Determinants III

Problem Set

The *cross product* of $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ is the "symbolic determinant" given by

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{bmatrix} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} & -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} & \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \end{bmatrix}^\intercal$$

Note that $\boldsymbol{v} \times \boldsymbol{w}$ is a *vector*.

Problem 1. Let $\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w}$ where $\boldsymbol{v} = \begin{bmatrix} 3 & -7 & 5 \end{bmatrix}^\intercal$ and $\boldsymbol{w} = \begin{bmatrix} 5 & 4 & 2 \end{bmatrix}^\intercal$. Find $\boldsymbol{n}$ and calculate the inner products $\langle \boldsymbol{n}, \boldsymbol{v} \rangle$ and $\langle \boldsymbol{n}, \boldsymbol{w} \rangle$.

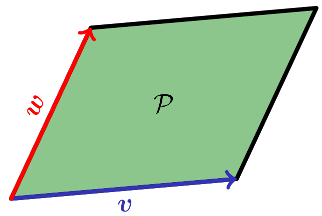
Problem 2. The *scalar triple product* states that any three vectors $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x} \in \mathbb{R}^3$ satisfy the equation

$$\langle \boldsymbol{x}, \boldsymbol{v} \times \boldsymbol{w} \rangle = \begin{vmatrix} x_1 & x_2 & x_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Use this equation to show that $\boldsymbol{v}$ is orthogonal to $\boldsymbol{v} \times \boldsymbol{w}$.

Hint. What is the rank of the matrix in this equation when $\boldsymbol{x} = \boldsymbol{v}$?

Problem 3. Consider the parallelogram $\mathcal{P}$ formed by two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$.



The area of this parallelogram is given by $\text{area}(\mathcal{P}) = \|\boldsymbol{v} \times \boldsymbol{w}\|$. Use this fact to calculate the area of the following parallelogram