Th9: The Poisson Process

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1 Introduction

A Poisson process is a model for a series of discrete events where the average time between events is known, but the exact timing of events is random. The arrival of an event is independent of the event before (waiting time between events is memoryless).

For example, suppose that from historical data, we know that earthquakes occur in a certain area with a rate of 2 per month. Other than this information, the timings of earthquakes seem to be completely random. Thus, we conclude that the Poisson process might be a good model for earthquakes. In practice, the Poisson process or its extensions have been used to model:

- The number of car accidents at a site or in an area;
- The location of users in a wireless network;
- The requests for individual documents on a web server;
- Photons landing on a photodiode.

2 Definition

Poisson random variable: a discrete random variable X is said to be a Poisson random variable with parameter μ, shown as $X \sim Poisson(\mu)$, if its range is $R_X = \{1, 2, 3, ...\}$, and its PMF is given by

$$P_X(k) = \begin{cases} \frac{e^{-\mu}\mu^k}{k!} & \text{for } k \in R_X \\ 0 & \text{otherwise} \end{cases}$$

Here are some useful facts that we have seen before:

1. If $X \sim Poisson(\mu)$, then $EX = \mu$, and $Var(X) = \mu$.
2. If $X_i \sim Poisson(\mu_i)$, for $i = 1, 2, ..., n$, and the X_i's are independent, then
$$X_1 + X_2 + ... + X_n \sim Poisson(\mu_1 + \mu_2 + ... + \mu_n) \quad .$$
3. The Poisson distribution can be viewed as the limit of binomial distribution.

Let $Y_n \sim Binomial(n, p = p(n))$. Let $\mu > 0$ be a fixed real number, and $\lim_{n\to\infty} np = \mu$. Then, the PMF of Y_n converges to a $Poisson(\mu)$ PMF, as $n \to \infty$. That is, for any $k \in \{0, 1, 2, ...\}$, we have

$$\lim_{n\to\infty} P_{Y_n}(k) = \frac{e^{-\mu}\mu^k}{k!}.$$

The resulting random process is called a Poisson process with rate (or intensity) λ. Here is a formal definition of the Poisson process.

The Poisson Process

Let $\lambda > 0$ be fixed. The counting process $\{N(t), t \in [0, \infty)\}$ is called a **Poisson process** with **rates** λ if all the following conditions hold:

1. $N(0) = 0$;
2. $N(t)$ has independent increments;
3. the number of arrivals in any interval of length $\tau > 0$ has $Poisson(\lambda\tau)$ distribution.

Note that from the above definition, we conclude that in a Poisson process, the distribution of the number of arrivals in any interval depends only on the length of the interval, and not on the exact location of the interval on the real line. Therefore the Poisson process has stationary increments.

2.1 Properties

The key properties of the Poisson process are:

1. **Stationarity:** The process is stationary, which means that the intensity or rate of events occurring in any time interval is constant.

2. **Independence:** The number of events in non-overlapping time intervals is independent. This implies that the occurrence of an event in one interval does not affect the occurrence of events in other intervals.

3. **Memorylessness:** The time until the next event is memoryless, meaning that the probability of an event occurring in the next infinitesimally small time interval is constant, regardless of the past.

4. **Discreteness:** The process is a counting process, meaning that it counts the number of events in a given time interval. The count can only take non-negative integer values.

5. **Homogeneity:** The process is homogeneous, meaning that the probability of observing k events in a time interval of length t is given by the Poisson distribution formula:

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

   where $N(t)$ is the number of events observed up to time t, and λ is the rate parameter, representing the average number of events per unit time.

6. **Interarrival Times:** The time between successive events, known as interarrival times, follows an exponential distribution with parameter λ.

7. **Sum of Poisson Processes:** If you have multiple independent Poisson processes with rates $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the sum of these processes is also a Poisson process with rate equal to the sum of the individual rates ($\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n$).

3 Simulation

Simulating a Poisson process on a computer is usually done in a bounded region of space, known as a simulation window, and requires two steps: appropriately creating a random number of points and then suitably placing the points in a random manner. Both these two steps depend on the specific Poisson point process that is being simulated.

3.1 Step 1: Number of points

The number of points N in the window, denoted here by W, needs to be simulated, which is done by using a (pseudo)-random number generating function capable of simulating Poisson random variables.

3.1.1 Homogeneous case

For the homogeneous case with the constant λ, the mean of the Poisson random variable N is set to $\lambda \mid W \mid$ where $\mid W \mid$ is the length, area or volume of W.

3.1.2 Inhomogeneous case

For the inhomogeneous case, $\lambda \mid W \mid$ is replaced with the volume integral

$$\mu(W) = \int_W \lambda(x)\, dx$$

3.2 Step 2: Positioning of points

The second stage requires randomly placing the N points in the window W.

3.2.1 Homogeneous case

For the homogeneous case in one dimension, all points are uniformly and independently placed in the window or interval W. For higher dimensions in a Cartesian coordinate system, each coordinate is uniformly and independently placed in the window W. If the window is not a subspace of Cartesian space (for example, inside a unit sphere or on the surface of a unit sphere), then the points will not be uniformly placed in W, and suitable change of coordinates (from Cartesian) are needed.

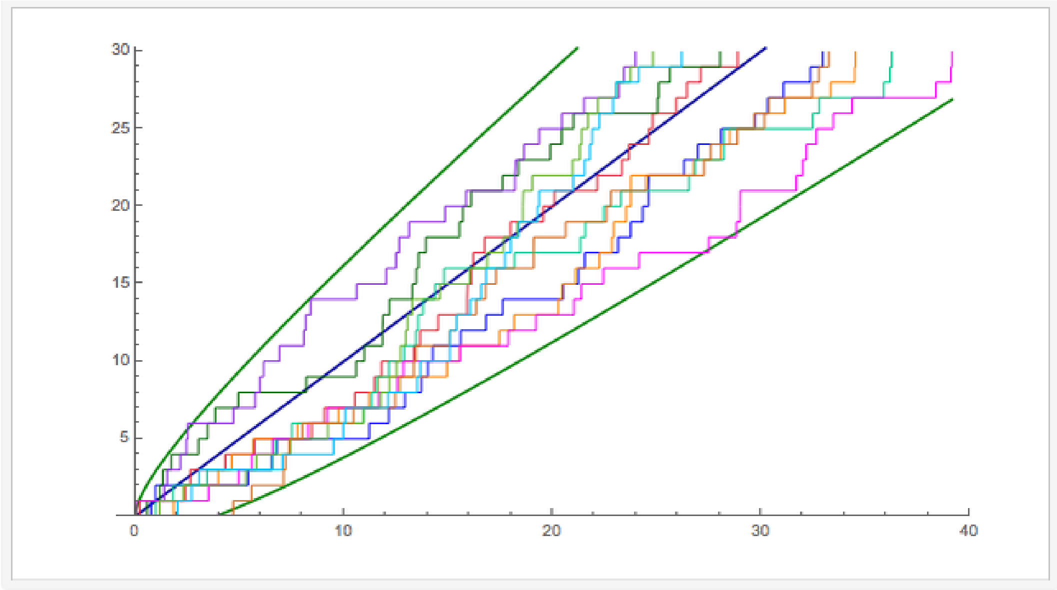
3.2.2 Inhomogeneous case

For the inhomogeneous case, a couple of different methods can be used depending on the nature of the intensity function $\lambda(x)$. If the intensity function is sufficiently simple, then independent and random non-uniform (Cartesian or other) coordinates of the points can be generated. For example, simulating a Poisson point process on a circular window can be done for an isotropic intensity function (in polar coordinates r and θ), implying it is rotationally variant or independent of θ but dependent on r, by a change of variable in r if the intensity function is sufficiently simple.

For more complicated intensity functions, one can use an acceptance-rejection method, which consists of using (or 'accepting') only certain random points and not using (or 'rejecting') the other points, based on the ratio:

$$\frac{\lambda(x_i)}{\mu(W)} = \frac{\lambda(x_i)}{\int_W \lambda(x)\, dx}.$$

where x_i is the point under consideration for acceptance or rejection.



Example of simulation of poisson process.

References

[1] https://en.wikipedia.org/wiki/Poisson_point_process#Simulation

[2] https://www.probabilitycourse.com/chapter11/11_1_2_basic_concepts_of_the_poisson_process.php

[3] https://builtin.com/data-science/poisson-process