Question 8

Mathematics and Statistics Research Competition

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October 24, 2022

The Situation

A particle generator emits X or Y particles into an empty tube, with equal probability. Shots are independent.

Problem 1

- What is the probability that no two X-particles are next to each other?

$$\Pr(\text{No consec X-particles}) = \frac{\#\text{Arrangements w/o consec X-particles}}{\#\text{Total arrangements}}$$

Let $f(n)$ denote the number of arrangements without two touching X particles after n shots.

Consider the first particle, which is either X or Y.

- If the first particle is Y, then it doesn't affect the number of arrangements giving us $f(n-1)$.
- Otherwise, the first particle is X and hence the next particle must be Y. Hence there are $f(n-2)$ arrangements.

Adding these two up, we get

$$f(n) = f(n-1) + f(n-2).$$

This satisfies the Fibonacci recursion, and since $f(1) = 2$ and $f(2) = 3$) have $f(n) = F_{n+2}$, where F_n is the nth Fibonacci number.

Since the total number of arrangements of n particles is 2^n, since each particle is either X or Y, the probabiliy that no two X particles are consecutive is

$$\frac{F_{n+2}}{2^n}.$$

Problem 2

Two consecutive X particles now collapse into one.

- Find the average number of particles after n shots.

This problem can be solved as a subcase of the more general following problem.

Problem 3

Continuing from the previous problem, let the probability of firing an X particle be some $p \in (0, 1)$. Suppose at each shot, the probability of firing an X particle is some $p \in (0, 1)$.

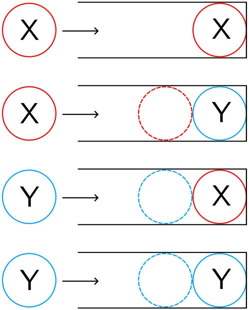
- Will the proportion of X particles in the tube stabilise as the number of shots goes to infinity?
- If so, is there a formula for this number?

We have:

$$\text{Proportion} = \frac{\#\text{X particles}}{\#\text{Particles}} = \frac{\#\text{Particles - Y particles}}{\#\text{Particles}}$$

The number of X particles stays the same if an X particle hits an X particle.

This happens with probability p^2.



Otherwise, the number of particles increases by 1 with probability $1 - p^2$. So,

$$T_{n+1} = T_n + 1 - p^2.$$

Solving, we get

$$T_n = (1 - p^2)n + p^2.$$

Setting $p = \frac{1}{2}$ yields the solution for Problem 2: $T_n = \frac{3}{4}n + \frac{1}{4}$.

After n shots, clearly the number of Y particles is

$$(1-p)n.$$

Hence, the proportion is

$$\text{Proportion} = \frac{(1-p^2)n + p^2 - (1-p)n}{(1-p^2)n + p^2}.$$

As n approaches infinity, the proportion approaches

$$\frac{p}{1+p}.$$

Generalisation

- What happens if m consecutive X particles collapsed into n particles?
- Everything else remains the same, i.e. Y particles don't collapse, probability of X particle is $p \in (0, 1)$
- What is the average number of particles after k shots?

Recursion

If an X particles hits $m-1$ consecutive X particles, then they collapse into n X particles, decreasing the number of particles by $m-1-n$.

Otherwise, the number of particles increases by 1.

Let the probability of having $m-1$ consecutive X particles be ϑ.

$$\begin{aligned} T_{k+1} &= T_k + 1 - p\vartheta + p\vartheta(n-m+1) \\ &= T_k + 1 - p\vartheta(m-n) \end{aligned}$$

Calculating ϑ

ϑ is the probability that there are $m-1$ X particles in a row.

The string of X particles must start with either a Y unless it is the beginning of the sequence.

$$\underbrace{\mathsf{X}\ldots\mathsf{X}}_{m-1}\mathsf{Y} \tag{A}$$

$$\underbrace{\mathsf{X}\ldots\mathsf{X}}_{m-1} \tag{B}$$

Configuration A

$$\underbrace{X\ldots X}_{m-1}Y \tag{A}$$

If the sequence has a Y particle at the end, then the probability is simply $(1-p)p^{m-1}$.

But that doesn't account for previous collapses.

$$\underbrace{X\ldots X}_{m+(m-n)-1}\quad Y \quad\longrightarrow\quad \underbrace{X\ldots X}_{n+(m-n)-1}\quad Y \quad=\quad \underbrace{X\ldots X}_{m-1}Y$$

Configuration A

Adding $m-n$ particles to $m-1$ reverts it back to $m-1$, which can happen $\left\lfloor \frac{k-m}{m-n} \right\rfloor$ times. Summing this up gives

$$\sum_{a=0}^{\left\lfloor \frac{k-m}{m-n} \right\rfloor} (1-p)p^{a(m-n)+m-1}.$$

Configuration B

$$\underbrace{\text{X}\ldots\text{X}}_{m-1} \tag{B}$$

Accounting for previous collapses, this configuration can obviously happen with a probability of p^k, if $k = a(m-n) + m - 1$ for some integer a. Hence, the probability is

$$\varepsilon = \begin{cases} p^k, & k - m + 1 \equiv 0 \bmod m - n \\ 0, & k - m + 1 \not\equiv 0 \bmod m - n. \end{cases}$$

Adding these two probabilities gives us

$$\vartheta = \sum_{a=0}^{\left\lfloor \frac{k-m}{m-n} \right\rfloor} (1-p)p^{a(m-n)+m-1} + \varepsilon,$$

where

$$\varepsilon = \begin{cases} p^k, & k-m+1 \equiv 0 \bmod m-n \\ 0, & k-m+1 \not\equiv 0 \bmod m-n. \end{cases}.$$

Putting it all together

Recall that we obtained a recursion for T_k earlier.

$$T_{k+1} = T_k + 1 - p\vartheta(m-n)$$

We simply sum the left hand side from $m-1$ to $k-1$ to get a closed expression:

$$T_k = m - 1 + \sum_{b=m-1}^{k-1} \left(1 - p\vartheta(m-n)\right).$$

The formula works for $k \geq m$ since $T_k = k$ for $k < m$.

We can even expand it, getting

$$T_k = m-1+\sum_{b=m-1}^{k-1}\left(1-p\left(\sum_{a=0}^{\left\lfloor\frac{b-m}{m-n}\right\rfloor}(1-p)p^{a(m-n)+m-1}+\varepsilon\right)(m-n)\right)$$

However, there is a notable case where this nasty formula simplifies dramatically.

Let us consider what happens when $n = m - 1 \iff m - n = 1$.

$$T_k = m - 1 + \sum_{b=m-1}^{k-1} \left(1 - p\left(\sum_{a=0}^{\left\lfloor \frac{b-m}{1} \right\rfloor} (1-p)p^{a(1)+m-1} + p^b\right)(1)\right)$$

Even ε becomes simplified into p^k since $z \equiv 0 \bmod 1$ for any integer z. Amazingly, we can simplify ϑ into

$$\vartheta = \sum_{a=0}^{k-m} (1-p)p^{a+m-1} + p^k = p^{m-1}$$

since it is a geometric series. Now ϑ is constant as it no longer depends on k.

Putting it back in, we get

$$\begin{aligned} T_k &= m - 1 + \sum_{b=m-1}^{k-1} (1 - p(p^{m-1})) \\ &= m - 1 + (k - m + 1)(1 - p^m) \end{aligned}$$

Setting $m = 2, n = 1, p = \frac{1}{2}$, the parameters of Problem 3, we arrive at the exact same formula:

$$T_k = (1 - p^2)k + p^2.$$

Use of computer simulation

We used C++ to quickly compute average number of particles after any number of shots.

Computer simulations helped us determine that a formula existed when finding the pattern.

They also helped us validate that our formula worked correctly.

Extra research ideas

There is still extra things that can be explored.

- What if both X and Y can collapse?
- What if there are more than 2 particles?
- What if there are n particles, k of which can collapse?