De Mathematics Competitions

2nd Annual

DMC 10C

Friday, June 11, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the DMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than writing utensils, blank scratch paper, rulers, compasses, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your non-existent proctor will not ask you to record certain information on the answer form.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have already taken the test in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

Students who score well on this DMC 10 may or may not be invited to the 2022 DIME (De Invitational Mathematics Examination). More details about the DIME and other information are on the back page of this test booklet.

1. What is the value of

$$4^1 - 3^2 + 2^3 - 1^4?$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2. Let a, b, c, and d be real numbers that satisfy

$$a + 1 = b, \quad b + 2 = c, \quad c + 3 = d, \quad d + 4 = 21.$$

What is $a + b + c + d$?

(A) 51 **(B)** 52 **(C)** 53 **(D)** 54 **(E)** 55

3. Given a right triangle with legs of lengths 5 and 6, a square is drawn with one side as its hypotenuse such that the triangle is completely inside the square. What is the area of the region inside the square but outside the triangle?

(A) 46 **(B)** 47 **(C)** 48 **(D)** 49 **(E)** 50

4. A group of 200 people were invited to see a movie, where each viewer at the movie either has first row, second row, or third row seats. It is given that four-fifths of the people invited chose to watch the movie, one-ninth of the viewers were not invited, one-fifth of the viewers had first row seats, and 60 of the viewers had third row seats. What is the probability that a randomly selected viewer had second row seats?

(A) $\frac{3}{10}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{7}{15}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{4}{5}$

5. What is the sum of all positive real numbers a such that the equation $x^2 + ax - 12 = 0$ has two distinct integer solutions x?

(A) 6 **(B)** 12 **(C)** 14 **(D)** 16 **(E)** 22

6. John is playing a game with 6 levels, each with 5 stages. After the third stage of each of the first five levels, John may choose whether or not to skip the remaining stages in the level and start at the first stage of the next level. If John finished the whole game, how many possible combinations of stages could John have played through?

(A) 5 **(B)** 10 **(C)** 16 **(D)** 30 **(E)** 32

7. Daniel has to walk one mile to complete his gym homework. He decides to split his path into quarters, where after each quarter, he randomly chooses to turn $90°$ clockwise or counterclockwise with equal probability. If Daniel walks in a straight line each quarter, what is the probability that he will end up where he started after walking the mile?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{8}$ **(E)** $\frac{3}{4}$

8. Let a and b be positive integers. If a is divisible by 2 but not 3, and b is divisible by 3 but not 2, what is the greatest possible three-digit value of $a+b$?

(A) 995 **(B)** 996 **(C)** 997 **(D)** 998 **(E)** 999

9. How many orderings of the six numbers 1, 1, 2, 2, 3, and 6 are there such that the sum of the first three numbers is twice the sum of the last three numbers?

(A) 9 **(B)** 18 **(C)** 27 **(D)** 36 **(E)** 72

10. Given a triangle, a line is drawn such that it intersects the triangle in exactly two points. Which of the following statements must always be true?

(A) The triangle is split into a smaller triangle and quadrilateral.

(B) The segment of the line in the triangle is shorter than every side of the triangle.

(C) At least two of the triangle's angle bisectors intersect the line in the triangle.

(D) The perimeters of each of the regions formed are less than that of the triangle.

(E) None of the above.

11. In a plane, eight rays emanate from a point P such that every two adjacent rays form an acute angle with measure $45°$. Next, a line segment with a finite length is drawn in the plane. If the line segment intersects exactly n of the rays, what is the sum of all possible values of n? (If the line segment passes through P, then $n = 8$.)

(A) 13 **(B)** 14 **(C)** 17 **(D)** 18 **(E)** 23

12. For certain real numbers x and y, the first 3 terms of a geometric progression are $x-2$, $2y$, and $x+2$ in that order, and the sum of these terms is 4. What is the fifth term?

(A) $\frac{64}{3}$ **(B)** $\frac{196}{9}$ **(C)** $\frac{512}{23}$ **(D)** $\frac{256}{11}$ **(E)** $\frac{128}{5}$

13. In the coordinate plane, let $P = (10, 0)$ and $Q = (a, a)$, for some real number a. It is given that when point P is reflected over point Q, the resulting point P' is 2 times as far from the origin as point P. What is the sum of all possible values of a?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

14. Draw two identical non-intersecting circles, a line tangent to both circles at distinct points A and B, where the circles are on the same side of the line, and a line tangent to both circles at distinct points C and D, where the circles are on opposite sides of the line. The lines intersect at point P. If $AB = 11$ and $CD = 5$, what is $AP \cdot BP$?

(A) 20 **(B)** 22 **(C)** 24 **(D)** 26 **(E)** 28

15. For positive integers n, let the nth triangular number be the sum of the first n positive integers. For how many integers n between 1 and 100, inclusive, does the nth triangular number have the same last digit as the product of the first n triangular numbers?

(A) 11 **(B)** 12 **(C)** 20 **(D)** 21 **(E)** 22

16. Let x and y be real numbers such that

$$|x - |y - x|| = 1,$$
$$|y - |x - y|| = 2.$$

What is the largest possible value of $x + y$?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

17. A car moves such that if there are n people in it, it moves at a constant rate of 4^n miles per hour. At noon, the car has 1 person in it and starts moving. After every mile, another person instantaneously gets in the car. How many people are in the car when the average speed the car has moved since noon reaches 17 miles per hour?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

18. A positive integer is called *fresh* if the sum of the squares of its digits in base-4 is greater than the integer itself. How many fresh positive integers are there?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

19. Two rays emanating from a point form a $60°$ angle. Circles ω_1 and ω_2 of radius 4 are such that ω_1 is tangent to both rays, and ω_2 is tangent to only one of the rays, on the same side of the ray as ω_1. If there exists a circle of radius 3, externally tangent to ω_2 and tangent to both rays, what is the distance between the centers of ω_1 and ω_2?

(A) $3\sqrt{3}$ **(B)** $4\sqrt{2}$ **(C)** 6 **(D)** $4\sqrt{3}$ **(E)** $3\sqrt{6}$

20. In the xy-plane are perpendicular lines $y = ax + d$ and $y = bx + c$, where a, b, c, and d are real numbers in a geometric progression in that order. If the two lines and the line $y = \frac{3}{2}x$ pass through a common point, what is the least possible value of $a+b+c+d$?

(A) $\frac{3}{2}$ **(B)** $\frac{51}{32}$ **(C)** $\frac{13}{8}$ **(D)** $\frac{111}{64}$ **(E)** $\frac{7}{4}$

21. Richard thinks of a positive integer n and writes the base ten representations of $n!$ and $(n+1)!$ on a board. He then erases the zeroes to the right of the last nonzero digit of each number (if any exist), resulting in two numbers a and b. If one of a and b is 4 times the other, what is the sum of all possible values of n less than 1000?

(A) 315 **(B)** 441 **(C)** 656 **(D)** 714 **(E)** 819

22. In rectangle $ABCD$, point A is reflected over diagonal $\overline{BD}$ to a point A'. If $A'B = A'C$ and $AA' = 6$, what is the area of rectangle $ABCD$?

(A) 18 **(B)** $8\sqrt{6}$ **(C)** $12\sqrt{3}$ **(D)** 21 **(E)** $9\sqrt{6}$

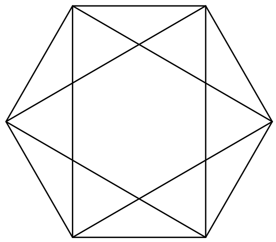
23. There are 15 people in a room, where everyone shakes hands with a positive number of other people in the room exactly once. If exactly 6 people shook 1 hand, exactly 5 people shook between 2 and 4 hands, inclusive, exactly 1 person shook 8 hands, and exactly 1 person shook 14 hands, what is the least possible total number of handshakes?

(A) 24 **(B)** 25 **(C)** 26 **(D)** 27 **(E)** 28

24. In right $\triangle ABC$ with $BC = 7$ and a right angle at A, let the midpoint of side $\overline{AB}$ be D. Suppose that there exists a point E on the circumference of the circumcircle of $\triangle ABC$ such that $\triangle CDE$ is equilateral. What is the side length of $\triangle CDE$?

(A) $\sqrt{21}$ **(B)** $2\sqrt{7}$ **(C)** $\frac{7\sqrt{21}}{6}$ **(D)** $\frac{7\sqrt{10}}{4}$ **(E)** $3\sqrt{6}$

25. Each of the six vertices of the regular hexagon shown below is labeled with either a 1 or a 2. Some diagonals of the hexagon are drawn, and each of the six points of intersection is labeled with either a 2, a 3, or a 4. In how many ways can the 12 points be labeled such that for every drawn diagonal of the hexagon, the sum of the numbers on its two endpoints is *not* equal to either of the numbers on the two points of intersection of the diagonal? Rotations and reflections are considered distinct.



(A) 502 **(B)** 514 **(C)** 526 **(D)** 538 **(E)** 550

2021 DMC 10C

DO NOT OPEN UNTIL FRIDAY, June 11, 2021

Questions and complaints about problems and solutions
for this exam should be sent by private message to:

DeToasty3.

The 2022 DIME may or may not be held. It would be a 15-question, 3-hour, integer-answer exam if it was to be held. You may or may not be invited to participate because this contest may or may not exist. *A complete listing of our previous publications may be found at our web site:*

`https://detoasty3.github.io/dmc.html`

****Try Administering This Exam On An Earlier Date. Oh Wait, You Can't.****

1. All the information needed to administer this exam is not contained in the non-existent DMC 10 Teacher's Manual.

2. YOU must not verify on the non-existent DMC 10 COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.

3. Send **DeToasty3**, **nikenissan**, **pog**, and **vsamc** a private message submitting your answers to the DMC 10. AoPS is the only way to submit your answers.

4. The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

The **2021 De Mathematics Competitions**
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AprаTrip, AT2005, Awesome_guy, DeToasty3, firebolt360, GammaZero, HrishiP, i3435, jayseemath, john0512, JustinLee2017, nikenissan, pog, richy, skyscraper, & vsamc

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