De Mathematics Competitions

2nd Annual

DIME

Friday, September 17, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a 15-question competition. All answers are integers ranging from 000 to 999, inclusive.
3. Mark your answer to each problem on the DIME Answer Form. Check your answers for accuracy. Only answers properly marked on the answer form will be graded; however, this mock will be graded by people.
4. SCORING: You will receive 1 point for each correct answer, 0 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the exam will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your non-existent proctor will not ask you to record certain information on the answer form.
8. You will have 3 hours to complete the competition once you decide to begin.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

The De Mathematics Competitions reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

The publication, reproduction or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

A combination of your DIME score and your DMC 10 score are not used to determine eligibility for participation in the De Junior Mathematical Olympiad (DJMO) because it will not exist anytime in the foreseeable future.

1. Charles has some marbles. Their colors are either red, green, or blue. The total number of red and green marbles is 38% more than that of blue marbles. The total number of green and blue marbles is 150% more than that of red marbles. If the total number of blue and red marbles is more than that of green marbles by n%, find n.

2. Let $P(x) = x^2 - 1$ be a polynomial, and let a be a positive real number satisfying
$$P(P(P(a))) = 99.$$
The value of a^2 can be written as $m + \sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

3. An up-right path from lattice points P and Q on the xy-plane is a path in which every move is either one unit right or one unit up. The probability that a randomly chosen up-right path from $(0, 0)$ to $(10, 3)$ does not intersect the graph of $y = x^2 + 0.5$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

4. Given a regular hexagon $ABCDEF$, let point P be the intersection of lines BC and DE, and let point Q be the intersection of lines AP and CD. If the area of $\triangle QEP$ is equal to 72, find the area of regular hexagon $ABCDEF$.

5. The four-digit base ten number $\underline{a}\ \underline{b}\ \underline{c}\ \underline{d}$ has all nonzero digits and is a multiple of 99. Additionally, the two-digit base ten number $\underline{a}\ \underline{b}$ is a divisor of 150, and the two-digit base ten number $\underline{c}\ \underline{d}$ is a divisor of 168. Find the remainder when the sum of all possible values of the number $\underline{a}\ \underline{b}\ \underline{c}\ \underline{d}$ is divided by 1000.

6. In $\triangle ABC$ with $AC > AB$, let D be the foot of the altitude from A to side $\overline{BC}$, and let M be the midpoint of side $\overline{AC}$. Let lines AB and DM intersect at a point E. If $AC = 8$, $AE = 5$, and $EM = 6$, find the square of the area of $\triangle ABC$.

7. Richard has an infinite row of empty boxes labeled $1, 2, 3, \ldots$ and an infinite supply of balls. Each minute, Richard finds the smallest positive integer k such that box k is empty. Then, Richard puts a ball into box k, and if $k \geq 3$, he removes one ball from each of boxes $1, 2, \ldots, k - 2$. Find the smallest positive integer n such that after n minutes, both boxes 9 and 10 have at least one ball in them.

8. Given a parallelogram $ABCD$, let $\mathcal{P}$ be a plane such that the distance from vertex A to $\mathcal{P}$ is 49, the distance from vertex B to $\mathcal{P}$ is 25, and the distance from vertex C to $\mathcal{P}$ is 36. Find the sum of all possible distances from vertex D to $\mathcal{P}$.

9. Let $a_1, a_2, \ldots, a_6$ be a sequence of integers such that for all $1 \le i \le 5$,

$$a_{i+1} = \frac{a_i}{3} \quad \text{or} \quad a_{i+1} = -2a_i.$$

Find the number of possible positive values of $a_1 + a_2 + \cdots + a_6$ less than 1000.

10. Let a and b be real numbers such that

$$\left(8^a + 2^{b+7}\right)\left(2^{a+3} + 8^{b-2}\right) = 4^{a+b+2}.$$

The value of the product ab can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

11. A positive integer n is called *un-two* if there does not exist an ordered triple of integers (a, b, c) such that exactly two of

$$\frac{7a+b}{n}, \quad \frac{7b+c}{n}, \quad \frac{7c+a}{n}$$

are integers. Find the sum of all un-two positive integers.

12. A sequence of polynomials is defined by the recursion $P_1(x) = x + 1$ and

$$P_n(x) = \frac{(P_{n-1}(x) + 1)^5 - (P_{n-1}(-x) + 1)^5}{2}$$

for all $n \ge 2$. Find the remainder when $P_{2022}(1)$ is divided by 1000.

13. A spinner has five sectors numbered -1.25, -1, 0, 1, and 1.25, each of which are equally likely to be spun. Ryan starts by writing the number 1 on a blank piece of paper. Each minute, Ryan spins the spinner randomly and overwrites the number currently on the paper with the number multiplied by the number the spinner lands on. The expected value of the largest number Ryan ever writes on the paper can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

14. Let $\triangle ABC$ be acute with $\angle BAC = 45^\circ$. Let $\overline{AD}$ be an altitude of $\triangle ABC$, let E be the midpoint of $\overline{BC}$, and let F be the midpoint of $\overline{AD}$. Let O be the center of the circumcircle of $\triangle ABC$, let K be the intersection of lines DO and EF, and let L be the foot of the perpendicular from O to line AK. If $BL = 6$ and $CL = 8$, find AL^2.

15. For positive integers n, let $f(n)$ denote the number of integers $1 \le a \le 130$ for which there exists some integer b such that $a^b - n$ is divisible by 131, and let $g(n)$ denote the sum of all such a. Find the remainder when

$$\sum_{n=1}^{130} [f(n) \cdot g(n)]$$

is divided by 131.

DIME

DO NOT OPEN UNTIL FRIDAY, September 17, 2021

****Administration on an earlier date will disqualify your results.****

- All the information needed to administer this exam is not contained in the non-existent DIME Teacher's Manual. PLEASE READ THE MANUAL BEFORE FRIDAY, SEPTEMBER 17, 2021.
- Send **DeToasty3**, **nikenissan**, **pog**, and **vsamc** a private message submitting your answers to the DIME. AoPS is the only way to submit your answers.
- The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

For more information about the DIME and our other competitions, please visit `https://detoasty3.github.io/dmc.html`.

Questions and comments about this competition should be sent to:

DeToasty3.