De Mathematics Competitions

2nd Annual

# DMC 10 E

Friday, January 21, 2022

## INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. For each question, only one answer choice is correct.
3. Mark your answer to each problem on the DMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, rulers, and erasers are allowed as aids. Prohibited materials include calculators, smartwatches, phones, computing devices, compasses, protractors, and graph paper. No problems on the competition will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the competition, your non-existent proctor will not ask you to record certain information on the answer form.
8. You will have 75 minutes to complete the competition. You can discuss only with people that have already taken the competition in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

---

The DMC Committee reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

The publication, reproduction or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

*Students who score well on this DMC 10 may or may not be invited to the 2022 DIME. More details about the DIME and other information are on the back page of this test booklet.*

1. The sum of the first five positive integers and the sum of the first six positive integers are multiplied. What is the resulting product?

   **(A)** 315 **(B)** 335 **(C)** 355 **(D)** 375 **(E)** 395

2. If $n$ is a positive integer such that $n \times 3^5 = 3^7 - 3^5$, what is $n$?

   **(A)** 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 9

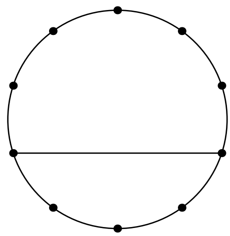
3. Bill writes all odd perfect squares from 1 to 100, inclusive, and Jill writes all even perfect squares from 1 to 100, inclusive. Who writes more digits, and by how many?

   **(A)** Bill, 1 **(B)** Bill, 2 **(C)** Jill, 1 **(D)** Jill, 2 **(E)** neither

4. Given a right triangle with legs of lengths 5 and 6, a square is drawn with one side as its hypotenuse such that the triangle is completely inside the square. What is the area of the region inside the square but outside the triangle?

   **(A)** 46 **(B)** 47 **(C)** 48 **(D)** 49 **(E)** 50

5. Ten points are equally spaced on the circumference of a circle, where two of the points are connected by a line segment, as shown below. Turner wants to choose two of the eight other points and draw a line segment connecting them so that this line segment does not intersect the other line segment. In how many ways can Turner do this?



   **(A)** 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

6. What is the smallest positive integer $n$ such that $n! + 1$ is not divisible by any integer between 2 and 9, inclusive?

   **(A)** 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

7. Four children and four adults are standing in a line, but every child insists on being in between a child and an adult. How many ways can the eight people be arranged to meet these demands? (Assume the children are identical and the adults are identical.)

   **(A)** 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

8. Justin has three weightless boxes and four pebbles, each of which has a weight of either 3, 4, or 5 ounces. He puts each pebble in one of the boxes such that each box has at least one pebble in it. If the weights of the boxes form an increasing arithmetic progression, what is the largest possible weight of the heaviest box, in ounces?

   **(A)** 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

9. Daniel has to walk one mile to complete his gym homework. He decides to split his path into quarters, where after each quarter, he randomly chooses to turn $90^\circ$ clockwise or counterclockwise with equal probability. If Daniel walks in a straight line each quarter, what is the probability that he will end up where he started after walking the mile?

   **(A)** $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{8}$ **(E)** $\frac{3}{4}$

10. How many orderings of the six numbers 1, 1, 2, 2, 3, and 6 are there such that the sum of the first three numbers is twice the sum of the last three numbers?

    **(A)** 9 **(B)** 18 **(C)** 27 **(D)** 36 **(E)** 72

11. In a plane, eight rays emanate from a point $P$ such that every two adjacent rays form an acute angle with measure $45^\circ$. Next, a line segment with a finite length is drawn in the plane. If the line segment intersects exactly $n$ of the rays, what is the sum of all possible values of $n$? (If the line segment passes through $P$, then $n = 8$.)

    **(A)** 13 **(B)** 14 **(C)** 17 **(D)** 18 **(E)** 23

12. At Test Academy, there are four classes, one on each of the four floors of the building. For each class, the class which is one floor above it has twice as many students and half the average grade of that class. If the average grade of all four classes combined is 20, what is the average grade of the class on the bottom floor?

    **(A)** 75 **(B)** 80 **(C)** 85 **(D)** 90 **(E)** 95

13. In regular hexagon $ABCDEF$, diagonals $\overline{AC}$ and $\overline{BF}$ intersect at a point $G$. If the area of $\triangle ABG$ is 2, what is the area of pentagon $CDEFG$?

    **(A)** 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

14. Each of 6 distinct positive integers is placed at each of 6 equally spaced points on the circumference of a circle. If the numbers on every two adjacent points are relatively prime, and the product of the numbers on every two diametrically opposite points is divisible by 3, what is the least possible sum of the 6 integers?

    **(A)** 12 **(B)** 25 **(C)** 26 **(D)** 29 **(E)** 32

15. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. A circle passes through $A$ and $B$ and intersects side $\overline{CD}$ at two points which trisect the side. What is the area of the circle?

    **(A)** $6\pi$ **(B)** $7\pi$ **(C)** $8\pi$ **(D)** $9\pi$ **(E)** $10\pi$

16. Let $x$ be a positive real number such that

$$\frac{1}{x - \frac{1}{x}} = \sqrt{x^2 + \frac{x^4}{4}}.$$

    What is the value of $x^2$?

    **(A)** $4 - 2\sqrt{2}$ **(B)** $\sqrt{2}$ **(C)** $2\sqrt{2} - 1$ **(D)** 2 **(E)** $\sqrt{2} + 1$

17. There exists a sequence $a_1, a_2, \ldots, a_6$ of positive integers such that for every term in the sequence, there exists another term in the sequence which is equal to that term. How many possible values of the product $a_1 a_2 \cdots a_6$ less than 1000 are there?

    **(A)** 36 **(B)** 37 **(C)** 38 **(D)** 39 **(E)** 40

18. Let $x$ and $y$ be real numbers such that

$$|x - |y - x|| = 1,$$
$$|y - |x - y|| = 2.$$

What is the largest possible value of $x + y$?

**(A)** 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

19. Six red balls and six blue balls are each numbered from 1 to 6. How many ways are there to form six pairs of one red ball and one blue ball such that the product of the two numbers on the balls in every pair is divisible by at least one of 2 and 3?

**(A)** 288 **(B)** 360 **(C)** 432 **(D)** 504 **(E)** 576

20. In trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$, $AB = 4$, and $AD = BC = 5$, let the angle bisector of $\angle ADC$ intersect the diagonal $\overline{AC}$ at a point $P$. If line $BP$ intersects the side $\overline{CD}$ at a point $Q$ such that $CQ = 8$, what is the area of trapezoid $ABCD$?

**(A)** 24 **(B)** 28 **(C)** 32 **(D)** 36 **(E)** 40

21. Bill and Ben each have 2 fair coins. In each turn, they flip all their coins at the same time, if they have any. If a coin lands heads, then the other person gets that coin. If a coin lands tails, then that coin stays with the same person. What is the probability that after Bill and Ben take exactly 3 turns, they each end up with 2 coins?

**(A)** $\frac{1}{4}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{7}{16}$ **(E)** $\frac{1}{2}$

22. In equilateral $\triangle ABC$, let points $D$ and $E$ be on lines $AB$ and $AC$, respectively, both on the opposite side of line $BC$ as $A$. If $CE = DE$, and the circumcircle of $\triangle CDE$ is tangent to line $AB$ at $D$, what is the degree measure of $\angle CDE$?

**(A)** 70 **(B)** 72 **(C)** 75 **(D)** 80 **(E)** 84

23. Let $a$, $b$, $c$, and $d$ be positive integers. If

$$\frac{a!}{b!} + \frac{c!}{d!} = \frac{2}{5},$$

what is the largest possible value of $a + b + c + d$?

**(A)** 10 **(B)** 18 **(C)** 26 **(D)** 34 **(E)** 42

24. Let $P(x)$ be a polynomial with degree 3 and real coefficients such that the coefficient of the $x^3$ term is 1, and $P(x)$ has roots $a$, $b$, and $c$ that satisfy

$$\frac{-(a+b)(b+c)(c+a)}{2022} = abc = 2021.$$

What is the minimum possible value of $|P(1)|$?

**(A)** 2019 **(B)** 2020 **(C)** 2021 **(D)** 2022 **(E)** 2023

25. Ryan has an infinite supply of slips and a spinner with letters $O$, $S$, and $T$, where each letter is equally likely to be spun. Each minute, Ryan spins the spinner randomly, writes on a blank slip the letter he spun, and puts it in a pile. Ryan continues until he has written all 3 letters at least once, at which point he stops. What is the probability that after he stops, he can form the words $OTSS$ and $TOST$ using 4 distinct slips from the pile? (Ryan may reuse slips he used for one word in forming the other.)

**(A)** $\frac{7}{54}$ **(B)** $\frac{13}{72}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{8}{27}$ **(E)** $\frac{1}{3}$

# DMC 10 E

## DO NOT OPEN UNTIL FRIDAY, January 21, 2022

****Administration on an earlier date will disqualify your results.****

- All the information needed to administer this exam is not contained in the non-existent DMC 10 Teacher's Manual. PLEASE READ THE MANUAL BEFORE FRIDAY, JANUARY 21, 2022.
- You may check your answers by using the answer key in the official opening post of this contest.
- The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

---

For more information about the DMC and our other competitions, please visit `https://detoasty3.github.io/dmc.html`.

Questions and comments about this competition should be sent to:

**DeToasty3**.