Toasty's Problems

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Credits

This document uses the package `shen.sty` by the AoPS user **TheUltimate123**.

Introduction

This handout contains every problem which I, DeToasty3, have proposed which has appeared in a released math competition, mock or official. Note that this also means that I will not be putting problems here which I posted outside of a contest or are currently not public.

The math competitions I have written problems for so far are:

- Lexington Math Tournament (LMT)
- Online Test Seasonal Series (OTSS)
- De Mathematics Competitions (DMC)
- Karate Masters Mathematics Competitions (KMMC)
- Geometry AMC (GAMC)
- Mock MATHCOUNTS States (hosted by the AoPS user **smartatmath**)

These problems will be listed in chronological order and in the order in which the problems are listed in their respective tests. Additionally, along the way I will be providing some remarks to certain problems[1], such as how I thought of the problem idea, how much I personally like the problem, and possibly other points that I would like to address. This document will not contain any solutions to my problems, but if you are curious, you can easily find most of the solutions to my problems by using the clickable links above. Finally, I will be updating this document as I continue to propose problems for math competitions.

Without further ado, please sit back and enjoy the problems! (and yes, this was a reference to the AoPS user **Binomial-theorem**)

[1] Not every problem!

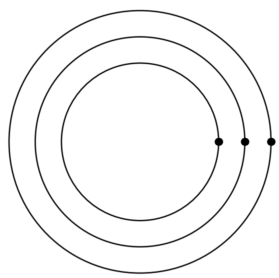
Problems (2019)

1. **(2019 Fall LMT Individual P6)**
   Find the minimum possible value of the expression $|x+1|+|x-4|+|x-6|$.

2. **(2019 Fall LMT Individual P11)**
   A two-digit number has the property that the difference between the number and the sum of its digits is divisible by the units digit. If the tens digit is 5, how many different possible values of the units digit are there?

3. **(2019 Fall LMT Theme P3)**
   Joe Quigley has 12 students in his math class. He will distribute N worksheets among the students. Find the smallest positive integer N for which any such distribution of the N worksheets among the 12 students results in at least one student having at least 3 worksheets.

4. **(2019 Fall LMT Theme P7)**
   Three planets with coplanar, circular, and concentric orbits are shown on the backside of this page[2]. The radii of the three circles are 3, 4, and 5. Initially, the three planets are collinear. Every hour, the outermost planet moves one-sixth of its full orbit, the middle planet moves one-fourth of its full orbit, and the innermost planet moves one-third of its full orbit (A full orbit occurs when a planet returns to its initial position). Moreover, all three planets orbit in the same direction. After three hours, what is the area of the triangle formed by the planets as its three vertices?



5. **(2019 Fall LMT Theme P11)**
   Festivus occurs every year on December 23rd. In 2019, Festivus will occur on a Monday. On what day will Festivus occur in the year 2029?

> **Remark.** This is the edited version of my problem; I do not have the original problem at hand.

6. **(2019 Fall LMT Theme P13)**
   How many permutations of the word $CHRISTMAS$ are there such that the S's are not next to each other and there is not a vowel anywhere between the two S's?

[2] In the official test sheet, the diagram was indeed on the backside of the page.

7. **(2019 Fall LMT Team P2)**
Determine the number of positive integers n with $1 \leq n \leq 400$ that satisfy the following:

- n is a square number.
- n is one more than a multiple of 5.
- n is even.

8. **(2019 Fall LMT Guts P9)**
A positive integer n is equal to one-third the sum of the first n positive integers. Find n.

9. **(2019 Fall LMT Guts P12)**
Define a sequence recursively by $F_0 = 0$, $F_1 = 1$, and for all $n \geq 2$, $F_n = \left\lceil \frac{F_{n-1}+F_{n-2}}{2} \right\rceil + 1$, where $\lceil r \rceil$ denotes the least integer greater than or equal to r. Find F_{2019}.

> **Remark.** The original problem asked for F_{13}, which is something that the AoPS user **GammaZero** will likely understand.

Problems (2020)

1. **(Season 1 TMC 10A P23/12A P20)**
In convex quadrilateral $ABCD$, $\angle A = 90^\circ$, $\angle C = 60^\circ$, $\angle ABD = 25^\circ$, and $\angle BDC = 5^\circ$. Given that $AB = 4\sqrt{3}$, find the area of quadrilateral $ABCD$.

   (A) 4 **(B)** $4\sqrt{3}$ **(C)** 8 **(D)** $8\sqrt{3}$ **(E)** $16\sqrt{3}$

   > **Remark.** This problem was originally proposed to the 2020 Spring LMT. However, due to certain changes, only the math team captains were able to propose problems for that contest, so I ended up moving this problem to OTSS.

2. **(Season 1 TMC 12B P7)**
Given that x and y are positive real numbers, with x and y each less than $\frac{\pi}{2}$, that satisfy the equations $x + y = \frac{\pi}{2}$ and $\sin(x) + 2\cos(y) = \frac{3\sqrt{3}}{2}$, what is $|x - y|$?

   (A) 0 **(B)** $\frac{\pi}{12}$ **(C)** $\frac{\pi}{6}$ **(D)** $\frac{\pi}{4}$ **(E)** $\frac{\pi}{3}$

3. **(Season 1 TMC 10B P13)**
For Color Day, 12 students in a class are to be randomly assigned a T-shirt to wear with one of three colors: red, blue, and yellow. A color may be worn by as few as 0 students. However, since the teacher wants color balance, there cannot be more than 9 students wearing the same color. In how many ways can this happen? Assume that the students are indistinguishable.

   (A) 71 **(B)** 73 **(C)** 79 **(D)** 85 **(E)** 91

   > **Remark.** This problem was originally proposed to the 2019 Fall LMT for the theme round, where instead of colors, it was which car of the train students would get put in to go to PUMaC. However, this problem was not used, so I ended up moving this problem to OTSS.

4. **(Season 1 TMC 10B P15)**
Let $\triangle ABC$ be isosceles with $AB = AC$. Let D be the reflection of B across the centroid of the triangle and M be the midpoint of $\overline{BC}$. If the area of quadrilateral $ADMB$ is 6 and $BC = 4$, then what is the square of length AB?

   (A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

   > **Remark.** A previous problem was written by the AoPS user **PCChess**, who used something with the centroid in his problem. However, I took it for a spin and made my own completely different problem which uses area ratios.

5. **(Season 1 TMC 12B P20)**
Richard writes the quadratic $f(x) = ax^2+bx+c$ on a whiteboard, where a, b, and c are distinct nonzero complex numbers. Matthew sees Richard's quadratic, and rearranges the order of the coefficients (i.e. permutes the order of a, b, and c) to make his own six distinct quadratics: $g_1(x)$, $g_2(x)$, $g_3(x)$, $g_4(x)$, $g_5(x)$, and $g_6(x)$ (one of which is equal to $f(x)$). What is the minimum number of possible distinct roots of
$$\prod_{k=1}^{6}(f(x)+g_k(x))?$$

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 10

> **Remark.** I originally wrote a version of this problem which was very guessable and not very full-fledged. However, thanks to the AoPS user **P_Groudon**, he was able to turn my problem into this pretty complete and nice problem, albeit a bit too hard for a P20 on an AMC 12.

6. **(Season 1 TMC 10B P23/12B P21)**
In a room with 10 people, each person knows exactly 4 different languages. A conversation is held between every pair of people with a language in common. If a total of 36 different languages are known throughout the room, and no two people have more than one language in common, what is the sum of all possible values of n such that a total of n conversations are held?

(A) 19 **(B)** 25 **(C)** 32 **(D)** 35 **(E)** 37

> **Remark.** A previous problem was written by the AoPS user **Qinghan04**, who used something with languages and conversations in her problem (her problem is not public, as it was an unused proposal to the 2019 Fall LMT). I expanded upon her promising problem idea and created what I personally believe to be one of the best problems which I have written thus far.

7. **(Season 1 TMC 10B P25/12B P24)**
Let O_1 be a circle with radius r. Let O_2 be a circle with radius between $\frac{r}{2}$ and r, exclusive, that goes through the center of circle O_1. Denote points X and Y as the intersections of the two circles. Let P be a point on the major arc $\widehat{XY}$ of O_1. Let $\overline{PX}$ intersect O_2 at A, strictly between P and X. Let $\overline{PY}$ intersect O_2 at B, strictly between P and Y. Let E be the midpoint of $\overline{PX}$ and F be the midpoint of $\overline{PY}$. If $AY = 100$, $AB = 65$, and $EF = 52$, what is BX?

(A) 104 **(B)** 105 **(C)** 106 **(D)** 107 **(E)** 108

> **Remark.** This problem was originally written by the AoPS user **jeteagle**, who intended this problem to be a problem with coordinate bash as its intended solution. However, me and the AoPS user **P_Groudon** worked our way to include some more interesting geometric concepts such as cyclic quadrilaterals, which mitigated the concern of this problem being too coordinate bash-able.

8. **(Season 1 OTIE P2)**
Jela and Benn are playing a game. Each round, Jela and Benn each flip a fair coin at the same time. Jela and Benn win if they flip heads together. However, they lose if they flip tails together for three rounds in a row. If neither event happens after the end of 4 rounds, they also lose. The probability that Jela and Benn win can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

> **Remark.** Originally, this problem was written by the AoPS user **ivyzheng**, but I came in and the two of us continued to improve the problem to its current state. I do believe, however, that it is a bit too casework-heavy and slightly bashy for a P2 on an AIME. Also, the AoPS user **I-_-I** suggested the names.

9. **(Season 1 OTIE P3)**
Two dogs, Otie and Amy, are each given an integer number of biscuits to eat, where Otie and Amy get x and y biscuits, respectively, and $0 < x < y < 72$. At the start, the numbers x, y, and 72 form an arithmetic progression, in that order. Each dog then eats N of their biscuits, where N is a positive integer less than x. After they finish eating, Amy now has exactly three times the number of biscuits left over as Otie. Find the number of possible values of N.

10. **(Mock MATHCOUNTS States Sprint P3)**
At his current speed, Michael runs 10 yards in 5 seconds. If he runs 0.3 yards per second faster, then how many yards can he run in 10 seconds?

11. **(2020 DMC 10 P1)**
What is the value of
$$\frac{(2^0-2^1)^{2020}}{(2\cdot 0+2^0)^{2021}}?$$

(A) -1 **(B)** $-\frac{1}{2}$ **(C)** 0 **(D)** $\frac{1}{2}$ **(E)** 1

12. **(2020 DMC 10 P2)**
If the ratio of males to females in a country club is exactly 9 to 5, and there are fewer than 100 people in the club, what is the largest possible number of people in the club? (Assume that all of the people in the club are either male or female.)

(A) 95 **(B)** 96 **(C)** 97 **(D)** 98 **(E)** 99

> **Remark.** This problem was originally proposed to the 2019 Fall LMT.

13. **(2020 DMC 10 P4)**
A dog has four legs, and a dug has three legs. Janelle has a whole number of dogs and dugs as pets, and she has no other pets. If there are 61 legs across all of Janelle's pets, what is the smallest possible number of dugs that Janelle could have?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

14. **(2020 DMC 10 P5)**
Rohan wants to distribute 25 slices of pizza to n people such that each person gets an equal number of slices, except for one person who gets one more slice than each of the other people. If n is greater than 1, how many different integer values of n exist?

(A) 2 **(B)** 5 **(C)** 7 **(D)** 8 **(E)** 9

> **Remark.** This problem was originally proposed to the Season 2 TMC 10.

15. **(2020 DMC 10 P6)**
Two distinct elements x and y are chosen from the set $\{1, 2, 3, 4\}$ at random. What is the probability that the line with slope $\frac{y}{x}$ passing through the point (x, y) also passes through the point $(2020, 1010)$?

(A) $\dfrac{1}{12}$ **(B)** $\dfrac{1}{6}$ **(C)** $\dfrac{1}{4}$ **(D)** $\dfrac{1}{3}$ **(E)** $\dfrac{1}{2}$

> **Remark.** This problem was originally proposed to the Season 2 TMC 10.

16. **(2020 DMC 10 P7)**
Anthony, Daniel, and Richard have 17, 20, and 26 trading cards, respectively. Every minute, one of the three boys gives away two of his trading cards such that the other two boys get one trading card each. What is the shortest amount of time, in minutes, that it could take for the three boys to each have an equal number of trading cards?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

> **Remark.** For some reason, I am actually quite proud of writing this problem. I believe that it is actually quite original for a problem in the first ten on an AMC 10.

17. **(2020 DMC 10 P8)**
Two distinct points A and B are chosen on the circumference of a circle with center O. Another point C, distinct from A and B, is chosen on the circumference. If $\angle AOB = 70^\circ$, what is the probability that $\triangle ABC$ is acute?

(A) $\dfrac{7}{36}$ **(B)** $\dfrac{7}{18}$ **(C)** $\dfrac{1}{2}$ **(D)** $\dfrac{11}{18}$ **(E)** $\dfrac{31}{36}$

> **Remark.** This problem was originally proposed to the Season 2 TMC 10.

18. **(2020 DMC 10 P9)**
Alice and Bob are racing each other on a track. Each of their lanes are 400 meters in length. Normally, Alice and Bob run at constant rates of a and b meters per minute, respectively, but Alice's lane has a 180-meter sand region in the middle, in which she runs at three-quarters of her normal speed. If Alice and Bob take the same amount of time to run through their lanes without stopping, what is $\frac{a}{b}$?

(A) 1.05 **(B)** 1.15 **(C)** 1.25 **(D)** 1.35 **(E)** 1.45

19. **(2020 DMC 10 P10)**
What is the largest integer n for which there exists an ordered triple (p, q, r) of distinct prime numbers such that $p^2(q^2 + r^2)$ is divisible by 2^n?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

20. **(2020 DMC 10 P12)**
Let A and B be two distinct points on a plane. Let $\mathcal{S}$ denote the set of all circles on the plane with a finite area such that A and B are on the circumference of the circle. What is the region of all points not on the circumference of any of the circles in $\mathcal{S}$?

(A) Every point on line AB excluding A and B
(B) Every point on segment $\overline{AB}$ excluding A and B
(C) Every point on line AB but not on segment $\overline{AB}$
(D) The midpoint of segment $\overline{AB}$
(E) None of the above

> **Remark.** This problem was originally proposed to the Season 2 TMC 10.

21. **(2020 DMC 10 P13)**
10 students are taking a final exam. Of the 10 students, 3 of them are guaranteed to pass. However, the other 7 students are lazy and are not guaranteed to pass, but each of them has the same probability of passing as one another, where the probability is nonzero. If Tomo is one of the 7 lazy students, and exactly 6 out of the 10 students passed the exam, what is the probability that Tomo was one of those 6 students?

(A) $\frac{5}{16}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{3}{7}$ **(E)** $\frac{5}{9}$

> **Remark.** This problem was originally proposed to the 2019 Fall LMT. In fact, this was the first problem I have ever written for LMT and the first problem I have ever written that was on a released math contest.

22. **(2020 DMC 10 P17)**
8 people randomly split into 2 groups of four to dance. After that, the 8 people randomly split into 4 pairs of two to talk. What is the probability that exactly 2 of the 4 pairs contain two people who have danced in the same group of four?

(A) $\frac{8}{35}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{4}{21}$ **(D)** $\frac{24}{35}$ **(E)** $\frac{6}{7}$

> **Remark.** A previous problem was written by the AoPS user **ivyzheng**, who used something with grouping and probability in her problem (her problem is not public, as it was an unused proposal to Season 2 OTSS). I simplified her idea to this problem, and I think it works fine as a P17 on an AMC 10.

23. **(2020 DMC 10 P18)**
A plane cuts into a sphere of radius 11 such that the area of the region of the plane inside the sphere is 108π. A perpendicular plane cuts into the sphere such that the area of the region of the plane inside the sphere is 94π. Given that the two planes intersect at a line, what is the length of the segment of the line inside the sphere?

$\textbf{(A)}\ 6\sqrt{3} \qquad \textbf{(B)}\ 12 \qquad \textbf{(C)}\ 11\sqrt{2} \qquad \textbf{(D)}\ 8\sqrt{5} \qquad \textbf{(E)}\ 18$

> **Remark.** For some reason, I am actually quite proud of writing this problem. I believe that it is actually quite original despite it seeming like something that could have easily appeared before. This is probably my second favorite problem on the test, only behind P23.

24. **(2020 DMC 10 P19)**
Let the sum of $n \geq 2$ consecutive integers be a positive prime number, where the smallest of the integers is a. If $a + n = 28$, what is the sum of all possible values of a?

$\textbf{(A)}\ -26 \qquad \textbf{(B)}\ -25 \qquad \textbf{(C)}\ -1 \qquad \textbf{(D)}\ 0 \qquad \textbf{(E)}\ 1$

> **Remark.** This problem was fun to make, and this was seen as many users' favorite problem on the test, alongside P23.

25. **(2020 DMC 10 P20)**
In trapezoid $ABCD$ with $\overline{AD} \parallel \overline{BC}$ and side lengths $AD = 18$, $BC = 20$, and $AB = CD = 8$, let X be the intersection of line AB and the bisector of $\angle ADC$, and let Y be the intersection of line CD and the bisector of $\angle DAB$. What is XY?

$\textbf{(A)}\ 22 \qquad \textbf{(B)}\ 24 \qquad \textbf{(C)}\ 25 \qquad \textbf{(D)}\ 27 \qquad \textbf{(E)}\ 28$

> **Remark.** I tried this problem after several months of not looking at it during a mini-event at the 2021 Spring LMT. I completely embarrassed myself by literally not knowing how to solve my own problem and looking like a fool in front of the AoPS users **pog** and **richy**.

26. **(2020 DMC 10 P21)**
A set of positive integers exists such that for any integer k in the set, all of the values $k^2 + 2$, $k^2 + 4$, and $k^2 + 8$ are prime numbers. Two distinct integers m and n are chosen from the set. Which of the following is a possible value of $m + n$?

$\textbf{(A)}\ 40 \qquad \textbf{(B)}\ 56 \qquad \textbf{(C)}\ 72 \qquad \textbf{(D)}\ 88 \qquad \textbf{(E)}\ 104$

> **Remark.** Let's just say that creating this problem was much harder than actually solving it. I actually had to manually verify that there exists m and n which satisfy the problem's conditions. Furthermore, I believe that this problem was misplaced on the test and should have been no later than P18 or so.

27. **(2020 DMC 10 P23)**
Joy picks an integer n from the interval $[1, 40]$. She tells Amy the remainder when n is divided by 7 and Sid the number of divisors of n. Amy and Sid both know n is in the interval $[1, 40]$, but they get confused and believe Amy was told the number of divisors and Sid was told the remainder. Amy says, "I know what n is." Sid replies, "If so, then I also know what n is." As it turns out, they thought of the same value but were wrong due to their confusion. If Amy and Sid tell the truth based on their beliefs and can reason perfectly, what is the sum of all possible actual values of n?

(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

> **Remark.** This was a problem which I originally thought was long, contrived, and annoying, but later got quite positive feedback, with many users calling this and P19 their two favorite problems on the test.

28. **(2020 DMC 10 P24)**
In triangle ABC, $AB = 16$ and $BC = 8$, with a right angle at C. Let M be the midpoint of side $\overline{AB}$, let N be a point on side $\overline{AC}$, and let P be the intersection of segments $\overline{BN}$ and $\overline{CM}$. If $BP = 7$, what is the sum of all possible values of $\frac{CN}{AN}$?

(A) $\frac{23}{21}$ **(B)** $\frac{21}{19}$ **(C)** $\frac{19}{17}$ **(D)** $\frac{17}{15}$ **(E)** $\frac{15}{13}$

29. **(KMMC 8 P1)**
What is the value of

$$20^2 - 0^2 + 0^2 \cdot 1?$$

(A) 0 **(B)** 1 **(C)** 20 **(D)** 400 **(E)** 401

30. **(KMMC 8 P2)**
How many lines of symmetry does my face have, shown below, if my right eye is six nanometers wider open than my left eye?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

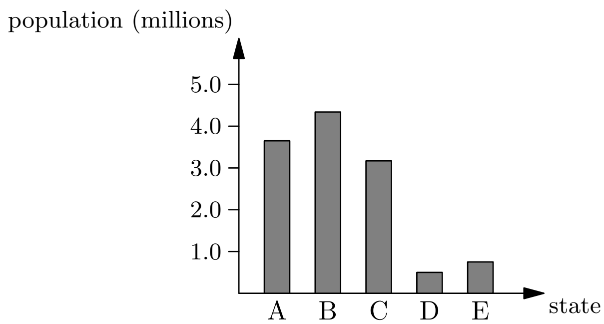
> **Remark.** OMG THIS IS THE MOST BEAUTIFUL PROBLEM I HAVE EVER LAID MY PUPILS UPON IN THE HISTORY OF MATHEMATICAL PROBLEMS!!!! /s

31. **(KMMC 8 P3)**
Bill says, "All prime numbers are odd." Ben replies, "No. The number n contradicts your statement." What is n?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

> **Remark.** This problem was originally proposed to the 2019 Fall LMT as a joke.

32. **(KMMC 8 P4)**
In the bar graph below, five states are compared in terms of their population. Which of the following is the closest to the difference in population between the most and least populated of the five states, in millions?



(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

> **Remark.** This problem was originally proposed to Season 2 OTSS for the now cancelled TMC 8.

33. **(KMMC 8 P5)**
What is the value of $1 + 3 + 5 + \cdots + 19 + 21$?

(A) 81 **(B)** 91 **(C)** 101 **(D)** 111 **(E)** 121

> **Remark.** This problem was originally proposed to Season 2 OTSS for the now cancelled TMC 8.

34. **(KMMC 8 P6)**
Karate has to feed the terrible Thompson triplets. He has 100 pieces of chicken nuggets in the freezer. The Thompson triplets insist on each having a whole number of pieces such that the ratio of the number of pieces each triplet gets is $3 : 4 : 5$. If the Thompson triplets are ravenous and will eat as many pieces as possible, how many pieces will not be eaten by the Thompson triplets?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

35. **(KMMC 8 P11)**
Two fair six-sided dice are rolled. What is the probability that the square of the sum of the numbers facing up on the dice is divisible by 8?

(A) $\frac{1}{6}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{5}{18}$ **(E)** $\frac{1}{3}$

36. **(KMMC 8 P14)**
If Karate travels at 78 miles per hour for 100 minutes going from his house to the hospital, how many miles per hour would he need to travel in 65 minutes to travel from the hospital back to his house?

(A) 100 **(B)** 110 **(C)** 120 **(D)** 130 **(E)** 140

37. **(KMMC 8 P15)**
Karate has a bag of marbles, where 3 of them are white, and the rest are black. He draws 4 marbles from the bag at random, all at once. If the probability of drawing 2 white marbles and 2 black marbles is equal to the probability of drawing 1 white marble and 3 black marbles, then how many marbles were in the bag at the start?

(A) 6 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 13

> **Remark.** This problem was originally proposed to the 2020 DMC 10.

38. **(KMMC 8 P16)**
How many ordered pairs of positive integers (x, y) satisfy

$$x^4 = y^2 + 8?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 8

> **Remark.** A previous problem was written by the AoPS user **GammaZero**, who used a somewhat similar equation in his problem.

39. **(KMMC 8 P17)**
How many permutations of the word $KARATE$ are there such that the two A's are not next to each other?

(A) 60 **(B)** 120 **(C)** 180 **(D)** 240 **(E)** 300

> **Remark.** This problem was originally proposed to the 2020 DMC 10.

40. **(KMMC 8 P18)**
Karate has n pieces of candy. Karate realizes that there are an odd number of values of k for which he can split the pieces of candy into k different groups such that each group has an equal number of pieces of candy. If n is between 5 and 500, inclusive, then how many values of n are possible?

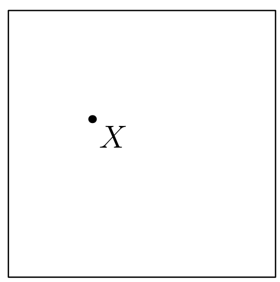
(A) 17 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 21

41. **(KMMC 8 P19)**
Karate has a whole number of cookies in his bag. If he had 5 more cookies than he currently does, he could give an equal number of cookies to 9 different people with none left over. If he had 2 fewer cookies than he currently does, he could give an equal number of cookies to 8 different people with none left over. Let N be the smallest possible number of cookies in Karate's bag. What is the sum of the digits of N?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

> **Remark.** This problem was originally proposed to the 2020 DMC 10.

42. **(KMMC 8 P20)**
A point X is randomly chosen from the interior of a square with side length 2. What is the probability that X is within 1 unit from the midpoints of at least two sides of the square?



(A) $\frac{1}{2}$ **(B)** $\frac{\pi-1}{4}$ **(C)** $\frac{\pi-2}{2}$ **(D)** $\frac{\pi+2}{8}$ **(E)** $\frac{2\pi-1}{8}$

> **Remark.** This problem was originally proposed to the 2020 DMC 10.

43. **(KMMC 8 P21)**
There exists a positive real number x such that

$$x^2 + 4x - 2020 = 0.$$

What is the sum of the digits of the nearest integer to x?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

44. **(KMMC 8 P23)**
If the number $8^a \cdot 9^b$ has 7800 positive integer divisors, where a and b are positive integers, what is the smallest possible value of a?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 8 \qquad \textbf{(D)}\ 13 \qquad \textbf{(E)}\ 17$

45. **(KMMC 8 P24)**
Karate writes the first 10 positive perfect squares on a whiteboard. He then uses as many of the digits that he wrote as possible to create a multiple of 9. For example, with the digits 9, 9, 8, 2, and 1, he can create the number 9189. How many digits does Karate use?

$\textbf{(A)}\ 14 \qquad \textbf{(B)}\ 15 \qquad \textbf{(C)}\ 16 \qquad \textbf{(D)}\ 17 \qquad \textbf{(E)}\ 18$

> **Remark.** This problem was originally proposed to Season 2 OTSS for the now cancelled TMC 8.

46. **(2020 Fall LMT Team Division A P1/B P9)**
Ben writes the string

$$\underbrace{111\ldots11}_{\text{2020 digits}}$$

on a blank piece of paper. Next, in between every two consecutive digits, he inserts either a plus sign $(+)$ or a multiplication sign $(\times)$. He then computes the expression using standard order of operations. Find the number of possible distinct values that Ben could have as a result.

47. **(2020 Fall LMT Team Division A P2/B P6)**
1001 marbles are drawn at random and without replacement from a jar of 2020 red marbles and n blue marbles. Find the smallest positive integer n such that the probability that there are more blue marbles chosen than red marbles is strictly greater than $\frac{1}{2}$.

> **Remark.** The original problem had 13 marbles drawn from a jar of 20 red marbles and n blue marbles, which is something that the AoPS user **GammaZero** will likely understand.

48. **(2020 Fall LMT Team Division A P10/B P18)**
Define a sequence $\{a_n\}_{n\geq 1}$ recursively by $a_1 = 1$, $a_2 = 2$, and for all integers $n \geq 2$, $a_{n+1} = (n+1)^{a_n}$. Determine the number of integers k between 2 and 2020, inclusive, such that $k+1$ divides $a_k - 1$.

49. **(2020 Fall LMT Team Division A P13)**
Find the number of integers n from 1 to 2020 inclusive such that there exists a multiple of n that consists of only 5's.

> **Remark.** This problem was originally written by the AoPS user **GammaZero**, but I worked with him to improve the problem, namely by fixing technical issues in his original solution.

50. **(2020 Fall LMT Guts P6)**
The number 2021 can be written as the sum of 2021 consecutive integers. What is the largest term in the sequence of 2021 consecutive integers?

> **Remark.** This was the first problem which I wrote for the 2020 Fall LMT.

51. **(2020 Fall LMT Guts P27)**
A list consists of all positive integers from 1 to 2020, inclusive, with each integer appearing exactly once. Define a move as the process of choosing four numbers from the current list and replacing them with the numbers $1, 2, 3, 4$. If the expected number of moves before the list contains exactly two 4's can be expressed as $\frac{a}{b}$ for relatively prime positive integers, evaluate $a + b$.

> **Remark.** This problem was originally proposed to the 2020 Spring LMT, with the AoPS user **richy** improving my first draft of the problem. However, only the math team captains were able to propose problems for that contest, so they ended up moving this problem to the 2020 Fall LMT.

52. **(Season 2 TMC 10 P5)**
15 students are to be randomly split into 5 groups of 3 to work on a project. Alice, Bob, and Cooper are three of the students. Given that Alice and Bob are in the same group, what is the probability that Cooper is not in that group?

(A) $\frac{4}{7}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{4}{5}$ **(D)** $\frac{8}{9}$ **(E)** $\frac{12}{13}$

> **Remark.** This problem was originally proposed to the now cancelled TMC 8.

53. **(Season 2 TMC 10 P10/12 P7)**
In a regular hexagon with side length 2, three of the sides are chosen at random. Next, the midpoints of each of the chosen sides are drawn. What is the probability that the triangle formed by the three midpoints has a perimeter which is an integer?

(A) $\frac{1}{10}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

54. **(Season 2 TMC 10 P12/12 P9)**
A group of people are in a room. It is given that 5 people have a pet dog, 6 people have a pet cat, 8 people have a pet fish, and 3 people have no pets. If no one has more than two pets, and no one has more than one of the same type of pet, what is the smallest possible number of people in the room?

(A) 10 **(B)** 13 **(C)** 14 **(D)** 16 **(E)** 19

> **Remark.** This problem was originally proposed to the 2020 DMC 10.

55. **(Season 2 TMC 12 P15)**
For how many positive integers $n \leq 15$ does there exist a positive integer k such that

$$\lfloor \log_2 k \rfloor + \lfloor \log_3 k \rfloor + \lfloor \log_4 k \rfloor + \cdots + \lfloor \log_8 k \rfloor = n?$$

(Here, $\lfloor r \rfloor$ denotes the largest integer less than or equal to r for all real numbers r.)

(A) 8 **(B)** 9 **(C)** 12 **(D)** 13 **(E)** 15

56. **(Season 2 TMC 12 P17)**
How many distinct cubic polynomials $P(x)$ with all integer coefficients and leading coefficient 1 exist such that $P(0) = 3$, $|P(1)| < 12$, and $P(x)$ has three (not necessarily real or distinct) roots whose squares sum to 34?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

> **Remark.** This problem was originally proposed to the Season 1 TMC 12B.

Problems (2021)

1. **(Season 2 OTIE P2)**
Let $\mathcal{A}$ and $\mathcal{B}$ be two sets. Suppose that $\mathcal{A}$ contains a distinct elements and $\mathcal{B}$ contains b distinct elements, where a and b are positive integers. For some positive integer n, if there exist 2021 distinct elements belonging to at least one of $\mathcal{A}$ and $\mathcal{B}$, and there exist n distinct elements belonging to both $\mathcal{A}$ and $\mathcal{B}$, then the number of possible ordered pairs (a, b) is $2n$. Find n.

2. **(Season 2 OTIE P6)**
Let $\mathcal{S}$ be the set of all positive integers less than and relatively prime to 49. Call a subset of $\mathcal{S}$ with 15 distinct numbers *great* if it can be divided into 3 pairwise disjoint groups of 5 numbers such that no two numbers in the same group leave the same remainder when divided by 7, and the product of the numbers in each group leaves a unique remainder when divided by 7. Let n be the number of great subsets of $\mathcal{S}$. Find the sum of the (not necessarily distinct) primes in the prime factorization of n.

3. **(Season 2 OTIE P7)**
Let a and b be positive real numbers such that $\log_a b = \log_{ab} a^2$, $17ab = 60b + 1$, and $a \neq b$. The difference between the largest and smallest possible values of ab can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

4. **(Season 2 OTIE P8)**
Find the sum of the three least positive integers that cannot be written as
$$\frac{a!}{b!} + \frac{c!}{d!} + \frac{e!}{f!}$$
for positive integers a, b, c, d, e, f less than or equal to 5.

5. **(Season 2 OTIE P9)**
A jar contains five slips labeled from 1 to 5, inclusive. In each turn, Kevin takes two different slips out of the jar at random. If Kevin selects slips with the numbers a and b, the numbers a and b are replaced with the numbers 0 and $a + b$, and both slips are put back in the jar. Kevin stops once he writes the number 12 on a slip or takes three turns. The probability that the number 12 has been written once Kevin stops is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

6. **(Season 2 OTIE P10)**
Consider the polynomial
$$P(x) = x^{21} - 364x^{20} + Q(x),$$
where $Q(x)$ is some polynomial of degree at most 19. If the roots of $P(x)$ are all integers and $P(21) = 2021$, find the remainder when $P(23)$ is divided by 1000.

7. **(KMMC 10 P4)**
At a buffet, Karate eats sushi, shrimp tacos, lo mein, and chicken tacos in some order. If he eats sushi sometime before he eats shrimp tacos, but eats lo mein sometime after he eats chicken tacos, in how many possible orders could he have eaten them?

(A) 3 **(B)** 6 **(C)** 12 **(D)** 15 **(E)** 24

8. **(KMMC 10 P9)**
There exist positive integers n which satisfy at least half of the following conditions:

- n is not a prime number.
- $n+1$ is equal to a perfect square.
- $n+2$ is a prime number.
- $n+3$ is equal to one more than a perfect square.

What is the sum of all n from 1 to 10, inclusive?

(A) 4 **(B)** 11 **(C)** 17 **(D)** 20 **(E)** 21

> **Remark.** This is probably my least favorite problem I have written that was on a released math contest. While I believe that this problem could have been a pretty good first ten problem, I was too lazy to tweak it so that it was less bashy and annoying than it currently is.

9. **(KMMC 10 P16)**
What is the smallest possible value of

$$\left|\frac{x}{3}-20\right|+\left|\frac{x}{2}-10\right|+\left|\frac{x}{3}+10\right|$$

over all real numbers x?

(A) 20 **(B)** 25 **(C)** 30 **(D)** 35 **(E)** 40

> **Remark.** Notice the similarity to the first problem in the 2019 section? Yeah, I was pretty worried that people would catch on, so I decided not to use this problem in the 2020 DMC 10. However, I was willing to put pretty much whatever we wanted in the KMMC 10, hence why this problem is on here.

10. **(KMMC 10 P21)**
Triangle ABC has $AB=8$, $BC=6$, $AC=11$. Let points D and E trisect side $\overline{BC}$ such that D is closer to B than C. Let F be the intersection of $\overline{AC}$ and the bisector of $\angle ABC$. Let X be the intersection of $\overline{AD}$ and $\overline{BF}$, and let Y be the intersection of $\overline{AE}$ and $\overline{BF}$. What is the ratio of the area of $\triangle AYF$ to the area of $\triangle AXB$?

(A) $\frac{25}{63}$ **(B)** $\frac{14}{33}$ **(C)** $\frac{16}{35}$ **(D)** $\frac{26}{55}$ **(E)** $\frac{10}{21}$

> **Remark.** This problem was proposed to the 2020 DMC 10, but I ultimately did not use this problem because we had too much geometry, and the AoPS user **richy** was concerned that the problem was a bit too standard. However, I was willing to put pretty much whatever we wanted in the KMMC 10, hence why this problem is on here.

11. **(KMMC 10 P22)**
Four people are sitting evenly spaced at a circular table. At once, each person chooses to sit at the seat to their left, the seat to their right, or their current seat, with each seat having a one-third chance of being chosen. If two or more people sit at the same seat, the people who chose that seat leave the table. The people who did not leave sit at their chosen seats. What is the probability that exactly two people are left sitting?

$\textbf{(A)}\ \frac{1}{3} \qquad \textbf{(B)}\ \frac{4}{9} \qquad \textbf{(C)}\ \frac{13}{27} \qquad \textbf{(D)}\ \frac{16}{27} \qquad \textbf{(E)}\ \frac{56}{81}$

> **Remark.** This problem was in storage for quite a while. This problem was actually proposed in late 2019 as a proposal to the 2019 Fall LMT, and it almost actually got used in the theme round. However, it was later shut down due to complaints from certain math team members about the problem being too annoying and/or bashy, especially for a middle school competition (which, frankly, I disagree with).
>
> After that unfortunate incident, I proposed this same problem to OTSS (Spring), where it was suggested that this problem would appear as OTIE P6 or P7, as dealing with the casework required a good deal of precision. However, it was not selected to appear on the final draft, as we already had a lot of combinatorics problems.
>
> Finally, I moved this problem to DMC as a possible candidate for the 2020 DMC 10. However, I grew paranoid due to this problem having been seen by many people up to this point. I did not want to start controversy, especially as this was my first released mock contest. But at long last, on February 2, 2021, this problem finally appeared on a released math contest, mock or real.

12. **(KMMC 10 P24)**
In triangle ABC with $AB = 13$, $BC = 14$, $AC = 15$, let O be the center of its circumcircle. Let P and Q be the feet of the perpendiculars from O to sides $\overline{AB}$ and $\overline{AC}$, respectively, and let E and F be the feet of the perpendiculars from B and C to line AO, respectively. What is $PE^2 + QF^2$?

$\textbf{(A)}\ 98.5 \qquad \textbf{(B)}\ 101.5 \qquad \textbf{(C)}\ 104.5 \qquad \textbf{(D)}\ 107.5 \qquad \textbf{(E)}\ 110.5$

> **Remark.** This problem was originally written by the AoPS user **i3435**, but I came in and kind of butchered the problem by having a really underwhelming solution. My bad!

13. **(2021 DIME P4)**
There are 7 balls in a jar, numbered from 1 to 7, inclusive. First, Richard takes a balls from the jar at once, where a is an integer between 1 and 6, inclusive. Next, Janelle takes b of the remaining balls from the jar at once, where b is an integer between 1 and the number of balls left, inclusive. Finally, Tai takes all of the remaining balls from the jar at once, if any are left. Find the remainder when the number of possible ways for this to occur is divided by 1000, if it matters who gets which ball.

> **Remark.** This problem was created with the AoPS user **firebolt360**.

14. **(2021 DIME P10)**
There exist complex numbers $z_1, z_2, \ldots, z_{10}$ which satisfy

$$|z_k i^k + z_{k+1} i^{k+1}| = |z_{k+1} i^k + z_k i^{k+1}|$$

for all integers $1 \le k \le 9$, where $i = \sqrt{-1}$. If $|z_1| = 9$, $|z_2| = 29$, and for all integers $3 \le n \le 10$, $|z_n| = |z_{n-1} + z_{n-2}|$, find the minimum value of $|z_1| + |z_2| + \cdots + |z_{10}|$.

> **Remark.** This problem was originally proposed to the Season 2 OTIE.

15. **(2021 Spring LMT Team Division A P11/B P17)**
In $\triangle ABC$ with $\angle BAC = 60^\circ$ and circumcircle ω, the angle bisector of $\angle BAC$ intersects side $\overline{BC}$ at point D, and line AD is extended past D to a point A'. Let points E and F be the feet of the perpendiculars of A' onto lines AB and AC, respectively. Suppose that ω is tangent to line EF at a point P between E and F such that $\frac{EP}{FP} = \frac{1}{2}$. Given that $EF = 6$, the area of $\triangle ABC$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m + n + p$.

> **Remark.** The original version of the problem said that A' was the reflection of A across D, when in reality, this is not the case.

16. **(2021 Spring LMT Team Division A P25/B P26)**
Chandler the Octopus is making a concoction to create the perfect ink. He adds 1.2 grams of melanin, 4.2 grams of enzymes, and 6.6 grams of polysaccharides. But Chandler accidentally added n grams of an extra ingredient to the concoction, Chemical X, to create glue. Given that Chemical X contains none of the three aforementioned ingredients, and the percentages of melanin, enzymes, and polysaccharides in the final concoction are all integers, find the sum of all possible positive integer values of n.

> **Remark.** The original problem was themed to the Powerpuff Girls, hence the mention of Chemical X. However, the LMT team decided to change the theme to Chandler the Octopus in order to match the actual LMT themes.

17. **(2021 Spring LMT Guts P22)**
A sequence $a_1, a_2, a_3, \ldots$ of positive integers is defined such that $a_1 = 4$, and for each integer $k \ge 2$,

$$2(a_{k-1} + a_k + a_{k+1}) = a_k a_{k-1} + 8.$$

Given that $a_6 = 488$, find $a_2 + a_3 + a_4 + a_5$.

18. **(2021 DMC 10A P1)**
The sum of the first five positive integers and the sum of the first six positive integers are multiplied. What is the resulting product?

(A) 315 **(B)** 335 **(C)** 355 **(D)** 375 **(E)** 395

> **Remark.** This problem was originally proposed to the Season 2 TMC 10.

19. **(2021 DMC 10A P13/11 P11)**
Let ω be the inscribed circle of a rhombus $ABCD$ with side length 4 and $\angle DAB = 60^\circ$. There exist two distinct lines which are parallel to line BD and tangent to ω. Given that the lines intersect sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ at points P, Q, R, and S, respectively, what is the area of quadrilateral $PQRS$?

$\textbf{(A)}\ 2\sqrt{3} \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 3\sqrt{3} \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 4\sqrt{3}$

> **Remark.** This problem was originally written by the AoPS user **AT2005**, but I modified the problem slightly. I believe that I could have done more with my modifications, though, because I believe that this problem was a bit standard.

20. **(2021 DMC 10A P14/11 P12)**
Alice goes cherry picking in a forest. For each tree Alice sees, she either picks one cherry or three cherries from the tree and puts them in her basket. Additionally, after every five trees Alice picks from, she finds an extra cherry on the ground and puts it in her basket. At the end, Alice has 45 cherries in her basket. If the smallest possible number of trees Alice could have picked from is n, what is the sum of the digits of n?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 7 \qquad \textbf{(E)}\ 8$

21. **(2021 DMC 10A P15/11 P13)**
Each of 6 distinct positive integers is placed at each of 6 equally spaced points on the circumference of a circle. If the numbers on every two adjacent points are relatively prime, and the product of the numbers on every two diametrically opposite points is divisible by 3, what is the least possible sum of the 6 integers?

$\textbf{(A)}\ 12 \qquad \textbf{(B)}\ 25 \qquad \textbf{(C)}\ 26 \qquad \textbf{(D)}\ 29 \qquad \textbf{(E)}\ 32$

> **Remark.** This problem was originally written by the AoPS user **AT2005**, but I modified the problem by adding more conditions to make the problem more tricky.

22. **(2021 DMC 10A P17/11 P15)**
Let $\triangle ABC$ have $AB = 20$, $AC = 21$, and a right angle at A. Let I be the center of the inscribed circle of $\triangle ABC$. Let point D be the reflection of point B over the line parallel to AB passing through I, and let point E be the reflection of point C over the line parallel to AC passing though I. What is the value of DE^2?

$\textbf{(A)}\ 145 \qquad \textbf{(B)}\ 149 \qquad \textbf{(C)}\ 153 \qquad \textbf{(D)}\ 157 \qquad \textbf{(E)}\ 161$

> **Remark.** This problem was originally written by the AoPS user **ApraTrip**, where it was a complicated AIME problem, but I came in and kind of butchered the problem by oversimplifying it. My bad!

23. **(2021 DMC 10A P20/11 P17)**
Ann rolls two fair six-sided dice. If the sum of the numbers she rolled is at least 7, she rolls the dice again (and does not roll after that). Otherwise, she does not. What is the probability she rolls a 5 on at least one of the dice, on at least one of the rolls?

(A) $\frac{1}{3}$ **(B)** $\frac{13}{36}$ **(C)** $\frac{29}{72}$ **(D)** $\frac{11}{27}$ **(E)** $\frac{4}{9}$

> **Remark.** This problem had two instances where a piece of information was omitted: the first was "on at least one of the dice," and the second was "on at least one of the rolls."

24. **(2021 DMC 10A P21/11 P18)**
In equilateral $\triangle ABC$, let points D and E be on lines AB and AC, respectively, both on the opposite side of line BC as A. If $CE = DE$, and the circumcircle of $\triangle CDE$ is tangent to line AB at D, what is the degree measure of $\angle CDE$?

(A) 70 **(B)** 72 **(C)** 75 **(D)** 80 **(E)** 84

> **Remark.** A previous problem was written by the AoPS user **i3435**, who used something with an equilateral triangle and length chasing. I experimented to create my own problem idea, and magically found this pretty interesting angle chasing problem with an unconventional answer. (No, I will not spoil the answer in this document.)

25. **(2021 DMC 11 P20)**
Richard has four identical balls labeled 1, and two identical balls labeled -1. He randomly places each ball into one of six different bins, where he is allowed to place multiple balls in the same bin. What is the probability that the sum of the numbers of the balls in each bin is nonnegative? (A bin with no balls in it has sum 0.)

(A) $\frac{61}{441}$ **(B)** $\frac{23}{147}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{3}{14}$ **(E)** $\frac{33}{98}$

26. **(2021 DMC 10A P23/11 P21)**
In pentagon $ABCDE$, where all interior angles have a positive degree measure less than $180°$, let M be the midpoint of side $\overline{DE}$. It is given that line BM splits $ABCDE$ into two isosceles trapezoids $ABME$ and $CDMB$ such that each one contains exactly three sides of equal length. If $AE = 3$ and $DE = 26$, what is the area of $ABCDE$?

(A) 216 **(B)** 234 **(C)** 288 **(D)** 312 **(E)** 330

> **Remark.** This problem idea was originally written by me for the 2021 Spring LMT, but it was too difficult and had too many things to keep track of. When I later proposed this problem to DMC, the AoPS user **jayseemath** suggested that the problem could give some side lengths and ask to find the area, and just like that, this problem came to be.

27. **(2021 DMC 11 P23)**
An acute $\triangle ABC$ has $BC = 30$, $\angle BAC = 60^\circ$, $AC > AB$, and circumcircle ω with center O. The line tangent to ω at A intersects line BC at a point D. It is given that the line passing through O, parallel to line BC, intersects line AD at a point P such that $AP : DP = 4 : 3$. The length AD can be written as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. What is $m + n$?

$\textbf{(A)}\ 19 \qquad \textbf{(B)}\ 22 \qquad \textbf{(C)}\ 25 \qquad \textbf{(D)}\ 28 \qquad \textbf{(E)}\ 31$

> **Remark.** I really wish that I had saved this problem for the second DIME.

28. **(2021 DMC 11 P24)**
There exist positive real numbers a and b such that $\lfloor ab^2 \rfloor = 1$, $\lfloor a^2b^2 \rfloor = 2$, and $\lfloor a^3b^2 \rfloor = 6$, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to a real number r. It is given that as a approaches its greatest lower bound A, the set of possible values of b approaches a single value B. What is the value of $A^4 + B^4$?

$\textbf{(A)}\ \frac{31}{3} \qquad \textbf{(B)}\ \frac{539}{48} \qquad \textbf{(C)}\ 13 \qquad \textbf{(D)}\ \frac{265}{16} \qquad \textbf{(E)}\ \frac{67}{4}$

> **Remark.** I really wish that I had saved this problem for the second DIME.

29. **(2021 DMC 10A P25/11 P25)**
Ryan has an infinite supply of slips and a spinner with letters O, S, and T, where each letter is equally likely to be spun. Each minute, Ryan spins the spinner randomly, writes on a blank slip the letter he spun, and puts it in a pile. Ryan continues until he has written all 3 letters at least once, at which point he stops. What is the probability that after he stops, he can form the words $OTSS$ and $TOST$ using 4 distinct slips from the pile? (Ryan may reuse slips he used for one word in forming the other.)

$\textbf{(A)}\ \frac{7}{54} \qquad \textbf{(B)}\ \frac{13}{72} \qquad \textbf{(C)}\ \frac{2}{9} \qquad \textbf{(D)}\ \frac{8}{27} \qquad \textbf{(E)}\ \frac{1}{3}$

30. **(Season 3 TIME P4)**
Find the number of positive integers $n \leq 1000$ such that

$$n(n+1)(n+\tfrac{1}{2})(n+\tfrac{1}{3})(n+\tfrac{1}{4})$$

is an integer.

> **Remark.** This problem was created with the AoPS user **Aathreyakadambi**.

31. **(Season 3 TIME P5)**
Let $P(x) = x^2 + ax + b$ be a quadratic with not necessarily distinct real roots r and s, where a and b are positive integers. If the quadratic $Q(x) = x^2 + 2ax + 3b$ has real roots r and $t \neq s$, find the maximum value of $P(1) + Q(1)$ less than 1000.

> **Remark.** I wrote this problem in around 10 minutes.

32. **(Season 3 TIME P7)**
During fencing practice, six people split into three pairs to spar. Two more people join afterwards. The eight people then randomly rearrange themselves into four pairs to spar. The probability that no one spars with someone that they previously sparred with is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

> **Remark.** This problem was created with the AoPS user **ivyzheng**.

33. **(Season 3 TIME P10)**
In triangle ABC with $AB = 26$, $BC = 28$, and $AC = 30$, let O and $\overline{AD}$ be the center and a diameter of the circumcircle of $\triangle ABC$, respectively. Two distinct lines pass through O, are parallel to $\overline{AB}$ and $\overline{AC}$, respectively, and meet side $\overline{BC}$ at points M and N, respectively. Let lines DM and DN meet the circumcircle of $\triangle ABC$ at points P and Q, respectively, both distinct from D. Find the area of $APDQ$.

> **Remark.** This problem was created with the AoPS user **NJOY**.