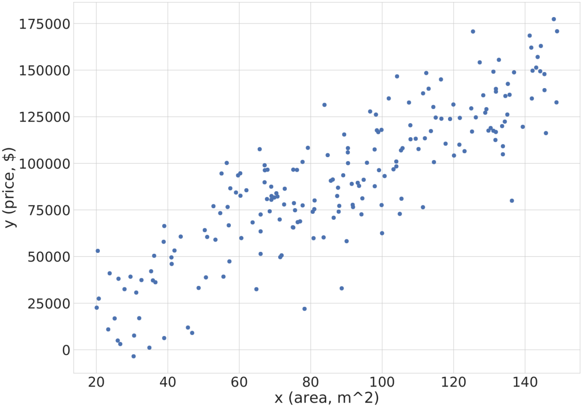
Section 1

Basic Linear Regression

Example: Housing price prediction

Given is a dataset $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^{N}$, of house areas x_i and corresponding prices y_i.

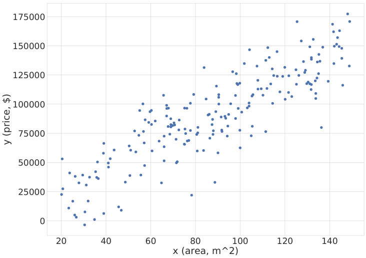


How do we estimate a price of a new house with area x_{new}?

Regression problem

Given

- observations [1]
  $\boldsymbol{X} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N\}$, $\boldsymbol{x}_i \in \mathbb{R}^D$
- targets
  $\boldsymbol{y} = \{y_1, y_2, \dots, y_N\}, \quad y_i \in \mathbb{R}$



Find

- Mapping $f(\cdot)$ from inputs to targets

$$y_i \approx f(\boldsymbol{x}_i)$$

[1] A common way to represent the samples is as a data matrix $\boldsymbol{X} \in \mathbb{R}^{N \times D}$, where each row represents one sample.

Linear model

Target y is generated by a deterministic function f of $\boldsymbol{x}$ plus noise

$$y_i = f(\boldsymbol{x}_i) + \epsilon_i, \qquad \epsilon_i \sim \mathcal{N}(0, \beta^{-1}) \tag{1}$$

Let's choose $f(\boldsymbol{x})$ to be a linear function

$$\begin{aligned} f_{\boldsymbol{w}}(\boldsymbol{x}_i) &= w_0 + w_1 x_{i1} + w_2 x_{i2} + \ldots + w_D x_{iD} \qquad (2) \\ &= w_0 + \boldsymbol{w}^T \boldsymbol{x}_i \qquad (3) \end{aligned}$$

From now we will always assume that the bias term is absorbed into the $\boldsymbol{x}$ vector

Absorbing the bias term

The linear function is given by

$$f_{\boldsymbol{w}}(\boldsymbol{x}) = w_0 + w_1 x_1 + w_2 x_2 + \ldots + w_D x_D \tag{4}$$
$$= w_0 + \boldsymbol{w}^T \boldsymbol{x} \tag{5}$$

Here w_0 is called bias or offset term. For simplicity, we can "absorb" it by prepending a 1 to the feature vector $\boldsymbol{x}$ and respectively adding w_0 to the weight vector $\boldsymbol{w}$:

$$\tilde{\boldsymbol{x}} = (1, x_1, \ldots, x_D)^T \qquad \tilde{\boldsymbol{w}} = (w_0, w_1, \ldots, w_D)^T$$

The function $f_{\boldsymbol{w}}$ can compactly be written as $f_{\boldsymbol{w}}(\boldsymbol{x}) = \tilde{\boldsymbol{w}}^T \tilde{\boldsymbol{x}}$.

To unclutter the notation, we will assume the bias term is always absorbed and write $\boldsymbol{w}$ and $\boldsymbol{x}$ instead of $\tilde{\boldsymbol{w}}$ and $\tilde{\boldsymbol{x}}$.

Now, how do we choose the "best" $\boldsymbol{w}$ that fits our data?

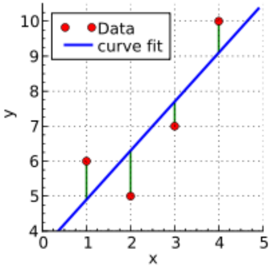
Error function

Error function gives a measure of "misfit" between our model (parametrized by $\boldsymbol{w}$) and observed data $\mathcal{D} = \{(\boldsymbol{x}_i, y_i)\}_{i=1}^{N}$.

Standard choice - least squares (LS) function

$$E_{\mathrm{LS}}(\boldsymbol{w}) = \frac{1}{2}\sum_{i=1}^{N}(f_{\boldsymbol{w}}(\boldsymbol{x}_i) - y_i)^2 \tag{6}$$

$$= \frac{1}{2}\sum_{i=1}^{N}(\boldsymbol{w}^T\boldsymbol{x}_i - y_i)^2 \tag{7}$$



Factor $\frac{1}{2}$ is for later convenience

Objective

Find the optimal weight vector $\boldsymbol{w}^\star$ that minimizes the error

$$\boldsymbol{w}^* = \arg\min_{\boldsymbol{w}} E_{\mathrm{LS}}(\boldsymbol{w}) \tag{8}$$

$$= \arg\min_{\boldsymbol{w}} \frac{1}{2} \sum_{i=1}^{N} (\boldsymbol{x}_i^T \boldsymbol{w} - y_i)^2 \tag{9}$$

By stacking the observations $\boldsymbol{x}_i$ as rows of the matrix $\boldsymbol{X} \in \mathbb{R}^{N \times D}$

$$= \arg\min_{\boldsymbol{w}} \frac{1}{2} (\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y})^T (\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y}) \tag{10}$$

Optimal solution

To find the minimum of the function $E(\boldsymbol{w})$, compute the gradient $\nabla_{\boldsymbol{w}} E(\boldsymbol{w})$:

$$
\begin{aligned}
\nabla_{\boldsymbol{w}} E_{\mathrm{LS}}(\boldsymbol{w}) &= \nabla_{\boldsymbol{w}} \frac{1}{2} (\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y})^T (\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y}) && (11) \\
&= \nabla_{\boldsymbol{w}} \frac{1}{2} \left(\boldsymbol{w}^T \boldsymbol{X}^T \boldsymbol{X} \boldsymbol{w} - 2\boldsymbol{w}^T \boldsymbol{X}^T \boldsymbol{y} + \boldsymbol{y}^T \boldsymbol{y} \right) && (12) \\
&= \boldsymbol{X}^T \boldsymbol{X} \boldsymbol{w} - \boldsymbol{X}^T \boldsymbol{y} && (13)
\end{aligned}
$$

See Equations (69), (81) from Matrix cookbook for details

Optimal solution

Now set the gradient to zero and solve for $\boldsymbol{w}$ to obtain the minimizer [2]

$$\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{w} - \boldsymbol{X}^T\boldsymbol{y} \stackrel{!}{=} 0 \tag{14}$$

This leads to the so-called normal equation of the least squares problem

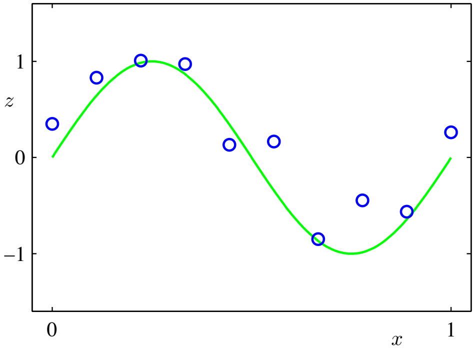
$$\boldsymbol{w}^* = \underbrace{(\boldsymbol{X}^\mathsf{T}\boldsymbol{X})^{-1}\boldsymbol{X}^\mathsf{T}}_{=\boldsymbol{X}^\dagger}\boldsymbol{y} \tag{15}$$

$\boldsymbol{X}^\dagger$ is called Moore-Penrose pseudo-inverse of $\boldsymbol{X}$ (because for an invertible square matrix, $\boldsymbol{X}^\dagger = \boldsymbol{X}^{-1}$).

[2] Because Hessian $\nabla_{\boldsymbol{w}}\nabla_{\boldsymbol{w}}E(\boldsymbol{w})$ is positive (semi)definite $\rightarrow$ see *Optimization*

Nonlinear dependency in data

What if the dependency between y and $\boldsymbol{x}$ is not linear?



Data generating process: $y_i = \sin(2\pi x_i) + \epsilon, \qquad \epsilon \sim \mathcal{N}(0, \beta^{-1})$

For this example assume that data dimensionality $D = 1$

Polynomials

Solution: Polynomials are universal function approximators, so we can define f as

$$f_{\boldsymbol{w}}(x) = w_0 + \sum_{j=1}^{M} w_j x^j \tag{16}$$

Or more generally

$$= w_0 + \sum_{j=1}^{M} w_j \phi_j(x) \tag{17}$$

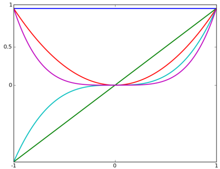
Define $\phi_0 = 1$

$$= \boldsymbol{w}^T \boldsymbol{\phi}(x) \tag{18}$$

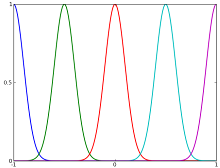
The function f is still linear in $\boldsymbol{w}$ (despite not being linear in $\boldsymbol{x}$)!

Typical basis functions

Polynomials $\phi_j(x) = x^j$

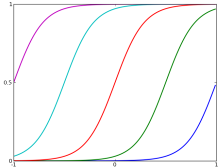


Gaussian $\phi_j(x) = \mathrm{e}^{\frac{-(x-\mu_j)^2}{2s^2}}$



Logistic Sigmoid $\phi_j(x) = \sigma(\frac{x-\mu_j}{s})$,

where $\sigma(a) = \frac{1}{1+\mathrm{e}^{-a}}$



Linear basis function model

Prediction for one sample

$$f_{\boldsymbol{w}}(\boldsymbol{x}) = w_0 + \sum_{j=1}^{M} w_j \phi_j(\boldsymbol{x}) = \boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}) \tag{19}$$

Using the same least squares error function as before

$$\begin{aligned} E_{\mathrm{LS}}(\boldsymbol{w}) &= \frac{1}{2} \sum^{N} \left(\boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i \right)^2 & (20) \\ &= \frac{1}{2} (\boldsymbol{\Phi} \boldsymbol{w} - \boldsymbol{y})^T (\boldsymbol{\Phi} \boldsymbol{w} - \boldsymbol{y}) & (21) \end{aligned}$$

with

$$\boldsymbol{\Phi} = \begin{pmatrix} \phi_0(\boldsymbol{x}_1) & \phi_1(\boldsymbol{x}_1) & \dots & \phi_M(\boldsymbol{x}_1) \\ \phi_0(\boldsymbol{x}_2) & \phi_1(\boldsymbol{x}_2) & & \vdots \\ \vdots & \vdots & \ddots & \\ \phi_0(\boldsymbol{x}_N) & \phi_1(\boldsymbol{x}_N) & \dots & \phi_M(\boldsymbol{x}_N) \end{pmatrix} \in \mathbb{R}^{N \times (M+1)}$$

being the design matrix of $\boldsymbol{\phi}$.

Optimal solution

Recall Equation 10 - we have the same expression except that data matrix $\boldsymbol{X} \in \mathbb{R}^{N \times D}$ is replaced by design matrix $\boldsymbol{\Phi} \in \mathbb{R}^{N \times (M+1)}$

$$E_{\mathrm{LS}}(\boldsymbol{w}) = \frac{1}{2}(\boldsymbol{\Phi w} - \boldsymbol{y})^T(\boldsymbol{\Phi w} - \boldsymbol{y}) \tag{22}$$

This means that the optimal weights $\boldsymbol{w}^*$ can be obtained in a similar way

$$\begin{aligned} \boldsymbol{w}^* &= (\boldsymbol{\Phi}^\mathsf{T}\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}^\mathsf{T}\boldsymbol{y} && (23) \\ &= \boldsymbol{\Phi}^\dagger \boldsymbol{y} && (24) \end{aligned}$$

Compare this to Equation 15:

$$\boldsymbol{w}^* = (\boldsymbol{X}^\mathsf{T}\boldsymbol{X})^{-1}\boldsymbol{X}^\mathsf{T}\boldsymbol{y} \tag{25}$$

Choosing degree of the polynomial

How do we choose the degree of the polynomial M?

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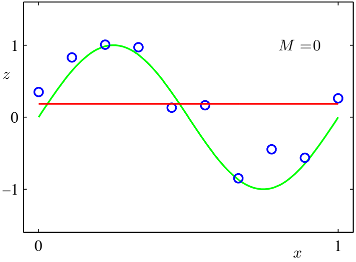

Choosing degree of the polynomial

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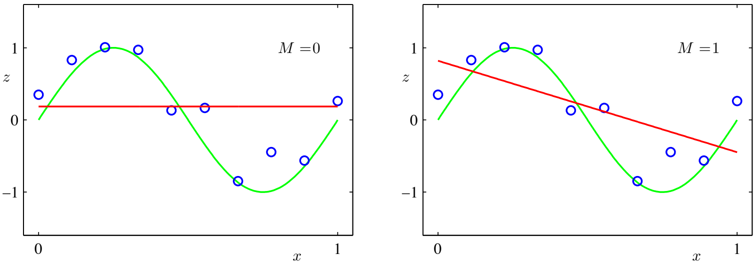

Choosing degree of the polynomial

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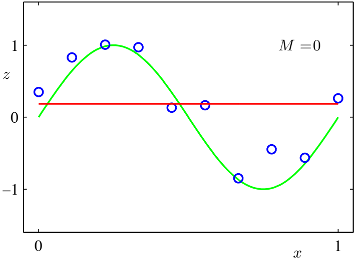

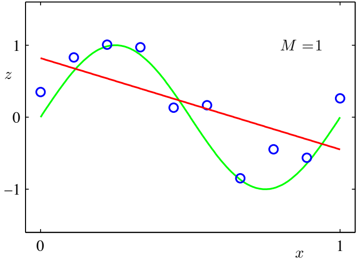

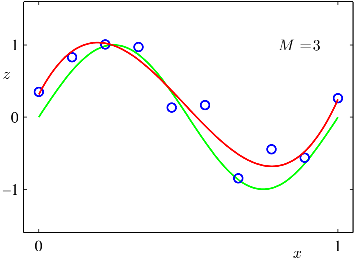

Choosing degree of the polynomial

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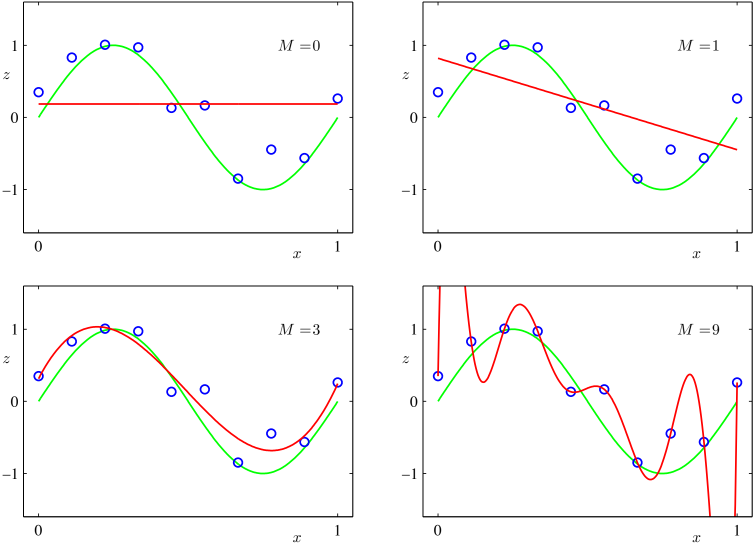
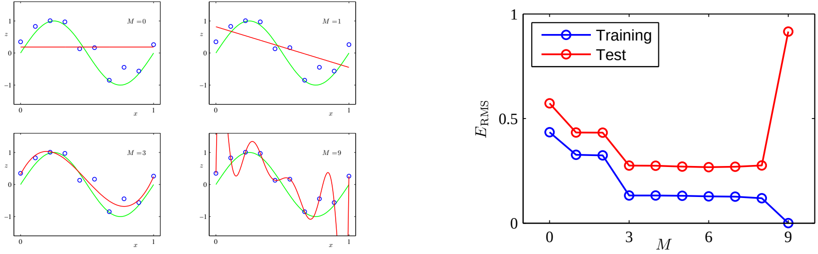
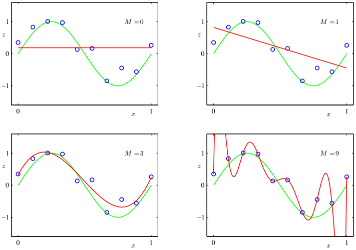


Choosing degree of the polynomial



One valid solution is to choose M using the standard train-validation split approach.

Choosing degree of the polynomial



	$M=0$	$M=1$	$M=6$	$M=9$
$w_0^\star$	0.19	0.82	0.31	0.35
$w_1^\star$		-1.27	7.99	232.37
$w_2^\star$			-25.43	-5321.83
$w_3^\star$			17.37	48568.31
$w_4^\star$				-231639.30
$w_5^\star$				640042.26
$w_6^\star$				-1061800.52
$w_7^\star$				1042400.18
$w_8^\star$				-557682.99
$w_9^\star$				125201.43

We also make another observation: overfitting occurs when the coefficients $\boldsymbol{w}$ become large.

What if we penalize large weights?

Controlling overfitting with regularization

Least squares loss function with L2 regularization
(also called ridge regression)

$$E_{\text{ridge}}(\boldsymbol{w}) = \frac{1}{2} \sum^{N} \left[\boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i\right]^2 + \frac{\lambda}{2} \|\boldsymbol{w}\|_2^2 \tag{26}$$

Where,

- $\|\boldsymbol{w}\|_2^2 \equiv \boldsymbol{w}^T \boldsymbol{w} = w_0^2 + w_1^2 + w_2^2 + \cdots + w_M^2$ - L2 norm
- $\lambda -$ regularization strength

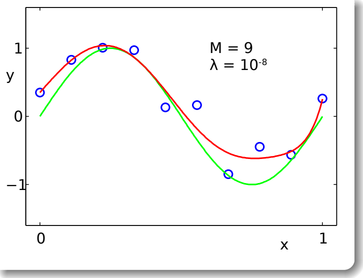
Controlling overfitting with regularization

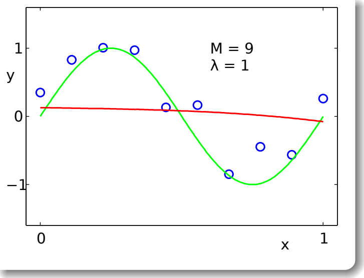
Least squares loss function with L2 regularization
(also called ridge regression)

$$E_{\text{ridge}}(\boldsymbol{w}) = \frac{1}{2} \sum^{N} \left[\boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i\right]^2 + \frac{\lambda}{2} \|\boldsymbol{w}\|_2^2 \qquad (26)$$

Where,

- $\|\boldsymbol{w}\|_2^2 \equiv \boldsymbol{w}^T \boldsymbol{w} = w_0^2 + w_1^2 + w_2^2 + \cdots + w_M^2$ - L2 norm
- λ – regularization strength





Larger regularization strength λ leads to smaller weights $\boldsymbol{w}$

Section 2

Probabilistic Linear Regression

Probabilistic formulation

Remember from our problem definition at the start of the lecture,

$$y_i = f_{\boldsymbol{w}}(\boldsymbol{x}_i) + \underbrace{\epsilon_i}_{\text{noise}}$$

Noise has zero-mean Gaussian distribution with a fixed precision $\beta = \frac{1}{\sigma^2}$

$$\epsilon_i \sim \mathcal{N}(0, \beta^{-1})$$

This implies that the distribution of the targets is

$$y_i \sim \mathcal{N}(f_{\boldsymbol{w}}(\boldsymbol{x}_i), \beta^{-1})$$

Remember: any function can be represented as $f_{\boldsymbol{w}}(\boldsymbol{x}_i) = \boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}_i)$

Maximum likelihood

Likelihood of a single sample

$$p(y_i \mid f_{\boldsymbol{w}}(\boldsymbol{x}_i), \beta) = \mathcal{N}(y_i \mid f_{\boldsymbol{w}}(\boldsymbol{x}_i), \beta^{-1}) \tag{27}$$

Assume that the samples are drawn i.i.d.
$\implies$ likelihood of the entire dataset $\mathcal{D} = \{\boldsymbol{X}, \boldsymbol{y}\}$ is

$$p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta) = \prod_{i=1}^{N} p(y_i \mid f_{\boldsymbol{w}}(\boldsymbol{x}_i), \beta) \tag{28}$$

We can now use the same approach we used in previous lecture - maximize the likelihood w.r.t. $\boldsymbol{w}$ and β

$$\boldsymbol{w}_{\mathrm{ML}}, \beta_{\mathrm{ML}} = \arg\max_{\boldsymbol{w}, \beta} p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta) \tag{29}$$

Maximum likelihood

Like in the coin flip example, we can make a few simplifications

$$\begin{aligned}\boldsymbol{w}_{\mathrm{ML}}, \beta_{\mathrm{ML}} &= \arg\max_{\boldsymbol{w},\beta} p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta) &(30)\\ &= \arg\max_{\boldsymbol{w},\beta} \ln p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta) &(31)\\ &= \arg\min_{\boldsymbol{w},\beta} - \ln p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta) &(32)\end{aligned}$$

Let's denote this quantity as maximum likelihood error function that we need to minimize

$$E_{\mathrm{ML}}(\boldsymbol{w}, \beta) = - \ln p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta) \tag{33}$$

Maximum likelihood

Simplify the error function

$$E_{\mathrm{ML}}(\boldsymbol{w}, \beta) = -\ln \left[\prod_{i=1}^{N} \mathcal{N}(y_i \mid f_{\boldsymbol{w}}(\boldsymbol{x}_i), \beta^{-1}) \right] \tag{34}$$

$$= -\ln \left[\prod_{i=1}^{N} \sqrt{\frac{\beta}{2\pi}} \exp\left(-\frac{\beta}{2} (\boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i)^2 \right) \right] \tag{35}$$

$$= -\sum_{i=1}^{N} \ln \left[\sqrt{\frac{\beta}{2\pi}} \exp\left(-\frac{\beta}{2} (\boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i)^2 \right) \right] \tag{36}$$

$$= \frac{\beta}{2} \sum_{i=1}^{N} (\boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i)^2 - \frac{N}{2} \ln \beta + \frac{N}{2} \ln 2\pi \tag{37}$$

Optimizing log-likelihood w.r.t. $\boldsymbol{w}$

$$\boldsymbol{w}_{\mathrm{ML}} = \arg\min_{\boldsymbol{w}} E_{\mathrm{ML}}(\boldsymbol{w}, \beta) \tag{38}$$

$$= \arg\min_{\boldsymbol{w}} \left[\frac{\beta}{2} \sum_{i=1}^{N} (\boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i)^2 \underbrace{- \frac{N}{2} \ln \beta + \frac{N}{2} \ln 2\pi}_{= \text{const}} \right] \tag{39}$$

$$= \arg\min_{\boldsymbol{w}} \underbrace{\frac{1}{2} \sum_{i=1}^{N} (\boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i)^2}_{\text{least squares error fn!}} \tag{40}$$

$$= \arg\min_{\boldsymbol{w}} E_{\mathrm{LS}}(\boldsymbol{w}) \tag{41}$$

Optimizing log-likelihood w.r.t. $\boldsymbol{w}$

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$$= \arg\min_{\boldsymbol{w}} E_{\mathrm{LS}}(\boldsymbol{w}) \tag{41}$$

Maximizing the likelihood is equivalent to minimizing the least squares error function!

$$\boldsymbol{w}_{\mathrm{ML}} = (\boldsymbol{\Phi}^{\mathsf{T}} \boldsymbol{\Phi})^{-1} \boldsymbol{\Phi}^{\mathsf{T}} \boldsymbol{y} = \boldsymbol{\Phi}^{\dagger} \boldsymbol{y} \tag{42}$$

Optimizing log-likelihood w.r.t. β

Plug in the estimate for $\boldsymbol{w}$ and minimize w.r.t. β

$$\beta_{\mathrm{ML}} = \arg\min_{\beta} E_{\mathrm{ML}}(\boldsymbol{w}_{\mathrm{ML}}, \beta) \tag{43}$$

$$= \arg\min_{\beta} \left[\frac{\beta}{2} \sum_{i=1}^{N} (\boldsymbol{w}_{\mathrm{ML}}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i)^2 - \frac{N}{2} \ln \beta + \frac{N}{2} \ln 2\pi \right] \tag{44}$$

Take derivative w.r.t. β and set it to zero

$$\frac{\partial}{\partial \beta} E_{\mathrm{ML}}(\boldsymbol{w}_{\mathrm{ML}}, \beta) = \frac{1}{2} \sum_{i=1}^{N} (\boldsymbol{w}_{\mathrm{ML}}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i)^2 - \frac{N}{2\beta} \stackrel{!}{=} 0 \tag{45}$$

Solving for β

$$\frac{1}{\beta_{\mathrm{ML}}} = \frac{1}{N} \sum_{i=1}^{N} (\boldsymbol{w}_{\mathrm{ML}}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i)^2 \tag{46}$$

Predicting for new data

Recall, that

$$y \sim \mathcal{N}(f_{\boldsymbol{w}}(\boldsymbol{x}), \beta^{-1}) \tag{47}$$

Plugging in the $\boldsymbol{w}_{\mathrm{ML}}$ and β_{ML} into our likelihood we get a predictive distribution that allows us to make prediction $\hat{y}_{new}$ for the new data $\boldsymbol{x}_{new}$.

$$p(\hat{y}_{new} \mid \boldsymbol{x}_{new}, \boldsymbol{w}_{\mathrm{ML}}, \beta_{\mathrm{ML}}) = \mathcal{N}(\hat{y}_{new} \mid \boldsymbol{w}_{\mathrm{ML}}^T \boldsymbol{\phi}(\boldsymbol{x}_{new}), \beta_{\mathrm{ML}}^{-1}) \tag{48}$$

Posterior distribution

Recall from the Lecture 3, that ML leads to overfitting (especially, when little training data is available).
Solution - consider the posterior distribution instead

$$p(\boldsymbol{w} \mid \boldsymbol{X}, \boldsymbol{y}, \beta, \cdot) = \frac{\overbrace{p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta)}^{\text{likelihood}} \cdot \overbrace{p(\boldsymbol{w} \mid \cdot)}^{\text{prior}}}{\underbrace{p(\boldsymbol{X}, \boldsymbol{y})}_{\text{normalizing constant}}} \tag{49}$$

$$\propto p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta) \cdot p(\boldsymbol{w} \mid \cdot) \tag{50}$$

Precision $\beta = 1/\sigma^2$ is treated as a known parameter to simplify the calculations.

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$$\propto p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta) \cdot p(\boldsymbol{w} \mid \cdot) \tag{50}$$

Connection to the coin flip example

	train data	likelihood	prior	posterior
coin:	$\mathcal{D} = \boldsymbol{X}$	$p(\mathcal{D} \mid \theta)$	$p(\theta \mid a, b)$	$p(\theta \mid \mathcal{D})$
regr.:	$\mathcal{D} = \{\boldsymbol{X}, \boldsymbol{y}\}$	$p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta)$	$p(\boldsymbol{w} \mid \cdot)$	$p(\boldsymbol{w} \mid \boldsymbol{X}, \boldsymbol{y}, \beta, \cdot)$

How do we choose the prior $p(\boldsymbol{w} \mid \cdot)$?

Precision $\beta = 1/\sigma^2$ is treated as a known parameter to simplify the calculations.

Prior for $\boldsymbol{w}$

We set the prior over $\boldsymbol{w}$ to an isotropic multivariate normal distribution with zero mean

$$p(\boldsymbol{w} \mid \alpha) = \mathcal{N}(\boldsymbol{w} \mid \mathbf{0}, \alpha^{-1}\mathbf{I}) = \left(\frac{\alpha}{2\pi}\right)^{\frac{M}{2}} \exp\left(-\frac{\alpha}{2}\boldsymbol{w}^T\boldsymbol{w}\right) \tag{51}$$

where,

α - precision of the distribution

M - number of elements in the vector $\boldsymbol{w}$

Motivation:

- Higher probability is assigned to small values of $\boldsymbol{w}$
  $\Longrightarrow$ prevents overfitting (recall slide 21)
- Likelihood is also Gaussian - simplified calculations

Maximum a posteriori (MAP)

We are looking for $\boldsymbol{w}$ that corresponds to the mode of the posterior

$$\boldsymbol{w}_{\text{MAP}} = \arg\max_{\boldsymbol{w}} \; p(\boldsymbol{w} \mid \boldsymbol{X}, \boldsymbol{y}, \alpha, \beta) \tag{52}$$

$$= \arg\max_{\boldsymbol{w}} \; \ln p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta) + \ln p(\boldsymbol{w} \mid \alpha) - \underbrace{\ln p(\boldsymbol{X}, \boldsymbol{y})}_{=\text{const}} \tag{53}$$

$$= \arg\min_{\boldsymbol{w}} - \ln p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta) - \ln p(\boldsymbol{w} \mid \alpha) \tag{54}$$

Similar to ML, define the MAP error function as negative log-posterior

$$E_{\text{MAP}}(\boldsymbol{w}) = - \ln p(\boldsymbol{w} \mid \boldsymbol{X}, \boldsymbol{y}, \alpha, \beta) \tag{55}$$

$$= - \ln p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta) - \ln p(\boldsymbol{w} \mid \alpha) + \text{const} \tag{56}$$

We ignore the constant terms in the error function, as they are independent of $\boldsymbol{w}$

MAP error function

Simplify the error function

$$
\begin{aligned}
E_{MAP} =& -\ln p(\boldsymbol{y} \mid \boldsymbol{X}, \boldsymbol{w}, \beta) - \ln p(\boldsymbol{w} \mid \alpha) \\
=& \frac{\beta}{2} \sum_{i=1}^{N} (\boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i)^2 - \frac{N}{2} \ln \beta + \frac{N}{2} \ln 2\pi \\
& - \ln \left(\frac{\alpha}{2\pi}\right)^{\frac{M}{2}} + \frac{\alpha}{2} \boldsymbol{w}^T \boldsymbol{w} \\
=& \frac{\beta}{2} \sum_{i=1}^{N} (\boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i)^2 + \frac{\alpha}{2} \|\boldsymbol{w}\|_2^2 + \text{ const} \\
=& \underbrace{\frac{1}{2} \sum_{i=1}^{N} (\boldsymbol{w}^T \boldsymbol{\phi}(\boldsymbol{x}_i) - y_i)^2 + \frac{\lambda}{2} \|\boldsymbol{w}\|_2^2}_{\text{ridge regression error fn!}} + \text{const} \qquad \text{where } \lambda = \frac{\alpha}{\beta} \\
=& E_{\text{ridge}}(\boldsymbol{w}) + \text{const}
\end{aligned}
\tag{57}
$$

MAP estimation with Gaussian prior is equivalent to ridge regression!

Predicting for new data

Recall, that

$$y \sim \mathcal{N}(f_{\boldsymbol{w}}(\boldsymbol{x}), \beta^{-1}) \tag{58}$$

Plugging in the $\boldsymbol{w}_{\text{MAP}}$ into our likelihood we get a predictive distribution that lets us make prediction $\hat{y}_{new}$ for the new data $\boldsymbol{x}_{new}$.

$$p(\hat{y}_{new} \mid \boldsymbol{x}_{new}, \boldsymbol{w}_{\text{MAP}}, \beta) = \mathcal{N}(\hat{y}_{new} \mid \boldsymbol{w}_{\text{MAP}}^T \boldsymbol{\phi}(\boldsymbol{x}_{new}), \beta^{-1}) \tag{59}$$

Recall, that we assume β to be known a priori (for simplified calculations).