Introductory Game Theory, Lecture 5

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Oligopoly Competition

The Cournot model was proposed in 1838 by French mathematician *Antoine Cournot* (1801-1877). It is the first game-theoretic model of duopoly with *quantity competition*.

A. Cournot (1801-1877)

Setup:

- Two firms A and B are in the market of a good, each firm i simultaneously chooses its output level Q_i, $i \in \{A, B\}$;
- Marginal cost is constantly equal to $c \geqslant 0$;
- The inverse demand function for the good is $P = a - b(Q_A + Q_B)$ (market determines the equilibrium price), where $a, b > 0$ and $a > c$;
- A firm's profit $\pi_i(Q_i, Q_{-i}) = [P(Q_i, Q_{-i}) - c]Q_i = [a - b(Q_i + Q_{-i}) - c]Q_i$.
- Each firm maximizes its (expected) profit.
- We want to predict the level of output for each firm.

- For every $Q_{-i} \geqslant 0$, we identify i's best response.
- The optimization problem for firm i is formulated as

$$\max_{Q_i \geqslant 0} \underbrace{[a - b(Q_i + Q_{-i}) - c]Q_i}_{\pi_i(Q_i, Q_{-i})}.$$

- The best response is characterized by the first-order condition:

$$\frac{\partial \pi_i(Q_i, Q_{-i})}{\partial Q_i} = a - bQ_i - bQ_{-i} - c - bQ_i = 0,$$

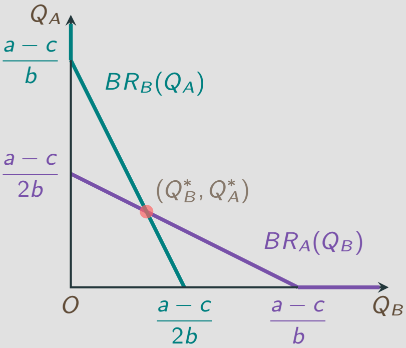
from which we solve

$$BR_i(Q_{-i}) = \max\left\{\frac{a - c - bQ_{-i}}{2b}, 0\right\}$$

- The pure-strategy Nash equilibrium (Q_A^*, Q_B^*) is characterized by

$$\begin{cases} Q_A^* = \dfrac{a - c - bQ_B^*}{2b} \\ Q_B^* = \dfrac{a - c - bQ_A^*}{2b} \end{cases} \Rightarrow \quad Q_A^* = Q_B^* = \frac{a - c}{3b}.$$

- **Question:** Can there be any mixed-strategy Nash equilibrium?



Some notable properties:

- It features *competition*: as compared with the case of monopoly, the quantity produced is higher.

  $$\text{Monopoly output } Q^m = \frac{a-c}{2b}; \quad \text{Total output in Cournot equilibrium } Q^* = Q_A^* + Q_B^* = \frac{2(a-c)}{3b}.$$

  In other words, there is *overproduction* as compared with the industry first best, and so the Cournot equilibrium is not Pareto efficient to the firms.

- It also features *market power*, and so is not *perfect competition*: the quantity produced is lower than the perfectly competitive level.

  $$\text{Output level in competitive equilibrium } Q^c = \frac{a-c}{b}; \text{ Equilibrium price } P(Q^*) = \frac{a+2c}{3} > c.$$

  In other words, there is *underproduction* as compared with the social first best.

- Each firm still receives a positive profit, but the total profit is lower than the monopoly profit.

  $$\text{Total profit in Cournot eqm: } \pi^* = \frac{2(a-c)^2}{9b}; \quad \text{Monopoly profit: } \pi^m = \frac{(a-c)^2}{4b}.$$

Now assume that there are $n \geqslant 2$ firms, other things being the same as before.

- Similar to the 2-firm case, we have

$$\pi_i(Q_i, Q_{-i}) = [a - b(Q_i + Q_{-i}) - c]Q_i, \text{ where } Q_{-i} = \sum_{j \neq i} Q_j.$$

- So firm i's best response is similar to the case with two firms (with a reinterpretation for Q_{-i}):

$$BR_i(Q_{-i}) = \max\left\{\frac{a - c - bQ_{-i}}{2b}, 0\right\}.$$

- So the Nash equilibrium $(Q_1^*, Q_2^*, \cdots, Q_n^*)$ is characterized by the system

$$\begin{cases} Q_1^* = \dfrac{a - c - b(Q_2^* + Q_3^* + \cdots + Q_n^*)}{2b} \\ Q_2^* = \dfrac{a - c - b(Q_1^* + Q_3^* + \cdots + Q_n^*)}{2b} \\ \qquad \cdots \quad \cdots \\ Q_n^* = \dfrac{a - c - b(Q_1^* + Q_2^* + \cdots + Q_{n-1}^*)}{2b} \end{cases} \Rightarrow \quad Q_1^* = Q_2^* = \cdots = Q_n^* = \frac{a - c}{(n+1)b}$$

How does the equilibrium depend on the number of firms?

- The total output in the n-firm Cournot equilibrium is

$$Q^* = \sum_{i=1}^{n} Q_i^* = n \cdot \frac{a-c}{(n+1)b} = \frac{n}{n+1}\frac{a-c}{b},$$

  and so we have

$$\lim_{n\to\infty} Q^* = \lim_{n\to\infty} \frac{n}{n+1}\frac{a-c}{b} = \frac{a-c}{b} = Q^c.$$

  Namely, as the number of firms increases, the total output will approach the competitive level.

- Also, the price in Cournot equilibrium will converge to the competitive level, c:

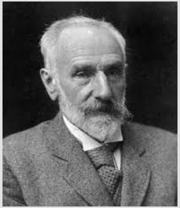
$$\lim_{n\to\infty} P(Q^*) = \lim_{n\to\infty} (a - bQ^*) = a - b \lim_{n\to\infty} Q^* = a - b \cdot \frac{a-c}{b} = c.$$

- **Conclusion:** Cournot equilibrium will be closer to a competitive equilibrium as the competition becomes more fierce (as the number of firms increases).

The idea was proposed in 1883 by French mathematician *Joseph Bertrand* (1822-1900) as an alternative model for duopoly competition.

The idea was later formally developed by Irish economist *Francis Edgeworth* (1845-1926) in 1889. It is the first game-theoretic model of duopoly with *price competition*.

J. Bertrand (1822-1900)



F. Edgeworth (1845-1926)

Setup:

- Two firms A and B are in the market of a good, each firm i simultaneously sets a price $p_i \geqslant 0$, $i \in \{A, B\}$;
- Marginal cost is constantly equal to $c \geqslant 0$;
- The firm with the lower price captures the whole market:

$$Q_i(p_i, p_{-i}) = \begin{cases} a - p_i, & \text{if } p_i < p_{-i} \\ \dfrac{a - p_i}{2}, & \text{if } p_i = p_{-i} \\ 0, & \text{if } p_i > p_{-i} \end{cases}, \quad \pi_i(p_i, p_{-i}) = \begin{cases} (a - p_i)(p_i - c), & \text{if } p_i < p_{-i} \\ \dfrac{(a - p_i)(p_i - c)}{2}, & \text{if } p_i = p_{-i} \\ 0, & \text{if } p_i > p_{-i} \end{cases}$$

  where $a > c$.
- Each firm maximizes its (expected) profit.

Bertrand model: analysis

Let us focus on pure-strategy Nash equilibria only.

- Let (p_A^*, p_B^*) be a pure-strategy Nash equilibrium of the Bertrand model.
- First, it is impossible that the two firms set different prices. If, say, $p_A^* > p_B^*$, then firm A receives a zero profit. We consider two cases.
  - If $p_B^* > c$, then firm A has an incentive to reduce its price to p_B^* to receive a positive profit.
  - If $p_B^* = c$, then firm B has an incentive to increase its price to p_A^* to receive a positive profit.
  - If $p_B^* < c$, then firm B has an incentive to increase its price to c.
- Second, it is impossible that $p_A^* = p_B^* > c$, since in this case each firm has an incentive to slightly undercut the other firm to capture the whole market:if $p_A^* = p_B^* = p^* > c$, then for sufficiently small $\varepsilon > 0$

$$\pi_A(p^*, p^*) = \frac{(a - p^*)(p^* - c)}{2} < \pi_A(p^* - \varepsilon, p^*) = (a - p^* + \varepsilon)(p^* - \varepsilon - c)$$

- So we are only left with one possibility: $p_A^* = p_B^* = c$, which is indeed a Nash equilibrium (a firm cannot get better off by raising its price, but will get strictly worse off by reducing its price).
- **Conclusion:** The unique pure-strategy Nash equilibrium of the Bertrand model is (c, c); that is, both firms set their prices at the level of the marginal cost.

Bertrand equilibrium

- The most striking prediction of Bertrand equilibrium is that the competitive equilibrium will be achieved even with two firms, as the equilibrium price equals the marginal cost c. This result is called the *Bertrand paradox*.
- Similar to Cournot equilibrium, the firms fail to achieve their first best, and so the equilibrium is not Pareto optimal for the two firms.
- This result crucially relies on the assumption that a firm can "slightly" undercut the other (called an *epsilon-undercut*). The conclusion may not hold if an epsilon-undercut is not possible.
- The result also crucially relies on the assumption that a firm with a lower price will be able to capture the whole market. The conclusion may not hold if firms are subject to *capacity constraints* or if the goods sold by the two firms are not *perfect substitutes*. The same is true for Cournot model.
- The conclusion may also change if firms do not have the same level of marginal cost.
- The conclusion will be significantly different if we consider the dynamic version of this model: firms may find it more profitable to jointly set a high price. This also applies to Cournot model.

Cournot and Bertrand: a comparison

- **Economic essence**: Cournot model better depicts the situation that firms have to choose their quantities first, while Bertrand model better depicts the situation that firms have to choose their prices first.
- **Types of externalities**: The *externality* a player causes to other players is the consequence one causes on other people but is not taken into account in one's decision making. Cournot model features continuous externality while Bertrand model features discontinuous externality.
- **Different relationship between players' strategies**: Cournot model features *strategic substitutes* (BR is downward sloping) while Bertrand model features *strategic complements* (BR is upward sloping).

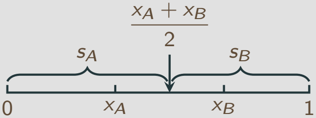
Hotelling model

The Hotelling model was proposed in 1929 by American economist *Harold Hotelling* (1895-1973), which captures firms' competition on *product differentiation*. The key idea is to model product differentiation as "location choice".

H. Hotelling (1895-1973)

A simplified setup that features only product differentiation:

- There are customers of total mass 1, who are located on a street $[0, 1]$;
- Customers' distribution is even/uniform: there is one customer at each point in $[0, 1]$;



- There are two firms A and B selling the same good. Each firm i simultaneously chooses a location $x_i \in [0, 1]$, $i \in \{A, B\}$;
- Each customer buys exactly one unit of the good. Customer at $x \in [0, 1]$ chooses the closer firm, and flips a fair coin to decide if both firms are equally close.
- Firm i's payoff equals its market share s_i (the fraction of customers who choose the firm).

Hotelling model : interpretation

For an obvious reason, Hotelling model is also called *spatial competition model*.

In different scenarios, the Hotelling model bears different interpretations. In the context of product differentiation, the following interpretation is adopted.

- Each point $x \in [0, 1]$ represents one possible type/version of the good.
- The customer at location x finds type-x good his/her most preferred type, and his/her utility from a particular good of type y is determined by the distance between x (his/her most preferred type) and y.
- The larger the gap $|y - x|$ is, the worse the type-y product is to the customer at x.
- Thus, in our context, each firm is choosing a particular type of the good to attract as many customers as possible.
- The model hence captures firms' incentives behind product differentiation: should the firms choose to produce the same type of the good (i.e., the same location, and so there is no product differentiation) or different types of the good (i.e., different locations, and so there is product differentiation)?

- We first derive the best response for each firm.
- Fix $x_{-i} \in [0,1]$. By choosing $x_i \in [0,1]$, firm i's payoff is

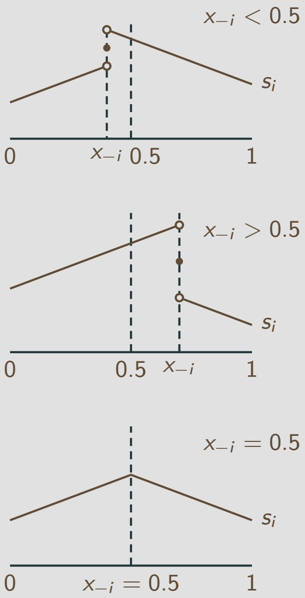
$$s_i = \begin{cases} \frac{x_i + x_{-i}}{2}, & \text{if } x_i < x_{-i} \\ 0.5, & \text{if } x_i = x_{-i} \\ 1 - \frac{x_i + x_{-i}}{2}, & \text{if } x_i > x_{-i} \end{cases}.$$

- So it is straightforward to see that

$$BR_i(x_{-i}) = \begin{cases} \varnothing, & \text{if } x_{-i} \neq 0.5 \\ 0.5, & \text{if } x_{-i} = 0.5 \end{cases}.$$

Note that the best response to $x_{-i} \neq 0.5$ is empty because we assume that a firm can choose any location, but firm i always has an incentive to choose a location arbitrarily close to x_{-i} from one side but choosing x_{-i} is not optimal.

- Thus, the model only admits a unique (pure-strategy) Nash equilibrium: $(0.5, 0.5)$.



- Our analysis reveals that in a duopoly market where firms compete on product differentiation, the firms will choose the same type of good to produce. This is called the *principle of minimum differentiation*.
- More precisely, the result says that if the only characteristic of the good firms can choose is its type and everything else (including the price) is the same, then firms have no incentive to pursue product differentiation.
- **Interpretation 1:** A duopoly market is likely to exhibit "product standardization" if product differentiation will not change consumers' willingness-to-pay.
- **Interpretation 2:** It also suggests that if there is product differentiation, then, typically, there should be other associated difference that is profitable to firm. In particular, product differentiation is favorable if it can enhance a firm's market power (so that it can charge a higher price).

Median voter theorem

An alternative interpretation of the Hotelling equilibrium is the famous *Median Voter Theorem* in political economy (Downs, 1957).

- We can reinterpret the two firms as two political candidates, and customers as voters who have different political positions in the spectrum (some are on the left, some are on the right, etc.).
- So a voter will vote for the candidate whose political position is the closest to his/hers.
- The problem is then transformed to the optimal choice of political platform in an election campaign.
- Then, if candidates would like to maximize their votes, the equilibrium choice will be for both candidates to choose the *median* political position rather than an extreme one.