De Mathematics Competitions

2nd Annual

DMC 10C

Friday, June 18, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the DMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than writing utensils, blank scratch paper, rulers, compasses, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your non-existent proctor will not ask you to record certain information on the answer form.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have already taken the test in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

Students who score well on this DMC 10 may or may not be invited to the 2022 DIME (De Invitational Mathematics Examination). More details about the DIME and other information are on the back page of this test booklet.

1. If n is a positive integer such that $n \times 3^5 = 3^7 - 3^5$, what is n?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 9

2. What is the smallest positive integer n such that $n! + 1$ is not divisible by any integer between 2 and 9, inclusive?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

3. A square has a perimeter which is twice the area of the circle inscribed in the square. What is the circumference of the circle?

(A) 8 **(B)** 16 **(C)** 32 **(D)** 64 **(E)** 128

4. For what values of k does the equation $k^2x + 2 = 4x + k$ have no real solutions x?

(A) -2 **(B)** 0 **(C)** 2 **(D)** -2 and 0 **(E)** -2 and 2

5. Justin has three weightless boxes and four pebbles, each of which has a weight of either 3, 4, or 5 ounces. He puts each pebble in one of the boxes such that each box has at least one pebble in it. If the weights of the boxes form an increasing arithmetic progression, what is the largest possible weight of the heaviest box, in ounces?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

6. At Test Academy, there are 4 classes in a row. For each class, the class to its right has twice as many students and half the average grade of that class. If the average grade of all 4 classes combined is 20, what is the average grade of the leftmost class?

(A) 75 **(B)** 80 **(C)** 85 **(D)** 90 **(E)** 95

7. How many ordered triples of integers (a, b, c) are there such that the product

$$(a - 2020)(2b - 2021)(3c - 2022)$$

is positive and has exactly three positive divisors?

(A) 3 **(B)** 9 **(C)** 12 **(D)** 24 **(E)** infinitely many

8. There are 18 members in a math club, where 4 members are wearing red and 14 members are wearing blue. The members are split into 6 groups of 3. For every two members in a group, if at least one of them is wearing red, they have a conversation. Otherwise, they do not. A person is called *social* if they have at least one conversation. What is the maximum possible number of social people in the club?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 12

9. John is on the infinite grid below, where every row and column of consecutive squares repeats the pattern 1, 2, 3, 4 from left to right and up to down. From any square, John may move up, down, left, or right one square. Starting at the center square labeled 1, in how many sequences of 4 moves can John land on squares labeled 1, 2, 3, and 4 in some order? (The starting square does not count unless he lands on it again.)

$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	1	2	3	4	1	$\cdots$
$\cdots$	2	3	4	1	2	$\cdots$
$\cdots$	3	4	1	2	3	$\cdots$
$\cdots$	4	1	2	3	4	$\cdots$
$\cdots$	1	2	3	4	1	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

(A) 16 **(B)** 32 **(C)** 48 **(D)** 64 **(E)** 96

10. In regular hexagon $ABCDEF$, diagonals $\overline{AC}$ and $\overline{BF}$ intersect at a point G. If the area of $\triangle ABG$ is 2, what is the area of pentagon $CDEFG$?

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

11. Two functions f and g are said to be *rivals* if there does not exist a real number x such that $f(x) = g(f(x))$. If there exist functions f and g which are both linear and rivals, which of the following sets contains all possible values that $g(1)$ can never take?

(A) $\{-1\}$ **(B)** $\{0\}$ **(C)** $\{1\}$ **(D)** $\{-1, 1\}$ **(E)** the empty set

12. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. A circle passes through A and B and intersects side $\overline{CD}$ at two points which trisect the side. What is the area of the circle?

(A) 6π **(B)** 7π **(C)** 8π **(D)** 9π **(E)** 10π

13. The degree measures of an interior angle of each of three regular polygons form an arithmetic progression of positive integers. If the polygon with the most sides has 360 sides, what is the smallest possible number of sides any of the polygons can have?

(A) 8 **(B)** 16 **(C)** 24 **(D)** 32 **(E)** 40

14. Let x and y be real numbers such that

$$|x - |y - x|| = 1,$$
$$|y - |x - y|| = 2.$$

What is the largest possible value of $x + y$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

15. Six red balls and six blue balls are each numbered from 1 to 6. How many ways are there to form six pairs of one red ball and one blue ball such that the product of the two numbers on the balls in every pair is divisible by at least one of 2 and 3?

(A) 288 **(B)** 360 **(C)** 432 **(D)** 504 **(E)** 576

16. A car moves such that if there are n people in it, the car moves at a constant rate of 4^n miles per hour. At noon, the car starts moving with 1 person in it. After every mile the car moves, another person instantaneously gets in it. How many people are in the car when the average speed it has moved since noon reaches 16 miles per hour?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

17. Two non-intersecting circles each have radius r. A line is drawn tangent to both circles at distinct points A and B, where the circles are on opposite sides of the line. Another line is drawn tangent to both circles, where the circles are on the same side of the line. Let the two lines intersect at P. If $AP = 3$ and $BP = 8$, what is r^2?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 21 **(E)** 24

18. At a motel, there are 15 rooms in a row. A visitor may rent 1 room for 5 dollars, or 2 adjacent rooms for 4 dollars each. At most 1 visitor may rent a given room at a time, and no 2 visitors may rent rooms adjacent to each other. If the leftmost and rightmost rooms must be rented, what is the largest dollar amount that the motel can earn?

(A) 40 **(B)** 41 **(C)** 42 **(D)** 43 **(E)** 44

19. Bill and Ben each have 2 fair coins. In each turn, they flip all their coins at the same time, if they have any. If a coin lands heads, then the other person gets that coin. If a coin lands tails, then that coin stays with the same person. What is the probability that after Bill and Ben take exactly 3 turns, they each end up with 2 coins?

(A) $\frac{1}{4}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{7}{16}$ **(E)** $\frac{1}{2}$

20. A convex quadrilateral $ABCD$ has $\angle ADC = \angle BAC = 90^\circ$ and side lengths $AB = 6$, $BC = 9$, and $CD = 5$. Let M be the midpoint of diagonal $\overline{BD}$. What is MC^2?

(A) 26 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 30

21. How many positive integers $n > 1$ are there such that when the number $57^n + 64^n$ is divided by $n!$, the remainder is equal to 1?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

22. What is the sum of the digits of the smallest positive integer n such that

$$\sqrt{5n-1} - \sqrt{5n-2} + \sqrt{5n-3} - \sqrt{5n-4}$$

is less than 0.05?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

23. In $\triangle ABC$ with $AB = 3$ and $AC = 6$, let D be the intersection of the angle bisector of $\angle BAC$ and $\overline{BC}$, and let M be the midpoint of $\overline{AC}$. Let the circumcircle of $\triangle DMC$ intersect line AD again at a point P, distinct from D. If $DM = 2$, what is CP^2?

(A) $\frac{72}{5}$ **(B)** $\frac{78}{5}$ **(C)** $\frac{84}{5}$ **(D)** 18 **(E)** $\frac{96}{5}$

24. Ada has a 35×35 square board, where each unit square can be either black or white. Initially, the squares in the middle column and middle row are black, and all other squares are white. In each move, Ada chooses one square to switch colors, and all squares adjacent to or sharing a corner with that square also switch colors. What is the fewest possible number of moves Ada can take to make the board entirely black?

(A) 120 **(B)** 128 **(C)** 132 **(D)** 144 **(E)** 160

25. A right $\triangle ABC$ has $AC > AB$ and a right angle at A. A circle is tangent to lines AB and AC at points D and E, respectively, and internally tangent to the circumcircle of $\triangle ABC$ at a point F. Let line EF and the angle bisector of $\angle ABC$ intersect at a point P. If $BP = 20$ and E is the midpoint of $\overline{AC}$, what is the area of $\triangle DEF$?

(A) 60 **(B)** $45\sqrt{2}$ **(C)** 80 **(D)** $60\sqrt{2}$ **(E)** $50\sqrt{3}$

2021 DMC 10C

DO NOT OPEN UNTIL FRIDAY, June 18, 2021

Questions and complaints about problems and solutions for this exam should be sent by private message to:

DeToasty3.

The 2022 DIME may or may not be held. It would be a 15-question, 3-hour, integer-answer exam if it was to be held. You may or may not be invited to participate because this contest may or may not exist. *A complete listing of our previous publications may be found at our web site:*

`https://detoasty3.github.io/dmc.html`

****Try Administering This Exam On An Earlier Date. Oh Wait, You Can't.****

1. All the information needed to administer this exam is not contained in the non-existent DMC 10 Teacher's Manual.
2. YOU must not verify on the non-existent DMC 10 COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.
3. Send **DeToasty3**, **nikenissan**, **pog**, and **vsamc** a private message submitting your answers to the DMC 10. AoPS is the only way to submit your answers.
4. The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

The **2021 De Mathematics Competitions**
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