

Report for the semester thesis “Development of a Monte Carlo algorithm for optimal control problems”

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Abstract—TODO *CRITICAL: Do Not Use Symbols, Special Characters, Footnotes, or Math in Paper Title or Abstract.
Index Terms—a, b, c

I. PROBLEM DESCRIPTION

The problem we want to solve is a variant of the *quarter five spot problem* in literature. We model a cross section of an oil field as a two dimensional square $\Omega := [0, 1]^2$. In the oilfield, there are two phases: water and oil. At the lower left corner $\mathbf{x}_{\text{drill}} := (0, 0)$, we know the pressure $p_{\text{drill}}(t)$. Opposite of that, at $\mathbf{x}_{\text{well}} := (1, 1)$ a well is located. There we can measure the pressure $p_{\text{well}}(t)$ as well as the total volumetric outflow

$$Q_{\text{tot}}(t) := Q_{\text{o}}(t) + Q_{\text{w}}(t)$$

per unit depth.

The flow rates for both phases are described by Darcy’s law

$$\mathbf{v}_{\text{w}} = -\frac{k k_{\text{rel, w}}}{\mu_{\text{w}}} \text{grad}(p), \quad (1)$$

for water and

$$\mathbf{v}_{\text{o}} = -\frac{k k_{\text{rel, o}}}{\mu_{\text{o}}} \text{grad}(p) \quad (2)$$

for oil. Here, $p(\mathbf{x}, t)$ is the pressure, $\mu_{\text{o}}, \mu_{\text{w}}$ are dynamic viscosities for oil and water, $k(\mathbf{x}, t)$ is the permeability. $k_{\text{rel, o}}(S), k_{\text{rel, w}}(S)$ are relative permeabilities and are assumed to depend quadratically on the saturation of water $S_{\text{w}} \in [0, 1]$ and the saturation of oil $S_{\text{o}} \in [0, 1]$:

$$k_{\text{rel, o}} = S_{\text{o}}^2 \quad (3)$$

$$k_{\text{rel, w}} = S_{\text{w}}^2. \quad (4)$$

The saturations are linked by the constitutive relation

$$S_{\text{o}} + S_{\text{w}} = 1. \quad (5)$$

$\mathbf{v}_{\text{o}}(\mathbf{x}, t), \mathbf{v}_{\text{w}}(\mathbf{x}, t)$ finally are the volumetric flow rates per unit area (Darcy velocities).

The saturation S is not assumed constant but instead is transported according to the equation

$$\phi \frac{\partial}{\partial t} S_{\text{w}} + \text{div}(\mathbf{v}_{\text{w}}) = q_{\text{w}}. \quad (6)$$

The term

$$q_{\text{w}} := Q_{\text{w}} \delta(\mathbf{x} - \mathbf{x}_{\text{well}}) \quad (7)$$

describes a line sink of water located at the well. Similarly, we use

$$q_{\text{o}} := Q_{\text{o}} \delta(\mathbf{x} - \mathbf{x}_{\text{well}}) \quad (8)$$

$$q_{\text{tot}} := q_{\text{o}} + q_{\text{w}}. \quad (9)$$

ϕ is the porosity of the rock which is assumed to be constant over the domain.

Conservation of the total mass then reads

$$\text{div}(\mathbf{v}_{\text{tot}}) = q_{\text{tot}}, \quad (10)$$

where

$$\mathbf{v}_{\text{tot}} := \mathbf{v}_{\text{o}} + \mathbf{v}_{\text{w}} \quad (11)$$

is the total Darcy velocity.

We then introduce the mobilities

$$\lambda_{\text{o}} := \frac{k k_{\text{rel, o}}}{\mu_{\text{o}}} \quad (12)$$

$$\lambda_{\text{w}} := \frac{k k_{\text{rel, w}}}{\mu_{\text{w}}} \quad (13)$$

$$\lambda_{\text{tot}} := \lambda_{\text{o}} + \lambda_{\text{w}}. \quad (14)$$

Substituting in the λ from (12) into Darcy’s law (2), (1) and adding both sides of the results we get the total Darcy’s law

$$\mathbf{v}_{\text{tot}} = -\lambda_{\text{tot}} \text{grad}(p). \quad (15)$$

Plugging this (15) into the conservation of mass (10) leads to the pressure equation

$$\boxed{\text{div}(\lambda_{\text{tot}} \text{grad}(p)) = -q_{\text{tot}}}. \quad (16)$$

Comparing the total Darcy’s law (15) and the Darcy’s law for water (1), we see that

$$\mathbf{v}_{\text{w}} = \frac{\lambda_{\text{w}}}{\lambda_{\text{tot}}} \mathbf{v}_{\text{tot}}. \quad (17)$$

We then plug in the model for the relative permeabilities in terms of the saturations (3), to get the final for of saturation transport equation

$$\boxed{\frac{\partial}{\partial t} S_w + \operatorname{div}(f(S_w) \mathbf{v}_{\text{tot}}) = \frac{q_w}{\phi}}, \quad (18)$$

where f is the flux function

$$\boxed{f(S_w) := \frac{S_w^2/\phi}{S_w^2 + (1 - S_w)^2 \mu_w/\mu_o}}. \quad (19)$$

A. Boundary and initial conditions

We assume the initial saturation of water to be given, which is $S_w(\mathbf{x}, 0)$. For the pressure equation we use homogeneous Neumann boundary conditions (no flow), i.e.

$$\operatorname{grad}(p) = p_0 \begin{pmatrix} \delta(x - x_w) \\ \delta(y - y_w) \end{pmatrix}, \quad (20)$$

where we choose p_0 such that the compatibility condition

$$\int_{\Omega} q_{\text{tot}} dA \stackrel{!}{=} \int_{\partial\Omega} \lambda_{\text{tot}} \operatorname{grad}(p)^T \mathbf{n} dl \quad (21)$$

is satisfied.

B. What to optimize?

To test the Monte-Carlo adjoint method, we want to match the pressure difference between drill and well, which is

$$c(T) := \int_0^T \left((p_{\text{drill}}(t) - p_{\text{well}}(t)) - (\tilde{p}_{\text{drill}}(t) - \tilde{p}_{\text{well}}(t)) \right)^2 dt, \quad (22)$$

where the variables with a tilde denote computed quantities and T is a final time.

C. What do we control?

We control the log-permeabilities $\ln(k)$, as these are hard to measure.

II. DISCRETIZATION

We discretize the square domain Ω with $n \times n$ square finite volumes, thus getting a mesh width of $h := 1/n$.

An overview of the discretization technique is given in the following procedure:

- 1) Solve the pressure equation (16) as detailed in subsection II-A
- 2) Compute the total Darcy velocity as in (15), using the same approximation of the gradient as in the first step
- 3) Consider the total Darcy velocity to be independent of the saturation S_w .
- 4) With this assumption, solve the saturation equation (18) as in subsection II-B
- 5) Update the relative permeabilities according to (3) and repeat.

A. Discretizing the pressure Poisson equation

Averaging the pressure Poisson equation (16) over such a finite volume K , and using the divergence theorem leads to

$$\frac{1}{h^2} \int_{\partial K} \lambda_{\text{tot}} \operatorname{grad}(p)^T \mathbf{n} dl = -\frac{1}{h^2} \int_K q_{\text{tot}} dA. \quad (23)$$

Using the four boundaries *North*, *East*, *South* and *West* of the finite volume K , we approximate (23) as

$$\begin{aligned} & \frac{1}{h} \left((\lambda_{\text{tot}} \frac{\delta}{\delta x} p)|_E - (\lambda_{\text{tot}} \frac{\delta}{\delta x} p)|_W \right. \\ & \quad \left. + (\lambda_{\text{tot}} \frac{\delta}{\delta y} p)|_N - (\lambda_{\text{tot}} \frac{\delta}{\delta y} p)|_S \right) \\ &= -\frac{Q_{\text{tot}}}{h^2} \cdot \begin{cases} 1, & \text{if } K \text{ is the finite volume nearest to the well} \\ 0, & \text{otherwise} \end{cases} \end{aligned} \quad (24)$$

$\frac{\delta}{\delta \cdot}$ denote the standard finite difference quotients, i.e.

$$\frac{\delta}{\delta x} p|_E = \frac{1}{h} (p_R - p_K) \quad (25)$$

$$\frac{\delta}{\delta x} p|_W = \frac{1}{h} (p_K - p_D) \quad (26)$$

$$\frac{\delta}{\delta y} p|_N = \frac{1}{h} (p_U - p_K) \quad (27)$$

$$\frac{\delta}{\delta y} p|_S = \frac{1}{h} (p_K - p_D). \quad (28)$$

Here, U stands for the upper neighbor of K , D for the lower (down), R for the right and L for the left. p_K is the pressure at the center of the volume, which is taken to be the same as the volume averaged \bar{p} , as our scheme is just first order.

The total mobilities λ_{tot} at the boundaries are approximated by the harmonic mean of the total mobilities inside the adjacent finite volumes as

$$\lambda_{\text{tot}}|_E \approx \operatorname{hm}(\lambda_{\text{tot}}|_K, \lambda_{\text{tot}}|_R) \quad (29)$$

$$\lambda_{\text{tot}}|_W \approx \operatorname{hm}(\lambda_{\text{tot}}|_K, \lambda_{\text{tot}}|_L) \quad (30)$$

$$\lambda_{\text{tot}}|_N \approx \operatorname{hm}(\lambda_{\text{tot}}|_K, \lambda_{\text{tot}}|_U) \quad (31)$$

$$\lambda_{\text{tot}}|_S \approx \operatorname{hm}(\lambda_{\text{tot}}|_K, \lambda_{\text{tot}}|_D), \quad (32)$$

where

$$\operatorname{hm}(a, b) = 2ab/(a + b). \quad (33)$$

Explain why the harmonic mean

The discretized pressure Poisson equation reads

$$\begin{aligned} & T_E(p_K - p_R) + T_W(p_K - p_L) \\ & \quad + T_N(p_K - p_U) + T_S(p_K - p_D) \\ &= Q_{\text{tot}} \cdot \begin{cases} 1, & \text{if } K \text{ is the finite volume nearest to the well} \\ 0, & \text{otherwise} \end{cases}, \end{aligned} \quad (34)$$

where

$$T_N = \operatorname{hm}(\lambda_{\text{tot}}|_K, \lambda_{\text{tot}}|_U), \quad (35)$$

and so on.

B. Discretizing the saturation equation

The finite volume reformulation of the saturation equation (18) leads to the following:

$$\begin{aligned} & \frac{\partial}{\partial t} \frac{1}{h^2} \int_K S_w dA \\ & + \frac{1}{h} \left((f(S_w) v_{\text{tot},x})|_E - (f(S_w) v_{\text{tot},x})|_W \right. \\ & \quad \left. + (f(S_w) v_{\text{tot},y})|_N - (f(S_w) v_{\text{tot},y})|_S \right) \\ & = \frac{Q_w}{h^2 \phi} \cdot \begin{cases} 1, & \text{if } K \text{ is the finite volume nearest to the well} \\ 0, & \text{otherwise} \end{cases} \end{aligned} \quad (36)$$

We identify the volume average

$$\frac{1}{h^2} \int_K S_w dA =: S_K, \quad (37)$$

with the saturation at the center of the volume S_K . This is legit, as our scheme is just first order.

For the total Darcy velocity v_{tot} we use the same discretization of the pressure gradients as in the pressure equation, (25). This makes it available at the boundaries (N , S , E , W), as required by (36).

For the flux function f at the boundaries, we use an upwind discretization. The standard formulation can be simplified by noting that

$$\frac{d}{dS_w} f > 0 \quad (38)$$

and so instead of the advection velocity

$$\frac{\partial}{\partial S_w} (f \cdot v_{\text{tot}}) \quad (39)$$

we can use the Darcy velocity v_{tot} . We remember that this requires fixing the total Darcy velocity v_{tot} to be independent of the saturation S_w .

For timestepping of the saturation equation (36), we use explicit Euler, as this simplifies the Jacobian used in the Monte-Carlo adjoint.

C. Discretizing the cost function

The cost function (22) is discretized as a sum of squares over the timesteps, where the computed quantities with a tilde are taken to be the quantities in the volumes nearest to the well and the drill.

$$\sum_{i=1}^n (\Delta p(i\Delta t) - \tilde{\Delta p}^{(i)})^2, \quad (40)$$

where

$$\Delta p(t) := p_{\text{well}}(t) - p_{\text{drill}}(t) \quad (41)$$

$$\tilde{\Delta p}^{(i)} := p_{\text{well cell}}^{(i)} - p_{\text{drill cell}}^{(i)} \quad (42)$$

and Δt is the timestep.

APPENDIX

A. Computing the quantities for the Monte-Carlo adjoint solver

First, we define the pressure residuals

$$\Pi_K^{(i)} := \begin{cases} p_K^{(i)} - p_{\text{well}}(i\Delta t), & \text{if } K \text{ is the cell nearest to the well} \\ T_E^{(i-1)}(p_K^{(i)} - p_R^{(i)}) \\ + T_W^{(i-1)}(p_K^{(i)} - p_L^{(i)}) \\ + T_N(p_K^{(i)} - p_U^{(i)}) \\ + T_S^{(i-1)}(p_K^{(i)} - p_D^{(i)}) - Q_{\text{tot}, K}^{(i-1)}, & \text{else} \end{cases}, \quad (43)$$

where

$$Q_{\text{tot}, K}^{(i)} := Q_{\text{tot}}(i\Delta t) \cdot \begin{cases} 1, & \text{if } K \text{ is the cell nearest to the drill} \\ 0, & \text{otherwise} \end{cases} \quad (44)$$

The saturation residuals are given by

$$\begin{aligned} \Sigma_K^{(i)} &:= S_w^{(i)}|_K - S_w^{(i-1)}|_K \\ &+ \frac{\Delta t}{h} \left((f(S_w^{(i-1)}) v_{\text{tot},x})|_E - (f(S_w^{(i-1)}) v_{\text{tot},x})|_W \right. \\ &\quad \left. + (f(S_w^{(i-1)}) v_{\text{tot},y})|_N - (f(S_w^{(i-1)}) v_{\text{tot},y})|_S \right) - \Delta t Q_{\text{tot}, K}^{(i)}. \end{aligned} \quad (45)$$

The nonzero derivatives for the diagonal blocks are given by

$$\frac{\partial}{\partial p_K^{(i)}} \Pi_K^{(i)} = T_E^{(i-1)} + T_W^{(i-1)} + T_N^{(i-1)} + T_S^{(i-1)} \quad (46)$$

$$\frac{\partial}{\partial p_R^{(i)}} \Pi_K^{(i)} = -T_E^{(i-1)} \quad (47)$$

$$\frac{\partial}{\partial p_L^{(i)}} \Pi_K^{(i)} = -T_W^{(i-1)} \quad (48)$$

$$\frac{\partial}{\partial p_N^{(i)}} \Pi_K^{(i)} = -T_U^{(i-1)} \quad (49)$$

$$\frac{\partial}{\partial p_S^{(i)}} \Pi_K^{(i)} = -T_D^{(i-1)} \quad (50)$$

for cells which are not nearest to the well

and by

$$\frac{\partial}{\partial S_K^{(i)}} \Sigma_K^{(i)} = 1 \quad (51)$$

$$\begin{aligned} \frac{\partial}{\partial S_{wK}^{(i-1)}} \Sigma_K^{(i)} = & -1 + f'(S_{wK}^{(i-1)}) \frac{\Delta t}{h} \cdot \left(\mathbb{1}(v_{\text{tot}, x}|_{E>0}) v_{\text{tot}, x}|_E \right. \\ & - \mathbb{1}(v_{\text{tot}, x}|_{W<0}) v_{\text{tot}, x}|_W + \mathbb{1}(v_{\text{tot}, y}|_{N>0}) v_{\text{tot}, y}|_N \\ & \left. - \mathbb{1}(v_{\text{tot}, y}|_{S<0}) v_{\text{tot}, y}|_S \right) \end{aligned} \quad (52)$$

$$\begin{aligned} & + \frac{\Delta t}{h} \cdot \left(-f(S_{wK}^{(i-1)})|_E \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}, R}, \lambda_{\text{tot}, K}) \frac{\partial}{\partial S_w} \lambda|_K \frac{\delta}{\delta x} p|_{E \pm \dots} \right) \\ \frac{\partial}{\partial S_{wR}^{(i-1)}} \Sigma_K^{(i)} = & f'(S_{wR}^{(i-1)}) \frac{\Delta t}{h} \cdot \mathbb{1}(v_{\text{tot}, x}|_{E<0}) v_{\text{tot}, x}|_E \\ & - \frac{\Delta t}{h} \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}, K}, \lambda_{\text{tot}, R}) \frac{\partial}{\partial S_w} \lambda|_R \frac{\delta}{\delta x} p|_E \end{aligned} \quad (53)$$

where

$$f'(S_w) = \frac{2\mu_o\mu_w(1-S_w)S_w}{\phi \cdot (\mu_w(1-S_w)^2 + \mu_o S_w^2)^2} \quad (54)$$

$$\mathbb{1}(b) = \begin{cases} 1, & \text{if } b \text{ true} \\ 0, & \text{otherwise.} \end{cases} \quad (55)$$

For the off diagonal blocks, we have

$$\begin{aligned} \frac{\partial}{\partial p_K^{(i)}} \Sigma_K^{(i)} = & + \frac{\Delta t}{h^2} \left(T_E^{(i-1)} f(S_w^{(i-1)})|_E \right. \\ & + T_W^{(i-1)} f(S_w^{(i-1)})|_W \\ & + T_N^{(i-1)} f(S_w^{(i-1)})|_N \\ & \left. + T_S^{(i-1)} f(S_w^{(i-1)})|_S \right) \end{aligned} \quad (56)$$

$$\frac{\partial}{\partial p_R^{(i)}} \Sigma_K^{(i)} = -\frac{\Delta t}{h^2} T_E^{(i-1)} f(S_w^{(i-1)})|_E \quad (57)$$

$$\frac{\partial}{\partial p_L^{(i)}} \Sigma_K^{(i)} = -\frac{\Delta t}{h^2} T_W^{(i-1)} f(S_w^{(i-1)})|_W \quad (58)$$

$$\frac{\partial}{\partial p_U^{(i)}} \Sigma_K^{(i)} = -\frac{\Delta t}{h^2} T_N^{(i-1)} f(S_w^{(i-1)})|_N \quad (59)$$

$$\frac{\partial}{\partial p_D^{(i)}} \Sigma_K^{(i)} = -\frac{\Delta t}{h^2} T_S^{(i-1)} f(S_w^{(i-1)})|_S, \quad (61)$$

for cells not nearest to the well and

$$\frac{\partial}{\partial S_{wK}^{(i-1)}} \Pi_K^{(i)} = 2 \frac{\partial}{\partial S_w} \lambda_{\text{tot}}^{(i-1)}|_K \quad (62)$$

$$\begin{aligned} \sum_{n \in \{U, D, L, R\}} \frac{(p_K^{(i-1)} - p_n^{(i-1)}) \cdot (\lambda_{\text{tot}, n}^{(i-1)})^2}{(\lambda_{\text{tot}, K}^{(i-1)} + \lambda_{\text{tot}, n}^{(i-1)})^2} \\ \frac{\partial}{\partial S_{wR}^{(i-1)}} \Pi_K^{(i)} = 2 \frac{\partial}{\partial S_w} \lambda_{\text{tot}}^{(i-1)}|_R \cdot \frac{(p_K^{(i-1)} - p_R^{(i-1)}) \cdot (\lambda_{\text{tot}, K}^{(i-1)})^2}{(\lambda_{\text{tot}, K}^{(i-1)} + \lambda_{\text{tot}, R}^{(i-1)})^2} \end{aligned} \quad (63)$$

$$\frac{\partial}{\partial S_{wL}^{(i-1)}} \Pi_K^{(i)} = 2 \frac{\partial}{\partial S_w} \lambda_{\text{tot}}^{(i-1)}|_L \cdot \frac{(p_K^{(i-1)} - p_L^{(i-1)}) \cdot (\lambda_{\text{tot}, K}^{(i-1)})^2}{(\lambda_{\text{tot}, K}^{(i-1)} + \lambda_{\text{tot}, L}^{(i-1)})^2} \quad (64)$$

$$\frac{\partial}{\partial S_{wU}^{(i-1)}} \Pi_K^{(i)} = 2 \frac{\partial}{\partial S_w} \lambda_{\text{tot}}^{(i-1)}|_U \cdot \frac{(p_K^{(i-1)} - p_U^{(i-1)}) \cdot (\lambda_{\text{tot}, K}^{(i-1)})^2}{(\lambda_{\text{tot}, K}^{(i-1)} + \lambda_{\text{tot}, U}^{(i-1)})^2} \quad (65)$$

$$\frac{\partial}{\partial S_{wD}^{(i-1)}} \Pi_K^{(i)} = 2 \frac{\partial}{\partial S_w} \lambda_{\text{tot}}^{(i-1)}|_D \cdot \frac{(p_K^{(i-1)} - p_D^{(i-1)}) \cdot (\lambda_{\text{tot}, K}^{(i-1)})^2}{(\lambda_{\text{tot}, K}^{(i-1)} + \lambda_{\text{tot}, D}^{(i-1)})^2} \quad (66)$$

where

$$\frac{\partial}{\partial S_w} \lambda_{\text{tot}} = 2k \left(\frac{S_w - 1}{\mu_o} + \frac{S_w}{\mu_w} \right). \quad (67)$$

For solving the adjoint equations, we also need the derivatives of the residuals with respect to the parameters, in our case $\ln(k)$.

For the pressure residuals and if K is not the cell at the well, those read

$$\frac{\partial}{\partial \ln(k)_K} \Pi_K^{(i)} = \lambda_{\text{tot}}^{(i-1)}|_K \sum_{n \in \{L, R, U, D\}} (p_K^{(i)} - p_n^{(i)}) \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}}^{(i-1)})|_n, \quad (68)$$

$$\begin{aligned} \lambda_{\text{tot}}^{(i-1)}|_K \\ \frac{\partial}{\partial \ln(k)_L} \Pi_K^{(i)} = (p_K^{(i)} - p_L^{(i)}) \lambda_{\text{tot}}^{(i-1)}|_L \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}}^{(i-1)}|_K, \lambda_{\text{tot}}^{(i-1)}|_L) \end{aligned} \quad (69)$$

$$\frac{\partial}{\partial \ln(k)_R} \Pi_K^{(i)} = (p_K^{(i)} - p_R^{(i)}) \lambda_{\text{tot}}^{(i-1)}|_R \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}}^{(i-1)}|_K, \lambda_{\text{tot}}^{(i-1)}|_R) \quad (70)$$

$$\frac{\partial}{\partial \ln(k)_U} \Pi_K^{(i)} = (p_K^{(i)} - p_U^{(i)}) \lambda_{\text{tot}}^{(i-1)}|_U \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}}^{(i-1)}|_K, \lambda_{\text{tot}}^{(i-1)}|_U) \quad (71)$$

$$\frac{\partial}{\partial \ln(k)_D} \Pi_K^{(i)} = (p_K^{(i)} - p_D^{(i)}) \lambda_{\text{tot}}^{(i-1)}|_D \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}}^{(i-1)}|_K, \lambda_{\text{tot}}^{(i-1)}|_D) \quad (72)$$

where

$$\frac{\partial}{\partial b} \text{hm}(a, b) := \frac{2a^2}{(a+b)^2}. \quad (73)$$

On the other hand, the saturation residuals Σ_K lead to the following derivatives

$$\begin{aligned}
\frac{\partial}{\partial \ln(k)_K} \Sigma_K &= \lambda_{\text{tot}}^{(i-1)}|_K \frac{\Delta t}{h} \left(\right. \\
&\quad (f(S_{\text{w}}^{(i-1)})v_{\text{tot},x}^{(i-1)})|_E \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}}^{(i-1)}|_E, \lambda_{\text{tot}}^{(i-1)}|_K) \left(-\frac{\delta}{\delta x} p|_E\right) \\
&\quad - (f(S_{\text{w}}^{(i-1)})v_{\text{tot},x}^{(i-1)})|_W \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}}^{(i-1)}|_W, \lambda_{\text{tot}}^{(i-1)}|_K) \left(-\frac{\delta}{\delta x} p|_W\right) \\
&\quad + (f(S_{\text{w}}^{(i-1)})v_{\text{tot},y}^{(i-1)})|_N \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}}^{(i-1)}|_N, \lambda_{\text{tot}}^{(i-1)}|_K) \left(-\frac{\delta}{\delta y} p|_N\right) \\
&\quad - (f(S_{\text{w}}^{(i-1)})v_{\text{tot},y}^{(i-1)})|_S \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}}^{(i-1)}|_E, \lambda_{\text{tot}}^{(i-1)}|_S) \left(-\frac{\delta}{\delta y} p|_S\right) \\
&\quad \left. \right)
\end{aligned} \tag{74}$$

$$\begin{aligned}
\frac{\partial}{\partial \ln(K)_R} \Sigma_K &= \lambda_{\text{tot}}^{(i-1)}|_R \frac{\Delta t}{h} f(S_{\text{w}}^{(i-1)})v_{\text{tot},x}^{(i-1)}|_E \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}}^{(i-1)}|_K, \lambda_{\text{tot}}^{(i-1)}|_E) \left(-\frac{\delta}{\delta x} p|_E\right) \\
&\tag{75}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \ln(K)_L} \Sigma_K &= -\lambda_{\text{tot}}^{(i-1)}|_L \frac{\Delta t}{h} f(S_{\text{w}}^{(i-1)})v_{\text{tot},x}^{(i-1)}|_W \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}}^{(i-1)}|_K, \lambda_{\text{tot}}^{(i-1)}|_L) \left(-\frac{\delta}{\delta x} p|_W\right) \\
&\tag{76}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \ln(K)_U} \Sigma_K &= \lambda_{\text{tot}}^{(i-1)}|_U \frac{\Delta t}{h} f(S_{\text{w}}^{(i-1)})v_{\text{tot},x}^{(i-1)}|_N \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}}^{(i-1)}|_K, \lambda_{\text{tot}}^{(i-1)}|_U) \left(-\frac{\delta}{\delta y} p|_N\right) \\
&\tag{77}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \ln(K)_D} \Sigma_K &= -\lambda_{\text{tot}}^{(i-1)}|_D \frac{\Delta t}{h} f(S_{\text{w}}^{(i-1)})v_{\text{tot},x}^{(i-1)}|_S \frac{\partial}{\partial b} \text{hm}(\lambda_{\text{tot}}^{(i-1)}|_K, \lambda_{\text{tot}}^{(i-1)}|_D) \left(-\frac{\delta}{\delta y} p|_S\right) \\
&\tag{78}
\end{aligned}$$