Report for the semester thesis “Development of a Monte Carlo algorithm for optimal control problems”

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***Abstract*—TODO *CRITICAL: Do Not Use Symbols, Special Characters, Footnotes, or Math in Paper Title or Abstract.**

***Index Terms*—a, b, c**

I. Problem description

We model a cross section of an oil field as a two dimensional square $\Omega := [0,1]^2$. In the oilfield, there are two phases: water and oil. At the lower left corner $(0,0)$, we know the pressure $p(t)$. Opposite of that, at $(1,1)$ a well is located. There we can measure the pressure $p_{\text{well}}(t)$ as well as the total volumetric outflow

$$Q_{\text{tot}}(t) := Q_{\text{o}}(t) + Q_{\text{w}}(t)$$

per unit depth.

The flow rates for both phases are described by Darcy's law

$$\boldsymbol{v}_{\text{w}} = -\frac{k k_{\text{rel, w}}}{\mu_{\text{w}}} \operatorname{grad}(p), \tag{1}$$

for water and

$$\boldsymbol{v}_{\text{o}} = -\frac{k k_{\text{rel, o}}}{\mu_{\text{o}}} \operatorname{grad}(p) \tag{2}$$

for oil. Here, $p(\boldsymbol{x},t)$ is the pressure, μ_{o}, μ_{w} are dynamic viscosities for oil and water, $k(\boldsymbol{x},t)$ is the permeability. $k_{\text{rel, o}}(S)$, $k_{\text{rel, w}}(S)$ are relative permeabilities and are assumed to depend quadratically on the saturation of water $S_{\text{w}} \in [0,1]$ and the saturation of oil $S_{\text{o}} \in [0,1]$:

$$k_{\text{rel, o}} = S_{\text{o}}^2 \tag{3}$$

$$k_{\text{rel, w}} = S_{\text{w}}^2. \tag{4}$$

The saturations are linked by the constitutive relation

$$S_{\text{o}} + S_{\text{w}} = 1. \tag{5}$$

$\boldsymbol{v}_{\text{o}}(\boldsymbol{x},t)$, $\boldsymbol{v}_{\text{w}}(\boldsymbol{x},t)$ finally are the volumetric flow rates per unit area (Darcy velocities).

The saturation S is not assumed constant but instead is transported according to the equation

$$\frac{\partial}{\partial t}\phi S_{\text{w}} + \operatorname{div}(\boldsymbol{v}_{\text{w}}) = q_{\text{w}}. \tag{6}$$

The term

$$q_{\text{w}} := Q_{\text{w}}\delta(\boldsymbol{x} - \begin{pmatrix}1\\1\end{pmatrix}) \tag{7}$$

describes a line sink of water located at the well. Similarly, we use

$$q_{\text{o}} := Q_{\text{o}}\delta(\boldsymbol{x} - \begin{pmatrix}1\\1\end{pmatrix}) \tag{8}$$

$$q_{\text{tot}} := q_{\text{o}} + q_{\text{w}}. \tag{9}$$

ϕ is the porosity of the rock which is assumed to be constant over the domain.

Conservation of the total mass then reads

$$\operatorname{div}(\boldsymbol{v}_{\text{tot}}) = q_{\text{tot}}, \tag{10}$$

where

$$\boldsymbol{v}_{\text{tot}} := \boldsymbol{v}_{\text{o}} + \boldsymbol{v}_{\text{w}} \tag{11}$$

is the total Darcy velocity.

We then introduce some additional simplifications: We set

$$\mu_{\text{w}} = \mu_{\text{o}} =: \mu, \tag{12}$$

and introduce the mobilities

$$\lambda_{\text{o}} := \frac{k k_{\text{rel, o}}}{\mu_{\text{o}}} \tag{13}$$

$$\lambda_{\text{w}} := \frac{k k_{\text{rel, w}}}{\mu_{\text{w}}} \tag{14}$$

$$\lambda_{\text{tot}} := \lambda_{\text{o}} + \lambda_{\text{w}}. \tag{15}$$

Substituting in the λ from (13) into Darcy's law (2), (1) and adding both sides of the results we get the total Darcy's law

$$\boxed{\boldsymbol{v}_{\text{tot}} = -\lambda_{\text{tot}} \operatorname{grad}(p).} \tag{16}$$

Plugging this (16) into the conservation of mass (10) leads to the pressure equation

$$\operatorname{div}(\lambda_{\text{tot}} \operatorname{grad}(p)) = q_{\text{tot}}. \tag{17}$$

Comparing the total Darcy's law (16) and the Darcy's law for water (1), we see that

$$\boldsymbol{v}_{\text{w}} = \frac{\lambda_{\text{w}}}{\lambda_{\text{tot}}}\boldsymbol{v}_{\text{tot}}. \tag{18}$$

We then plug in the model for the relative permeabilities in terms of the saturations (3), to get the final for of saturation transport equation

$$\boxed{\frac{\partial}{\partial t} S_{\mathrm{w}} + \operatorname{div}\left(\boldsymbol{F}(S_{\mathrm{w}}, \boldsymbol{v}_{\mathrm{tot}})\right) = \frac{q_{\mathrm{w}}}{\phi},} \tag{19}$$

where $\boldsymbol{F}$ is the flux function

$$\boxed{\boldsymbol{F}(S_{\mathrm{w}}, \boldsymbol{v}_{\mathrm{tot}}) := \frac{S_{\mathrm{w}}^2/\phi}{S_{\mathrm{w}}^2 + (1 - S_{\mathrm{w}})^2 \mu_{\mathrm{w}}/\mu_{\mathrm{o}}} \boldsymbol{v}_{\mathrm{tot}}.} \tag{20}$$