KMMC

Karate Masters Mathematics Competitions

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2nd Annual

KMMC 10

Karate Masters Mathematics Competition 10

Thursday, October 14, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the KMMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than writing utensils, blank scratch paper, rulers, compasses, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your non-existent proctor will not ask you to record certain information on the answer form.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have already taken the test in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

The Committee on the Karate Masters Mathematics Competitions reserves the right to re-examine students before deciding whether to grant official status to their scores. The Committee also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score well on this KMMC 10 will not be invited to anything because the KMIME (Karate Masters Invitational Mathematics Examination) does not exist. More details about the KMIME and other information are not on the back page of this test booklet.

1. What is the value of

$$2^0 - (21 - 20) + 2^1?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

2. What is the smallest positive integer which can be expressed as the sum of a positive perfect square and a distinct positive perfect cube?

(A) 2 **(B)** 3 **(C)** 5 **(D)** 9 **(E)** 10

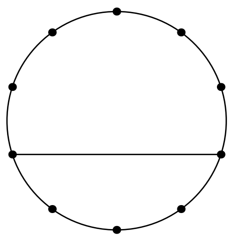
3. Squares $\mathcal{P}$ and $\mathcal{Q}$ are such that half the area of $\mathcal{Q}$ lies within $\mathcal{P}$. If the areas of $\mathcal{P}$ and $\mathcal{Q}$ are 8 and 2, respectively, what is the area of the region inside $\mathcal{P}$ and outside $\mathcal{Q}$?

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

4. A positive integer n is divisible by 5 but not 4. For which of the following values will adding it to n never result in a sum that is divisible by 20?

(A) 45 **(B)** 50 **(C)** 55 **(D)** 60 **(E)** 65

5. Ten points are equally spaced on the circumference of a circle, where two of the points are connected by a line segment, as shown below. Karate wants to choose two of the eight other points and draw a line segment connecting them so that this line segment does not intersect the other line segment. In how many ways can Karate do this?



(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

6. One of the side lengths of an obtuse isosceles triangle with integer side lengths is 2. What is the perimeter of this triangle?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

7. Let a be a positive integer. A geometric sequence b, c, d in that order satisfies

$$ab = 15, \quad bd = 16, \quad \text{and} \quad ac = 120.$$

What is the value of $a + d$?

(A) 56 **(B)** 58 **(C)** 60 **(D)** 62 **(E)** 64

8. Let n be the number of factors of 3 that the product

$$(1 + 2 + 3)(4 + 5 + 6)(7 + 8 + 9) \cdots (94 + 95 + 96)(97 + 98 + 99)$$

contains. What is the sum of the digits of n?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

9. A rectangle $ABCD$ has $AB = 8$ and $BC = 4$. Points P and Q lie on sides $\overline{AB}$ and $\overline{BC}$, respectively, such that $AP = CQ$ and the area of $\triangle BPQ$ is 6. What is PQ^2?

(A) 32 **(B)** 34 **(C)** 36 **(D)** 38 **(E)** 40

10. There exist 4-digit numbers such that the 4-digit number is divisible by 9, the first 3 digits in order form a 3-digit number divisible by 9, and the last 3 digits in order form a 3-digit number divisible by 9. How many such 4-digit numbers exist?

(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

11. In the xy-plane, the distance between the points (r, r) and $(13, 7)$ is equal to d, where r is a real number satisfying the equation $r^2 - 20r + 21 = 0$. What is d^2?

(A) 78 **(B)** 106 **(C)** 150 **(D)** 176 **(E)** 218

12. Karate has some shirts, pairs of pants, and socks. An outfit consists of one shirt, one pair of pants, and two socks. Karate can currently wear 60 possible different outfits, but if he were to get either four more shirts or four more socks, then Karate could wear 180 possible different outfits. How many pairs of pants does Karate have? (The socks are distinguishable, and the order in which he wears the socks does not matter.)

(A) 2 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 15

13. Karate writes the number 2021 on a blackboard. He then repeatedly erases and writes a new number on the blackboard, where if the current number on the blackboard is odd, he will erase the number and write $8n$ on the blackboard, and if the current number on the blackboard is even, he will erase the number and write $n + 2019$ on the blackboard. Eventually, Karate will have written 2020 numbers on the board (including the initial 2021). What is the remainder when his 2020th number is divided by 9?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 7

14. In the xy-plane, the point $(24, 7)$ is reflected over the line $y = ax$ and then shifted up b units to the point $(20, 21)$, where a and b are positive real numbers. What is $a \cdot b$?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 12

15. Karate repeatedly rolls two fair six-sided dice. After every roll, Karate writes the sum of the numbers on the two dice. Karate stops once he writes the number 11. What is the probability that Karate writes the number 5 at least once before stopping?

(A) $\frac{1}{3}$ **(B)** $\frac{5}{12}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

16. A pyramid $ABCDE$ has rectangular base $ABCD$ with $AB = 8$ and $BC = 6$ and apex E located 8 units above $ABCD$. A plane parallel to $\triangle ACE$ hits segments $\overline{AB}$, $\overline{BC}$, and $\overline{BE}$ at F, G, and H, respectively. If $FG = \frac{15}{2}$, what is the volume of $DFGH$?

(A) 27 **(B)** 36 **(C)** 45 **(D)** 48 **(E)** 60

17. A positive divisor of 2021^{14} is chosen at random. What is the probability that when it is divided by 7, the remainder is equal to 1?

(A) $\frac{1}{9}$ **(B)** $\frac{2}{15}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{5}$ **(E)** $\frac{1}{3}$

18. An isosceles trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$, $AB = AD = BC = 3$, and $CD = 7$ is inscribed in a circle. A point E on the circle satisfies $\overline{AC} \perp \overline{BE}$. What is DE^2?

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

19. Let a, b, c, and d be positive integers. If

$$\frac{a!}{b!} + \frac{c!}{d!} = \frac{2}{5},$$

what is the largest possible value of $a + b + c + d$?

(A) 10 **(B)** 18 **(C)** 26 **(D)** 34 **(E)** 42

20. Let $\triangle ABC$ with $AB = 14$ be inscribed in a circle with center O. Let P be a point on side $\overline{BC}$, and let line AP intersect the circle at a point D, distinct from A. If $\triangle ABC$ is acute and $ABDO$ is a rhombus, what is the largest possible integer value of BP^2?

(A) 33 **(B)** 48 **(C)** 65 **(D)** 78 **(E)** 95

21. Consider the polynomial

$$P(x) = (x - a)(x - b)(x - c)(x - d),$$

where a, b, c, and d are fixed positive real numbers less than 1. What is the maximum possible number of distinct real numbers r that can satisfy $|P(r)| = 1$?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

22. Let S be the set of all positive integers which are relatively prime to 42. What is the smallest positive integer n such that the smallest number in S with exactly 2^n positive divisors is divisible by a perfect square which is greater than 1?

(A) 2 **(B)** 3 **(C)** 6 **(D)** 7 **(E)** 8

23. Karate has the string $AABBABBB$. Judo is told the string has 8 letters, each an A or a B, but not the string itself or how many of each letter are in it. Judo is told that each move, Karate will pick the ith letter of the existing string, that letter and every letter to its right will switch from A to B and vice versa, and Judo will learn what i is and how many of each letter are in the new string. If Karate moves optimally and Judo reasons well, what is the fewest number of moves for Judo to know the string?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

24. In right $\triangle ABC$ with $AB = 8$, $AC = 6$, and a right angle at A, let M be the midpoint of side $\overline{BC}$, and let D be the reflection of C over line AM. Line segments $\overline{AC}$ and $\overline{DM}$ are extended to meet at a point E. What is the length of $\overline{CE}$?

(A) $\dfrac{32}{5}$ **(B)** $\dfrac{120}{17}$ **(C)** $\dfrac{50}{7}$ **(D)** $\dfrac{95}{13}$ **(E)** $\dfrac{84}{11}$

25. What is the largest positive integer k for which there exist positive integers a and b, where $a < b$, such that $a + b > 48$ and $1 < \sqrt[3]{a^3 + kb^2} - a < 2$?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

2021 FALL KMMC 10

DO NOT OPEN UNTIL THURSDAY, October 14, 2021

KMMC
KARATE MASTERS MATHEMATICS COMPETITIONS

Questions and complaints about problems and solutions for this exam should be sent by private message to:

DeToasty3, **karate7800**, and **pandabearcat**.

The 2022 KMIME will never be held. It would be a 15-question, 3-hour, integer-answer exam if it was held. You will not be invited to participate because this contest does not exist. *A complete listing of our previous publications may be found at our web site:*

Wait, we don't have a website!

****Try Administering This Exam On An Earlier Date. Oh Wait, You Can't.****

1. All the information needed to administer this exam is not contained in the non-existent KMMC 10 Teacher's Manual.
2. YOU must not verify on the non-existent KMMC 10 COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.
3. Send **DeToasty3**, **karate7800**, and **pandabearcat** a private message submitting your answers to the KMMC 10. AoPS is the only way to submit your answers.
4. The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

The **2021 Karate Masters Mathematics Competitions**
was made possible by the contributions of the following people:

bobthegod78, dc495, DeToasty3, HrishiP, karate7800, math31415926535, MathPirate101, Michael595, pandabearcat, PhunsukhWangdu, pog, themathboi101, & treemath

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