2021 GAMC Solutions

Geometry AMC Series

July 17, 2021, to August 21, 2021

AOPSmathematics, ApraTrip, depsilon0, DeToasty3, & Radio2

Credit goes to Online Test Seasonal Series (OTSS) for the booklet template.

Answer Key:

1. (B)	2. (B)	3. (C)	4. (A)	5. (C)
6. (B)	7. (C)	8. (B)	9. (E)	10. (E)
11. (D)	12. (B)	13. (D)	14. (A)	15. (E)
16. (D)	17. (B)	18. (E)	19. (C)	20. (D)
21. (C)	22. (B)	23. (D)	24. (E)	25. (C)

Solutions:

1. What is the area of the largest circle which can fit entirely within the interior of a semicircle with diameter 24?

   $\textbf{(A)}\ 24\pi \qquad \textbf{(B)}\ 36\pi \qquad \textbf{(C)}\ 48\pi \qquad \textbf{(D)}\ 72\pi \qquad \textbf{(E)}\ 144\pi$

   Answer (B): We see that the largest circle has a diameter perpendicular to the diameter of the semicircle at its midpoint. The radius of the semicircle is $\frac{24}{2} = 12$, and so the radius of the circle is $\frac{12}{2} = 6$. Thus, our answer is $\pi(6)^2 = \boxed{\textbf{(B)}\ 36\pi}$. ■

2. Line ℓ intersects circle ω at points A and B such that $AB = 8$. If the radius of ω is 8, what is the measure of the minor arc $\widehat{AB}$ of ω? (Recall that the measure of a minor arc is the measure of $\angle AOB \leq 180^\circ$, where O is the center of the circle.)

   $\textbf{(A)}\ 30^\circ \qquad \textbf{(B)}\ 60^\circ \qquad \textbf{(C)}\ 90^\circ \qquad \textbf{(D)}\ 120^\circ \qquad \textbf{(E)}\ 180^\circ$

   Answer (B): If O is the center of ω, then since $AB = BO = AO$, we have that $\triangle OAB$ is equilateral. Thus, we have that $\angle AOB = \boxed{\textbf{(B)}\ 60^\circ}$. ■

3. Two circles ω_1 and ω_2 have the same center and radii r_1 and r_2, respectively. Suppose that the area of the region between the circles is 28π, and $\frac{r_2}{r_1} = \frac{4}{3}$. What is r_1?

   (A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 12

   Answer (C): The area of the region between the circles is $\pi(r_2^2 - r_1^2)$. Since $\frac{r_2}{r_1} = \frac{4}{3}$, we have that

$$\pi\left(\left(\frac{4}{3}r_1\right)^2 - r_1^2\right) = 28\pi.$$

   Solving the equation (or guessing) gives $r_1 = \boxed{\textbf{(C) } 6}$. ■

4. Rectangle $ABCD$ has $AB = 10$ and $AD = 24$. Point P is inside $ABCD$ so that the areas of $\triangle PAB$, $\triangle PBC$, $\triangle PCD$, and $\triangle PAD$ are all equal. What is the length AP?

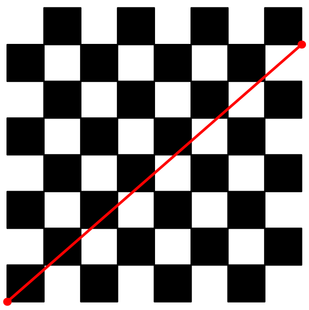
   (A) 13 **(B)** 15 **(C)** 20 **(D)** 24 **(E)** 26

   Answer (A): Since $[\triangle PAB] = [\triangle PCD]$, the height from P to AB and the height from P to CD must be equal. Similarly, since $[\triangle PBC] = [\triangle PAD]$, the height from P to BC and from P to AD must be equal. Thus P must be at the center of the square, meaning that $PA = \frac{1}{2}AD = \frac{1}{2}\sqrt{10^2 + 24^2} = \boxed{\textbf{(A) } 13}$. ■

5. Consider an 8×8 checkerboard with side length 8, in which each unit square is either black or white, and no two unit squares sharing an edge are the same color. What is the greatest distance between a point in a black square and a point in a white square?

   (A) $7\sqrt{2}$ **(B)** 10 **(C)** $\sqrt{113}$ **(D)** $8\sqrt{2}$ **(E)** $\sqrt{130}$

   Answer (C): The longest distance between any two points on the checkerboard is $8\sqrt{2}$, achieved when the two points are on opposite corners. However, the squares on opposite corners are the same color (say, black). So, we tweak the arrangement, moving one of the points to the appropriate corner of the adjacent white square, as shown below. From here, it follows that our answer is $\sqrt{7^2 + 8^2} = \boxed{\textbf{(C) } \sqrt{113}}$. ■



6. Points A, B, C, D on the circumference of a circle satisfy $AB = 15$, $AD = 20$, $CD = 24$, and $BD = 25$. What is the length BC?

   (A) $3\sqrt{3}$ **(B)** 7 **(C)** 8 **(D)** 10 **(E)** 12

   Answer (B): Note that $AB^2 + AD^2 = BC^2$, so $\angle BAD = 90^\circ$. Since $ABCD$ is cyclic, $\angle BCD = 90^\circ$. Hence $BC = \sqrt{BD^2 - BC^2} = \boxed{\textbf{(B) } 7}$. ■

7. Let $\overline{AB}$ be a diameter of circle ω with radius 1. Let P be a point on ω. What is the maximum possible value of $AP \cdot BP$?

   (A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** $2\sqrt{3}$ **(E)** 4

   Answer (C): Since $\triangle ABP$ is right, $AP \cdot BP = 2[\triangle ABP]$. Using AB as the base, it suffices to maximize the height from P to AB. This occurs when P is at the midpoint of arc $\widehat{AB}$, at which point $AP = BP = \sqrt{2}$, giving the answer of $\boxed{\textbf{(C) } 2}$. ■

8. In triangle ABC, let P be the foot of the perpendicular from B to side $\overline{AC}$. Let D be a point on the extension of segment $\overline{BP}$ past B. If $AB = \sqrt{55}$, $AD = \sqrt{61}$, and $CD = \sqrt{79}$, what is the length BC?

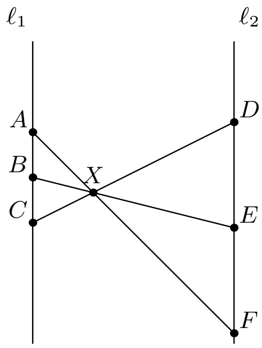
   (A) $\sqrt{37}$ **(B)** $\sqrt{73}$ **(C)** $\sqrt{85}$ **(D)** $\sqrt{116}$ **(E)** $\sqrt{195}$

   Answer (B): We observe that

$$\begin{aligned} AD^2 + BC^2 &= (AP^2 + DP^2) + (BP^2 + CP^2) \\ &= (AP^2 + BP^2) + (CP^2 + DP^2) \\ &= AB^2 + CD^2. \end{aligned}$$

   So $BC^2 = AB^2 + CD^2 - AD^2 = 73$, and $BC = \boxed{\textbf{(B) } \sqrt{73}}$. ■

9. Point X lies between parallel lines ℓ_1 and ℓ_2 such that it is $\frac{1}{5}$ of the way from ℓ_1 to ℓ_2. Segments $\overline{AF}$, $\overline{BE}$, and $\overline{CD}$ pass through X, with A, B, C on ℓ_1 and D, E, F on ℓ_2. If the areas of $\triangle ABX$ and $\triangle XDE$ are 2 and 48, respectively, what is $\frac{BC}{EF}$?



(A) $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{3}{8}$

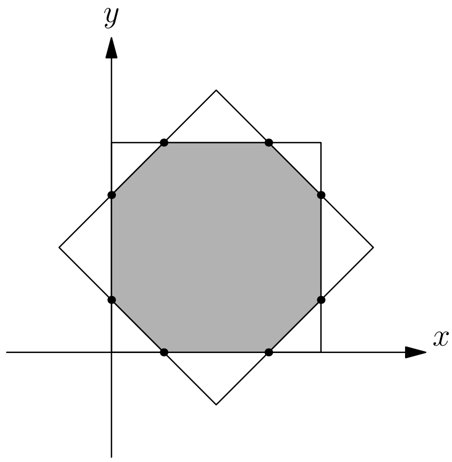
Answer (E): The parallel lines ℓ_1 and ℓ_2 induce many similar triangles. Since X is $\frac{1}{5}$ of the way from ℓ_1 to ℓ_2, the ratio of similarity between $\triangle ABX$ and $\triangle FEX$ is $\frac{1}{4}$. Thus, the area of $\triangle XEF$ is $2 \cdot 4^2 = 32$. Similarly, $\triangle BCX \sim \triangle EDX$ in ratio $1 : 4$, so the area of $\triangle BCX$ is $\frac{48}{4^2} = 3$. Since $\triangle ABX$ and $\triangle BCX$ have the same height from X, and the ratio of their areas is $2 : 3$, the ratio $AB : BC = 2 : 3$. Since $\frac{EF}{AB} = 4$ (through similar triangles), we have $\dfrac{BC}{EF} = \boxed{\textbf{(E)}\ \frac{3}{8}}$. ■

10. Two real numbers x and y are chosen uniformly at random from the interval $[0, 1]$. What is the probability that

$$\left|\left|x - \frac{1}{2}\right| + \left|y - \frac{1}{2}\right|\right| \leq \frac{3}{4}?$$

(A) $\frac{1}{2}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{4}{5}$ **(E)** $\frac{7}{8}$

Answer (E): We use geometric probability. First note that $\left|x - \frac{1}{2}\right| + \left|y - \frac{1}{2}\right| \geq 0$, so the given condition is equivalent to just $\left|x - \frac{1}{2}\right| + \left|y - \frac{1}{2}\right| \leq \frac{3}{4}$ (without the outer absolute values). The graph of this is a diamond bounded by the lines $\pm\left(x - \frac{1}{2}\right) \pm \left(y - \frac{1}{2}\right) = \frac{3}{4}$, by casework on the signs of $x - \frac{1}{2}$ and $y - \frac{1}{2}$. This diamond intersects the boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$ at points distance $\frac{1}{4}$ from the nearest vertex (for example, $\left(0, \frac{1}{4}\right)$ and $\left(\frac{1}{4}, 0\right)$). Therefore, the area of the desired region can be computed by subtracting off the areas of the 4 triangles, each of which have area $\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{32}$, giving a final answer of $1 - \dfrac{4}{32} = \boxed{\textbf{(E)}\ \frac{7}{8}}$. ■

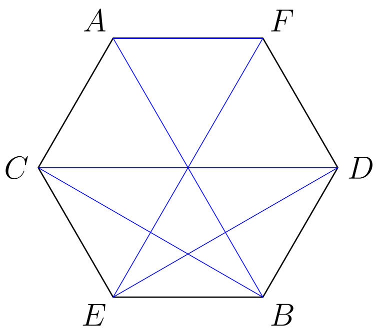


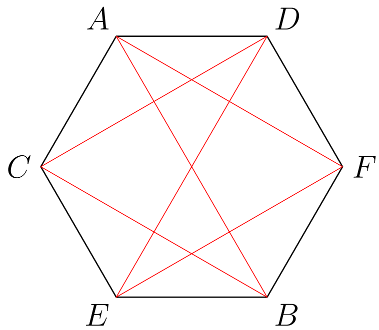
11. In a regular hexagon with side length 1, label the vertices with A, B, C, D, E, and F, not necessarily in that order. What is the maximum possible value of

$$AB + BC + CD + DE + EF + FA?$$

(A) 6 **(B)** $3 + 4\sqrt{3}$ **(C)** $7 + 2\sqrt{3}$ **(D)** $4 + 4\sqrt{3}$ **(E)** $6 + 3\sqrt{3}$

Answer (D): Each of the lengths $AB, BC, \ldots$ is a diagonal or a side of the hexagon, and therefore has length either 1, $\sqrt{3}$, or 2. Through some experimentation, it is not hard to see that there are only two arrangements that might be maximal, shown below. It is not hard to see that the one on the left gives $7 + 2\sqrt{3}$ while the one on the right gives $4 + 4\sqrt{3}$, and the latter is larger. Thus, the answer is $\boxed{\textbf{(D) } 4 + 4\sqrt{3}}$. ■





12. Bela is out for a walk on a field. She starts from her house, walks 1 mile straight, then turns 60° clockwise as viewed from above. After walking another mile in this

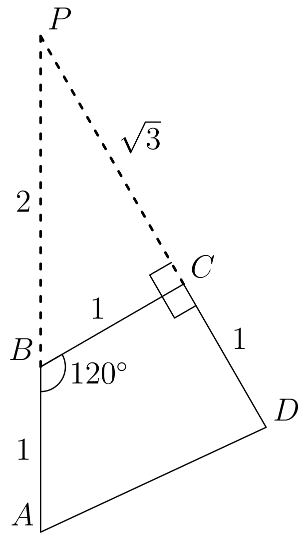
direction, she turns 90° clockwise as viewed from above. She then walks another mile in this direction. At her current position, how many miles away from home is Bela?

(A) $\sqrt{3}$ **(B)** $\sqrt{4-\sqrt{3}}$ **(C)** $\sqrt{3}+1$ **(D)** $\sqrt{10-\sqrt{3}}$ **(E)** $3+\sqrt{3}$

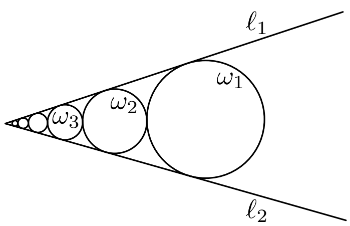
Answer (B): Let Bela's starting point be A, her position after walking 1 mile be B, her position after walking the next mile be C, and her position after walking the final mile be D. Extend AB past B and CD past C until they meet at P. Note that $\angle PCB = 180^\circ - \angle BCD = 90^\circ$, and $\angle PBC = 180^\circ - \angle ABC = 120^\circ$, so $\triangle PBC$ is a $30-60-90$ triangle. Since $BC = 1$, $PC = \sqrt{3}$ and $PB = 2$. Then, using Law of Cosines on $\triangle PAD$ gives

$$AD^2 = 3^2 + (\sqrt{3}+1)^2 - 2(3)(\sqrt{3}+1)\cos 30^\circ = 4-\sqrt{3},$$

and $AD = \boxed{\textbf{(B) } \sqrt{4-\sqrt{3}}}$. ■



13. An infinite sequence of circles $\omega_1, \omega_2, \ldots$ is such that ω_k is externally tangent to ω_{k+1} for $k = 1, 2, \ldots$, and the circles are all tangent to two lines ℓ_1 and ℓ_2. Suppose that the radius of ω_1 is 6 and the radius of ω_3 is 3. What is the total area of the circles?



(A) 36π **(B)** 48π **(C)** 60π **(D)** 72π **(E)** 84π

Answer (D): The main idea is that the figure is self-similar. Let ℓ_1 and ℓ_2 intersect at P, and let circle ω_i have radius r_i for each integer i. Then applying a homothety (dilation) at P with ratio $\frac{r_1}{r_2}$ maps ω_2 to ω_1. The image of ω_3 must be tangent to the image of ω_2 and the lines ℓ_1 and ℓ_2, but there is only one circle tangent to ω_1 and the lines ℓ_1 and ℓ_2 that is smaller than ω_1. Therefore, the image of ω_3 is ω_2, and similarly the image of ω_4 is ω_3, and so on. Since the radius of the image of ω_3 is $\frac{r_1}{r_2}(r_3)$, this gives $\frac{r_1}{r_2} = \frac{r_2}{r_3}$, and similarly for the rest of the circles. Thus the radii of the circles are in a geometric sequence. Since $r_1 = 6$ and $r_3 = 3$, the areas of the circles are in a geometric sequence with ratio 2 and first term 36π, giving a total area of $\boxed{\textbf{(D) } 72\pi}$. ■

14. In quadrilateral $ABCD$, triangles ABC and ACD are similar, with the vertices in that order. Let the diagonals $\overline{AC}$ and $\overline{BD}$ intersect at P. Suppose that $\frac{AB}{AD} = \frac{4}{9}$ and $\frac{AP}{CP} = \frac{1}{5}$. If the area of $\triangle PDC$ is 15, what is the area of quadrilateral $ABCD$?

    (A) 26 **(B)** $\dfrac{82}{3}$ **(C)** 28 **(D)** 30 **(E)** $\dfrac{94}{3}$

    Answer (A): Since $\triangle ABC \sim \triangle ACD$, $\angle BAP = \angle DAP$, so by the angle bisector theorem, $\frac{BP}{PD} = \frac{4}{9}$. Since $\triangle DPC$ and $\triangle BPC$ have the same height, $[\triangle BPC] = \frac{20}{3}$. Using the fact that $\frac{AP}{CP} = \frac{1}{5}$ and repeating a similar procedure gives that $[\triangle BAP] = \frac{4}{3}$ and $[\triangle DAP] = 3$. So the total area is $\frac{4}{3} + \frac{20}{3} + 15 + 3 = \boxed{\textbf{(A) } 26}$. ■

15. In isosceles triangle ABC, $AC = BC = 15$. Let D be the foot of the perpendicular from C to side $\overline{AB}$. Point E is on side $\overline{BC}$ such that if segment $\overline{AE}$ intersects segment $\overline{CD}$ at P, then $AP = 3\sqrt{10}$ and $EP = \sqrt{10}$. What is the area of $\triangle ABC$?

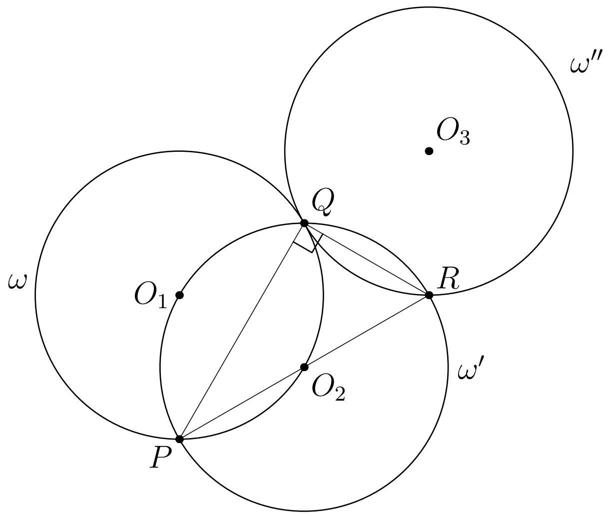
    (A) 30 **(B)** $\dfrac{21\sqrt{10}}{2}$ **(C)** $12\sqrt{10}$ **(D)** 60 **(E)** 90

    Answer (E): Since $\triangle ABC$ is isosceles, D is the midpoint of AB. We also know that $\frac{AP}{PE} = 3$. Using mass points, we can find that $\frac{CE}{EB} = \frac{1}{2}$, so $CE = 5$. (Alternatively, one could note that $[\triangle CPA] = 3[\triangle CPE]$, and $[\triangle CPA] = [\triangle CPB]$, so $[\triangle PEB] = 2[\triangle CPE]$.) Either way, we find $[\triangle ACE] = 3$ using Heron's. Since $\frac{CE}{CB} = \frac{1}{3}$, $[\triangle ABC] = 3[\triangle ACE]$, giving an answer of $\boxed{\textbf{(E) } 90}$. ■

16. Let ω be a circle with radius 4, and suppose that P is a point on the circumference of ω. Circle ω is rotated $60°$ clockwise about P to create a new circle ω'. Let $Q \neq P$ be the intersection of ω and ω'. Circle ω' is rotated $120°$ counterclockwise about Q to create a new circle ω'', which intersects ω' at R. What is the area of $\triangle PQR$?

(A) $\sqrt{3}$ **(B)** $2\sqrt{3}$ **(C)** $4\sqrt{3}$ **(D)** $8\sqrt{3}$ **(E)** $12\sqrt{3}$

Answer (D): Let the centers of ω, ω', and ω'' be O_1, O_2, and O_3 respectively. Note that $\angle O_1PO_2 = 60$, so $\triangle O_1PO_2$ is equilateral. By symmetry, $\triangle O_1QO_2$ is equilateral as well, so $\angle PQO_2 = 30$. Similarly, note that $\angle O_2QO_3 = 120$, and since $QO_2 = QO_3$, $\angle O_2QR = 60$. Thus $\angle PQR = \angle PQO_2 + \angle O_2QR = 90$. Dropping a perpendicular, we see that $PQ = 4\sqrt{3}$ since $\angle PO_2Q = 120$. Similarly, $QR = 4$ since $\angle QO_2R = 60$. Thus, the area of $\triangle PQR$ is $\frac{1}{2} \cdot 4\sqrt{3} \cdot 4 = \boxed{\textbf{(D) } 8\sqrt{3}}$. ■



17. Circles ω_1 and ω_2 intersect at A and B. Point P is on ω_1 and point Q is on ω_2 so that segment $\overline{PQ}$ intersects segment $\overline{AB}$ at X. Let segment $\overline{PQ}$ intersect ω_2 at $R \neq Q$ and ω_1 at $S \neq P$. If $PR = 5$, $RX = 1$, and $SX = 2$, what is the length QS?

(A) 5 **(B)** 10 **(C)** 12 **(D)** 13 **(E)** 17

Answer (B): By Power of a Point on X with respect to ω_1, we have $AX \cdot XB = PX \cdot XS$. By Power of a Point on X with respect to ω_2, $AX \cdot XB = QX \cdot XR$. So $PX \cdot XS = QX \cdot XR$, and $XQ = 2 \cdot 6 = 12$, so $QS = \boxed{\textbf{(B) } 10}$. ■

18. In octagon $ABCDEFGH$, the side lengths alternate between 7 and 2 (i.e. $AB = 7$, $BC = 2$, $CD = 7$, etc.), opposite sides are parallel (i.e. $\overline{AB} \parallel \overline{EF}$, $\overline{BC} \parallel \overline{FG}$, etc.), and the sum of any two adjacent interior angles is 270°. If the area of quadrilateral $ACEG$ is 60, then the area of quadrilateral $BDFH$ is $m + n\sqrt{p}$, where m, n, and p are integers and p is not divisible by the square of any prime. What is $m + n + p$?

(A) 47 **(B)** 60 **(C)** 61 **(D)** 72 **(E)** 75

Answer (E): We claim $ACEG$ is a square. Note that $\triangle ABC \cong \triangle CDE$ by SAS. So $AC = CE$, and similarly we can find that all the sides of $ACEG$ are the same. Also $\angle ACE = \angle BCD - \angle BCA - \angle DCE = \angle BCD - \angle BCA - \angle BAC = \angle BCD - (180^\circ - \angle ABC)$. Since the sum of adjacent angles is 270°, $\angle BCD + \angle ABC = 270^\circ$, so $\angle ACE = 90^\circ$. Similar reasoning for the other angles proves that all the angles are 90°, so $ACEG$ is a square. Similarly, $BDFH$ is a square.

So the area of $ACEG$ is $60 = AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cos\angle ABC = 53 - 28\cos\angle ABC$ by Law of Cosines. Solving gives $\cos\angle ABC = -\frac{1}{4}$. So $\cos\angle BCD = \cos(270^\circ - \angle ABC) = \sin\angle ABC = \frac{\sqrt{15}}{4}$. Thus the area of $BDFH$ is

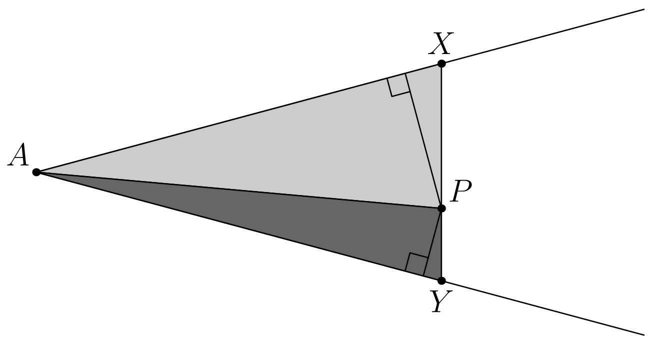
$$CE^2 = CD^2 + ED^2 - 2 \cdot CD \cdot ED \cos\angle BCD = 53 - 28\cos\angle BCD = 53 + 7\sqrt{15},$$

giving us the answer $53 + 7 + 15 = \boxed{\textbf{(E) } 75}$. ■

19. Rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are such that $\angle BAC = 30^\circ$. Let $\mathcal{R}$ be the region consisting of all points P that lie between the two rays such that the sum of the perpendicular distances from P to each of the two rays is at most 4. What is the area of $\mathcal{R}$?

(A) 8 **(B)** $\dfrac{4\pi\sqrt{3}+8\pi}{3}$ **(C)** 16 **(D)** $3\pi\sqrt{3}$ **(E)** $\dfrac{16\pi}{3}$

Answer (C): We claim that the locus of points P such that the sum of the perpendicular distances from P to the two rays is constant is a line perpendicular to the angle bisector of $\angle BAC$. To see this, let P be an arbitrary point and let X be on $\overrightarrow{AB}$ and Y be on $\overrightarrow{AC}$ such that XY passes through P and XY is perpendicular to the angle bisector of $\angle BAC$. Then $AX = AY$. Using $\frac{1}{2}bh$, the perpendicular distance from P to AB is $\frac{2}{AX}[\triangle AXP]$, and the perpendicular distance from P to AC is $\frac{2}{AY}[\triangle AYP] = \frac{2}{AX}[\triangle AYP]$. So the sum of the distances is $\frac{2}{AX}([\triangle AXP] + [\triangle AYP]) = \frac{2}{AX}[\triangle AXY]$, which is independent of the location of P along XY.



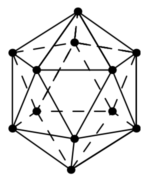
Since $[\triangle AXY] = \frac{1}{2}(AX)(AY)\sin 30^\circ = \frac{1}{4}AX^2$, our expression further simplifies to $\frac{1}{2}AX$. Since this is increasing as P becomes further away from A, the locus of points where the sum of the perpendicular distances is at most 4 is simply an isosceles triangle, bounded on the right by the locus of points where the perpendicular distance is exactly 4. If this line has endpoints D and E, then $AD = AE = 8$, since by the above reasoning the sum of the perpendicular distances is $\frac{1}{2}AE$. So the area of the triangle is $\frac{1}{2}(8^2)\sin 30° = \boxed{\textbf{(C) } 16}$. ■

20. In the coordinate plane, circle ω passes through the origin O. For some positive real number a, let ω intersect the line $y = ax$ at a point A and the line $y = -\frac{1}{a}x$ at a point B. Let the tangent lines to ω at A and O intersect at a point P, and let the tangent lines to ω at B and O intersect at a point Q. If a is chosen such that the coordinates of P are $(6, -2)$ and the coordinates of Q are $(-3, 1)$, what is the radius of ω?

(A) $\sqrt{5}$ **(B)** $\sqrt{10}$ **(C)** 4 **(D)** $2\sqrt{5}$ **(E)** $2\sqrt{10}$

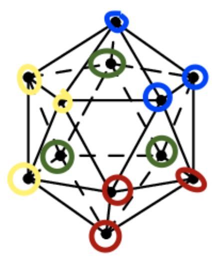
Answer (D): Let T be the center of the circle. Note that AB passes through T since the lines $y = ax$ and $y = -\frac{1}{a}x$ are perpendicular. Since OQ and BQ are tangent lines, QT bisects $\angle BTO$. Similarly PT bisects $\angle ATO$. So $\angle PTQ = \frac{1}{2}\angle BTA = 90°$. Thus OT is an altitude in right triangle QPT, and by similar triangles $OT = \sqrt{OP \cdot OQ}$. But we can compute OP and OQ, which gives $\sqrt{4\sqrt{5}\sqrt{5}} = \boxed{\textbf{(D) } 2\sqrt{5}}$. ■

21. A *regular icosahedron* is a 20-faced solid where each face is an equilateral triangle and five triangles meet at every vertex. John picks n faces of a regular icosahedron so that none of the faces meet, even at a vertex. What is the largest possible value of n?



(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Answer (C): There are 12 vertices total, and since each triangle has 3 vertices, and no two triangles can share a vertex, there can be a maximum of $\boxed{\textbf{(C) } 4}$ triangles. Such a construction is shown below (credit to vvluo). ■



22. Let $z_1 = 7 + 8i$ and $z_2 = 1 - 4i$ be two complex numbers. Suppose that z_3 is another complex number such that

$$|z_3 - z_1|^2 + |z_2 - z_3|^2 = 180.$$

What is the smallest possible value of $|z_3|$?

(A) 2 **(B)** $\sqrt{5}$ **(C)** $\sqrt{41} - 4$ **(D)** $\sqrt{29} - 2$ **(E)** $2\sqrt{5}$

Answer (B): We note that $|z_1 - z_2|^2 = 180$. By the Pythagorean Theorem, the condition is equivalent to saying that the triangle formed by z_1, z_2, and z_3 must be right, with the endpoints of the hypotenuse being z_1 and z_2. So z_3 lies on a circle with diameter with endpoints z_1 and z_2.

Let P be the center of this circle, let r be the radius of the circle, and let O be the origin. Then by triangle inequality on the triangle formed by P, O, and z_3, we have that $|z_3| \geq r - OP$. Computing $r = \frac{|z_1 - z_2|}{2} = 3\sqrt{5}$ and $OP = 2\sqrt{5}$ gives that $|z_3| \geq \boxed{\textbf{(B) } \sqrt{5}}$. ■

23. In triangle ABC with $AC = 5$, $BC = 7$, and $AB = 8$, let I be the center of the inscribed circle of $\triangle ABC$. Let points P and Q lie on sides $\overline{AB}$ and $\overline{AC}$, respectively, such that $\angle PIQ = 120°$. If $\frac{AQ}{CQ} = \frac{\sqrt{21}}{3}$, what is the area of quadrilateral $APIQ$?

(A) $\dfrac{3\sqrt{3}}{2}$ **(B)** $\dfrac{5\sqrt{3}}{2}$ **(C)** 5 **(D)** $3\sqrt{3}$ **(E)** $2\sqrt{7}$

Answer (D): The original problem was as follows: "In $\triangle ABC$, $AC = 5$, $BC = 7$, and $AB = 8$. Let I be the center of the inscribed circle of $\triangle ABC$. Suppose that regular hexagon $IJKLMN$ has side length 10, and line IJ intersects segment $\overline{AC}$ at P and line IN intersects segment $\overline{AB}$ at Q. If $\frac{CP}{PA} = \sqrt{\frac{3}{7}}$, what is the area of quadrilateral $APIQ$?" Due to laziness, we will be answering this version of the problem, which will give the same answer anyways.

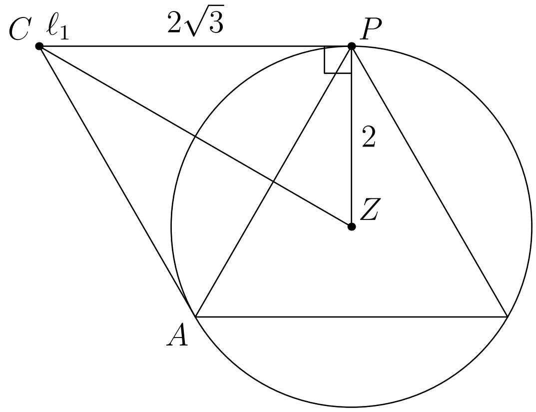
Let the foot of the perpendicular from I to AC be X and the foot of the perpendicular from I to AB be Y. Note that $\angle A = 60°$ (by Law of Cosines) and therefore $\angle XIY = 120°$, since $\angle AXI = 90° = \angle AYI$. In addition, since $IJKLMN$ is regular, $\angle JIN = 120°$. So, $\angle PIX = \angle YIQ$. Since $IX = IY$, this implies $\triangle IXP \cong \triangle IYQ$, and therefore the areas of the two triangles are the same. So the area of quadrilateral $APIQ$ is the same as the area of quadrilateral $AIXY$. It is easy to compute that the inradius of $\triangle ABC$ is $\sqrt{3}$, so $IX = IY = \sqrt{3}$, and since $\triangle AXI$ is $30 - 60 - 90$, $AX = 3$. So the area of quadrilateral $AIXY$ is $\boxed{\textbf{(D) } 3\sqrt{3}}$. ■

24. A right circular cone with base ω and apex P is inscribed in a sphere so that both P and the circumference of ω lie on the surface of the sphere. Points A and B are on ω so that if O is the center of ω, then $\angle AOB = 90°$. Suppose that point Q is such that lines AQ, BQ, and PQ are all tangent to the sphere. If the radius of ω is $\sqrt{3}$ and the radius of the sphere is 2, what is the distance from Q to the center of the sphere?

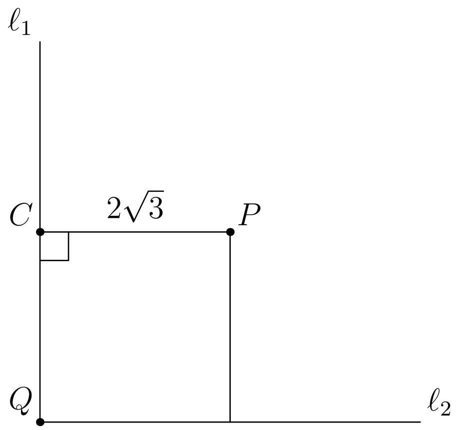
(A) $2\sqrt{3}$ **(B)** $2\sqrt{5}$ **(C)** $2\sqrt{6}$ **(D)** $3\sqrt{3}$ **(E)** $2\sqrt{7}$

Answer (E): Note that the locus of points Q such that PQ is tangent to the sphere is a plane tangent to the sphere at P, and similarly for A and B. Let the plane tangent to the sphere at P and the plane tangent to the sphere at A intersect in line ℓ_1, and let the plane tangent to the sphere at P and the plane tangent to the sphere at B intersect in line ℓ_2. Point Q then lies at the intersection ℓ_1 and ℓ_2.

We take a cross section of the diagram passing through P, A, and the center of the sphere Z. The plane tangent to P and the plane tangent to A then appear as lines, intersecting at a point C which is the projection of line ℓ_1. The cone appears as an isosceles triangle. Since the base of the triangle has length $2\sqrt{3}$ (twice the radius of ω), and the radius of the sphere is 2, this triangle is equilateral. So $\triangle CPZ$ is $30-60-90$, and $PZ = 2$, $PC = 2\sqrt{3}$.



Now we examine the plane tangent at P, where the projection of Z onto the plane becomes point P. The intersection of lines ℓ_1 and ℓ_2 can be clearly seen, and this is the point Q. Since $\angle AOB = 90$, $\ell_1 \perp \ell_2$, and the distance from P to ℓ_1 is the same as the distance from P to ℓ_2, namely $PC = 2\sqrt{3}$.



Since PZ, PC, and CQ are mutually orthogonal, the distance from Q to Z is just $\sqrt{PZ^2 + PC^2 + PQ^2} = \boxed{\textbf{(E) } 2\sqrt{7}}$. ■

25. Let triangle ABC be a right triangle with $\angle ACB = 90°$. Suppose that $PQRS$ is a square such that P is on side $\overline{AC}$, Q is on side $\overline{BC}$, and R is on side $\overline{AB}$. If $AC = 5$ and $BC = 7$, what is the smallest possible side length of the square?

(A) $\dfrac{35}{17}$ **(B)** $\dfrac{35\sqrt{193}}{193}$ **(C)** $\dfrac{35}{13}$ **(D)** $\dfrac{35\sqrt{2}}{17}$ **(E)** 7

Answer (C): Put $\triangle ABC$ on the coordinate plane with C at the origin, and AC along the positive y-axis and BC along the positive x-axis. Let the y-coordinate of P be b and let the x-coordinate of Q be a. If the foot of the perpendicular from R to BC is X, then note $\triangle PQC \cong \triangle QRX$. So R is at $(a+b, a)$. Since $\triangle RXB \sim \triangle ACB$, this means $XB = \frac{7}{5}a$. Thus $BC = a + b + \frac{7}{5}a = 7$.

Simplifying this gives $12a + 5b = 35$. We now want to minimize $\sqrt{a^2 + b^2}$. Note that the graph of $12a + 5b = 35$ is a line in the plane, so in order to minimize $\sqrt{a^2 + b^2}$ we wish to find the closest this line gets to the origin. If O denotes the origin in this new coordinate system, then this occurs at a point Z such that OZ is perpendicular to the line. Since the x and y intercepts of the line form a right triangle with O, we simply want to find the length of the altitude to the hypotenuse in this right triangle. This is not hard, and gives $\boxed{\textbf{(C) } \dfrac{35}{13}}$. ■