Th12: Ito Integration and Calculus

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1 Introduction

Itô's famous work on stochastic integrals and stochastic differential equations started in 1942, when mathematicians in Japan were completely isolated from the world because of the war. During that year, he wrote a colloquium report in Japanese, where he presented basic ideas for the study of diffusion processes and jump–diffusion processes. Most of them were completed and published in English during 1944–1951.

Itô's work was observed with keen interest in the 1960s both by probabilists and applied mathematicians. Itô's stochastic integrals based on Brownian motion were extended to those based on martingales, through Doob–Meyer's decomposition of positive submartingales. The famous Itô formula was extended to semimartingales.

2 Definition

The central concept is the Itô stochastic integral, a stochastic generalization of the Riemann–Stieltjes integral in analysis. The integrands and the integrators are now stochastic processes:

$$Y_t = \int_0^t H_s \, dX_s$$

where H is a locally square-integrable process adapted to the filtration generated by X (Revuz and Yor 1999, Chapter IV), which is a Brownian motion or, more generally, a semimartingale. The result of the integration is then another stochastic process. Concretely, the integral from 0 to any particular t is a random variable, defined as a limit of a certain sequence of random variables.

As Itô calculus is concerned with continuous-time stochastic processes, it is assumed that an underlying filtered probability space is given

$$(\Omega, F, (F_t)_{t\geq 0}, P)$$

The σ-algebra F_t represents the information available up until time t, and a process X is adapted if X_t is F_t-measurable. A Brownian motion B is understood to be an F_t-Brownian motion, which is just a standard Brownian motion with the properties that B_t is F_t-measurable and that $B_{t+s} - B_t$ is independent of F_t for all $t, s \geq 0$.

3 Integration

Consider a stochastic process X_t and its differential dX_t with respect to Brownian motion W_t. The differential is given by the Itô integral:

$$dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t$$

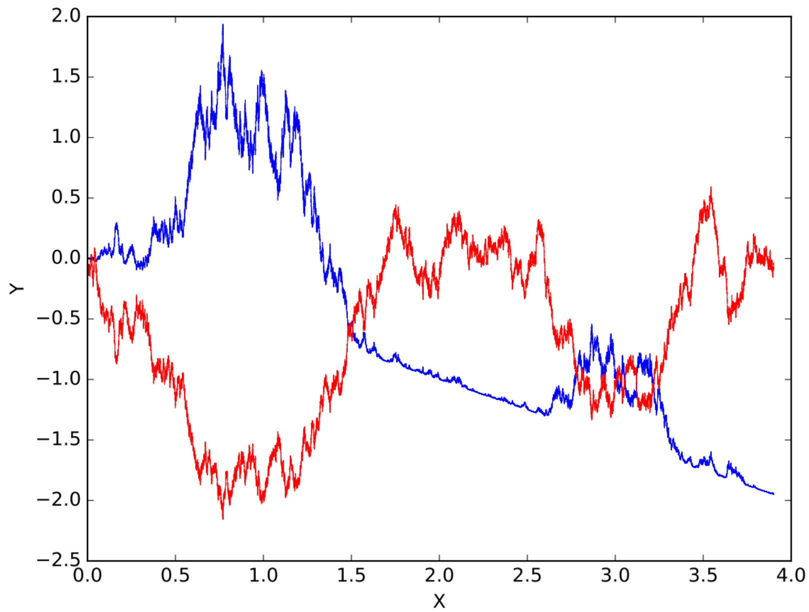
where $\mu(t, X_t)$ is the drift term, $\sigma(t, X_t)$ is the diffusion term, dt is the differential of time, and dW_t is the differential of Brownian motion.

To integrate this differential equation over a time interval $[a, b]$, we can use the Itô integral notation:

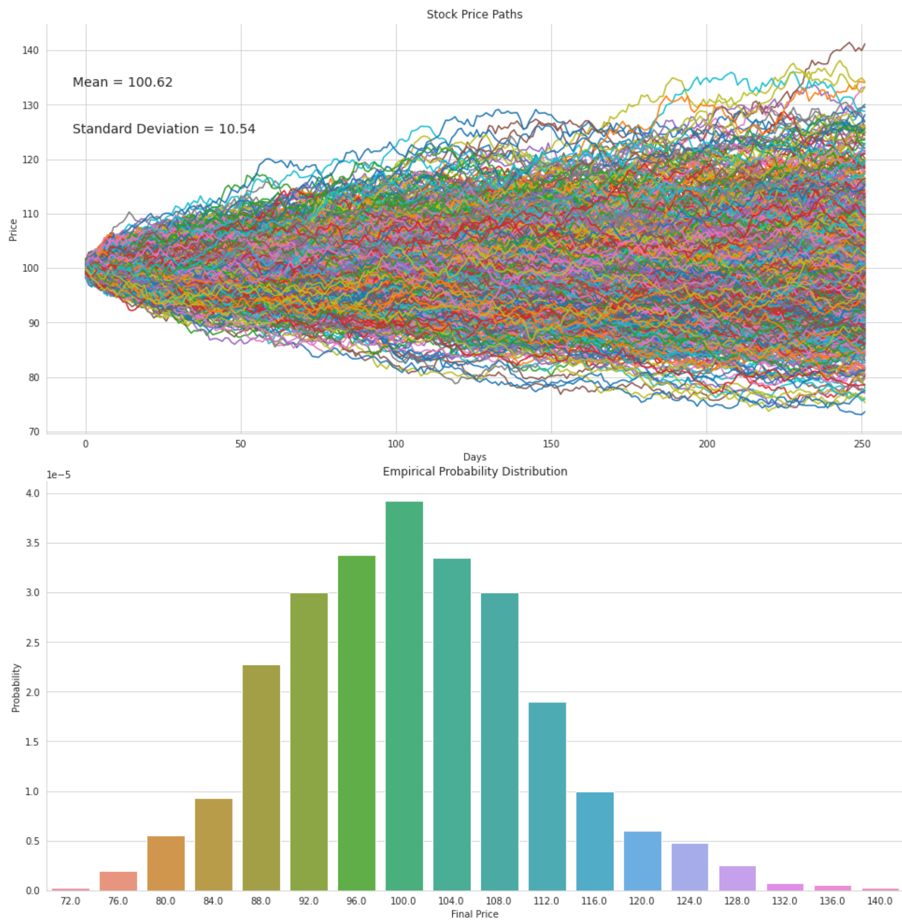
$$X_b - X_a = \int_a^b \mu(t, X_t)\, dt + \int_a^b \sigma(t, X_t)\, dW_t$$

This represents the integral of the drift term with respect to time plus the Itô integral of the diffusion term with respect to Brownian motion.

4 Simulations



Itô integral Yt(B) (blue) of a Brownian motion B (red) with respect to itself, i.e., both the integrand and the integrator are Brownian. It turns out Yt(B) = (B2 - t)/2.



Application of Ito Calculus: Monte Carlo Simulation.

References

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