Question 8
Mathematics and Statistics Research Competition

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The Situation

A particle generator emits X or Y particles into an empty tube, with equal probability. Shots are independent.

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Problem 1

▶ What is the probability that no two X-particles are next to each other?

$$\Pr(\text{No consec X-particles}) = \frac{\#\text{Arrangements w/o consec X-particles}}{\#\text{Total arrangements}}$$

Let $g(n)$ denote the number of arrangements without two touching X particles after n shots.

Consider the first particle, which is either X or Y.

▶ If the first particle is Y, then it doesn't affect the number of arrangements giving us $g(n-1)$.

▶ Otherwise, the first particle is X and hence the next particle must be Y. Hence there are $g(n-2)$ arrangements.

Adding these two up, we get

$$g(n) = g(n-1) + g(n-2).$$

This satisfies the Fibonacci recursion, and since $f(1) = 2$ and $f(2) = 3$ have $g(n) = F_{n+2}$, where F_n is the nth Fibonacci number.

Since the total number of arrangements of n particles is 2^n, since each particle is either X or Y, the probability that no two X particles are consecutive is

$$\frac{F_{n+2}}{2^n}.$$

Problem 2

Two consecutive X particles now collapse into one.

▶ Find the average number of particles after *n* shots.

This problem can be solved as a subcase of the more general following problem.

Problem 3

Two touching X particles now collapse into 1.

Continuing from the previous problem, let the probability of firing an X particle be some $p \in (0,1)$. Suppose at each shot, the probability of firing an X particle is some $p \in (0,1)$.
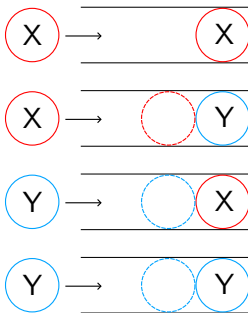
▶ Will the proportion of X particles in the tube stabilise as the number of shots goes to infinity?

▶ If so, is there a formula for this number?

We have:

$$\text{Proportion} = \frac{\#\text{X particles}}{\#\text{Particles}} = \frac{\#\text{Particles - Y particles}}{\#\text{Particles}}$$

The total number of particles stays the same if an X particle hits an X particle.

This happens with probability p^2.

Otherwise, the number of particles increases by 1 with probability $1 - p^2$. So,

$$T_{n+1} = T_n + 1 - p^2.$$

Solving, we get

$$T_n = (1 - p^2)n + p^2.$$

Setting $p = \frac{1}{2}$ yields the solution for Problem 2: $T_n = \frac{3}{4}n + \frac{1}{4}$.

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After n shots, clearly the number of Y particles is

$$(1 - p)n.$$

Hence, the proportion is

$$\text{Proportion} = \frac{(1 - p^2)n + p^2 - (1 - p)n}{(1 - p^2)n + p^2}.$$

As n approaches infinity, the proportion approaches

$$\frac{p}{1 + p}.$$

Generalisation

▶ What happens if m consecutive X particles collapsed into n particles?

▶ Everything else remains the same, i.e. Y particles don't collapse, probability of X particle is $p \in (0, 1)$

▶ What is the average number of particles after k shots?

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Recursion

If an X particles hits $m - 1$ consecutive X particles, then they collapse into n X particles, decreasing the number of particles by $m - 1 - n$.

Otherwise, the number of particles increases by 1.

Let the probability of having $m - 1$ consecutive X particles be ϑ.

$$T_{k+1} = T_k + 1 - p\vartheta + p\vartheta(n - m + 1)$$
$$= T_k + 1 - p\vartheta(m - n)$$

Calculating ϑ

ϑ is the probability that there are $m - 1$ X particles in a row.

The string of X particles must start with either a Y unless it is the beginning of the sequence.

$$\underbrace{X \ldots X}_{m-1} Y \qquad\qquad\text{(A)}$$

$$\underbrace{X \ldots X}_{m-1} \qquad\qquad\text{(B)}$$

Configuration A

$$\underbrace{X \ldots X}_{m-1} Y \tag{A}$$

If the sequence has a Y particle at the end, then the probability is simply $(1-p)p^{m-1}$.

But that doesn't account for previous collapses.

$$\underbrace{X \ldots X}_{m+(m-n)-1} Y \quad \longrightarrow \quad \underbrace{X \ldots X}_{n+(m-n)-1} Y \quad = \quad \underbrace{X \ldots X}_{m-1} Y$$

Configuration A

Adding $m - n$ particles to $m - 1$ reverts it back to $m - 1$, which can happen $\left\lfloor \dfrac{k - m}{m - n} \right\rfloor$ times. Summing this up gives

$$\sum_{a=0}^{\left\lfloor \frac{k-m}{m-n} \right\rfloor} (1 - p)p^{a(m-n)+m-1}.$$

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Configuration B

$$\underbrace{X\ldots X}_{m-1} \tag{B}$$

Accounting for previous collapses, this configuration can obviously happen with a probability of p^k, if $k = a(m-n) + m - 1$ for some integer a. Hence, the probability is

$$\varepsilon = \begin{cases} p^k, & k - m + 1 \equiv 0 \bmod m - n \\ 0, & k - m + 1 \not\equiv 0 \bmod m - n. \end{cases}$$

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Adding these two probabilities gives us

$$\vartheta = \sum_{a=0}^{\left\lfloor \frac{k-m}{m-n} \right\rfloor} (1-p)p^{a(m-n)+m-1} + \varepsilon,$$

where

$$\varepsilon = \begin{cases} p^k, & k-m+1 \equiv 0 \bmod m-n \\ 0, & k-m+1 \not\equiv 0 \bmod m-n. \end{cases}$$

Putting it all together

Recall that we obtained a recursion for T_k earlier.

$$T_{k+1} = T_k + 1 - p\vartheta(m - n)$$

We simply sum the left hand side from $m - 1$ to $k - 1$ to get a closed expression:

$$T_k = m - 1 + \sum_{b=m-1}^{k-1} (1 - p\vartheta(m - n)).$$

The formula works for $k \geq m$ since $T_k = k$ for $k < m$.

We can even expand it, getting

$$
T_k = m-1+ \sum_{b=m-1}^{k-1} \left(1 - p \left(\sum_{a=0}^{\left\lfloor \frac{b-m}{m-n} \right\rfloor} (1-p)p^{a(m-n)+m-1} + \varepsilon \right) (m-n) \right)
$$

However, there is a notable case where this nasty formula simplifies dramatically.

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Let us consider what happens when $n = m - 1 \iff m - n = 1$.

$$T_k = m - 1 + \sum_{b=m-1}^{k-1} \left(1 - p \left(\sum_{a=0}^{\lfloor \frac{b-m}{1} \rfloor} (1-p)p^{a(1)+m-1} + p^b \right) (1) \right)$$

Even ε becomes simplified into p^k since $z \equiv 0 \mod 1$ for any integer z. Amazingly, we can simplify ϑ into

$$\vartheta = \sum_{a=0}^{k-m} (1-p)p^{a+m-1} + p^k = p^{m-1}$$

since it is a geometric series. Now ϑ is constant as it no longer depends on k.

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Putting it back in, we get

$$T_k = m - 1 + \sum_{b=m-1}^{k-1} (1 - p(p^{m-1}))$$

$$= m - 1 + (k - m + 1)(1 - p^m)$$

Setting $m = 2, n = 1, p = \frac{1}{2}$, the parameters of Problem 3, we arrive at the exact same formula:

$$T_k = (1 - p^2)k + p^2.$$

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Use of computer simulation

We used C++ to quickly compute average number of particles after any number of shots.

Computer simulations helped us determine that a formula existed when finding the pattern.

They also helped us validate that our formula worked correctly.

Extra research ideas

There is still extra things that can be explored.

▶ What if both X and Y can collapse?

▶ What if there are more than 2 particles?

▶ What if there are n particles, k of which can collapse?