Introductory Game Theory, Lecture 4

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Rationalizability (cont.)

Setup:

- There are $n \geqslant 2$ players, each player i simultaneously submits a number $x_i \in [0, 100]$;
- For each profile of submitted numbers $(x_1, x_2, \cdots, x_n)$, player i's payoff is

$$u_i(x_1, x_2, \cdots, x_n) = -\left(x_i - \frac{\overline{x}}{2}\right)^2,$$

  where $\overline{x}$ is the mean of the numbers players' submitted.
- **Interpretation:** People would like to "match" half of the average play.

Example 2: Keynesian beauty contest (cont.)

Analysis: The payoff function for a typical player i

$$u_i(x_1, x_2, \cdots, x_n) = -\left(x_i - \frac{\overline{x}}{2}\right)^2, \text{ where } \overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

- We first transform the payoff function to

$$u_i(x_1, x_2, \cdots, x_n) = -\left[\left(1 - \frac{1}{2n}\right) x_i - \frac{1}{2n} \sum_{j \neq i} x_j\right]^2.$$

  to investigate the best response of player i.
- The utility function is quadratic, and the (unique) best response is

$$x_i^*(x_{-i}) = \frac{1}{2n-1} \sum_{j \neq i} x_j.$$

- Observe that the closer player i's choice is to $x_i^*(x_{-i})$, the better the choice is.

Keynesian beauty contest (cont.)

When every $x_j \in [0, 100]$, we must have

$$0 \leqslant \frac{1}{2n-1} \sum_{j \neq i} x_j \leqslant \frac{n-1}{2n-1} \cdot 100 < 100,$$

which implies that all numbers above $\frac{100(n-1)}{2n-1}$ is strictly dominated by $\frac{100(n-1)}{2n-1}$.

- **Round 1:** We can eliminate all numbers above $\frac{100(n-1)}{2n-1}$ and so we can reduce the pure strategy space for each player to $\left[0, \frac{100(n-1)}{2n-1}\right]$.
- **Round 2:** Since each player will only submit a number in $\left[0, \frac{100(n-1)}{2n-1}\right]$, then

$$0 \leqslant \frac{1}{2n-1} \sum_{j \neq i} x_j \leqslant \left(\frac{n-1}{2n-1}\right)^2 \cdot 100,$$

  which implies that we can eliminate all numbers above $100\left(\frac{n-1}{2n-1}\right)^2$ and reduce the pure strategy space for each player to $\left[0, 100\left(\frac{n-1}{2n-1}\right)^2\right]$.
- **Round k:** Proceeding in this way, after k rounds of elimination, we will be left with $\left[0, 100\left(\frac{n-1}{2n-1}\right)^k\right]$.
- Since $\lim_{k \to \infty} \left(\frac{n-1}{2n-1}\right)^k = 0$, every positive number will eventually be eliminated. Thus, IESDS yields a unique prediction $(0, 0, \cdots, 0)$.

The original definition for rationalizability

- The solution concept of rationalizability is not originally defined as what we have seen. Instead, in each round, all *never-best responses* in pure strategy are eliminated (see *Berheim* (1984) and *Pearce* (1984)).
- A strategy is rationalizable if it can survive iterative elimination of never-best responses. Note that less strategies will survive under this algorithm.
- The two definitions are identical for two-player finite games.

Another related algorithm

- Another algorithm that involves iterative elimination is called *iterative elimination of weakly dominated strategies*, in which we eliminate all weakly dominated strategies iteratively.
- The knowledge underpinning this solution concept is not that simple (not just common knowledge of rationality). Cautiousness/Simplicity/low rationality or information demand?

Nash Equilibrium

Big idea

- Nash equilibrium is a solution concept based on all players' rationality and the extra assumption about players' knowledge: *every player has "correct" conjecture/belief about others' strategies*.
- The requirement can be equivalently restated as *every player knows the strategy choices of the others*.
- **Implication:**
  - If player i is rational and knows that others will play the strategy σ_{-i}, then his choice will be some $\sigma_i \in BR_i(\sigma_{-i})$.
  - Since this property holds for every player $i \in N$, the property that $\sigma_i \in BR_i(\sigma_{-i})$ must hold for every player.
  - In words, the strategy profile $\sigma = (\sigma_i)_{i \in N}$ has to be *mutually optimal*; namely, *every* player is playing a best response to others' strategies.
  - A strategy profile with this property is called a *Nash equilibrium*, the most commonly used solution concept in game theory.

Fix a (normal-form) game $\big(N, (S_i)_{i\in N}, (u_i)_{i\in N}\big)$.

- **Definition.** A strategy profile $\sigma^* = (\sigma_i^*)_{i\in N}$, where each $\sigma_i^* \in \triangle S_i$, is called a *Nash equilibrium* if for each $i \in N$, $\sigma_i^* \in BR_i(\sigma_{-i}^*)$; that is, for every player i, and for every feasible strategy σ_i of player i,

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geqslant U_i(\sigma_i, \sigma_{-i}^*).$$

- Notice how the two essential assumptions are featured in the definition:
  - Every player i has to choose a *best response* to his/her conjecture/belief about others' strategies: $\sigma_i^* \in BR_i(\cdot)$;
  - Every player's conjecture/beliief about others' strategies is *correct*: the dot in $BR_i(\cdot)$ has to be σ_{-i}^*, the actual strategies others will play.
- **Definition.** A Nash equilibrium is called a *pure-strategy Nash equilibrium* or a *Nash equilibrium in pure strategies* if every player plays a pure strategy in the Nash equilibrium; otherwise it is called a *mixed-strategy Nash equilibrium* or a *Nash equilibrium in mixed strategies*.

- **Stable state/point**: No player has any incentive to deviate *unilaterally* from a Nash equilibrium, because it is optimal for him/her to play the strategy as specified in the Nash equilibrium, holding others' strategies fixed. This is also called *self-binding agreement*.
- **State without regrets**: If players end up with playing a Nash equilibrium, no one can get strictly better off by changing his/her strategy, and hence will not regret his/her choice.
- **A realization of mutual optimality**: In a Nash equilibrium, every player is indeed choosing the best option, not just to his/her conjecture, but to the real strategies of others.
- **Self-fulfilling beliefs**: If every player believes that a strategy profile will be played, then it will be actually carried out only if it is a Nash equilibrium.

- According to the definition, σ^* is a Nash equilibrium if and only if it is located at the *intersection* of every player's best response correspondence.
- When every player's best response is unique, a Nash equilibrium $\sigma^* = (\sigma_1^*, \sigma_2^*, \cdots, \sigma_n^*)$ solves the system

$$\begin{cases} \sigma_1^* = BR_1(\sigma_{-1}^*) \\ \sigma_2^* = BR_2(\sigma_{-2}^*) \\ \quad \cdots \\ \sigma_n^* = BR_n(\sigma_{-n}^*) \end{cases}$$

- Define the *grand best response correspondence* in the following way: for each strategy profile of players σ,

$$BR(\sigma) = \prod_{i \in N} BR_i(\sigma_{-i}).$$

  For example, if $\sigma = (\sigma_A, \sigma_B)$, then $BR(\sigma) = BR_A(\sigma_B) \times BR_B(\sigma_A)$.
- **A fixed-point characterization.** A strategy profile σ^* is a Nash equilibrium if $\sigma^* \in BR(\sigma^*)$. In words, σ^* has to be a *fixed point* of the grand best response correspondence.

Example 1. Consider the following normal-form game.

		Player B		
		ℓ	m	r
	U	$1, 1$	$0, 4$	$2, 2$
Player A	C	$2, 4$	$2, 1$	$1, 2$
	D	$1, 0$	$0, 1$	$0, 2$

- *Restricting to pure strategies only*, our analysis shows that there is a unique intersections between the two best response correspondences: (C, ℓ).
- **Conclusion:** There is a unique pure-strategy Nash equilibrium (C, ℓ).

Example 2. Battle of the sexes (a model of coordination with conflict)

		Betty	
		P	B
Andy	P	$2, 1$	$0, 0$
	B	$0, 0$	$1, 2$

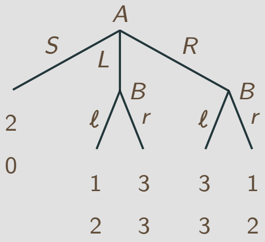
- Restricting to pure strategies only, our analysis shows that there are two intersections between the two best response correspondences: (P, P) and (B, B).
- **Conclusion:** There are two pure-strategy Nash equilibria, (P, P) and (B, B).

Example 3. Prisoners' dilemma

		Prisoner B	
		D	C
Prisoner A	D	$-0.5, -0.5$	$-10, 0$
	C	$0, -10$	$-5, -5$

- Restricting to pure strategies only, our analysis shows that there is only one intersection between the two best response correspondences: (C, C).
- **Conclusion:** There is a unique pure-strategy Nash equilibrium (C, C).
- **Question:** Is there any Nash equilibrium in mixed strategies? Why?

Example 4. An extensive-form game:



The normal-form representation of this game:

		B			
		$(\ell \mid L, \ell \mid R)$	$(\ell \mid L, r \mid R)$	$(r \mid L, \ell \mid R)$	$(r \mid L, r \mid R)$
A	L	1, 2	1, 2	3, 3	3, 3
	R	3, 3	1, 2	3, 3	1, 2
	S	2, 0	2, 0	2, 0	2, 0

- Therefore, there are five pure-strategy Nash equilibria for this game:

$$\big(R, (\ell \mid L, \ell \mid R)\big),\ \big(S, (\ell \mid L, r \mid R)\big),\ \big(L, (r \mid L, \ell \mid R)\big),\ \big(R, (r \mid L, \ell \mid R)\big),\ \big(L, (r \mid L, r \mid R)\big).$$

- **Question:** Is there any Nash equilibrium among the five that looks “weird” to you?

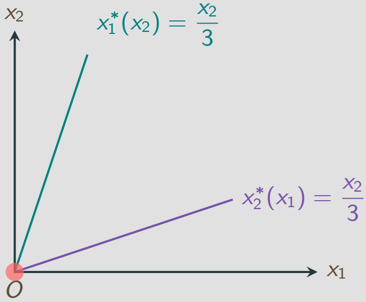
Example 5. Revisit Keynesian beauty contest with $n = 2$:

- Each player i simultaneously submits a number $x_i \in [0, 100]$;
- For each profile of submitted numbers (x_1, x_2), i's payoff is

$$u_i(x_1, x_2) = -\left(x_i - \frac{\overline{x}}{2}\right)^2,$$

  where $\overline{x}$ is the mean of the numbers players' submitted.
- Best responses:

$$x_1^*(x_2) = \frac{x_2}{3}, \quad x_2^*(x_1) = \frac{x_1}{3}.$$



- As can be seen from the figure, there is only one intersection between the two best response correspondences $(0, 0)$, which is the unique pure-strategy Nash equilibrium of this game.
- We can also find the pure-strategy Nash equilibrium (x_1^N, x_2^N) by solving the system

$$\begin{cases} x_1^N = BR_1(x_2^N) = \dfrac{x_2^N}{3} \\ x_2^N = BR_2(x_1^N) = \dfrac{x_1^N}{3} \end{cases} \Rightarrow \quad x_1^N = x_2^N = 0.$$

Example 6. Matching pennies

		B	
		H	T
A	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

- It should be clear that there is no pure-strategy Nash equilibrium for this game. Why?
- Also, there is no Nash equilibrium in which one randomizes but the other plays a pure strategy. Why?
- If there is a Nash equilibrium, it must be in (strictly) mixed strategies. Let it be

$$(\sigma_A^N, \sigma_B^N) = (p_A H \oplus (1 - p_A)T, p_B H \oplus (1 - p_B)T), \text{ where } p_A, p_B \in (0, 1).$$

- A necessary condition for one to optimally randomize is that every pure strategy that will be played with positive probability yields the same expected utility: $U_A(H, \sigma_B^N) = U_A(T, \sigma_B^N)$, $U_B(\sigma_A^N, H) = U_B(\sigma_A^N, T)$,i.e.,

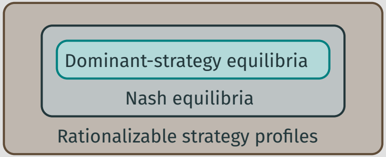
$$\begin{cases} p_B \cdot 1 + (1 - p_B) \cdot (-1) = p_B \cdot (-1) + (1 - p_B) \cdot 1 & (A \text{ is indifferent between } H \text{ and } T) \\ p_A \cdot (-1) + (1 - p_A) \cdot 1 = p_A \cdot 1 + (1 - p_A) \cdot (-1) & (B \text{ is indifferent between } H \text{ and } T) \end{cases}$$

from which we solve $p_A = p_B = \frac{1}{2}$.

- Thus, the game admits a unique Nash equilibrium in which every player randomizes between H and T with equal probability $1/2$.

Connection with previous solution concepts

- **Observation 1:** Every dominant-strategy equilibrium is also a Nash equilibrium, but the converse is not true.
- **Observation 2:** In each Nash equilibrium, every player must play a rationalizable strategy, and so IESDS will preserve all Nash equilibria. However, not all profiles of rationalizable strategies are Nash equilibria.



- **Observation 3:** There is a unique Nash equilibrium if every player has only one rationalizable strategy.
- **Observation 4:** In no Nash equilibrium will any player ever play a strictly dominated strategy or a never-best response.

Why Nash equilibrium?

- **Pre-play communication:** If players can discuss and coordinate their strategies before playing the game, then any agreement they can reach has to be a Nash equilibrium, since otherwise some player(s) will find it profitable not to honor the agreement.
- **Learning:** People may learn from others about how people are supposed to behave in certain scenarios. The behavior patterns (*stylized facts*) are typically Nash equilibria.
- **Dynamic adjustment/Evolution:** If a game is played repeatedly, players can retrospect and learn from their "mistakes" in the past, which generate a dynamic of adjustment. The dynamic can only stop at a Nash equilibrium (as no one wants to further modify his/her decision) (*stylized facts*).
- **Pervasive existence** (a technical reason) In almost all social interactions there is Nash equilibrium, and so the solution concept is widely applicable.

Theorem (Nash, 1950) Every *finite game* (i.e., a game with finitely many players, and each player has finitely many feasible actions) admits a Nash equilibrium (possibly in mixed strategies).

Main string of thoughts and key techniques:

- The key question: when does there exist a strategy profile σ such that $\sigma \in BR(\sigma)$?
- John Nash noticed that this is the same as asking: when does the grand best response correspondence $BR(\cdot)$ have a *fixed point*?
- Show that the grand best response correspondence always possesses a fixed point (*Kakutani's fixed point theorem*) in a finite game.