De Mathematics Competitions

Official Solutions

De Mathematics Competitions

2nd Annual

DIME

De Invitational Mathematics Examination

Friday, September 17, 2021

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

Questions and complaints about this competition should be sent by private message to

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The problems and solutions for this DIME were prepared by the DIME Editorial Board under the direction of

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Answer Key:

1. (140)	2. (012)	3. (248)	4. (324)	5. (108)
6. (252)	7. (089)	8. (220)	9. (142)	10. (271)
11. (660)	12. (616)	13. (017)	14. (048)	15. (054)

Problem 1:

(stayhomedomath) Charles has some marbles. Their colors are either red, green, or blue. The total number of red and green marbles is 38% more than that of blue marbles. The total number of green and blue marbles is 150% more than that of red marbles. If the total number of blue and red marbles is more than that of green marbles by $n\%$, find n.

Answer (140):

The fraction of the total number of marbles that are blue marbles is

$$\frac{1}{1+(1+38\%)}=\frac{50}{119},$$

and the fraction of the total number of marbles that are red marbles is

$$\frac{1}{1+(1+150\%)}=\frac{2}{7}.$$

Thus, the fraction of the total number of marbles that are green marbles is

$$1-\frac{50}{119}-\frac{2}{7}=\frac{5}{17}.$$

Thus, the percentage of the number of red and blue marbles more than that of green marbles is

$$\left(\frac{1-\frac{5}{17}}{\frac{5}{17}}-1\right)\cdot 100\%=\frac{7}{5}\cdot 100\%=140\%.$$

Thus, the requested answer is $n=\boxed{140}$. ■

Problem 2:

(HrishiP) Let $P(x)=x^2-1$ be a polynomial, and let a be a positive real number satisfying

$$P(P(P(a)))=99.$$

The value of a^2 can be written as $m+\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

Answer (012):

First, we obtain $P(a) = a^2 - 1$. Upon plugging in this value into the polynomial again, we obtain

$$P(P(a)) = (a^2-1)^2 - 1 = (a^2-1+1)(a^2-1-1) = a^2(a^2-2) = a^4 - 2a^2.$$

Finally, upon plugging in this value into the polynomial again, we obtain

$$\begin{aligned} P(P(P(a))) &= (a^4-2a^2)^2 - 1 \\ &= (a^4-2a^2+1)(a^4-2a^2-1) \\ &= (a^2-1)^2((a^2-1)^2-2) \\ &= (a^2-1)^4 - 2(a^2-1)^2. \end{aligned}$$

Setting this equal to 99 and letting $y = a^2-1$, we get $y^4 - 2y^2 = 99$. Adding 1 to both sides of the equation gives us

$$y^4 - 2y^2 + 1 = 100 \implies (y^2-1)^2 = 100 \implies (a^4-2a^2)^2 = 100 \implies a^4 - 2a^2 = \pm 10.$$

Next, by the quadratic formula, we obtain

$$a^2 = \frac{2 \pm \sqrt{4 \pm 40}}{2} = 1 \pm \sqrt{11}.$$

Since a^2 is positive, we have that $a^2 = 1+\sqrt{11}$, so the requested answer is $1+11 = \boxed{012}$.
■

Problem 3:

(HrishiP) An up-right path from lattice points P and Q on the xy-plane is a path in which every move is either one unit right or one unit up. The probability that a randomly chosen up-right path from $(0,0)$ to $(10,3)$ does not intersect the graph of $y = x^2 + 0.5$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer (248):

We count the number of acceptable paths. The first move is forced to be $(0,0) \to (1,0)$. Now, there are two cases.

Case 1: If we go from $(1,0) \to (1,1)$, then the next move is forced to be $(1,1) \to (2,1)$ from which there are $\binom{8+2}{2} = 45$ paths.

Case 2: If we go from $(1,0) \to (2,0)$, then there are $\binom{8+3}{3} = 165$ paths.

> The number of possible paths disregarding the restriction is $\binom{13}{3} = 286$. Since there are $45 + 165 = 210$ acceptable paths, the probability is $\frac{210}{286} = \frac{105}{143}$. Thus, the requested answer is $105 + 143 = \boxed{248}$. ■

Problem 4:

(DeToasty3) Given a regular hexagon $ABCDEF$, let point P be the intersection of lines BC and DE, and let point Q be the intersection of lines AP and CD. If the area of $\triangle QEP$ is equal to 72, find the area of regular hexagon $ABCDEF$.

> **Answer (324):**
>
> Let lines AB and CD intersect at a point R. Notice that $DE = DP = AB = BR$. This is because all six sides of the regular hexagon have equal length, and we are forming equilateral triangles $\triangle BCR$ and $\triangle CDP$ with one side as a side of the regular hexagon. This means that the areas of $\triangle QED$ and $\triangle PDQ$ are equal, or 36 each.
>
> Next, we see that $\triangle ARQ \sim \triangle PDQ$ because $\overline{AR} \parallel \overline{DP}$ and lines AP and DR intersect at point Q. This means that $\frac{QR}{DQ} = 2$, which means that the area of $\triangle ARQ$ is four times the area of $\triangle PDQ$, or 144.
>
> Notice that $AB = BR$, which means that the areas of $\triangle ABQ$ and $\triangle ARQ$ are equal, or 72 each. Next, notice that $\frac{RC}{CQ} = 3$, so the area of $\triangle ABC$ is $\frac{3}{4}$ the area of $\triangle ABQ$, or 54. Finally, notice that this area is $\frac{1}{6}$ the area of the regular hexagon because, letting O be the center of the regular hexagon, we have that the areas of $\triangle ABC$ and $\triangle ABO$ are equal.
>
> Thus, the requested answer is $54 \cdot 6 = \boxed{324}$. ■

Problem 5:

(treemath) The four-digit base ten number $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$ has all nonzero digits and is a multiple of 99. Additionally, the two-digit base ten number $\underline{a}\,\underline{b}$ is a divisor of 150, and the two-digit base ten number $\underline{c}\,\underline{d}$ is a divisor of 168. Find the remainder when the sum of all possible values of the number $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$ is divided by 1000.

> **Answer (108):**
>
> Let $x = \underline{a}\,\underline{b}$ and $y = \underline{c}\,\underline{d}$. Then $100x + y$ is divisible by 99, so $x + y$ is divisible by 99. However, since x and y are two-digit numbers that are not 99, we must have $x + y = 99$. To quickly find all solutions, we can list out the divisors of 150 and 168 that are at least 50, and see whether 99 minus those divisors is a divisor of the other number.

For 150, we get $99 - 75 = 24$ and $99 - 50 = 49$, and since 24 is a divisor of 168, the four-digit number could be 7524. For 168, we get $99 - 84 = 15$ and $99 - 56 = 43$, and since 15 is a divisor of 150, the four-digit number could also be 1584.

Thus, the sum is $7524 + 1584 = 9108$, so the requested answer is $\boxed{108}$. ■

Problem 6:

(DeToasty3) In $\triangle ABC$ with $AC > AB$, let D be the foot of the altitude from A to side $\overline{BC}$, and let M be the midpoint of side $\overline{AC}$. Let lines AB and DM intersect at a point E. If $AC = 8$, $AE = 5$, and $EM = 6$, find the square of the area of $\triangle ABC$.

Answer (252):

Extend $\overline{DM}$ past M to a point P such that $\overline{AP} \parallel \overline{CD}$. We see that since $DM = AM = CM = 4$, we must have that $PM = 4$. Thus, $APCD$ is a rectangle. Given that $EM = 6$, we have that $DE = 2$ and $DP = 8$, so $EP = 10$. Since $\triangle BED \sim \triangle AEP$, we have that $BD : AP = 1 : 5$. Since $AP = CD$, we have that $BD : CD = 1 : 5$ as well. Given that $AE = 5$, we have that $AB = \frac{4}{5} \cdot 5 = 4$. Let $BD = x$. By the Pythagorean Theorem, we have

$$\begin{aligned} AD^2 = AB^2 - BD^2 &= AC^2 - CD^2 \\ 16 - x^2 &= 64 - 25x^2 \\ 24x^2 &= 48 \\ x &= \sqrt{2}. \end{aligned}$$

This also gives us that $AD = \sqrt{14}$. Thus, the square of the area of $\triangle ABC$ is

$$\left(\frac{1}{2} \cdot 6\sqrt{2} \cdot \sqrt{14}\right)^2 = (6\sqrt{7})^2 = \boxed{252},$$

as requested. ■

Problem 7:

(vvluo & richy) Richard has an infinite row of empty boxes labeled $1, 2, 3, \ldots$ and an infinite supply of balls. Each minute, Richard finds the smallest positive integer k such that box k is empty. Then, Richard puts a ball into box k, and if $k \geq 3$, he removes one ball from each of boxes $1, 2, \ldots, k-2$. Find the smallest positive integer n such that after n minutes, both boxes 9 and 10 have at least one ball in them.

Answer (089):

Let $f(n)$ denote the smallest positive integer such that after $f(n)$ minutes, both boxes n and $n-1$ have at least one ball in it, where $f(1)=1$. We want to find $f(10)$. Note that every box can only have at most one ball in it at a time. This is because in order for a ball to be placed in a box, it must already be empty, according to the problem statement. Also, note that the first time a box, say, box k, first gets filled is when all of the boxes $1,2,\ldots,k-1$ are filled, since this is the smallest numbered box which is empty, and in this minute, boxes $1,2,\ldots,k-2$ will become empty, but boxes k and $k-1$ will have a ball in it. Suppose that boxes $k-1$ and $k-2$ have a ball in it, which takes $f(k-1)$ minutes. Then, in order to get to where boxes k and $k-1$ have a ball in it, we must first fill up boxes $1,2,\ldots,k-3$, which are currently all empty, and then add one minute for adding a ball to box k. Suppose that the number of minutes to fill up boxes $1,2,\ldots,k-3$ is M. Then we have that

$$f(k)=f(k-1)+M+1.$$

Note that M is the same number of minutes as if the boxes were all completely empty. Thus, we biject to this setting, and in the $M+1$th minute, box $k-2$ will be filled, leaving only boxes $k-2$ and $k-3$ filled, and so

$$f(k-2)=M+1.$$

Combining our results, we get that

$$f(k)=f(k-1)+f(k-2).$$

Seeing that $f(1)=1$ and $f(2)=2$, we work our way up to find that the requested number of minutes is $f(10)=\boxed{089}$. ■

Problem 8:

(HrishiP) Given a parallelogram $ABCD$, let $\mathcal{P}$ be a plane such that the distance from vertex A to $\mathcal{P}$ is 49, the distance from vertex B to $\mathcal{P}$ is 25, and the distance from vertex C to $\mathcal{P}$ is 36. Find the sum of all possible distances from vertex D to $\mathcal{P}$.

Answer (220):

Credit for this solution goes to **asdf334** for their solution sketch.

Without loss of generality, assume that the plane $\mathcal{P}$ is the xy-plane. Let z_A, z_B, z_C, and z_D denote the z-coordinates of vertices A, B, C, and D, respectively, in three dimensional space. From the given conditions, we have that $|z_A|=49$, $|z_B|=25$, and $|z_C|=36$.

Notice that the diagonals of a parallelogram bisect each other. This means that

$$z_A + z_C = z_B + z_D.$$

Rearranging, we get that $z_D = z_A + z_C - z_B$. Our remaining step is to find all possible values of $|z_D|$.

Note that from the equation $z_D = z_A + z_C - z_B$, we get that

$$z_D = \pm 49 \pm 36 \mp 25 \implies |z_D| \in \{12, 38, 60, 110\},$$

so the requested answer is $12 + 38 + 60 + 110 = \boxed{220}$. ■

Problem 9:

(stayhomedomath) Let $a_1, a_2, \ldots, a_6$ be a sequence of integers such that for all $1 \le i \le 5$,

$$a_{i+1} = \frac{a_i}{3} \quad \text{or} \quad a_{i+1} = -2a_i.$$

Find the number of possible positive values of $a_1 + a_2 + \cdots + a_6$ less than 1000.

Answer (142):

For each $1 \le i \le 5$, either $\frac{a_{i+1}}{a_i} = \frac{1}{3}$ or $\frac{a_{i+1}}{a_i} = -2$. Note that $-2 \equiv \frac{1}{3} \equiv 5 \pmod 7$, so

$$a_1 + a_2 + \cdots + a_6 \equiv a_1(1 + 5 + 5^2 + \cdots + 5^5) \equiv 0 \pmod 7,$$

which means that $a_1 + a_2 + \cdots + a_6$ must be a multiple of 7.

Next, we show that all multiples of 7 are possible values of $a_1 + a_2 + \cdots + a_6$. Setting

$$(a_1, a_2, a_3, a_4, a_5, a_6) = (9k, 3k, k, -2k, 4k, -8k)$$

where k is any integer, we have that $a_1 + a_2 + \cdots + a_6 = 7k$.

Thus, the requested answer is equal to the number of multiples of 7 greater than 0 but less than 1000, which is $\boxed{142}$. ■

Problem 10:

(stayhomedomath) Let a and b be real numbers such that

$$\left(8^a + 2^{b+7}\right)\left(2^{a+3} + 8^{b-2}\right) = 4^{a+b+2}.$$

The value of the product ab can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer (271):

By the AM-GM inequality, we have the following:

$$8^a + 2^{b+7} = 2^{3a} + 2^{b+7} \geq 2 \cdot \sqrt{2^{3a} \cdot 2^{b+7}} = 2^{\frac{3a+b+9}{2}}.$$

Equality holds only when $2^{3a} = 2^{b+7}$, or $3a = b + 7$.

By the AM-GM inequality, we have the following:

$$2^{a+3} + 8^{b-2} = 2^{a+3} + 2^{3b-6} \geq 2 \cdot \sqrt{2^{a+3} \cdot 2^{3b-6}} = 2^{\frac{a+3b-1}{2}}.$$

Equality holds only when $2^{a+3} = 2^{3b-6}$, or $a + 3 = 3b - 6$.

Multiplying the above two inequalities together, we have

$$\left(8^a + 2^{b+7}\right)\left(2^{a+3} + 8^{b-2}\right) \geq 2^{\frac{3a+b+9}{2}} \cdot 2^{\frac{a+3b-1}{2}} = 4^{a+b+2}.$$

Equality holds only when $3a = b + 7$ and $a + 3 = 3b - 6$. Solving for a and b, we get that $a = \frac{15}{4}$ and $b = \frac{17}{4}$. As we need the equality to hold, the only possible value of the product ab is $\frac{255}{16}$, so the requested answer is $255 + 16 = \boxed{271}$. ■

Problem 11:

(stayhomedomath) A positive integer n is called *un-two* if there does not exist an ordered triple of integers (a, b, c) such that exactly two of

$$\frac{7a+b}{n}, \frac{7b+c}{n}, \frac{7c+a}{n}$$

are integers. Find the sum of all un-two positive integers.

Answer (660):

First, assume that n is an un-two positive integer. If $a = 1$, $b = -7$, and $c = 7^2$, then both $7a+b$ and $7b+c$ are equal to 0, which must be divisible by n. Thus, $7c+a = 7^3+1$ must also be divisible by n, so n must be a divisor of $7^3 + 1$.

Next, assume that integers a, b, and c fulfill that there is at least two of $7a + b$, $7b + c$, and $7c + a$ are divisible by n. We claim that all of them are divisible by n if $n \mid 7^3 + 1$.

Without loss of generality, we assume that $7a+b$ and $7b+c$ are divisible by n. Then,

$$\begin{aligned}7c+a &= 7c-7^3a+(7^3+1)a \\ &= (7c+7^2b)-(7^2b+7^3a)+(7^3+1)a \\ &= 7(7b+c)-7^2(7a+b)+(7^3+1)a.\end{aligned}$$

As all three of $7b+c$, $7a+b$, and 7^3+1 are divisible by n, we have that $7c+a$ must also be divisible by n.

Therefore, a positive integer is un-two if and only if it is a divisor of 7^3+1. Our remaining step is to find the sum of the divisors of 7^3+1.

We have that

$$7^3+1=(7+1)(7^2-7+1)=8\cdot 43=2^3\cdot 43,$$

so the sum of the divisors is

$$(1+2+4+8)(1+43)=15\cdot 44=660.$$

Thus, the requested answer is $\boxed{660}$. ■

Problem 12:

(treemath) A sequence of polynomials is defined by the recursion $P_1(x)=x+1$ and

$$P_n(x)=\frac{(P_{n-1}(x)+1)^5-(P_{n-1}(-x)+1)^5}{2}$$

for all $n\geq 2$. Find the remainder when $P_{2022}(1)$ is divided by 1000.

Answer (616):

The recursion is applying the second roots of unity filter to $(P_n(x)+1)^5$ that only returns terms with odd degree. For the first step, $P_2(x)$ is the odd-degree terms of $(x+2)^5$, which is x^5+40x^3+80x. For all subsequent steps, we again apply the second roots of unity filter to $(P_n(x)+1)^5$ that only returns terms with odd degree.

In general, if a polynomial $P(x)$ has only terms of odd degree, then $(P(x))^n$ will only have terms of odd degree if n is odd and will only have terms of even degree if n is even. The proof for this is left as an exercise to the reader. Since $P_n(x)$ will have all odd coefficients for all $n\geq 2$, we can treat $P_n(x)$ as a single term with odd degree. Thus, by the Binomial Theorem, we get that

$$P_{n+1}(x)=(P_n(x))^5+10(P_n(x))^3+5(P_n(x))$$

for all $n \geq 2$.

Next, we see that $P_2(1) = 121$, which is equivalent to 1 modulo 8. We see that

$$P_3(1) \equiv 1^5 + 10(1)^3 + 5(1) \equiv 0 \pmod{8},$$

so $P_n(1) \equiv 0 \pmod{8}$ for all $n \geq 3$.

Next, we see that $121 \equiv -4 \pmod{125}$. Computing, we get

$$P_3(1) \equiv (-4)^5 + 10(-4)^3 + 5(-4) \equiv 66 \pmod{125}.$$

We can rewrite 66 as $50 + 16$. From here, we get that

$$\begin{aligned} P_4(1) &\equiv (50+16)^5 + 10(50+16)^3 + 5(50+16) \\ &\equiv 16^5 + 10(16)^3 + 5(16) \\ &\equiv 116 \pmod{125}. \end{aligned}$$

Note that $116 = 100 + 16$, so for any $n \geq 5$, we see that

$$\begin{aligned} P_n(1) &\equiv (100+16)^5 + 10(100+16)^3 + 5(100+16) \\ &\equiv 16^5 + 10(16)^3 + 5(16) \\ &\equiv 116 \pmod{125}. \end{aligned}$$

Thus, we have that $P_{2022}(1)$ is equivalent to 0 modulo 8 and 116 modulo 125. By the Chinese Remainder Theorem, the requested answer is $\boxed{616}$. ■

Problem 13:

(treemath) A spinner has five sectors numbered -1.25, -1, 0, 1, and 1.25, each of which are equally likely to be spun. Ryan starts by writing the number 1 on a blank piece of paper. Each minute, Ryan spins the spinner randomly and overwrites the number currently on the paper with the number multiplied by the number the spinner lands on. The expected value of the largest number Ryan ever writes on the paper can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer (017):

To approach this problem, we will find the expected contribution of the nth term; that is, the probability that the nth term is the largest number (last positive number) times the expected absolute value of that term. This simplifies things since all numbers until the first 0 are equally likely to be positive or negative. Additionally, multiplication of the absolute value by 1 or 1.25 happens with a $\frac{2}{5}$ probability each, and a 0 occurs with probability $\frac{1}{5}$. Let $a_0 = 1$ be the 0th term.

In order for the nth term ($n \geq 1$) to be the last positive number, the first n terms must not be 0, the nth term must be positive, and all subsequent terms until the first 0 must be negative. This probability is

$$\left(\frac{4}{5}\right)^n \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}\left(\frac{4}{5}\right)^n.$$

The 0th term has a $\frac{1}{3}$ probability of being the last positive number, so the probabilities all add up properly. Using the Binomial Theorem, the expected absolute value of the last positive term (given that it is not 0) is

$$\sum_{i=0}^{n} 1.25^i \cdot \frac{\binom{n}{i}}{2^n} = \left(\frac{9}{8}\right)^n.$$

Finally, the expected contribution from the nth term is

$$\frac{1}{6}\left(\frac{9}{8} \cdot \frac{4}{5}\right)^n = \frac{1}{6}\left(\frac{9}{10}\right)^n.$$

The total expected value happily converges:

$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{9}{10}\right)^n = \frac{1}{3} + \frac{1}{6} \cdot \frac{\frac{9}{10}}{1 - \frac{9}{10}} = \frac{11}{6}.$$

Thus, the requested answer is $11 + 6 = \boxed{017}$. ■

Problem 14:

(Awesome_guy) Let $\triangle ABC$ be acute with $\angle BAC = 45^\circ$. Let $\overline{AD}$ be an altitude of $\triangle ABC$, let E be the midpoint of $\overline{BC}$, and let F be the midpoint of $\overline{AD}$. Let O be the center of the circumcircle of $\triangle ABC$, let K be the intersection of lines DO and EF, and let L be the foot of the perpendicular from O to line AK. If $BL = 6$ and $CL = 8$, find AL^2.

Answer (048):

Reflect O over side $\overline{BC}$ to a point P. Due to reflections, we have that E is the midpoint of $\overline{OP}$, and we are given that E is also the midpoint of $\overline{BC}$, so $BOCP$ is a parallelogram. We also have that $\angle BAC = 45^\circ$, so $\angle BOC = 90^\circ$. Since $BO = CO$, we have that $BOCP$ is a square.

Next, we claim that points A, K, and P are collinear. We have that $\angle ADB = \angle OEB = 90^\circ$, so $AD \parallel OP$. Thus, $\triangle DKF \sim \triangle OKE$. We also have that $AF : DF = OE :$

$PE = 1 : 1$, so $\triangle AKD \sim \triangle PKO$ by spiral similarity. It then follows that points A, K, and P are collinear, as desired.

Now, we see that $\angle ALO = \angle PLO = 90°$, so L lies on the circumcircle of $BOCP$. By the Pythagorean Theorem,

$$BC^2 = BL^2 + CL^2 = 64 + 36 = 100 \implies BC = 10.$$

By Ptolemy's Theorem on quadrilateral $BCLO$,

$$\begin{aligned} BC \cdot LO + CL \cdot BO &= BL \cdot CO \\ \implies 10 \cdot LO + 6 \cdot 5\sqrt{2} &= 8 \cdot 5\sqrt{2} \\ \implies LO &= \sqrt{2}. \end{aligned}$$

Finally, we have that $AO = 5\sqrt{2}$ (radius of the circumcircle of $\triangle ABC$), $LO = \sqrt{2}$, and $\angle ALO = 90°$, so by the Pythagorean Theorem,

$$AL^2 = AO^2 - LO^2 = (5\sqrt{2})^2 - (\sqrt{2})^2 = 50 - 2 = \boxed{048},$$

as requested. ■

Problem 15:

(ApraTrip) For positive integers n, let $f(n)$ denote the number of integers $1 \le a \le 130$ for which there exists some integer b such that $a^b - n$ is divisible by 131, and let $g(n)$ denote the sum of all such a. Find the remainder when

$$\sum_{n=1}^{130} [f(n) \cdot g(n)]$$

is divided by 131.

Answer (054):

Let all sums in this solution be taken modulo 131, and let $\text{ord}_{131}(x)$ be the order of x modulo 131.

We claim that there exists an integer s such that $r^s \equiv x \pmod{131}$ if and only if $\text{ord}_{131}(x) \mid \text{ord}_{131}(r)$. Let g be a primitive root $\pmod{131}$. Notice that when $\text{ord}_{131}(x) \mid \text{ord}_{131}(r)$, then $\frac{130}{\text{ord}_{131}(r)} \mid \frac{130}{\text{ord}_{131}(x)}$. Thus, since $r \equiv g^{\frac{130a}{\text{ord}_{131}(r)}}$ and $x \equiv g^{\frac{130b}{\text{ord}_{131}(x)}}$ for some a and b where $\gcd(a, 130) = 1$ and $\gcd(b, 130) = 1$, if we let $c \equiv \frac{b}{a}$

(mod 130) and $d = \frac{\frac{130}{\text{ord}_{131}(x)}}{\frac{130}{\text{ord}_{131}(r)}}$, then

$$r^{cd} = g^{\frac{130ac}{\text{ord}_{131}(x)}} = g^{\frac{130b}{\text{ord}_{131}(x)}} = x.$$

Thus, there exists an integer s such that $r^s \equiv x \pmod{131}$ if $\text{ord}_{131}(x) \mid \text{ord}_{131}(r)$.

Now, notice that if $r^s \equiv x \pmod{131}$, then

$$x^{\text{ord}_{131}(r)} \equiv r^{s \cdot \text{ord}_{131}(r)} \equiv 1 \pmod{131},$$

which implies that $\text{ord}_{131}(x) \mid \gcd(\text{ord}_{131}(r), 130) = \text{ord}_{131}(r)$, as $\text{ord}_{131}(r) \mid 130$. Thus, our claim is true.

Thus, $f(x)$ is the number of integers $1 \le r \le 130$ such that $\text{ord}_{131}(x) \mid \text{ord}_{131}(r)$, and $g(x)$ is the sum of all integers $1 \le r \le 130$ such that $\text{ord}_{131}(x) \mid \text{ord}_{131}(r)$. We will deal with both functions separately, and put them together at the end.

$f(x)$: It's well known that there are $\varphi(n)$ numbers (mod p) with order n where p is prime and $n \mid p-1$. Thus,

$$\begin{aligned} f(x) &= \sum_{\substack{\text{ord}_{131}(x) \mid d, \\ d \mid 130}} \varphi(d) = \sum_{d \mid \frac{130}{\text{ord}_{131}(x)}} [\varphi(\text{ord}_{131}(x)) \cdot \varphi(d)] \\ &= \varphi(\text{ord}_{131}(x)) \cdot \sum_{d \mid \frac{130}{\text{ord}_{131}(x)}} \varphi(d) = \frac{130\varphi(\text{ord}_{131}(x))}{\text{ord}_{131}(x)}, \end{aligned}$$

where the second equality follows as $\varphi(x)$ is multiplicative and any two factors of 130 which multiply to be a factor of 130 are relatively prime (as 130 isn't divisible by any perfect powers).

$g(x)$: It's known that for all primes p such that that $p-1$ isn't divisible by any perfect powers, the sum of all numbers with order d (for some $d \mid p-1$) is $(-1)^{\text{the number of prime factors of } d}$ (see 2021 DIME #14). Thus, in order to find $g(x)$, we'll do casework on the number of factors of $\text{ord}_{131}(x)$, while utilizing the fact that 130 has three prime factors.

Case 1: $\text{ord}_{131}(x)$ has 0 prime factors.

In this case, there would be 1 multiple of $\text{ord}_{131}(x)$ with 0 factors that divides 130, 3 multiples of $\text{ord}_{131}(x)$ with one factor that divide 130, 3 multiples of $\text{ord}_{131}(x)$ with two factors that divide 130, and 1 multiple of $\text{ord}_{131}(x)$ with three factors that divides 130. Thus, $g(x) = 1 - 3 + 3 - 1 = 0$ when $\text{ord}_{131}(x)$ has 0 prime factors.

Case 2: $\text{ord}_{131}(x)$ has 1 prime factor.

In this case, there would be 1 multiple of $\text{ord}_{131}(x)$ with one factor that divides 130, 2 multiples of $\text{ord}_{131}(x)$ with two factors that divide 130, and 1 multiple of $\text{ord}_{131}(x)$ with three factors that divides 130. Thus, $g(x) = -1 + 2 - 1 = 0$ when $\text{ord}_{131}(x)$ has 1 prime factor.

Case 3: $\text{ord}_{131}(x)$ has 2 prime factors.

In this case, there would be 1 multiple of $\text{ord}_{131}(x)$ with two factors that divides 130, and 1 multiple of $\text{ord}_{131}(x)$ with three factors that divides 130. Thus, $g(x) = 1 - 1 = 0$ when $\text{ord}_{131}(x)$ has 2 prime factors. At this point, it might seem that $g(x)$ is 0 for all x (which would make this a very trolly problem), but this is not true.

Case 4: $\text{ord}_{131}(x)$ has 3 prime factors.

In this case, there would be 1 multiple of $\text{ord}_{131}(x)$ with three factors that divides 130. Thus, $g(x) = -1$ when $\text{ord}_{131}(x)$ has three prime factors.

Thus, $g(x) = 0$ when x is not a primitive root, and $g(x) = -1$ when x is a primitive root. Thus, our sum becomes

$$\begin{aligned}\sum_{\text{ord}_{131}(x)=130} [f(x) \cdot g(x)] &= -\sum_{\text{ord}_{131}(x)=130} f(x) \\ &= -\sum_{\text{ord}_{131}(x)=130} \frac{130\varphi(130)}{130} \\ &= -\varphi(130)^2 \\ &= -2304.\end{aligned}$$

Thus, the requested remainder when our sum is divided by 131 is $\boxed{054}$. ■