De Mathematics Competitions

2nd Annual

DMC 10B

Friday, June 11, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the DMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than writing utensils, blank scratch paper, rulers, compasses, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your non-existent proctor will not ask you to record certain information on the answer form.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have already taken the test in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

Students who score well on this DMC 10 may or may not be invited to the 2022 DIME (De Invitational Mathematics Examination). More details about the DIME and other information are on the back page of this test booklet.

1. What is the value of
$$4^1 - 3^2 + 2^3 - 1^4?$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2. Let a, b, c, and d be real numbers that satisfy
$$a+1=b, \quad b+2=c, \quad c+3=d, \quad d+4=21.$$
What is $a+b+c+d$?

(A) 51 **(B)** 52 **(C)** 53 **(D)** 54 **(E)** 55

3. Given a right triangle with legs of lengths 5 and 6, a square is drawn with one side as its hypotenuse such that the triangle is completely inside the square. What is the area of the region inside the square but outside the triangle?

(A) 46 **(B)** 47 **(C)** 48 **(D)** 49 **(E)** 50

4. A group of 200 people were invited to see a movie, where each person either had first row seats, second row seats, or third row seats. It is given that four-fifths of the people invited chose to watch the movie, one-ninth of the viewers were not invited, one-fifth of the viewers had first row seats, and 60 of the viewers had third row seats. What is the probability that a randomly selected viewer had second row seats?

(A) $\frac{3}{10}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{7}{15}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{4}{5}$

5. What is the sum of all positive real numbers a such that the equation $x^2 + ax - 12 = 0$ has two distinct integer solutions x?

(A) 6 **(B)** 12 **(C)** 14 **(D)** 16 **(E)** 22

6. Joe is playing a game with 6 levels, each with 8 stages. After the sixth stage of each of the first five levels, Joe may choose whether or not to skip the remaining stages in the level and start at the first stage of the next level. If Joe finished the last stage of the last level, how many possible total number of stages could Joe have played?

(A) 3 **(B)** 5 **(C)** 6 **(D)** 10 **(E)** 12

7. How many orderings of the six numbers 1, 1, 2, 2, 3, and 6 are there such that the sum of the first three numbers is twice the sum of the last three numbers?

(A) 9 **(B)** 18 **(C)** 27 **(D)** 36 **(E)** 72

8. Let a and b be positive integers. If a is divisible by 2 but not 3, and b is divisible by 3 but not 2, what is the greatest possible three-digit value of $a + b$?

(A) 995 **(B)** 996 **(C)** 997 **(D)** 998 **(E)** 999

9. Given a triangle, a line is drawn such that it intersects the triangle in exactly two points. Which of the following statements is necessarily true?

(A) The triangle is split into a smaller triangle and quadrilateral.

(B) The segment of the line in the triangle is shorter than every side of the triangle.

(C) At least two of the triangle's angle bisectors intersect the line in the triangle.

(D) The perimeters of each of the regions formed are less than that of the triangle.

(E) None of the above.

10. In a plane, eight rays emanate from a point P such that every two adjacent rays form an acute angle with measure $45°$. Next, a line segment with a finite length is drawn in the plane. If the line segment intersects exactly n of the rays, what is the sum of all possible values of n? (If the line segment passes through P, then $n = 8$.)

(A) 13 **(B)** 14 **(C)** 17 **(D)** 18 **(E)** 23

11. For certain real numbers x and y, the first 3 terms of a geometric progression are $x-2$, $2y$, and $x+2$ in that order, and the sum of these terms is 4. What is the fifth term?

(A) $\frac{64}{3}$ **(B)** $\frac{196}{9}$ **(C)** $\frac{512}{23}$ **(D)** $\frac{256}{11}$ **(E)** $\frac{128}{5}$

12. There exists a sequence $a_1, a_2, \ldots, a_6$ of positive integers such that for every term in the sequence, there exists another term in the sequence which is equal to that term. How many possible values of the product $a_1a_2\cdots a_6$ less than 1000 are there?

(A) 36 **(B)** 37 **(C)** 38 **(D)** 39 **(E)** 40

13. Let x be a positive real number such that

$$\frac{1}{x-\frac{1}{x}}=\sqrt{x^2+\frac{x^4}{4}}.$$

What is the value of x^4+2x^2?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

14. For all positive integers n, let the nth triangular number be defined as the sum of all positive integers between 1 and n, inclusive. For how many positive integers n between 1 and 100, inclusive, does the nth triangular number have the same last digit as the product of the first n triangular numbers when expressed in base ten?

(A) 11 **(B)** 12 **(C)** 20 **(D)** 21 **(E)** 22

15. Four people are in a tournament where every person duels each other person exactly once. Every duel ends in one person winning and the other losing (no ties). After the tournament, each person counts the number of wins they have and adds one, and then their numbers are multiplied. What is the minimum possible resulting product?

(A) 15 **(B)** 16 **(C)** 18 **(D)** 24 **(E)** 27

16. Ryan has 6 cards numbered 1 through 6. In each turn, Ryan chooses a card at random, and then throws out the card as well as all cards with a number that is *not* relatively prime to the number on the card. He continues until he has thrown out all his cards. What is the probability that Ryan takes four turns to throw out all his cards?

(A) $\frac{1}{2}$ **(B)** $\frac{7}{12}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{5}{6}$

17. Two non-intersecting circles have radius r and centers A and B. A line is drawn tangent to both circles, where the circles are on opposite sides of the line. Another line is drawn tangent to both circles, where the circles are on the same side of the line. Let the two lines intersect at P. If $AP=2$ and $BP=3$, then r^2 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

(A) 49 **(B)** 53 **(C)** 57 **(D)** 61 **(E)** 65

18. At a motel, there are 15 rooms in a row. A visitor may rent 1 room for 5 dollars, or 2 adjacent rooms for 4 dollars each. At most 1 visitor may rent a given room at a time, and no 2 visitors may rent rooms adjacent to each other. If the leftmost and rightmost rooms must be rented, what is the largest dollar amount that the motel can earn?

(A) 40 **(B)** 41 **(C)** 42 **(D)** 43 **(E)** 44

19. In the xy-plane are perpendicular lines $y = ax + d$ and $y = bx + c$, where a, b, c, and d are real numbers in a geometric progression in that order. If the two lines and the line $y = \frac{3}{2}x$ pass through a common point, what is the least possible value of $a+b+c+d$?

(A) $\frac{3}{2}$ **(B)** $\frac{51}{32}$ **(C)** $\frac{13}{8}$ **(D)** $\frac{111}{64}$ **(E)** $\frac{7}{4}$

20. Richard thinks of a positive integer n and writes the base ten representations of $n!$ and $(n+1)!$ on a board. He then erases the zeroes to the right of the last nonzero digit of each number (if any exist), resulting in two numbers a and b. If one of a and b is 4 times the other, what is the sum of all possible values of n less than 1000?

(A) 315 **(B)** 441 **(C)** 656 **(D)** 714 **(E)** 819

21. In rectangle $ABCD$, point A is reflected over diagonal $\overline{BD}$ to a point A'. If $A'B = A'C$ and $AA' = 6$, what is the area of rectangle $ABCD$?

(A) 18 **(B)** $8\sqrt{6}$ **(C)** $12\sqrt{3}$ **(D)** 21 **(E)** $9\sqrt{6}$

22. Janelle sees a right triangle with all integer side lengths and computes the sum of the squares of the lengths. She then writes down her sum in base-2 and base-3. Janelle realizes her two expressions have the same second-to-last digit but a different last digit. What is the sum of all possible remainders when Janelle's sum is divided by 36?

(A) 20 **(B)** 28 **(C)** 34 **(D)** 42 **(E)** 48

23. There are 15 people in a room, where everyone shakes hands with a positive number of other people in the room exactly once. If exactly 6 people shook 1 hand, exactly 5 people shook between 2 and 4 hands, inclusive, exactly 1 person shook 8 hands, and exactly 1 person shook 14 hands, what is the least possible total number of handshakes?

(A) 25 **(B)** 26 **(C)** 27 **(D)** 28 **(E)** 29

24. In right $\triangle ABC$ with $BC = 7$ and a right angle at A, let the midpoint of side $\overline{AB}$ be D. Suppose that there exists a point E on the circumference of the circumcircle of $\triangle ABC$ such that $\triangle CDE$ is equilateral. What is the side length of $\triangle CDE$?

(A) $\sqrt{21}$ **(B)** $2\sqrt{7}$ **(C)** $\frac{7\sqrt{21}}{6}$ **(D)** $\frac{7\sqrt{10}}{4}$ **(E)** $3\sqrt{6}$

25. Let x and y be distinct real numbers chosen at random from the interval $[-100, 100]$, excluding 0. What is the probability that

$$\left\lfloor \frac{|x|}{|y|} \right\rfloor \geq \left\lfloor \frac{|x+y|}{|x-y|} \right\rfloor,$$

where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to a real number r?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{9}$ **(C)** $\frac{9}{16}$ **(D)** $\frac{7}{12}$ **(E)** $\frac{5}{8}$

2021 DMC 10B

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Questions and complaints about problems and solutions for this exam should be sent by private message to:

DeToasty3.

The 2022 DIME may or may not be held. It would be a 15-question, 3-hour, integer-answer exam if it was to be held. You may or may not be invited to participate because this contest may or may not exist. *A complete listing of our previous publications may be found at our web site:*

`https://detoasty3.github.io/dmc.html`

****Try Administering This Exam On An Earlier Date. Oh Wait, You Can't.****

1. All the information needed to administer this exam is not contained in the non-existent DMC 10 Teacher's Manual.
2. YOU must not verify on the non-existent DMC 10 COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.
3. Send **DeToasty3**, **nikenissan**, **pog**, and **vsamc** a private message submitting your answers to the DMC 10. AoPS is the only way to submit your answers.
4. The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

The **2021 De Mathematics Competitions**
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