Conditional Probability and Bayes' Rule

Lab Assignment

(10) 1. Consider the experiment of flipping three fair coins. Let $X = 1$ if the first flip is H and -1 if not. Let $Y = 1$ if the second flip is H and -1 if not. Let Z be the number of heads on the last two flips. Let W be the number of heads on the first two flips.

(a) Explain what it means for two random variables to be *independent*.

(b) For each pairwise combination of these four random variables, explain why the pair is independent or not independent.
(For four random variables, there are 6 pair-wise combinations of random variables.)

(20) 2. Consider the experiment of rolling two six-sided die, one of them red and one of them black. Let the event A be "the red die came up 5" and let the event B be the event "the total roll came up 7".

(a) What is the sample space Ω?

(b) Compute $P(A)$.

(c) Compute $P(B)$.

(d) Compute $P(A|B)$.

(e) Compute $P(B|A)$.

(f) Are events A and B independent?

(10) 3. Suppose a box contains three coins. Two of the coins are fair coins with H and T as sides. One of the coins has H on both sides!

(a) Pick a coin uniformly at random (don't look at it!) and toss it. What's the probability that it lands H?

(b) Suppose you picked a coin at random, tossed it, and got H. What is the probability that you chose the double-H coin?

(10) 4. Here we want you to actually *prove* Bayes' Rule. It's just a small amount of algebra, but as we've seen, the result is really powerful!

Suppose that we run some experiment, and that A and B are two events, each with non-zero probability.

(a) Write down the definitions of the conditional probabilities $P(A|B)$ and $P(B|A)$.

(b) Using the formulae above, and some basic algebra and reasoning, prove Bayes' Rule.

(20) 5. Suppose you go to the doctor for a routine checkup. At this checkup, you are randomly selected to be tested for Very Bad Syndrome (VBS). You test positive.

(a) You want to know how worried you should be. As a reasonable consumer of medical information, what questions should you now ask? (Don't turn the page yet, or it'll spoil this question!)

(b) Suppose that current studies tell us that every 1 in $10,000$ people has VBS. The false positive rate for this particular VBS test is .01, and the true positive rate is 0.999. After hearing the news about your positive test, what's your estimate of the probability that you have VBS?

(c) A different person comes into the office and also tests positive. It turns out that this person was recently on a cruise ship, where there was a VBS outbreak. Estimates from the ship's doctors tell us that 1 in 20 of the passengers on that ship disembarked with VBS. How worried should that person be before the positive test? After the positive test?

(d) Why are the probabilities of actually having VBS in these two scenarios so different?

(20) 6. (a) Write pseudo-code for Monte Carlo simulations to experimentally verify your answers to both previous parts of this question.

(b) Code and run your Monte Carlo simulations. How do the probabilities estimated via Monte Carlo simulation compare to your hand-calculated probabilities? (Be sure to upload your code as part of submitting this lab.)

(20) 7. The color distribution in a bag of M&M's has changed over the years, no doubt in response to intense market research! Your friend comes to you with a 1994 bag and a 1996 bag. She gives you one yellow M&M and one green M&M, and tells you that exactly one came from each bag.

In 1994, the bags contained the following colors with percentages in parentheses: brown (30), yellow (20), red (20), green (10), orange (10), and tan[1] (10).

By 1996, tan was gone and had been replaced by blue. The new color distribution was brown (13), yellow (14), red (13), green (20), orange (16), and blue (24).

What is the probability that your friend picked the yellow candy from the 1994 bag? (Hint: this problem is not as easy as it might look! Make sure that you are in fact using all of the evidence available to you!)

[1] Yes, Tan!