Th11: The functional CLT (Donsker's invariance principle)

Federico Detomaso

1 Introduction

In 1952, Monroe D. Donsker formally stated what is now known as Donsker's invariance principle, which formalized this idea of the Wiener process as a limit. Donsker's invariance principle shows that the Wiener process is a universal scaling limit of a large class of random walks. This means that the exact distributions of the steps of the random walk do not matter when we take the limit as time and step distance shrink to zero.

In probability theory, is a functional extension of the central limit theorem.

2 Definition

Let $X_1, X_2, X_3, ...$ be a sequence of independent and identically distributed (i.i.d.) random variables with mean 0 and variance 1. Let $S_n := \sum_{i=1}^{n} X_i$. The stochastic process $S := (S_n)_{n \in N}$ is known as a random walk. Define the diffusively rescaled random walk (partial-sum process) by

$$W^{(n)}(t) := \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}},\ t \in [0,1].$$

The central limit theorem asserts that $W^{(n)}(1)$ converges in distribution to a standard Gaussian random variable $W(1)$ as $n \longrightarrow \infty$. Donsker's invariance principle extends this convergence to the whole function $W^{(n)} := (W^{(n)}(t))_{t\in[0,1]}$. More precisely, in its modern form, Donsker's invariance principle states that: As random variables taking values in the Skorokhod space $D[0,1]$, the random function $W^{(n)}$ converges in distribution to a standard Brownian motion $W := (W(t))_{t\in[0,1]}$ as $n \longrightarrow \infty$.

3 Proof

We will prove this result by showing the convergence of characteristic functions. Let $\phi_X(t)$ be the characteristic function of X_1.

The characteristic function of Z_n is given by

$$\phi_{Z_n}(t) = E\left[e^{itZ_n}\right] = E\left[e^{it\frac{S_n - n\mu}{\sigma\sqrt{n}}}\right] = \left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n.$$

Now, let $\phi(t)$ be the characteristic function of the standard normal distribution. We know that $\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)$ converges pointwise to $\phi(t)$ as n approaches infinity.

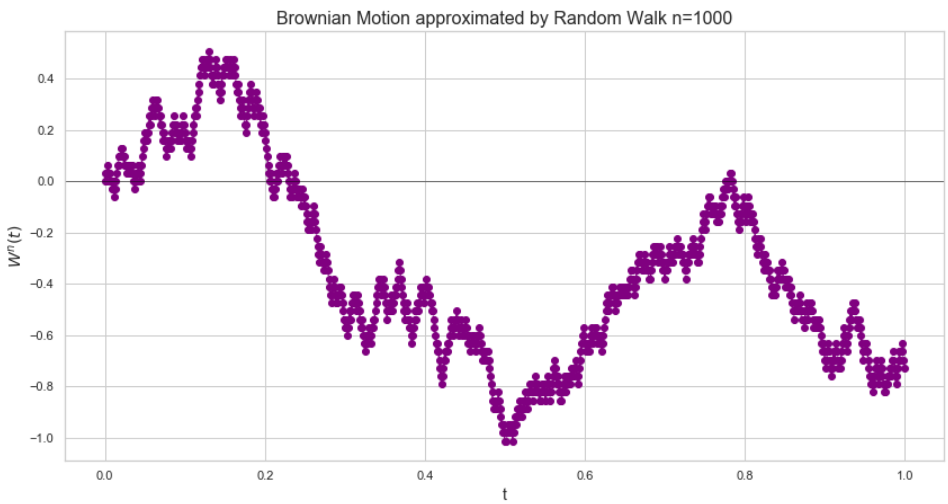
Using dominated convergence, we can interchange the limit and the expectation:

$$\lim_{n\to\infty} \phi_{Z_n}(t) = \lim_{n\to\infty} \left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n = \left(\lim_{n\to\infty} \phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n = e^{-\frac{t^2}{2}}.$$

The limit is the characteristic function of the standard normal distribution, which implies convergence in distribution by the Levy Continuity Theorem. Therefore, Z_n converges in distribution to the standard Brownian motion.

4 Simulations

Donsker's Theorem (also known as Donsker's Invariance Principle, or the Functional Central Limit Theorem) allows to simulate paths of the one-dimensional standard Brownian Motion using different kinds of random walks.



References

[1] https://en.wikipedia.org/wiki/Wiener_process

[2] https://en.wikipedia.org/wiki/Geometric_Brownian_motion

[3] https://www.simtrade.fr/blog_simtrade/monte-carlo-simulation-method/

[4] https://en.wikipedia.org/wiki/Stochastic_differential_equation