Official Solutions

De Mathematics Competitions

1st Annual

DMC 11

Friday, May 7, 2021

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

Questions and complaints about this competition should be sent by private message to

DeToasty3.

Answer Key:

1. (B)	2. (E)	3. (D)	4. (C)	5. (C)
6. (C)	7. (E)	8. (C)	9. (B)	10. (C)
11. (E)	12. (D)	13. (C)	14. (D)	15. (A)
16. (B)	17. (D)	18. (D)	19. (E)	20. (C)
21. (D)	22. (D)	23. (B)	24. (D)	25. (A)

Problem 1:

(HrishiP) What is the value of

$$\frac{1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4!}{(2 \times 0 \times 2 \times 1) + (2^3 + 0^3 + 2^3 + 1^3)}?$$

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Answer (B): It is an identity that

$$1 \cdot 1! + \cdots + n \cdot n! = (n+1)! - 1,$$

so the numerator is equal to $5! - 1 = 119$. (Computation suffices as well.) Obviously, 0 times anything is 0, so $2 \times 0 \times 2 \times 1 = 0$. Finally, we have that

$$2^3 + 0^3 + 2^3 + 1^3 = 17.$$

Thus, the requested value is equal to $\frac{119}{0+17} = \frac{119}{17} = \boxed{\text{(B) } 7}$. ■

Problem 2:

(pog) Let n be a positive integer less than 2021. It is given that if a regular hexagon is rotated n degrees clockwise about its center, the resulting hexagon coincides with the original hexagon. How many possible values of n are there?

(A) 8 **(B)** 16 **(C)** 17 **(D)** 32 **(E)** 33

Answer (E): Note that the 6 vertices of the hexagon are equidistant, so rotating the hexagon a multiple of $\frac{360°}{6} = 60°$ will give a new hexagon that coincides with the original.

Thus, we wish to find how many positive multiples of 60 are less than 2021; since $2021 = 60 \times 33 + 41$, we have that the $\boxed{\text{(E) } 33}$ possible values of n are 60×1, 60×2, 60×3, $\cdots$, and 60×33, as requested. ■

Problem 3:

(HrishiP) Today is Toasty's birthday. It is given that the square of his age is 32 less than the number of months he is old. What is the sum of Toasty's possible ages?

(A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 14

Answer (D): Let x be Toasty's age, in years. Then, Toasty is $12x$ months old. Now, we have that $x^2 = 12x - 32$, so

$$x^2 - 12x + 32 = 0$$
$$(x-4)(x-8) = 0.$$

Thus, Toasty can be either 4 years old or 8 years old, and the requested sum is equal to $4 + 8 = \boxed{\text{(D) } 12}$. ■

Problem 4:

(pog) If the product of three distinct positive real numbers forming a geometric progression is equal to 2197, what is the median of the three numbers?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

Answer (C): Let the first term of the geometric progression be equal to a, and let the common ratio between two consecutive terms of the geometric progression be equal to r. Then, the first term of the geometric progression is equal to a, the second term (or the median, since r is clearly positive) of the geometric progression is equal to ar, and the third term of the geometric progression is equal to ar^2. Thus, the product of the terms of the geometric progression is equal to

$$a \times ar \times ar^2 = a^3r^3.$$

Since the product of the terms in the geometric progression is equal to 2197, we have that $a^3r^3 = 2197$, so since we wish to find ar (the median), we can take the cube root of both sides, giving the requested median as $\sqrt[3]{2197} = \boxed{\text{(C) } 13}$. ■

Problem 5:

(HrishiP) Four children and four adults are standing in a line, but every child insists on being in between a child and an adult. How many ways can the eight people be arranged to meet these demands? (Assume the children are identical and the adults are identical.)

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer (C): Note that children will come in blocks of two, CC. Then, there must be adults around this block, so we have ACCA. Note that the ends must be adults. If the second person in line is an adult, then there is only 1 way to fill the rest. If the second person is a child, then there are two ways to fill the rest. Thus, the requested answer is $\boxed{\text{(C) } 3}$. ■

Problem 6:

(vsamc) A sequence is defined such that the first term is equal to 1, and every subsequent term is equal to 2021 more than 5 times the preceding term. For example, the second term is $5 \cdot 1 + 2021 = 2026$, and the third term is $5 \cdot 2026 + 2021 = 12151$. What is the value of the 2021st term divided by the 2020th term, rounded to the nearest integer?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Answer (C): The key observation is that 5^{2020} is much, much larger than 2021. We can use this fact by noting that as we reach the 2020th term of the sequence, our number becomes greater than 5^{2020}. To create the 2021st term of the sequence, we have to multiply 5 by a number that is much, much larger than 2021, so the addition of 2021 at the end is negligible. Therefore, the 2021st term divided by the 2020th term is very close to $\boxed{\text{(C) } 5}$, as requested. ■

Problem 7:

(HrishiP) A positive integer n exists such that n^3 has four times as many divisors as n. What is the sum of the three smallest values of n with this property?

(A) 26 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 30

Answer (E): Testing out numbers starting from 1, we see that 2 through 5 do not work, ruling out all numbers of the form p and p^2 for a prime p. We see that $6 = 2 \cdot 3$, which has $(1+1)(1+1) = 4$ divisors, and we have that $6^3 = 2^3 \cdot 3^3$, which has $(3+1)(3+1) = 16 = 4 \cdot 4$ divisors. This motivates us to look for numbers of the form $p_1 \cdot p_2$ for distinct primes p_1 and p_2. We see that the next two smallest numbers of this form are $10 = 2 \cdot 5$ and $14 = 2 \cdot 7$. All we have to check now are $8 = 2^3$ and $12 = 2^2 \cdot 3$.

We see that $8 = 2^3$ has $3 + 1 = 4$ divisors, and $(2^3)^3 = 2^9$ has $9 + 1 = 10$ divisors, but $10 \neq 4 \cdot 4$. We see that $12 = 2^2 \cdot 3$ has $(2 + 1)(1 + 1) = 6$ divisors, and $(2^2 \cdot 3)^3 = 2^6 \cdot 3^3$ has $(6 + 1)(3 + 1) = 28$ divisors, but $28 \neq 4 \cdot 6$.

Thus, the requested answer is $6 + 10 + 14 = \boxed{\text{(E) } 30}$. ■

Problem 8:

(pog) In the coordinate plane, let $\mathcal{P}$ be the figure formed by the set of points with coordinates satisfying $0.5x + y = 1$, and let $\mathcal{Q}$ be the figure formed by the set of points with coordinates satisfying $0.25x^2 + y^2 = 1$. How many points lie on both $\mathcal{P}$ and $\mathcal{Q}$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** infinitely many

Answer (C): Squaring both sides of the first equation gives

$$0.25x^2 + y^2 + xy = 1,$$

so since $0.25x^2 + y^2 = 1$, we have that xy must be equal to 0. However, squaring both sides of an equation may produce extraneous solutions, so we must check the solutions for $xy = 0$ against the solutions for $0.5x + y = 1$, giving either $x = 0$, $y = 1$ or $y = 0$, $x = 2$; there are thus $\boxed{\text{(C) } 2}$ requested points that lie on both $\mathcal{P}$ and $\mathcal{Q}$, namely $(0, 1)$ and $(2, 0)$.

OR

Note that the equation for $\mathcal{Q}$ can be rewritten as

$$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1,$$

which, when graphing, will give an ellipse with center $(0, 0)$, horizontal major axis 4, and vertical minor axis 2, so an approximate graph of the line $\mathcal{P}$ formed by the equation $0.5x + y = 1$ will intersect the ellipse twice, for a total of $\boxed{\text{(C) } 2}$ requested points that lie on both $\mathcal{P}$ and $\mathcal{Q}$.

Remark. Briefly checking the infinitely many solutions to $xy = 0$ by the graph in the second solution will make it easier to spot that there are extraneous solutions. ■

Problem 9:

(pog) How many real numbers x satisfy the equation

$$9^x + 3^{3x} = 3^{x+1} + 3?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer (B): Let $3^x = n$. Then

$$\begin{aligned} 9^x + 3^{3x} &= 3^{x+1} + 3 \\ (3^2)^x + (3^x)^3 &= 3 \times 3^x + 3 \\ (3^x)^2 + n^3 &= 3n + 3 \\ n^2 + n^3 &= 3n + 3. \end{aligned}$$

We can factor this as $n^2(n+1) = 3(n+1)$, so subtracting $3(n+1)$ from both sides gives $(n^2 - 3)(n+1) = 0$. Thus, either $n^2 - 3 = 0$ or $n + 1 = 0$, so the solutions for n are

$$n = -\sqrt{3}, \sqrt{3}, -1.$$

Since x is real, $3^x = n$ cannot be negative, so our only solution is $n = \sqrt{3}$ (where $x = \frac{1}{2}$), so $\boxed{\text{(B) } 1}$ real number x satisfies the equation $9^x + 3^{3x} = 3^{x+1} + 3$, as requested. ■

Problem 10:

(HrishiP) How many of the following statements are always true?

- The product of the lengths of the diagonals of a rectangle is equal to the area of the rectangle.
- The figure formed by the midpoints of the sides of a rectangle has half the area of the rectangle.
- The figure formed by the intersections of the internal angle bisectors of a rectangle with unequal side lengths is a rectangle with unequal side lengths.
- Consider any point inside a rectangle. If the point is reflected over all its sides, the figure formed by the reflection points has twice the area of the rectangle.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer (C): Consider a rectangle with side lengths a and b. The diagonal has length $\sqrt{a^2+b^2}$. The product of the diagonals is $a^2+b^2 \neq ab$, so statement 1 is not true.

Now, if we connect the midpoints, we have a rhombus of diagonals a, b. So, its area is $\frac{ab}{2}$, half the area the rectangle, which is ab.

Rectangles have all angles at $90°$, so the angle bisectors split it into $45-45$. Noticing similar triangles, we can see the quadrilateral at the center has 4 right angles. However, we see that the perpendicular distance between any two parallel angle bisectors is equal because the rectangle is symmetric across the x-axis and the y-axis, so it is also a square, which has equal side lengths. So statement 3 is not true.

Without the loss of generality, consider a side AB of the rectangle, and P' be the reflection of P over AB. Also, let AB intersect PP' at F. Then, $\triangle FBP \cong \triangle FBP'$ and $\triangle AFP \cong \triangle AFP'$. So, the area of $\triangle ABP$ is the same as the area of $\triangle ABP'$. Applying similar arguments to all sides, this statement is true.

Thus, the requested answer is $\boxed{\text{(C) } 2}$. ■

Problem 11:

(AT2005 & DeToasty3) Let ω be the inscribed circle of a rhombus $ABCD$ with side length 4 and $\angle DAB = 60°$. There exist two distinct lines which are parallel to line BD and tangent to ω. Given that the lines intersect sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ at points P, Q, R, and S, respectively, what is the area of quadrilateral $PQRS$?

(A) $2\sqrt{3}$ **(B)** 4 **(C)** $3\sqrt{3}$ **(D)** 6 **(E)** $4\sqrt{3}$

Answer (E): It is easy to see that $PQRS$ is a rectangle by symmetry. Note that line BD splits rhombus $ABCD$ into two equilateral triangles $\triangle ABD$ and $\triangle CBD$. We see that each equilateral triangle will contain exactly half of the area of $PQRS$ by symmetry, so we will focus on $\triangle ABD$ without loss of generality.

We see that $\triangle ABD$ has an inscribed semicircle, which is half of ω. We must find the radius of this semicircle. To do this, we note that the center of the semicircle is the midpoint M of side $\overline{BD}$, so dropping a perpendicular from M to one of the other two sides, and calling the foot point F, gives us the radius of the semicircle. Using similar triangles, we see that $\triangle AFM \sim \triangle AMD$, where the radius is equal to the length FM. We get

$$\frac{FM}{AM} = \frac{CM}{AD} \implies FM = 2 \cdot 2\sqrt{3} \cdot \frac{1}{4} = \sqrt{3}.$$

Thus, the length of the altitude of $\triangle APS$ is $2\sqrt{3} - \sqrt{3} = \sqrt{3}$, so the side length of $\triangle APS$ (which is equilateral) is 2, so $PS = QR = 2$. Also, since the length of the radius of the semicircle is $\sqrt{3}$, we have that the diameter of ω, or $PQ = RS$, is $2\sqrt{3}$.

Thus, the requested area of $PQRS$ is $2 \cdot 2\sqrt{3} = \boxed{\textbf{(E) } 4\sqrt{3}}$. ■

Problem 12:

(DeToasty3) Alice goes cherry picking in a forest. For each tree Alice sees, she either picks one cherry or three cherries from the tree and puts them in her basket. Additionally, after every five trees Alice picks from, she finds an extra cherry on the ground and puts it in her basket. At the end, Alice has 45 cherries in her basket. If the smallest possible number of trees Alice could have picked from is n, what is the sum of the digits of n?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Answer (D): Let x be the number of trees where Alice picked one cherry, and let y be the number of trees where Alice picked three cherries. We have the equation

$$\left\lfloor \frac{x+y}{5} \right\rfloor + 3x + y = 45,$$

which we can rewrite to be

$$\left\lfloor \frac{x+y}{5} \right\rfloor + x + y + 2x = 45.$$

Note that since $x + y$ is an integer, we can move it inside the floors to get

$$\left\lfloor \frac{6x+6y}{5} \right\rfloor + 2x = 45.$$

We want to find the smallest value of $x + y$ such that x is both an integer and is less than $x + y$. Testing values of $x + y$, we see that when $x + y = 15$, we have that $2x = 45 - 18 = 27 \implies x = 13.5 < 15$. However, 13.5 is not an integer. However, when $x + y = 16$, we have that $2x = 45 - 19 = 26 \implies x = 13 < 16$ and 13 is an integer. Therefore, the requested digit sum is $1 + 6 = \boxed{\textbf{(D) } 7}$. ■

Problem 13:

(AT2005 & DeToasty3) Each of 6 distinct positive integers is placed at each of 6 equally spaced points on the circumference of a circle. If the numbers on every two adjacent points are relatively prime, and the product of the numbers on every two diametrically opposite points is divisible by 3, what is the least possible sum of the 6 integers?

(A) 12 **(B)** 25 **(C)** 26 **(D)** 29 **(E)** 32

Answer (C): First, we must have that the points on the circle must alternate between multiple of 3 and not a multiple of 3. First, suppose that there existed a diameter where the numbers on both points of the diameter are multiples of 3. Then, the numbers adjacent to each of the two points must not be multiples of 3 to satisfy the relatively prime condition. However, this means that the products of the numbers of the other two diameters will both not be divisible by 3 because they contain no multiples of 3. Thus, we want every diameter to have exactly one point with a multiple of 3, so we should alternate between multiple of 3 and not a multiple of 3 between adjacent points.

Next, to find the smallest sum, we should try to minimize the value of the multiples of 3. This can be achieved by having three of the numbers be 3, 6, and 9. Now, the other three numbers must not be divisible by 3, giving us 1 and 2. However, we note that we cannot place a 4. This is because we already have two numbers divisible by 2, and casework tells us that the 2, 4, and 6 cannot all be placed on the same circle. Thus, the next smallest number is 5, which we can easily verify to work. (Note that if we try to create 1, 2, and 4 with the non-multiples of 3, then we have to find a larger multiple of 3, so our net gain will be positive.)

Thus, the requested sum is $1+2+3+5+6+9 = \boxed{\textbf{(C) } 26}$. ■

Problem 14:

(pog) If a and b are the distinct roots of the polynomial $x^2+2021x+2019$, then

$$\frac{1}{a^2+2019a+2019} + \frac{1}{b^2+2019b+2019} = \frac{m}{n},$$

where m and n are relatively prime positive integers. What is $m+n$?

(A) 2020 **(B)** 2021 **(C)** 4040 **(D)** 6059 **(E)** 6061

Answer (D): Note that since a is a root of $x^2+2021x+2019$, we have that $a^2+2021a+2019 = 0$ and

$$a^2+2019a+2019 = (a^2+2021a+2019) - 2a = 0 - 2a = -2a.$$

Similarly, note that since b is a root of $x^2+2021x+2019$, we have that $b^2+2021b+2019 = 0$ and

$$b^2 + 2019b + 2019 = (b^2 + 2021b + 2019) - 2b = 0 - 2b = -2b.$$

Thus, $\dfrac{1}{a^2 + 2019a + 2019} + \dfrac{1}{b^2 + 2019b + 2019} = \dfrac{1}{-2a} + \dfrac{1}{-2b} = \dfrac{-2a - 2b}{4ab}$.

Since a and b are the roots of $x^2 + 2021x + 2019$, we have that $(x - a)(x - b) = x^2 - ax - bx + 2019 = x^2 - (a + b)x + 2019 = x^2 + 2021x + 2019$, so $a + b = -2021$ and $ab = 2019$.

Finally, we get that $\dfrac{-2a - 2b}{4ab} = -\dfrac{a + b}{2ab}$, so since $a + b = -2021$ and $ab = 2019$,

$$\frac{1}{a^2 + 2019a + 2019} + \frac{1}{b^2 + 2019b + 2019} = -\frac{-2021}{2 \times 2019} = \frac{2021}{4038}$$

and the requested sum is equal to $2021 + 4038 = \boxed{\textbf{(D) } 6059}$. ■

Problem 15:

(ApraTrip & DeToasty3) Let $\triangle ABC$ have $AB = 20$, $AC = 21$, and a right angle at A. Let I be the center of the inscribed circle of $\triangle ABC$. Let point D be the reflection of point B over the line parallel to AB passing through I, and let point E be the reflection of point C over the line parallel to AC passing though I. What is the value of DE^2?

(A) 145 **(B)** 149 **(C)** 153 **(D)** 157 **(E)** 161

Answer (A): Let lines BD and CE intersect at a point P. We see that $ABPC$ is a rectangle because $\angle BAC = 90°$, and reflections over a line parallel to the side create right angles $\angle ABD$ and $\angle ACE$. With three right angles, we can be certain that $ABPC$ is a rectangle.

Next, we see that $BD = CE = 2r$, where r is the radius of the inscribed circle of $\triangle ABC$. This is because D is the reflection of a line parallel to side $\overline{AB}$ passing through I, so this line and line AB have the same perpendicular distance throughout, which is also the perpendicular from I to line AB (the inradius of $\triangle ABC$). The same can be said about E with respect to side $\overline{AC}$.

The inradius of $\triangle ABC$ can be found by dividing its area by its semiperimeter. The area of $\triangle ABC$ is $\frac{1}{2} \cdot 20 \cdot 21 = 210$, and the semiperimeter of $\triangle ABC$ is $\frac{20+21+29}{2} = 35$, where the 29 comes from the Pythagorean Theorem. Thus,

$$r = \frac{210}{35} = 6.$$

Then, we have that $BD = CE = 12$, so $PD = 21 - 12 = 9$, and $PE = 20 - 12 = 8$. Since $\angle DPE = 90^\circ$, we can use the Pythagorean Theorem to get the requested answer of

$$DE^2 = 9^2 + 8^2 = 81 + 64 = \boxed{\textbf{(A) } 145}.$$

■

Problem 16:

(HrishiP) Let $b > 6$ be an integer. There exist base-b and base-$(b+1)$ numbers such that

$$8\gcd(600_b, 660_b) = 45\gcd(100_{b+1}, 110_{b+1}).$$

What is the sum of the digits of b?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (B): Using the base expansions, we have by Euclidean Algorithm

$$8\gcd(600_b, 660_b) = 8\gcd(6b^2, 6b^2 + 6b) = 8\gcd(6b^2, 6b) = 48b.$$

Similarly, we have

$$\begin{aligned} 45\gcd(110_{b+1}, 100_{b+1}) &= 45\gcd((b+1)^2 + (b+1), (b+1)^2) \\ &= 45(b+1) = 45b + 45. \end{aligned}$$

So, $48b = 45b + 45$ which implies $b = 15$. The requested sum of the digits is $\boxed{\textbf{(B) } 6}$. ■

Problem 17:

(DeToasty3) Hanami rolls two standard six-sided dice. If the sum of the numbers she rolled is at least 7, she rolls both dice again (and does not roll again thereafter). Otherwise, she does not roll again. What is the probability that she rolls a 5 on at least one of the dice, on at least one of the rolls?

(A) $\frac{1}{3}$ **(B)** $\frac{13}{36}$ **(C)** $\frac{29}{72}$ **(D)** $\frac{11}{27}$ **(E)** $\frac{4}{9}$

Answer (D): Let us find the probability that Hanami does not roll the number 5 at all.

The number of possible dice rolls with at least one 5 is all the rolls with one 5 and one non-5, and a roll with two 5s, for $2 \cdot 5 + 1 = 11$ possibilities. Of the $36 - 11 = 25$ other possibilities, we have that the 13 rolls $1-1$, $1-2$, $2-1$, $2-2$, $1-3$, $3-1$, $1-4$, $4-1$, $2-3$, $3-2$, $2-4$, $4-2$, and $3-3$ have a sum less than 7. Thus, the other $25-13=12$ have a sum at least 7. We now perform casework:

If Hanami rolled the two dice once, she has a $\frac{13}{36}$ chance of rolling no 5s.

If Hanami rolled the two dice twice, then her first roll has a $\frac{12}{36}=\frac{1}{3}$ chance of having a sum at least 7 and without any 5s, and her second roll has a $\frac{25}{36}$ chance of having no 5s.

This contributes a

$$\frac{13}{36}+\frac{1}{3}\cdot\frac{25}{36}=\frac{16}{27}$$

complementary probability.

Subtracting from 1 gives us the requested answer of

$$1-\frac{16}{27}=\boxed{\textbf{(D) } \frac{11}{27}}.$$

■

Problem 18:

(DeToasty3) In equilateral $\triangle ABC$, let points D and E be on lines AB and AC, respectively, both on the opposite side of line BC as A. If $CE=DE$, and the circumcircle of $\triangle CDE$ is tangent to line AB at D, what is the degree measure of $\angle CDE$?

(A) 70 **(B)** 72 **(C)** 75 **(D)** 80 **(E)** 84

Answer (D): First, we note that $\angle CED=\angle CDB$. This is because the chord $\overline{CD}$ shares point D with the point of tangency of the circumcircle of $\triangle CDE$ to line AB, so the measure of the acute inscribed angle of chord $\overline{CD}$ is equal to the measure of the angle formed between the chord $\overline{CD}$ and the tangent line AB.

Next, letting $\angle CED=\angle CDB=2x$, we get that

$$\angle DCE=\frac{1}{2}\cdot(180^\circ-2x)=90^\circ-x,$$

so

$$\angle BCD=180^\circ-\angle ACB-\angle DCE=30^\circ+x.$$

Additionally, by using $\triangle BCD$, we also get that

$$\angle BCD=180^\circ-\angle BDC-\angle DBC=60^\circ-2x.$$

Solving the equation $30° + x = 60° - 2x$, we get that $x = 10°$, so

$$\angle CDE = 90° - 10° = 80°,$$

so the requested answer is $\boxed{\textbf{(D) } 80}$.

Problem 19:

(HrishiP) If the sum of the digits of the base-three representation of

$$\frac{(3^0+1)^3+1}{(3^0)^2+3^0+1} + \frac{(3^1+1)^3+1}{(3^1)^2+3^1+1} + \cdots + \frac{(3^{15}+1)^3+1}{(3^{15})^2+3^{15}+1}$$

is equal to S, what is the value of S when expressed in base-ten?

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Answer (E): Observe that

$$\frac{(x+1)^3+1}{x^2+x+1} = \frac{x^3+3x+3x+2}{x^2+x+1} = x+2.$$

Then, our sum is

$$3^0 + 3^1 + \cdots + 3^{15} + 32.$$

We know that $32 = 3^3 + 3 + 2(3^0)$. Then, if we add to the rest, we have

$$3(3^0) + 2(3^1) + 3^2 + 2(3^3) + \cdots + 3^{15}.$$

However, $3(3^0) = 3^1$ gets carried over to 3^1, so we have $3(3^1) = 3^2$ which gets carried over to 3^2. Thus, our sum is expressed as

$$2(3^2) + 2(3^3) + 3^4 + \cdots + 3^{15}.$$

This is equal to 1111111111112200_3, so the requested digit sum is $12 \times 1 + 2 \times 2 + 2 \times 0 = \boxed{\textbf{(E) } 16}$.

Problem 20:

(DeToasty3) Richard has four identical balls labeled 1, and two identical balls labeled -1. He randomly places each ball into one of six different bins, where he is allowed to place multiple balls in the same bin. What is the probability that the sum of the numbers of the balls in each bin is nonnegative? (A bin with no balls in it has sum 0.)

(A) $\frac{61}{441}$ **(B)** $\frac{23}{147}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{3}{14}$ **(E)** $\frac{33}{98}$

Answer (C): Note that since the balls are identical, to find the denominator, we look at each of the balls labeled 1 independently of each of the balls labeled -1 by using stars and balls.

With four identical balls labeled 1, we have four balls and five dividers (because six bins), so this gives us $\binom{4+5}{4} = 126$. With two identical balls labeled -1, we have two balls and five dividers, so this gives us $\binom{2+5}{2} = 21$. Next, we perform casework.

Case 1: All four positive balls are in the same bin. Then, the only way for each bin to have a nonnegative sum is if the two negative balls are in that bin. Since there are six bins in total, this contributes 6 cases.

Case 2: Three positive balls are in one bin, and one positive ball is in a different bin. There are $\binom{6}{2} = 15$ ways to choose the two bins, and 2 ways to decide the ordering, which gives us $15 \cdot 2 = 30$ ways to put the positive balls in the bins. Now, for the negative balls, either both of them go in the bin with three positive balls, or one ball goes in the bin with three positive balls, and the other goes in the bin with one positive ball. This gives us 2 ways to place the negative balls for each placing of the positive balls, for $30 \cdot 2 = 60$ cases.

Case 3: Two positive balls are in one bin, and two positive balls are in a different bin. There are $\binom{6}{2} = 15$ ways to choose the two bins. Now, the negative balls can either go 2 and 0, 1 and 1, or 0 and 2 among the two occupied bins, from left to right. This gives us 3 ways to place the negative balls for each placing of the positive balls, for $15 \cdot 3 = 45$ cases.

Case 4: Two positive balls are in one bin, one is in another bin, one is in a third bin. There are $\binom{6}{3} = 20$ ways to choose the three bins, and 3 ways to decide the ordering, which gives us $20 \cdot 3 = 60$ ways to put the positive balls in the bins. Now, the negative balls can either both go in the bin with two balls, one goes in the one with two balls, and the other goes in one of the 2 bins with one ball, or each goes in one of the bins with one ball, for a total of $1 + 2 + 1 = 4$ possibilities for each placing of the positive balls, for $60 \cdot 4 = 240$ cases.

Case 5: All four positive balls are in different bins. There are $\binom{6}{4} = 15$ ways to choose the four bins. Now, the negative balls must go in different bins with a positive ball in it, so there are $\binom{4}{2} = 6$ possibilities for each placing of the positive balls, for $15 \cdot 6 = 90$ cases.

The requested answer is equal to

$$\frac{6 + 60 + 45 + 240 + 90}{126 \cdot 21} = \frac{441}{21^2 \cdot 6} = \boxed{\text{(C) } \frac{1}{6}}.$$

OR

Note that since the balls are identical, to find the denominator, we look at each of the balls labeled 1 independently of each of the balls labeled -1 by using stars and balls.

With four identical balls labeled 1, we have four balls and five dividers (because six bins), so this gives us $\binom{4+5}{4} = 126$. With two identical balls labeled -1, we have two balls and five dividers, so this gives us $\binom{2+5}{2} = 21$.

Note that each -1 ball must be paired with a 1 ball in order for the sum of the numbers in each bin to be nonnegative. After we pair, we have two more 1 balls left to distribute. By Stars and Bars, we have $\binom{2+5}{2} = 21$ ways to distribute the two -1 and 1 pairs, and $\binom{2+5}{2} = 21$ ways to distribute the remaining two 1 balls. Therefore, our requested probability is

$$\frac{21 \cdot 21}{126 \cdot 21} = \boxed{\textbf{(C) } \frac{1}{6}}.$$

■

Problem 21:

(DeToasty3) In pentagon $ABCDE$, where all interior angles have a positive degree measure less than 180°, let M be the midpoint of side $\overline{DE}$. It is given that line BM splits $ABCDE$ into two isosceles trapezoids $ABME$ and $CDMB$ such that each one contains exactly three sides of equal length. If $AE = 3$ and $DE = 26$, what is the area of $ABCDE$?

(A) 216 **(B)** 234 **(C)** 288 **(D)** 312 **(E)** 330

Answer (D): Given the side lengths, we find that $EM = 13$. Thus, in isosceles trapezoid $ABME$, either $AE = BM = AB = 3$ and $EM = 13$ or $EM = BM = AB = 13$ and $AE = 3$.

Case 1: $AE = BM = AB = 3$ and $EM = 13$. Then $DM = 13$ and $BM = 3$. Note that in trapezoid $CDMB$, we must either have $BM = BC = CD = 3$ and $DM = 13$ or $DM = BC = CD = 13$ and $BM = 3$. Note that in the former case, the greatest possible length of BD is $3+3 = 6$. However, $\triangle BMD$ would have side lengths 3, 6, and 13, which is impossible because, by the triangle inequality, $3 + 6 = 9 < 13$. In the second case, note that $BM \parallel CD$ since $DM = BC = 13$ are the legs. But note that $\angle ABM > 90^\circ$ (intuitively) and $\angle MBC > 90^\circ$ (intuitively), so $\angle ABC = \angle ABM + \angle MBC > 180^\circ$, which contradicts the interior angles having a positive degree measure less than 180° condition. Thus, nothing is possible in this case.

Case 2: $EM = BM = AB = 13$ and $AE = 3$. Then $DM = 13$ and $BM = 13$. Note that we do not want to have $BC = 13$ because that makes $BM \parallel CD$, and $\angle ABC = 180^\circ$, which we do not want. Therefore, $CD = 13$ and $DM \parallel BC$. This is a working pentagon.

Finally, to find the area of this pentagon, note that we can drop perpendiculars from E and A to segment $\overline{BM}$. Then, we have that segment $\overline{BM}$ is split in lengths $\frac{13-3}{2} = 5$, 3, and 5. Thus, the height is $\sqrt{13^2 - 5^2} = 12$, so the area of $ABME$ is $\frac{1}{2} \cdot (3+13) \cdot 12 = 96$. Similarly, in isosceles trapezoid $CDMB$, we use similarity to find that its height is also 12. Then, the base $\overline{BC}$ has length $5 + 13 + 5 = 23$, and $DM = 13$. Therefore, the area of $CDMB$ is $\frac{1}{2} \cdot (13 + 23) \cdot 12 = 216$. Finally, the requested area of pentagon $ABCDE$ is $96 + 216 = \boxed{\textbf{(D) } 312}$. ■

Problem 22:

(HrishiP) Let $P(x)$ be a polynomial with degree 3 and real coefficients such that the coefficient of the x^3 term is 1, and $P(x)$ has roots a, b, and c that satisfy

$$\frac{-(a+b)(b+c)(c+a)}{2022} = abc = 2021.$$

What is the minimum possible value of $|P(1)|$?

(A) 2019 **(B)** 2020 **(C)** 2021 **(D)** 2022 **(E)** 2023

Answer (D): Let $P(x) = x^3 + wx^2 + zx - 2021$. We have

$$P(a+b+c = -w) = (-w-a)(-w-b)(-w-c) = (a+b)(b+c)(c+a) = 2021(-2022).$$

So, we know that $wz = 2021^2$. Note that

$$|P(1)| = |w + z - 2020|$$

thus we must either minimize w and z when they are both positive or maximize w and z when they are both negative.

By AM-GM,

$$\frac{\frac{2021^2}{w} + w}{2} \geq \sqrt{2021^2} = 2021,$$

and

$$\frac{\frac{-2021^2}{w} - w}{2} \leq -\sqrt{2021^2} = -2021.$$

Thus, $w+z \leq -4042$ or $w+z \geq 4042$. Note that $|P(1)|$ is minimized when $w+z = 4042$. The requested minimum possible value of $|P(1)|$ is

$$|P(1)| = 1^3 + 4042 - 2021 = \boxed{\textbf{(D) } 2022}.$$

■

Problem 23:

(DeToasty3) An acute $\triangle ABC$ has $BC = 30$, $\angle BAC = 60^\circ$, $AC > AB$, and circumcircle ω with center O. The line tangent to ω at A intersects line BC at a point D. It is given that the line passing through O, parallel to line BC, intersects line AD at a point P such that $AP : DP = 4 : 3$. The length AD can be written as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. What is $m+n$?

(A) 19 **(B)** 22 **(C)** 25 **(D)** 28 **(E)** 31

Answer (B): Let M be the midpoint of $\overline{BC}$. It is known that the perpendicular to $\overline{BC}$ through M passes through O. Since $AP : DP = 4 : 3$, we also have that the line parallel to $\overline{BC}$ through A is $\frac{4}{3}$ times as far from the line parallel to $\overline{BC}$ through O as line BC is from the line parallel to $\overline{BC}$ through O. Let the intersection of line MO and the line parallel to $\overline{BC}$ through A be N, and let the foot of the perpendicular from A to the line parallel to $\overline{BC}$ through O be Q. We see that $AD \perp AO$, so by angle chasing, we see that $\triangle ANO \sim \triangle AQP$.

We see that $MO = 5\sqrt{3}$ by $30-60-90$ properties on $\triangle BMO$, so $NO = AQ = \frac{20\sqrt{3}}{3}$. We know that $AO = BO = 10\sqrt{3}$, so by the Pythagorean Theorem, we see that

$$AN = \sqrt{AO^2 - NO^2} = \frac{10\sqrt{15}}{3}.$$

By similarity, we have that

$$\frac{AQ}{AP} = \frac{AN}{AO} \implies \frac{20\sqrt{3}}{3} \cdot 10\sqrt{3} = \frac{10\sqrt{15}}{3} \cdot AP \implies AP = 4\sqrt{15}.$$

Thus,

$$AD = \frac{7}{4} \cdot 4\sqrt{15} = 7\sqrt{15},$$

so the requested answer is $m+n = 7+15 = \boxed{\textbf{(B) } 22}$. ■

Problem 24:

(DeToasty3) There exist positive real numbers a and b such that $\lfloor ab^2 \rfloor = 1$, $\lfloor a^2b^2 \rfloor = 2$, and $\lfloor a^3b^2 \rfloor = 6$, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to a real number r. It is given that as a approaches its greatest lower bound A, the set of possible values of b approaches a single value B. What is the value of $A^4 + B^4$?

(A) $\frac{31}{3}$ **(B)** $\frac{539}{48}$ **(C)** 13 **(D)** $\frac{265}{16}$ **(E)** $\frac{67}{4}$

Answer (D): Given that $\lfloor ab^2 \rfloor = 1$ and $\lfloor a^2b^2 \rfloor = 2$, we find that the smallest interval containing all possible values of a is $(\frac{2}{1+1}, \frac{2+1}{1}) = (1, 3)$.

Given that $\lfloor a^2b^2 \rfloor = 2$ and $\lfloor a^3b^2 \rfloor = 6$, we find that the smallest interval containing all possible values of a is $(\frac{6}{2+1}, \frac{6+1}{2}) = (2, \frac{7}{2})$.

Given that $\lfloor ab^2 \rfloor = 1$ and $\lfloor a^3b^2 \rfloor = 6$, we find that the smallest interval containing all possible values of a^2 is $(\frac{6}{1+1}, \frac{6+1}{1}) = (3, 7)$.

We want a to be in all three of these intervals. So, a^2 must be in all of $(1, 9)$, $(4, \frac{49}{4})$, and $(3, 7)$, which gives us $a^2 \in (4, 7)$.

Thus, we find that $A^2 = 4 \implies A^4 = 16$. To find B, we note that this maximum lower bound of A occurs when $a^2b^2 = 3$ and $a^3b^2 = 6$. Cubing the first equation and squaring the second equation gives us $a^6b^6 = 27$ and $a^6b^4 = 36$, so

$$\frac{b^6}{b^4} = b^2 = B^2 = \frac{3}{4} \implies B^4 = \frac{9}{16}.$$

Thus, the requested answer is

$$A^4 + B^4 = 16 + \frac{9}{16} = \boxed{\textbf{(D)}\ \frac{265}{16}}.$$

■

Problem 25:

(DeToasty3) Ryan has an infinite supply of slips and a spinner with letters O, S, and T, where each letter is equally likely to be spun. Each minute, Ryan spins the spinner randomly, writes on a blank slip the letter he spun, and puts it in a pile. Ryan continues until he has written all 3 letters at least once, at which point he stops. What is the probability that after he stops, he can form the words $OTSS$ and $TOST$ using 4 distinct slips from the pile? (Ryan may reuse slips he used for one word in forming the other.)

(A) $\frac{7}{54}$ **(B)** $\frac{13}{72}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{8}{27}$ **(E)** $\frac{1}{3}$

Answer (A): Note that the condition is equivalent to saying that we need one O, two Ss, and two Ts. Instead of using normal counting, we will use complementary counting. Note that in order to not satisfy the condition, we require Ryan to write only one S, only one T, or both. Also, note that the only way for Ryan to write at least two Ss and at least two Ts is if the letter O is the last letter written.

The probability of the letter O being the last letter written is $\frac{1}{3}$. This is because each letter is equally likely to be last, and there are no other restrictions. Now, we find the probabilities that given that Ryan writes the letter O last, he has also written only one S, only one T, or both. For this, we will use PIE.

Case 1: Ryan writes only one S. Consider the first n letters that he writes. Note that the S can go in one of these n spots, and the rest can be filled with Ts. Including the last letter O, this gives us $n+1$ letters in total. This gives us a $\frac{n}{3^{n+1}}$ probability for $n+1$ letters. Summing over all integers $n \geq 2$, this gives us

$$\frac{2}{3^3} + \frac{3}{3^4} + \frac{4}{3^5} + \cdots .$$

We can compute this by letting

$$S = \frac{2}{3^3} + \frac{3}{3^4} + \frac{4}{3^5} + \cdots .$$

We then get that

$$\frac{S}{3} = \frac{2}{3^4} + \frac{3}{3^5} + \frac{4}{3^6} + \cdots ,$$

and subtracting the latter equation from the former equation gives

$$\frac{2S}{3} = \frac{2}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6} + \cdots .$$

Evaluating the infinite series, we get

$$\frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6} + \cdots = \frac{1}{81} \cdot \frac{3}{2} = \frac{1}{54},$$

so

$$\frac{2S}{3} = \frac{2}{27} + \frac{1}{54} = \frac{5}{54} \implies S = \frac{5}{36}.$$

Case 2: Ryan writes only one T. By symmetry, this case has the same probability as Case 1, or $\frac{5}{36}$.

Case 3: Ryan writes only one S and only one T. There are only two cases: STO and TSO, where each has a $\frac{1}{27}$ probability, for a total of $\frac{2}{27}$ for this case.

We find that our overall complementary probability is

$$2 \cdot \frac{5}{36} - \frac{2}{27} = \frac{11}{54},$$

so the requested answer is

$$\frac{1}{3} - \frac{11}{54} = \boxed{\textbf{(A) } \frac{7}{54}}.$$

■