Another Mock Contest

Problems 1-25

Name ______________________________

DO NOT BEGIN UNTIL YOU ARE INSTRUCTED TO SO.

This section of the competition consists of 25 problems. You will have 75 minutes to complete all the problems. You are not allowed to use calculators, books or other aids during this round. Calculations may be done on scratch paper. All answers must be complete, legible and simplified to lowest terms. Record only final answers in the blanks in the lefthand column of the competition booklet. If you complete the problems before time is called, use the remaining time to check your answers.

In each written round of the competition, the required unit for the answer is included in the answer blank. The plural form of the unit is always used, even if the answer appears to require the singular form of the unit. The unit provided in the answer blank is the only form of the answer that will be accepted.

The problems and solutions for this competition were prepared by the AMC Editorial Board under the direction of:

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1. ________ Find the value of $2^0 + 2^1 + 2^0 + 2^2$.

2. ________ ways Penny has 2022 indistinguishable pennies. She colors some (but not all) of the pennies red, and she colors the rest of the pennies blue. In how many ways can she do this such that she has more red pennies than blue pennies?

3. ________ Find the sum of all two-digit multiples of 5 whose digits sum to 8.

4. ________ hits Karate and Judo are fighting in the parking lot. The moment a person gets hit 50 times, they lose the fight. If Karate has already hit Judo 37 times, and Judo has already hit Karate 40 times, what is the largest number of total additional hits possible without one person losing the fight after the last of these hits?

5. ________ ways How many ways can 2022 be written as the sum of two nonnegative integers whose digits are only 0s and 1s?

6. ________ gallons Every minute, 3 gallons of water are drained from a pool. After 40 minutes, the amount of water that has been drained is twice the amount of water still in the pool. How many gallons of water were in the pool before any water was drained?

7. ________ Find

$$(2022^2 - 2022) - (2021^2 - 2021).$$

8. ________ Two rectangles with whole number side lengths have equal areas and perimeters of 18 and 22. What is the area of one of the rectangles?

9. ________ Rectangle $ABCD$ has side lengths $AB = 52$ and $BC = 10$. Let M be the midpoint of AB, and let N be the midpoint of DM. Find the length of $\overline{AN}$.

10. ________ Let R, A, C, K, E, and T be distinct single-digit positive integers. If

$$A \cdot T = 20,$$
$$R \cdot E \cdot K \cdot T = 30,$$
$$R \cdot A \cdot C \cdot K \cdot E \cdot T = 720,$$

what is $A + C + K$?

11. ________ milliliters A 2-liter mixture of water and acid contains 60% acid. When some of the water is removed, the mixture now contains 80% acid. How many milliliters of water were removed?

12. ________ Rectangle $ABCD$ has perimeter 1820. Rectangle $EFGH$ is constructed such that $EF = AB - 1$ and $FG = BC - 1$. Find the difference between the areas of $ABCD$ and $EFGH$.

13. ________ ordered triples How many ordered triples of (not necessarily distinct) primes (p, q, r) each less than 25 are there such that $p + q = r$?

14. ________ An arithmetic sequence of positive even integers with 100 terms has exactly four multiples of 66. Find the least possible value of the median of the terms in this sequence.

15. ________ Bill has one penny. He plays a game where he wagers a penny on a positive integer from 1 to 6 (inclusive). Then, he rolls three fair standard six-sided dice. If Bill's number appears on 1, 2, or 3 dice, he will win 1, 2, or 3 pennies, respectively (along with his original penny). Otherwise, if Bill's number appears on none of the three dice, he will not get his original penny back. The expected number of pennies Bill will have after the game is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

16. ________ solutions How many real solutions (x, y) are there to the following system?

$$(x^2 - 4y^2)(4x^2 - y^2) = 0$$

$$2x + 4y = 7$$

17. ________ Let f be a function such that for any real number x,

$$f(x) + f(2x) = f(x) \cdot f(2x).$$

If $f(256) = 24$, find $f(1)$.

18. ________ Let $\mathcal{P}(S)$ denote the product of the elements of a set S. Let

$$A_1, A_2, A_3, \ldots$$

denote the distinct non-empty subsets of set $A = \{a, b, c\}$, where a, b, and c are positive integers and $a < b < c$. Given that

$$\mathcal{P}(A_1) + \mathcal{P}(A_2) + \mathcal{P}(A_3) + \cdots = 99,$$

what is the value of c?

19. ________ Find the largest positive integer n such that $\dfrac{101^n + 1}{n!}$ is an integer.

20. ________ values Let (a, b) be an ordered pair of positive real numbers such that

$$ab^3 + a^3b = 78.$$

How many possible integer values of $a \cdot b$ are there?

21. ________ A ray from a point P goes through a circle with center O at points A and B, with $PB > PA$. Line segment $\overline{OP}$ intersects the circle at point C. If $PA = 93$, $PC = 51$, and $\angle PBO = 60°$, find the radius of the circle.

22. ________ Circles ω_1 and ω_2 have centers O_1 and O_2, respectively, where the radius of ω_2 is 8, and $O_1O_2 = 20$. The circles intersect at two distinct points, one of which is P. The line PO_2 intersects ω_2 at A, distinct from P. Given that there exists a point B on ω_1 such that $BPAO_1$ is a parallelogram, the area of $BPAO_1$ is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

23. ________ colorings Six distinct points lie in a straight line. First, Cindy draws a red line segment, choosing two of the six points to be the endpoints. Next, Sophia draws a blue line segment, choosing two of the six points to be the endpoints (these can include Cindy's points). Finally, Michael shades any points not on either line segment yellow. If any point on only the red segment is red, any point on only the blue segment is blue, and any point on both the red and blue line segments is purple, find the number of possible colorings of the points. (One coloring is red, purple, blue, blue, yellow, yellow.)

24. ________ Lines with slopes n, $\frac{1}{n}$, and $-\frac{1}{n}$ have y-intercepts $(0, 75)$, $(0, 602)$, and $(0, 700)$, respectively, and all meet at a single point. The value of n can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

25. ________ Let acute $\triangle ABC$ have $AB = 9$, $AC = 16$, and $\angle ACB = 30^\circ$. Let O be the center of the circumcircle of $\triangle ABC$, and let A' be the reflection of A over point O. Construct line ℓ which passes through O and is parallel to side $\overline{AB}$. Let P be the intersection of ℓ and the bisector of $\angle A'AB$. The perimeter of $ABPA'C$ can be written as $m\sqrt{n} + p$, where m, n, and p are positive integers, and n is not divisible by the square of any prime. Find $m + n + p$.