De Mathematics Competitions

2nd Annual

DMC 10 D

Thursday, October 14, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the DMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than writing utensils, blank scratch paper, rulers, compasses, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your non-existent proctor will not ask you to record certain information on the answer form.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have already taken the test in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

Students who score well on this DMC 10 may or may not be invited to the 2022 DIME (De Invitational Mathematics Examination). More details about the DIME and other information are on the back page of this test booklet.

1. What is the value of
$$2^0 - (21 - 20) + 2^1?$$

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

2. What is the smallest positive integer which can be expressed as the sum of a positive perfect square and a distinct positive perfect cube?

   (A) 2 **(B)** 3 **(C)** 5 **(D)** 9 **(E)** 10

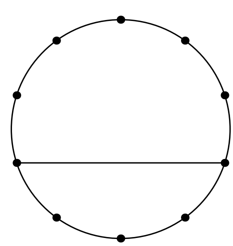
3. Squares $\mathcal{P}$ and $\mathcal{Q}$ are such that half the area of $\mathcal{Q}$ lies within $\mathcal{P}$. If the areas of $\mathcal{P}$ and $\mathcal{Q}$ are 8 and 2, respectively, what is the area of the region inside $\mathcal{P}$ and outside $\mathcal{Q}$?

   (A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

4. A positive integer n is divisible by 5 but not 4. For which of the following values will adding it to n never result in a sum that is divisible by 20?

   (A) 45 **(B)** 50 **(C)** 55 **(D)** 60 **(E)** 65

5. Ten points are equally spaced on the circumference of a circle, where two of the points are connected by a line segment, as shown below. Turner wants to choose two of the eight other points and draw a line segment connecting them so that this line segment does not intersect the other line segment. In how many ways can Turner do this?



   (A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

6. One of the side lengths of an obtuse isosceles triangle with integer side lengths is 2. What is the perimeter of this triangle?

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

7. Let a be a positive integer. A geometric sequence b, c, d in that order satisfies
$$ab = 15, \quad bd = 16, \quad \text{and} \quad ac = 120.$$
What is the value of $a + d$?

(A) 56 **(B)** 58 **(C)** 60 **(D)** 62 **(E)** 64

8. Let n be the number of factors of 3 that the product
$$(1 + 2 + 3)(4 + 5 + 6)(7 + 8 + 9) \cdots (94 + 95 + 96)(97 + 98 + 99)$$
contains. What is the sum of the digits of n?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

9. A rectangle $ABCD$ has $AB = 8$ and $BC = 4$. Points P and Q lie on sides $\overline{AB}$ and $\overline{BC}$, respectively, such that $AP = CQ$ and the area of $\triangle BPQ$ is 6. What is PQ^2?

(A) 32 **(B)** 34 **(C)** 36 **(D)** 38 **(E)** 40

10. There exist 4-digit numbers such that the 4-digit number is divisible by 9, the first 3 digits in order form a 3-digit number divisible by 9, and the last 3 digits in order form a 3-digit number divisible by 9. How many such 4-digit numbers exist?

(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

11. In the xy-plane, the distance between the points (r, r) and $(13, 7)$ is equal to d, where r is a real number satisfying the equation $r^2 - 20r + 21 = 0$. What is d^2?

(A) 78 **(B)** 106 **(C)** 150 **(D)** 176 **(E)** 218

12. Sheldon has some shirts, pairs of pants, and socks. An outfit consists of one shirt, one pair of pants, and two socks. Sheldon can currently wear 60 possible different outfits, but if he were to get either four more shirts or four more socks, then Sheldon could wear 180 possible different outfits. How many pairs of pants does Sheldon have? (The socks are distinguishable, and the order in which he wears the socks does not matter.)

(A) 2 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 15

13. Trixie writes the number 2021 on a blackboard. She then repeatedly erases and writes a new number on the blackboard, where if the current number on the blackboard is odd, she will erase the number and write $8n$ on the blackboard, and if the current number on the blackboard is even, she will erase the number and write $n + 2019$ on the blackboard. Eventually, Trixie will have written 2020 numbers on the board (including the initial 2021). What is the remainder when her 2020th number is divided by 9?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 7

14. In the xy-plane, the point $(24, 7)$ is reflected over the line $y = ax$ and then shifted up b units to the point $(20, 21)$, where a and b are positive real numbers. What is $a \cdot b$?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 12

15. Daniel repeatedly rolls two fair six-sided dice. After every roll, Daniel writes the sum of the numbers on the two dice. Daniel stops once he writes the number 11. What is the probability that Daniel writes the number 5 at least once before stopping?

(A) $\frac{1}{3}$ **(B)** $\frac{5}{12}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

16. A pyramid $ABCDE$ has rectangular base $ABCD$ with $AB = 8$ and $BC = 6$ and apex E located 8 units above $ABCD$. A plane parallel to $\triangle ACE$ hits segments $\overline{AB}$, $\overline{BC}$, and $\overline{BE}$ at F, G, and H, respectively. If $FG = \frac{15}{2}$, what is the volume of $DFGH$?

(A) 27 **(B)** 36 **(C)** 45 **(D)** 48 **(E)** 60

17. A positive divisor of 2021^{14} is chosen at random. What is the probability that when it is divided by 7, the remainder is equal to 1?

(A) $\frac{1}{9}$ **(B)** $\frac{2}{15}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{5}$ **(E)** $\frac{1}{3}$

18. An isosceles trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$, $AB = AD = BC = 3$, and $CD = 7$ is inscribed in a circle. A point E on the circle satisfies $\overline{AC} \perp \overline{BE}$. What is DE^2?

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

19. Let a, b, c, and d be positive integers. If

$$\frac{a!}{b!} + \frac{c!}{d!} = \frac{2}{5},$$

what is the largest possible value of $a + b + c + d$?

(A) 10 **(B)** 18 **(C)** 26 **(D)** 34 **(E)** 42

20. Let $\triangle ABC$ with $AB = 14$ be inscribed in a circle with center O. Let P be a point on side $\overline{BC}$, and let line AP intersect the circle at a point D, distinct from A. If $\triangle ABC$ is acute and $ABDO$ is a rhombus, what is the largest possible integer value of BP^2?

(A) 33 **(B)** 48 **(C)** 65 **(D)** 78 **(E)** 95

21. Consider the polynomial

$$P(x) = (x - a)(x - b)(x - c)(x - d),$$

where a, b, c, and d are fixed positive real numbers less than 1. What is the maximum possible number of distinct real numbers r that can satisfy $|P(r)| = 1$?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

22. Let S be the set of all positive integers which are relatively prime to 42. What is the smallest positive integer n such that the smallest number in S with exactly 2^n positive divisors is divisible by a perfect square which is greater than 1?

(A) 2 **(B)** 3 **(C)** 6 **(D)** 7 **(E)** 8

23. Joseph has the string $AABBABBB$. John is told the string has 8 letters, each an A or a B, but not the string itself or how many of each letter are in it. John is told that each move, Joseph will pick the ith letter of the existing string, that letter and every letter to its right will switch from A to B and vice versa, and John will learn what i is and how many of each letter are in the new string. If Joseph moves optimally and John reasons well, what is the fewest number of moves for John to know the string?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

24. In right $\triangle ABC$ with $AB = 8$, $AC = 6$, and a right angle at A, let M be the midpoint of side $\overline{BC}$, and let D be the reflection of C over line AM. Line segments $\overline{AC}$ and $\overline{DM}$ are extended to meet at a point E. What is the length of $\overline{CE}$?

(A) $\dfrac{32}{5}$ **(B)** $\dfrac{120}{17}$ **(C)** $\dfrac{50}{7}$ **(D)** $\dfrac{95}{13}$ **(E)** $\dfrac{84}{11}$

25. What is the largest positive integer k for which there exist positive integers a and b, where $a < b$, such that $a + b > 48$ and $1 < \sqrt[3]{a^3 + kb^2} - a < 2$?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

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For more information about the DMC and our other competitions, please visit `https://detoasty3.github.io/dmc.html`.

Questions and comments about this competition should be sent to:

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The problems and solutions for this DMC 10 were prepared by the DMC Editorial Board under the direction of: