Official Solutions

De Mathematics Competitions

2nd Annual

DMC 10B

Friday, June 11, 2021

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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Questions and complaints about this competition should be sent by private message to

DeToasty3.

Answer Key:

1. (C)	2. (B)	3. (D)	4. (B)	5. (A)
6. (A)	7. (D)	8. (A)	9. (B)	10. (C)
11. (D)	12. (B)	13. (C)	14. (E)	15. (C)
16. (C)	17. (B)	18. (B)	19. (A)	20. (C)
21. (D)	22. (C)	23. (B)	24. (A)	25. (E)

Problem 1:

(DeToasty3) What is the value of

$$2^0 \times 2^1 + 2^0 \times 2^2?$$

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Answer (C): The requested answer is

$$2^0 \times 2^1 + 2^0 \times 2^2 = 2 + 4 = \boxed{\text{(C) } 6}.$$

■

Problem 2:

(DeToasty3) How many single-digit positive integers n are there such that $2n$ is a perfect square?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer (B): We see that finding all single-digit positive integers n such that $2n$ is a perfect square is equivalent to finding all even perfect squares between 2 and 18, inclusive. We see that the only even perfect squares in this range are 4 and 16, so the requested answer is $\boxed{\text{(B) } 2}$. ■

Problem 3:

(pog) If n is a positive integer such that $n \times 3^5 = 3^7 - 3^5$, what is n?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 9

Answer (D): We can rewrite the right-hand side of the equation as

$$(3^2 - 1) \times 3^5 = 8 \times 3^5,$$

so the requested answer is $\boxed{\textbf{(D) } 8}$. ■

Problem 4:

(DeToasty3) What is the smallest positive integer n such that $n! + 1$ is not divisible by any integer between 2 and 9, inclusive?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Answer (B): Testing values of n, we see that $1! + 1 = 2$, $2! + 1 = 3$, $3! + 1 = 7$, $4! + 1 = 25$, and $5! + 1 = 121$. We see that $121 = 11^2$, so $5! + 1$ is the first listed number which is not divisible by any integer between 2 and 9, inclusive. Thus, the requested answer is $\boxed{\textbf{(B) } 5}$. ■

Problem 5:

(HrishiP) A square has a perimeter which is twice the area of the circle inscribed in the square. What is the circumference of the circle?

(A) 8 **(B)** 16 **(C)** 32 **(D)** 64 **(E)** 128

Answer (A): Let the radius of the inscribed circle be equal to r. It follows that the side length of the square is equal to $2r$, so the perimeter of the square is equal to $8r$. The area of the circle is equal to πr^2, so we have

$$8r = 2\pi r^2 \implies r = \frac{4}{\pi}.$$

Finally, the circumference of the circle is equal to $2\pi r$, so plugging in r gives us

$$2\pi r = 2\pi \cdot \frac{4}{\pi} = \boxed{\textbf{(A) } 8},$$

as requested. ■

Problem 6:

(DeToasty3) For what values of k does the equation

$$k^2x + 2 = 4x + k$$

have no real solutions x?

(A) -2 **(B)** 0 **(C)** 2 **(D)** -2 and 0 **(E)** -2 and 2

Answer (A): Rearranging the given equation, we get

$$k^2 + 2 = 4x + k \implies (k^2 - 4)x = k - 2.$$

Note that this linear equation in x will always have a real solution x unless $k^2 - 4 = 0$ and $k - 2 \neq 0$. This happens only when $k = -2$, so the requested answer is $\boxed{\textbf{(A) } -2}$. ■

Problem 7:

(DeToasty3) Justin has three weightless boxes and four pebbles, each of which has a weight of either 3, 4, or 5 ounces. He puts each pebble in one of the boxes such that each box has at least one pebble in it. If the weights of the boxes form an increasing arithmetic progression, what is the largest possible weight of the heaviest box, in ounces?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Answer (D): Note that if all of the boxes have at least one pebble in it, then it follows that one box has two pebbles, and the other two boxes have one pebble each. Also, note that the smallest possible weight of the box with two pebbles in it, in ounces, is $3 + 3 = 6 > 5$, so the median weight of the three boxes has to be with one of the two boxes with one pebble in it.

The maximum possible weight of a box with one pebble in it, in ounces, is 5, and the minimum possible weight, in ounces, is 3. Thus, the largest possible weight, in ounces, of the heaviest box, or the box with two pebbles in it, is $\boxed{\textbf{(D) } 7}$, as requested. We see that this can be accomplished by having the heaviest box have two pebbles of weights 3 and 4 ounces, and the other two boxes have one pebble with weights 3 and 5 ounces. ■

Problem 8:

(pog & DeToasty3) At Test Academy, there are four classes, one on each of the four floors of the building. For each class, the class which is one floor above it has twice as many students and half the average grade of that class. If the average grade of all four classes combined is 20, what is the average grade of the class on the bottom floor?

(A) 75 **(B)** 80 **(C)** 85 **(D)** 90 **(E)** 95

Answer (A): Let n be the number of students in the class on the bottom floor, and let g be the average grade of the class on the bottom floor. Our requested answer is

$$\frac{gn + \frac{g}{2} \cdot 2n + \frac{g}{4} \cdot 4n + \frac{g}{8} \cdot 8n}{n + 2n + 4n + 8n} = 20 \implies \frac{4gn}{15n} = 20 \implies g = \boxed{\textbf{(A)}\ 75}.$$

■

Problem 9:

(jayseemath) What is the value of

$$\frac{9^{1010} + 3^{2021}}{9^{1009} + 3^{2020}}?$$

(A) 3.5 **(B)** 3.6 **(C)** 3.7 **(D)** 3.8 **(E)** 3.9

Answer (B): The requested answer is

$$\frac{9^{1010} + 3^{2021}}{9^{1009} + 3^{2020}} = \frac{3^{2020} + 3^{2021}}{3^{2018} + 3^{2020}} = \frac{3^{2018} \cdot (9 + 27)}{3^{2018} \cdot (1 + 9)} = \boxed{\textbf{(B)}\ 3.6}.$$

■

Problem 10:

(pog & DeToasty3) How many ordered triples of integers (a, b, c) are there such that the product

$$(a - 2020)(2b - 2021)(3c - 2022)$$

is positive and has exactly three positive divisors?

(A) 3 **(B)** 9 **(C)** 12 **(D)** 24 **(E)** infinitely many

Answer (C): The key observation is that 2022 is divisible by 3, namely $2022 = 3 \cdot 674$. This means that our product becomes

$$3(a-2020)(2b-2021)(c-674).$$

This means that our product is divisible by 3. Now, in order for our product to have three positive divisors, we must have that the product is equal to $3^2 = 9$, which has $2+1=3$ factors. Note that this is the only possibility since 3 is prime.

This means that for each of the three factors $a-2020$, $2b-2021$, and $c-674$, we must have that they multiply to 3. We can either choose two of the three to be negative or all three to be positive, for 4 possibilities, and there are three ways to choose which one has the factor of 3. This gives us a total of $4 \cdot 3 = \boxed{\text{(C) } 12}$ ordered triples, as requested. Note that since -3, -1, 1, and 3 are all odd, and $2b-2021$ is always odd, we do not over count. ■

Problem 11:

(richy) John is on the infinite grid below, where every row and column of consecutive squares repeats the pattern 1, 2, 3, 4 from left to right and up to down. From any square, John may move up, down, left, or right one square. Starting at the center square labeled 1, in how many sequences of 4 moves can John land on squares labeled 1, 2, 3, and 4 in some order? (The starting square does not count unless he lands on it again.)

...	...	...	...	...	...	...
...	1	2	3	4	1	...
...	2	3	4	1	2	...
...	3	4	1	2	3	...
...	4	1	2	3	4	...
...	1	2	3	4	1	...
...	...	...	...	...	...	...

(A) 16 **(B)** 32 **(C)** 48 **(D)** 64 **(E)** 96

Answer (D): Upon inspection, we see that there are four possible sequences of numbers that John can take, in order: $(4,3,2,1)$, $(4,1,2,3)$, $(2,3,4,1)$, and $(2,1,4,3)$. We also see that for each of the four sequences, there are two squares that John can land on for each of the four numbers. Thus, the requested answer is $2^4 \cdot 4 = \boxed{\text{(D) } 64}$. ■

Problem 12:

(john0512) In regular hexagon $ABCDEF$, diagonals $\overline{AC}$ and $\overline{BF}$ intersect at a point G. If the area of $\triangle ABG$ is 2, what is the area of pentagon $CDEFG$?

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

Answer (B): Since $\triangle ABG$ is similar to $\triangle FCG$, and $FC = 2AB$, the area of $\triangle FCG$ is four times the area of $\triangle ABG$. Since the two side triangles in a trapezoid are each the geometric mean of the top and bottom triangles, the areas of $\triangle AFG$ and $\triangle BCG$ are both twice that of $\triangle ABG$. This means that the areas of $ABCF$ and $CDEF$ are both nine times that of $\triangle ABG$. We get that the areas of $\triangle FCG$ and $CDEF$ are $4 \cdot 2 = 8$ and $9 \cdot 2 = 18$, respectively, so the requested answer is $8 + 18 = \boxed{\textbf{(B)}\ 26}$. ■

Problem 13:

(DeToasty3) Two functions f and g, in that order, are said to be *rivals* if there does not exist a real number x such that $f(x) = g(f(x))$. If f and g are linear, non-constant, and rivals, which of the following sets contains all possible values that $g(1)$ can never take?

(A) $\{-1\}$ **(B)** $\{0\}$ **(C)** $\{1\}$ **(D)** $\{-1, 1\}$ **(E)** the empty set

Answer (C): Let $f(x) = ax+b$ and $g(x) = cx+d$ for real numbers a, b, c, and d. Then, we get that

$$g(f(x)) = c(ax + b) + d = acx + bc + d.$$

In order for the equation $f(x) = g(f(x))$ to not have any real solutions, we must have that the graphs of the lines $f(x)$ and $g(f(x))$ are parallel and are not the same line. This means that both $a = ac \implies c = 1$ (because $a \neq 0$) and

$$b \neq bc + d \implies b \neq b + d \implies d \neq 0.$$

Finally, we see that $g(1) = c + d$, so if $c = 1$ and $d \neq 0$, then $g(1) \neq 1$. Thus, the requested answer is $\boxed{\textbf{(C)}\ \{1\}}$. ■

Problem 14:

(DeToasty3) Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. A circle passes through A and B and intersects side $\overline{CD}$ at two points which trisect the side. What is the area of the circle?

(A) 6π **(B)** 7π **(C)** 8π **(D)** 9π **(E)** 10π

Answer (E): Let O be the center of the circle, and let E and F be the two distinct points of intersection between the circle and side $\overline{CD}$. We see that points A, B, E, and F are all equidistant from O; in particular, the distance between any of the four points to O is the radius of the circle, which we will denote as r. Since $\overline{AB} \parallel \overline{EF}$, drop a perpendicular from the midpoint of side $\overline{AB}$ to $\overline{EF}$. It is clear that this perpendicular passes through O and the midpoint of $\overline{EF}$, and the distance from the midpoints is 4 (the length of side $\overline{BC}$).

Next, let the midpoints of $\overline{AB}$ and $\overline{EF}$ be M and N, respectively. Let $MO = x$ and $NO = 4 - x$. Next, we use the Pythagorean Theorem to obtain

$$3^2 + x^2 = r^2 \quad \text{and} \quad 1^2 + (4-x)^2 = r^2,$$

so we have the equation

$$9 + x^2 = 1 + x^2 - 8x + 16 \implies x = 1.$$

Then, we get

$$9 + 1^2 = r^2 \implies r^2 = 10.$$

Thus, the requested area of the circle is $\pi r^2 = \boxed{\textbf{(E) } 10\pi}$. ■

Problem 15:

(HrishiP & DeToasty3) Let x be a positive real number such that

$$\frac{1}{x - \frac{1}{x}} = \sqrt{x^2 + \frac{x^4}{4}}.$$

What is the value of x^2?

(A) $4 - 2\sqrt{2}$ **(B)** $\sqrt{2}$ **(C)** $2\sqrt{2} - 1$ **(D)** 2 **(E)** $\sqrt{2} + 1$

Answer (C): Rewriting both sides of the equation, we get

$$\frac{1}{x - \frac{1}{x}} = \sqrt{\frac{x^4 + 4x^2}{4}}.$$

Flipping the denominators of both fractions, we get

$$x - \frac{1}{x} = \sqrt{\frac{4}{x^4 + 4x^2}}.$$

Next, squaring both sides of the equation, we get

$$x^2 + \frac{1}{x^2} - 2 = \frac{4}{x^4 + 4x^2} \implies \left(x^2 + \frac{1}{x^2} - 2\right)(x^4 + 4x^2) = 4.$$

Expanding, we get

$$\begin{aligned} x^6 + 4x^4 + x^2 + 4 - 2x^4 - 8x^2 &= 4 \\ \implies x^6 + 2x^4 - 7x^2 &= 0 \\ \implies x^4 + 2x^2 - 7 &= 0. \end{aligned}$$

Now, by the quadratic formula, we get

$$x^2 = \frac{-2 \pm \sqrt{4+28}}{2} = -1 \pm 2\sqrt{2}.$$

Since we given that x is a positive real number, our only solution is $x^2 = 2\sqrt{2} - 1$. It is easy to check that this works by plugging it into the original equation. Thus, the requested answer is $\boxed{\textbf{(C) } 2\sqrt{2} - 1}$. ■

Problem 16:

(john0512) The degree measures of an interior angle of each of three regular polygons form an arithmetic progression of positive integers. If the polygon with the most sides has 360 sides, what is the smallest possible number of sides any of the polygons can have?

(A) 8 **(B)** 16 **(C)** 24 **(D)** 32 **(E)** 40

Answer (C): We have that the measure of one angle of a regular n-gon is $\frac{180^\circ(n-2)}{n}$. This means that the measure of one angle of the regular polygon with 360 sides is $\frac{180^\circ \cdot 358}{360} = 179^\circ$. Note that if all three measures are integers, then the smallest measure must be an odd integer. Therefore, we must have that n is divisible by 8 but not 16.

Testing $n = 8$, we get that the smallest measure is $\frac{180^\circ \cdot 6}{8} = 135^\circ$, which means that the middle measure is 157°. However,

$$\frac{180^\circ \cdot (n-2)}{n} = 157^\circ \implies 23n = 360,$$

which means that n will not be an integer.

We then test $n = 24$, which gives $\frac{180^\circ \cdot 22}{24} = 165^\circ$, so the middle measure is 172°. Then,

$$\frac{180^\circ \cdot (n-2)}{n} = 172^\circ \implies 8n = 360,$$

which gives $n = 45$, so this works.

Thus, the requested answer is $\boxed{\textbf{(C) } 24}$. ■

Problem 17:

(DeToasty3) There exists a sequence $a_1, a_2, \ldots, a_6$ of positive integers such that for every term in the sequence, there exists another term in the sequence which is equal to that term. How many possible values of the product $a_1 a_2 \cdots a_6$ less than 1000 are there?

(A) 36 **(B)** 37 **(C)** 38 **(D)** 39 **(E)** 40

Answer (B): Note that the condition implies that for any unique value in the sequence, there must be at least two terms with that value. Testing cases, we may have 3 of one value and 3 of another value, all 6 of the same value, 4 of one value and 2 of another value, and 2 of one value, 2 of another value, and 2 of a third value. Note that the set of all possible values of $a_1 a_2 \cdots a_6$ of these cases is precisely the set of all perfect squares and perfect cubes (or both). Thus, the problem boils down to:

"How many positive integers between 1 and 999, inclusive, are perfect squares or perfect cubes (or both)?"

We see that in this range, there are 31 perfect squares $(1^2, 2^2, \ldots, 31^2)$, 9 perfect cubes $(1^3, 2^3, \ldots, 9^3)$, and 3 perfect sixths $(1^6, 2^6, 3^6)$. Thus, by the Principle of Inclusion and Exclusion, we get the requested answer of $31 + 9 - 3 = \boxed{\textbf{(B) } 37}$. ■

Problem 18:

(DeToasty3) In trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$, $AB = 4$, and $AD = BC = 5$, let the angle bisector of $\angle ADC$ intersect the diagonal $\overline{AC}$ at a point P. If line BP intersects the side $\overline{CD}$ at a point Q such that $CQ = 8$, what is the area of trapezoid $ABCD$?

(A) 24 **(B)** 28 **(C)** 32 **(D)** 36 **(E)** 40

Answer (B): First, we see that $\overline{AB} \parallel \overline{CQ}$, which means that $\angle BAP = \angle PCQ$ and $\angle ABP = \angle PQC$, so we get that $\triangle PQC \sim \triangle PBA$. Then, we have that

$$\frac{AP}{CP} = \frac{AB}{CQ} = \frac{1}{2}.$$

Next, let $DQ = x$. We see that segment $\overline{DP}$ bisects $\angle ADC$. By the angle bisector theorem, we have that

$$\frac{AP}{CP} = \frac{AD}{CD} = \frac{5}{x+8}.$$

Then, we have that

$$\frac{1}{2} = \frac{5}{x+8} \implies x = 2.$$

This means that $CD = 10$. Drop a perpendicular from A to side $\overline{CD}$ and call the foot X. We see that $AD = 5$ and $DX = \frac{10-4}{2} = 3$. By the Pythagorean Theorem, we get that $AX = 4$. Thus, the requested answer is $\frac{1}{2} \cdot 4 \cdot (4+10) = \boxed{\textbf{(B) } 28}$. ■

Problem 19:

(DeToasty3) Six red balls and six blue balls are each numbered from 1 to 6. How many ways are there to form six pairs of one red ball and one blue ball such that the product of the two numbers on the balls in every pair is divisible by at least one of 2 and 3?

(A) 288 **(B)** 360 **(C)** 432 **(D)** 504 **(E)** 576

Answer (A): We see that in order for the product of the two numbers on the balls in every pair to be divisible by at least one of 2 and 3, we need every ball numbered 1 and 5 to be paired with a ball of the opposite color with one of the other four numbers. For the red ball numbered 1, there are four choices for the blue ball for it to be paired with. Next, for the red ball numbered 5, there are three remaining choices for the blue ball for it to be paired with. Finally, there are $4! = 24$ ways to pair the remaining four red balls with a blue ball. Thus, the requested answer is $4 \cdot 3 \cdot 24 = \boxed{\textbf{(A) } 288}$. ■

Problem 20:

(DeToasty3) At a motel, there are 15 rooms in a row. A visitor may rent 1 room for 5 dollars, or 2 adjacent rooms for 4 dollars each. At most 1 visitor may rent a given room at a time, and no 2 visitors may rent rooms adjacent to each other. If the leftmost and rightmost rooms must be rented, what is the largest dollar amount that the motel can earn?

(A) 40 **(B)** 41 **(C)** 42 **(D)** 43 **(E)** 44

Answer (C): We will use a "density" argument. In other words, for each of the two options (one room or two adjacent rooms), we will find the amount of money the motel earns divided by the number of rooms it takes up (including the one empty room to its right. For the single room option, we have $\frac{5}{2}$, and for the two adjacent rooms option, we have $\frac{8}{3}$. Since $\frac{8}{3} > \frac{5}{2}$, we wish to maximize the number of two adjacent rooms. This gives us 5 pairs of two adjacent rooms, but since we also want to minimize the number of empty rooms, for the last group of three, we can do single room, empty room, and single room in order from left to right. Thus, the requested answer is $4 \cdot 8 + 2 \cdot 5 = \boxed{\text{(C) } 42}$. ■

Problem 21:

(DeToasty3) A convex quadrilateral $ABCD$ has $\angle ADC = \angle BAC = 90^\circ$ and side lengths $AB = 6$, $BC = 9$, and $CD = 5$. Let M be the midpoint of diagonal $\overline{BD}$. What is MC^2?

(A) 26 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 30

Answer (D): Extend side $\overline{AD}$ past A and drop a perpendicular from B to line AD and call the foot point P. Then, using the Pythagorean Theorem, we find that $AC = 3\sqrt{5}$ and $AD = 2\sqrt{5}$. Note that

$$\angle CAD = 180^\circ - \angle BAP \implies \angle CAD = \angle ABP$$

and $\angle CDA = \angle BPA = 90^\circ$, so $\triangle CAD \sim \triangle ABP$, so $AP = 2\sqrt{5}$ and $BP = 4$. Thus, we see that A is the midpoint of segment $\overline{DP}$. Since M is the midpoint of diagonal $\overline{BD}$, we have that $2AM = BP$ and so $\overline{AM} \parallel \overline{CD}$. Thus, the answer is

$$MC^2 = (CD - AM)^2 + AD^2 = (5-2)^2 + (2\sqrt{5})^2 = \boxed{\text{(D) } 29},$$

as requested. ■

Problem 22:

(HrishiP & DeToasty3) Bill and Ben each have 2 fair coins. In each turn, they flip all their coins at the same time, if they have any. If a coin lands heads, then the other person gets that coin. If a coin lands tails, then that coin stays with the same person. What is the probability that after Bill and Ben take exactly 3 turns, they each end up with 2 coins?

(A) $\frac{1}{4}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{7}{16}$ **(E)** $\frac{1}{2}$

Answer (C): Note that no matter where each coin ends up on minute $n-1$, each coin has a $\frac{1}{2}$ chance of going to each person on minute n. Thus, the probability is

$$\frac{\binom{4}{2}}{2^4} = \boxed{\textbf{(C) } \frac{3}{8}},$$

as requested. ■

Problem 23:

(DeToasty3) What is the sum of the digits of the smallest positive integer n such that

$$\sqrt{5n-1} - \sqrt{5n-2} + \sqrt{5n-3} - \sqrt{5n-4}$$

is less than 0.05?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

Answer (B): Note that we can rewrite the expression as

$$\frac{1}{\sqrt{5n-1}+\sqrt{5n-2}} + \frac{1}{\sqrt{5n-3}+\sqrt{5n-4}}.$$

Now, we just have to bound.

Note that since the four square root values are very close to each other, suppose that they were all equal to some value k. Then, we have

$$\frac{1}{2k} + \frac{1}{2k} < 0.05 \implies \frac{1}{k} < 0.05 \implies k > 20.$$

This means that we should bound $\sqrt{5n}$ so that it is close to 20, or $5n \approx 400$, or $n \approx 80$.

Testing out $n = 80$, we get

$$\frac{1}{\sqrt{399}+\sqrt{398}} + \frac{1}{\sqrt{397}+\sqrt{396}}.$$

However, we see that all four square root values are less than $\sqrt{400} = 20$, so the resulting fractions will each be greater than $\frac{1}{40}$, so the whole expression will be greater than $\frac{1}{20} = 0.05$, which we do not want.

On the other hand, if $n = 81$, we get

$$\frac{1}{\sqrt{404}+\sqrt{403}} + \frac{1}{\sqrt{402}+\sqrt{401}}.$$

In this case, we see that all four square root values are greater than $\sqrt{400} = 20$, so the resulting fractions will each be less than $\frac{1}{40}$, so the whole expression will be less than $\frac{1}{20} = 0.05$.

This means that $n = 81$ is the smallest positive integer that works, so the requested digit sum is $8 + 1 = \boxed{\textbf{(B)}\ 9}$. ■

Problem 24:

(DeToasty3) In $\triangle ABC$ with $AB = 3$ and $AC = 6$, let D be the intersection of the angle bisector of $\angle BAC$ and $\overline{BC}$, and let M be the midpoint of $\overline{AC}$. Let the circumcircle of $\triangle DMC$ intersect line AD again at P, distinct from D. If $DM = 2$, what is PC^2?

(A) $\dfrac{72}{5}$ **(B)** $\dfrac{78}{5}$ **(C)** $\dfrac{84}{5}$ **(D)** 18 **(E)** $\dfrac{96}{5}$

Answer (A): We see that $AB = AM = 3$, $AD = AD$, and $\angle BAD = \angle MAD$, so we have that $\triangle BAD \cong \triangle MAD$. This means that $BD = DM = 2$. Next, by the Angle Bisector Theorem, we get that

$$AB : AC = BD : CD \implies CD = 4.$$

Next, drop perpendiculars from M and C to line AP and call the feet Q and R, respectively. We see that Q is the midpoint of segment $\overline{AR}$. This is because M is the midpoint of side $\overline{AC}$, and since $\angle AQM = \angle ARC = 90^\circ$ and $\angle MAD = \angle CAR$, we see that $\triangle AMQ \sim \triangle ACR$. Next, we see that D is two-thirds of the way from R to Q on segment $\overline{QR}$. To show this, we have that $\angle QBD = \angle RCD$ because $\overline{BM} \parallel \overline{CR}$, and $\angle BQD = \angle CRD = 90^\circ$, so $\triangle BQD \sim \triangle CRD$, and so

$$DQ : DR = BD : CD = 1 : 2.$$

With all this in mind, let $DQ = x$. It then follows that $DR = 2x$, $AQ = QR = 3x$, and $AD = 4x$. By the Pythagorean Theorem on $\triangle AMD$, we have

$$QM^2 = AM^2 - AQ^2 = DM^2 - DQ^2 \implies 9 - 9x^2 = 4 - x^2 \implies AD = \sqrt{10}.$$

Next, by Power of a Point, we have that

$$AD \cdot AP = AM \cdot AC \implies \frac{AD}{AM} = \frac{AC}{AP} \implies \triangle MAD \sim \triangle PAC,$$

which means that

$$\frac{PC}{DM} = \frac{AC}{AD} \implies PC = \frac{AC \cdot DM}{AD} = \frac{12}{\sqrt{10}}.$$

Thus, the answer is

$$PC^2 = \left(\frac{12}{\sqrt{10}}\right)^2 = \frac{144}{10} = \boxed{\textbf{(A) } \frac{72}{5}},$$

as requested. ■

Problem 25:

(HrishiP) Let x and y be distinct real numbers chosen at random from the interval $[-1, 1]$, excluding 0. What is the probability that

$$\left\lfloor \frac{|x|}{|y|} \right\rfloor \geq \left\lfloor \frac{|x+y|}{|x-y|} \right\rfloor,$$

where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to a real number r?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{9}$ **(C)** $\frac{9}{16}$ **(D)** $\frac{7}{12}$ **(E)** $\frac{5}{8}$

Answer (E): We will split this into two cases: x and y have the same sign (positive or negative), and x and y have opposite signs.

Case 1: If x and y have the same sign, then by symmetry, we can assume that x and y are both positive without loss of generality. We see that we can merely scale x and y proportionally (in other words, this case will form a line), so suppose that $y = kx$ for some constant k. We then have that

$$\left\lfloor \frac{|x|}{|kx|} \right\rfloor \geq \left\lfloor \frac{|x+kx|}{|x-kx|} \right\rfloor \implies \left\lfloor \frac{1}{k} \right\rfloor \geq \left\lfloor \frac{|1+k|}{|1-k|} \right\rfloor.$$

We see that as k approaches 0, the left hand side will approach infinity, so we have to find the smallest possible value of k such that the inequality does not hold. We see that as we increase k, the left hand side decreases and the right hand side increases. Also note that as k approaches 0, the right hand side approaches 1. Our strategy is to find the smallest values of k when the value of $\left\lfloor \frac{|1+k|}{|1-k|} \right\rfloor$ changes. First, we try

$$\frac{|1+k|}{|1-k|} = 2 \implies k = \frac{1}{3},$$

which makes the left hand side equal to 3, so the inequality still holds. Next, we try

$$\frac{|1+k|}{|1-k|} = 3 \implies k = \frac{1}{2},$$

which makes the left hand side equal to 2, so the inequality stops holding at $k = \frac{1}{2}$. We see that as we increase k to approach 1, the right hand side approaches infinity, then as k keeps increasing, the value of $|1+k|$ will still be greater than $|1-k|$ (intuitively). Meanwhile, the value of $\left\lfloor \frac{1}{k} \right\rfloor$ will be 0 as k increases from 1. Therefore, we have that $k \leq \frac{1}{2}$, and so this forms the line $y = \frac{x}{2}$. To find the area of the region in this case, we look at the first and third quadrants and see that the area of the region bounded by the line $y = \frac{x}{2}$ and the x-axis is $\frac{1}{4}$ of the area of the whole quadrant.

Case 2: If x and y have opposite signs, then by symmetry, we can assume that x is positive and y is negative without loss of generality. Note that if $|y| > |x|$, then the left hand side of the inequality will be equal to 0, and so will the right hand side since $|x-y|$ is adding two positives and $|x+y|$ is subtracting $|x|$ from $|y|$. Otherwise, if $|x| > |y|$, then the left hand side will be greater than or equal to 1, and $|x-y|$ will be greater than $|x+y|$ for the same reason as above, so the right hand side will be equal to 0. Therefore, this case is valid for any real x and y with opposite signs. (The case $x = -y$ is obvious and is left for the reader to show that it works.) To find the area of the region in this case, we see that the second and fourth quadrants will be completely covered.

Thus, the probability is

$$\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} = \boxed{\textbf{(E)}\ \frac{5}{8}},$$

as requested. ∎