Introductory Game Theory, Lecture 2

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Normal-form (Strategic Form) Representation

Definition

A *normal-form representation* of a strategic situation, called a *normal-form game*, is a list

$$G = \left(N, (S_i)_{i \in N}, (u_i)_{i \in N}\right)$$

such that

- N is the set of players;
- S_i is the set of *pure strategies* for player i, $i \in N$, with each $s_i \in S_i$ being called a *pure strategy* of player i;
- $u_i : S \to \mathbf{R}$ is player i's utility function that is used to evaluate each possible outcome of the game for each $i \in N$, where

  $$S = \prod_{i \in N} S_i$$

  is the set of all *pure strategy profiles*. Each element in S is a complete description about every player's pure strategy.

Payoff matrix

- When there are two players, a normal-form game can be conveniently expressed using a *payoff matrix*:

		Player B	
		L	R
Player A	U	$u_A(U, L),\ u_B(U, L)$	$u_A(U, R),\ u_B(U, R)$
	D	$u_A(D, L),\ u_B(D, L)$	$u_A(D, R),\ u_B(D, R)$

- Here player A is called the "*row player*", who chooses among pure strategies denoted by rows of the matrix; player B is called the "*column player*", who chooses among pure strategies denoted by columns of the matrix.
- In each cell, the left component is the payoff to the row player, and the right component is the payoff to the column player.
- Payoff matrix can also be used to represent 3-player normal-form game, where multiple payoff matrices are needed: one player chooses among rows, one player chooses among columns, and the third player chooses among matrices.

- *Matching pennies*:
  - Two players, A and B.
  - Each has a penny and covertly turns the penny to H or T, after which the pennies are revealed.
  - If pennies match (i.e., both H or both T), then A wins. Otherwise B wins.
  - Winner obtains a payoff of 1 and loser -1.

		B: H	B: T
A	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

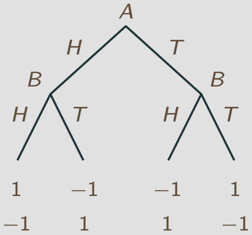
Normal-form game for matching pennies

- Rock-paper-scissors

		Player 2: R	Player 2: P	Player 2: S
Player 1	R	$0, 0$	$-1, 1$	$1, -1$
	P	$1, -1$	$0, 0$	$-1, 1$
	S	$-1, 1$	$1, -1$	$0, 0$

Normal-form game for rock-paper-scissors

- *Matching pennies with perfect information*:



Top number being A's payoff and bottom being B's

		B			
		$(H \mid H, H \mid T)$	$(H \mid H, T \mid T)$	$(T \mid H, H \mid T)$	$(T \mid H, T \mid T)$
A	H	$1, -1$	$1, -1$	$-1, 1$	$-1, 1$
	T	$-1, 1$	$1, -1$	$-1, 1$	$1, -1$

Normal-form game for matching pennies with perfect information

Strategies in Normal-form Games

Definition

- The concept of *pure strategy* in a normal-form game has already been informally defined. Basically, it specifies a deterministic action in every contingency of the game.
- For normal-form games, the only possible randomization is in the form of a *mixed strategy*, which is, like in extensive-form games, a probability distribution over the set of pure strategies.
- For example, in matching pennies, a mixed strategy for A is $pH \oplus (1-p)T$ with $p \in [0,1]$;

  in matching pennies with perfect information, a mixed strategy for B (second mover) is

  $$p(H \mid H, H \mid T) \oplus q(H \mid H, T \mid T) \oplus r(T \mid H, H \mid T) \oplus (1-p-q-r)(T \mid H, T \mid T),$$

  where $p, q, r \geqslant 0$ and $p + q + r \leqslant 1$.
- If S_i is the set of pure strategies of player i, then the set of mixed strategies of i is $\triangle S_i$.
- The set of all pure strategies that will be chosen with positive probability in a mixed strategy is called the *support* of the mixed strategy.

  Example: if $S_i = \{a, b, c, d\}$ and $\sigma_i = 0.5a \oplus 0.2b \oplus 0.3d$, then $\mathrm{Supp}(\sigma_i) = \{a, b, d\}$.

Calculating Payoffs in the Presence of Randomization or Chance

Probability distribution over payoffs

- When some players (including the Nature) adopt a strictly mixed strategy or a behavior strategy that involves randomization, a probability distribution over terminal nodes (for extensive-form game) or pure strategy profiles (for normal-form games) will be generated.
- Unless declared otherwise, we assume that players randomize *independently* across each other.
- Once we have obtained the probability distribution over terminal nodes/pure strategy profiles, we can calculate the expected payoff to each player using the payoff associated with each terminal node/pure strategy profile and the probability distribution.

For an extensive-form game, let σ_i be a behavior strategy of player i.

- As we have explained last week, $\sigma_i(h)$ is player i's strategy at the information set led by the history h.
- For each action a_i that is is feasible at the information set led by h, $\sigma_i(a_i \mid h)$ is the probability that player i will choose a_i at the information set.

Let $\sigma = (\sigma_i)_{i \in N}$ be the (behavior) strategy profile of all players.

- For each terminal history $h \in Z$, we denote by $\sigma(h)$ be the probability that h will be reached under the strategy profile σ.
- Player i's expected payoff from σ is denoted by $U_i(\sigma)$, where

$$U_i(\sigma) = \sum_{h \in Z} \sigma(h) u_i(h).$$

- The expected payoff profile/vector for all players is written as $U(\sigma)$, where

$$U(\sigma) = (U_i(\sigma))_{i \in N}.$$

  For example, if $N = \{1, 2, \dots, n\}$, then $U(\sigma) = (U_1(\sigma), U_2(\sigma), \cdots, U_n(\sigma))$.

Some notations (cont.)

Consider a normal-form game $G = \left(N, (S_i)_{i \in N}, (u_i)_{i \in N}\right)$ and let σ_i be a mixed strategy of player i.

- We use $\sigma_i(s_i)$ to denote the probability the mixed strategy assigns to the pure strategy s_i.
- For example, in rock-paper-scissor, if player A's strategy is $\sigma_A = 0.2R \oplus 0.5P \oplus 0.3S$, then

$$\sigma_A(R) = 0.2, \quad \sigma_A(P) = 0.5, \quad \sigma_A(S) = 0.3.$$

Let $\sigma = (\sigma_i)_{i \in N}$ be a *mixed strategy profile* for all players.

- For each pure strategy profile $s = (s_i)_{i \in N} \in S$, denote by $\sigma(s)$ the probability that s will be played, that is,

$$\sigma(s) = \prod_{i \in N} \sigma_i(s_i).$$

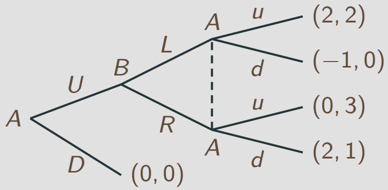
- Player i's expected payoff from σ is denoted by $U_i(\sigma)$, where

$$U_i(\sigma) = \sum_{s \in S} \sigma(s) u_i(s).$$

- The expected payoff profile/vector for all players is also written as $U(\sigma)$, where

$$U(\sigma) = (U_i(\sigma))_{i \in N}.$$

Consider the extensive-form game in the right figure.



- Suppose player A's (behavior) strategy is σ_A, where

$$\sigma_A(\varnothing) = \frac{1}{4} D \oplus \frac{3}{4} U, \quad \sigma_A(UL) = \sigma_A(UR) = \frac{1}{2} u \oplus \frac{1}{2} d.$$

  Suppose player B's mixed strategy is $\frac{1}{3} L \oplus \frac{2}{3} R$.

- Then the corresponding probabilities for terminal nodes are

$$\sigma(ULu) = \frac{3}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{8}, \quad \sigma(ULd) = \frac{3}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{8}, \quad \sigma(URu) = \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{4}$$
$$\sigma(URd) = \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{4}, \quad \sigma(D) = \frac{1}{4}.$$

- Thus, provided the strategy profile (σ_A, σ_B), the expected payoff vector for the two players is

$$\frac{1}{8}(2,2) + \frac{1}{8}(-1,0) + \frac{1}{4}(0,3) + \frac{1}{4}(2,1) + \frac{1}{4}(0,0) = \left(\frac{5}{8}, \frac{5}{4}\right).$$

  That is, the strategy profile (σ_A, σ_B) yields an expected payoff of $5/8$ to A and $5/4$ to B.

- Consider a strategic situation represented by the following normal-form game:

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{B} \\
 & & H & T \\
\hline
\multirow{2}{*}{A} & H & 1,-1 & -1,1 \\
\hline
 & T & -1,1 & 1,-1 \\
\hline
\end{array}
$$

- Suppose $\sigma_A = pH \oplus (1-p)T$, $\sigma_B = qH \oplus (1-q)T$, where $p, q \in [0,1]$.
- Then the probability distribution over the set of pure strategy profiles is

$$\sigma(H,H) = pq, \quad \sigma(H,T) = p(1-q), \quad \sigma(T,H) = (1-p)q, \quad \sigma(T,T) = (1-p)(1-q).$$

- Thus, the expected payoff vector to the players is

$$pq(1,-1) + p(1-q)(-1,1) + (1-p)q(-1,1) + (1-p)(1-q)(1,-1) = \big((1-2p)(1-2q), (2p-1)(1-2q)\big).$$

Therefore, the strategy profile (σ_A, σ_B) yields an expected payoff of $(1-2p)(1-2q)$ to A and $(2p-1)(1-2q)$ to B.

A useful variant for normal-form games

- For analytical purpose, it is often useful to write a player's expected payoff in a way such that this player's strategy is singled out.
- To single out a player, we introduce the following notations:
  - For each player i, we use $-i$ to label all other players.
  - If $\sigma = (\sigma_i)_{i \in N}$ is a strategy profile for all players, then σ_{-i} is the profile of strategies of all players except player i.
  - **Examples**
    - If $N = \{A, B, C, D\}$, then $\sigma_{-B} = (\sigma_A, \sigma_C, \sigma_D)$;
    - If $N = \{1, 2, \ldots, n\}$, then $\sigma_{-i} = (\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_n)$ for every $i \in N$.
  - When we want to single out the strategy for player i, we can write the strategy profile of all players, σ, as (σ_i, σ_{-i}).

- We group the terms $u_i(s)\sigma(s)$ according to the pure strategy player i ends up with playing.
- In particular, the sum of all the terms in which player i's action is s_i is

$$\begin{aligned}\sum_{s_{-i}\in S_{-i}} \sigma(s_i, s_{-i})u_i(s_i, s_{-i}) &= \sum_{s_{-i}\in S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i})u(s_i, s_{-i}) \\ &= \sigma_i(s_i) \sum_{s_{-i}\in S_{-i}} \sigma_{-i}(s_{-i})u_i(s_i, s_{-i}) \\ &= \sigma_i(s_i)\, U_i(s_i, \sigma_{-i})\end{aligned}$$

- Thus, we have

$$U_i(\sigma) = \sum_{s_i\in S_i} \sigma_i(s_i)\, U_i(s_i, \sigma_{-i}),$$

  where $U_i(s_i, \sigma_{-i})$ is i's expected payoff from playing the pure strategy s_i when others play σ_{-i}.
- Similarly, we can also group according to other players' pure strategies s_{-i} and end up with

$$U_i(\sigma) = \sum_{s_{-i}\in S_{-i}} \sigma_{-i}(s_{-i})\, U_i(\sigma_i, s_{-i}).$$

Example

- A has two pure strategies $S_A = \{a, b\}$, B has three pure strategies $S_B = \{x, y, z\}$.
- The strategies are $\sigma_A = p_a a \oplus p_b b$, $\sigma_B = p_x x \oplus p_y y \oplus p_z z$.
- Then A's expected payoff under the strategy profile $\sigma = (\sigma_A, \sigma_B)$ is

$$\begin{aligned}
U_A(\sigma) &= p_a p_x u_A(a, x) + p_a p_y u_A(a, y) + p_a p_z u_A(a, z) + p_b p_x u_A(b, x) + p_b p_y u_A(b, y) + p_b p_z u_A(b, z) \\
&= p_a \Big[p_x u_A(a, x) + p_y u_A(a, y) + p_z u_A(a, z) \Big] + p_b \Big[p_x u_A(b, x) + p_y u_A(b, y) + p_z u_A(b, z) \Big] \\
&= p_a U_A(a, \sigma_B) + p_b U_A(b, \sigma_B).
\end{aligned}$$

- We can also group according to B's pure strategy:

$$\begin{aligned}
U_A(\sigma) &= p_x \Big[p_a u_A(a, x) + p_b u_A(b, x) \Big] + p_y \Big[p_a u_A(a, y) + p_b u_A(b, y) \Big] + p_z \Big[p_a u_A(a, z) + p_b u_A(b, z) \Big] \\
&= p_x U_A(\sigma_A, x) + p_y U_A(\sigma_A, y) + p_z U_A(\sigma_A, z).
\end{aligned}$$

Best Response

Definition

- A strategy σ_i is a *best response* to the strategies of the other players σ_{-i} if it is the best choice for player i provided (or if he believes) that the other players will play strategy σ_{-i}.
- More formally, σ_i is a best response to σ_{-i} if

  $$U_i(\sigma_i, \sigma_{-i}) \geqslant U_i(\sigma_i', \sigma_{-i}) \text{ for every strategy } \sigma_i' \text{ that is feasible to player } i,$$

  namely, among all strategies feasible to player i, σ_i yields the highest possible (expected) payoff to player i if other players play σ_{-i}.
- If σ_i is a best response to σ_{-i}, we can write $\sigma_i \in BR_i(\sigma_{-i})$, where $BR_i(\cdot)$ is called the *best response correspondence* of player i, which is a set-valued function.
- **Principle of Rationality 1:** If a player is rational and has a *belief* about other players' strategies, then he/she should choose a best response to those strategies.

Example 1. Matching pennies

		B: H	B: T
A	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

- $BR_A(H) = \{H\}$, $BR_A(T) = \{T\}$;
- $BR_B(H) = \{T\}$, $BR_B(T) = \{H\}$;
- in general, if B plays a mixed strategy $pH \oplus (1-p)T$, then

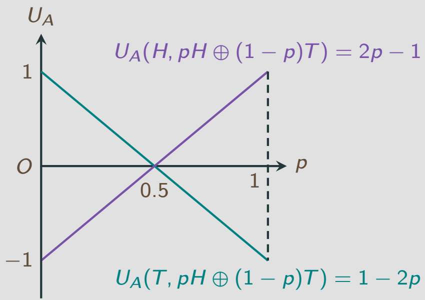
$$U_A(H, pH \oplus (1-p)T) = p \cdot 1 + (1-p) \cdot (-1) = 2p - 1;$$
$$U_A(T, pH \oplus (1-p)T) = p \cdot (-1) + (1-p) \cdot 1 = 1 - 2p.$$

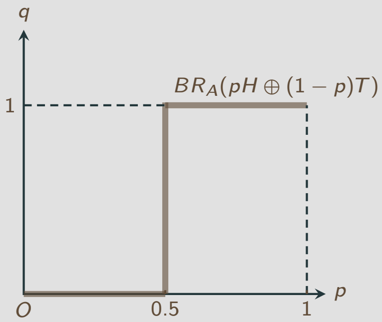
So H is *the* best response if $2p - 1 > 1 - 2p$, or $p > 0.5$; T is *the* best response if $1 - 2p > 2p - 1$, or $p < 0.5$; if, however, $2p - 1 = 1 - 2p$, or $p = 0.5$, then every mixed strategy of A is *a* best response.

$$BR_A(pH \oplus (1-p)T) = \begin{cases} \{H\}, & \text{if } p > 0.5 \\ \{qH \oplus (1-q)T \mid q \in [0,1]\}, & \text{if } p = 0.5 \\ \{T\}, & \text{if } p < 0.5 \end{cases}$$

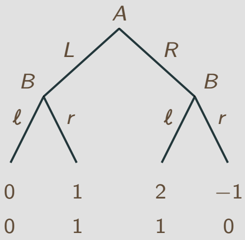
Graphical illustration of the best response

- Clearly, A's strategy can be identified with the probability that he chooses H (denoted by q), and B's strategy can be identified with p, i.e., the probability that she chooses H.





Example 2. Consider the following extensive-form game.



- We have $BR_B(pL \oplus (1-p)R) = \{(r \mid L, \ell \mid R)\}$ for every $p \in [0,1]$;
- if B plays the behavior strategy $\sigma_B(q,q') = (q\ell \oplus (1-q)r \mid L, q'\ell \oplus (1-q')r \mid R)$ (where $q, q' \in [0,1]$), then

$$U_A(L, \sigma_B(q,q')) = q \cdot 0 + (1-q) \cdot 1 = 1 - q,$$

$$U_A(R, \sigma_B(q,q')) = q' \cdot 2 + (1-q')(-1) = 3q' - 1.$$

  Therefore, A's best response is

$$BR_A(\sigma_B(q,q')) = \begin{cases} \{L\}, & \text{if } 3q' + q < 2 \\ \{pL \oplus (1-p)R \mid p \in [0,1]\}, & \text{if } 3q' + q = 2 \\ \{R\}, & \text{if } 3q' + q > 2 \end{cases}$$

A property of best response

- As we have seen in the previous two examples, sometimes there is only best response in pure strategy, but sometimes there is also best response in strictly mixed strategy.
- A notable feature, however, is that we always have a best response in pure strategy (if there is indeed a best response at all).
- **Theorem (pure strategy property).** For each player i and every strategy profile of other players σ_{-i}, if $BR_i(\sigma_{-i}) \neq \varnothing$, then $BR_i(\sigma_{-i})$ must contain some pure strategy. Moreover, $BR_i(\sigma_{-i}) = \triangle BR_i^{ps}(\sigma_{-i})$, where $BR_i^{ps}(\sigma_{-i})$ is the set of all pure strategies that are best responses to σ_{-i}.
- **Logic:** It is a good idea to randomize over pure strategies only when they are equally the best, namely,
    - all of them yield the same level of expected payoff; and moreover,
    - such a payoff is the highest possible level that can be achieved.

  This then necessarily implies that each of the pure strategies is also a best response.

A formal but simplified argument

- Suppose $\sigma_i = p_1 s_i^1 \oplus p_2 s_i^2$, where $p_1, p_2 > 0$, and $\sigma_i \in BR_i(\sigma_{-i})$.
- We want to argue that $U_i(\sigma_i, \sigma_{-i}) = U(s_i^1, \sigma_{-i}) = U(s_i^2, \sigma_{-i})$, and this is implied by $U(s_i^1, \sigma_{-i}) = U(s_i^2, \sigma_{-i})$.
- Suppose, to the contrary, that $U_i(s_i^1, \sigma_{-i}) > U_i(s_i^2, \sigma_{-i})$.
- Using the variant to rewrite player i's expected payoff, we have

$$\begin{aligned} U_i(\sigma_i, \sigma_{-i}) &= p_1 U_i(s_i^1, \sigma_{-i}) + p_2 U_i(s_i^2, \sigma_{-i}) \\ &< p_1 U_i(s_i^1, \sigma_{-i}) + p_2 U_i(s_i^1, \sigma_{-i}) \\ &= U_i(s_i^1, \sigma_{-i}), \end{aligned}$$

which means that the pure strategy s_i^1 outperforms σ_i when others play σ_{-i}, contradicting the fact that $\sigma_i \in BR_i(\sigma_{-i})$.

Dominance

- As we have seen, one's best response typically varies in other players' strategies, which makes it hard to tell which strategy is a *best* strategy.
- However, there is one particular case in which the answer to this question should be unambiguously clear.
- If there is a strategy that is *always* a best response no matter what strategy others will play, then the strategy is naturally considered a best strategy.
- Such a strategy is called a *dominant strategy*.

- A strategy σ_i is a (*weakly*) *dominant strategy* for player i if it is a best response to every strategies of other players', that is, $\sigma_i \in BR_i(\sigma_{-i})$ for all feasible strategies of other players σ_{-i}.

- Equivalently, σ_i is a dominant strategy for player i if for every feasible strategy σ_i' of player i and every feasible strategy profile of other players' σ_{-i} we have

$$U_i(\sigma_i, \sigma_{-i}) \geqslant U_i(\sigma_i', \sigma_{-i}).$$

- A strategy σ_i is a *strictly dominant strategy* for player i if it is the *unique* best response to all feasible strategies of others', namely, $\{\sigma_i\} = BR_i(\sigma_{-i})$ for all feasible strategies of other players σ_{-i}.
- Equivalently, for every *other* feasible strategy σ_i' of player i and every feasible strategy profile of other players' σ_{-i} we have

$$U_i(\sigma_i, \sigma_{-i}) > U_i(\sigma_i', \sigma_{-i}).$$

Example

Consider the following normal-form game:

		B	
		L	R
A	T	$2, -1$	$1, 1$
	B	$0, 1$	$1, -1$

- We argue that T is a dominant strategy for A.
- To this end, we pick an arbitrary strategy of B, $\sigma_B(p) = pL \oplus (1-p)R$ ($p \in [0,1]$), and also an arbitrary strategy of A, $\sigma_A(q) = qT \oplus (1-q)B$.

$$U_A(T, \sigma_B(p)) = p \cdot 2 + (1-p) \cdot 1 = 1 + p,$$

$$U_A(\sigma_A(q), \sigma_B(p)) = pq \cdot 2 + q(1-p) \cdot 1 + (1-q)p \cdot 0 + (1-q)(1-p) \cdot 1 = 1 - p + 2pq.$$

- Then

$$U_A(T, \sigma_B(p)) - U_A(\sigma_A(q), \sigma_B(p)) = (1+p) - (1-p+2pq) = 2p(1-q) \geqslant 0,$$

  which implies that T is a dominant strategy for A.
- **Quick question:** Is T a strictly dominant strategy for A?

A useful trick of finding dominant strategies

A *dominant pure strategy* is a strategy that is both dominant and a pure. The following two theorems guarantee that it suffices to first identify dominant pure strategies and to restrict attention to pure strategies only.

- **Theorem.** A pure strategy s_i is a dominant strategy for i if and only if for every pure strategy of i, s_i', and every possible strategy profile of others' σ_{-i} we always have

$$U_i(s_i, \sigma_{-i}) \geqslant U_i(s_i', \sigma_{-i}).$$

  Interpretation: A pure strategy is dominant if and only if it is always a best choice among all pure strategies.

- **Theorem.** A mixed strategy is a dominant strategy if and only if it is a probability distribution over dominant pure strategies.

The following theorem guarantees that we only need to check the situations in which the other players are all playing pure strategies.

- **Theorem.** A strategy σ_i is a dominant strategy for player i if for any feasible strategy σ_i' of player i and any feasible *pure* strategies of others' s_{-i} we always have

$$U_i(\sigma_i, s_{-i}) \geqslant U_i(\sigma_i', s_{-i}).$$

Putting the previous three theorems together, we reach the following result:

- **Theorem.** A pure strategy s_i is a dominant pure strategy for player i if and only if for every pure strategy $s_i' \in S_i$ and every pure strategies of others $s_{-i} \in S_{-i}$, we always have

$$U_i(s_i, s_{-i}) \geqslant U_i(s_i', s_{-i}).$$

By this result, we can identify dominant strategies for a player in the following way.

- **Step 1.** Identify all dominant pure strategies for the player. For each possible pure strategies of other players, we see if there is a pure strategy of the player which always outperforms his/her other pure strategies. Identify all those pure strategies and denote by D the set of them.
- **Step 2.** If $D = \varnothing$, then we can stop and conclude that there is no dominant strategy for this player. Otherwise, the set of all dominant strategies of this player is $\triangle D$.