Official Solutions

De Mathematics Competitions

1st Annual

DMC 9

De Mathematics Competition 9

Thursday, December 30, 2021

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

Questions and complaints about this competition should be sent by private message to

DeToasty3, **karate7800**, **pandabearcat**, and **pog**.

The problems and solutions for this DMC 9 were prepared by the DMC 9 Editorial Board under the direction of

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Answer Key:

1. (E)	2. (A)	3. (E)	4. (A)	5. (B)
6. (C)	7. (D)	8. (B)	9. (A)	10. (D)
11. (B)	12. (B)	13. (E)	14. (C)	15. (B)
16. (B)	17. (C)	18. (B)	19. (D)	20. (C)
21. (C)	22. (A)	23. (B)	24. (D)	25. (E)

Problem 1:

(DeToasty3) Maxim spent \$1.75 on candy, \$2.15 on soda, and \$4.25 on popcorn. If Maxim pays with only 1-dollar bills, what is the least possible amount of change he could get back?

(A) \$0.15 **(B)** \$0.40 **(C)** \$0.55 **(D)** \$0.60 **(E)** \$0.85

Answer (E):

Maxim will get the least possible amount of change back if he pays as little as possible. Since he must pay at least

$$\$1.75 + \$2.15 + \$4.25 = \$8.15$$

in 1-dollar bills, the least he can pay is \$9.00. Consequently, the least possible amount of change he can get back is $\$9.00 - \$8.15 = \boxed{\textbf{(E)}\ \$0.85}$.

Problem 2:

(pog) What is the value of the expression $\dfrac{4 \cdot 5 \cdot 6}{2(0+2+1)}$?

(A) 20 **(B)** 28 **(C)** 30 **(D)** 36 **(E)** 40

Answer (A):

We have that the denominator of the given expression is $2(0+2+1) = 2(3) = 6$, so the requested answer is equal to

$$\frac{4 \cdot 5 \cdot 6}{2(0+2+1)} = \frac{4 \cdot 5 \cdot 6}{6} = 4 \cdot 5 = \boxed{\textbf{(A)}\ 20}.$$

Problem 3:

(pog) Samuel is running forwards. At some point, he turns around and runs backwards. If Samuel ran a total of 60 equal steps and ended up 24 steps behind where he started running, for how many steps was Samuel running backwards?

(A) 18 **(B)** 36 **(C)** 38 **(D)** 40 **(E)** 42

Answer (E):

For every step Samuel runs backwards, he will end up 2 steps behind if he had ran forwards. Let b be how many steps Samuel was running backwards. Then $60-2b=-24$, so $-2b=-84$ and solving gives $b=\boxed{\textbf{(E) } 42}$. ■

Problem 4:

(pog) Karate and Judo are playing a game with 100 rounds. Each round, either Karate wins, Judo wins, or they tie. If Karate wins 3 rounds, and the number of rounds Judo wins is a perfect square, what is the least possible number of rounds where they tie?

(A) 16 **(B)** 19 **(C)** 22 **(D)** 25 **(E)** 28

Answer (A):

Since Bill won 3 rounds, Ben can win at most 97 rounds. To minimize the number of rounds where nobody wins, we want Ben to win as many rounds as possible. The largest perfect square that is less than 97 is 81, so Ben won 81 rounds and there are $97-81=\boxed{\textbf{(A) } 16}$ rounds where they tie. ■

Problem 5:

(treemath) Julia's Ice Cream Shop sells ice cream cones where each wafer cone and each scoop of ice cream costs the same amount for all of the orders. If an ice cream cone with four scoops costs the same as three ice cream cones with one scoop each, what is the ratio between the cost of a wafer cone and the cost of a scoop of ice cream?

(A) $1:3$ **(B)** $1:2$ **(C)** $2:3$ **(D)** $1:1$ **(E)** $2:1$

Answer (B):

Let the price of an ice cream cone be c, and let the price of a scoop of ice cream be s. We get that $c + 4s = 3c + 3s$, so $2c = s$. Thus, the requested ratio is

$$\frac{c}{s} = \frac{c}{2c} = \boxed{\textbf{(B) } 1:2}.$$

■

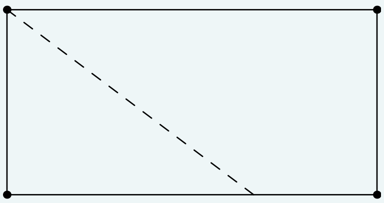
Problem 6:

(treemath) A line intersects a rectangle and divides it into two shapes. If one of the shapes has 4 sides, what are all of the numbers of sides the other shape could possibly have?

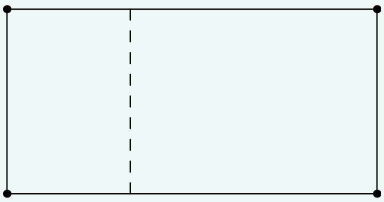
(A) 3 **(B)** 4 **(C)** 3 and 4 **(D)** 4 and 5 **(E)** $3, 4,$ and 5

Answer (C):

Note that the other shape must have at least three sides or else it would not be a shape. We see that the other shape can have three sides if the line intersects the rectangle at a vertex and another side (not at another vertex).



The other shape can also have four sides if the line intersects the rectangle at two opposite sides (not at any vertices).



Finally, the other shape cannot have more than four sides because the line can intersect the rectangle at most twice, creating at most two new sides, as well as the segment of the line inside the rectangle, which creates two new sides, one for each shape. This creates at most $4 + 2 + 2 = 8$ total sides between the two shapes, so hence the other shape has at most $8 - 4 = 4$ sides.

Thus, our answer is $\boxed{\textbf{(C) } 3 \text{ and } 4}$. ■

Problem 7:

(treemath) How many 5-digit numbers evenly divide 100,000?

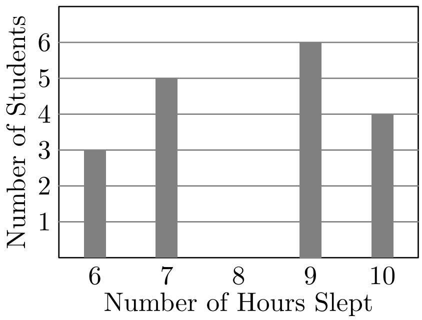
(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Answer (D):

Note that all factors of 100,000 are in the form $\frac{100{,}000}{\gamma}$, where γ is the corresponding factor in the factor pair $\left(\frac{100{,}000}{\gamma}, \gamma\right)$. Note that $\frac{100{,}000}{\gamma}$ is only a 5-digit number when $2 \leqslant \gamma \leqslant 10$. Checking values of γ, we get that $\gamma \in \{2, 4, 5, 8, 10\}$. Each value of γ corresponds with a 5-digit number that evenly divides 100,000, for an answer of $\boxed{\textbf{(D) } 5}$. ■

Problem 8:

(DeToasty3 & pandabearcat) In a survey, Richard asked the students of his third grade class how many hours of sleep each student got last night. The bar graph below shows the results of Richard's survey. However, the bar representing 8 hours has been mysteriously erased. If the median number of hours slept is 8.5, how many students slept for 8 hours?



(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer (B):

Since the median number of hours slept is 8.5 and some number of students slept for 9 hours, the two middle numbers of the survey are 8 and 9. Consequently, there must be the same number of students who slept for 9 or 10 hours as the number of students who slept for 6, 7, or 8 hours.

Let x be the number of students who slept for 8 hours. Then $3 + 5 + x = 6 + 4$. Solving gives $x = \boxed{\text{(B) } 2}$. ■

Problem 9:

(treemath) Will has 100 bronze coins. As many times as he wants, he can exchange 4 bronze coins for a silver coin or 6 silver coins for a gold coin. What is the smallest number of coins that Will can have after making some number of exchanges?

(A) 5 **(B)** 7 **(C)** 8 **(D)** 10 **(E)** 12

Answer (A):

If we don't exchange for as many gold coins as possible, we will be left with at least 6 other coins that we could've exchanged for a gold coin, so it is ideal to exchange for as many gold coins as possible.

Since a gold coin is worth $4 \cdot 6 = 24$ bronze coins and we have 100 bronze coins, we want to exchange for 4 gold coins, so thus we now have $100 - 24 \cdot 4 = 4$ bronze coins left. Now it is clearly optimal to exchange for a silver coin, so thus we have 4 gold coins and 1 silver coin for an answer of $\boxed{\text{(A) } 5}$ coins.

Remark: This method is called the greedy algorithm, and it worked here because of the specific denominations of the coins—in some cases, the greedy algorithm does not produce an optimal solution. For example, if we had 30 pennies and we could exchange for a dime or a quarter, the greedy algorithm would exchange for 1 quarter and 5 pennies, while the optimal strategy is to exchange for 3 dimes. ■

Problem 10:

(treemath & pog) What is the probability that a randomly chosen arrangement of the letters of the word $KARATE$ will have an A as the first letter or the last letter (or both)?

(A) $\frac{7}{15}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{5}{9}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{2}{3}$

Answer (D):

Note that

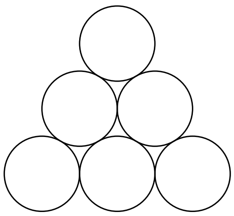
$$P(A \text{ is first letter or last letter}) + P(A \text{ is neither first letter nor last letter}) = 1,$$

so the desired probability is equal to $1 - P(A \text{ is neither first letter nor last letter})$. Hence, we can start by finding the probability that A is neither the first nor the last letter of a randomly chosen arrangement of the letters of the word $KARATE$ and subtract it from 1.

There is a $\frac{4}{6}$ probability that the first A is neither first nor last. Then, after this A is placed, there is a $\frac{3}{5}$ probability that the second A is neither first nor last. Thus, the complementary probability is $\frac{4}{6} \cdot \frac{3}{5} = \frac{2}{5}$. Hence, the requested probability is $1 - \frac{2}{5} = \boxed{\textbf{(D) } \frac{3}{5}}$. ■

Problem 11:

(HrishiP) What is the minimum number of distinct colors needed to color each of the six circles below such that no two circles with the same color are touching?



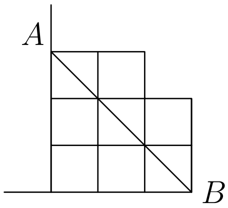
(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Answer (B):

First, color the top circle red. Next, we need the two circles below the red circle to be different colors, say, blue and green, from left to right. Finally, if we color the bottom three circles green, red, and blue, from left to right, then this coloring works. We have used three colors and we must use at least three colors in order to color the top three circles that are all touching, so the answer is $\boxed{\textbf{(B) } 3}$. ■

Problem 12:

(DeToasty3) Peter pushes n unit squares to a wall. Then, he puts three unit squares on top and pushes them to the wall. Finally, he puts two unit squares on top and pushes them to the wall. The figure below shows the resulting shape for $n = 3$. If points A and B represent the top-left and bottom-right vertices of the shape, respectively, what is the largest value of n such that segment $\overline{AB}$ does not go outside of the shape?



(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Answer (B):

Since $A = (0, 3)$ and $B = (n, 0)$, segment $\overline{AB}$ has slope $-\frac{3}{n}$. Hence, the farthest out segment $\overline{AB}$ will extend in the top row is $\frac{n}{3}$ and the farthest out segment $\overline{AB}$ will extend in the middle row is $\frac{2n}{3}$.

Consequently, segment $\overline{AB}$ will stay inside of the shape if $\frac{n}{3} \leqslant 2$ and $\frac{2n}{3} \leqslant 3$. Solving each inequality, we get that $n \leqslant 6$ and $n \leqslant 4.5$, so thus the largest possible value of n is $\boxed{\text{(B) } 4}$. ■

Problem 13:

(treemath) Anthony has 55 toys that he is packing into boxes. For every box, he wants the number of toys in that box to be equal to the number of boxes (including itself) that have the same number of toys as it does. How many boxes does Anthony need?

(A) 6 **(B)** 8 **(C)** 10 **(D)** 13 **(E)** 15

Answer (E):

We can consider every box with n toys as part of a collection of n boxes. For convenience, denote a set of boxes as $(a_1, a_2, \cdots)$, where each positive integer term of the set corresponds with a collection of n boxes (each with n toys), and $a_1 > a_2 > \cdots$.

Note that you need $(a_1)^2 + (a_2)^2 + \cdots$ toys to pack a set of boxes $(a_1, a_2, \cdots)$. Hence we want to express 55 as a sum of distinct perfect squares. Evidently $a_1 \leqslant 7$, since $7^2 < 55 < 8^2$.

Case 1: $a_1 = 7$

Then we want a sum of distinct perfect squares to be equal to $55 - 49 = 6$. However, this cannot happen, so $a_1 \leqslant 6$.

Case 2: $a_1 = 6$

Then we want a sum of distinct perfect squares to be equal to $55 - 36 = 19$. However, this also cannot happen, so $a_1 \leqslant 5$.

Case 3: $a_1 = 5$

Then we want a sum of distinct perfect squares to be equal to $55 - 25 = 30$. Since $4^2 + 3^2 + 2^2 + 1^2 = 30$, this works—and is the only possible solution, as we already proved that $a_1 \leqslant 5$, but if we hadn't used any of 5, 4, 3, 2, or 1, we would have less than 55 boxes.

Hence, $(a_1, a_2, a_3, a_4, a_5) = (5, 4, 3, 2, 1)$ and Anthony must use $5+4+3+2+1 = \boxed{\textbf{(E) } 15}$ boxes. ■

Problem 14:

(treemath) A karate outfit consists of either a beginner robe and a beginner belt or an advanced robe and an advanced belt. Alex has 5 robes and 13 belts in his collection, each of which are beginner or advanced. Alex can currently make 31 possible karate outfits. If he adds a beginner belt and an advanced belt, how many outfits can he then make?

(A) 33 **(B)** 35 **(C)** 36 **(D)** 38 **(E)** 39

Answer (C):

If Alex has r_b beginner robes, r_a advanced robes, b_b beginner belts, and b_a advanced robes, then by the Fundamental Counting Principle, he can make $r_b b_b + r_a b_a$ outfits. Hence

$$r_b b_b + r_a b_a = 31.$$

If we add a beginner belt and an advanced belt, we can then make

$$r_b(b_b + 1) + r_a(b_a + 1) = (r_b b_b + r_a b_a) + (r_b + r_a)$$

outfits. We are given that $r_b b_b + r_a b_a = 31$. Since Alex has 5 robes, $r_b + r_a = 5$, so hence Alex can then make $31 + 5 = \boxed{\textbf{(C) } 36}$ outfits. ■

Problem 15:

(treemath) Each of the vertices of a 10-sided regular polygon is labeled with a positive digit from 1 to 9, inclusive. If the sum of the labels of any four consecutive vertices is constant, what is the greatest possible sum of all of the distinct digits used?

(A) 9 **(B)** 17 **(C)** 24 **(D)** 30 **(E)** 45

Answer (B):

Suppose that the labels are a_1, a_2, …, a_{10} in that order. For five consecutive labels a_1, a_2, a_3, a_4, a_5, we have that

$$a_1 + a_2 + a_3 + a_4 = a_2 + a_3 + a_4 + a_5,$$

so $a_1 = a_5$.

Hence, for any five consecutive vertices, the first label is equal to the last label. Continuing this pattern, $a_1 = a_5 = a_9 = a_3 = a_7 = a_1 = \cdots$ (note that vertices a_9, a_{10}, a_1, a_2, a_3 are consecutive).

As well, $a_2 = a_6 = a_{10} = a_4 = a_8 = a_2 = \cdots$, so

$$a_1 = a_3 = a_5 = a_7 = a_9$$
$$a_2 = a_4 = a_6 = a_8 = a_{10}.$$

Therefore, there are at most two distinct digits used. The greatest possible sum occurs when the digits are as large as possible, so they are equal to 8 and 9 (in some order). Hence, our answer is $8 + 9 = \boxed{\textbf{(B) } 17}$. ■

Problem 16:

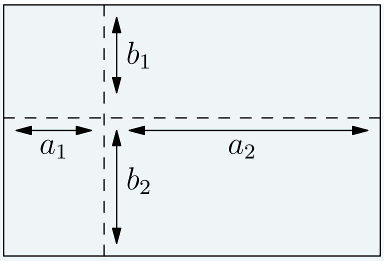
(treemath) A rectangle with a perimeter of 40 is split into four smaller rectangles by two lines that are parallel to the sides of the rectangle. If three of the rectangles have perimeters of 19, 21, and 25, what is the perimeter of the fourth rectangle?

(A) 9 **(B)** 15 **(C)** 17 **(D)** 23 **(E)** 27

Answer (B):

If the two lines are parallel to the same side of the rectangle, the rectangle will only be split into 3 smaller rectangles. Hence the two lines are parallel to different sides of the rectangle.

Let the sides of the smaller rectangles be a_1, a_2, b_1, and b_2, where a_1 and a_2 are their horizontal lengths and b_1 and b_2 are their vertical lengths.



Hence, the perimeters of the four smaller rectangles are then $2(a_1 + b_1)$, $2(b_1 + a_2)$, $2(a_1 + b_2)$, and $2(b_2 + a_2)$ in some order.

Note that the sum of the perimeters of the four smaller rectangles is equal to

$$2(a_1 + b_1 + b_1 + a_2 + a_1 + b_2 + b_2 + a_2) = 2(2a_1 + 2a_2 + 2b_1 + 2b_2).$$

Since $a_1 + a_2$ is equal to the base of the rectangle and $b_1 + b_2$ is equal to the height of the rectangle, the perimeter of the rectangle is equal to $2a_1 + 2a_2 + 2b_1 + 2b_2$.

Consequently, $2a_1 + 2a_2 + 2b_1 + 2b_2 = 40$ and thus the sum of the perimeters of the four smaller rectangles is equal to $2 \cdot 40 = 80$. Let the perimeter of the fourth rectangle be x. Then $19 + 21 + 25 + x = 80$, so $x = \boxed{\textbf{(B)}\ 15}$. ■

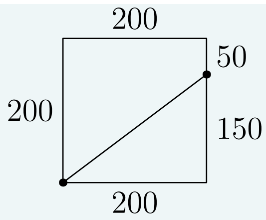
Problem 17:

(treemath) Starting from the same corner of a square city block, Jack and Jill both start walking around its 800-meter long perimeter in opposite directions. Jack walks at 1.4 meters per second, while Jill walks at 1.8 meters per second. What is the direct distance from their starting point to the point where they first meet again, in meters?

(A) 200 **(B)** $175\sqrt{2}$ **(C)** 250 **(D)** $200\sqrt{2}$ **(E)** 300

Answer (C):

Note that Jack and Jill will get $1.4 + 1.8 = 3.2$ meters closer every second. Thus $800 \div 3.2 = 250$ seconds will pass before they meet again. At that point, Jack will have walked $250 \cdot 1.4 = 350$ meters and Jill will have walked $250 \cdot 1.8 = 450$ meters. Each side of the rectangle is 200 meters long, so Jack will have walked one side and 150 meters of his second side, while Jill will have walked two sides and 50 meters of her third side.



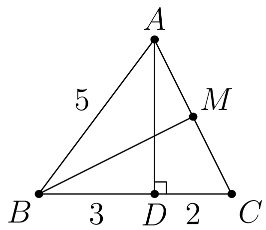
Consequently, by the Pythagorean theorem, the direct distance from their starting point to the point where they first meet again is

$$\sqrt{200^2+150^2} = 50\sqrt{3^2+4^2} = 50 \cdot 5 = \boxed{\textbf{(C) } 250}.$$

■

Problem 18:

(PhunsukhWangdu & DeToasty3) In $\triangle ABC$, let D be the foot of the altitude from A to side $\overline{BC}$, and let M be the midpoint of side $\overline{AC}$. If $BD = 3$, $CD = 2$, and $AB = 5$, what is the length of $\overline{BM}$?



(A) 4 **(B)** $2\sqrt{5}$ **(C)** $2\sqrt{6}$ **(D)** $3\sqrt{3}$ **(E)** $4\sqrt{2}$

Answer (B):

By the Pythagorean Theorem, we have that $AD = \sqrt{AB^2 - BD^2} = \sqrt{25-9} = 4$. Next, drop a perpendicular from M to side $\overline{BC}$ and call its foot point P. By similarity, we have that $MP = \frac{AD}{2} = 2$, and $BP = BD + DP = 3 + \frac{CD}{2} = 4$. Finally, by the Pythagorean Theorem on $\triangle BPM$, we have that

$$BM = \sqrt{MP^2 + BP^2} = \sqrt{4+16} = \boxed{\textbf{(B) } 2\sqrt{5}},$$

as desired.

OR

Note that because M is the midpoint of $\overline{AC}$ and $AB = BC$, we have that $\triangle AMB \cong$

$\triangle CMB$ by SSS congruence. Hence, $\angle AMB = \angle CMB$. Since $\overline{AC}$ is a line, $\angle AMB + \angle CMB = 180^\circ$. Thus, $\angle AMB = \angle CMB = 90^\circ$ and $\overline{BM} \perp \overline{AC}$.

Hence, $\overline{BM}$ is the altitude from B of $\triangle ABC$. Next, by the Pythagorean Theorem, we have that $AD = \sqrt{AB^2 - BD^2} = \sqrt{25-9} = 4$. By the Pythagorean Theorem again, we have that $AC = \sqrt{AD^2 + CD^2} = \sqrt{16+4} = 2\sqrt{5}$. By the base-area formula, we have that

$$\frac{AD \cdot BC}{2} = \frac{AC \cdot BM}{2}.$$

This equation becomes $4 \cdot 5 = 2\sqrt{5} \cdot BM$, giving $BM = \boxed{\textbf{(B) } 2\sqrt{5}}$. ∎

Problem 19:

(pog) If m and n are positive integers and $m^2 - n^2$ is equal to a prime number p, which of the following statements must always be true?

(A) $m+n$ is divisible by 3 **(B)** $p = m-n$ **(C)** $p-m-n$ is odd
(D) $p^2+m^2+n^2$ is not prime **(E)** $p-n^2$ is even

Answer (D):

First, note that $m > n$. By differences of squares, $m^2 - n^2 = p$ is equal to

$$(m+n)(m-n) = p,$$

and since $m > n$, we have that $m+n$ and $m-n$ are both positive integers. Seeing as p is prime, it will have no divisors other than 1 and itself. Since b is positive, $m+n$ must be the larger divisor, so thus $m+n = p$ and $m-n = 1$. Now, we look at the answer choices.

Since $m+n = (n+1)+n = 2b+1$, it can be any odd prime depending on the value of b. In this case, $m+n$ is only divisible by 3 when $(m,n) = (2,1)$, and thus **(A)** is not always true.

Since $m-n = 1$, which is not prime, it cannot be equal to p and thus **(B)** is not always true.

Note that $p-m-n = p-(m+n) = p-p = 0$, so $p-m-n$ is never odd and thus **(C)** is not always true.

Since $p-n^2$ is only even when n is odd and n can be odd or even, **(E)** is not always true.

Finally, seeing as $m = n + 1$, we get that

$$(m, n) = (\text{even}, \text{odd}) \quad \text{or} \quad (\text{odd}, \text{even}),$$

so $m^2 + n^2$ is always $(\text{even}) + (\text{odd})$ in some order. Consequently, $m^2 + n^2$ is always odd. Thus, $p^2 + m^2 + n^2$ is always even and greater than 2, so $\boxed{\textbf{(D)}\ p^2 + m^2 + n^2 \text{ is not prime}}$ is always true.

Remark: Alternatively, we can provide a counterexample for every other option, such as $(m, n, p) = (3, 2, 5)$. ■

Problem 20:

(pog) How many different real numbers x satisfy the equation

$$\left|5|x| - x^2\right| = 10?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 8

Answer (C):

For convenience, substitute $y = |x|$.

Hence, $\left|5y - x^2\right| = 10$. Note that $x^2 = |x|^2 = y^2$, so $\left|5y - x^2\right| = \left|5y - y^2\right| = 10$.

Consequently, either

$$5y - y^2 = 10 \quad \text{or} \quad 5y - y^2 = -10.$$

Case 1: $5y - y^2 = 10$

If $5y - y^2 = 10$, then $y^2 - 5y - 10 = 0$. This equation has a discriminant of

$$-5 - 4(1)(-10) = 35,$$

so in this case there are two real solutions for y. However, since the product of these two real solutions is -10, exactly one of the solutions for y here is negative. Since $y = |x|$, only the positive solution for y works, so thus there is only one solution for y in this case.

Case 2: $5y - y^2 = -10$

If $5y - y^2 = -10$, then $y^2 - 5y + 10 = 0$. This equation has a discriminant of

$$-5 - 4(1)(10) = -45,$$

so in this case there are no real solutions for y.

Thus, $|x|$ must be the positive root of $y^2 - 5y - 10$, so hence $x \in \{|x|, -|x|\}$ and the requested answer is $\boxed{\text{(C) } 2}$. ■

Problem 21:

(treemath) Ryan, Emily, and Anna play a game, where each pair of players plays one match. In each match, there is a one-third chance of a tie, and each player has a one-third chance of winning. If players earn two points for winning and one point for a tie, what is the probability that each player has the same number of points after the game?

(A) $\frac{1}{27}$ **(B)** $\frac{2}{27}$ **(C)** $\frac{1}{9}$ **(D)** $\frac{7}{27}$ **(E)** $\frac{1}{3}$

Answer (C):

Regardless of if the games are tied or won, each game will give out 2 points. Hence, players get a total of $2 \cdot 3 = 6$ points across the three games, and thus each player must have $\frac{6}{3} = 2$ points.

Each player plays 2 games. There are 2 ways to get 2 points across 2 games: either both games are tied, or one game is won and the other game is lost. Note that this implies all players must tie both of their games or all players must win exactly one game.

Case 1: All games are tied.

The probability that a given game is tied is $\frac{1}{3}$, so the probability that all three games are tied is $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$ probability.

Case 2: All players win exactly one game.

If all three games result in someone winning and someone losing, and all three winners are different, then each player will have the same number of points after the game. Suppose that Ryan beat Emily. Then, Emily must have beat Anna, and Anna must have beat Ryan. Otherwise, if Emily beat Ryan, then Ryan must have beat Anna, and Anna must have beat Emily. This gives us 2 ways, for a probability of $\frac{2}{27}$.

Thus, the total probability is $\dfrac{1}{27} + \dfrac{2}{27} = \boxed{\text{(C) } \dfrac{1}{9}}$. ■

Problem 22:

(pog) Derek thinks of a two-digit number. Dean then asks the following questions in order:

- "Is the number a multiple of 2?"
- "Is the number a multiple of 3?"
- "Is the number a multiple of 4?"

Derek answers "yes," "yes," and "no" to each of the questions, but Dean forgot which order Derek said his responses in. How many possible values of the number are there?

(A) 21 **(B)** 24 **(C)** 29 **(D)** 33 **(E)** 37

Answer (A):

Let Derek's number be n. We can split the possible values of n into cases, as follows.

Case 1: n is a multiple of 4

If n is a multiple of 4, then it is also a multiple of 2. Seeing as Derek said no to one of the questions, it cannot also be a multiple of 3. Thus, n can be any two-digit multiple of 4 that is not also a multiple of 3. The smallest two-digit multiple of 4 is $3 \cdot 4$ and the largest two-digit multiple of 4 is $24 \cdot 4$, so there are $24 - 3 + 1 = 22$ two-digit multiples of 4. Of these, $\{12, 24, 36, 48, 60, 72, 84, 96\}$ are also multiples of 3, which don't work, so there are $22 - 8 = 14$ possibilities for this case.

Case 2: n is not a multiple of 4

Seeing as Derek said yes to two of the questions, n must be a multiple of both 2 and 3, so n can be any two-digit multiple of 6 that is not also a multiple of 4. The smallest two-digit multiple of 6 is $2 \cdot 6$ and the largest two-digit multiple of 6 is $16 \cdot 6$, so there are $16 - 2 + 1 = 15$ two-digit multiples of 6. Of these, $\{12, 24, 36, 48, 60, 72, 84, 96\}$ are also multiples of 4, which don't work, so there are $15 - 8 = 7$ possibilities for this case.

There are no overlaps between the two groups as the first group cannot contain multiples of 3 and the second group can only contain multiples of 3, so our answer is $14 + 7 = \boxed{\textbf{(A)}\ 21}$. ■

Problem 23:

(treemath & pog) Define the operation $a \ominus b = a + b - \sqrt{4ab}$. If N is a two-digit number such that

$$(4 \ominus N) \ominus (81 \ominus 196) = 4,$$

what is the sum of the possible values of N?

(A) 73 **(B)** 106 **(C)** 113 **(D)** 130 **(E)** 145

Answer (B):

Note that $a+b-\sqrt{4ab}=a+b-2\sqrt{ab}$, which we notice is equal to $\left(\sqrt{a}-\sqrt{b}\right)^2$.

By the given equation, we have that

$$\begin{aligned}(4\ominus N)\ominus(81\ominus 196)&=4\\ \left(2-\sqrt{N}\right)^2\ominus\left(\sqrt{81}-\sqrt{196}\right)^2&=4\\ \left(2-\sqrt{N}\right)^2\ominus 25&=4\\ \left(\sqrt{\left(2-\sqrt{N}\right)^2}-\sqrt{25}\right)^2&=4\\ \sqrt{\left(2-\sqrt{N}\right)^2}-5&=\pm 2.\end{aligned}$$

Note that $\sqrt{\left(2-\sqrt{N}\right)^2}=\left|2-\sqrt{N}\right|$. Since N is a two-digit number, $\sqrt{N}>2$, so thus $2-\sqrt{N}$ is negative.

Hence, $\left|2-\sqrt{N}\right|=-(2-\sqrt{N})=\sqrt{N}-2$. Thus, $\sqrt{N}-2-5=\pm 2$, so either $\sqrt{N}=9$ or $\sqrt{N}=5$. Consequently, the possible values of N are 81 and 25, for an answer of $25+81=\boxed{\textbf{(B)}\ 106}$.

OR

Let $4\ominus N=\gamma$. Then $\gamma\ominus(81\ominus 196)=4$. By computation, we get $81\ominus 196=81+196-\sqrt{4\cdot 81\cdot 196}=277-2\cdot 9\cdot 14=25$, so $\gamma\ominus 25=4$.

Thus, $\gamma+25-\sqrt{4\cdot 25\cdot\gamma}=4$. Rearranging, we get $\gamma-10\sqrt{\gamma}+21=0$, which factors as $\left(\sqrt{\gamma}-3\right)\left(\sqrt{\gamma}-7\right)=0$.

Hence, $\sqrt{\gamma}=3$ or $\sqrt{\gamma}=7$, so either $\gamma=9$ or $\gamma=49$.

Case 1: $\gamma=9$

We get that $4+N-\sqrt{4\cdot 4\cdot N}=9$. Rearranging, we get $N-4\sqrt{N}-5=0$, which factors as $\left(\sqrt{N}+1\right)\left(\sqrt{N}-5\right)=0$. Since $\sqrt{N}$ must be positive, we get that $\sqrt{N}=5$, so if $\gamma=9$, then $N=25$.

Case 2: $\gamma=49$

We get that $4 + N - \sqrt{4 \cdot 4 \cdot N} = 49$. Rearranging, we get $N - 4\sqrt{N} - 45 = 0$, which factors as $\left(\sqrt{N} + 5\right)\left(\sqrt{N} - 9\right) = 0$. Since $\sqrt{N}$ must be positive, we get that $\sqrt{N} = 9$, so if $\gamma = 49$, then $N = 81$.

Consequently, the possible values of N are 25 and 81, for an answer of $81 + 25 = \boxed{\textbf{(B) } 106}$. ■

Problem 24:

(pog) Daniel's favorite positive integer A has B positive integer factors. If the product of A and B is equal to 13,500, what is the sum of the digits of the sum of A and B?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

Answer (D):

Note that $13500 = 2^2 \cdot 3^3 \cdot 5^3$, and thus $A \cdot B$ contains 2 powers of 2, 3 powers of 3, and 3 powers of 5.

Let $A = 2^x \cdot 3^y \cdot 5^z$, where $x \in \{0, 1, 2\}$, $y \in \{0, 1, 2, 3\}$, and $z \in \{0, 1, 2, 3\}$. Then $B = (x + 1)(y + 1)(z + 1)$.

If A is a multiple of 5, then since 5 is prime, $x + 1$, $y + 1$, or $z + 1$ would have to contain a multiple of 5, which is impossible as x, y, and z cannot be greater than 3, so B cannot be a multiple of 5.

However, $A \cdot B$ has exactly three powers of 5, so A must contain all three powers of 5, and thus $z = 3$.

Consequently, $B = (x + 1)(y + 1)(3 + 1)$, so B must contain two powers of 2. Since $A \cdot B$ has exactly two powers of 2, we get that A contains no powers of 2, so $x = 0$.

Hence, $A = 2^0 \cdot 3^y \cdot 5^3$, where $y \in \{0, 1, 2, 3\}$.

If y is odd, then $B = (0 + 1)(\text{odd} + 1)(3 + 1)$. However, then B would contain at least three powers of 2, and since $A \cdot B$ has exactly two powers of 2, we get that y cannot be odd.

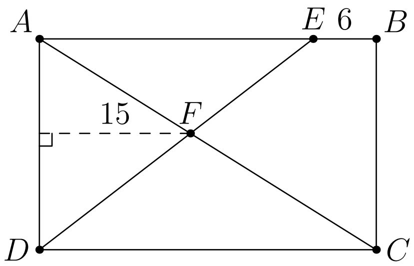
If $y = 0$, then A has no powers of 3 and $B = (0 + 1)(0 + 1)(3 + 1) = 4$ has no powers of 3 either. However, $A \cdot B$ must have exactly three powers of 3, so y cannot be 0.

If $y = 2$, then A has two powers of 3 and $B = (0 + 1)(2 + 1)(3 + 1) = 12$ has one power of 3, which works.

Therefore, $(A, B) = (1125, 12)$ and $A + B = 1125 + 12 = 1137$. Hence, our answer is $1 + 1 + 3 + 7 = \boxed{\textbf{(D) } 12}$. ■

Problem 25:

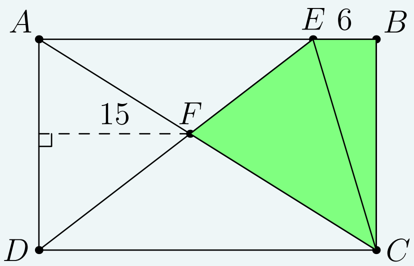
(stayhomedomath) Let $ABCD$ be a rectangle. Let E be on side $\overline{AB}$ such that $BE = 6$, and let F be the intersection of $\overline{AC}$ and $\overline{DE}$. If the distance from F to side $\overline{AD}$ is 15, and the absolute difference between the areas of $\triangle AEF$ and $\triangle CDF$ is 40, what is the area of $BCFE$?



$\textbf{(A)}$ 56 $\textbf{(B)}$ 88 $\textbf{(C)}$ 100 $\textbf{(D)}$ 124 $\textbf{(E)}$ 140

Answer (E):

We denote $[WXYZ]$ as the area of polygon $WXYZ$. Join EC.



As $\triangle ABC$ and $\triangle CDE$ has same base and same height, their areas must be same. Note that $[\triangle ABC] = [\triangle AEF] + [\triangle CEF] + [\triangle BCE]$ and $[\triangle CDE] = [\triangle CDF] + [\triangle CEF]$. Therefore,

$$\begin{aligned}
[\triangle ABC] &= [\triangle CDE] \\
[\triangle AEF] + [\triangle CEF] + [\triangle BCE] &= [\triangle CDF] + [\triangle CEF] \\
[\triangle AEF] + [\triangle BCE] &= [\triangle CDF] \\
[\triangle BCE] &= [\triangle CDF] - [\triangle AEF] \\
&= 40
\end{aligned}$$

So the area of $\triangle BCE$ is 40. As $AD = BC$, $\triangle ADF$ has same height as $\triangle BCE$. So the ratio of their areas is equal to the ratio of their heights. Therefore $[\triangle ADF] = 40 \cdot \frac{15}{6} = 100$. Next, we notice that $\triangle ACD$ and $\triangle CDE$ also has same base and same height. So

$$\begin{aligned}
[\triangle ACD] &= [\triangle CDE] \\
[\triangle ADF] + [\triangle CDF] &= [\triangle CEF] + [\triangle CDF] \\
[\triangle ADF] &= [\triangle CEF] \\
[\triangle CEF] &= 100
\end{aligned}$$

So $[BCFE] = [\triangle BCE] + [\triangle CEF] = 40 + 100 = \boxed{\textbf{(E)}\ 140}$.

OR

Let $CD = x$. Then, we have $AE = x - 6$. Let $AD = h$. Due to parallel lines AB and CD and transversals AC and DE, we have that $\triangle AFE \sim \triangle CFD$. Thus, if we draw a common altitude passing through F and intersecting lines AB and CD and P and Q, respectively, we have that $FP = \frac{x-6}{2x-6} \cdot h$ and $FQ = \frac{x}{2x-6} \cdot h$. Clearly, the area of $\triangle CFD$ is larger than the area of the area of $\triangle AFE$, so we have the equation

$$\left(\frac{1}{2} \cdot x \cdot \frac{x}{2x-6} \cdot h\right) - \left(\frac{1}{2} \cdot (x-6) \cdot \frac{x-6}{2x-6} \cdot h\right) = 40.$$

This equation rearranges to

$$\frac{x^2 - (x-6)^2}{2x-6} \cdot h = 80 \implies \frac{12x - 36}{2x-6} \cdot h = 80 \implies h = \frac{40}{3}.$$

Next, we have that the area of $\triangle BCE$ is equal to $\frac{1}{2} \cdot 6 \cdot h = 40$ by using the fact that $\angle EBC = 90^\circ$. Next, we see that the areas of $\triangle AFD$ and $\triangle EFC$ are equal. This is because the areas of $\triangle ADC$ and $\triangle ECD$ are equal by sharing common base $\overline{CD}$ and distance from $\overline{AE}$ to $\overline{CD}$, so subtracting the area of $\triangle DFC$ gives us the desired result. The area of $\triangle AFC$ is equal to $\frac{1}{2} \cdot 15 \cdot h = 100$, so the area of $\triangle EFC$ is also equal to 100. Thus, the area of $BCFE$ is equal to the sum of the areas of $\triangle BCE$ and $\triangle EFC$, which is $40 + 100 = \boxed{\textbf{(E)}\ 140}$. ■