



Official Solutions

De Mathematics Competitions

1st Annual

DMC 8A

De Mathematics Competition 8A

Thursday, September 9, 2021



This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

People are encouraged to share copies of the problem booklet and official solutions with their friends and acquaintances to make the DMC Committee feel happy. All problems should be credited to the DMC (for example, "2020 DMC 10, Problem #21"). The publication, reproduction, or communication of the competition's problems or solutions for revenue-generating purposes is illegal.

Questions and complaints about this competition should be
sent by private message to

DeToasty3, karate7800, and pandabearcat.

The problems and solutions for this DMC 8 were prepared by the
DMC 8 Editorial Board under the direction of

bobthegod78, dc495, DeToasty3, HrishiP, karate7800, math31415926535, MathPirate101,
Michael595, pandabearcat, PhunsukhWangdu, pog, themathboi101, & treemath

Answer Key:

1. (B)	2. (E)	3. (B)	4. (B)	5. (D)
6. (C)	7. (B)	8. (A)	9. (C)	10. (E)
11. (D)	12. (C)	13. (D)	14. (C)	15. (A)
16. (B)	17. (E)	18. (B)	19. (E)	20. (C)
21. (D)	22. (E)	23. (C)	24. (C)	25. (A)

Problem 1:

(DeToasty3) Brenda's daily training starts at 6:10 PM and ends at 7:45 PM. If Brenda takes a 20-minute break in the middle of her training, for how many minutes does she train?

- (A) 65 (B) 75 (C) 85 (D) 95 (E) 105

Answer (B): There are 95 minutes from 6:10 PM to 7:45 PM. Brenda does not train for 20 of them, so the requested answer is $95 - 20 = \boxed{\text{(B) } 75}$ minutes. ■

Problem 2:

(HrishiP) What is the value of the expression $\frac{1^3 + 2^2 + 3^1}{1^2 + 2^1}$?

- (A) 1 (B) $\frac{6}{5}$ (C) 2 (D) $\frac{5}{2}$ (E) $\frac{8}{3}$

Answer (E): We have that $\frac{1^3 + 2^2 + 3^1}{1^2 + 2^1} = \frac{1 \times 1 \times 1 + 2 \times 2 + 3}{1 \times 1 + 2}$, so computation gives $\frac{1 + 4 + 3}{1 + 2} = \boxed{\text{(E) } \frac{8}{3}}$. ■

Problem 3:

(dc495) Jackie and Joe are shopping for school supplies at a store where pencils cost \$0.10 each, and pens cost \$0.25 each. Given that Jackie buys five pencils and two pens, and Joe spends the same amount of money as Jackie by buying only pens, how many pencils and pens did Jackie and Joe buy all together?

- (A) 10 (B) 11 (C) 12 (D) 13 (E) 14

Answer (B): Jackie spends $5 \times \$0.10 + 2 \times \$0.25 = \$0.50 + \$0.50 = \$1.00$, so Joe buys \$1.00 in pens. Since pens cost \$0.25, Joe buys $\$1.00 \div \$0.25 = 4$ pens and Jackie and Joe bought $5 + 2 + 4 =$ **(B) 11** all together. ■

Problem 4:

(pog & DeToasty3) Johnny has a recipe for hot chocolate which requires 2 grams of cocoa powder and 5 grams of milk. After adding 5 grams of milk, Johnny accidentally adds 3 grams of cocoa powder, so he adds additional milk in the same proportion as the recipe to balance the cocoa powder out. How many grams of additional milk does he add?

- (A) 1 (B) 2.5 (C) 5 (D) 7.5 (E) 10

Answer (B): Since Johnny adds 50% too much cocoa powder, he needs to add 50% more grams of milk, or $50\% \cdot 5 =$ **(B) 2.5**. ■

Problem 5:

(pog) Let $P = 2^2 + 3^2 + 4^2 + \cdots + 10^2$ and $Q = 1^2 + 2^2 + 3^2 + \cdots + 9^2$. What is $P - Q$?

- (A) 0 (B) 1 (C) 2 (D) 99 (E) 100

Answer (D): We can pair the terms from 2^2 to 9^2 , giving

$$10^2 + (9^2 - 9^2) + (8^2 - 8^2) + \cdots + (2^2 - 2^2) - 1^2 = 10^2 - 1^2,$$

so $P - Q =$ **(D) 99**. ■

Problem 6:

(pog & DeToasty3) Mike has a bag of sweets consisting of 30% pieces of chocolate, 45% pieces of toffee, and the rest pieces of caramel. After giving half of his caramel to his wife, he has 15 pieces of caramel left. How many pieces of toffee does he have?

- (A) 27 (B) 36 (C) 54 (D) 60 (E) 120

Answer (C): First, we find that Mike has $100\% - (30\% + 45\%) = 25\%$ pieces of caramel in his bag. Next, we have that half of the total pieces of caramel is equal to 15 pieces of caramel, so letting p be the total number of pieces of sweets in Mike's bag, we have that $12.5\% \cdot p = 15 \implies p = 120$. Finally, we find that the number of pieces of toffee in Mike's bag is equal to $45\% \cdot 120 = \boxed{\text{(C) } 54}$. ■

Problem 7:

(MathPirate101) A right triangle with a hypotenuse of length 5 inches and a leg of length 3 inches has the same area as a square. What is the side length in inches of the square?

- (A) 2 (B) $\sqrt{6}$ (C) 3 (D) $2\sqrt{3}$ (E) 4

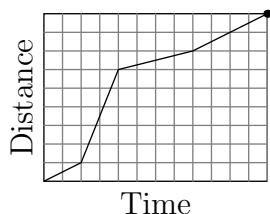
Answer (B): We start by finding the area of the triangle. Let the unknown length of the triangle be x . By the Pythagorean theorem, we have that $3^2 + x^2 = 5^2$, so $x^2 = 5^2 - 3^2 = 16$. Clearly x is positive, so $x = \sqrt{16} = 4$ and the area of the triangle is

$$\frac{3 \cdot 4}{2} = \frac{12}{2} = 6.$$

Let the side length in inches of the square be s , so $s^2 = 6$. Clearly s is positive as well, so $s = \boxed{\text{(B) } \sqrt{6}}$ inches. ■

Problem 8:

(DeToasty3) One day, Joseph hiked through a forest for two hours. The graph below shows his hike, indicating the general time and distance hiked. Which of the following represents a time m in minutes after the hike started where Joseph is moving the fastest?



- (A) 22 (B) 44 (C) 66 (D) 88 (E) 110

Answer (A): We start by noticing that the grid is split into 12 equal columns. We also know that Joseph's hike lasted for two hours, or 120 minutes. Thus, we have that each column is equal to 10 minutes. Next, we know that Joseph is moving the fastest when the slope of the graph is the greatest (which means that Joseph is traveling the largest distance in the shortest amount of time). We see that this happens between the second and fourth columns, so m is between 20 and 40. We see that the only answer which satisfies this is $m = \boxed{\text{(A) } 22}$. ■

Problem 9:

(DeToasty3) Given that $a = 0.78$ and $b = 78$, which of the following is equal to 7800?

- (A) $\frac{a}{b^2}$ (B) $\frac{a^2}{b}$ (C) $\frac{b^2}{a}$ (D) $\frac{b^3}{a}$ (E) $\frac{b^3}{a^2}$

Answer (C): Note that $7800 = 78 \times 100$. We have that $b = 78$ and $\frac{b}{a} = 100$, so $b \cdot \frac{b}{a} = 7800$, for an answer of $\boxed{\text{(C) } \frac{b^2}{a}}$. ■

Problem 10:

(treemath) At a station, 12 people board the front car and 40 people board the rear car of a two-car train. Later, some people switch from the rear car to the front car. At the next station, 16 people exit the front car and 20 people exit the rear car, after which there is an equal number of people in both cars. How many people switched cars?

- (A) 4 (B) 6 (C) 8 (D) 10 (E) 12

Answer (E): Note that there are $12 + 40 - 16 - 20 = 16$ people in the train, so there are 8 people in each car. Thus, before people exited the cars, there were $8 + 16 = 24$ people in the front car, and $24 - 12 = \boxed{\text{(E) } 12}$ people switched from the rear car to the front car.

OR

We have that after 20 people exit the rear car and 16 people exit the front car, there is an equal number of people in both cars. Thus, there were $20 - 16 = 4$ more people in the rear car than the front car after switching.

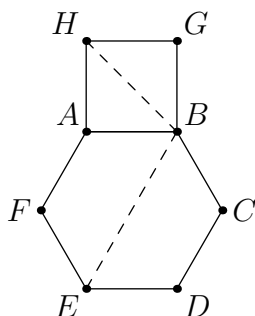
Let x be the number of people that switched from the rear car to the front car, so after switching there are $x + 12$ people in the front car and $40 - x$ in the rear car. Thus,

$$(40 - x) - (x + 12) = 4 \implies 28 - 2x = 4,$$

and solving for x gives $x = \text{(E) } 12$. ■

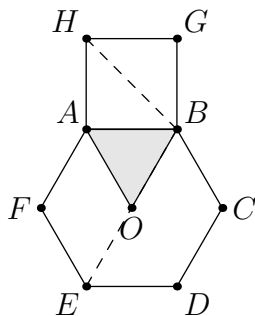
Problem 11:

(treemath) In the diagram below, square $ABGH$ lies outside of regular hexagon $ABCDEF$. What is the degree measure of $\angle EBH$?



- (A) 90 (B) 95 (C) 100 (D) 105 (E) 110

Answer (D): Note that $\angle EBH = \angle EBA + \angle ABH$. Since BH is the diagonal of square $ABGH$, we get that $\angle ABH = \frac{90^\circ}{2} = 45^\circ$. Let the center of the hexagon be O . Since $ABCDEF$ is a regular hexagon, triangle ABO is equilateral, so $\angle EBA = 60^\circ$. Thus, $\angle EBH = 60^\circ + 45^\circ$, so its degree measure is **(D) 105**. ■



Problem 12:

(dc495) While driving to work, Zach drove at a constant rate of 60 miles per hour on the highway and a constant rate of 40 miles per hour while driving through the local streets. If Zach drove for 160 miles, and he drove for twice as long on the highway as on the local streets, for how many minutes did he drive on the highway?

- (A) 90 (B) 105 (C) 120 (D) 135 (E) 150

Answer (C): If m is the number of hours that Zach drove for on the local streets, then he drove $40 \cdot m = 40m$ miles on the local streets and $60 \cdot 2m = 120m$ miles on the highway, for a total of $160m$ miles. We are given that he drove 160 miles, so $160m = 160 \implies m = 1$. Thus, Zach drove on the highway for 2 hours, or **(C) 120** minutes. ■

Problem 13:

(DeToasty3) Daniel has 3 coins, each of which is either a penny (a 1-cent coin), a nickel (a 5-cent coin), or a dime (a 10-cent coin). If Daniel trades one of his coins for a quarter (a 25-cent coin), the total value of his 3 coins will then be twice the total value of his original 3 coins. What is the least possible total value in cents of his original 3 coins?

- (A) 14 (B) 15 (C) 17 (D) 20 (E) 24

Answer (D): Let the coin Daniel trades for a quarter be worth p cents, and let his other coins be worth q and r cents, so

$$2(p + q + r) = 25 + q + r$$

$$2p + 2q + 2r = 25 + q + r$$

$$2p + q + r = 25.$$

If $p = 1$, then $q + r = 25 - 5 \times 1 = 23$, which is greater than the maximum possible value of $q + r$ —two dimes—and thus unattainable.

If $p = 5$, then $q + r = 25 - 2 \times 5 = 15$, which can be attained with a nickel and a dime.

If $p = 10$, then $q + r = 25 - 2 \times 10 = 5$. If q or r is a nickel or dime, then $q + r$ will be too large, but if q and r are both pennies, then $q + r$ will be too small, so this is unattainable.

Thus, our coins are two nickels and a dime in some order, so our answer is $5 + 5 + 10 =$ **(D) 20** cents.

Remark: A majority of the answer choices are unattainable with the types of coins given. ■

Problem 14:**(DeToasty3)** What is the value of the expression

$$\left(\frac{2^2}{1+3} - \frac{4^2}{3+5}\right) + \left(\frac{6^2}{5+7} - \frac{8^2}{7+9}\right) + \cdots + \left(\frac{98^2}{97+99} - \frac{100^2}{99+101}\right)?$$

- (A) -100 (B) -50 (C) -25 (D) -10 (E) -5

Answer (C): Note that, for any n , we have that

$$\frac{n^2}{(n-1) + (n+1)} = \frac{n^2}{2n} = \frac{n}{2}.$$

Thus, our expression is equal to $(1-2) + (3-4) + \cdots + (49-50)$. Each group in parentheses is equal to -1 . Since there are 50 numbers and 2 numbers in each group, there are 25 groups and the requested answer is equal to $25(-1) = \boxed{\text{(C)} -25}$. ■

Problem 15:

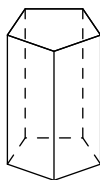
(treemath) Elaine takes a positive integer and divides it by 12, and the remainder is 4. Without knowing Elaine's integer, Hans infers that Elaine could have also divided by one of the following numbers and get a remainder of 4 with certainty. Which number is this?

- (A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Answer (A): Note that Elaine's positive integer is equal to $12x+4$, where x is a positive integer. Looking at the answer choices, we see that $12x+4 = 6(2x)+4$, so dividing Elaine's positive integer by 6 would give a quotient of $2x$ and a remainder of 4. Thus, our answer is $\boxed{\text{(A)} 6}$. ■

Problem 16:

(treemath) A pentagonal prism has two pentagonal faces as bases and five rectangles as faces around the prism. Billy wants to color each of the seven faces of a pentagonal prism so that any two faces that share an edge have different colors. What is the smallest possible number of distinct colors that Billy needs to do this?



(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Answer (B): We will first show that it is impossible to color the pentagonal prism with only three colors. For each of the five rectangular faces, we will choose one of them and color it red, for instance. Then, the two rectangular faces adjacent to that face must not be red. If the two faces are different colors, then we have used up three colors and the two pentagonal faces must be a fourth color. Otherwise, if the two faces are both blue, for instance, then the remaining two rectangular faces must both be non-blue and different colors. Once again, we need a third color to accomplish this, and there needs to be a fourth color for each of the two pentagonal faces. Thus, the smallest possible number of colors needed is **(B) 4**. ■

Problem 17:

(pog) Suppose that $a \clubsuit b$ means $a^2 - ab + b^2$. What is the value of x^2 if

$$(x + 4) \clubsuit (x - 4) = 75?$$

(A) 18 (B) 21 (C) 24 (D) 25 (E) 27

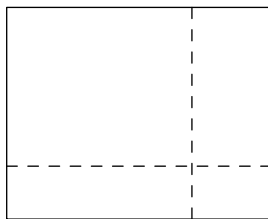
Answer (E): Note that $(a - b)^2 = a^2 - 2ab + b^2$, so $a \clubsuit b = (a - b)^2 + ab$. Thus, we get that

$$\begin{aligned} (x + 4) \clubsuit (x - 4) &= ([x + 4] - [x - 4])^2 + (x + 4)(x - 4) \\ &= 8^2 + (x + 4)(x - 4) \\ &= 64 + (x + 4)(x - 4) \\ &= 75. \end{aligned}$$

By difference of squares, $(a + b)(a - b) = a^2 - b^2$, so $(x + 4)(x - 4) = x^2 - 4^2 = x^2 - 16$. From earlier, we have that $64 + (x + 4)(x - 4) = 75$, so $64 + (x^2 - 16) = 75$. Subtracting 64 from both sides of $64 + (x^2 - 16) = 75$ gives $x^2 - 16 = 11$, so $x^2 = \mathbf{(E) 27}$. ■

Problem 18:

(DeToasty3) A rectangle is cut using only vertical or horizontal cuts, like the cuts shown below. What is the smallest possible number of cuts needed to cut the rectangle into 36 pieces?



- (A) 5 (B) 10 (C) 12 (D) 18 (E) 35

Answer (B): Notice that if we have m horizontal cuts and n vertical cuts, then we cut the rectangle into $m + 1$ rows and $n + 1$ columns, giving a total of $(m + 1)(n + 1)$ pieces. Thus, we must have that $(m + 1)(n + 1) = 36$, where m and n are non-negative integers. We wish to find the smallest possible value of $m + n$. Checking factor pairs of 36, we see that $m + 1 = n + 1 = 6$ gives the smallest possible value of $m + n$, where $m = n = 5$. Thus, our answer is $5 + 5 =$ **(B) 10**. ■

Problem 19:

(treemath) A certain four-digit number satisfies the following properties:

- The thousands digit is equal to 4.
- The sum of the last two digits is 17.
- The number is divisible by 18.

What is the sum of the hundreds digit and the ones digit of this number?

- (A) 6 (B) 8 (C) 10 (D) 12 (E) 14

Answer (E): We are given that the sum of our number's last two digits is 17, so the only possibilities for our number are $4\diamond 98$ or $4\diamond 89$, where \diamond is the unknown digit.

If a number is divisible by 18, that means it is also divisible by 2 and 9. Since our number is divisible by 2, it is even, so our number must be $4\diamond 98$.

Since our number is divisible by 9, the sum of its digits is divisible by 9, so

$$4 + \diamond + 9 + 8 = 21 + \diamond$$

is a multiple of 9. Since \diamond is a digit, we get that $21 + \diamond = 27$, so solving gives $\diamond = 6$ and our number is 4698. Thus, our answer is $6 + 8 =$ **(E) 14**. ■

Problem 20

(DeToasty3) Three boys named Ryu, Kai, and Naruto, as well as two girls named Haruka and Ayaka, sit in a straight line in a randomly chosen order. What is the probability that exactly one boy and one girl are sitting in between Ryu and Kai?

- (A) $\frac{1}{15}$ (B) $\frac{1}{10}$ (C) $\frac{2}{15}$ (D) $\frac{1}{6}$ (E) $\frac{1}{5}$

Answer (C): There must be two spots between Ryu and Kai, so we have

$R _ _ K _$

$_ R _ _ K$

$K _ _ R _$

$_ K _ _ R$

as our possible seatings. For each of the $3! \cdot 4 = 24$ seatings where there are two spots between Ryu and Kai, there is a $\frac{2}{3}$ probability that exactly one boy and one girl are sitting in between Ryu and Kai, since if Naruto sits in either of the two spots between Ryu and Kai (out of three possible spots), the other person sitting between Ryu and Kai will be a girl no matter what. Thus, there are $24 \cdot \frac{2}{3} = 16$ possible seatings and our final probability is $\frac{16}{5!} = \frac{16}{120} = \text{(C)} \frac{2}{15}$. ■

Problem 21

(DeToasty3) Kyle has five blocks in a row, as shown below, where each letter represents a number, and not all five numbers are equal.

[A] [B] [C] [D] [E]

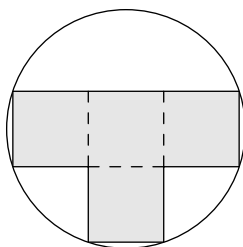
Kyle notices that the sums of the numbers on any three consecutive blocks are equal. In which of the following arrangements of the five blocks is it necessarily true that the sums of the numbers on any three consecutive blocks are equal?

- (A) [A] [C] [D] [B] [E] (B) [B] [A] [C] [D] [E] (C) [C] [A] [B] [E] [D]
 (D) [D] [E] [C] [A] [B] (E) [E] [A] [B] [D] [C]

Answer (D): Inspecting the given arrangement, we must have that the sum of the numbers on blocks A, B, and C is equal to the sum of the numbers on blocks B, C, and D, which means that the numbers on blocks A and D are equal. Similarly, we must have that the sum of the numbers on blocks B, C, and D is equal to the sum of the numbers on blocks C, D, and E, which means that the numbers on blocks B and E are equal. From here, it becomes clear that the only arrangement in the list of answers which works is **(D)** **D** **E** **C** **A** **B**.

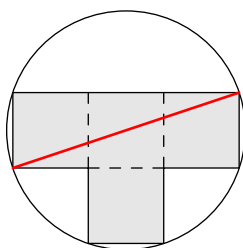
Problem 22

(DeToasty3) A T-shaped block is inscribed in a circle, as shown below. The T-shaped block is made up of four congruent squares. Given that the area of the circle is 78π , which of the following is closest to the area of the T-shaped block?



- (A) 105 (B) 110 (C) 115 (D) 120 (E) 125

Answer (E): Since the area of the circle is 78π , the radius of the circle is $\sqrt{78}$. The red line below is a diameter of the circle, so it is equal to $2\sqrt{78}$.



Letting the side length of one of the squares be s , by the Pythagorean theorem, we get

$$\begin{aligned}(3s)^2 + s^2 &= (2\sqrt{78})^2 \\ 9s^2 + s^2 &= (2\sqrt{78})^2 \\ 10s^2 &= 312.\end{aligned}$$

Thus, since the T-shaped block is made of four squares, the area of the T-shaped block is equal to $4s^2$. Note that $4s^2 = \frac{4}{10} \cdot 10s^2 = \frac{4}{10} \cdot 312 = 124.8$, which is closest to **(E) 125**. ■

Problem 23

(dc495) At one of Sarah's parties, each man shook hands exactly once with everyone other than his own wife, and no two women shook hands with each other. Given that 6 married couples attended, how many handshakes took place among these 12 people?

- (A) 15 (B) 36 (C) 45 (D) 60 (E) 66

Answer (C): We have two types of valid pairs: two men shake hands with each other, and a man and a woman who are not married shake hands with each other. For the first type of pair, we have six men in total, so this gives us 6 choices for the first man and 5 choices for the second man. However, each handshake is counted twice, so we divide by 2, for a total of $\frac{6 \cdot 5}{2} = 15$ handshakes. For the second type of pair, we have six men in total, and for each man, there are five women who are not the man's wife, so this gives us $6 \cdot 5 = 30$ handshakes. Adding these up, we find that our answer is $15 + 30 =$ **(C) 45** handshakes. ■

Problem 24

(treemath) Don and Doug each draw 5 cards from a deck of 10 cards numbered from 1 to 10. The sum of Don's cards is 1 less than the sum of Doug's cards, and Doug drew all of the perfect squares. What is the sum of all cards Doug could **not** have drawn?

- (A) 17 (B) 20 (C) 27 (D) 28 (E) 32

Answer (C): Note that all 10 cards sum to $1 + 2 + 3 + \cdots + 10 = 55$. Let the sum of Don's cards be x , so the sum of Doug's cards is $x + 1$. Then $x + (x + 1) = 55$, so solving for x gives $x = 27$. Thus, Don's cards sum to 27 and Doug's cards sum to 28.

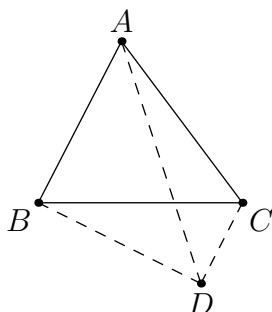
We know that Doug has cards $1^2 = 1$, $2^2 = 4$, and $3^2 = 9$, so his other two cards sum to $28 - (1 + 4 + 9) = 14$. The possible pairs of cards that sum to 14 are

$$(10, 4), (9, 5), (8, 6) \quad \text{and} \quad (7, 7).$$

However, Doug already used the 4 and 9, and he cannot use 7 twice, so his last two cards must be (6, 8). Thus, Doug's cards are (1, 4, 6, 8, 9), so the cards Doug could not have drawn have a sum of $2 + 3 + 5 + 7 + 10 =$ **(C) 27**. ■

Problem 25

(DeToasty3) In triangle ABC , let D be the point on the opposite side of line BC as A such that $\overline{AB} \parallel \overline{CD}$. Given that $AB = BD = 8$, $BC = 9$, and $CD = 4$, what is AD^2 ?



- (A) 126 (B) 130 (C) 134 (D) 138 (E) 142

Answer (A): Let E be the intersection of segments \overline{AD} and \overline{BC} . Since $\overline{AB} \parallel \overline{CD}$ and $\angle AEB = \angle DEC$, we have that $\triangle AEB \sim \triangle DEC$. We are also given that $BC = 9$, so by similarity ratios, we find that $BE = 6$ and $CE = 3$. Let M be the midpoint of segment \overline{AD} . Since $AB = BD$, we have that $\triangle ABD$ is isosceles, so $\overline{AD} \perp \overline{BM}$. Note that $AE : DE = 2 : 1$ and $AM : DM = 1 : 1$, so it follows that $AM : EM = 3 : 1$. Let $EM = x$. By the Pythagorean Theorem, we have that

$$AB^2 - AM^2 = BE^2 - EM^2 \implies 64 - 9x^2 = 36 - x^2 \implies x^2 = \frac{7}{2}.$$

Finally, since $AM = 3x$, we have that $AD = 6x$, so $AD^2 = 36x^2 = 36 \cdot \frac{7}{2} = \boxed{\text{(A) } 126}$.

