De Mathematics Competitions

1st Annual

DIME

De Invitational Mathematics Examination

Sunday, February 14, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor a penalty for wrong answers.
3. No aids other than writing utensils, blank scratch paper, rulers, compasses, and erasers are permitted. In particular, **calculators, calculating devices, graph paper, protractors, smartphones or smartwatches, and computers are not permitted.**
4. Figures are not necessarily drawn to scale.
5. A combination of the DIME and the De Mathematics Contest (DMC) 10 are not used to determine eligibility for participation in the De Junior Mathematical Olympiad (DJMO) because it will not exist anytime in the foreseeable future.
6. Record all your answers, but not identification information, on the DIME answer form. Only the answer form will be collected from you.

2021 DIME

DO NOT OPEN UNTIL SUNDAY, February 14, 2021

Questions and complaints about problems and solutions for this exam should be sent by private message to:

DeToasty3, **firebolt360**, and **nikenissan**.

The 2021 DJMO will never be held. It would be a 6-question, 9-hour, explanation-based exam if it was held. You will not be invited to participate because this contest does not exist. *A complete listing of our previous publications may be found at our web site:*

`http://detoasty3.gq/DMC`

****Try Administering This Exam On An Earlier Date. Oh Wait, You Can't.****

1. All the information needed to administer this exam is not contained in the non-existent DIME Teacher's Manual.
2. YOU must not verify on the non-existent DIME COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.
3. Send **DeToasty3**, **firebolt360**, **nikenissan**, **pog**, and **vsamc** a private message submitting your answers to the DIME. AoPS is the only way to submit your answers.
4. The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

The **2021 De Mathematics Competitions**
were made possible by the contributions of the following people:

Aathreyakadambi, ApraTrip, AT2005, Awesome_guy, DeToasty3, firebolt360, GammaZero, i3435, jayseemath, karate7800, nikenissan, pog, richy, skyscraper, & vsamc

Credit goes to Online Test Seasonal Series (OTSS) for the booklet template.

1. Find the remainder when the number of positive divisors of the value

$$(3^{2020}+3^{2021})(3^{2021}+3^{2022})(3^{2022}+3^{2023})(3^{2023}+3^{2024})$$

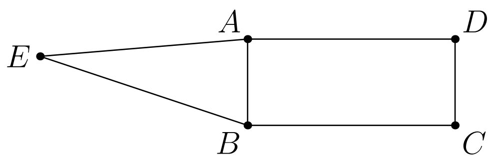
   is divided by 1000.

2. If x is a real number satisfying the equation

$$9\log_3 x - 10\log_9 x = 18\log_{27} 45,$$

   then the value of x is equal to $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

3. In the diagram below, rectangle $ABCD$ has $AB = 5$ and $AD = 12$. Also, E is a point in the same plane outside $ABCD$ such that the perpendicular distances from E to the lines AB and AD are 12 and 1, respectively, and $\triangle ABE$ is acute. There exists a line passing through E which splits $ABCD$ into two figures of equal area. Suppose that this line intersects $\overline{AB}$ at a point F and $\overline{CD}$ at a point G. Find FG^2.



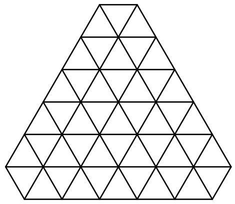
4. There are 7 balls in a jar, numbered from 1 to 7, inclusive. First, Richard takes a balls from the jar at once, where a is an integer between 1 and 6, inclusive. Next, Janelle takes b of the remaining balls from the jar at once, where b is an integer between 1 and the number of balls left, inclusive. Finally, Tai takes all of the remaining balls from the jar at once, if any are left. Find the remainder when the number of possible ways for this to occur is divided by 1000, if it matters who gets which ball.

5. Let $\mathcal{S}$ be the set of all positive integers which are both a multiple of 3 and have at least one digit that is a 1. For example, 123 is in $\mathcal{S}$ and 450 is not. The probability that a randomly chosen 3-digit positive integer is in $\mathcal{S}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

6. Let ABC be a right triangle with right angle at A and side lengths $AC = 8$ and $BC = 16$. The lines tangent to the circumcircle of $\triangle ABC$ at points A and B intersect at D. Let E be the point on side $\overline{AB}$ such that $\overline{AD} \parallel \overline{CE}$. Then DE^2 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

7. In a game, Jimmy and Jacob each randomly choose to either roll a fair six-sided die or to automatically roll a 1 on their die. If the product of the two numbers face up on their dice is even, Jimmy wins the game. Otherwise, Jacob wins. The probability Jimmy wins 3 games before Jacob wins 3 games can be written as $\frac{p}{2^q}$, where p and q are positive integers, and p is odd. Find the remainder when $p+q$ is divided by 1000.

8. In the diagram below, a group of equilateral triangles are joined together by their sides. A parallelogram in the diagram is defined as a parallelogram whose vertices are all at the intersection of two grid lines and whose sides all travel along the grid lines. Find the number of distinct parallelograms in the diagram below.



9. Real numbers a, b, c, and d satisfy the system of equations

$$\begin{aligned} -a - 27b - 8d &= 1, \\ 8a + 64b + c + 27d &= 0, \\ 27a + 125b + 8c + 64d &= 1, \\ 64a + 216b + 27c + 125d &= 8. \end{aligned}$$

Find $12a + 108b + 48d$.

10. There exist complex numbers $z_1, z_2, \ldots, z_{10}$ which satisfy

$$|z_k i^k + z_{k+1} i^{k+1}| = |z_{k+1} i^k + z_k i^{k+1}|$$

for all integers $1 \leq k \leq 9$, where $i = \sqrt{-1}$. If $|z_1| = 9$, $|z_2| = 29$, and for all integers $3 \leq n \leq 10$, $|z_n| = |z_{n-1} + z_{n-2}|$, find the minimum value of $|z_1| + |z_2| + \cdots + |z_{10}|$.

11. Call a positive integer k *pretty* if for every positive integer a, there exists an integer n such that $n^2 + n + k$ is divisible by 2^a but not 2^{a+1}. Find the remainder when the 2021st pretty number is divided by 1000.

12. Let $\omega_1, \omega_2, \omega_3, \ldots, \omega_{2020!}$ be the distinct roots of $x^{2020!} - 1$. Suppose that n is the largest integer such that 2^n divides the value

$$\sum_{k=1}^{2020!} \frac{2^{2019!} - 1}{\omega_k^{2020} + 2}.$$

Then n can be written as $a! + b$, where a and b are positive integers, and a is as large as possible. Find the remainder when $a + b$ is divided by 1000.

13. Let $\triangle ABC$ have side lengths $AB = 7$, $BC = 8$, and $CA = 9$. Let D be the projection from A to $\overline{BC}$ and D' be the reflection of D over the perpendicular bisector of $\overline{BC}$. Let P and Q be distinct points on the line through D' parallel to $\overline{AC}$ such that $\angle APB = \angle AQB = 90^\circ$. The value of $AP + AQ$ can be written as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that b and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.

14. For a positive integer n not divisible by 211, let $f(n)$ denote the smallest positive integer k such that $n^k - 1$ is divisible by 211. Find the remainder when

$$\sum_{n=1}^{210} nf(n)$$

is divided by 211.

15. Let right $\triangle ABC$ have $AC = 3$, $BC = 4$, and right angle at C. Let D be the projection from C to $\overline{AB}$. Let ω be a circle with center D and radius $\overline{CD}$, and let E be a variable point on the circumference of ω. Let F be the reflection of E over point D, and let O be the center of the circumcircle of $\triangle ABE$. Let H be the intersection of the altitudes of $\triangle EFO$. As E varies, the path of H traces a region $\mathcal{R}$. The area of $\mathcal{R}$ can be written as $\frac{m\pi}{n}$, where m and n are relatively prime positive integers. Find $\sqrt{m} + \sqrt{n}$.