3, 4$ and $\sum_{i=1}^4 p_i = 1$. Here p_i is the probability that the pure strategy σ_B^i will be chosen by B .
- A pure strategy is a mixed strategy that assigns a probability of 1 to the pure strategy.
- A *strictly mixed strategy* is a mixed strategy that does not choose any pure strategy with probability 1. A *totally mixed strategy* is a strategy that chooses every pure strategy with a positive probability.

- Both behavior strategy and mixed strategy may involve *randomization*, but they involve different ways to randomize.
- A mixed strategy is a randomization over all pure strategies at the beginning of the play.
- A behavior strategy randomizes independently at each information set as the play progresses.
- An interesting analogy:
 - A mixed strategy is like to pick a book randomly from a library;
 - A behavior strategy is like to pick a random page from every book in a library.
- For games with perfect information, the two types of strategies are equivalent in the sense that every distribution over terminal nodes that can be generated by a mixed strategy can also be achieved with a behavior strategy, and vice versa. (a weaker version of *Kuhn* (1953))