De Mathematics Competitions

3rd Annual

DMC 12 B

Friday, October 28, 2022

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. For each question, only one answer choice is correct.
3. Mark your answer to each problem on the DMC 12 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, rulers, compasses, and erasers are allowed as aids. Prohibited materials include calculators, smartwatches, phones, computing devices, protractors, and graph paper. No problems on the competition will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the competition, your non-existent proctor will not ask you to record certain information on the answer form.
8. You will have 75 minutes to complete the competition. You can discuss only with people that have already taken the competition in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

The DMC Committee reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

The publication, reproduction or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

Students who score well on this DMC 12 may or may not be invited to the 2023 DIME. More details about the DIME and other information are on the back page of this test booklet.

1. What is the value of $\dfrac{2022! \cdot 2019!}{2020! \cdot 2021!}$?

   (A) $\dfrac{1009}{1010}$ **(B)** $\dfrac{2020}{2021}$ **(C)** $\dfrac{2021}{2022}$ **(D)** $\dfrac{1011}{1010}$ **(E)** $\dfrac{2023}{2022}$

2. Taiki and Richard are playing frisbee. Each round, players earn 3 points for winning and 1 point for a tie. The game ends when someone gets 20 points. At the end of the game, Richard had lost 3 more rounds than Taiki. How many points did Richard have at the end of the game?

   (A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 16

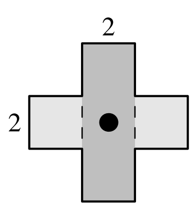
3. Ken writes 10 positive integers onto a sheet of paper. Joe then asks the following questions:

   - How many of the numbers on your paper are less than 2?
   - How many of the numbers on your paper are greater than 2?
   - How many of the numbers on your paper are equal to 4?

   Ken truthfully answers 3 to every question. What is the sum of the numbers on Ken's paper?

   (A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

4. Two congruent rectangles each with height 2 are stacked on top of each other. The top sheet is then rotated 90° about its center, resulting in the 12-sided polygon as shown.



   If the area of this polygon is 30, what is the area of one of the rectangles?

   (A) 12 **(B)** $6\sqrt{6}$ **(C)** 15 **(D)** $5\sqrt{10}$ **(E)** 17

5. Let a, b, and c be consecutive positive integers, and let p, q, and r also be consecutive positive integers, both not necessarily in order. Given that $a \cdot p = 161$ and $b \cdot q = 189$, what is $c \cdot r$?

   (A) 128 **(B)** 144 **(C)** 160 **(D)** 176 **(E)** 192

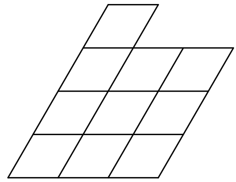
6. If x and y are real numbers such that

$$\sqrt{x\sqrt[3]{y}} = 4 \quad \text{and} \quad \sqrt{y\sqrt[3]{x}} = 9,$$

   what is xy?

   (A) 36 **(B)** 72 **(C)** 108 **(D)** 144 **(E)** 216

7. How many parallelograms are in the diagram below?



   (A) 18 **(B)** 31 **(C)** 36 **(D)** 39 **(E)** 40

8. Ayaka and Judo are each given a number greater than 23 and less than 35. Ayaka is also told the units digit of Judo's number, while Judo forgets the units digit of his own number. They are told that both of their numbers are greater than 23 and less than 35. They know where each digit they remember is located in their two numbers and that they could have the same number.

   - **Ayaka:** I don't know whether my number is less than your number.
   - **Judo:** Oh, then I know whose number is larger.
   - **Ayaka:** Then, the positive difference between our numbers is 5.

   What is the sum of Ayaka and Judo's numbers?

   (A) 53 **(B)** 55 **(C)** 59 **(D)** 61 **(E)** 63

9. The number of positive divisors of the number $(12!)^{26}$ can be written as

$$\underline{1}\,\underline{3}\,\underline{2}\,\underline{A}\,\underline{0}\,\underline{3}\,\underline{7}\,\underline{6}\,\underline{6}\,\underline{B},$$

where A and B are digits. What is the ordered pair (A, B)?

(A) $(1,7)$ **(B)** $(2,6)$ **(C)** $(6,3)$ **(D)** $(7,6)$ **(E)** $(9,8)$

10. A set S consists of 20 consecutive even integers, one of which is equal to 20. Ryan is finding the sum of the 20 elements of S, but he subtracts 20 instead of adding 20, resulting in an incorrect sum that is a multiple of 100. What is the sum of all possible values of the largest element of S?

(A) 108 **(B)** 122 **(C)** 136 **(D)** 150 **(E)** 164

11. Aki has 24 identical apples, 5 of which are rotten. He randomly shares them with Yui, Yuzu, and Yuka so that each of the four people gets at least one non-rotten apple. What is the probability someone gets all 5 rotten apples?

(A) $\frac{1}{32}$ **(B)** $\frac{1}{28}$ **(C)** $\frac{1}{21}$ **(D)** $\frac{1}{18}$ **(E)** $\frac{1}{14}$

12. A line passes through the point $A(5,4)$ and has slope -8. A second line passes through the point $B(1,6)$ and intersects the first line at a point C, equidistant from A and B. What is the slope of the second line?

(A) $\frac{1}{3}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{6}{11}$ **(E)** $\frac{4}{7}$

13. Given that

$$\log_2 5 \cdot \log_3 6 \cdot \log_4 7 \cdot \log_5 8 = \log_2 a + \log_3 a,$$

what is the nearest integer to a?

(A) 15 **(B)** 17 **(C)** 19 **(D)** 21 **(E)** 23

14. A rectangle has perimeter 36. The rectangle is split into three smaller rectangles with dimensions 9-by-4, 6-by-5, and m-by-n. What is $m + n$?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

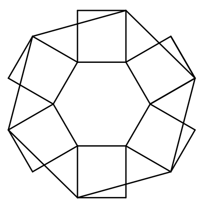
15. Let $w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$. How many ordered pairs of positive integers (a, b) each at most 6 are there such that $|3w^a + 4z^b| = 5$?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 12 **(E)** 16

16. There are 2022 members in a math tournament, where 999 members are girls, and the rest are boys. The members are split into 674 groups of 3. For every two members in a group, if at least one of them is a girl, they shake hands. Otherwise, they do not. A member is *handy* if they shake hands with both members in their group. Let N be the maximum number of handy members in the math tournament. What is the sum of the digits of N?

(A) 6 **(B)** 15 **(C)** 16 **(D)** 22 **(E)** 23

17. In the figure below, six congruent rectangles are glued to each of the sides of a regular hexagon with side length 2, and six of the vertices of the rectangles are connected to form a regular hexagon with side length 4. The length of a side of one of the rectangles not equal to 2 can be written as $\sqrt{m} - \sqrt{n}$, where m and n are positive integers. What is $m + n$?



(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

18. Let $\triangle ABC$ be inscribed in a circle with radius 6. If

$$\sin(\angle ABC) \cdot \sin(\angle BCA) \cdot \sin(\angle CAB) = \frac{17}{24},$$

what is the area of $\triangle ABC$?

(A) 48 **(B)** 51 **(C)** 54 **(D)** 57 **(E)** 60

19. Let $P(x)$ be a polynomial with degree 3 and roots r, s, and t with sum 25 such that the coefficient of the x^3 term is 1, and

$$(r+s)(s+t)(t+r) = 2500 \quad \text{and} \quad \left(\frac{1}{r}+\frac{1}{s}\right)\left(\frac{1}{s}+\frac{1}{t}\right)\left(\frac{1}{t}+\frac{1}{r}\right) = 100$$

are satisfied. If the constant term of $P(x)$ is positive, the value of $P(1)$ is equal to $\frac{m}{n}$ for relatively prime positive integers m and n. What is $m + n$?

(A) 406 **(B)** 407 **(C)** 408 **(D)** 409 **(E)** 410

20. Let parallelogram $ABCD$ have $BC = 5$, $\angle ABC < 90^\circ$, and $\angle ACB > 90^\circ$. Let line AD and side $\overline{CD}$ intersect the circle passing though A, B, and C at $P \neq A$ and $Q \neq C$, respectively. If $CP = 10$ and $CQ = 4$, what is AP?

(A) $\frac{48}{7}$ **(B)** 7 **(C)** $\frac{36}{5}$ **(D)** $\frac{15}{2}$ **(E)** 8

21. For each positive integer n, let $f_1(n) = n!$, and for $k \geq 2$, let

$$f_k(n) = f_{k-1}(1) \cdot f_{k-1}(2) \cdot \ldots \cdot f_{k-1}(n).$$

Let N be the largest integer such that $f_4(10)$ is divisible by 2^N. What is the sum of the digits of N?

(A) 4 **(B)** 15 **(C)** 16 **(D)** 20 **(E)** 21

22. Freida writes down the number 1 on a blackboard. Then, she repeatedly picks a written number n at random, and she writes down either $2n$, $3n$ or $4n$, where each number is equally likely to be written. On average, what is the sum of the first 20 numbers she writes down?

(A) 1330 **(B)** 1540 **(C)** 1600 **(D)** 1760 **(E)** 1960

23. In acute $\triangle ABC$ with $AC < BC$, the perpendicular bisector of $\overline{AB}$ meets lines AB, BC, and AC at D, E, and F, respectively. If $AD = 5$, $BE = 13$, and the area of $\triangle ABC$ is 14 units greater than that of $\triangle ADF$, what is AF^2?

(A) 650 **(B)** 701 **(C)** 754 **(D)** 809 **(E)** 866

24. Define a sequence of real numbers $a_1, a_2, \ldots$ by $a_1 = 2$, $a_2 = 9$, and $a_{n+2} = 2a_{n+1} + a_n$ for all positive integers n. If

$$\sum_{k=1}^{\infty} \frac{a_k a_{k+1}}{(a_{k+1}^2 - a_k^2)^2} = \frac{1}{m},$$

where m is a positive integer, what is the sum of the digits of m?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

25. Convex quadrilateral $ABCD$ has acute angles $\angle A$ and $\angle D$, obtuse angles $\angle B$ and $\angle C$, $AB = BC = CD = 10$, and $AC + BD = 24\sqrt{2}$. Let M and N be the midpoints of $\overline{AC}$ and $\overline{BD}$, respectively. If $MN = 5$, what is AD^2?

(A) 288 **(B)** 320 **(C)** 384 **(D)** 432 **(E)** 486

DMC 12 B

DO NOT OPEN UNTIL FRIDAY, October 28, 2022

****Administration on an earlier date will disqualify your results.****

- All the information needed to administer this exam is not contained in the non-existent DMC 12 Teacher's Manual. PLEASE READ THE MANUAL BEFORE FRIDAY, OCTOBER 28, 2022.
- Send **DeToasty3**, **HrishiP**, and **pog** a private message on Art of Problem Solving submitting your answers to the DMC 12. Alternatively, you may submit your answers via a Google Form linked in the opening post.
- The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

For more information about the DMC and our other competitions, please visit `https://detoasty3.github.io/dmc.html`.

Questions and comments about this competition should be sent to:

DeToasty3.