Th8: Stochastic processes and SDE's

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1 Introduction

A stochastic process is any process describing the evolution in time of a random phenomenon. From a mathematical point of view, is an object usually defined as a sequence of random variables, where the index of the sequence has the interpretation of time.
Stochastic processes are widely used as mathematical models of systems and phenomena that appear to vary in a random manner. Stochastic processes have applications in many disciplines such as biology, chemistry, ecology, neuroscience, physics, image processing, signal processing, control theory, information theory, computer science, and telecommunications. Furthermore, seemingly random changes in financial markets have motivated the extensive use of stochastic processes in finance.

2 Definition

A stochastic process is defined as a collection of random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is a sample space, $\mathcal{F}$ is a σ-algebra, $\mathcal{P}$ is a probability measure; and the random variables, indexed by some set T, all take values in the same mathematical space S, which must be measurable with respect to some σ-algebra Σ.

In other words, for a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a measurable space (S, Σ), a stochastic process is a collection of S-valued random variables, which can be written as:

$$\{X(t) : t \in T\}$$

3 SDE

A SDE is comprised of a differential equation that includes a stochastic process.

SDEs have many applications throughout pure mathematics and are used to model various behaviours of stochastic models such as stock prices, random growth models or physical systems that are subjected to thermal fluctuations.

A Stochastic Differential Equation (SDE) is a differential equation in which one or more terms contain a stochastic process. A standard form for an SDE is given by:

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t \tag{1}$$

where:

- X_t is the stochastic process,
- $\mu(t, X_t)$ is the drift term,
- $\sigma(t, X_t)$ is the diffusion term,
- dW_t is the differential of a Wiener process.

This equation represents the infinitesimal change in the stochastic process X_t over a small time interval dt. The drift term $\mu(t, X_t)$ governs the deterministic part of the motion, while the diffusion term $\sigma(t, X_t)$ introduces randomness through the Wiener process dW_t.

There are many examples of SDE:

- The SDE for Geometric Brownian Motion is given by:

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t \tag{2}$$

  where:

  - X_t is the asset price at time t,
  - μ is the drift coefficient,
  - σ is the diffusion coefficient,
  - dW_t is the differential of a Wiener process.

- The SDE for an Ornstein-Uhlenbeck process is given by:

$$dX_t = \theta(\mu - X_t) \, dt + \sigma \, dW_t \tag{3}$$

  where:

  - X_t is the process variable at time t,
  - θ is the speed of mean reversion,
  - μ is the mean to which the process reverts,
  - σ is the volatility,
  - dW_t is the differential of a Wiener process.

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- The Black-Scholes Equation for a financial option pricing is a well-known SDE given by:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \tag{4}$$

  where:

  - S_t is the price of the underlying asset at time t,
  - μ is the drift coefficient (expected return),
  - σ is the volatility,
  - dW_t is the differential of a Wiener process.

- The CIR Model, used in interest rate modeling, is described by the following SDE:

$$dr_t = \kappa(\theta - r_t) \, dt + \sigma \sqrt{r_t} \, dW_t \tag{5}$$

  where:

  - r_t is the short-term interest rate at time t,
  - κ is the speed of mean reversion,
  - θ is the long-term mean,
  - σ is the volatility,
  - dW_t is the differential of a Wiener process.

References

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