De Mathematics Competitions

2nd Annual

DMC 10 B

Friday, June 11, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. For each question, only one answer choice is correct.
3. Mark your answer to each problem on the DMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, rulers, and erasers are allowed as aids. Prohibited materials include calculators, smartwatches, phones, computing devices, compasses, protractors, and graph paper. No problems on the competition will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the competition, your non-existent proctor will not ask you to record certain information on the answer form.
8. You will have 75 minutes to complete the competition. You can discuss only with people that have already taken the competition in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

The De Mathematics Competitions reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

The publication, reproduction or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

Students who score well on this DMC 10 may or may not be invited to the 2022 DIME. More details about the DIME and other information are on the back page of this test booklet.

1. What is the value of
$$2^0 \times 2^1 + 2^0 \times 2^2?$$

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

2. How many single-digit positive integers n are there such that $2n$ is a perfect square?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

3. If n is a positive integer such that $n \times 3^5 = 3^7 - 3^5$, what is n?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 9

4. What is the smallest positive integer n such that $n! + 1$ is not divisible by any integer between 2 and 9, inclusive?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

5. A square has a perimeter which is twice the area of the circle inscribed in the square. What is the circumference of the circle?

(A) 8 **(B)** 16 **(C)** 32 **(D)** 64 **(E)** 128

6. For what values of k does the equation
$$k^2 x + 2 = 4x + k$$
have no real solutions x?

(A) -2 **(B)** 0 **(C)** 2 **(D)** -2 and 0 **(E)** -2 and 2

7. Justin has three weightless boxes and four pebbles, each of which has a weight of either 3, 4, or 5 ounces. He puts each pebble in one of the boxes such that each box has at least one pebble in it. If the weights of the boxes form an increasing arithmetic progression, what is the largest possible weight of the heaviest box, in ounces?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

8. At Test Academy, there are four classes, one on each of the four floors of the building. For each class, the class which is one floor above it has twice as many students and half the average grade of that class. If the average grade of all four classes combined is 20, what is the average grade of the class on the bottom floor?

(A) 75 **(B)** 80 **(C)** 85 **(D)** 90 **(E)** 95

9. What is the value of

$$\frac{9^{1010}+3^{2021}}{9^{1009}+3^{2020}}?$$

(A) 3.5 **(B)** 3.6 **(C)** 3.7 **(D)** 3.8 **(E)** 3.9

10. How many ordered triples of integers (a, b, c) are there such that the product

$$(a-2020)(2b-2021)(3c-2022)$$

is positive and has exactly three positive divisors?

(A) 3 **(B)** 9 **(C)** 12 **(D)** 24 **(E)** infinitely many

11. John is on the infinite grid below, where every row and column of consecutive squares repeats the pattern 1, 2, 3, 4 from left to right and up to down. From any square, John may move up, down, left, or right one square. Starting at the center square labeled 1, in how many sequences of 4 moves can John land on squares labeled 1, 2, 3, and 4 in some order? (The starting square does not count unless he lands on it again.)

$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	1	2	3	4	1	$\cdots$
$\cdots$	2	3	4	1	2	$\cdots$
$\cdots$	3	4	1	2	3	$\cdots$
$\cdots$	4	1	2	3	4	$\cdots$
$\cdots$	1	2	3	4	1	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

(A) 16 **(B)** 32 **(C)** 48 **(D)** 64 **(E)** 96

12. In regular hexagon $ABCDEF$, diagonals $\overline{AC}$ and $\overline{BF}$ intersect at a point G. If the area of $\triangle ABG$ is 2, what is the area of pentagon $CDEFG$?

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

13. Two functions f and g, in that order, are said to be *rivals* if there does not exist a real number x such that $f(x) = g(f(x))$. If f and g are linear, non-constant, and rivals, which of the following sets contains all possible values that $g(1)$ can never take?

(A) $\{-1\}$ **(B)** $\{0\}$ **(C)** $\{1\}$ **(D)** $\{-1, 1\}$ **(E)** the empty set

14. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. A circle passes through A and B and intersects side $\overline{CD}$ at two points which trisect the side. What is the area of the circle?

(A) 6π **(B)** 7π **(C)** 8π **(D)** 9π **(E)** 10π

15. Let x be a positive real number such that

$$\frac{1}{x - \frac{1}{x}} = \sqrt{x^2 + \frac{x^4}{4}}.$$

What is the value of x^2?

(A) $4 - 2\sqrt{2}$ **(B)** $\sqrt{2}$ **(C)** $2\sqrt{2} - 1$ **(D)** 2 **(E)** $\sqrt{2} + 1$

16. The degree measures of an interior angle of each of three regular polygons form an arithmetic progression of positive integers. If the polygon with the most sides has 360 sides, what is the smallest possible number of sides any of the polygons can have?

(A) 8 **(B)** 16 **(C)** 24 **(D)** 32 **(E)** 40

17. There exists a sequence $a_1, a_2, \ldots, a_6$ of positive integers such that for every term in the sequence, there exists another term in the sequence which is equal to that term. How many possible values of the product $a_1 a_2 \cdots a_6$ less than 1000 are there?

(A) 36 **(B)** 37 **(C)** 38 **(D)** 39 **(E)** 40

18. In trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$, $AB = 4$, and $AD = BC = 5$, let the angle bisector of $\angle ADC$ intersect the diagonal $\overline{AC}$ at a point P. If line BP intersects the side $\overline{CD}$ at a point Q such that $CQ = 8$, what is the area of trapezoid $ABCD$?

(A) 24 **(B)** 28 **(C)** 32 **(D)** 36 **(E)** 40

19. Six red balls and six blue balls are each numbered from 1 to 6. How many ways are there to form six pairs of one red ball and one blue ball such that the product of the two numbers on the balls in every pair is divisible by at least one of 2 and 3?

(A) 288 **(B)** 360 **(C)** 432 **(D)** 504 **(E)** 576

20. At a motel, there are 15 rooms in a row. A visitor may rent 1 room for 5 dollars, or 2 adjacent rooms for 4 dollars each. At most 1 visitor may rent a given room at a time, and no 2 visitors may rent rooms adjacent to each other. If the leftmost and rightmost rooms must be rented, what is the largest dollar amount that the motel can earn?

(A) 40 **(B)** 41 **(C)** 42 **(D)** 43 **(E)** 44

21. A convex quadrilateral $ABCD$ has $\angle ADC = \angle BAC = 90^\circ$ and side lengths $AB = 6$, $BC = 9$, and $CD = 5$. Let M be the midpoint of diagonal $\overline{BD}$. What is MC^2?

(A) 26 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 30

22. Bill and Ben each have 2 fair coins. In each turn, they flip all their coins at the same time, if they have any. If a coin lands heads, then the other person gets that coin. If a coin lands tails, then that coin stays with the same person. What is the probability that after Bill and Ben take exactly 3 turns, they each end up with 2 coins?

(A) $\frac{1}{4}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{7}{16}$ **(E)** $\frac{1}{2}$

23. What is the sum of the digits of the smallest positive integer n such that

$$\sqrt{5n-1}-\sqrt{5n-2}+\sqrt{5n-3}-\sqrt{5n-4}$$

is less than 0.05?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

24. In $\triangle ABC$ with $AB = 3$ and $AC = 6$, let D be the intersection of the angle bisector of $\angle BAC$ and $\overline{BC}$, and let M be the midpoint of $\overline{AC}$. Let the circumcircle of $\triangle DMC$ intersect line AD again at P, distinct from D. If $DM = 2$, what is PC^2?

(A) $\frac{72}{5}$ **(B)** $\frac{78}{5}$ **(C)** $\frac{84}{5}$ **(D)** 18 **(E)** $\frac{96}{5}$

25. Let x and y be distinct real numbers chosen at random from the interval $[-1, 1]$, excluding 0. What is the probability that

$$\left\lfloor \frac{|x|}{|y|} \right\rfloor \geq \left\lfloor \frac{|x+y|}{|x-y|} \right\rfloor,$$

where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to a real number r?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{9}$ **(C)** $\frac{9}{16}$ **(D)** $\frac{7}{12}$ **(E)** $\frac{5}{8}$

DMC 10 B

DO NOT OPEN UNTIL FRIDAY, June 11, 2021

****Administration on an earlier date will disqualify your results.****

- All the information needed to administer this exam is not contained in the non-existent DMC 10 Teacher's Manual. PLEASE READ THE MANUAL BEFORE FRIDAY, JUNE 11, 2021.
- Send **DeToasty3**, **nikenissan**, **pog**, and **vsamc** a private message submitting your answers to the DMC 10. AoPS is the only way to submit your answers.
- The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

For more information about the DMC and our other competitions, please visit `https://detoasty3.github.io/dmc.html`.

Questions and comments about this competition should be sent to:

DeToasty3.

The problems and solutions for this DMC 10 were prepared by the DMC Editorial Board under the direction of:

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