Dynamic Programming, Part Deux

- Our mid-term is coming!
- Spring Break's also coming!

- You'll make it! ☺ Hang in there!!!

Midterm

- Wednesday, March 4 in class
  - SDAC: Please schedule with SDAC for Wednesday
  - Mostly in-class with a (required) take-home portion
  - Take-home “bonus” – If you do better on take-home than on its ”starter” question on the in-class, you can earn back half the difference.
- Practice Midterm <u>and Solutions</u> on Collab
- Review Session on Panopto
- More office hours by me! See Piazza

Today's Keywords

- Dynamic Programming
- Longest Common Subsequence
- Seam Carving

CLRS Readings

- Chapter 15
  - Section 15.1, Log/Rod cutting, optimal substructure property
    - Note: r_i in book is called Cut() or C[] in our slides. We use their example.
  - Section 15.3, More on elements of DP, including optimal substructure property
  - **Section 15.2, matrix-chain multiplication**
  - **Section 15.4, longest common subsequence**

Dynamic Programming

- Requires Optimal Substructure
  – Solution to larger problem contains the solutions to smaller ones
- Avoid extra work due to overlapping subproblems
- Idea:
  1. Identify the recursive structure of the problem
     - What is the “last thing” done?
  2. Save the solution to each subproblem in memory
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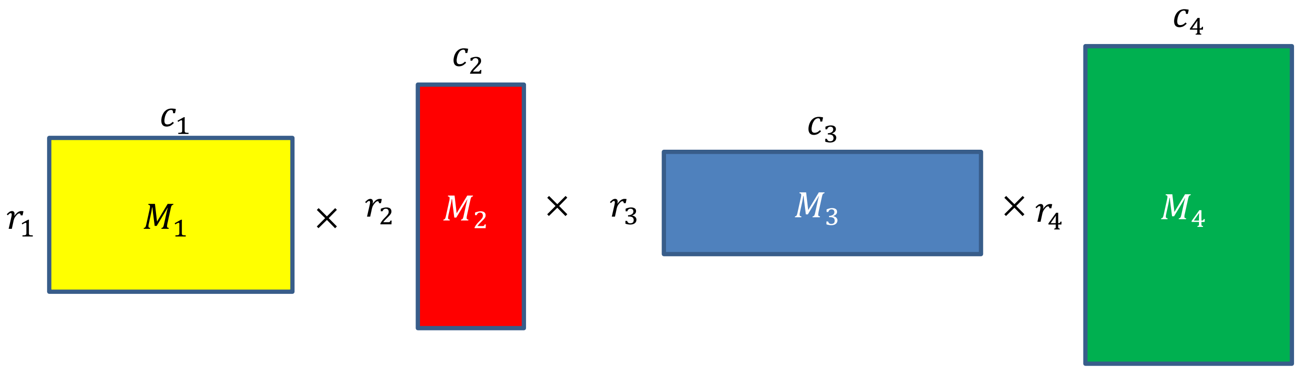
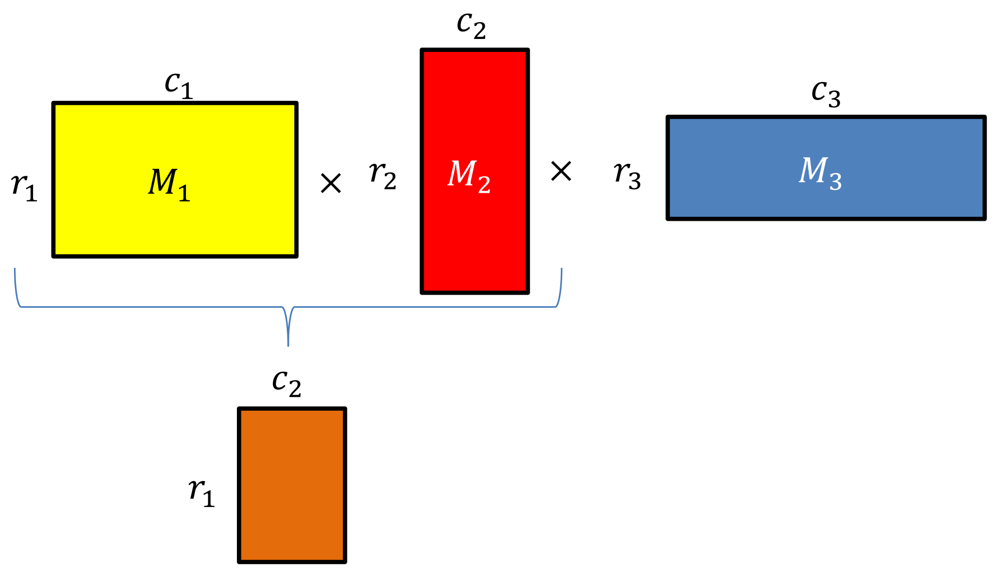
Matrix Chaining

- Given a sequence of Matrices $(M_1, \ldots, M_n)$, what is the most efficient way to multiply them?



Order Matters!

$c_1 = r_2$
$c_2 = r_3$



- $(M_1 \times M_2) \times M_3$
  - uses $(c_1 \cdot r_1 \cdot c_2) + c_2 \cdot r_1 \cdot c_3$ operations

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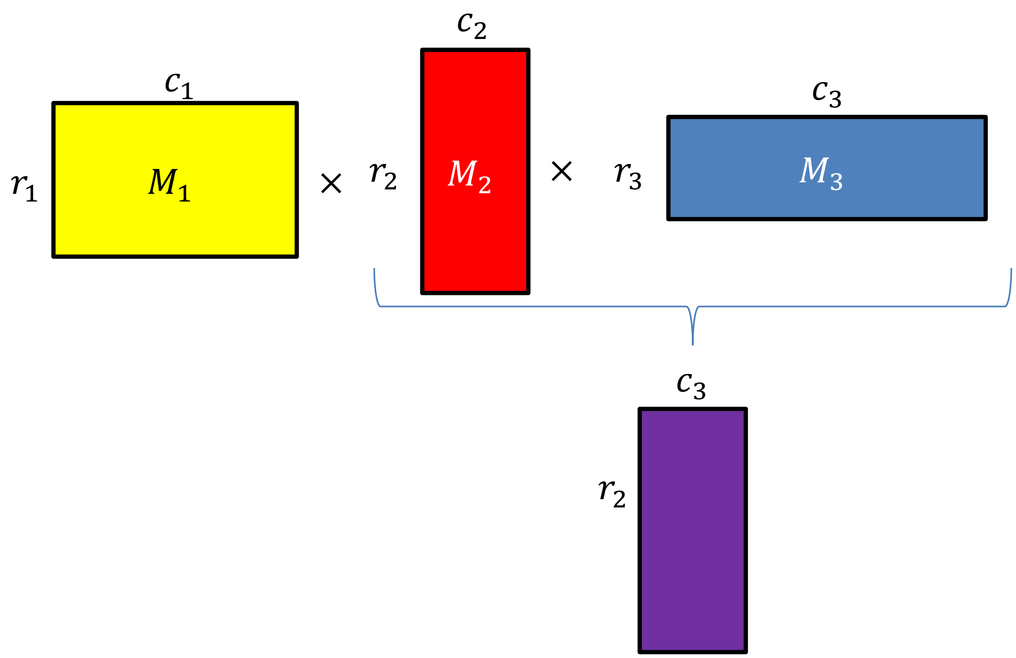

- $M_1 \times (M_2 \times M_3)$
  - uses $c_1 \cdot r_1 \cdot c_3 + (c_2 \cdot r_2 \cdot c_3)$ operations

Order Matters!

$c_1 = r_2$
$c_2 = r_3$

- $(M_1 \times M_2) \times M_3$
  - uses $(c_1 \cdot r_1 \cdot c_2) + \mathrm{c}_2 \cdot r_1 \cdot c_3$ operations
  - $(10 \cdot 7 \cdot 20) + 20 \cdot 7 \cdot 8 = 2520$
- $M_1 \times (M_2 \times M_3)$
  - uses $c_1 \cdot r_1 \cdot c_3 + (\mathrm{c}_2 \cdot r_2 \cdot c_3)$ operations
  - $10 \cdot 7 \cdot 8 + (20 \cdot 10 \cdot 8) = 2160$

$M_1 = 7 \times 10$
$M_2 = 10 \times 20$
$M_3 = 20 \times 8$

$c_1 = 10$
$c_2 = 20$
$c_3 = 8$
$r_1 = 7$
$r_2 = 10$
$r_3 = 20$

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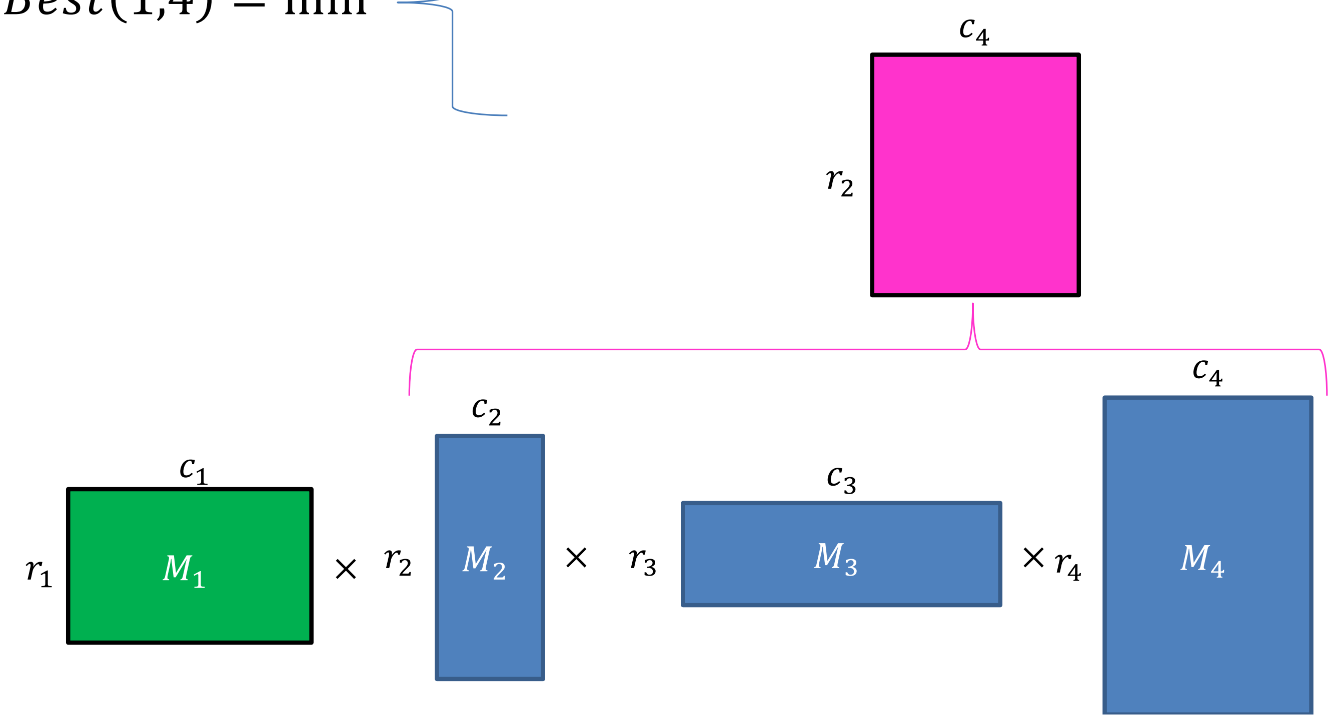
1. Identify the Recursive Structure of the Problem

$Best(1,n)$ = cheapest way to multiply together M_1 through M_n

$$Best(1,4) = \min \begin{cases} Best(2,4) + r_1 r_2 c_4 \\ \\ \end{cases}$$



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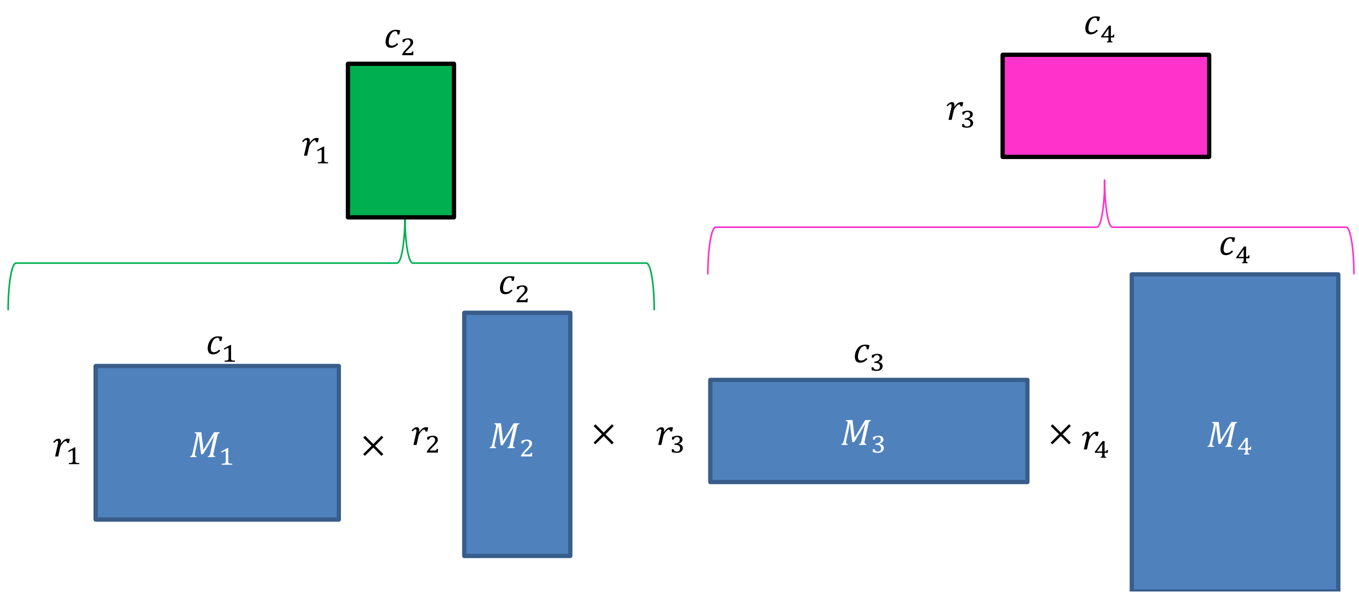

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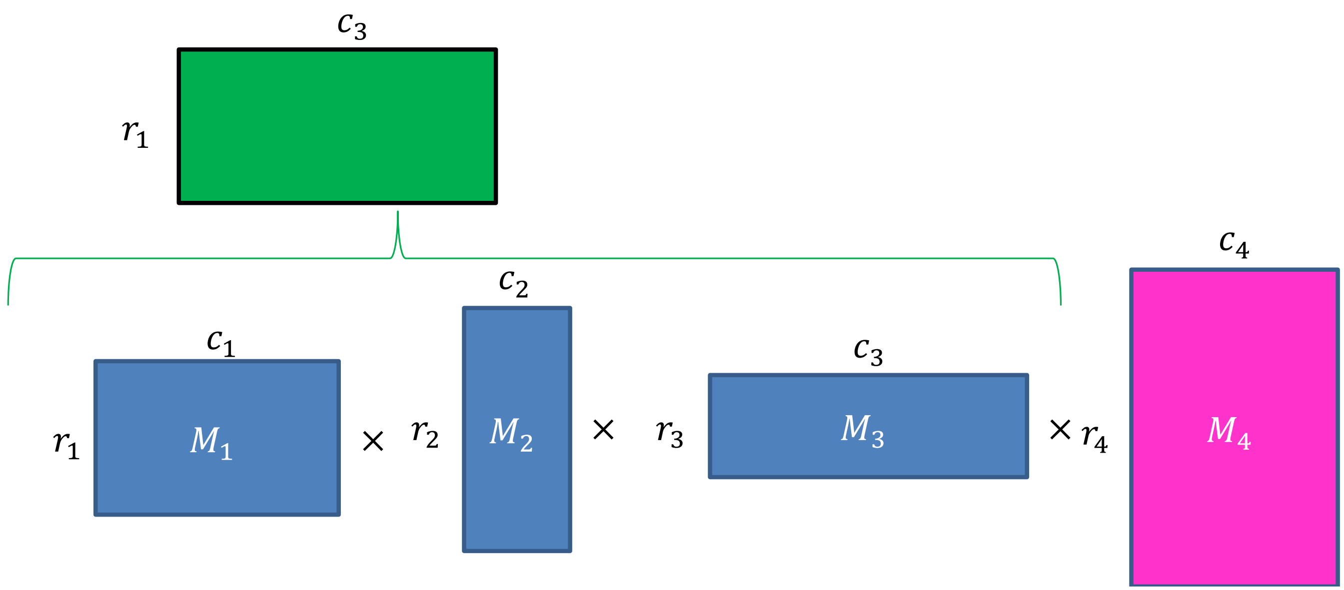

1. Identify the Recursive Structure of the Problem

- In general:

$Best(i,j) =$ cheapest way to multiply together M_i through M_j

$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$$Best(1,n) = \min \begin{cases} Best(2,n) + r_1 r_2 c_n \\ Best(1,2) + Best(3,n) + r_1 r_3 c_n \\ Best(1,3) + Best(4,n) + r_1 r_4 c_n \\ Best(1,4) + Best(5,n) + r_1 r_5 c_n \\ \quad \dots \\ Best(1,n-1) + r_1 r_n c_n \end{cases}$$

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2. Save Subsolutions in Memory

- In general:

$Best(i, j)$ = cheapest way to multiply together M_i through M_j

$$Best(i, j) = \min_{k=i}^{j-1}\left(Best(i, k) + Best(k+1, j) + r_i r_{k+1} c_j\right)$$

$$Best(i, i) = 0$$

Read from M[n] if present

Save to M[n]

$$Best(1, n) = \min \begin{cases} Best(2, n) + r_1 r_2 c_n \\ Best(1,2) + Best(3, n) + r_1 r_3 c_n \\ Best(1,3) + Best(4, n) + r_1 r_4 c_n \\ Best(1,4) + Best(5, n) + r_1 r_5 c_n \\ \ldots \\ Best(1, n-1) + r_1 r_n c_n \end{cases}$$

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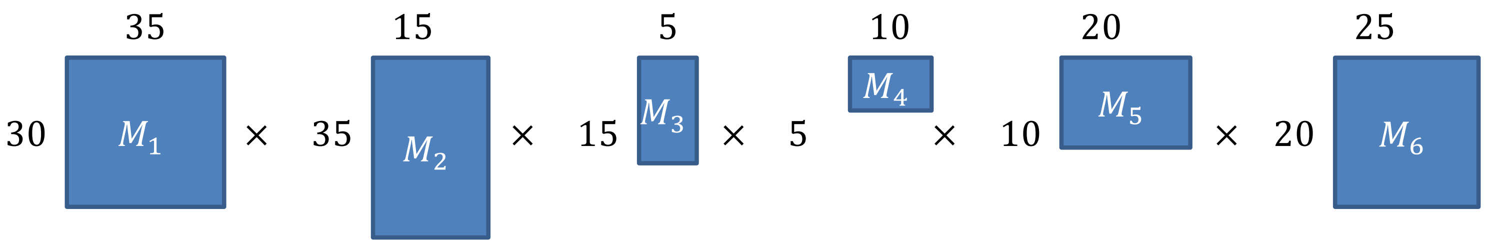
3. Select a good order for solving subproblems

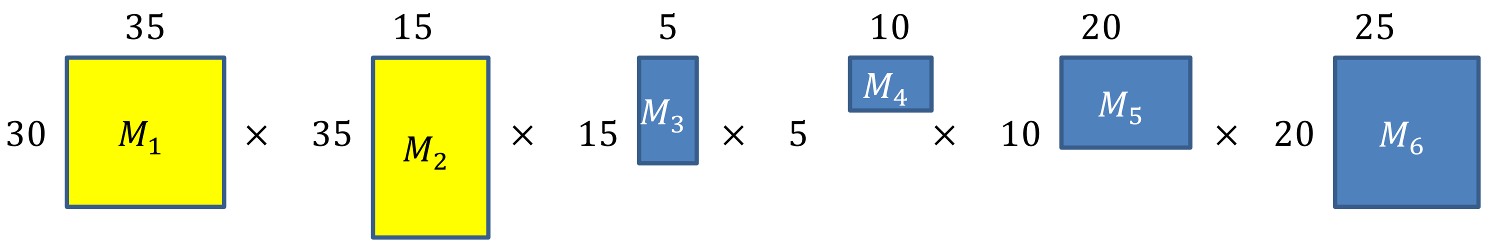


$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

$j =$	1	2	3	4	5	6	$= i$
	0						1
		0					2
			0				3
				0			4
					0		5
						0	6

3. Select a good order for solving subproblems



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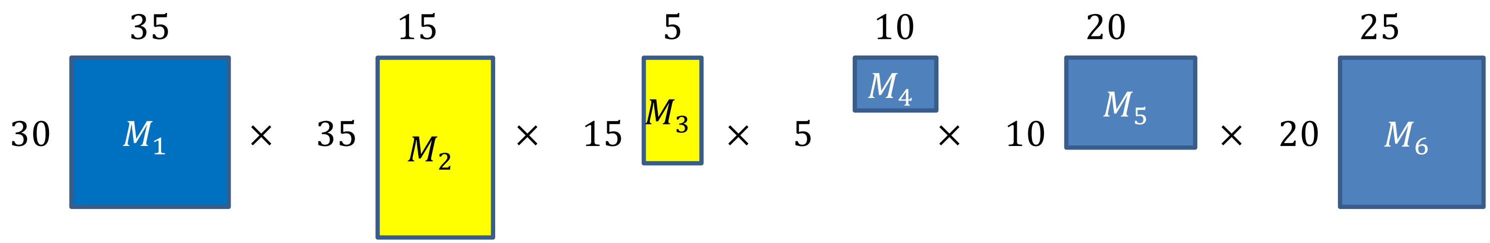
$$Best(i,i) = 0$$

$j =$ 1	2	3	4	5	6	$= i$
0	15750					1
	0					2
		0				3
			0			4
				0		5
					0	6

$$Best(1,2) = \min\left\{\begin{array}{l} Best(1,1) + Best(2,2) + r_1 r_2 c_2 \end{array}\right.$$

3. Select a good order for solving subproblems



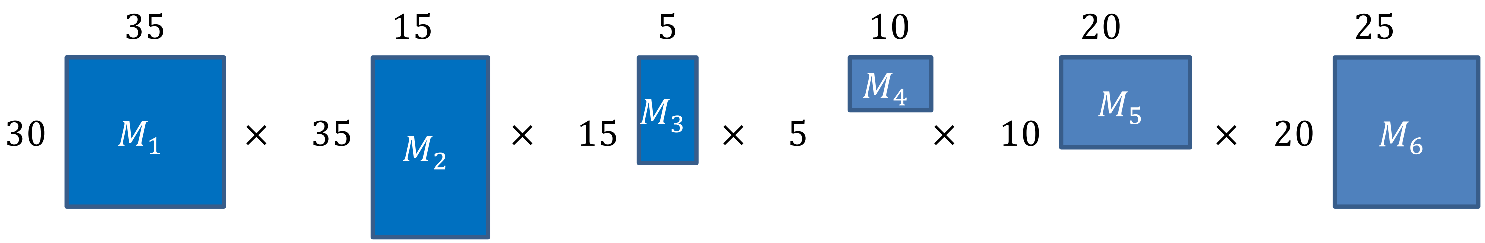
$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

$$Best(i,i) = 0$$

i \ $j =$	1	2	3	4	5	6
1	0	15750				
2		0	2625			
3			0			
4				0		
5					0	
6						0

$$Best(2,3) = \min \left\{ Best(2,2) + Best(3,3) + r_2 r_3 c_3 \right.$$

3. Select a good order for solving subproblems



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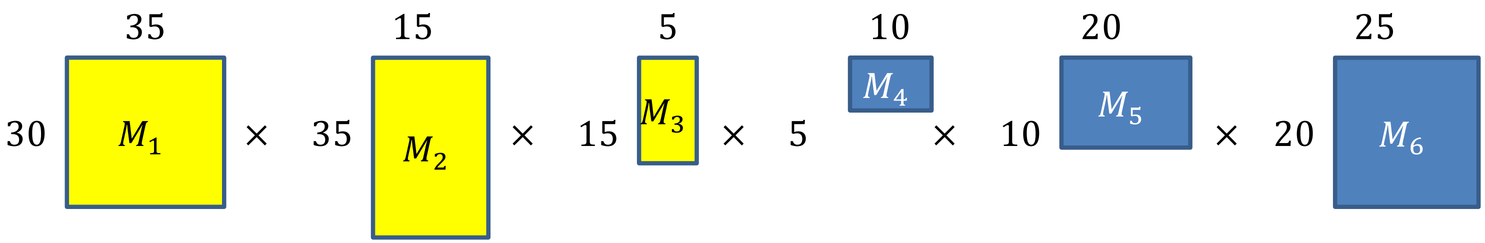
$$Best(i,i) = 0$$

$j =$ 1	2	3	4	5	6	$i =$
0	15750					1
	0	2625				2
		0	750			3
			0	1000		4
				0	5000	5
					0	6

3. Select a good order for solving subproblems



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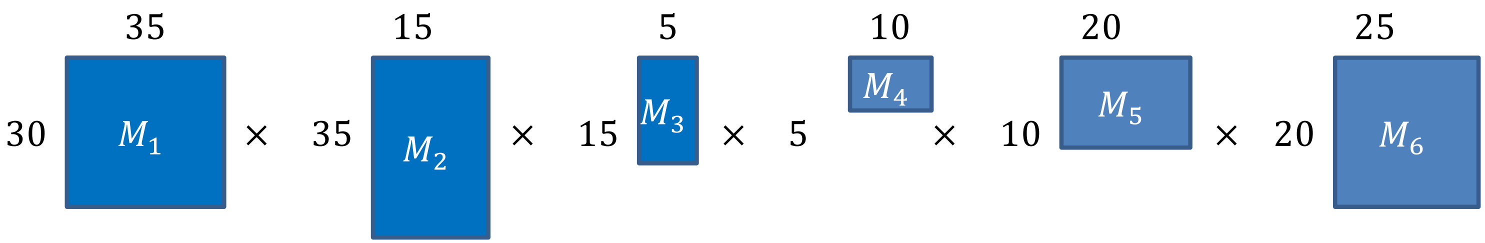
$$Best(i,i) = 0$$

$$r_1 r_2 c_3 = 30 \cdot 35 \cdot 5 = 5250$$
$$r_1 r_3 c_3 = 30 \cdot 15 \cdot 5 = 2250$$

$j =$ 1	2	3	4	5	6	$i =$
0	15750	7875				1
	0	2625				2
		0	750			3
			0	1000		4
				0	5000	5
					0	6

$$Best(1,3) = \min\begin{cases} \overset{0}{Best(1,1)} + \overset{2625}{Best(2,3)} + r_1 r_2 c_3 \\ \underset{15750}{Best(1,2)} + \underset{0}{Best(3,3)} + r_1 r_3 c_3 \end{cases}$$

3. Select a good order for solving subproblems



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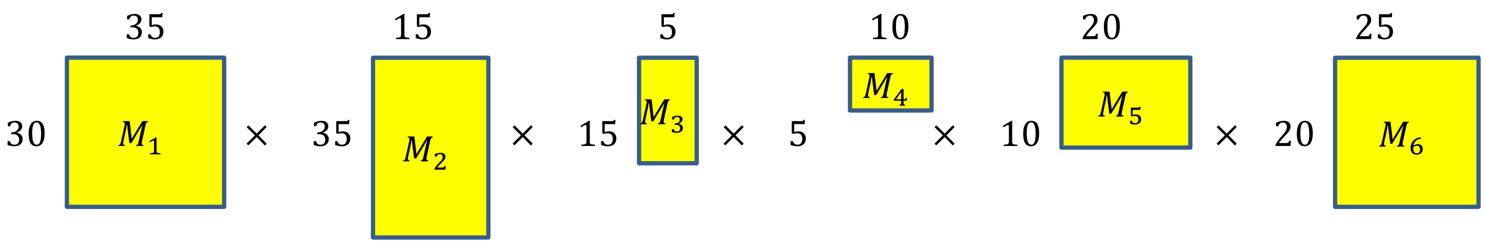
$$Best(i,i) = 0$$

$i =$ \ $j =$	1	2	3	4	5	6
1	0	15750	7875			
2		0	2625			
3			0	750		
4				0	1000	
5					0	5000
6						0

To find $Best(i,j)$: Need all preceding terms of row i and column j

Conclusion: solve in order of diagonal

Matrix Chaining



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$$Best(i,i) = 0$$

$j =$	1	2	3	4	5	6	$= i$
	0	15750	7875	9375	11875	15125	1
		0	2625	4375	7125	10500	2
			0	750	2500	5375	3
				0	1000	3500	4
					0	5000	5
						0	6

$$Best(1,6) = \min \begin{cases} Best(1,1) + Best(2,6) + r_1 r_2 c_6 \\ Best(1,2) + Best(3,6) + r_1 r_3 c_6 \\ Best(1,3) + Best(4,6) + r_1 r_4 c_6 \\ Best(1,4) + Best(5,6) + r_1 r_5 c_6 \\ Best(1,5) + Best(6,6) + r_1 r_6 c_6 \end{cases}$$

Run Time

1. Initialize $Best[i,i]$ to be all 0s $\Theta(n^2)$ cells in the Array
2. Starting at the main diagonal, working to the upper-right, fill in each cell using:
   1. $Best[i,i] = 0$
   2. $Best[i,j] = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$

$\Theta(n)$ options for each cell

Each "call" to Best() is a O(1) memory lookup

$\Theta(n^3)$ overall run time

Backtrack to find the best order

"Remember" which choice of k was the minimum at each cell.
Intuitively this was the best place to "split" for that range (i,j).

$$Best(i,j) = \min_{k=i}^{j-1}\left(Best(i,k) + Best(k+1,j) + r_i r_{k+1} c_j\right)$$

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		0	750	2500	5375	3
			0	1000	3500 5	4
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					0	6

$$Best(1,6) = \min \begin{cases} Best(1,1) + Best(2,6) + r_1 r_2 c_6 \\ Best(1,2) + Best(3,6) + r_1 r_3 c_6 \\ \boxed{Best(1,3) + Best(4,6) + r_1 r_4 c_6} \\ Best(1,4) + Best(5,6) + r_1 r_5 c_6 \\ Best(1,5) + Best(6,6) + r_1 r_6 c_6 \end{cases}$$

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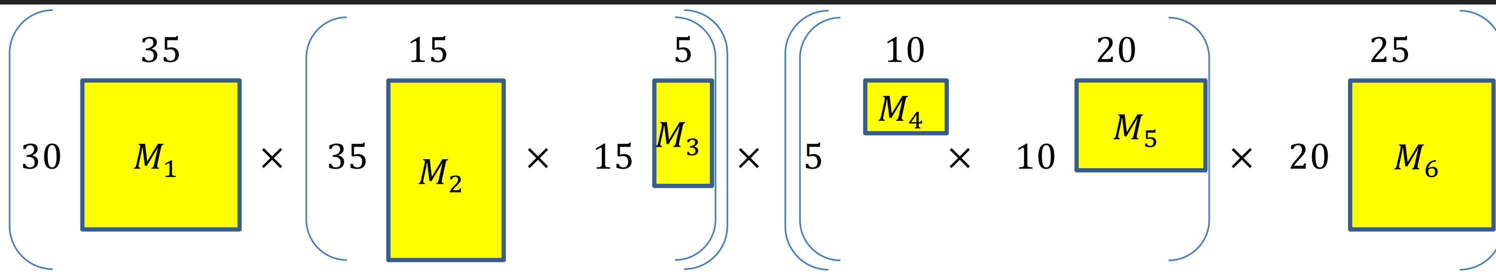
Matrix Chaining



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$$Best(1,6) = \min \begin{cases} Best(1,1) + Best(2,6) + r_1 r_2 c_6 \\ Best(1,2) + Best(3,6) + r_1 r_3 c_6 \\ \boxed{Best(1,3) + Best(4,6) + r_1 r_4 c_6} \\ Best(1,4) + Best(5,6) + r_1 r_5 c_6 \\ Best(1,5) + Best(6,6) + r_1 r_6 c_6 \end{cases}$$

Storing and Recovering Optimal Solution

- Maintain table Choice[i,j] in addition to Best table
  - Choice[i,j] = k means the best "split" was right after M_k
  - Work backwards from value for whole problem, Choice[1,n]
  - Note: Choice[i,i+1] = i because there are just 2 matrices
- From our example:
  - Choice[1,6] = 3. So $[M_1\ M_2\ M_3]\ [M_4\ M_5\ M_6]$
  - We then need Choice[1,3] = 1. So $[(M_1)\ (M_2\ M_3)]$
  - Also need Choice[4,6] = 5. So $[(M_4\ M_5)\ M_6]$
  - Overall: $[(M_1)\ (M_2\ M_3)]\ [(M_4\ M_5)\ M_6]$

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- Slides on Longest Common Subsequence (LCS) problem moved to next slide set.