Question 8

Mathematics and Statistics Research Competition

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The Situation

A particle generator emits X or Y particles into an empty tube, with equal probability. Shots are independent.

Problem 1

- What is the probability that no two X-particles are next to each other?

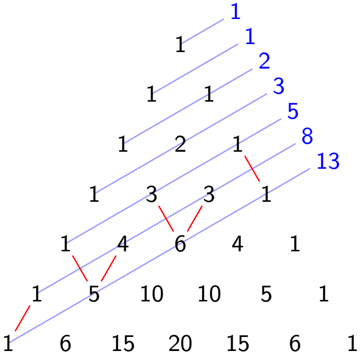
$$\text{Pr(No consec X-particles)} = \frac{\#\text{Arrangements w/o consec X-particles}}{\#\text{Total arrangements}}$$

Consider a tube with n particles, k of them are X, $n-k$ are Y.

$$\underbrace{\text{YY}\ldots\text{YY}}_{n-k}$$

Placing all k of the X particles in the $n-k+1$ gaps will ensure no consecutive X's. Hence, the number of arrangements without consecutive X's is

$$\sum_{k=0}^{n}\binom{n-k+1}{k}.$$



Observe the diagram of Pascal's triangle.

The formula represents the sum of each diagonal, which is made up of the sum of the previous two diagonals, satisfying the Fibonacci recursion.

Hence,

$$\sum_{k=0}^{n} \binom{n-k+1}{k} = F_{n+2}.$$

Since each of the n particles is X or Y, the number of arrangements is 2^n.

Hence the probability that no two X-particles are consecutive after n shots is

$$\frac{F_{n+2}}{2^n}.$$

Problem 2

Two consecutive X particles now collapse into one.

- Find the average number of particles after n shots.

$$X \to X : X$$
$$Y \to X : YX$$
$$X \to Y : XY$$
$$Y \to Y : YY$$

Out of 4 possible events, 3 increase the number of particles by 1. Hence, the average number of particles increases by $\frac{3}{4}$ each shot, so the formula is

$$T_n = \frac{3}{4}n + \frac{1}{4}.$$

Problem 3

Continuing from the previous problem, let the probability of firing an X particle be some $p \in (0, 1)$.

- What is the proportion of X particles in the tube as the number of shots goes to infinity?

$$\text{Proportion} = \frac{\#\text{X particles}}{\#\text{Particles}} = \frac{\#\text{Particles - Y particles}}{\#\text{Particles}}$$

The number of X particles stays the same if an X particle hits an X particle.

This happens with probability p^2.

Hence the average number of particles increases by $1 - p^2$ per shot, so after n shots,

$$T_n = (1 - p^2)n + p^2.$$

The average number of Y particles is clearly

$$(1-p)n.$$

Hence, the proportion is

$$\text{Proportion} = \frac{(1-p^2)n + p^2 - (1-p)n}{(1-p^2)n + p^2}.$$

As n approaches infinity, the proportion approaches

$$\frac{p}{1+p}.$$