Geometry AMC Series

1st Annual

GAMC

Saturday, July 17, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem in your GAMC Private Message with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than writing utensils, blank scratch paper, rulers, compasses, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your non-existent proctor will not ask you to record certain information on the answer form.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only after the problems have been posted in the public discussion forum after the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

The Committee on the Geometry AMC Series (CGAMCS) reserves the right to re-examine students before deciding whether to grant official status to their scores. The Committee also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score well on this GAMC will not be invited to the 2021 GIME (Geometry Invitational Mathematics Examination). More details about the GIME and other information are on the back page of this test booklet.

The publication, reproduction or communication of the problems or solutions of the GAMC during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

1. What is the area of the largest circle which can fit entirely within the interior of a semicircle with diameter 24?

(A) 24π **(B)** 36π **(C)** 48π **(D)** 72π **(E)** 144π

2. Line ℓ intersects circle ω at points A and B such that $AB = 8$. If the radius of ω is 8, what is the measure of the minor arc $\widehat{AB}$ of ω? (Recall that the measure of a minor arc is the measure of $\angle AOB \leq 180^\circ$, where O is the center of the circle.)

(A) 30° **(B)** 60° **(C)** 90° **(D)** 120° **(E)** 180°

3. Two circles ω_1 and ω_2 have the same center and radii r_1 and r_2, respectively. Suppose that the area of the region between the circles is 28π, and $\frac{r_2}{r_1} = \frac{4}{3}$. What is r_1?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 12

4. Rectangle $ABCD$ has $AB = 10$ and $AD = 24$. Point P is inside $ABCD$ so that the areas of $\triangle PAB$, $\triangle PBC$, $\triangle PCD$, and $\triangle PAD$ are all equal. What is the length AP?

(A) 13 **(B)** 15 **(C)** 20 **(D)** 24 **(E)** 26

5. Consider an 8×8 checkerboard with side length 8, in which each unit square is either black or white, and no two unit squares sharing an edge are the same color. What is the greatest distance between a point in a black square and a point in a white square?

(A) $7\sqrt{2}$ **(B)** 10 **(C)** $\sqrt{113}$ **(D)** $8\sqrt{2}$ **(E)** $\sqrt{130}$

6. Points A, B, C, D on the circumference of a circle satisfy $AB = 15$, $AD = 20$, $CD = 24$, and $BD = 25$. What is the length BC?

(A) $3\sqrt{3}$ **(B)** 7 **(C)** 8 **(D)** 10 **(E)** 12

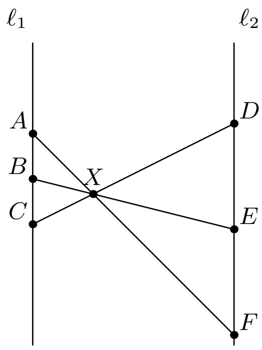
7. Let $\overline{AB}$ be a diameter of circle ω with radius 1. Let P be a point on ω. What is the maximum possible value of $AP \cdot BP$?

(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** $2\sqrt{3}$ **(E)** 4

8. In triangle ABC, let P be the foot of the perpendicular from B to side $\overline{AC}$. Let D be a point on the extension of segment $\overline{BP}$ past B. If $AB = \sqrt{55}$, $AD = \sqrt{61}$, and $CD = \sqrt{79}$, what is the length BC?

(A) $\sqrt{37}$ **(B)** $\sqrt{73}$ **(C)** $\sqrt{85}$ **(D)** $\sqrt{116}$ **(E)** $\sqrt{195}$

9. Lines ℓ_1 and ℓ_2 are parallel. Point X lies between ℓ_1 and ℓ_2 such that it is $\frac{1}{5}$ from ℓ_1 to ℓ_2. Segments $\overline{AF}$, $\overline{BE}$, and $\overline{CD}$ pass through X, with A, B, C on ℓ_1 and D, E, F on ℓ_2. If the areas of $\triangle ABX$ and $\triangle XDE$ are 2 and 48, respectively, what is $\frac{BC}{EF}$?



(A) $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{3}{8}$

10. Two real numbers x and y are chosen uniformly at random from the interval $[0, 1]$. What is the probability that

$$\left|\left|x - \frac{1}{2}\right| + \left|y - \frac{1}{2}\right|\right| \leq \frac{3}{4}?$$

(A) $\frac{1}{2}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{4}{5}$ **(E)** $\frac{7}{8}$

11. In a regular hexagon with side length 1, label the vertices with A, B, C, D, E, and F, not necessarily in that order. What is the maximum possible value of

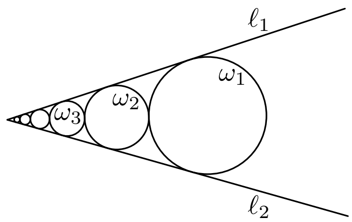
$$AB + BC + CD + DE + EF + FA?$$

(A) 6 **(B)** $3 + 4\sqrt{3}$ **(C)** $7 + 2\sqrt{3}$ **(D)** $4 + 4\sqrt{3}$ **(E)** $6 + 3\sqrt{3}$

12. Bela is out for a walk on a field. She starts from her house, walks 1 mile straight, then turns 60° clockwise as viewed from above. After walking another mile in this direction, she turns 90° clockwise as viewed from above. She then walks another mile in this direction. At her current position, how many miles away from home is Bela?

(A) $\sqrt{3}$ **(B)** $\sqrt{4-\sqrt{3}}$ **(C)** $\sqrt{3}+1$ **(D)** $\sqrt{10-\sqrt{3}}$ **(E)** $3+\sqrt{3}$

13. An infinite sequence of circles $\omega_1, \omega_2, \ldots$ is such that ω_k is externally tangent to ω_{k+1} for $k = 1, 2, \ldots$, and the circles are all tangent to two lines ℓ_1 and ℓ_2. Suppose that the radius of ω_1 is 6 and the radius of ω_3 is 3. What is the total area of the circles?



(A) 36π **(B)** 48π **(C)** 60π **(D)** 72π **(E)** 84π

14. In quadrilateral $ABCD$, triangles ABC and ACD are similar, with the vertices in that order. Let the diagonals $\overline{AC}$ and $\overline{BD}$ intersect at P. Suppose that $\frac{AB}{AD} = \frac{4}{9}$ and $\frac{AP}{CP} = \frac{1}{5}$. If the area of $\triangle PDC$ is 15, what is the area of quadrilateral $ABCD$?

(A) 26 **(B)** $\dfrac{82}{3}$ **(C)** 28 **(D)** 30 **(E)** $\dfrac{94}{3}$

15. In isosceles triangle ABC, $AC = BC = 15$. Let D be the foot of the perpendicular from C to side $\overline{AB}$. Point E is on side $\overline{BC}$ such that if segment $\overline{AE}$ intersects segment $\overline{CD}$ at P, then $AP = 3\sqrt{10}$ and $EP = \sqrt{10}$. What is the area of $\triangle ABC$?

(A) 30 **(B)** $\dfrac{21\sqrt{10}}{2}$ **(C)** $12\sqrt{10}$ **(D)** 60 **(E)** 90

16. Let ω be a circle with radius 4, and suppose that P is a point on the circumference of ω. Circle ω is rotated 60° clockwise about P to create a new circle ω'. Let $Q \neq P$ be the intersection of ω and ω'. Circle ω' is rotated 120° counterclockwise about Q to create a new circle ω'', which intersects ω' at R. What is the area of $\triangle PQR$?

(A) $\sqrt{3}$ **(B)** $2\sqrt{3}$ **(C)** $4\sqrt{3}$ **(D)** $8\sqrt{3}$ **(E)** $12\sqrt{3}$

17. Circles ω_1 and ω_2 intersect at A and B. Point P is on ω_1 and point Q is on ω_2 so that segment $\overline{PQ}$ intersects segment $\overline{AB}$ at X. Let segment $\overline{PQ}$ intersect ω_2 at $R \neq Q$ and ω_1 at $S \neq P$. If $PR = 5$, $RX = 1$, and $SX = 2$, what is the length QS?

(A) 5 **(B)** 10 **(C)** 12 **(D)** 13 **(E)** 17

18. In octagon $ABCDEFGH$, the side lengths alternate between 7 and 2 (i.e. $AB = 7$, $BC = 2$, $CD = 7$, etc.), opposite sides are parallel (i.e. $\overline{AB} \parallel \overline{EF}$, $\overline{BC} \parallel \overline{FG}$, etc.), and the sum of any two adjacent interior angles is $270°$. If the area of quadrilateral $ACEG$ is 60, then the area of quadrilateral $BDFH$ is $m + n\sqrt{p}$, where m, n, and p are integers and p is not divisible by the square of any prime. What is $m + n + p$?

(A) 47 **(B)** 60 **(C)** 61 **(D)** 72 **(E)** 75

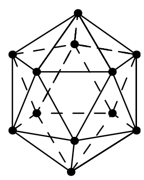
19. Rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are such that $\angle BAC = 30°$. Let $\mathcal{R}$ be the region consisting of all points P that lie between the two rays such that the sum of the perpendicular distances from P to each of the two rays is at most 4. What is the area of $\mathcal{R}$?

(A) 8 **(B)** $\dfrac{4\pi\sqrt{3} + 8\pi}{3}$ **(C)** 16 **(D)** $3\pi\sqrt{3}$ **(E)** $\dfrac{16\pi}{3}$

20. In the coordinate plane, circle ω passes through the origin O. For some positive real number a, let ω intersect the line $y = ax$ at a point A and the line $y = -\frac{1}{a}x$ at a point B. Let the tangent lines to ω at A and O intersect at a point P, and let the tangent lines to ω at B and O intersect at a point Q. If a is chosen such that the coordinates of P are $(6, -2)$ and the coordinates of Q are $(-3, 1)$, what is the radius of ω?

(A) $\sqrt{5}$ **(B)** $\sqrt{10}$ **(C)** 4 **(D)** $2\sqrt{5}$ **(E)** $2\sqrt{10}$

21. A *regular icosahedron* is a 20-faced solid where each face is an equilateral triangle and five triangles meet at every vertex. John picks n faces of a regular icosahedron so that none of the faces meet, even at a vertex. What is the largest possible value of n?



(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

22. Let $z_1 = 7 + 8i$ and $z_2 = 1 - 4i$ be two complex numbers. Suppose that z_3 is another complex number such that

$$|z_3 - z_1|^2 + |z_2 - z_3|^2 = 180.$$

What is the smallest possible value of $|z_3|$?

(A) 2 **(B)** $\sqrt{5}$ **(C)** $\sqrt{41} - 4$ **(D)** $\sqrt{29} - 2$ **(E)** $2\sqrt{5}$

23. In triangle ABC with $AC = 5$, $BC = 7$, and $AB = 8$, let I be the center of the inscribed circle of $\triangle ABC$. Let points P and Q lie on sides $\overline{AB}$ and $\overline{AC}$, respectively, such that $\angle PIQ = 120°$. If $\frac{AQ}{CQ} = \frac{\sqrt{21}}{3}$, what is the area of quadrilateral $APIQ$?

(A) $\dfrac{3\sqrt{3}}{2}$ **(B)** $\dfrac{5\sqrt{3}}{2}$ **(C)** 5 **(D)** $3\sqrt{3}$ **(E)** $2\sqrt{7}$

24. A right circular cone with base ω and apex P is inscribed in a sphere so that both P and the circumference of ω lie on the surface of the sphere. Points A and B are on ω so that if O is the center of ω, then $\angle AOB = 90°$. Suppose that point Q is such that lines AQ, BQ, and PQ are all tangent to the sphere. If the radius of ω is $\sqrt{3}$ and the radius of the sphere is 2, what is the distance from Q to the center of the sphere?

(A) $2\sqrt{3}$ **(B)** $2\sqrt{5}$ **(C)** $2\sqrt{6}$ **(D)** $3\sqrt{3}$ **(E)** $2\sqrt{7}$

25. Let triangle ABC be a right triangle with $\angle ACB = 90°$. Suppose that $PQRS$ is a square such that P is on side $\overline{AC}$, Q is on side $\overline{BC}$, and R is on side $\overline{AB}$. If $AC = 5$ and $BC = 7$, what is the smallest possible side length of the square?

(A) $\dfrac{35}{17}$ **(B)** $\dfrac{35\sqrt{193}}{193}$ **(C)** $\dfrac{35}{13}$ **(D)** $\dfrac{35\sqrt{2}}{17}$ **(E)** 7

2021 GAMC

DO NOT OPEN UNTIL SATURDAY, July 17, 2021

Questions and comments about problems and solutions
for this exam should be sent by private message to:

depsilon0 and **DeToasty3**.

The 2021 GIME might be held. It would be a 15-question, 3-hour, integer-answer exam if it was to be held. You will not be invited to participate because this contest currently does not exist. *A complete listing of our previous publications may be found at our web site:*

Wait, we don't have one!

****Try Administering This Exam On An Earlier Date. Oh Wait, You Can't.****

1. All the information needed to administer this exam is not contained in the non-existent GAMC Teacher's Manual.
2. YOU must not verify on the non-existent GAMC COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.
3. Send **depsilon0** and **DeToasty3** a private message submitting your answers to the GAMC. AoPS is the only way to submit your answers.
4. The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

The **2021 Geometry AMC Series**
was made possible by the contributions of the following people:

AOPSmathematics, ApraTrip, depsilon0, DeToasty3, & Radio2

Credit goes to Online Test Seasonal Series (OTSS) for the booklet template.