De Mathematics Competitions

2nd Annual

DMC 10 F

Friday, January 21, 2022

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. For each question, only one answer choice is correct.
3. Mark your answer to each problem on the DMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, rulers, and erasers are allowed as aids. Prohibited materials include calculators, smartwatches, phones, computing devices, compasses, protractors, and graph paper. No problems on the competition will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the competition, your non-existent proctor will not ask you to record certain information on the answer form.
8. You will have 75 minutes to complete the competition. You can discuss only with people that have already taken the competition in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

Students who score well on this DMC 10 may or may not be invited to the 2022 DIME. More details about the DIME and other information are on the back page of this test booklet.

1. How many single-digit positive integers n are there such that $2n$ is a perfect square?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2. Squares $\mathcal{P}$ and $\mathcal{Q}$ are such that half the area of $\mathcal{Q}$ lies within $\mathcal{P}$. If the areas of $\mathcal{P}$ and $\mathcal{Q}$ are 8 and 2, respectively, what is the area of the region inside $\mathcal{P}$ and outside $\mathcal{Q}$?

   (A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

3. A dog has four legs, and a dug has three legs. Janelle has a whole number of dogs and dugs as pets, and she has no other pets. If there are 61 legs across all of Janelle's pets, what is the smallest possible number of dugs that Janelle could have?

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

4. A positive integer n is divisible by 5 but not 4. For which of the following values will adding it to n never result in a sum that is divisible by 20?

   (A) 45 **(B)** 50 **(C)** 55 **(D)** 60 **(E)** 65

5. Today is Toasty's birthday. It is given that the square of his age is 32 less than the number of months he is old. What is the sum of Toasty's possible ages?

   (A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 14

6. One of the side lengths of an obtuse isosceles triangle with integer side lengths is 2. What is the perimeter of this triangle?

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

7. For what values of k does the equation

$$k^2x + 2 = 4x + k$$

   have no real solutions x?

   (A) -2 **(B)** 0 **(C)** 2 **(D)** -2 and 0 **(E)** -2 and 2

8. Anthony, Daniel, and Richard have 17, 20, and 26 trading cards, respectively. Every minute, one of the three boys gives away two of his trading cards such that the other two boys get one trading card each. What is the shortest amount of time, in minutes, that it could take for the three boys to each have an equal number of trading cards?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

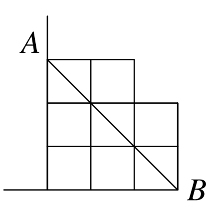
9. Alice and Bob are racing each other on a track. Each of their lanes are 400 meters in length. Normally, Alice and Bob run at constant rates of a and b meters per minute, respectively, but Alice's lane has a 180-meter sand region in the middle, in which she runs at three-quarters of her normal speed. If Alice and Bob take the same amount of time to run through their lanes without stopping, what is $\frac{a}{b}$?

   (A) 1.05 **(B)** 1.15 **(C)** 1.25 **(D)** 1.35 **(E)** 1.45

10. Let a and b be positive integers. If a is divisible by 2 but not 3, and b is divisible by 3 but not 2, what is the greatest possible three-digit value of $a + b$?

    (A) 995 **(B)** 996 **(C)** 997 **(D)** 998 **(E)** 999

11. Peter pushes n unit squares to a wall. Then, he puts three unit squares on top and pushes them to the wall. Finally, he puts two unit squares on top and pushes them to the wall. The figure below shows the resulting shape for $n = 3$. If points A and B represent the top-left and bottom-right vertices of the shape, respectively, what is the largest value of n such that segment $\overline{AB}$ does not go outside of the shape?



    (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

12. What is the largest integer n for which there exists an ordered triple (p, q, r) of distinct prime numbers such that $p^2(q^2 + r^2)$ is divisible by 2^n?

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

13. Let A and B be two distinct points on a plane. Let $\mathcal{S}$ denote the set of all circles on the plane with a finite area such that A and B are on the circumference of the circle. What is the region of all points not on the circumference of any of the circles in $\mathcal{S}$?

    (A) Every point on line AB excluding A and B

    (B) Every point on segment $\overline{AB}$ excluding A and B

    (C) Every point on line AB but not on segment $\overline{AB}$

    (D) The midpoint of segment $\overline{AB}$

    (E) None of the above

14. Alice goes cherry picking in a forest. For each tree Alice sees, she either picks one cherry or three cherries from the tree and puts them in her basket. Additionally, after every five trees Alice picks from, she finds an extra cherry on the ground and puts it in her basket. At the end, Alice has 45 cherries in her basket. If the smallest possible number of trees Alice could have picked from is n, what is the sum of the digits of n?

    (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

15. At a motel, there are 15 rooms in a row. A visitor may rent 1 room for 5 dollars, or 2 adjacent rooms for 4 dollars each. At most 1 visitor may rent a given room at a time, and no 2 visitors may rent rooms adjacent to each other. If the leftmost and rightmost rooms must be rented, what is the largest dollar amount that the motel can earn?

    (A) 40 **(B)** 41 **(C)** 42 **(D)** 43 **(E)** 44

16. A set of positive integers exists such that for any integer k in the set, all of the values k^2+2, k^2+4, and k^2+8 are prime numbers. Two distinct integers m and n are chosen from the set. Which of the following is a possible value of $m+n$?

    (A) 40 **(B)** 56 **(C)** 72 **(D)** 88 **(E)** 104

17. Draw two identical non-intersecting circles, a line tangent to both circles at distinct points A and B, where the circles are on the same side of the line, and a line tangent to both circles at distinct points C and D, where the circles are on opposite sides of the line. The lines intersect at point P. If $AB = 11$ and $CD = 5$, what is $AP \cdot BP$?

    (A) 20 **(B)** 22 **(C)** 24 **(D)** 26 **(E)** 28

18. 8 people randomly split into 2 groups of four to dance. After that, the 8 people randomly split into 4 pairs of two to talk. What is the probability that exactly 2 of the 4 pairs contain two people who have danced in the same group of four?

(A) $\frac{8}{35}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{4}{21}$ **(D)** $\frac{24}{35}$ **(E)** $\frac{6}{7}$

19. A plane cuts into a sphere of radius 11 such that the area of the region of the plane inside the sphere is 108π. A perpendicular plane cuts into the sphere such that the area of the region of the plane inside the sphere is 94π. Given that the two planes intersect at a line, what is the length of the segment of the line inside the sphere?

(A) $6\sqrt{3}$ **(B)** 12 **(C)** $11\sqrt{2}$ **(D)** $8\sqrt{5}$ **(E)** 18

20. Richard thinks of a positive integer n and writes the base ten representations of $n!$ and $(n+1)!$ on a board. He then erases the zeroes to the right of the last nonzero digit of each number (if any exist), resulting in two numbers a and b. If one of a and b is 4 times the other, what is the sum of all possible values of n less than 1000?

(A) 315 **(B)** 441 **(C)** 656 **(D)** 714 **(E)** 819

21. In trapezoid $ABCD$ with $\overline{AD} \parallel \overline{BC}$ and side lengths $AD = 18$, $BC = 20$, and $AB = CD = 8$, let X be the intersection of line AB and the bisector of $\angle ADC$, and let Y be the intersection of line CD and the bisector of $\angle DAB$. What is XY?

(A) 22 **(B)** 24 **(C)** 25 **(D)** 27 **(E)** 28

22. Define a sequence recursively by $T_1 = 1$ and

$$T_n = \frac{n! \cdot T_{n-1}}{(n-1)! + n \cdot T_{n-1}}$$

for all integers $n \geq 2$. The value of T_{2020} can be written as $\frac{2020!}{m}$, where m is a positive integer. What is the sum of the distinct prime divisors of m?

(A) 198 **(B)** 199 **(C)** 200 **(D)** 201 **(E)** 202

23. Joy picks an integer n from the interval $[1, 40]$. She tells Amy the remainder when n is divided by 7 and Sid the number of divisors of n. Amy and Sid both know n is in the interval $[1, 40]$, but they get confused and believe Amy was told the number of divisors and Sid was told the remainder. Amy says, "I know what n is." Sid replies, "If so, then I also know what n is." As it turns out, they thought of the same value but were wrong due to their confusion. If Amy and Sid tell the truth based on their beliefs and can reason perfectly, what is the sum of all possible actual values of n?

(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

24. What is the sum of the digits of the smallest positive integer n such that

$$\sqrt{5n-1} - \sqrt{5n-2} + \sqrt{5n-3} - \sqrt{5n-4}$$

is less than 0.05?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

25. In $\triangle ABC$ with $AB = 3$ and $AC = 6$, let D be the intersection of the angle bisector of $\angle BAC$ and $\overline{BC}$, and let M be the midpoint of $\overline{AC}$. Let the circumcircle of $\triangle DMC$ intersect line AD again at P, distinct from D. If $DM = 2$, what is PC^2?

(A) $\frac{72}{5}$ **(B)** $\frac{78}{5}$ **(C)** $\frac{84}{5}$ **(D)** 18 **(E)** $\frac{96}{5}$

DMC 10 F

DO NOT OPEN UNTIL FRIDAY, January 21, 2022

****Administration on an earlier date will disqualify your results.****

- All the information needed to administer this exam is not contained in the non-existent DMC 10 Teacher's Manual. PLEASE READ THE MANUAL BEFORE FRIDAY, JANUARY 21, 2022.
- You may check your answers by using the answer key in the official opening post of this contest.
- The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

For more information about the DMC and our other competitions, please visit `https://detoasty3.github.io/dmc.html`.

Questions and comments about this competition should be sent to:

DeToasty3.