De Mathematics Competitions

De Mathematics Competitions

1st Annual

DMC 10

Wednesday, September 30, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the DMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your non-existent proctor will not ask you to record certain information on the answer form.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have already taken the test in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

The Committee on the De Mathematics Competitions reserves the right to re-examine students before deciding whether to grant official status to their scores. The Committee also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score well on this DMC 10 will not be invited to anything because the DIME (De Invitational Mathematics Examination) does not exist. More details about the DIME and other information are not on the back page of this test booklet.

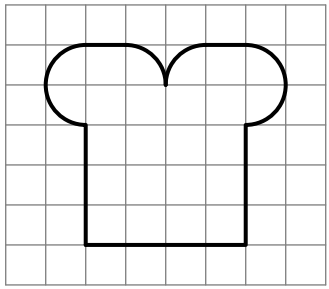
1. What is the value of
$$\frac{(2^0 - 2^1)^{2020}}{(2 \cdot 0 + 2^0)^{2021}}?$$

(A) -1 **(B)** $-\frac{1}{2}$ **(C)** 0 **(D)** $\frac{1}{2}$ **(E)** 1

2. If the ratio of males to females in a country club is exactly 9 to 5, and there are fewer than 100 people in it, then what is the largest possible number of people in the club? (Assume that there are only two genders.)

(A) 95 **(B)** 96 **(C)** 97 **(D)** 98 **(E)** 99

3. The figure below has been drawn on 1 cm $\times$ 1 cm graph paper, where every square not completely covered or empty is filled with a quarter circle. What is the area of the figure in cm^2?



(A) $12 + \frac{3\pi}{2}$ **(B)** $8 + 3\pi$ **(C)** $18 + \pi$ **(D)** $12 + 3\pi$ **(E)** $18 + \frac{3\pi}{2}$

4. Rohan wants to distribute 25 slices of pizza to n people such that each person gets an equal number of slices, except for one person who gets one more slice than each of the other people. If n is greater than 1, how many different integer values of n exist?

(A) 2 **(B)** 5 **(C)** 7 **(D)** 8 **(E)** 9

5. A dog has four legs, and a dug has three legs. Janelle has a certain number of dogs and dugs as pets, and she has no other pets. If there are 61 legs across all of Janelle's pets, then what is the smallest possible number of dugs that Janelle could have?

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

6. Two distinct elements x and y are chosen from the set $\{1, 2, 3, 4\}$ at random. What is the probability that the line with slope $\frac{y}{x}$ passing through the point (x, y) also passes through the point $(2020, 1010)$?

   (A) $\frac{1}{12}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

7. Anthony, Daniel, and Richard have 17, 20, and 26 trading cards, respectively. Every minute, one of the three boys gives away two of his trading cards such that the other two boys get one trading card each. What is the shortest amount of time, in minutes, that it could take for the three boys to each have an equal number of trading cards?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

8. Two distinct points A and B are chosen on the circumference of a circle with center O. Another point C, distinct from A and B, is chosen on the circumference. If $\angle AOB = 70^\circ$, what is the probability that $\triangle ABC$ is acute?

   (A) $\frac{7}{36}$ **(B)** $\frac{7}{18}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{11}{18}$ **(E)** $\frac{31}{36}$

9. Alice and Bob are racing each other on a track. Each of their lanes are 400 meters in length. Normally, Alice and Bob run at constant rates of a and b meters per minute, respectively, but Alice's lane has a 180-meter puddle in the middle, in which she runs at three-quarters of her normal speed. If Alice and Bob take the same amount of time to run through their lanes without stopping, then what is $\frac{a}{b}$?

   (A) 1.05 **(B)** 1.15 **(C)** 1.25 **(D)** 1.35 **(E)** 1.45

10. In triangle ABC, $AB = 9$ and $BC = 40$, with a right angle at B. Point P lies on side $\overline{BC}$ such that the line perpendicular to $\overline{BC}$ passing through point P splits the area of $\triangle ABC$ exactly in half. What is PC?

(A) $\frac{9\sqrt{2}}{2}$ **(B)** 20 **(C)** $\frac{41}{2}$ **(D)** $20\sqrt{2}$ **(E)** $\frac{41\sqrt{2}}{2}$

11. What is the largest integer n for which there exists an ordered triple (p, q, r) of distinct prime numbers such that $p^2(q^2 + r^2)$ is divisible by 2^n?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

12. Let $f(x) = x^2 - x - 3$ and $g(x) = 2x + 3$ for all real numbers x. What is the sum of all real values of x such that $f(g(x)) = g(f(x))$?

(A) -6 **(B)** $-\frac{9}{2}$ **(C)** -3 **(D)** $-\frac{3}{2}$ **(E)** 0

13. Let A and B be two distinct points on a plane. Let $\mathcal{S}$ denote the set of all circles on the plane with a finite area such that A and B are on the circumference of the circle. What is the region of all points not on the circumference of any of the circles in $\mathcal{S}$?

(A) Every point on line AB excluding A and B

(B) Every point on segment $\overline{AB}$ excluding A and B

(C) Every point on line AB but not on segment $\overline{AB}$

(D) The midpoint of segment $\overline{AB}$

(E) None of the above

14. Let n be the 2020th smallest positive integer such that the sum of the digits in its base-nine representation is divisible by 8. What is the sum of the digits of the base-ten representation of n, in base-ten?

(A) 14 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 22

15. 10 students are taking a math test. Of the 10 students, 3 of them are guaranteed to solve the last problem. However, the other 7 students are not guaranteed to solve the last problem, but each of them are equally likely to solve it. If Sean is one of the other 7 students, and exactly 6 students out of the 10 solved the last problem, what is the probability that Sean was one of the 6 students who solved it?

(A) $\frac{5}{16}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{3}{7}$ **(E)** $\frac{5}{9}$

16. A plane cuts into a sphere of radius 11 such that the area of the region of the plane inside the sphere is 108π. A perpendicular plane cuts into the sphere such that the area of the region of the plane inside the sphere is 94π. Given that the two planes intersect at a line, what is the length of the segment of the line inside the sphere?

(A) $6\sqrt{3}$ **(B)** 12 **(C)** $11\sqrt{2}$ **(D)** $8\sqrt{5}$ **(E)** 18

17. There exist three positive integers such that the sum of the integers is 14, the sum of the squares of the integers is 78, and the sum of the cubes of the integers is 476. The sum of the reciprocals of the integers can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

(A) 126 **(B)** 127 **(C)** 128 **(D)** 129 **(E)** 130

18. Eight people randomly split into two groups of four to dance. After that, the eight people randomly split into four groups of two to talk. What is the probability that exactly two of the four groups contain people who have danced in a group of four?

(A) $\frac{8}{35}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{4}{21}$ **(D)** $\frac{24}{35}$ **(E)** $\frac{6}{7}$

19. In trapezoid $ABCD$ with $\overline{AD} \parallel \overline{BC}$ and side lengths $AD = 18$, $BC = 20$, and $AB = CD = 8$, let X be the intersection of line AB and the bisector of $\angle ADC$, and let Y be the intersection of line CD and the bisector of $\angle DAB$. What is XY?

(A) 22 **(B)** 24 **(C)** 25 **(D)** 27 **(E)** 28

20. A set of positive integers exists such that for any integer k in the set, all of the values k^2+2, k^2+4, and k^2+8 are prime numbers. Two distinct integers m and n are chosen from the set. Which of the following is a possible value of $m+n$?

(A) 40 **(B)** 56 **(C)** 72 **(D)** 88 **(E)** 104

21. In triangle ABC with $AB = 8$, $BC = 6$, and $AC = 11$, let points D and E trisect side $\overline{BC}$ such that D is closer to B than C. Let F be the intersection of $\overline{AC}$ and the bisector of $\angle ABC$. Let X be the intersection of $\overline{AD}$ and $\overline{BF}$, and let Y be the intersection of $\overline{AE}$ and $\overline{BF}$. What is the ratio of the areas of $\triangle AYF$ and $\triangle AXB$?

(A) $\frac{25}{63}$ **(B)** $\frac{14}{33}$ **(C)** $\frac{16}{35}$ **(D)** $\frac{26}{55}$ **(E)** $\frac{10}{21}$

22. Define a sequence recursively by $T_1 = 1$ and

$$T_n = \frac{n! \cdot T_{n-1}}{(n-1)! + n \cdot T_{n-1}}$$

for all integers $n \geq 2$. What is the sum of the digits of T_9?

(A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

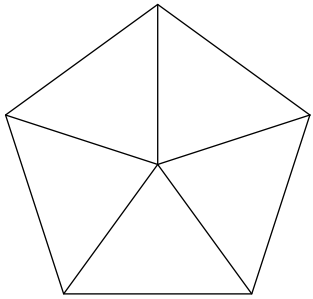
23. An integer n is chosen from the interval $[1, 50]$, where n is not a perfect square. Amy learns the remainder when n is divided by 7, and Sid learns the number of divisors of n, but they both believe n is a perfect square (in the interval $[1, 50]$), Amy learned the number of divisors, and Sid learned the remainder upon division by 7. Amy says, "I know what n is." Sid replies, "Then, I also know what n is." It turns out that they were thinking of the same value of n, but it was incorrect due to their misconceptions. If Amy and Sid always tell the truth based on what they believe and can reason perfectly, then what is the sum of all possible actual values of n?

(A) 26 **(B)** 37 **(C)** 45 **(D)** 64 **(E)** 71

24. In triangle ABC, $AB = 16$ and $BC = 8$, with a right angle at C. Let M be the midpoint of side $\overline{AB}$, let N be a point on side $\overline{AC}$, and let P be the intersection of segments $\overline{BN}$ and $\overline{CM}$. If $BP = 7$, then what is the sum of all possible values of $\frac{CN}{AN}$?

(A) $\frac{23}{21}$ **(B)** $\frac{21}{19}$ **(C)** $\frac{19}{17}$ **(D)** $\frac{17}{15}$ **(E)** $\frac{15}{13}$

25. The diagram below is composed of five triangles forming a regular pentagon. How many ways are there to color the 10 edges in the diagram with red, blue, or green in such a way that no vertex of a triangle has 3 or more edges connected to it that are the same color? Rotations and reflections are considered distinct colorings.



(A) 4860 **(B)** 9650 **(C)** $11,790$ **(D)** $12,130$ **(E)** $14,580$

2021 SPRING DMC 10

DO NOT OPEN UNTIL WEDNESDAY, September 30, 2020

Questions and complaints about problems and solutions for this exam should be sent by private message to:

DeToasty3.

The 2021 DIME may or may not be held. It would be a 15-question, 3-hour, integer-answer exam if it was to be held. You may or may not be invited to participate because this contest may or may not exist. *A complete listing of our previous publications may be found at our web site:*

`http://detoasty3.github.io`

****Try Administering This Exam On An Earlier Date. Oh Wait, You Can't.****

1. All the information needed to administer this exam is not contained in the non-existent DMC 10 Teacher's Manual.
2. YOU must not verify on the non-existent DMC 10 COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.
3. Send **DeToasty3** a private message submitting your answers to the DMC 10. AoPS is the only way to submit your answers.
4. The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

The **2020-2021 De Mathematics Competitions**
was made possible by the contributions of the following people:

DeToasty3, jayseemath, karate7800, nikenissan, & richy

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