Marco Discacciati

Coupling free and porous-media flows: models and numerical approximation

Abstract: In this paper, we provide an overview of mathematical and numerical models for the coupling of the Navier–Stokes and the Darcy equations to describe the filtration of incompressible flows through porous media. After introducing suitable interface conditions, we set up the coupled problem and, using domain decomposition techniques, we present possible algorithms to compute its finite element approximation through the independent solution of the Navier–Stokes and of the Darcy equations in their respective subdomains.

Keywords: Navier–Stokes Equation, Darcy's Law, Interface Condition, Domain Decomposition, Finite Elements Method.

Mathematics Subject Classifications 2010: 76D05, 76S05, 65M60, 65M55

Marco Discacciati: Laboratori de Càlcul Numèric (LaCàN), Departament de Matemàtica Aplicada III (MA3), Universitat Politècnica de Catalunya (UPC BarcelonaTech), Barcelona, Spain, marco.discacciati@upc.edu

1 Introduction

The filtration of fluids through porous media is an interesting research subject with many relevant applications. To quote some examples, these phenomena occur in physiology when studying the filtration of blood through arterial vessel walls, in industrial processes involving, e.g. air or oil filters, in cross-flow filtration procedures, and in the environment with the percolation of waters of hydrological basins through rocks and sand.

The modeling of such physical processes requires considering different systems of partial differential equations in each subregion of the domain of interest. Typically, the motion of incompressible free fluids is described by the Navier–Stokes equations, whereas Darcy equations are adopted to model the filtration process. These equations must be linked through suitable conditions that describe the motion of the fluid across the surface of the porous media where the filtration occurs.

The resulting system may be possibly completed by including the transport of passive scalars in the main field and in the porous medium to represent, e.g. so-

The support of the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement n. 294229 is kindly acknowledged.

lutes, chemical pollutants, etc. For example, in hydrological environmental applications, we can model the transport of contaminants in coastal areas, rivers, basins, or lakes. In this case, the coupling of the Navier–Stokes equations for free-surface flows with the groundwater flow in the porous media, together with a numerical model of transport-diffusion for chemical pollutants in the two regions, would help assess the short and medium-term effects of polluting agents. On the other hand, in bioengineering applications, blood oxygenators and hemodialysis devices are based on the transport of chemicals from the main blood stream in the arteries through a porous membrane. Similar problems occur within human arteries when chemical substances (such as lipoproteins, drugs, or oxygen) are carried through the vessel wall from the main blood stream. Here, the problem is made more difficult by the complex mechanical behavior of the material constituting the several layers of a vessel wall. In both cases we are facing a coupling between fluid flow in heterogeneous media and transport-diffusion (and possibly reaction) phenomena.

Those coupled problems have received an increasing attention during the past years both from the mathematical and the numerical point of view. Starting from the original experimental works of Beavers and Joseph on the coupling conditions between a fluid and a porous medium, mathematical investigations have been carried out in [32, 42–44, 48, 51]. Under these conditions, the analysis of the coupled Stokes/Darcy problem has been studied in [19, 21, 24, 25, 28, 29, 34–38, 45, 47, 62, 63, 68]. Moreover, extensions to the Navier–Stokes equations [2, 25, 39] and to the shallow water equations [28, 53, 54] have been considered. Applications in the biomedical context have been investigated as well. Let us mention, e.g. [5, 41, 59, 68]. A vast literature on approximation methods, as well as on numerical algorithms for the solutions of the associated systems is available.

In this chapter, we give an overview on the coupled free/porous-media flow problem, on its finite element approximation and on possible algorithms to compute its solution using domain decomposition methods. We will focus mainly on a linear steady model involving the Stokes and the Darcy equations. This problem will allow us to introduce some numerical techniques that may be extended to the more general setting.

2 Setting of the problem

In our setting, we consider that the fluid has no free surface. This is the case in applications like the filtration of blood through the arterial wall, or the filtration of aqueous solutions through membranes in cross-flow filtration, but it may not occur when considering the coupling between surface and groundwater flows. If so, we consider a computational domain close enough to the porous medium and we impose a suitable boundary condition on the top artificial boundary to simulate the presence of

a volume of water above. The extension of our approach to the free-surface case can be found in [28, 53, 54] and references therein.

The computational domain will be a region naturally split into two parts: the free flow region and the porous medium. More precisely, let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain, partitioned into two nonintersecting subdomains Ω_f and Ω_p separated by an interface Γ, i.e. $\bar{\Omega} = \bar{\Omega}_\mathrm{f} \cup \bar{\Omega}_\mathrm{p}$, $\Omega_\mathrm{f} \cap \Omega_\mathrm{p} = \varnothing$ and $\bar{\Omega}_\mathrm{f} \cap \bar{\Omega}_\mathrm{p} = \Gamma$. We suppose the boundaries $\partial\Omega_\mathrm{f}$ and $\partial\Omega_\mathrm{p}$ to be Lipschitz continuous. From the physical point of view, Γ is a surface separating the domain Ω_f filled by a fluid, from a domain Ω_p formed by a porous medium. As mentioned above, we assume that Ω_f has a fixed surface. The fluid in Ω_f, that will be referred to as *free fluid* in the following, can filtrate through the adjacent porous medium. In Figure 4.1, we show a schematic representation of the computational domain.

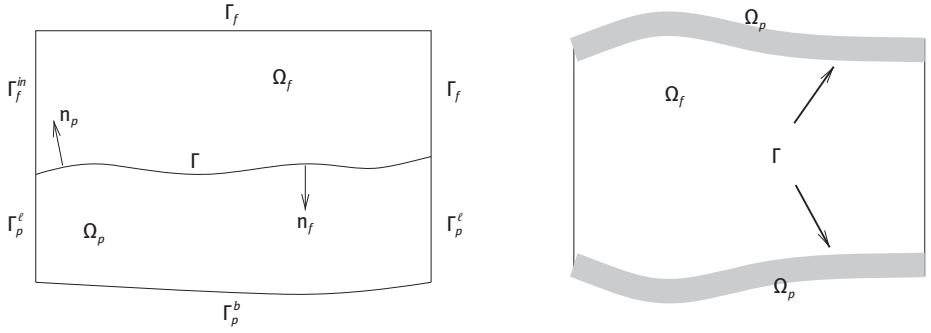

Figure 4.1: Schematic representation of a 2D section of two possible computational domains: the surface-groundwater setting on the left, and the cross-flow filtration or blood-flow problem on the right.

2.1 The surface-groundwater flow problem

In order to describe the motion of the fluid in Ω_f, we introduce the Navier–Stokes equations: $\forall t > 0$,

$$\partial_t \mathbf{u}_\mathrm{f} - \boldsymbol{\nabla} \cdot \mathsf{T}(\mathbf{u}_\mathrm{f}, p_\mathrm{f}) + (\mathbf{u}_\mathrm{f} \cdot \nabla)\mathbf{u}_\mathrm{f} = \mathbf{f} \quad \text{in } \Omega_\mathrm{f}\,, \tag{4.1}$$

$$\nabla \cdot \mathbf{u}_\mathrm{f} = 0 \quad \text{in } \Omega_\mathrm{f}\,, \tag{4.2}$$

where $\mathsf{T}(\mathbf{u}_\mathrm{f}, p_\mathrm{f}) = \nu(\nabla\mathbf{u}_\mathrm{f} + \nabla^\mathrm{T}\mathbf{u}_\mathrm{f}) - p_\mathrm{f}\mathsf{I}$ is the Cauchy stress tensor, I being the identity tensor. $\nu > 0$ is the kinematic viscosity of the fluid, $\mathbf{f}$ a given volumetric force, while $\mathbf{u}_\mathrm{f}$ and p_f are the fluid velocity and pressure, respectively. ∂_t denotes the time derivative, $\partial_i = \partial/\partial x_i$ is the spatial derivative with respect to the coordinate x_i, while ∇ and $\nabla\cdot$ are, respectively, the gradient and the divergence operator with respect to the

space coordinates. Moreover,

$$\boldsymbol{\nabla} \cdot \mathsf{T} = \left(\sum_{j=1}^{d} \partial_j \mathsf{T}_{ij} \right)_{i=1,\dots,d}.$$

Finally, we recall that

$$(\mathbf{v} \cdot \nabla)\mathbf{w} = \sum_{i=1}^{d} v_i \partial_i \mathbf{w},$$

for all vector functions $\mathbf{v} = (v_1, \dots, v_d)$ and $\mathbf{w} = (w_1, \dots, w_d)$.

The filtration of an incompressible fluid through porous media is often described using Darcy's law. The latter provides the simplest linear relation between velocity and pressure in porous media under the physically reasonable assumption that fluid flows are usually very slow and all the inertial (nonlinear) terms may be neglected.

We introduce a fictitious flow velocity, the *Darcy velocity* or *specific discharge* $\mathbf{q}$ through a given cross section of the porous medium, rather than the true velocity $\mathbf{u}_\mathrm{p}$ with respect to the porous matrix:

$$\mathbf{q} = \mathbf{u}_\mathrm{p} n, \tag{4.3}$$

with n being the *volumetric porosity*, defined as the ratio between the volume of void space and the total volume of the porous medium.

Moreover, we define a scalar quantity φ called *piezometric head* which essentially represents the fluid pressure in Ω_p:

$$\varphi = z + \frac{p_\mathrm{p}}{g}, \tag{4.4}$$

where z is the elevation from a reference level, accounting for the potential energy per unit weight of fluid, p_p is the ratio between the fluid pressure in Ω_p and its density and g is the gravity acceleration.

Then, Darcy's law can be written as

$$\mathbf{q} = -K \nabla \varphi, \tag{4.5}$$

where K is a symmetric positive definite tensor $K = (K_{ij})_{i,j=1,\dots,d}$, $K_{ij} \in L^\infty(\Omega_\mathrm{p})$, $K_{ij} > 0$, $K_{ij} = K_{ji}$, called *hydraulic conductivity tensor*, which depends on the properties of the fluid as well as on the characteristics of the porous medium. The hydraulic conductivity K is a macroscopic quantity characterizing porous media; in Table 4.1 we report some typical values that it may assume (see [7]). The hydraulic conductivity tensor K can be diagonalized by introducing three mutually orthogonal axes called *principal directions of anisotropy*. In the following, we will always suppose that the principal axes are in the x, y, and z directions so that the tensor will be considered diagonal: $K = \mathrm{diag}(K_1, K_2, K_3)$. Moreover, let us denote $\mathsf{K} = K/n$.

Table 4.1: Typical values of hydraulic conductivity K.

K (m/s):	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}
Permeability	Pervious				Semipervious				Impervious				
Soil	Clean gravel		Clean sand or sand and gravel			Very fine sand, silt, loam							
					Peat			Stratified clay		Unweathered clay			
Rocks					Oil rocks			Sandstone		Good limestone, dolomite		Breccia, granite	

In conclusion, the motion of an incompressible fluid through a saturated porous medium is described by the following equations:

$$\mathbf{u}_{\mathrm{p}} = -\mathrm{K}\nabla\varphi \quad \text{in } \Omega_{\mathrm{p}}\,, \tag{4.6}$$

$$\nabla \cdot \mathbf{u}_{\mathrm{p}} = 0 \qquad \text{in } \Omega_{\mathrm{p}}\,. \tag{4.7}$$

Note that the system (4.6) and (4.7) can be rewritten as an elliptic equation for the scalar unknown φ:

$$-\nabla \cdot (\mathrm{K}\nabla\varphi) = 0 \quad \text{in } \Omega_{\mathrm{p}}\,. \tag{4.8}$$

Extensions of Darcy's law are given, e.g. by the Forchheimer or Brinkman equations when the Reynolds number in Ω_{p} is not small (see [17, 22, 33, 51]), or by more complicated models like Richards' equations apt to describe saturated-unsaturated fluid flows (see, e.g. [11] and references therein).

2.2 Interface conditions to couple surface and groundwater flows

We now consider the issue of finding effective coupling conditions across the interface Γ which separates the fluid flow in Ω_{f} and the porous medium. This is a classical problem that has been investigated from both a physical and a rigorous mathematical point of view. A mathematical difficulty arises from the fact that we need to couple two different systems of partial differential equations: Darcy equations (4.6) and (4.7) contain second-order derivatives for the pressure and first order for the velocity, while in the Navier–Stokes system the opposite situation occurs.

In the following, $\mathbf{n}_{\mathrm{p}}$ and $\mathbf{n}_{\mathrm{f}}$ denote the unit outward normal vectors to the surfaces $\partial\Omega_{\mathrm{p}}$ and $\partial\Omega_{\mathrm{f}}$, respectively, and we have $\mathbf{n}_{\mathrm{f}} = -\mathbf{n}_{\mathrm{p}}$ on Γ. We suppose $\mathbf{n}_{\mathrm{f}}$ and $\mathbf{n}_{\mathrm{p}}$ to be regular enough. Moreover, we shall indicate $\mathbf{n} = \mathbf{n}_{\mathrm{f}}$ for simplicity of notation, and denote by ∂_n the partial derivative along $\mathbf{n}$.

Three conditions are to be prescribed on Γ.

(1) An obvious condition is the continuity of the normal velocity, which is a consequence of the incompressibility of the fluid:

$$\mathbf{u}_{\mathrm{f}} \cdot \mathbf{n} = \mathbf{u}_{\mathrm{p}} \cdot \mathbf{n} \quad \text{on } \Gamma\,. \tag{4.9}$$

(2) Moreover, a suitable condition relating the pressures of the fluid on either side of Γ has to be prescribed. A common choice (see, e.g. [25, 39, 47]) reads

$$-\mathbf{n}\cdot\mathsf{T}(\mathbf{u}_{\mathrm{f}},p_{\mathrm{f}})\cdot\mathbf{n}=g\varphi\quad\text{on }\Gamma\,. \tag{4.10}$$

This condition, which actually allows the pressure to be discontinuous across Γ, is well-suited for the analysis of the coupled fluid-groundwater flow problem. Indeed, it can be naturally incorporated in its weak formulation as it is a Neumann-type boundary condition on Γ for the Navier–Stokes equations (4.1) and (4.2). Let us note that if one adopted the divergence form of the Navier–Stokes momentum equation:

$$\partial_t\mathbf{u}_{\mathrm{f}}-\boldsymbol{\nabla}\cdot\mathsf{T}(\mathbf{u}_{\mathrm{f}},p_{\mathrm{f}})+\nabla\cdot(\mathbf{u}_{\mathrm{f}}\,\mathbf{u}_{\mathrm{f}}^{\mathrm{T}})=\mathbf{f}\quad\text{in }\Omega_{\mathrm{f}}, \tag{4.11}$$

then, equation (4.10) should be replaced by the following one (see [39])

$$-\mathbf{n}\cdot\mathsf{T}(\mathbf{u}_{\mathrm{f}},p_{\mathrm{f}})\cdot\mathbf{n}+\frac{1}{2}(\mathbf{u}_{\mathrm{f}}\cdot\mathbf{u}_{\mathrm{f}})=g\varphi\quad\text{on }\Gamma\,, \tag{4.12}$$

that is pressure p_{f} in equation (4.10) has to be replaced by the total pressure $p_{\mathrm{f}}+\frac{1}{2}|\mathbf{u}_{\mathrm{f}}|^2$ in equation (4.12).

(3) Finally, in order to have the flow of the free fluid determined completely, we have to specify a further condition on the tangential component of the fluid velocity at the interface. A classical condition was proposed by Beavers and Joseph [8] who found experimentally that the difference between the slip velocity of the free fluid and the tangential component of the velocity through the porous medium is proportional to the shear rate of the free fluid. The proportionality constant depends linearly on the square root of the permeability

$$-(\partial_n\mathbf{u}_{\mathrm{f}})_\tau=\frac{\alpha_{\mathrm{BJ}}}{\sqrt{\mathsf{K}}}(\mathbf{u}_{\mathrm{f}}-\mathbf{u}_{\mathrm{p}})_\tau\quad\text{on }\Gamma\,, \tag{4.13}$$

where α_{BJ} is a dimensionless constant that depends only on the microstructure of the porous medium and by $(\mathbf{v})_\tau$ we indicate the tangential component to the interface of any vector $\mathbf{v}$: $(\mathbf{v})_\tau=\mathbf{v}-\mathbf{v}\cdot\mathbf{n}$ on Γ.

A simplification of (4.13) that we will use in this paper is due to Saffman (see [64]) who pointed out that for $\mathsf{K}\ll 1$ or for large ν the velocity $\mathbf{u}_{\mathrm{p}}$ was much smaller than the other quantities appearing in equation (4.13) and that, in fact, it could be dropped. The new proposed interface condition reads therefore

$$-(\partial_n\mathbf{u}_{\mathrm{f}})_\tau=\frac{\alpha_{\mathrm{BJ}}}{\sqrt{\mathsf{K}}}(\mathbf{u}_{\mathrm{f}})_\tau\quad\text{on }\Gamma\,. \tag{4.14}$$

A first attempt towards an analytical study of the interface conditions between a free fluid and a porous medium can be found in [58]; a mathematical investigation using homogenization theory has been conducted by Jäger and Mikelić (see [42–44]). Note that, strictly speaking, equation (4.14) is not a coupling condition in the sense that it does not relate quantities from the two subdomains Ω_{f} and Ω_{p}, but it is actually

a boundary condition on Γ for the fluid problem. Moreover, we remark that some authors impose the vanishing of the tangential velocity at the interface, $(\mathbf{u}_\mathrm{f})_\tau = 0$ on Γ instead of (4.14) (see, e.g. [68] for blood flow simulations).

We point out that the conditions studied by Jäger and Mikelić have been adopted also, e.g. in [19, 47, 63].

3 Weak formulation and analysis

We now focus on the steady Darcy/Navier–Stokes coupled problem where we replace equation (4.1) by

$$-\boldsymbol{\nabla}\cdot\mathsf{T}(\mathbf{u}_\mathrm{f},p_\mathrm{f})+(\mathbf{u}_\mathrm{f}\cdot\nabla)\mathbf{u}_\mathrm{f}=\mathbf{f}\quad\text{in }\Omega_\mathrm{f}. \tag{4.15}$$

(A similar kind of "steady" problem can be found when using an implicit time-advancing scheme on the time-dependent problem (4.1).)

As concerns boundary conditions, for the Darcy equation we assign the piezometric head $\varphi=\varphi_\mathrm{p}$ on Γ_p^b (Figure 4.1, left); moreover, we impose that the normal component of the velocity vanishes on the lateral surface, i.e. $\mathbf{u}_\mathrm{p}\cdot\mathbf{n}_\mathrm{p}=0$ on Γ_p^ℓ. This condition may be rewriten as $\mathsf{K}\partial_n\varphi=0$ on Γ_p.

For the Navier–Stokes problem, several combinations of boundary conditions could be considered, representing different kinds of flow problems. Here we consider a non-null inflow $\mathbf{u}_\mathrm{f}=\mathbf{u}_{in}$ on the left-hand boundary Γ_f^{in} and a no-slip condition $\mathbf{u}_\mathrm{f}=\mathbf{0}$ on the remaining boundary Γ_f.

We introduce the following functional spaces:

$$H_{0,\Gamma_\mathrm{f}}=\left\{v\in H^1(\Omega_\mathrm{f})\ :\ v=0\text{ on }\Gamma_\mathrm{f}\right\}, \tag{4.16}$$

$$H_{0,\Gamma_\mathrm{f}\cup\Gamma_\mathrm{f}^{in}}=\left\{v\in H_{0,\Gamma_\mathrm{f}}\ :\ v=0\text{ on }\Gamma_\mathrm{f}^{in}\right\},\quad H_\mathrm{f}=\left(H_{0,\Gamma_\mathrm{f}\cup\Gamma_\mathrm{f}^{in}}\right)^d, \tag{4.17}$$

$$H_\mathrm{f}^0=\left\{\mathbf{v}\in H_\mathrm{f}\ :\ \mathbf{v}\cdot\mathbf{n}_\mathrm{f}=0\text{ on }\Gamma\right\}, \tag{4.18}$$

$$\widetilde{H}_\mathrm{f}=\left\{\mathbf{v}\in(H^1(\Omega_\mathrm{f}))^d\ :\ \mathbf{v}=\mathbf{0}\text{ on }\Gamma_\mathrm{f}\cup\Gamma\right\}, \tag{4.19}$$

$$Q=L^2(\Omega_\mathrm{f}),\quad Q_0=\left\{q\in Q:\int_{\Omega_\mathrm{f}}q=0\right\}, \tag{4.20}$$

$$H_\mathrm{p}=\left\{\psi\in H^1(\Omega_\mathrm{p})\ :\ \psi=0\text{ on }\Gamma_\mathrm{p}^b\right\},\quad H_\mathrm{p}^0=\left\{\psi\in H_\mathrm{p}\ :\ \psi=0\text{ on }\Gamma\right\}. \tag{4.21}$$

We denote by $|\cdot|_1$ and $\|\cdot\|_1$ the H^1–seminorm and norm, respectively, and by $\|\cdot\|_0$ the L^2–norm; it will always be clear from the context whether we are referring to spaces on Ω_f or Ω_p. The space $W=H_\mathrm{f}\times H_\mathrm{p}$ is a Hilbert space with the norm

$$\|\underline{w}\|_W=\left(\|\mathbf{w}\|_1^2+\|\psi\|_1^2\right)^{1/2}\quad\forall\,\underline{w}=(\mathbf{w},\psi)\in W.$$

Finally, we consider on Γ the trace space $\Lambda=H_{00}^{1/2}(\Gamma)$ and denote its norm by $\|\cdot\|_\Lambda$ (see [49]).

We introduce a continuous extension operator

$$E_{\mathrm{f}}\colon (H^{1/2}(\Gamma_{\mathrm{f}}^{in}))^d \to \widetilde{H}_{\mathrm{f}}\,. \tag{4.22}$$

Then $\forall \mathbf{u}_{in} \in (H_{00}^{1/2}(\Gamma_{\mathrm{f}}^{in}))^d$ we can construct a vector function $E_{\mathrm{f}}\mathbf{u}_{in} \in \widetilde{H}_{\mathrm{f}}$ such that $E_{\mathrm{f}}\mathbf{u}_{in\,|\Gamma_{\mathrm{f}}^{in}} = \mathbf{u}_{in}$.

We introduce another continuous extension operator

$$E_{\mathrm{p}}\colon H^{1/2}(\Gamma_{\mathrm{p}}^{b}) \to H^1(\Omega_{\mathrm{p}}) \quad \text{such that } E_{\mathrm{p}}\varphi_{\mathrm{p}} = 0 \text{ on } \Gamma\,. \tag{4.23}$$

Then, for all $\varphi \in H^1(\Omega_{\mathrm{p}})$ we define the function $\varphi_0 = \varphi - E_{\mathrm{p}}\varphi_{\mathrm{p}}$.

Finally, we define the following bilinear forms:

$$a_{\mathrm{f}}(\mathbf{v},\mathbf{w}) = \int_{\Omega_{\mathrm{f}}} \frac{\nu}{2}(\nabla\mathbf{v} + \nabla^{\mathrm{T}}\mathbf{v}) \cdot (\nabla\mathbf{w} + \nabla^{\mathrm{T}}\mathbf{w}) \quad \forall \mathbf{v},\mathbf{w} \in (H^1(\Omega_{\mathrm{f}}))^d\,, \tag{4.24}$$

$$b_{\mathrm{f}}(\mathbf{v},q) = -\int_{\Omega_{\mathrm{f}}} q\,\nabla\cdot\mathbf{v} \quad \forall \mathbf{v} \in (H^1(\Omega_{\mathrm{f}}))^d\,, \quad \forall q \in Q\,, \tag{4.25}$$

$$a_{\mathrm{p}}(\varphi,\psi) = \int_{\Omega_{\mathrm{p}}} \nabla\psi : \mathsf{K}\nabla\varphi \quad \forall \varphi,\psi \in H^1(\Omega_{\mathrm{p}})\,, \tag{4.26}$$

and, for all $\mathbf{v},\mathbf{w},\mathbf{z} \in (H^1(\Omega_{\mathrm{f}}))^d$, the trilinear form

$$c_{\mathrm{f}}(\mathbf{w};\mathbf{z},\mathbf{v}) = \int_{\Omega_{\mathrm{f}}} [(\mathbf{w}\cdot\nabla)\mathbf{z}]\cdot\mathbf{v} = \sum_{i,j=1}^{d} \int_{\Omega_{\mathrm{f}}} w_j \frac{\partial z_i}{\partial x_j}\, v_i\,. \tag{4.27}$$

Now, if we multiply (4.15) by $\mathbf{v} \in H_{\mathrm{f}}$ and integrate by parts we obtain

$$a_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}},\mathbf{v}) + c_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}) + b_{\mathrm{f}}(\mathbf{v},p_{\mathrm{f}}) - \int_{\Gamma} \mathbf{n}\cdot\mathsf{T}(\mathbf{u}_{\mathrm{f}},p_{\mathrm{f}})\,\mathbf{v} = \int_{\Omega_{\mathrm{f}}} \mathbf{f}\cdot\mathbf{v}\,.$$

Note that we can write

$$-\int_{\Gamma} \mathbf{n}\cdot\mathsf{T}(\mathbf{u}_{\mathrm{f}},p_{\mathrm{f}})\,\mathbf{v} = -\int_{\Gamma} [\mathbf{n}\cdot\mathsf{T}(\mathbf{u}_{\mathrm{f}},p_{\mathrm{f}})\cdot\mathbf{n}](\mathbf{v}\cdot\mathbf{n}) - \int_{\Gamma} (\mathsf{T}(\mathbf{u}_{\mathrm{f}},p_{\mathrm{f}})\cdot\mathbf{n})_\tau(\mathbf{v})_\tau,$$

so that we can incorporate in the weak form the interface conditions (4.10) and (4.14) as follows:

$$-\int_{\Gamma} \mathbf{n}\cdot\mathsf{T}(\mathbf{u}_{\mathrm{f}},p_{\mathrm{f}})\,\mathbf{v} = \int_{\Gamma} g\varphi(\mathbf{v}\cdot\mathbf{n}) + \int_{\Gamma} \frac{\nu\alpha_{\mathrm{BJ}}}{\sqrt{\mathsf{K}}}(\mathbf{u}_{\mathrm{f}})_\tau(\mathbf{v})_\tau\,.$$

Finally, we consider the lifting $E_{\mathrm{f}}\mathbf{u}_{in}$ of the boundary datum and we split $\mathbf{u}_{\mathrm{f}} = \mathbf{u}_{\mathrm{f}}^0 + E_{\mathrm{f}}\mathbf{u}_{in}$ with $\mathbf{u}_{\mathrm{f}}^0 \in H_{\mathrm{f}}$; we recall that $E_{\mathrm{f}}\mathbf{u}_{in} = 0$ on Γ and we get

$$\begin{aligned} &a_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}^0,\mathbf{v}) + c_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}^0 + E_{\mathrm{f}}\mathbf{u}_{in};\mathbf{u}_{\mathrm{f}}^0 + E_{\mathrm{f}}\mathbf{u}_{in},\mathbf{v}) + b_{\mathrm{f}}(\mathbf{v},p_{\mathrm{f}}) \\ &\qquad + \int_{\Gamma} g\varphi(\mathbf{v}\cdot\mathbf{n}) + \int_{\Gamma} \frac{\nu\alpha_{\mathrm{BJ}}}{\sqrt{\mathsf{K}}}(\mathbf{u}_{\mathrm{f}})_\tau(\mathbf{v})_\tau = \int_{\Omega_{\mathrm{f}}} \mathbf{f}\cdot\mathbf{v} - a_{\mathrm{f}}(E_{\mathrm{f}}\mathbf{u}_{in},\mathbf{v}). \end{aligned} \tag{4.28}$$

From equation (4.2) we find

$$b_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}^0, q) = -b_{\mathrm{f}}(E_{\mathrm{f}}\mathbf{u}_{in}, q) \quad \forall q \in Q\,. \tag{4.29}$$

On the other hand, if we multiply equation (4.8) by $\psi \in H_{\mathrm{p}}$ and integrate by parts we get

$$a_{\mathrm{p}}(\varphi, \psi) + \int_\Gamma (\mathsf{K}\partial_n \varphi)\psi = 0\,.$$

Now we incorporate the interface condition (4.9) in the weak form as

$$a_{\mathrm{p}}(\varphi, \psi) - \int_\Gamma (\mathbf{u}_{\mathrm{f}} \cdot \mathbf{n})\psi = 0,$$

and, considering the splitting $\varphi = \varphi_0 + E_{\mathrm{p}}\varphi_{\mathrm{p}}$ we obtain

$$a_{\mathrm{p}}(\varphi_0, \psi) - \int_\Gamma (\mathbf{u}_{\mathrm{f}} \cdot \mathbf{n})\psi = -a_{\mathrm{p}}(E_{\mathrm{p}}\varphi_{\mathrm{p}}, \psi)\,. \tag{4.30}$$

We multiply (4.30) by g and sum to (4.28) and (4.29); then, we define

$$\begin{aligned}\mathcal{A}(\underline{v}, \underline{w}) = a_{\mathrm{f}}(\mathbf{v}, \mathbf{w}) + g\, a_{\mathrm{p}}(\varphi, \psi) + \int_\Gamma g\,\varphi(\mathbf{w}\cdot\mathbf{n}) - \int_\Gamma g\,\psi(\mathbf{v}\cdot\mathbf{n}) \\ + \int_\Gamma \frac{\nu\alpha_{\mathrm{BJ}}}{\sqrt{\mathsf{K}}}(\mathbf{w})_\tau(\mathbf{v})_\tau\,,\end{aligned} \tag{4.31}$$

$$C(\underline{v}; \underline{w}, \underline{u}) = c_{\mathrm{f}}(\mathbf{v}; \mathbf{w}, \mathbf{u})\,, \tag{4.32}$$

$$\mathcal{B}(\underline{w}, q) = b_{\mathrm{f}}(\mathbf{w}, q)\,, \tag{4.33}$$

for all $\underline{v} = (\mathbf{v}, \varphi)$, $\underline{w} = (\mathbf{w}, \psi)$, $\underline{u} = (\mathbf{u}, \xi) \in W$, $q \in Q$. Finally, we introduce the following linear functionals:

$$\langle \mathcal{F}, \underline{w}\rangle = \int_{\Omega_{\mathrm{f}}} \mathbf{f}\cdot\mathbf{w} - a_{\mathrm{f}}(E_{\mathrm{f}}\mathbf{u}_{in}, \mathbf{w}) - g\, a_{\mathrm{p}}(E_{\mathrm{p}}\varphi_{\mathrm{p}}, \psi)\,, \tag{4.34}$$

$$\langle \mathcal{G}, q\rangle = -b_{\mathrm{f}}(E_{\mathrm{f}}\mathbf{u}_{in}, q)\,, \tag{4.35}$$

for all $\underline{w} = (\mathbf{w}, \psi) \in W$, $q \in Q$.

Adopting these notations, the weak formulation of the steady coupled Navier–Stokes/Darcy problem reads: find $\underline{u} = (\mathbf{u}_{\mathrm{f}}^0, \varphi_0) \in W$, $p_{\mathrm{f}} \in Q$ such that

$$\mathcal{A}(\underline{u}, \underline{v}) + C(\underline{u} + \underline{u}^*; \underline{u} + \underline{u}^*, \underline{v}) + \mathcal{B}(\underline{v}, p_{\mathrm{f}}) = \langle \mathcal{F}, \underline{v}\rangle \quad \forall \underline{v} = (\mathbf{v}, \psi) \in W \tag{4.36}$$

$$\mathcal{B}(\underline{u}, q) = \langle \mathcal{G}, q\rangle \quad \forall q \in Q\,, \tag{4.37}$$

with $\underline{u}^* = (E_{\mathrm{f}}\mathbf{u}_{in}, 0) \in \widetilde{H}_{\mathrm{f}} \times H^1(\Omega_{\mathrm{p}})$.

Note that the interface conditions have been incorporated in the above weak model as natural conditions on Γ: in particular, (4.10) and (4.14) are natural conditions for

the Navier–Stokes problem, while (4.9) becomes a natural condition for Darcy's problem.

In the case of the Darcy/Stokes problem (which can be obtained by neglecting the nonlinear term $C(\cdot;\cdot,\cdot)$ in (4.36)), one can easily prove that the coupled problem is well-posed using the theory of Brezzi [12] for saddle-point problems. See [29] for details.

3.1 Mixed formulation of Darcy's equation

The formulation (4.36) and (4.37) allows to compute the pressure head φ in the porous medium domain, but not the velocity field $\mathbf{u}_p$ that can be recovered only at a later stage through the gradient of φ (see equation (4.5)). At the numerical level, this approach may lead to low-order approximations for $\mathbf{u}_p$ and may not guarantee mass conservation, unless proper gradient reconstruction techniques are used (see, e.g. [50]). A more popular approach is based on mixed methods that allow to compute at once both φ and $\mathbf{u}_p$, as we will discuss in Section 5. Here we present briefly two possible mathematical formulations to account for the Darcy equations in the mixed form (4.6) and (4.7). For more details we refer, e.g. to [34, 38, 47, 57, 67].

In [47] the coupling with the Stokes equations is realized via the Lagrange multiplier $\ell = -\mathbf{n}\cdot\mathsf{T}(\mathbf{u}_f,p_f)\cdot\mathbf{n} = g\varphi$ on Γ. Defining the duality pairing $b_\Gamma : (H_f \times X_2)\times\Lambda \to \mathbb{R}$, $b_\Gamma(\underline{v},\ell) = \langle \mathbf{v}_1\cdot\mathbf{n} + \mathbf{v}_2\cdot\mathbf{n}, \ell\rangle$ where X_2 is a suitable subspace of $H(div;\Omega_p)$ and $\underline{v} = (\mathbf{v}_1,\mathbf{v}_2)$, the following global mixed problem is considered: find $\underline{u} = (\mathbf{u}_f,\mathbf{u}_p) \in H_f\times X_2$, $\underline{p} = (p_f,\varphi) \in M \subset Q\times L^2(\Omega_p)$, $\ell\in\Lambda$:

$$
\begin{aligned}
a(\underline{u},\underline{v}) + b(\underline{v},\underline{p}) + b_\Gamma(\underline{v},\ell) &= f(\underline{v}) \quad && \forall \underline{v}\in H_f\times X_2\,,\\
b(\underline{u},\underline{q}) &= g(\underline{q}) && \forall \underline{q}\in M,\\
b_\Gamma(\underline{u},\sigma) &= 0 && \forall \sigma\in\Lambda\,,
\end{aligned}
$$

with

$$
\begin{aligned}
a(\underline{u},\underline{v}) &= a_f(\mathbf{u}_f,\mathbf{v}_1) + \int_\Gamma \frac{\nu\alpha_{BJ}}{\sqrt{\mathsf{K}}}(\mathbf{u}_f)_\tau(\mathbf{v}_1)_\tau + \int_{\Omega_p}\mathsf{K}^{-1}\mathbf{u}_p\,\mathbf{v}_2\,,\\
b(\underline{v},\underline{p}) &= b_f(\mathbf{v}_1,p_f) - \int_{\Omega_p}\varphi\nabla\cdot\mathbf{v}_2\,,
\end{aligned}
$$

and f, g are suitably defined linear continuous functionals.

An alternative formulation based on a primal-mixed form of the Darcy equations can be found in [67]. In this form, all the interface conditions can be imposed naturally without introducing Lagrange multipliers. More precisely, assuming homogeneous boundary conditions for simplicity of notation, the weak formulation reads:

find $\mathbf{u}_{\mathrm{f}}^0 \in H_{\mathrm{f}}$, $p_{\mathrm{f}} \in Q$, $\mathbf{u}_{\mathrm{p}} \in L^2(\Omega_{\mathrm{p}})$, $\varphi_0 \in H_{\mathrm{p}}$ such that

$$\begin{aligned} a_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}^0, \mathbf{v}) + b_{\mathrm{f}}(\mathbf{v}, p_{\mathrm{f}}) + \int_\Gamma \frac{\nu \alpha_{\mathrm{BJ}}}{\sqrt{\mathsf{K}}} (\mathbf{u}_{\mathrm{f}}^0)_\tau (\mathbf{v})_\tau + \int_\Gamma \varphi_0 (\mathbf{v} \cdot \mathbf{n}) &= \int_{\Omega_{\mathrm{f}}} \mathbf{f} \cdot \mathbf{v} && \forall \mathbf{v} \in H_{\mathrm{f}} \\ b_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}^0, q) &= 0 && \forall q \in Q \\ a_d(\mathbf{u}_{\mathrm{p}}, \mathbf{w}) + b_{\mathrm{p}}(\mathbf{w}, \varphi_0) &= 0 && \forall \mathbf{w} \in L^2(\Omega_{\mathrm{p}}) \\ b_{\mathrm{p}}(\mathbf{u}_{\mathrm{p}}, \psi) - \frac{1}{2} a_{\mathrm{p}}(\varphi_0, \psi) + \int_\Gamma (\mathbf{u}_{\mathrm{f}}^0 \cdot \mathbf{n}) \psi &= 0 && \forall \psi \in H_{\mathrm{p}}. \end{aligned}$$

The bilinear forms are defined as

$$\begin{aligned} a_d(\mathbf{v}, \mathbf{w}) &= \frac{1}{2} \int_{\Omega_{\mathrm{p}}} \mathsf{K}^{-1} \mathbf{v} \cdot \mathbf{w} \\ b_{\mathrm{p}}(\mathbf{v}, q) &= \frac{1}{2} \int_{\Omega_{\mathrm{p}}} \nabla q \cdot \mathbf{v}. \end{aligned}$$

The bilinear form $a_{\mathrm{p}}(\cdot, \cdot)$ provides a stabilization term that guarantees the well-posedness of the problem. The interest of this formulation is that, apart from avoiding introducing an additional unknown, it allows to perform a finite element approximation with classical Lagrangian elements (see [52, 67]).

3.2 Time-dependent Stokes/Darcy model

The analysis of a time-dependent Stokes/Darcy system has been carried out in [21]. In particular, equation (4.7) has been replaced by the saturated flow model

$$s \partial_t \varphi + \nabla \cdot \mathbf{u}_{\mathrm{p}} = 0 \quad \text{in } \Omega_{\mathrm{p}}, \tag{4.38}$$

where s denotes the mass storativity coefficient that gives the mass of water added to storage (or released from it) in the porous medium depending on the rise (or decline) of the potential φ. Combining equations (4.38) and (4.6), the following time-dependent equation for the piezometric head is obtained:

$$s \partial_t \varphi - \nabla \cdot (\mathsf{K} \nabla \varphi) = 0 \quad \text{in } \Omega_{\mathrm{p}}. \tag{4.39}$$

The Beavers–Joseph condition (4.13) is used for the coupling. Consider now the bilinear form $\mathcal{A}_\eta(\underline{v}, \underline{w}) : W \times W \to \mathbb{R}$,

$$\begin{aligned} \mathcal{A}_\eta(\underline{v}, \underline{w}) = a_{\mathrm{f}}(\mathbf{v}, \mathbf{w}) + \frac{\eta}{s} a_{\mathrm{p}}(\varphi, \psi) + \int_\Gamma g \varphi (\mathbf{w} \cdot \mathbf{n}) - \frac{\eta}{s} \int_\Gamma \psi (\mathbf{v} \cdot \mathbf{n}) \\ + \int_\Gamma \frac{\nu \alpha_{\mathrm{BJ}}}{\sqrt{\mathsf{K}}} (\mathbf{v} + \mathsf{K} \nabla \varphi)_\tau (\mathbf{w})_\tau, \end{aligned}$$

for all $\underline{v} = (\mathbf{v}, \varphi)$, $\underline{w} = (\mathbf{w}, \psi) \in W$, where η is a suitable scaling parameter, and the following duality pairing associated with the time derivative:

$$\langle \underline{v}_t, \underline{w} \rangle = \langle \partial_t \mathbf{v}, \mathbf{w} \rangle + \eta \langle \partial_t \varphi, \psi \rangle .$$

Then, the weak form of the coupled time-dependent Stokes/Darcy problem can be written as: find $\underline{u} = (\mathbf{u}_\mathrm{f}, \varphi)$ and p_f such that

$$\begin{aligned} \langle \underline{u}_t, \underline{v} \rangle + \mathcal{A}_\eta(\underline{u}, \underline{v}) + \mathcal{B}(\underline{v}, p_\mathrm{f}) &= \langle \widetilde{\mathcal{F}}, \underline{v} \rangle \quad && \forall \underline{v} = (\mathbf{v}, \psi) \in W , \\ \mathcal{B}(\underline{u}, q) &= 0 && \forall q \in Q , \end{aligned}$$

where $\widetilde{\mathcal{F}}$ is a linear continuous functional defined similarly to equation (4.34) and $\mathcal{B}$ is the bilinear form equation (4.33).

The problem is studied firstly in the steady case showing its well-posedness for small enough values of the coefficient α_BJ. Then, a backward-Euler discretization in time is introduced and the convergence to the continuous solution as the time step tends to zero is proved. Finally, the convergence of the fully discretized system is guaranteed.

These results rely on the choice of a suitably large parameter η. Notice that the choice of a large rescaling parameter η makes sense since the flow in porous media evolves on a relatively slow time scale compared to that of the flow in the domain Ω_f, and the re-scaling essentially brings them to the same time scale.

An interesting approach is also presented in [65] where the fully evolutionary time-dependent Darcy/Stokes problem is studied (see also Section 8).

4 Multidomain formulation of the coupled problem

A possible approach to study the Navier–Stokes/Darcy problem is to exploit its naturally decoupled structure keeping separated the free-flow region and the porous medium and exchanging information between surface and groundwater flows only through boundary conditions at the interface. From the computational point of view, this strategy is useful at the stage of setting up methods to solve the problem numerically. As we shall illustrate in Section 5, a discretization of this problem using, e.g. finite elements leads to a large sparse ill-conditioned linear system which requires a suitable preconditioning strategy to be solved. We would like to exploit the intrinsic decoupled structure of the problem to design iterative procedures requiring at each step to compute independently the solution of the fluid and of the groundwater problems.

To achieve this aim, in the next sections we shall apply a domain decomposition technique at the differential level to study the Navier–Stokes/Darcy coupled problem. Our aim will be to reformulate it solely in terms of interface unknowns. This reinterpretation will be crucial to set up iterative procedures between the subdomains Ω_f and Ω_p, that will be applied at the discrete level.

4.1 The Stokes/Darcy problem

In this section, we consider on the interface the condition $(\mathbf{u}_\mathrm{f})_\tau = 0$ on Γ instead of (4.14). This simplification is acceptable from the physical viewpoint in certain regimes, as discussed in Section 2.2, and it does not dramatically influence the coupling of the two subproblems since, as we have already pointed out, condition (4.14) is not strictly a coupling condition but only a boundary condition for the fluid problem in Ω_f. We will return to the more general boundary condition (4.14) in Section 4.2.

The Stokes/Darcy problem can be rewritten in a multidomain formulation and, in particular, the following result holds (see [29]).

Proposition 4.1. *Let Λ be the space of traces introduced in Section 3. Neglecting the nonlinear term $C(\cdot;\cdot,\cdot)$, problem (4.36) and (4.37) can be reformulated in an equivalent way as follows: find $\mathbf{u}_\mathrm{f}^0 \in H_\mathrm{f}^\tau$, $p_\mathrm{f} \in Q$, $\varphi_0 \in H_\mathrm{p}$ such that*

$$a_\mathrm{f}(\mathbf{u}_\mathrm{f}^0 + E_\mathrm{f}\mathbf{u}_{in}, \mathbf{w}) + b_\mathrm{f}(\mathbf{w}, p_\mathrm{f}) = \int_{\Omega_\mathrm{f}} \mathbf{f}\cdot\mathbf{w} \quad \forall \mathbf{w} \in (H_0^1(\Omega_\mathrm{f}))^d\,, \tag{4.40}$$

$$b_\mathrm{f}(\mathbf{u}_\mathrm{f}^0 + E_\mathrm{f}\mathbf{u}_{in}, q) = 0 \quad \forall q \in Q\,, \tag{4.41}$$

$$a_\mathrm{p}(\varphi_0 + E_\mathrm{p}\varphi_\mathrm{p}, \psi) = 0 \quad \forall \psi \in H_\mathrm{p}^0\,, \tag{4.42}$$

$$\int_\Gamma (\mathbf{u}_\mathrm{f}^0\cdot\mathbf{n})\mu = a_\mathrm{p}(\varphi_0 + E_\mathrm{p}\varphi_\mathrm{p}, R_2\mu) \quad \forall \mu \in \Lambda\,, \tag{4.43}$$

$$\int_\Gamma g\varphi_0\mu = \int_{\Omega_\mathrm{f}} \mathbf{f}\cdot(R_1^\tau\mu) - a_\mathrm{f}(\mathbf{u}_\mathrm{f}^0 + E_\mathrm{f}\mathbf{u}_{in}, R_1^\tau\mu) - b_\mathrm{f}(R_1^\tau\mu, p_\mathrm{f}) \quad \forall \mu \in \Lambda\,, \tag{4.44}$$

where R_2 is any possible continuous extension operator from $H^{1/2}(\Gamma)$ to H_p such that $R_2\mu = \mu$ on Γ for all $\mu \in H^{1/2}(\Gamma)$ and $R_1^\tau : \Lambda \to H_\mathrm{f}^\tau$ is any possible continuous extension operator from Λ to H_f^τ such that $R_1^\tau\mu\cdot\mathbf{n} = \mu$ on Γ for all $\mu \in \Lambda$, with

$$H_\mathrm{f}^\tau = \left\{\mathbf{v} \in H_\mathrm{f} : (\mathbf{v})_\tau = 0 \text{ on } \Gamma\right\}. \tag{4.45}$$

4.1.1 The interface equation associated to the Stokes/Darcy problem

We now choose a suitable governing variable on the interface Γ. Considering the interface conditions (4.9) and (4.10), we can foresee two different strategies to select the interface variable:

(1) we can set the interface variable λ as the trace of the normal velocity on the interface:

$$\lambda = \mathbf{u}_\mathrm{f}\cdot\mathbf{n} = -\mathsf{K}\partial_n\varphi\,; \tag{4.46}$$

(2) we can define the interface variable σ as the trace of the piezometric head on Γ:

$$\sigma = g\varphi = -\mathbf{n}\cdot\mathsf{T}(\mathbf{u}_\mathrm{f}, p_\mathrm{f})\cdot\mathbf{n}\,. \tag{4.47}$$

Both choices are suitable from the mathematical viewpoint since they guarantee well-posed subproblems in the fluid and the porous medium part. We shall consider here the interface equation corresponding to λ. We refer the reader to [25] for the study of the equation associated to σ.

We consider as governing variable on the interface Γ the normal component of the velocity field $\lambda = \mathbf{u}_\mathrm{f} \cdot \mathbf{n}$ as indicated in equation (4.46). Should we know a priori the value of λ on Γ, from equation (4.46) we would obtain a Dirichlet boundary condition for the Stokes system in Ω_f ($\mathbf{u}_\mathrm{f} \cdot \mathbf{n} = \lambda$ on Γ) and a Neumann boundary condition for the Darcy equation in Ω_p ($-\mathrm{K}\partial_n \varphi = \lambda$ on Γ). Jointly with equation (4.14) for the fluid problem, these conditions allow to recover (independently) the solutions $(\mathbf{u}_\mathrm{f}, p_\mathrm{f})$ of the Stokes problem in Ω_f and the solution φ of the Darcy problem in Ω_p.

We define the continuous extension operator

$$E_\Gamma : H^{1/2}(\Gamma) \to H^\tau_\mathrm{f}, \quad \eta \to E_\Gamma \eta \quad \text{such that} \quad E_\Gamma \eta \cdot \mathbf{n} = \eta \quad \text{on } \Gamma\,. \tag{4.48}$$

We consider the (unknown) interface variable $\lambda = \mathbf{u}_\mathrm{f} \cdot \mathbf{n}$ on Γ, $\lambda \in \Lambda$, and we split it as $\lambda = \lambda_0 + \lambda_*$ where $\lambda_* \in \Lambda$ depends on the inflow data and satisfies

$$\int_\Gamma \lambda_* = -\int_{\Gamma_\mathrm{f}^{in}} \mathbf{u}_{in} \cdot \mathbf{n}\,, \tag{4.49}$$

whereas $\lambda_0 \in \Lambda_0$, with

$$\Lambda_0 = \{\mu \in \Lambda \,:\, \textstyle\int_\Gamma \mu = 0\} \subset \Lambda\,. \tag{4.50}$$

Then, we introduce two auxiliary problems whose solutions (which depend on the problem data) are related to that of the global problem (4.40)–(4.44), as we will see later on:

(P1) find $\boldsymbol{\omega}_0^* \in (H_0^1(\Omega_\mathrm{f}))^d$, $\pi^* \in Q_0$ such that

$$a_\mathrm{f}(\boldsymbol{\omega}_0^* + E_\mathrm{f}\mathbf{u}_{in} + E_\Gamma\lambda_*, \mathbf{v}) + b_\mathrm{f}(\mathbf{v}, \pi^*) = \int_{\Omega_\mathrm{f}} \mathbf{f}\cdot\mathbf{v} \quad \forall \mathbf{v} \in (H_0^1(\Omega_\mathrm{f}))^d\,, \tag{4.51}$$

$$b_\mathrm{f}(\boldsymbol{\omega}_0^* + E_\mathrm{f}\mathbf{u}_{in} + E_\Gamma\lambda_*, q) = 0 \qquad \forall q \in Q_0\,; \tag{4.52}$$

(P2) find $\varphi_0^* \in H_\mathrm{p}$ such that

$$a_\mathrm{p}(\varphi_0^* + E_\mathrm{p}\varphi_\mathrm{p}, \psi) = \int_\Gamma \lambda_*\psi \quad \forall \psi \in H_\mathrm{p}\,. \tag{4.53}$$

Now we define the following extension operators:

$$R_\mathrm{f}\colon \Lambda_0 \to H^\tau_\mathrm{f} \times Q_0, \quad \eta \to R_\mathrm{f}\eta = (R^1_\mathrm{f}\eta, R^2_\mathrm{f}\eta)$$

such that $(R^1_\mathrm{f}\eta)\cdot\mathbf{n} = \eta$ on Γ and

$$a_\mathrm{f}(R^1_\mathrm{f}\eta, \mathbf{v}) + b_\mathrm{f}(\mathbf{v}, R^2_\mathrm{f}\eta) = 0 \quad \forall \mathbf{v} \in (H_0^1(\Omega_\mathrm{f}))^d\,, \tag{4.54}$$

$$b_\mathrm{f}(R^1_\mathrm{f}\eta, q) = 0 \quad \forall q \in Q_0\,; \tag{4.55}$$

and

$$R_{\mathrm{p}}\colon \Lambda \to H_{\mathrm{p}}\,, \quad \eta \to R_{\mathrm{p}}\eta$$

such that

$$a_{\mathrm{p}}(R_{\mathrm{p}}\eta, R_2\mu) = \int_\Gamma \eta\mu \quad \forall\,\mu \in H^{1/2}(\Gamma)\,. \tag{4.56}$$

We define the *Steklov–Poincaré* operator S as follows: for all $\eta \in \Lambda_0$, $\mu \in \Lambda$,

$$\langle S\eta, \mu\rangle = a_{\mathrm{f}}(R_{\mathrm{f}}^1\eta, R_1^\tau\mu) + b_{\mathrm{f}}(R_1^\tau\mu, R_{\mathrm{f}}^2\eta) + \int_\Gamma g(R_{\mathrm{p}}\eta)\mu\,, \tag{4.57}$$

which can be split as the sum of two suboperators $S = S_{\mathrm{f}} + S_{\mathrm{p}}$:

$$\langle S_{\mathrm{f}}\eta, \mu\rangle = a_{\mathrm{f}}(R_{\mathrm{f}}^1\eta, R_1^\tau\mu) + b_{\mathrm{f}}(R_1^\tau\mu, R_{\mathrm{f}}^2\eta)\,, \tag{4.58}$$

$$\langle S_{\mathrm{p}}\eta, \mu\rangle = \int_\Gamma g\,(R_{\mathrm{p}}\eta)\mu\,, \tag{4.59}$$

for all $\eta \in \Lambda_0$ and $\mu \in \Lambda$. Moreover, we define the functional $\chi : \Lambda_0 \to \mathbb{R}$,

$$\langle \chi, \mu\rangle = \int_{\Omega_{\mathrm{f}}} \mathbf{f}\,(R_1^\tau\mu) - a_{\mathrm{f}}\left(\boldsymbol{\omega}_0^* + E_{\mathrm{f}}\mathbf{u}_{in} + E_\Gamma\lambda_*, R_1^\tau\mu\right) - b_{\mathrm{f}}(R_1^\tau\mu, \pi^*) - \int_\Gamma g\,\varphi_0^*\mu \qquad \forall\,\mu \in \Lambda\,. \tag{4.60}$$

Now we can express the solution of the coupled problem in terms of the interface variable λ_0; precisely, we can prove the following result (see [29]).

Theorem 4.2. *The solution of equations* (4.40)–(4.44) *can be characterized as follows:*

$$\mathbf{u}_{\mathrm{f}}^0 = \boldsymbol{\omega}_0^* + R_{\mathrm{f}}^1\lambda_0 + E_\Gamma\lambda_*\,, \quad p_{\mathrm{f}} = \pi^* + R_{\mathrm{f}}^2\lambda_0 + \hat{p}_{\mathrm{f}}\,, \quad \varphi_0 = \varphi_0^* + R_{\mathrm{p}}\lambda_0\,, \tag{4.61}$$

where $\hat{p}_{\mathrm{f}} = (\mathrm{meas}(\Omega_{\mathrm{f}}))^{-1}\int_{\Omega_{\mathrm{f}}} p_{\mathrm{f}}$ *and* $\lambda_0 \in \Lambda_0$ *is the solution of the following Steklov–Poincaré problem:*

$$\langle S\lambda_0, \mu_0\rangle = \langle \chi, \mu_0\rangle \quad \forall\,\mu_0 \in \Lambda_0\,. \tag{4.62}$$

Moreover, $\hat{p}_{\mathrm{f}}$ *can be obtained from* λ_0 *by solving the algebraic equation*

$$\hat{p}_{\mathrm{f}} = \frac{1}{\mathrm{meas}(\Gamma)}\langle S\lambda_0 - \chi, \zeta\rangle\,, \tag{4.63}$$

where $\zeta \in \Lambda$ *is a fixed function such that*

$$\frac{1}{\mathrm{meas}(\Gamma)}\int_\Gamma \zeta = 1\,. \tag{4.64}$$

Finally, the interface equation (4.62) *has a unique solution.*

4.2 The Navier–Stokes/Darcy problem

The multidomain formulation introduced for the Darcy/Stokes problem can also be applied in the case of the Navier–Stokes equations. We present hereafter the main results in the nonlinear case and we refer to [2] for details and proofs. In this section, for the sake of simplicity, we adopt homogeneous boundary conditions, i.e. we will take $\mathbf{u}_{in} = \mathbf{0}$ on Γ_{f}^{in} and $\varphi_{\mathrm{p}} = 0$ on Γ_{p}^{b}.

We consider a linear continuous extension operator $R_1 : \Lambda \to H_{\mathrm{f}}$ such that $R_1\mu \cdot \mathbf{n} = \mu$ on Γ, for all $\mu \in \Lambda$, while we will let R_2 be the operator introduced in Proposition 4.1. Since $H_{\mathrm{f}} = H_{\mathrm{f}}^0 + \{R_1\mu : \mu \in \Lambda\}$, we can prove the following result (see [25]).

Proposition 4.3. *The coupled Navier–Stokes/Darcy problem can be equivalently reformulated in the multidomain form: find* $\mathbf{u}_{\mathrm{f}} \in H_{\mathrm{f}}$, $p_{\mathrm{f}} \in Q$, $\varphi \in H_{\mathrm{p}}$ *such that*

$$a_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}, \mathbf{v}) + c_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}; \mathbf{u}_{\mathrm{f}}, \mathbf{v}) + b_{\mathrm{f}}(\mathbf{v}, p_{\mathrm{f}}) + \int_{\Gamma} \frac{\nu\alpha_{BJ}}{\sqrt{\mathrm{K}}} (\mathbf{u}_{\mathrm{f}})_\tau (\mathbf{v})_\tau = \int_{\Omega_{\mathrm{f}}} \mathbf{f} \cdot \mathbf{v} \quad \forall \mathbf{v} \in H_{\mathrm{f}}^0\,, \tag{4.65}$$

$$b_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}, q) = 0 \qquad \forall q \in Q\,, \tag{4.66}$$

$$a_{\mathrm{p}}(\varphi, \psi) = 0 \qquad \forall \psi \in H_{\mathrm{p}}^0\,, \tag{4.67}$$

$$\int_{\Gamma} (\mathbf{u}_{\mathrm{f}} \cdot \mathbf{n})\mu = a_{\mathrm{p}}(\varphi, R_2\mu) \quad \forall \mu \in \Lambda\,, \tag{4.68}$$

$$\begin{aligned} \int_{\Gamma} g\,\varphi\mu = \int_{\Omega_{\mathrm{f}}} \mathbf{f} \cdot (R_1\mu) - a_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}, R_1\mu) - c_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}; \mathbf{u}_{\mathrm{f}}, R_1\mu) \\ - b_{\mathrm{f}}(R_1\mu, p_{\mathrm{f}}) - \int_{\Gamma} \frac{\nu\alpha_{BJ}}{\sqrt{\mathrm{K}}} (\mathbf{u}_{\mathrm{f}})_\tau (R_1\mu)_\tau \quad \forall \mu \in \Lambda\,. \end{aligned} \tag{4.69}$$

Similarly to the case of Stokes/Darcy (see Section 4.1.1), we now want to rewrite equation (4.65)–(4.69) as an interface equation for the interface variable $\lambda = (\mathbf{u}_{\mathrm{f}} \cdot \mathbf{n})_{|\Gamma}$. The incompressibility constraint in Ω_{f} and the boundary conditions imposed on $\partial\Omega_{\mathrm{f}} \setminus \Gamma$ imply that $\lambda \in \Lambda_0$.

4.2.1 The interface equation associated to the Navier–Stokes/Darcy problem

We formally define the *nonlinear* pseudodifferential operator $S : \Lambda_0 \to \Lambda_0'$,

$$\begin{aligned} \langle S(\eta), \mu \rangle = a_{\mathrm{f}}(\mathcal{R}_f^1(\eta), R_1\mu) + c_{\mathrm{f}}(\mathcal{R}_f^1(\eta); \mathcal{R}_f^1(\eta), R_1\mu) + b_{\mathrm{f}}(R_1\mu, \mathcal{R}_f^2(\eta)) \\ + \int_{\Gamma} \frac{\nu\alpha_{\mathrm{BJ}}}{\sqrt{\mathrm{K}}} (\mathcal{R}_f^1(\eta))_\tau (R_1\mu)_\tau - \int_{\Omega_{\mathrm{f}}} \mathbf{f} \cdot (R_1\mu) \\ + \int_{\Gamma} g(R_{\mathrm{p}}\eta)\mu \quad \forall \eta \in \Lambda_0, \forall \mu \in \Lambda\,, \end{aligned} \tag{4.70}$$

where $\mathcal{R}_f$ is the *nonlinear* extension operator:

$$\mathcal{R}_f : \Lambda_0 \to H_f \times Q_0, \quad \eta \to \mathcal{R}_f(\eta) = (\mathcal{R}_f^1(\eta), \mathcal{R}_f^2(\eta))$$

such that $\mathcal{R}_f^1(\eta) \cdot \mathbf{n} = \eta$ on Γ, and, for all $\mathbf{v} \in H_{\mathrm{f}}^0$, $q \in Q_0$,

$$a_{\mathrm{f}}(\mathcal{R}_f^1(\eta), \mathbf{v}) + c_{\mathrm{f}}(\mathcal{R}_f^1(\eta); \mathcal{R}_f^1(\eta), \mathbf{v}) + b_{\mathrm{f}}(\mathbf{v}, \mathcal{R}_f^2(\eta)) + \int_\Gamma \frac{\nu \alpha_{\mathrm{BJ}}}{\sqrt{\mathrm{K}}} (\mathcal{R}_f^1(\eta))_\tau (\mathbf{v})_\tau = \int_{\Omega_{\mathrm{f}}} \mathbf{f} \cdot \mathbf{v}\,, \tag{4.71}$$

$$b_{\mathrm{f}}(\mathcal{R}_f^1(\eta), q) = 0\,. \tag{4.72}$$

The operator S is composed of two parts: a nonlinear component associated to the fluid problem in Ω_{f} (the terms in the first two lines), and the linear part S_{p} related to the problem in the porous media that we already defined in (4.59). The fluid part extends the operator S_{f} in (4.58) to the nonlinear case and, similarly to S_{f}, it plays the role of a nonlinear Dirichlet-to-Neumann map that associates at any given normal velocity η on Γ the normal component of the corresponding Cauchy stress tensor on Γ.

We have the following equivalence result (see [2]).

Theorem 4.4. *The solution of equation* (4.65)–(4.69) *can be characterized as follows:*

$$\mathbf{u}_{\mathrm{f}} = \mathcal{R}_f^1(\lambda), \quad p_{\mathrm{f}} = \mathcal{R}_f^2(\lambda) + \hat{p}_{\mathrm{f}}, \quad \varphi = R_{\mathrm{p}}\lambda\,, \tag{4.73}$$

where $\hat{p}_{\mathrm{f}} = (\mathrm{meas}(\Omega_{\mathrm{f}}))^{-1} \int_{\Omega_{\mathrm{f}}} p_{\mathrm{f}}$, *and* $\lambda \in \Lambda_0$ *is the solution of the nonlinear interface problem:*

$$\langle S(\lambda), \mu \rangle = 0 \quad \forall \mu \in \Lambda_0\,. \tag{4.74}$$

Moreover, $\hat{p}_{\mathrm{f}}$ *can be obtained from* λ *by solving the algebraic equation*

$$\hat{p}_{\mathrm{f}} = (\mathrm{meas}(\Gamma))^{-1} \langle S(\lambda), \varepsilon \rangle\,,$$

where $\varepsilon \in \Lambda$ *is a fixed function such that*

$$\frac{1}{\mathrm{meas}(\Gamma)} \int_\Gamma \varepsilon = 1\,. \tag{4.75}$$

4.3 Well-posedness of the interface problems

Concerning the well-posedness of the interface problems, in the case of the linear equation (4.62) it is guaranteed since we proved that such equation is equivalent to the original Darcy/Stokes problem whose well-posedness is in its turn guaranteed by Brezzi's theory (see Section 3). For the nonlinear interface problem (4.74), we can prove the following result (see [2]).

Theorem 4.5. *If $\|\mathbf{f}\|_{L^2(\Omega_f)} \le C\nu^2$ for a suitable constant $C > 0$, there exists a suitable positive radius $r_M = R_M(\nu, \|\mathbf{f}\|_{L^2(\Omega_f)})$ such that equation (4.74) has a unique solution λ in the set $S_{r_M} = \{\eta \in \Lambda_0 : |\widetilde{R}_f^1 \eta|_1 < r_M\} \subset \Lambda_0$, where $\widetilde{R}_f^1$ is the linear extension operator:*

$$\widetilde{R}_f : \Lambda_0 \to H_f \times Q_0, \quad \eta \to \widetilde{R}_f \eta = (\widetilde{R}_f^1 \eta, \widetilde{R}_f^2 \eta), \tag{4.76}$$

satisfying $\widetilde{R}_f^1 \eta \cdot \mathbf{n} = \eta$ on Γ, and, for all $\mathbf{v} \in H_f^0$, $q \in Q_0$,

$$a_f(\widetilde{R}_f^1 \eta, \mathbf{v}) + b_f(\mathbf{v}, \widetilde{R}_f^2 \eta) + \int_\Gamma \frac{\nu \alpha_{BJ}}{\sqrt{K}} (\widetilde{R}_f^1 \eta)_\tau (\mathbf{v})_\tau = 0, \tag{4.77}$$

$$b_f(\widetilde{R}_f^1 \eta, q) = 0. \tag{4.78}$$

Note that the condition $\|\mathbf{f}\|_{L^2(\Omega_f)} \le C\nu^2$ is analogous to that usually required to prove existence and uniqueness of the solution of the Navier–Stokes equations. Moreover, Theorem 4.5 states that the solution is unique in S_{r_M}. This means that, in contrast to the linear case, the solution is unique only for sufficiently small normal velocities λ across the interface Γ.

Remark 4.6. The steady Navier–Stokes/Darcy problem has also been studied in [39] where no multidomain approach is considered and the analysis is carried out following the classical theory for nonlinear systems.

The multidomain setting that we have illustrated in this section for the steady problem can also be extended to the time-dependent case. An introductory discussion is provided in Section 8.

5 Finite element approximation of free and porous-media flows

We consider a triangulation $\mathcal{T}_h$ of the domain $\overline{\Omega}_f \cup \overline{\Omega}_p$, depending on a positive parameter $h > 0$, made of triangles if $d = 2$, or tetrahedra in the three-dimensional case. We assume that the triangulations $\mathcal{T}_{fh}$ and $\mathcal{T}_{ph}$ induced on the subdomains Ω_f and Ω_p are compatible on Γ, that is they share the same edges (if $d = 2$) or faces (if $d = 3$) therein. (The case of nonmatching grids across the interface Γ for the Stokes–Darcy coupling has been studied, e.g. in [10, 24, 34, 35, 37, 62, 63].)

For the finite element approximation of the (Navier-)Stokes equations we consider *inf-sup* stable [12] methods featuring either discontinuous pressure finite elements (e.g. the $\mathbb{P}_2 - \mathbb{P}_0$ elements or the Crouzeix–Raviart elements defined using cubic bubble functions) or continuous pressure finite elements. Among the latter we recall the Taylor–Hood (or $\mathbb{P}_2 - \mathbb{P}_1$) elements and the $(\mathbb{P}_1 iso \mathbb{P}_2) - \mathbb{P}_1$ elements (see, e.g. [60, Chapter 9] or [16, Chapter VI].)

Concerning the solution of the Darcy problem, either we solve the elliptic problem for the piezometric head φ and then we recover the velocity computing its gradient,

or we use a mixed approach. This latter (and more popular) approach permits to recover simultaneously both the primal unknown and its gradient with the same order of convergence. Moreover, mass is locally conserved and the continuity of fluxes is preserved.

This approach comprises the so-called mixed (MFE) and mixed-hybrid (MHFE) finite elements, among which we recall the Raviart–Thomas (RT) elements, the Brezzi–Douglas–Marini (BDM) and the Brezzi–Douglas–Fortin–Marini (BDFM) elements, only to quote the most classical ones (see [13–16, 55, 56, 61, 66]). In this context we mention also [52] which presents a new stabilized MFE method without mesh-dependent parameters, and the comparative study [40] concerning the numerical reliability of MFE and MHFE methods applied to porous media flows under the influence of mesh parameters and medium heterogeneity. The method proposed in [52] has been used in [67] to set up a globally stable method for the Darcy/Stokes system using Lagrangian polynomials for both subproblems.

Other approaches are based on the discontinuous Galerkin (DG) methods (see [1, 23]) which are attractive for porous media flow due to their high order convergence property, local conservation of mass, flexibility with respect to meshing and hp-adaptive refinement, and their robustness with respect to strongly discontinuous coefficients. A numerical comparison between DG and MFE for porous media can be found in [6].

MFE and DG have also been adopted in the works [38, 45, 47, 62, 63] for the Stokes/Darcy coupling, and in [39] for the Navier–Stokes/Darcy problem. In particular, in [47] a coupling between inf-sup-stable finite elements for Stokes and MFE for Darcy equations is realized using hanging nodes on the interface Γ. The analysis developed shows that optimal error bounds can be obtained in both the fluid and the porous region.

DG methods based on Interior Penalty are considered in [39, 62] for both the fluid and the groundwater problem and all unknowns are approximated by totally discontinuous polynomials of different orders.

The two approaches are combined in [63] where the fluid velocity and pressure are obtained by MFE in the porous media region, while they are approximated by DG in the incompressible flow region. Error estimates are derived for two-dimensional problems and the authors point out that nonmatching grids on the interface can be used, with the space of discrete normal velocities on Γ playing the role of a mortar space.

The issue of adopting different meshes in the two subdomains has been also considered in [19], where $\mathbb{P}_1 - \mathbb{P}_0$ finite elements, stabilized through a generalization of the Brezzi–Pitkäranta penalization, have been used for both the fluid and the porous medium, realizing the coupling via a Nitsche method.

A finite element scheme for the approximation of multidomain heterogeneous problems like Stokes/Darcy has been proposed in [24]. This approach exploits stabilized mixed finite elements together with Nitsche-type matching conditions that

automatically adapt to the coupling of different subproblem combinations (see also [18, 20, 69]).

We also refer to [57] for the analysis of several finite element methods for the coupled Darcy/Stokes problem.

Finally, stabilized finite element methods for both Stokes and Darcy problems based on the variational multiscale method can be found in [3, 4].

5.1 Galerkin finite-element approximation of the Stokes/Darcy problem

In this section, we introduce a possible conforming finite element discretization of the filtration problem focusing for simplicity of exposition on the Stokes/Darcy model and adopting the single field formulation (4.8) for Darcy equation. This approach allows to treat the interface conditions as natural conditions for both the fluid and the porous media, and it will perfectly serve our purpose to characterize iterative substructuring methods to solve the coupled problem. In the following, for the sake of exposition, we will consider the special choice of piecewise quadratic elements for the velocity components and piecewise linear for the pressure in the fluid domain ($\mathbb{P}_2 - \mathbb{P}_1$ finite elements), while we shall consider quadratic $\mathbb{P}_2$ elements for the piezometric head in the porous media domain. For the sake of clarity let us illustrate the degrees of freedom we are considering and how they match across the interface Γ: in Figure 4.2 we sketch two triangles of a conforming regular mesh and we indicate the degrees of freedom corresponding to the velocity $\mathbf{u}_\mathrm{f}$ and the pressure p_f in Ω_f, and to the piezometric head φ in Ω_p.

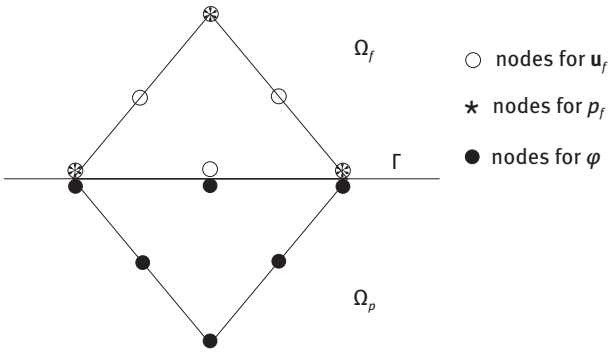

Figure 4.2: Degrees of freedom of the finite elements used for approximating velocity, pressure and piezometric head.

Introducing suitable finite element spaces (see [30] for details), we obtain the algebraic system associated to the Darcy–Stokes problem

$$\begin{pmatrix} \mathrm{A}_{ff} & \mathrm{B}^{\mathrm{T}} & \mathrm{A}_{f\Gamma} & 0 & 0 \\ \mathrm{B}_1 & 0 & \mathrm{B}_{f\Gamma} & 0 & 0 \\ \mathrm{A}_{\Gamma f} & \mathrm{B}_{f\Gamma}^{\mathrm{T}} & \mathrm{A}_{\Gamma\Gamma}^{f} & \mathrm{M}_\Gamma & 0 \\ 0 & 0 & -\mathrm{M}_\Gamma^{\mathrm{T}} & \mathrm{A}_{\Gamma\Gamma}^{p} & \mathrm{A}_{p\Gamma}^{\mathrm{T}} \\ 0 & 0 & 0 & \mathrm{A}_{p\Gamma} & \mathrm{A}_{pp} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{\mathrm{int}} \\ \boldsymbol{p} \\ \boldsymbol{u}_\Gamma \\ \boldsymbol{\varphi}_\Gamma \\ \boldsymbol{\varphi}_i \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_{\mathrm{f}} \\ \boldsymbol{f}_{in} \\ \boldsymbol{f}_\Gamma \\ \boldsymbol{f}_{p\Gamma} \\ \boldsymbol{f}_{pi} \end{pmatrix} \tag{4.79}$$

where $\boldsymbol{u}_{\mathrm{int}}$ is the vector of the nodal values of the velocity in Ω_{f}, except those of the normal velocity on Γ which are stored in the vector $\boldsymbol{u}_\Gamma$. On the other hand, $\boldsymbol{\varphi}_\Gamma$ and $\boldsymbol{\varphi}_i$ are the vectors of the nodal values of the piezometric head corresponding, respectively, to the nodes on the interface Γ and to the remaining nodes of the domain Ω_{p}. Finally, $\boldsymbol{p}$ is the vector of the nodal values of the fluid pressure in Ω_{f}.

The matrix of the linear system (4.79) is nonsingular, and generally it is large and sparse. To effectively solve this system using an iterative method, a preconditioning strategy is thus in order. The characterization of suitable preconditioners will be the focus of the next sections.

Remark that the coupling between Stokes and Darcy equations is realized at this algebraic stage through the third and the fourth rows of the global matrix. Indeed, the submatrices M_Γ and $-\mathrm{M}_\Gamma^{\mathrm{T}}$ impose the algebraic counterpart of the coupling conditions (4.10) and (4.9), respectively. More precisely, the matrix M_Γ corresponds to the Galerkin approximation of the term

$$\int_\Gamma g\varphi(\mathbf{v}\cdot\mathbf{n})$$

in (4.36), i.e.

$$(\mathrm{M}_\Gamma)_{ij} = \int_\Gamma g\phi_j^h(\boldsymbol{v}_i^h\cdot\mathbf{n}) \quad i=1,\dots,n_v,\ j=1,\dots,n_\varphi,$$

where $\{\phi_j^h\}_{j=1,\dots,n_\varphi}$ and $\{\boldsymbol{v}_i^h\}_{i=1,\dots,n_v}$ are the basis functions for the finite element discretization of the spaces of the piezometric head and of the fluid velocity, respectively. On the other hand, $-\mathrm{M}_\Gamma^{\mathrm{T}}$ corresponds to the discretization of the interface term $-\int_\Gamma g\psi(\mathbf{u}\cdot\mathbf{n})$ in (4.36).

6 Algebraic formulation of the linear interface problem and solution techniques

In this section, we introduce the algebraic counterpart of the interface equation (4.62) and we characterize suitable iterative methods to compute separately the solution of the Stokes and of the Darcy problems.

We rewrite the system (4.79) as follows, introducing a variable $\boldsymbol{u}_i = (\boldsymbol{u}_{\text{int}}, \boldsymbol{p})^{\text{T}}$ which groups the nodal values of the internal velocity and of the pressure in Ω_{f}:

$$\begin{pmatrix} \text{F} & \text{D} & 0 & 0 \\ \text{D}^{\text{T}} & \text{A}_{\Gamma\Gamma}^{f} & \text{M}_{\Gamma} & 0 \\ 0 & -\text{M}_{\Gamma}^{\text{T}} & \text{A}_{\Gamma\Gamma}^{p} & \text{A}_{p\Gamma}^{\text{T}} \\ 0 & 0 & \text{A}_{p\Gamma} & \text{A}_{pp} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{u}_{\Gamma} \\ \boldsymbol{\varphi}_{\Gamma} \\ \boldsymbol{\varphi}_i \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_{fi} \\ \boldsymbol{f}_{f\Gamma} \\ \boldsymbol{f}_{p\Gamma} \\ \boldsymbol{f}_{pi} \end{pmatrix}. \tag{4.80}$$

The algebraic counterpart of (4.62) corresponds to the Schur complement system with respect to the unknown $\boldsymbol{u}_{\Gamma}$ (see [30]):

$$\Sigma_{\text{f}} \boldsymbol{u}_{\Gamma} + \Sigma_{\text{p}} \boldsymbol{u}_{\Gamma} = \boldsymbol{\chi}_{fp} \tag{4.81}$$

where $\Sigma_{\text{f}} = \text{A}_{\Gamma\Gamma}^{f} - \text{D}^{\text{T}}\text{F}^{-1}\text{D}$, $\Sigma_{\text{p}} = \text{M}_{\Gamma}\Sigma_{d}^{-1}\text{M}_{\Gamma}^{\text{T}}$ with $\Sigma_d = \text{A}_{\Gamma\Gamma}^{p} - \text{A}_{p\Gamma}^{\text{T}}\text{A}_{pp}^{-1}\text{A}_{p\Gamma}$, while

$$\boldsymbol{\chi}_{fp} = \boldsymbol{f}_{f\Gamma} - \text{D}^{\text{T}}\text{F}^{-1}\boldsymbol{f}_{fi} - \begin{pmatrix} \text{M}_{\Gamma} & 0 \end{pmatrix} \begin{pmatrix} \text{A}_{\Gamma\Gamma}^{p} & \text{A}_{p\Gamma}^{\text{T}} \\ \text{A}_{p\Gamma} & \text{A}_{pp} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{f}_{p\Gamma} \\ \boldsymbol{f}_{pi} \end{pmatrix}.$$

The matrices Σ_{f} and Σ_{p} are, respectively, the algebraic counterpart of the finite element approximation of the Steklov–Poincaré operator S_{f} (4.58) and S_{p} (4.59). Indeed, applying S_{f} to λ (at the discrete level, Σ_{f} to $\boldsymbol{u}_{\Gamma}$) implies solving the Stokes problem (4.54) and (4.55) in Ω_{f} imposing that the normal velocity on Γ is equal to λ (at the discrete level, to $\boldsymbol{u}_{\Gamma}$), and computing the corresponding normal stress across the interface Γ. On the other hand, applying S_{p} to λ (at the discrete level, Σ_{p} to $\boldsymbol{u}_{\Gamma}$) implies solving the Laplace problem (4.56) in Ω_{p} imposing that the conormal derivative of the piezometric head $-\text{K}\nabla\varphi \cdot \mathbf{n}$ is equal to λ on Γ (at the discrete level, to $\boldsymbol{u}_{\Gamma}$) and computing the corresponding value of φ on the interface. See [25] for more details.

If we denote by S_{fh} and S_{ph} the finite element approximations of S_{f} and S_{p}, respectively, we can prove the following result (see [25, 30, 46]).

Lemma 4.7. *Let Λ_{0h} be a conforming finite element approximation of Λ_0. The discrete Steklov–Poincaré operators S_{fh} and S_{ph} enjoy the following properties:*
(1) S_{fh} and S_{ph} are linear continuous operators on Λ_{0h};
(2) S_{fh} is symmetric and coercive on Λ_{0h};
(3) S_{ph} is symmetric and coercive on Λ_{0h}, with coercivity constant dependent on h.
Moreover, assuming that ν and K are constant in Ω_{f} and Ω_{p}, we can provide the following bounds for the eigenvalues of their algebraic representation:

$$h\nu \preceq \sigma(\Sigma_{\text{f}}) \preceq \nu \quad \textit{and} \quad h^2\text{K}^{-1} \preceq \sigma(\Sigma_{\text{p}}) \preceq h\text{K}^{-1} \tag{4.82}$$

and also

$$h\text{K} \preceq \sigma(\Sigma_d) \preceq \text{K} \quad \textit{and} \quad h^2\nu^{-1} \preceq \sigma(\Sigma_s) \preceq h\nu^{-1} \tag{4.83}$$

where $\Sigma_s = \text{M}_{\Gamma}^{\text{T}}\Sigma_{\text{f}}^{-1}\text{M}_{\Gamma}$. ($\preceq$ indicates that the inequalities hold up to constants independent of h, ν, K.)

As a consequence of Lemma 4.7, the matrices Σ_f and $(\Sigma_\mathrm{f}+\Sigma_\mathrm{p})$ are symmetric and positive definite. Moreover, there holds

$$[\Sigma_f \boldsymbol{\mu}, \boldsymbol{\mu}] \leq \left[(\Sigma_\mathrm{f}+\Sigma_\mathrm{p})\boldsymbol{\mu}, \boldsymbol{\mu}\right] \leq \left(1+\frac{\beta_\mathrm{p}}{\alpha_\mathrm{f}}\right)[\Sigma_f \boldsymbol{\mu}, \boldsymbol{\mu}] \quad \forall \boldsymbol{\mu} \in \mathbb{R}^{N_\Gamma}, \tag{4.84}$$

where $[\cdot,\cdot]$ is the Euclidean scalar product in $\mathbb{R}^{N_\Gamma}$ and α_f and β_p are, respectively, the coercivity and continuity constants of S_{fh} and S_{ph}. Thus, the spectral condition number χ_{sp} of the matrix $\Sigma_f^{-1}(\Sigma_\mathrm{f}+\Sigma_\mathrm{p})$ is bounded independently of h:

$$\chi_{sp}\left(\Sigma_f^{-1}(\Sigma_\mathrm{f}+\Sigma_\mathrm{p})\right) \leq 1+\frac{\beta_\mathrm{p}}{\alpha_\mathrm{f}}, \tag{4.85}$$

and Σ_f is an optimal preconditioner for $(\Sigma_\mathrm{f}+\Sigma_\mathrm{p})$. (Remark, however, that the bound (4.85) of the condition number depends on the values of ν and K.) Therefore, should we use Σ_f as preconditioner to solve the symmetric linear system (4.81) using a PCG method with Σ_f as preconditioner, we would expect to get convergence with a rate independent of h (but not of ν and K).

Remark 4.8. One may also consider the Schur complement of (4.80) with respect to $\boldsymbol{\varphi}_\Gamma$ instead of $\boldsymbol{u}_\Gamma$. In that case, one would obtain the following system:

$$\Sigma_s \boldsymbol{\varphi}_\Gamma + \Sigma_d \boldsymbol{\varphi}_\Gamma = \boldsymbol{\chi}_{sd} \tag{4.86}$$

where the matrices Σ_s and Σ_d have been defined previously and $\boldsymbol{\chi}_{sd}$ is the corresponding right-hand side. This strategy has been studied in [25] and we refer the reader to that reference for details. We only mention here that in this case it can be proved that the matrices Σ_d and $(\Sigma_s+\Sigma_d)$ are symmetric positive definite and that Σ_d is an optimal preconditioner for $(\Sigma_s+\Sigma_d)$ in the sense that it may yield convergence in a number of iterations that does not depend on h (but may depend on the values of ν and K).

6.1 Numerical results

We consider the computational domain $\Omega \subset \mathbb{R}^2$ with $\Omega_\mathrm{f} = (0,1)\times(1,2)$, $\Omega_\mathrm{p} = (0,1)\times(0,1)$ and the interface $\Gamma = (0,1)\times\{1\}$. We impose suitable boundary conditions in such a way that the exact solution of the coupled Stokes/Darcy problem is $(\mathbf{u}_\mathrm{f})_1 = y^2 - 2y + 1$, $(\mathbf{u}_\mathrm{f})_2 = x^2 - x$, $p_\mathrm{f} = 2\nu(x+y-1) + (3\mathsf{K})^{-1}$, $\varphi = \mathsf{K}^{-1}\left(x(1-x)(y-1) + y^3/3 - y^2 + y\right) + 2\nu x$. The most relevant physical quantities for the coupling are the fluid viscosity ν and the hydraulic conductivity K. We start by considering the PCG method for (4.81) and (4.86) with preconditioners Σ_f and Σ_d, respectively. We use a convergence test based on the relative residual with tolerance 10^{-10}. In Table 4.2, we report the number of iterations for several choices

Table 4.2: PCG iterations for (4.81) with preconditioner Σ_{f} (left) and (4.86) with preconditioner Σ_d (right).

ν	K	$h = 1/7$	$h = 1/14$	$h = 1/28$	$h = 1/56$	$h = 1/7$	$h = 1/14$	$h = 1/28$	$h = 1/56$
1	1	5	5	5	5	6	6	6	6
10^{-1}	10^{-1}	11	11	10	10	10	10	9	9
10^{-2}	10^{-1}	15	19	18	17	15	15	14	14
10^{-3}	10^{-2}	20	54	73	56	19	46	52	43
10^{-4}	10^{-3}	20	59	#	#	22	55	82	88
10^{-6}	10^{-4}	20	59	148	#	41	78	102	123

of ν and K (the symbol # indicates that the method did not converge within 150 iterations). We can see that the convergence of the algorithm is troublesome when the values of ν and K decrease. In fact, in that case the method converges in a large number of iterations which increases when h decreases, losing its optimality properties. The subdomain iterative method that we have proposed is then effective only when the product νK is sufficiently large, while dealing with small values causes severe difficulties. (Remark that the latter are the values of interest in real-life applications: see, for example, the values of K reported in Table 4.1 and recall that water has kinematic viscosity $\nu = 10^{-6}\,\mathrm{m}^2/\mathrm{s}$.)
A more detailed study of the dependence of the behavior of the iterative methods on the physical parameters can be found in [25, 27].

6.2 Other preconditioning methods

The results presented in the previous section show that other preconditioning strategies should be devised for the cases where the fluid viscosity ν and the hydraulic conductivity K are small. These are indeed situations of interest for most practical applications. In [31] a Robin–Robin method was proposed to solve effectively (4.81). Here we present a different approach based on the generalized Hermitian/skew-Hermitian splitting method of [9] for (4.81) and (4.86). We start considering (4.81).

The matrix $\Sigma_{\mathrm{f}} + \Sigma_{\mathrm{p}}$ has no skew-symmetric component being symmetric positive definite, but thanks to the estimates (4.82) we can mimick the splitting proposed in [9] considering Σ_{f} as a matrix multiplied by a coefficient (ν) which may become small. Thus, we can characterize the preconditioner

$$P_1 = (\Sigma_{\mathrm{f}} + \alpha_1 I)(\Sigma_{\mathrm{p}} + \alpha_1 I) \tag{4.87}$$

where α_1 is a suitable acceleration parameter. Following [9], we set $\alpha_1 \simeq \sqrt{\nu}$ and we use P_1 as preconditioner for GMRES iterations for (4.81).

Table 4.3: Number of iterations to solve (4.81) and (4.86) using preconditioners P_1 and P_2 (four computational meshes and several values of ν and K have been considered).

CG iterations and GMRES iterations with preconditioner $\boldsymbol{P_1}$ for (4.81).

	$\nu = 10^{-4}$, K $= 10^{-3}$			$\nu = 10^{-6}$, K $= 10^{-5}$			$\nu = 10^{-6}$, K $= 10^{-8}$		
	CG	GMRES + P_1		CG	GMRES + P_1		CG	GMRES + P_1	
h_1	9	5	$(\alpha_1 = 10^{-2})$	9	4	$(\alpha_1 = 10^{-3})$	9	4	$(\alpha_1 = 10^{-3})$
h_2	20	7	$(\alpha_1 = 10^{-2})$	20	4	$(\alpha_1 = 10^{-3})$	20	4	$(\alpha_1 = 10^{-3})$
h_3	42	9	$(\alpha_1 = 10^{-3})$	42	4	$(\alpha_1 = 10^{-3})$	42	4	$(\alpha_1 = 10^{-3})$
h_4	64	9	$(\alpha_1 = 10^{-3})$	66	4	$(\alpha_1 = 10^{-3})$	66	4	$(\alpha_1 = 10^{-3})$

CG iterations and GMRES iterations with preconditioner $\boldsymbol{P_2}$ for (4.86).

	$\nu = 10^{-4}$, K $= 10^{-3}$			$\nu = 10^{-6}$, K $= 10^{-5}$			$\nu = 10^{-6}$, K $= 10^{-8}$		
	CG	GMRES + P_2		CG	GMRES + P_2		CG	GMRES + P_2	
h_1	11	8	$(\alpha_2 = 10^{-2})$	13	5	$(\alpha_2 = 10^{-3})$	-	3	$(\alpha_2 = 10^{-3})$
h_2	22	9	$(\alpha_2 = 10^{-2})$	24	5	$(\alpha_2 = 10^{-3})$	-	4	$(\alpha_2 = 10^{-3})$
h_3	47	10	$(\alpha_2 = 10^{-2})$	52	6	$(\alpha_2 = 10^{-3})$	57	4	$(\alpha_2 = 10^{-3})$
h_4	84	10	$(\alpha_2 = 10^{-2})$	108	6	$(\alpha_2 = 10^{-3})$	124	4	$(\alpha_2 = 10^{-3})$

Proceeding analogously for (4.86), we can characterize the preconditioner

$$P_2 = (\Sigma_d + \alpha_2 I)(\Sigma_s + \alpha_2 I) \tag{4.88}$$

that we use for preconditioned GMRES iterations setting $\alpha_2 \simeq \sqrt{\mathrm{K}}$.

Remark that P_1 and P_2 can be regarded as generalizations of the Robin–Robin method introduced in [25, 31].

For the numerical tests we consider $\Omega_\mathrm{f} = (0,1) \times (1,2)$, $\Omega_\mathrm{p} = (0,1)^2$ with interface $\Gamma = (0,1) \times \{1\}$ and the analytic solution: $\mathbf{u} = ((y-1)^2 + (y-1) + 1, x(x-1))$, $p = 2\nu(x + y - 1)$, $\varphi = \mathrm{K}^{-1}(x(1-x)(y-1) + (y-1)^3/3) + 2\nu x$. Tolerance is 10^{-9} and we use four computational meshes with $h_j = 2^{-(j+1)}$. The numerical results reported in Table 4.3 show that these preconditioners are more effective in case of low values of the physical coefficients. A detailed analysis can be found in [27].

7 Iterative methods for the Navier–Stokes/Darcy problem

In this section, we present some classical iterative methods to compute the solution of a conforming finite element approximation of the nonlinear Navier–Stokes/Darcy problem (4.36) and (4.37). In particular, we illustrate classical fixed-point and Newton methods that require to solve at each iteration a coupled linearized Navier–Stokes/Darcy problem. An example of an iterative scheme based on Richardson

iterations that permits to treat in a decoupled way the Navier–Stokes equations and the Darcy ones is given in [2]. However, the convergence of such method is quite slow compared to the Newton and fixed-point algorithms. To our knowledge, other effective methods decoupling the two subproblems are not available yet. In the rest of this section, we describe the fixed-point and the Newton schemes. For easiness of notation, we will write them in the continuous form.

Fixed-point iterations

Fixed-point iterations to solve the coupled problem (4.36) and (4.37) can be written as follows. Given $\mathbf{u}_{\mathrm{f}}^0 \in H_{\mathrm{f}}$, for $k \geq 1$, find $\mathbf{u}_{\mathrm{f}}^k \in H_{\mathrm{f}}$, $p_{\mathrm{f}}^k \in Q$, $\varphi^k \in H_{\mathrm{p}}$ such that, for all $\mathbf{v} \in H_{\mathrm{f}}$, $q \in Q$, $\psi \in H_{\mathrm{p}}$,

$$a_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}^k, \mathbf{v}) + c_{\mathrm{f}}\left(\mathbf{u}_{\mathrm{f}}^{k-1}; \mathbf{u}_{\mathrm{f}}^k, \mathbf{v}\right) + b_{\mathrm{f}}(\mathbf{v}, p_{\mathrm{f}}^k) + \int_\Gamma g\,\varphi^k(\mathbf{v}\cdot\mathbf{n}) + \int_\Gamma \frac{\nu\alpha_{\mathrm{BJ}}}{\sqrt{\mathrm{K}}}(\mathbf{u}_{\mathrm{f}}^k)_\tau(\mathbf{v})_\tau = \int_{\Omega_{\mathrm{f}}} \mathbf{f}\cdot\mathbf{v}\,, \tag{4.89}$$

$$b_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}^k, q) = 0\,, \tag{4.90}$$

$$a_{\mathrm{p}}(\varphi^k, \psi) = \int_\Gamma \psi(\mathbf{u}_{\mathrm{f}}^k\cdot\mathbf{n})\,. \tag{4.91}$$

Algorithm (4.89)–(4.91) requires to solve at each iteration a linear coupled problem, and it can be reinterpreted as a fixed-point method to solve the interface problem (4.74).

Newton-like methods

Let us consider now the following Newton method to solve equation (4.36) and (4.37): let $\mathbf{u}_{\mathrm{f}}^0 \in H_{\mathrm{f}}$ be given; then, for $k \geq 1$, find $\mathbf{u}_{\mathrm{f}}^k \in H_{\mathrm{f}}$, $p_{\mathrm{f}}^k \in Q$, $\varphi^k \in H_{\mathrm{p}}$ such that, for all $\mathbf{v} \in H_{\mathrm{f}}$, $q \in Q$, $\psi \in H_{\mathrm{p}}$,

$$a_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}^k, \mathbf{v}) + c_{\mathrm{f}}\left(\mathbf{u}_{\mathrm{f}}^k; \mathbf{u}_{\mathrm{f}}^{k-1}, \mathbf{v}\right) + c_{\mathrm{f}}\left(\mathbf{u}_{\mathrm{f}}^{k-1}; \mathbf{u}_{\mathrm{f}}^k, \mathbf{v}\right) + b_{\mathrm{f}}\left(\mathbf{v}, p_{\mathrm{f}}^k\right) + \int_\Gamma g\varphi^k\,(\mathbf{v}\cdot\mathbf{n}) + \int_\Gamma \frac{\nu\alpha_{\mathrm{BJ}}}{\sqrt{\mathrm{K}}}\left(\mathbf{u}_{\mathrm{f}}^k\right)_\tau(\mathbf{v})_\tau = c_{\mathrm{f}}\left(\mathbf{u}_{\mathrm{f}}^{k-1}; \mathbf{u}_{\mathrm{f}}^{k-1}, \mathbf{v}\right) + \int_{\Omega_{\mathrm{f}}} \mathbf{f}\cdot\mathbf{v}\,, \tag{4.92}$$

$$b_{\mathrm{f}}\left(\mathbf{u}_{\mathrm{f}}^k, q\right) = 0\,, \tag{4.93}$$

$$a_{\mathrm{p}}\left(\varphi^k, \psi\right) = \int_\Gamma \psi\left(\mathbf{u}_{\mathrm{f}}^k\cdot\mathbf{n}\right)\,. \tag{4.94}$$

To reduce the computational cost, we might consider the modified Newton method: find $\mathbf{u}_{\mathrm{f}}^k \in H_{\mathrm{f}}$, $p_{\mathrm{f}}^k \in Q$, $\varphi^k \in H_{\mathrm{p}}$ such that, for all $\mathbf{v} \in H_{\mathrm{f}}$, $q \in Q$, $\psi \in H_{\mathrm{p}}$,

$$a_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}^k, \mathbf{v}) + c_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}^k; \mathbf{u}_{\mathrm{f}}^0, \mathbf{v}) + c_{\mathrm{f}}\left(\mathbf{u}_{\mathrm{f}}^0; \mathbf{u}_{\mathrm{f}}^k, \mathbf{v}\right) + b_{\mathrm{f}}\left(\mathbf{v}, p_{\mathrm{f}}^k\right) + \int_\Gamma g\varphi^k(\mathbf{v}\cdot\mathbf{n})$$
$$+ \int_\Gamma \frac{\nu\alpha_{\mathrm{BJ}}}{\sqrt{\mathsf{K}}}\left(\mathbf{u}_{\mathrm{f}}^k\right)_\tau(\mathbf{v})_\tau = c_{\mathrm{f}}\left(\mathbf{u}_{\mathrm{f}}^{k-1}; \mathbf{u}_{\mathrm{f}}^0, \mathbf{v}\right) + c_{\mathrm{f}}\left(\mathbf{u}_{\mathrm{f}}^0 - \mathbf{u}_{\mathrm{f}}^{k-1}; \mathbf{u}_{\mathrm{f}}^{k-1}, \mathbf{v}\right) + \int_{\Omega_{\mathrm{f}}} \mathbf{f}\cdot\mathbf{v}, \quad (4.95)$$

$$b_{\mathrm{f}}(\mathbf{u}_{\mathrm{f}}^k, q) = 0, \quad (4.96)$$

$$a_{\mathrm{p}}(\varphi^k, \psi) = \int_\Gamma \psi(\mathbf{u}_{\mathrm{f}}^k \cdot \mathbf{n}). \quad (4.97)$$

Like for fixed-point iterations, we have to solve a linearized coupled problem at each iteration of the Newton algorithms. The method (4.95)–(4.97) corresponds to using an inexact Jacobian in which we have dropped the terms arising from the linearized trilinear convective terms. The matrix associated to this inexact Jacobian is now independent of the iteration level, hence it can be factorized once and for all (offline) at the very first iteration.

The Newton methods (4.92)–(4.94) and (4.95)–(4.97) can be interpreted as iterative schemes for the interface equation (4.74). For more details and convergence proofs we refer to [2].

Table 4.4 shows the behavior of these methods on a test with computational domain $\Omega_{\mathrm{f}} = (0,1)\times(1,2)$, $\Omega_{\mathrm{p}} = (0,1)^2$ and interface $\Gamma = (0,1)\times\{1\}$. The boundary conditions and the forcing terms are set so that the problem has the same analytic solution indicated at the end of Section 6.2.

Table 4.4: Convergence behavior of the fixed-point and Newton methods with respect to the parameters ν and K.

		Iterations for the fixed-point method			Iterations for the Newton method		
ν	K	$h = 1/7$	$h = 1/14$	$h = 1/28$	$h = 1/7$	$h = 1/14$	$h = 1/28$
1	1	7	7	7	4	4	4
1	10^{-4}	5	5	5	4	4	4
10^{-1}	10^{-1}	10	10	10	5	5	5
10^{-2}	10^{-1}	15	15	15	6	6	6
10^{-2}	10^{-3}	13	13	13	6	6	6

8 Subdomain iterative methods for the time-dependent (Navier–)Stokes/Darcy problem

In this section, we sketch some possible methods to solve the evolutive (Navier–) Stokes/Darcy problem.

A simple algorithm to solve the coupled problem formed by the time-dependent Stokes equations and the Darcy equation is presented in [26]. The method reads as follows. Let $[0,T]$ be a characteristic time interval. Consider for the sake of simplicity the first-order backward Euler scheme, and denote by $\Delta t > 0$ the time step and $N = T/\Delta t$. For $n = 0,\dots,N-1$, do

(0) Choose an initial guess $(\mathbf{u}_{f,n+1}^0)\cdot\mathbf{n}$ for the normal velocity on Γ at the $(n+1)$th time level;

For $k \geq 0$ until convergence, do

(1) Solve Darcy equation

$$a_{\mathrm{p}}(\varphi_{n+1}^{k+1},\psi) = \int_\Gamma ((\mathbf{u}_{f,n+1}^k)\cdot\mathbf{n})\psi \quad \forall\psi;$$

(2) Solve the Stokes problem

$$\frac{1}{\Delta t}\int_{\Omega_{\mathrm{f}}} \mathbf{u}_{f,n+1}^{k+\frac{1}{2}}\cdot\mathbf{v} + a_{\mathrm{f}}(\mathbf{u}_{f,n+1}^{k+\frac{1}{2}},\mathbf{v}) + \int_\Gamma \frac{\nu\alpha_{\mathrm{BJ}}}{\sqrt{\mathrm{K}}}(\mathbf{u}_{f,n+1}^{k+\frac{1}{2}})_\tau(\mathbf{v})_\tau + b_{\mathrm{f}}(\mathbf{v},p_{f,n+1}^{k+\frac{1}{2}}) + \int_\Gamma g\varphi_{n+1}^{k+1}(\mathbf{v}\cdot\mathbf{n}) = \int_{\Omega_{\mathrm{f}}}\mathbf{f}^{n+1}\cdot\mathbf{v} + \frac{1}{\Delta t}\int_{\Omega_{\mathrm{f}}}\mathbf{u}_{f,n}\cdot\mathbf{v} \quad \forall\mathbf{v},$$

$$b_{\mathrm{f}}(\mathbf{u}_{f,n+1}^{k+\frac{1}{2}},q) = 0 \quad \forall q.$$

(3) Update: $(\mathbf{u}_{f,n+1}^{k+1})\cdot\mathbf{n} = \theta\,(\mathbf{u}_{f,n+1}^{k+\frac{1}{2}})\cdot\mathbf{n} + (1-\theta)\,(\mathbf{u}_{f,n+1}^{k})\cdot\mathbf{n}$ on Γ, $\theta\in(0,1)$.

This algorithm requires solving at each time step the Darcy and the Stokes equations separately performing subiterations at each time step. (For numerical results see [26]).

A similar approach for the coupled time-dependent Navier–Stokes/Darcy problem has been proposed in [22]. In that case, since the application of interest required computing the steady state solution, a backward Euler scheme is applied with explicit treatment of the nonlinear convective term but without performing subiterations at each time step. More precisely, the scheme reads as follows.

Let $\tilde{\varphi}^0$ and ψ^0 be suitable approximations at the initial time of the piezometric head φ on Γ and of the normal velocity $\mathbf{u}_f\cdot\mathbf{n}$ on Γ, respectively. Moreover, let $0 \leq \alpha,\beta \leq 1$ be two suitably chosen relaxation parameters. Then, for $n \geq 0$

(1) find $\mathbf{u}_f^{n+1}$ and p_f^{n+1} such that

$$\frac{1}{\Delta t}\int_{\Omega_f}\mathbf{u}_f^{n+1}\cdot\mathbf{v}+a_f(\mathbf{u}_f^{n+1},\mathbf{v})+c_f(\mathbf{u}_f^{n+1};\mathbf{u}_f^n,\mathbf{v})+\int_\Gamma\frac{\nu\alpha_{BJ}}{\sqrt{K}}(\mathbf{u}_f^{n+1})_\tau\cdot(\mathbf{v})_\tau$$
$$+b_f(\mathbf{v},p_f^{n+1})\int_\Gamma\tilde{\varphi}^n(\mathbf{v}\cdot\mathbf{n})=\int_{\Omega_f}\mathbf{f}^{n+1}\cdot\mathbf{v}+\frac{1}{\Delta t}\int_{\Omega_f}\mathbf{u}_f^n\cdot\mathbf{v}\quad\forall\mathbf{v},$$
$$b_f(\mathbf{u}_f^{n+1},q)=0\qquad\forall q.$$

(2) Update the normal velocity of the fluid across Γ:

$$\psi^{n+1}=(1-\beta)\psi^n+\beta\mathbf{u}_f^{n+1}\cdot\mathbf{n}\quad\text{on }\Gamma.$$

(3) Find φ^{n+1} such that

$$a_p(\varphi^{n+1},q_p)=\int_\Gamma\psi^{n+1}q_p\quad\forall q_p.$$

(4) Compute the new pressure across Γ:

$$\tilde{\varphi}^{n+1}=(1-\alpha)\varphi^n+\alpha\tilde{\varphi}^{n+1}\quad\text{on }\Gamma.$$

(5) Increment n and go back to step (1).

At each step this algorithm requires solving separately and in a sequential fashion the Navier–Stokes equations in Ω_f and the Darcy equations in Ω_p. Numerical results are shown in [22].

An interesting method to solve the fully evolutionary Stokes/Darcy problem is presented in [65] where the authors propose a partitioned scheme using different time steps for each subproblem. This allows to select better time discretizations adapted to the different time scales of the two problems.

References

[1] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J. Numer. Anal.* 39 (2002), 1749–1779.

[2] L. Badea, M. Discacciati, and A. Quarteroni, Numerical analysis of the Navier-Stokes/Darcy coupling, *Numer. Math* 115 (2010), 195–227.

[3] S. Badia and R. Codina, Unified stabilized finite element formulations for the Stokes and the Darcy problems, *SIAM J. Numer. Anal.* 47 (2009), 1971–2000.

[4] S. Badia and R. Codina, Stokes, Maxwell and Darcy: a single finite element approximation for three model problems, *Appl. Numer. Math.* 62 (2012), 246–263.

[5] S. Badia, A. Quaini, and A. Quarteroni, Coupling Biot and Navier-Stokes equations for modelling fluid-poroelastic media interaction, *J. Comput. Phys.* 228 (2009), 7986–8014.

[6] P. Bastian, Higher order discontinuous Galerkin methods for flow and transport in porous media, in: *Challenges in Scientific Computing CISC2002. Lecture Notes in Computational Science and Engineering (35)* (E. Baensch, ed.), pp. 1–22, Springer, Berlin and Heidelberg, 2003.

[7] J. Bear, *Hydraulics of Groundwater*, McGraw-Hill, New York, 1979.
[8] G. S. Beavers and D. D. Joseph, Boundary conditions at a naturally permeable wall, *J. Fluid Mech.* 30 (1967), 197–207.
[9] M. Benzi, A generalization of the Hermitian and skew-Hermitian splitting iteration, *SIAM J. Matrix Anal. Appl.* 31 (2009), 360–374.
[10] C. Bernardi, T. C. Rebollo, F. Hecht, and Z. Mghazli, Mortar finite element discretization of a model coupling Darcy and Stokes equations, *M2AN Math. Model. Numer. Anal.* 42 (2008), 375–410.
[11] H. Berninger, *Domain Decomposition Methods for Elliptic Problems with Jumping Nonlinearities and Application to the Richards Equation*, Ph.D. thesis, Freie Universität Berlin, 2008.
[12] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers, *Rev. Française Automat. Informat. Recherche Opérationnelle, sér. Rouge* 8 (1974), 129–151.
[13] F. Brezzi, J. Douglas Jr., R. Durán, and M. Fortin, Mixed finite elements for second order elliptic problems in three variables, *Numer. Math.* 51 (1987), 237–250.
[14] F. Brezzi, J. Douglas Jr., M. Fortin, and L. D. Marini, Efficient rectangular mixed finite elements in two and three space variables, *RAIRO Modél. Math. Anal. Numér.* 21 (1987), 581–604.
[15] F. Brezzi, J. Douglas Jr., and L. D. Marini, Two families of mixed finite elements for second order elliptic problems, *Numer. Math.* 47 (1985), 217–235.
[16] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Method*, Springer, New York, 1991.
[17] H. C. Brinkman, A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles, *Appl. Sci. Res. A* 1 (1947), 27–34.
[18] E. Burman, Pressure projection stabilizations for Galerkin approximations of Stokes' and Darcy's problems, *Numer. Methods Partial Differential Equations* 24 (2008), 127–143.
[19] E. Burman and P. Hansbo, A unified stabilized method for Stokes' and Darcy's equations, *J. Comput. Appl. Math.* 198 (2007), 35–51.
[20] E. Burman and P. Zunino, A domain decomposition method based on weighted interior penalties for advection-diffusion-reaction problems, *SIAM J. Numer. Anal.* 44 (2006), 1612–1638.
[21] Y. Cao, M. Gunzburger, F. Hua, and X. Wang, Coupled Stokes-Darcy model with Beavers-Joseph interface boundary conditions, *Comm. Math. Sci.* 8 (2010), 1–25.
[22] F. Cimolin and M. Discacciati, *Navier-Stokes/Forchheimer models for filtration through porous media*, MATHICSE - Ecole Polytechnique Fédérale de Lausanne, Report, 2011.
[23] B. Cockburn, G. E. Karniadakis, and C.-W. Shu, *The development of discontinuous Galerkin methods*, Discontinuous Galerkin methods (Newport, RI, 1999), Lect. Notes Comput. Sci. Eng. 11, Springer, Berlin, 2000, pp. 3–50.
[24] C. D'Angelo and P. Zunino, A finite element method based on weighted interior penalties for heterogenenous incompressible flows, *SIAM J. Numer. Anal.* 47 (2009), 3990–4020.
[25] M. Discacciati, *Domain Decomposition Methods for the Coupling of Surface and Groundwater Flows*, Ph.D. thesis, École Polytechnique Fédérale de Lausanne, Switzerland, 2004.
[26] M. Discacciati, Iterative methods for Stokes/Darcy coupling, in: *Domain Decomposition Methods in Science and Engineering. Lecture Notes in Computational Science and Engineering (40)* (R. et al. Kornhuber, ed.), pp. 563–570, Springer, Berlin and Heidelberg, 2004.
[27] M. Discacciati, *Preconditioning methods for the coupled Darcy/Stokes system*, Universitat Politècnica de Catalunya, Report, 2013, In preparation.
[28] M. Discacciati, E. Miglio, and A. Quarteroni, Mathematical and numerical models for coupling surface and groundwater flows, *Appl. Numer. Math.* 43 (2002), 57–74.
[29] M. Discacciati and A. Quarteroni, Analysis of a domain decomposition method for the coupling of Stokes and Darcy equations, in: *Numerical Mathematics and Advanced Applications,*

ENUMATH 2001 (F. Brezzi, A. Buffa, S. Corsaro, and A. Murli, eds.), pp. 3–20, Springer. Milan, 2003.

[30] M. Discacciati and A. Quarteroni, Convergence analysis of a subdomain iterative method for the finite element approximation of the coupling of Stokes and Darcy equations, *Comput. Visual. Sci.* 6 (2004), 93–103.

[31] M. Discacciati, A. Quarteroni, and A. Valli, Robin-Robin domain decomposition methods for the Stokes-Darcy coupling, *SIAM J. Numer. Anal.* 45 (2007), 1246–1268.

[32] H. I. Ene and E. Sanchez-Palencia, Équations et phénomenès de surface pour l'écoulement dans un modèle de milieu poreux, *J. Mécanique* 14 (1975), 73–108.

[33] P. Forchheimer, Wasserbewegung durch Boden, *Z. Ver. Deutsch. Ing.* 45 (1901), 1782–1788.

[34] J. Galvis and M. Sarkis, Non-matching mortar discretization analysis for the coupling Stokes-Darcy equations, *Electon. Trans. Numer. Anal.* 26 (2007), 350–384.

[35] J. Galvis and M. Sarkis, FETI and BDD preconditioners for Stokes-Mortar-Darcy systems, *Commun. Appl. Math. Comput. Sci.* 5 (2010), 1–30.

[36] J. C. Galvis and M. Sarkis, Inf-sup for coupling Stokes-Darcy, in: *XXV Iberian Latin American Congress in Computational Methods in Engineering, 2005, Recife. Proceedings of the XXV Iberian Latin American Congress in Computational Methods in Engineering, Recife, Brazil, 2004* (A. et al. Loula, ed.), Universidade Federal de Pernambuco, 2004.

[37] J. C. Galvis and M. Sarkis, Balancing domain decomposition methods for mortar coupling Stokes-Darcy systems, in: *Domain Decomposition Methods in Science and Engineering XVI* (D. E. Keyes and O. B. Widlund, eds.), pp. 373–380, Springer, Berlin and Heidelberg, 2006.

[38] G. Gatica, R. Oyarzúa, and F.-J. Sayas, Convergence of a family of Galerkin discretizations for the Stokes-Darcy coupled problem, *Numer. Methods Partial Differential Equations* 27 (2011), 721–748.

[39] V. Girault and B. Rivière, DG approximation of coupled Navier-Stokes and Darcy equations by Beaver-Joseph-Saffman interface condition, *SIAM J. Numer. Anal.* 47 (2009), 2052–2089.

[40] H. Hoteit, J. Erhel, R. Mosé, B. Philippe, and Ph. Ackerer, Numerical reliability for mixed methods applied to flow problems in porous media, *Comput. Geosciences* 6 (2002), 161–194.

[41] O. Iliev, A. Mikelić, and P. Popov, On upscaling certain flows in deformable porous media, *Multiscale Model. Simul.* 7 (2008), 93–123.

[42] W. Jäger and A. Mikelić, On the boundary conditions at the contact interface between a porous medium and a free fluid, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 23 (1996), 403–465.

[43] W. Jäger and A. Mikelić, On the interface boundary condition of Beavers, Joseph and Saffman, *SIAM J. Appl. Math.* 60 (2000), 1111–1127.

[44] W. Jäger, A. Mikelić, and N. Neuss, Asymptotic analysis of the laminar viscous flow over a porous bed, *SIAM J. Sci. Comput.* 22 (2001), 2006–2028.

[45] G. Kanschat and B. Rivière, A strongly conservative finite element method for the coupling of Stokes and Darcy flow, *J. Comput. Phys.* 229 (2010), 5933–5943.

[46] L. Lakatos, *Numerical Analysis of Iterative Substructuring Methods for the Stokes/Darcy Problem*, Master's thesis, Ecole Polytechnique Fédérale de Lausanne, Switzerland, 2010.

[47] W. L. Layton, F. Schieweck, and I. Yotov, Coupling fluid flow with porous media flow, *SIAM J. Num. Anal.* 40 (2003), 2195–2218.

[48] T. Levy and E. Sanchez-Palencia, On the boundary conditions for fluid flow in porous media, *Int. J. Engng. Sci.* 13 (1975), 923–940.

[49] J.-L. Lions and E. Magenes, *Problèmes aux Limites Non Homogènes et Applications*, 1, Dunod, Paris, 1968.

[50] A. F. D. Loula, F. A. Rochinha, and M. A. Murad, High-order gradient post-processings for second-order elliptic problems, *Comput. Meth. Appl. Mech. Engrg.* 128 (1995), 361–381.

[51] E. Marušić-Paloka and A. Mikelić, The derivation of a nonlinear filtration law including the inertia effects via homogenization, *Nonlin. Anal.* 42 (2000), 97–137.
[52] A. Masud and T. J. R. Hughes, A stabilized mixed finite element method for Darcy flow, *Comput. Meth. Appl. Mech. Engrg.* 191 (2002), 4341–4370.
[53] E. Miglio, *Mathematical and Numerical Modeling for Enviromental Applications*, Ph.D. thesis, Università degli Studi di Milano, Politecnico di Milano, 2000.
[54] E. Miglio, A. Quarteroni, and F. Saleri, Coupling of free surface and groundwater flows, *Comput. Fluids* 32 (2003), 73–83.
[55] J.-C. Nédélec, Mixed finite elements in $\mathbb{R}^3$, *Numer. Math.* 35 (1980), 119–136.
[56] J.-C. Nédélec, A new family of mixed finite elements in $\mathbb{R}^3$, *Numer. Math.* 50 (1986), 57–81.
[57] R. E. Oyarzúa Vargas, *Métodos de Elementos Finitos Mixtos para el Problema Acoplado de Stokes-Darcy*, Ph.D. thesis, Universidad de Concepción, Chile, 2010.
[58] L. E. Payne and B. Straughan, Analysis of the boundary condition at the interface between a viscous fluid and a porous medium and related modelling questions, *J. Math. Pures Appl.* 77 (1998), 317–354.
[59] A. Quaini, *Algorithms for Fluid-Structure Interaction Problems Arising in Hemodynamics*, Ph.D. thesis, École Polytechnique Fédérale de Lausanne, Switzerland, 2008.
[60] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*, Springer, Berlin, 1994.
[61] P.-A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, Mathematical Aspects of Finite Element Methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), Springer, Berlin, 1977, pp. 292–315. Lecture Notes in Math., Vol. 606.
[62] B. Rivière, Analysis of a discontinuous finite element method for the coupled Stokes and Darcy problems, *J. Sci. Comput.* 22 (2005), 479–500.
[63] B. Rivière and I. Yotov, Locally conservative coupling of Stokes and Darcy flows, *SIAM J. Numer. Anal.* 42 (2005), 1959–1977.
[64] P. G. Saffman, On the boundary condition at the interface of a porous medium, *Stud. Appl. Math.* 1 (1971), 93–101.
[65] L. Shan, H. Zheng, and W. J. Layton, *Decoupled scheme with different time step sizes for the evolutionary Stokes-Darcy model*, University of Pittsburgh, Report, 2011.
[66] J. M. Thomas, *Sur l'Analyse Numérique des Méthodes d'Eléments Finis Hybrides et Mixtes*, Ph.D. thesis, Université Pierre et Marie Curie, 1977.
[67] J. M. Urquiza, D. N'Dri, A. Garon, and M. C. Delfour, Coupling Stokes and Darcy equations, *Appl. Numer. Math.* 58 (2008), 525–538.
[68] P. Zunino, *Mathematical and Numerical Modeling of Mass Transfer in the Vascular System*, Ph.D. thesis, École Polytechnique Fédérale de Lausanne, Switzerland, 2002.
[69] P. Zunino, Mortar and discontinuous Galerkin methods based on weighted interior penalties, in: *Domain Decomposition Methods in Science and Engineering XVII* (U. et al. Langer, ed.), Lect. Notes Comput. Sci. Eng. 60, pp. 321–327, Springer, Berlin, 2008.