Official Solutions

De Mathematics Competitions

2nd Annual

DMC 10D

Thursday, October 14, 2021

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

Questions and complaints about this competition should be
sent by private message to

DeToasty3, **karate7800**, and **pandabearcat**.

The problems and solutions for this DMC 10 were prepared by the
DMC 10 Editorial Board under the direction of

bobthegod78, dc495, DeToasty3, HrishiP, karate7800, math31415926535, MathPirate101, Michael595, pandabearcat, PhunsukhWangdu, pog, themathboi101, & treemath

Answer Key:

1. (C)	2. (C)	3. (B)	4. (D)	5. (C)
6. (C)	7. (D)	8. (B)	9. (E)	10. (C)
11. (D)	12. (A)	13. (A)	14. (A)	15. (D)
16. (C)	17. (D)	18. (A)	19. (D)	20. (C)
21. (B)	22. (D)	23. (A)	24. (E)	25. (B)

Problem 1:

(treemath) What is the value of

$$2^0 - (21 - 20) + 2^1?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

> **Answer (C):**
>
> Using order of operations, the requested answer is $1 - 1 + 2 = \boxed{\text{(C) } 2}$. ■

Problem 2:

(DeToasty3) What is the smallest positive integer which can be expressed as the sum of a positive perfect square and a distinct positive perfect cube?

(A) 2 **(B)** 3 **(C)** 5 **(D)** 9 **(E)** 10

> **Answer (C):**
>
> We see that 1 is both a positive perfect square and a positive perfect cube. Now, we want to find the smallest number greater than 1 which is either a positive perfect square or a positive perfect cube (or both). This is $2^2 = 4$. Thus, the requested answer is $1 + 4 = \boxed{\text{(C) } 5}$. ■

Problem 3:

(HrishiP) Squares $\mathcal{P}$ and $\mathcal{Q}$ are such that half the area of $\mathcal{Q}$ lies within $\mathcal{P}$. If the areas of $\mathcal{P}$ and $\mathcal{Q}$ are 8 and 2, respectively, what is the area of the region inside $\mathcal{P}$ and outside $\mathcal{Q}$?

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

Answer (B):

Since we are given that the area of $\mathcal{Q}$ is 2 square units and half of $\mathcal{Q}$ lies within $\mathcal{P}$, we get that 1 square unit of $\mathcal{Q}$ lies within $\mathcal{P}$. Thus, the area of the region inside $\mathcal{P}$ but not $\mathcal{Q}$ is $8 - 1 = \boxed{\textbf{(B) } 7}$ square units. ■

Problem 4:

(DeToasty3) A positive integer n is divisible by 5 but not 4. For which of the following values will adding it to n never result in a sum that is divisible by 20?

(A) 45 **(B)** 50 **(C)** 55 **(D)** 60 **(E)** 65

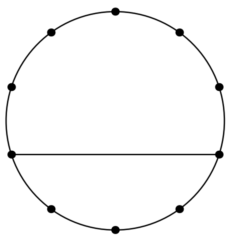
Answer (D):

We want our sum to never be divisible by 20, so it should be not a multiple of 5 or not a multiple of 4. Since all of the answers are multiples of 5 and n is divisible by 5, the sum will always be a multiple of 5.

Thus, we want our sum to not be divisible by 4. Note that, since n is not a multiple of 4, the sum when n is added to a multiple of 4 will also not be a multiple of 4. Thus, $n + 60$ is never divisible by 20, so the requested answer is $\boxed{\textbf{(D) } 60}$. ■

Problem 5:

(DeToasty3) Ten points are equally spaced on the circumference of a circle, where two of the points are connected by a line segment, as shown below. Turner wants to choose two of the eight other points and draw a line segment connecting them so that this line segment does not intersect the other line segment. In how many ways can Turner do this?



(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

Answer (C):

Note that each line segment must be completely within the top part of the circle or completely within the bottom part of the circle.

In the top part of the circle, there are 5 ways to choose the point to start a line segment and 4 ways to choose the point to end that line segment, so there are $5 \cdot 4 = 20$ line segments. However, each line segment is identical to another line segment (i.e. the line from point A to point B is the same as the line from point B to point A), so thus each distinct line segment is counted twice. Hence, we divide by 2, for a total of 10 lines in the top part of the circle.

In the bottom part of the circle, there are once again $3 \cdot 2 = 6$ line segments, but each distinct line segment is counted twice. Hence, we divide by 2, for a total of 3 lines in the bottom part of the circle.

Thus, the requested answer is $10 + 3 = \boxed{\text{(C) } 13}$. ■

Problem 6:

(treemath) One of the side lengths of an obtuse isosceles triangle with integer side lengths is 2. What is the perimeter of this triangle?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (C):

Let c be the longest side of a triangle. If $a^2+b^2 > c^2$, the triangle will be acute, whereas if $a^2 + b^2 < c^2$, the triangle will be obtuse.

Case 1: If the base side is 2, then the other sides must be equal, say s and s. However, $2^2 + s^2 > s^2$, so the resulting triangle will be acute.

Case 2: If the leg of the isosceles triangle is 2, then the sides of the triangle are 2, 2, and s, and by the triangle inequality, $0 < s < 4$.

If $s \leq 2$, then $s^2+2^2 > 2^2$, so the resulting triangle will be acute. Finally, if $s = 3$, then $2^2 + 2^2 < 3^2$, so the resulting triangle will be obtuse and the perimeter of the triangle is $2 + 2 + 3 = \boxed{\text{(C) } 7}$. ■

Problem 7:

(pog) Let a be a positive integer. A geometric sequence b, c, d in that order satisfies

$$ab = 15, \quad bd = 16, \quad \text{and} \quad ac = 120.$$

What is the value of $a + d$?

(A) 56 **(B)** 58 **(C)** 60 **(D)** 62 **(E)** 64

Answer (D):

By the first and third equations, $8ab = ac$. Dividing both sides by a, we get that $8b = c$, so the common ratio of the sequence is 8.

Thus, $c = 8b$ and $d = 8c = 8(8b) = 64b$. By the second equation, we have that $b \cdot 64b = 16$, so $64b^2 = 16$. Thus, $b^2 = \frac{1}{4}$. Since a is a positive integer and $ab = 15$, b must be positive, so thus $b = \sqrt{\frac{1}{4}} = \frac{1}{2}$.

Finally, we have that $a \cdot \dfrac{1}{2} = 15$, so $a = 30$, and thus

$$a + d = a + 64b = 30 + 32 = \boxed{\textbf{(D) } 62},$$

as requested. ■

Problem 8:

(DeToasty3) Let n be the number of factors of 3 that the product

$$(1 + 2 + 3)(4 + 5 + 6)(7 + 8 + 9) \cdots (94 + 95 + 96)(97 + 98 + 99)$$

contains. What is the sum of the digits of n?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (B):

Let the center number in a given expression be x. Thus the group of parantheses with center number x is equal to $(x - 1) + x + (x + 1) = 3x$. Thus, our product is equal to

$$2(3) \times 5(3) \times 8(3) \times \cdots \times 95(3) \times 98(3).$$

Note that the terms of 2, 5, 8, $\ldots$, 95, 98 will always have a remainder of 2 when divided by 3, as the first number is 2 and each new term is 3 more than the previous.

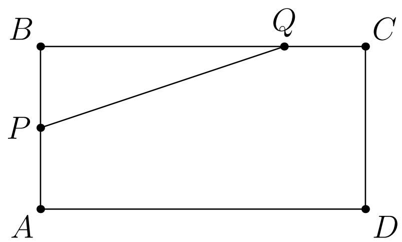
Thus, each term in the product will always contribute one factor of 3. The kth term of 2, 5, 8, $\ldots$, 95, 98 is equal to $2 + 3(k - 1)$, so 98 is the 33rd term of the sequence and there are 33 factors of 3 in the product, for an answer of $3 + 3 = \boxed{\textbf{(B) } 6}$. ■

Problem 9:

(DeToasty3) A rectangle $ABCD$ has $AB = 8$ and $BC = 4$. Points P and Q lie on sides $\overline{AB}$ and $\overline{BC}$, respectively, such that $AP = CQ$ and the area of $\triangle BPQ$ is 6. What is PQ^2?

(A) 32 **(B)** 34 **(C)** 36 **(D)** 38 **(E)** 40

Answer (E):



Since the area of $\triangle BPQ$ is 6, we get that $\frac{BP \cdot BQ}{2} = 6$. Thus, $BP \cdot BQ = 12$. Let $AP = QC = x$. Then $BP = 4 - x$ and $BQ = 8 - x$, so $(4 - x)(8 - x) = 12$. Expanding and factoring gives $(x - 2)(x - 10) = 0$, so either $x = 2$ or $x = 10$.

If $x = 10$, then $BP = -6$ and $BQ = -2$, which is impossible, so thus $x = 2$. This gives $BP = 2$ and $BQ = 6$. Since $ABCD$ is a rectangle, $\angle B = 90°$, so applying the Pythagorean Theorem on $\triangle BPQ$ gives $2^2 + 6^2 = PQ^2$. Thus, $PQ^2 = 4 + 36 = \boxed{\textbf{(E) } 40}$.
■

Problem 10:

(bobthegod78) There exist 4-digit numbers such that the 4-digit number is divisible by 9, the first 3 digits in order form a 3-digit number divisible by 9, and the last 3 digits in order form a 3-digit number divisible by 9. How many such 4-digit numbers exist?

(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

Answer (C):

Let the given number be $\overline{PQRS}$. We are given that $P + Q + R + S$ is a multiple of 9, $Q + R + S$ is a multiple of 9, and $P + Q + R$ is a multiple of 9.

Since $P + Q + R$ and $(P + Q + R) + S$ are both multiples of 9, S must be a multiple of 9. Similarly, since $Q+R+S$ and $P+(Q+R+S)$ are both multiples of 9, P must also be a multiple of 9. Thus, since P and S are digits, $P \in \{0, 9\}$ and $S \in \{0, 9\}$.

Since $\overline{PQR}$ and $\overline{QRS}$ are 3-digit numbers, P and Q are nonzero. Thus, $P = 9$, and our number is equal to $9\overline{QRS}$, where $S \in \{0, 9\}$ and Q is nonzero. Since $Q+R+S$ is a multiple of 9 and S is a multiple of 9, $Q+R$ must also be a multiple of 9.

Thus, noting that Q must be nonzero, we can have

$$(Q, R) = (1, 8), (2, 7), (3, 6), (4, 5), (5, 4), (6, 3), (7, 2), (8, 1), (9, 0), \text{ and } (9, 9).$$

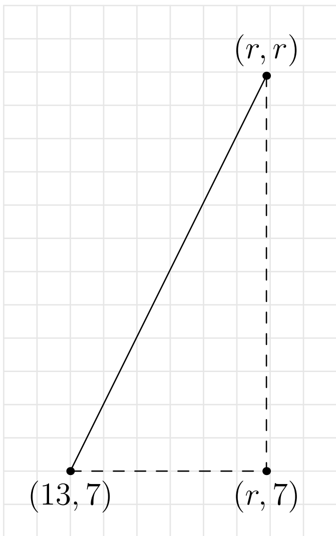
For each of these 10 pairs, there are two choices for S, so there are $10 \cdot 2 = \boxed{\textbf{(C) } 20}$ of the requested numbers.

Problem 11:

(math31415926535) In the xy-plane, the distance between the points (r, r) and $(13, 7)$ is equal to d, where r is a real number satisfying the equation $r^2 - 20r + 21 = 0$. What is d^2?

(A) 78 **(B)** 106 **(C)** 150 **(D)** 176 **(E)** 218

Answer (D):



By the Pythagorean Theorem, we can form a right triangle from $(13, 7)$ to (r, r), so $d^2 = (r-13)^2 + (r-7)^2$. Expanding gives $d^2 = 2r^2 - 40r + 218$. We are given that

$r^2 - 20r + 21 = 0$; note that

$$2r^2 - 40r + 218 = 2(r^2 - 20r + 109) = 2\left((r^2 - 20r + 21) + 88\right),$$

so $d^2 = 2(0 + 88) = \boxed{\textbf{(D)}\ 176}$. ■

Problem 12:

(DeToasty3) Sheldon has some shirts, pairs of pants, and socks. An outfit consists of one shirt, one pair of pants, and two socks. Sheldon can currently wear 60 possible different outfits, but if he were to get either four more shirts or four more socks, then Sheldon could wear 180 possible different outfits. How many pairs of pants does Sheldon have? (The socks are distinguishable, and the order in which he wears the socks does not matter.)

(A) 2 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 15

Answer (A):

Let Sheldon have a shirts, b pairs of pants, and c socks. There are c ways to choose the first sock and $c - 1$ ways to choose the second sock; however, each distinct pair of socks is counted twice (i.e. wearing sock A and sock B is the same as wearing sock B and sock A). Hence, we divide by 2, for a total of $\frac{c(c-1)}{2}$ ways to choose the socks.

Similarly, note that with $c + 4$ socks, there are $c + 4$ ways to choose the first sock and $c + 3$ ways to choose the second sock; however, each distinct pair of socks is counted twice (i.e. wearing sock A and sock B is the same as wearing sock B and sock A). Hence, we divide by 2, for a total of $\frac{(c+4)(c+3)}{2}$ ways to choose the socks. Thus, by the fundamental counting principle, we can create the following equations based on the information in the problem:

$$a \cdot b \cdot \frac{c(c-1)}{2} = 60, \tag{1}$$

$$(a+4) \cdot b \cdot \frac{c(c-1)}{2} = 180, \tag{2}$$

$$a \cdot b \cdot \frac{(c+4)(c+3)}{2} = 180. \tag{3}$$

First, dividing (2) by (1), we get that $a + 4 = 3a$, so solving gives $a = 2$. Next, dividing (3) by (1), we get that $(c+4)(c+3) = 3c(c-1)$. Expanding gives $2c^2 - 10c - 12 = 0$, and factoring gives $2(c+1)(c-6) = 0$. Clearly Sheldon has a positive number of socks, so $c = 6$.

Substitution gives $2 \cdot b \cdot \frac{6 \cdot 5}{2} = 60$, so $2 \cdot b \cdot 15 = 60$. Solving for b, we get that Sheldon has $\boxed{\textbf{(A) } 2}$ pairs of pants. ■

Problem 13:

(pog) Trixie writes the number 2021 on a blackboard. She then repeatedly erases and writes a new number on the blackboard, where if the current number on the blackboard is odd, she will erase the number and write $8n$ on the blackboard, and if the current number on the blackboard is even, she will erase the number and write $n + 2019$ on the blackboard. Eventually, Trixie will have written 2020 numbers on the board (including the initial 2021). What is the remainder when her 2020th number is divided by 9?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 7

Answer (A):

Note that if a number is odd, then the succeeding number will be even, as it will be a multiple of 8. As well, if a number is even, then the succeeding number will be odd, as it will be added to 2019.

Consider the terms on the board when taken mod 9. Note that $8n \equiv -n \mod 9$ and $n + 2019 \equiv 3 \mod 9$. We start with an odd number, and each succeeding number on the board will switch between even and odd. Computing, we get

$$
\begin{align}
2021 &\equiv 5 \pmod 9, \\
-5 &\equiv 4 \pmod 9, \\
4+3 &\equiv 7 \pmod 9, \\
-7 &\equiv 2 \pmod 9, \\
2+3 &\equiv 5 \pmod 9, \\
-5 &\equiv 4 \pmod 9, \\
4+3 &\equiv 7 \pmod 9, \\
-7 &\equiv 2 \pmod 9, \\
2+3 &\equiv 5 \pmod 9, \\
&\cdots.
\end{align}
$$

Since every other number on the board is odd, the fifth term is odd and has a remainder of 5 when divided by 9. Note that, as Trixie will apply the same process of addition and multiplication on the next 4 terms, this process of remainders will repeat every 4 terms.

> Thus, the 4th term, 8th term, 16th term, and so forth, all the way to the 2020th term, will all have a remainder of $\boxed{\textbf{(A) } 2}$ when divided by 9. ■

Problem 14:

(DeToasty3) In the xy-plane, the point $(24, 7)$ is reflected over the line $y = ax$ and then shifted up b units to the point $(20, 21)$, where a and b are positive real numbers. What is $a \cdot b$?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 12

> **Answer (A):**
>
> Note that the line $y = ax$ passes through the point $(0, 0)$. Thus, due to reflections, we have that the point after $(24, 7)$ is reflected has the same distance from the origin as $(24, 7)$. By the Pythagorean Theorem, this length is $\sqrt{24^2 + 7^2} = 25$. Now, since a shift up results in the point $(20, 21)$, we must have that the point after reflection also has x-coordinate 20. Thus, by the Pythagorean Theorem, the y-coordinate of this point is $\pm\sqrt{25^2 - 20^2} = \pm 15$. However, we throw away -15 because then a would be negative. We know that the line $y = ax$ passes through the midpoint of the segment connecting $(20, 15)$ and $(24, 7)$, or $(22, 11)$, so the slope of this line, or the value of a, is $\frac{1}{2}$, and $b = 21 - 15 = 6$. Thus, the requested value of $a \cdot b$ is $\frac{1}{2} \cdot 6 = \boxed{\textbf{(A) } 3}$. ■

Problem 15:

(DeToasty3) Daniel repeatedly rolls two fair six-sided dice. After every roll, Daniel writes the sum of the numbers on the two dice. Daniel stops once he writes the number 11. What is the probability that Daniel writes the number 5 at least once before stopping?

(A) $\frac{1}{3}$ **(B)** $\frac{5}{12}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

> **Answer (D):**
>
> Note that the complementary case is that he does not write a 5 before writing an 11. Thus, the question can be rewritten as: "What is the probability that Daniel first writes the number 5 before first writing the number 11?" Note that Daniel can roll a sum of 5 in four ways - $(1, 4)$, $(2, 3)$, $(3, 2)$, and $(4, 1)$, and a sum of 11 in two ways - $(5, 6)$ and $(6, 5)$. Since Daniel is twice as likely to roll a 5 as he is to roll an 11, the answer is $\dfrac{2}{2+1} = \boxed{\textbf{(D) } \frac{2}{3}}$. ■

Problem 16:

(DeToasty3) A pyramid $ABCDE$ has rectangular base $ABCD$ with $AB = 8$ and $BC = 6$ and apex E located 8 units above $ABCD$. A plane parallel to $\triangle ACE$ hits segments $\overline{AB}$, $\overline{BC}$, and $\overline{BE}$ at F, G, and H, respectively. If $FG = \frac{15}{2}$, what is the volume of $DFGH$?

(A) 27 **(B)** 36 **(C)** 45 **(D)** 48 **(E)** 60

Answer (C):

We see that $AC = \sqrt{AB^2 + BC^2} = 10$, so $FG : AC = 3 : 4$. Since the plane is parallel to $\triangle ACE$, we have that H is $\frac{3}{4} \cdot 8 = 6$ units above $ABCD$. Now, we just need to compute the area of $\triangle DFG$. This can be found by subtracting the areas of right triangles ADF, CDG, and BFG, from the area of $ABCD$. By similarity, we see that the lengths of the legs of $\triangle ADF$ are $AD = 6$ and $AF = 2$, the lengths of the legs of $\triangle CDG$ are $CD = 8$ and $CG = \frac{3}{2}$, and the lengths of the legs of $\triangle BFG$ are $BF = 6$ and $BG = \frac{9}{2}$. Thus, the areas of $\triangle ADF$, $\triangle CDG$, and $\triangle BFG$ are 6, 6, and $\frac{27}{2}$, respectively. Thus, the area of $\triangle DFG$ is equal to

$$48 - 6 - 6 - \frac{27}{2} = \frac{45}{2},$$

so the volume of $DFGH$ is $\frac{1}{3} \cdot \frac{45}{2} \cdot 6 = \boxed{\textbf{(C)}\ 45}$. ■

Problem 17:

(treemath) A positive divisor of 2021^{14} is chosen at random. What is the probability that when it is divided by 7, the remainder is equal to 1?

(A) $\frac{1}{9}$ **(B)** $\frac{2}{15}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{5}$ **(E)** $\frac{1}{3}$

Answer (D):

First, note that $2021 = 43 \cdot 47$. Since $43 \equiv 1 \mod 7$, it will not affect the probability that a positive divisor $43^m \cdot 47^n$, where $0 \le m \le 14$ and $0 \le n \le 14$, has a remainder of 1 when divided by 7, so we want $47^n \equiv 5^n$ to have a remainder of 1 when divided by 1. The powers of 5 cycle through $1 - 5 - 4 - 6 - 2 - 3$, so there are 3 choices of n from 0 to 14, namely 0, 6, and 12. Since there are a total of 15 choices of n, our answer is $\dfrac{3}{15} = \boxed{\textbf{(D)}\ \frac{1}{5}}$. ■

Problem 18:

(DeToasty3) An isosceles trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$, $AB = AD = BC = 3$, and $CD = 7$ is inscribed in a circle. A point E on the circle satisfies $\overline{AC} \perp \overline{BE}$. What is DE^2?

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

Answer (A):

Note that since $\overline{AC} \perp \overline{BE}$ and $AB = BC$, chord $\overline{AC}$ is bisected by and perpendicular to $\overline{BE}$. This is sufficient information to deduce that $\overline{BE}$ is a diameter of the circle. This also means that $\angle BAE = 90°$, and since $\overline{AB} \parallel \overline{CD}$, we have that $\overline{AE} \perp \overline{CD}$. Let F be the intersection of $\overline{AE}$ and $\overline{CD}$. We have that $\triangle AFC \sim \triangle DFE$ by intercepted arcs, and since AF is an altitude of $ABCD$, we have that $DF = \frac{7-3}{2} = 2$, $CF = 7 - 2 = 5$, and $AF = \sqrt{3^2 - 2^2} = \sqrt{5}$. Thus, we have

$$\frac{DF}{EF} = \frac{AF}{CF} \implies EF = 2\sqrt{5}.$$

Finally, by the Pythagorean Theorem, our answer is $(2\sqrt{5})^2 + 2^2 = \boxed{\textbf{(A) } 24}$. ■

Problem 19:

(DeToasty3) Let a, b, c, and d be positive integers. If

$$\frac{a!}{b!} + \frac{c!}{d!} = \frac{2}{5},$$

what is the largest possible value of $a + b + c + d$?

(A) 10 **(B)** 18 **(C)** 26 **(D)** 34 **(E)** 42

Answer (D):

Note that since $\frac{2}{5} < 1$, we must have that $a < b$ and $c < d$. Also, note that since this is the case, the fractions $\frac{a!}{b!}$ and $\frac{c!}{d!}$ will both be of the form $\frac{1}{n}$ for some positive integer $n > 1$. This is because all factors of $a!$ will cancel with factors of $b!$, and all factors of $c!$ will cancel with factors of $d!$.

Note that by letting $\frac{1}{p} + \frac{1}{q} = \frac{2}{5}$, we have

$$2pq - 5p - 5q = 0 \implies 4pq - 10p - 10q + 25 = 25 \implies (2p - 5)(2q - 5) = 25.$$

We have that $\frac{a!}{b!}$ and $\frac{c!}{d!}$ are $\frac{1}{p}$ and $\frac{1}{q}$ in some order. It should be clear that in order to maximize $a+b+c+d$, we should let $\frac{a}{b}$ and $\frac{c}{d}$ be $\frac{p-1}{p}$ and $\frac{q-1}{q}$ in some order. Now, we want to make p and q as far away as possible in order to give us the maximum value of $p+q$. This is accomplished with $2p-5=1$ and $2q-5=25$, which gives us $p=3$ and $q=15$. This gives us that $\frac{a}{b}$ and $\frac{c}{d}$ are $\frac{2}{3}$ and $\frac{14}{15}$ in some order, so our answer is $2+3+14+15=\boxed{\textbf{(D) } 34}$. ■

Problem 20:

(DeToasty3) Let $\triangle ABC$ with $AB=14$ be inscribed in a circle with center O. Let P be a point on side $\overline{BC}$, and let line AP intersect the circle at a point D, distinct from A. If $\triangle ABC$ is acute and $ABDO$ is a rhombus, what is the largest possible integer value of BP^2?

(A) 33 **(B)** 48 **(C)** 65 **(D)** 78 **(E)** 95

Answer (C):

We see that the upper bound of the length of $\overline{BP}$ happens where $\angle ABC = 90^\circ$. Letting this be the case, we see that $ABDC$ is half of a regular hexagon with diagonals $\overline{AD}$ and $\overline{BC}$ which intersect at P. To find BP^2, we use the fact that $BP:CP = 1:2$ (due to similar triangles $\triangle APB$ and $\triangle CPD$), and the length of $\overline{BC}$ is equal to twice the altitude of an equilateral triangle with side length 14. This gives us that

$$BP = \frac{1}{3}\cdot 2\cdot 7\sqrt{3} \implies BP^2 = \frac{196}{3}.$$

Finally, we want to find the greatest integer less than or equal to this value, so our answer is $\boxed{\textbf{(C) } 65}$. ■

Problem 21:

(DeToasty3) Consider the polynomial

$$P(x) = (x-a)(x-b)(x-c)(x-d),$$

where a, b, c, and d are fixed positive real numbers less than 1. What is the maximum possible number of distinct real numbers r that can satisfy $|P(r)| = 1$?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

Answer (B):

Without loss of generality, suppose that $a \le b \le c \le d$. Then, by graphing $P(x)$, it should be clear that there exist some $r < a$ and $r > d$ such that $P(r) = 1$ ($P(r)$ will be positive since by then all four factors will be either positive or negative). However, when r is between any roots, we know that the absolute value of each of the four factors will be strictly less than 1 because a, b, c, and d are all between 0 and 1, exclusive, and so is r. Thus, there will not be any real numbers r which satisfy $|P(r)| = 1$ when r is between any roots. In total, there will always be 2 distinct real solutions r, so the requested maximum possible number of distinct real numbers r is also $\boxed{\textbf{(B) } 2}$. ■

Problem 22:

(treemath) Let S be the set of all positive integers which are relatively prime to 42. What is the smallest positive integer n such that the smallest number in S with exactly 2^n positive divisors is divisible by a perfect square which is greater than 1?

(A) 2 **(B)** 3 **(C)** 6 **(D)** 7 **(E)** 8

Answer (D):

We can express any number in S in the form $p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdots p_k^{e_k}$, where $p_1, p_2, p_3, \ldots, p_k$ are distinct primes not equal to 2, 3, or 7, and $e_1, e_2, e_3, \ldots, e_k$ are positive integers. In order for a number in S to have a number of positive divisors which is equal to a power of two, we must have that $e_1, e_2, e_3, \ldots, e_k$ are each one less than a power of two. Optimally, to minimize the value of such a number with a given power of two number of divisors, we should let the exponents be equal to 1, so the number of positive divisors is

$$(e_1 + 1)(e_2 + 1)(e_3 + 1) \cdots (e_k + 1) = 2^k.$$

However, for some large enough n, there will exist an even smaller value where some exponent will be greater than 1; e.g. it equals 3. The smallest prime we can use is 5, so if the exponent of 5 in the number is equal to 3, it would be equivalent to multiplying a number with 2^{k-1} positive divisors by 25 to result in a number with 2^k positive divisors. Thus, we look for all of the primes less than 25, greater than 5, and not equal to 2, 3, or 7:

$$11, 13, 17, 19, 23.$$

With the number $5 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$, which is the smallest number in S with 2^6 positive divisors, we see that instead of multiplying by 29, the next smallest prime, to create a number with 2^7 positive divisors, we can instead multiply by 25 to also create a number with 2^7 positive divisors. This number will be divisible by 25, which is a perfect square which is greater than 1. Thus, the answer is $\boxed{\textbf{(D) } 7}$. ■

Problem 23:

(DeToasty3) Joseph has the string $AABBABBB$. John is told the string has 8 letters, each an A or a B, but not the string itself or how many of each letter are in it. John is told that each move, Joseph will pick the ith letter of the existing string, that letter and every letter to its right will switch from A to B and vice versa, and John will learn what i is and how many of each letter are in the new string. If Joseph moves optimally and John reasons well, what is the fewest number of moves for John to know the string?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Answer (A):

In order to minimize the number of moves, Joseph should choose the first letter of each time a new substring of consecutive letters with the same letter starts; i.e. the 3rd, 5th, and 6th letters. However, we do not have to choose the first letter at all. This is because if Joseph only chooses the third, fifth, and sixth letters, the entire string will be As, but John will be told $i = 3$, $i = 5$, and $i = 6$, so John will know how many times each letter switches, including the first two, which he knows do not switch at all. This is sufficient for John to know the string. Thus, the answer is $\boxed{\textbf{(A) } 3}$.

To show that it cannot be less, suppose that after the first move, John was told the value of i, as well as the fact that there are a_1 As and b_1 Bs in the new string. Then, if after the second move, John was told a greater value of i, say j, as well the fact that there are a_2 As and b_2 Bs in the new string, then the number of letters to the left of the jth letter will stay the same, but the string of letters from the jth letter to the 8th letter will change. However, since John is told a_1, a_2, b_1, and b_2, John will know that $|a_1 - a_2| = |b_1 - b_2|$, which are both equal to the absolute difference between the number of As and the number of Bs in the string of letters from the jth letter to the 8th letter (John will also know which letter there is more of). Combined with the fact that John knows what j is, he can figure out how many of each letter are in the string of letters from the jth letter to the 8th letter. From here, John can also figure out how many of each letter are in the string of letters from the 1st letter to the $j-1$th letter.

In the best case scenario, in the new string, all of the letters in the first $j-1$ are the same, and all of the letters from j to 8 are the same. Then, given i and j, John could figure out the string from 1 to $i-1$, i to $j-1$, and j to 8. However, this is an impossible scenario with the given string. A similar proof is possible for $i > j$. ■

Problem 24:

(DeToasty3) In right $\triangle ABC$ with $AB = 8$, $AC = 6$, and a right angle at A, let M be the midpoint of side $\overline{BC}$, and let D be the reflection of C over line AM. Line segments $\overline{AC}$ and $\overline{DM}$ are extended to meet at a point E. What is the length of $\overline{CE}$?

(A) $\frac{32}{5}$ **(B)** $\frac{120}{17}$ **(C)** $\frac{50}{7}$ **(D)** $\frac{95}{13}$ **(E)** $\frac{84}{11}$

Answer (E):

We see that, due to reflections, $CM = DM$. However, we also know that $CM = BM$. Thus, we have that $\angle BDC = 90^\circ$. However, since $\angle BAC = 90^\circ$ as well, we know that points A and D lie on the circle with diameter $\overline{BC}$. Let this circle intersect line DM at a point F, distinct from D. We have that since $\overline{DF}$ passes through M, $\overline{DF}$ is another diameter of this circle. Thus, since $\overline{BC}$ and $\overline{DF}$ are both diameters of the circle, we must have that $\overline{BD} \parallel \overline{CF}$ and have equal lengths. We also see that due to reflections, $\overline{AM} \perp \overline{CD}$, so $\overline{AM}$ is also parallel to $\overline{BD}$ and $\overline{CF}$. Then, we have that $\triangle EFC \sim \triangle EMA$. Since $FM = 5$ and $AC = 6$, we must have that $CE = x$ and $EF = \frac{5x}{6}$. By Power of a Point, we have

$$EF \cdot DE = CE \cdot AE \implies \frac{5x}{6} \cdot \left(\frac{5x}{6} + 10\right) = x \cdot (x + 6).$$

Finally, solving for $x = CE$, we get that our answer is $\boxed{\textbf{(E) } \frac{84}{11}}$. ■

Problem 25:

(DeToasty3) What is the largest positive integer k for which there exist positive integers a and b, where $a < b$, such that $a + b > 48$ and $1 < \sqrt[3]{a^3 + kb^2} - a < 2$?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (B):

We first see that we want to minimize b because if the expression $\sqrt[3]{a^3 + kb^2} - a$ is greater than or equal to 2 when b is minimal, then increasing b will only increase the expression even more. Thus, we let $b = a + 1$ because $a < b$. We then have

$$1 < \sqrt[3]{a^3 + k(a+1)^2} - a < 2 \implies (a+1)^3 < a^3 + k(a+1)^2 < (a+2)^3.$$

For now, we only concern ourselves with

$$a^3 + k(a+1)^2 < (a+2)^3.$$

Expanding both sides of the inequality, we get

$$a^3 + ka^2 + 2ka + k < a^3 + 6a^2 + 12a + 8 \implies (6-k)a^2 + (12-2k)a + (8-k) > 0.$$

Note that by letting $k = 6$, we see that this inequality boils down to $2 > 0$, which we know is always true. However, by letting $k = 7$, we get $a^2 + 2a - 1 < 0$, which is never satisfied when a is a positive integer. It is clear that continuing to increase k will not do us any good.

Finally, testing $k = 6$ on $(a+1)^3 < a^3 + k(a+1)^2$, we get

$$a^3 + 3a^2 + 3a + 1 < a^3 + 6a^2 + 12a + 6,$$

which clearly holds for all positive integers a. Thus, the answer is $\boxed{\textbf{(B) } 6}$.

Remark. Of course, the $a+b > 48$ condition was never used here. It was simply added to avoid fakesolving the problem. ■