Th10: The Wiener process and the GBM

Federico Detomaso

1 Introduction

In mathematics, the Wiener process is a real-valued continuous-time stochastic process named in honor of American mathematician Norbert Wiener for his investigations on the mathematical properties of the one-dimensional Brownian motion. It is often also called Brownian motion due to its historical connection with the physical process of the same name originally observed by Scottish botanist Robert Brown. It is one of the best known Lévy processes (càdlàg stochastic processes with stationary independent increments) and occurs frequently in pure and applied mathematics, economics, quantitative finance, evolutionary biology, and physics.

2 Definition

A Wiener process, also known as Brownian motion, is a continuous-time stochastic process that satisfies the following properties:

1. $W_0 = 0$ (It starts at 0).
2. The increments $W_{t_2} - W_{t_1}$ are normally distributed with mean 0 and variance $t_2 - t_1$ for any $0 \leq t_1 < t_2$.
3. The process has independent increments, meaning that the increment in any time interval is independent of the past values.
4. The paths of the process are continuous, but the process is not differentiable.

Mathematically, a Wiener process is often denoted as $\{W_t : t \geq 0\}$ and satisfies the following properties:

$$W_t \sim \mathcal{N}(0, t) \quad \text{for all } t \geq 0 \tag{1}$$

where W_t is normally distributed with mean 0 and variance t.

3 GBM

A geometric Brownian motion (GBM) (also known as exponential Brownian motion) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion (also called a Wiener process) with drift. It is an important example of stochastic processes satisfying a stochastic differential equation (SDE); in particular, it is used in mathematical finance to model stock prices in the Black–Scholes model.

Definition:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \tag{2}$$

where:

- S_t is the price of the asset at time t,
- μ is the drift coefficient (expected return),
- σ is the volatility,
- dW_t is the differential of a Wiener process.

The solution to this SDE is given by the exponential process:

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \tag{3}$$

where:

- S_0 is the initial price of the asset at time $t = 0$.

3.1 Derivations

When deriving further properties of GBM, use can be made of the SDE of which GBM is the solution, or the explicit solution given above can be used. For example, consider the stochastic process $\log(S_t)$. This is an interesting process, because in the Black–Scholes model it is related to the log return of the stock price. Using Itô's lemma with $f(S) = \log(S)$ gives

$$\begin{aligned} d\log(S) &= f'(S)dS + \tfrac{1}{2}f''(S)S^2\sigma^2 dt \\ &= \tfrac{1}{S}(\sigma S dW_t + \mu S dt) - \tfrac{1}{2}\sigma^2 dt \\ &\sigma dW_t + (\mu - \tfrac{\sigma^2}{2})dt \end{aligned} \quad .$$

It follows that $E\log(S_t) = \log(S_0) + (\mu - \frac{\sigma^2}{2})t$.

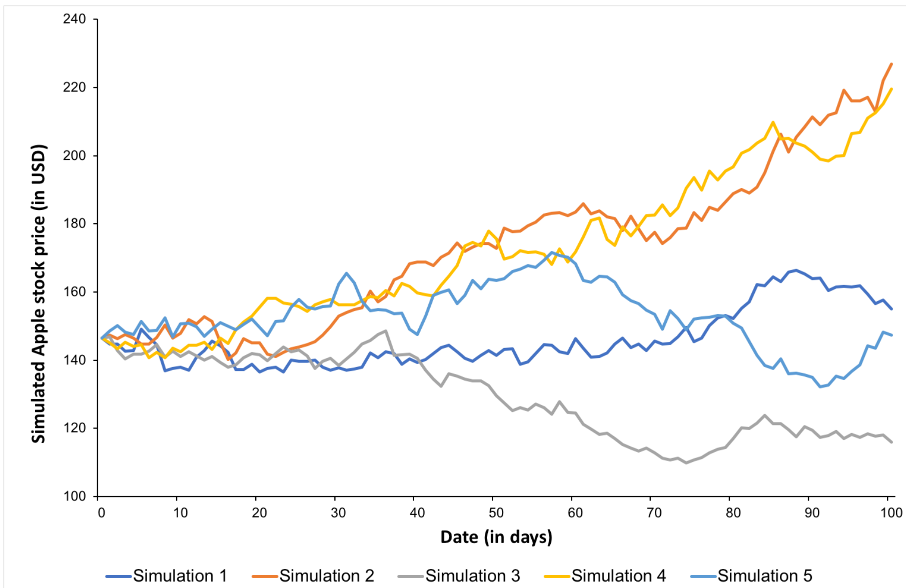
This result can also be derived by applying the logarithm to the explicit solution of GBM:

$$\begin{aligned} \log(S_t) &= \log(S_0 \exp((\mu - \tfrac{\sigma^2}{2})t + \sigma W_t)) \\ &= \log(S_0) + (\mu - \tfrac{\sigma^2}{2})t + \sigma W_t. \end{aligned}$$

Taking the expectation yields the same result as above: $E\log(S_t) = \log(S_0) + (\mu - \frac{\sigma^2}{2})t$.

3.2 Simulations

A Monte Carlo simulation applies a selected model (that specifies the behavior of an instrument) to a large set of random trials in an attempt to produce a plausible set of possible future outcomes. In regard to simulating stock prices, the most common model is geometric Brownian motion (GBM). GBM assumes that a constant drift is accompanied by random shocks. While the period returns under GBM are normally distributed, the consequent multi-period (for example, ten days) price levels are lognormally distributed.



References

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[2] https://en.wikipedia.org/wiki/Geometric_Brownian_motion

[3] https://www.simtrade.fr/blog_simtrade/monte-carlo-simulation-method/

[4] https://en.wikipedia.org/wiki/Stochastic_differential_equation