

# Report for the semester thesis “Development of a Monte Carlo algorithm for optimal control problems”

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**Abstract—TODO \*CRITICAL: Do Not Use Symbols, Special Characters, Footnotes, or Math in Paper Title or Abstract.**  
**Index Terms—**a, b, c

## I. PROBLEM DESCRIPTION

We model a cross section of an oil field as a two dimensional square  $\Omega := [0, 1]^2$ . In the oilfield, there are two phases: water and oil. At the lower left corner  $(0, 0)$ , we know the pressure  $p(t)$ . Opposite of that, at  $(1, 1)$  a well is located. There we can measure the pressure  $p_{\text{well}}(t)$  as well as the total volumetric outflow

$$Q_{\text{tot}}(t) := Q_o(t) + Q_w(t)$$

per unit depth.

The flow rates for both phases are described by Darcy’s law

$$\mathbf{v}_w = -\frac{k k_{\text{rel}, w}}{\mu_w} \text{grad}(p), \quad (1)$$

for water and

$$\mathbf{v}_o = -\frac{k k_{\text{rel}, o}}{\mu_o} \text{grad}(p) \quad (2)$$

for oil. Here,  $p(\mathbf{x}, t)$  is the pressure,  $\mu_o, \mu_w$  are dynamic viscosities for oil and water,  $k(\mathbf{x}, t)$  is the permeability.  $k_{\text{rel}, o}(S), k_{\text{rel}, w}(S)$  are relative permeabilities and are assumed to depend quadratically on the saturation of water  $S_w \in [0, 1]$  and the saturation of oil  $S_o \in [0, 1]$ :

$$k_{\text{rel}, o} = S_o^2 \quad (3)$$

$$k_{\text{rel}, w} = S_w^2. \quad (4)$$

The saturations are linked by the constitutive relation

$$S_o + S_w = 1. \quad (5)$$

$\mathbf{v}_o(\mathbf{x}, t), \mathbf{v}_w(\mathbf{x}, t)$  finally are the volumetric flow rates per unit area (Darcy velocities).

The saturation  $S$  is not assumed constant but instead is transported according to the equation

$$\frac{\partial}{\partial t} \phi S_w + \text{div}(\mathbf{v}_w) = q_w. \quad (6)$$

The term

$$q_w := Q_w \delta(\mathbf{x} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \quad (7)$$

describes a line sink of water located at the well. Similarly, we use

$$q_o := Q_o \delta(\mathbf{x} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \quad (8)$$

$$q_{\text{tot}} := q_o + q_w. \quad (9)$$

$\phi$  is the porosity of the rock which is assumed to be constant over the domain.

Conservation of the total mass then reads

$$\text{div}(\mathbf{v}_{\text{tot}}) = q_{\text{tot}}, \quad (10)$$

where

$$\mathbf{v}_{\text{tot}} := \mathbf{v}_o + \mathbf{v}_w \quad (11)$$

is the total Darcy velocity.

We then introduce some additional simplifications: We set

$$\mu_w = \mu_o =: \mu, \quad (12)$$

and introduce the mobilities

$$\lambda_o := \frac{k k_{\text{rel}, o}}{\mu_o} \quad (13)$$

$$\lambda_w := \frac{k k_{\text{rel}, w}}{\mu_w} \quad (14)$$

$$\lambda_{\text{tot}} := \lambda_o + \lambda_w. \quad (15)$$

Substituting in the  $\lambda$  from (13) into Darcy’s law (2), (1) and adding both sides of the results we get the total Darcy’s law

$$\boxed{\mathbf{v}_{\text{tot}} = -\lambda_{\text{tot}} \text{grad}(p)}. \quad (16)$$

Plugging this (16) into the conservation of mass (10) leads to the pressure equation

$$\text{div}(\lambda_{\text{tot}} \text{grad}(p)) = q_{\text{tot}}. \quad (17)$$

Comparing the total Darcy’s law (16) and the Darcy’s law for water (1), we see that

$$\mathbf{v}_w = \frac{\lambda_w}{\lambda_{\text{tot}}} \mathbf{v}_{\text{tot}}. \quad (18)$$

We then plug in the model for the relative permeabilities in terms of the saturations (3), to get the final for of saturation transport equation

$$\begin{aligned} \frac{\partial}{\partial t} S_w + \operatorname{div}(\mathbf{F}(S, \mathbf{v}_{\text{tot}})) &= \frac{q_w}{\phi} \\ \mathbf{F}(S, \mathbf{v}_{\text{tot}}) &:= \frac{1}{\phi} \frac{S_w^2}{S_w^2 + (1 - S_w)^2 \mu_w / \mu_o} \mathbf{v}_{\text{tot}}, \end{aligned} \tag{19}$$

$$\tag{20}$$

where  $\mathbf{F}$  is the flux function.