Question 8
Mathematics and Statistics Research Competition

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October 24, 2022

The Situation

A particle generator emits X or Y particles into an empty tube, with equal probability. Shots are independent.

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Problem 1

▶ What is the probability that no two X-particles are next to each other?

$$\Pr(\text{No consec X-particles}) = \frac{\#\text{Arrangements w/o consec X-particles}}{\#\text{Total arrangements}}$$

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Consider a tube with n particles, k of them are X, $n - k$ are Y.

$$\underbrace{YY \ldots YY}_{n-k}$$

Placing all k of the X particles in the $n - k + 1$ gaps will ensure no consecutive X's. Hence, the number of arrangements without consecutive X's is

$$\sum_{k=0}^{n} \binom{n - k + 1}{k}.$$

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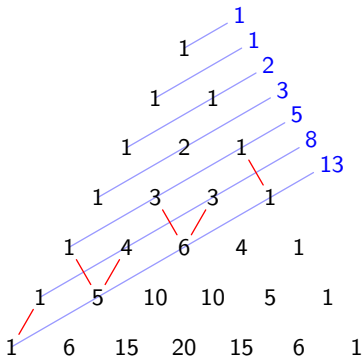
Observe the diagram of Pascal's triangle.

The formula represents the sum of each diagonal, which is made up of the sum of the previous two diagonals, satisfying the Fibonacci recursion.

Hence,

$$\sum_{k=0}^{n} \binom{n-k+1}{k} = F_{n+2}.$$

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Since each of the n particles is X or Y, the number of arrangements is 2^n.

Hence the probability that no two X-particles are consecutive after n shots is

$$\frac{F_{n+2}}{2^n}.$$

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Problem 2

Two consecutive X particles now collapse into one.

▶ Find the average number of particles after *n* shots.

This problem can be solved as a subcase of the more general following problem.

Problem 3

Continuing from the previous problem, let the probability of firing an X particle be some $p \in (0, 1)$. Suppose at each shot, the probability of firing an X particle is some $p \in (0, 1)$.
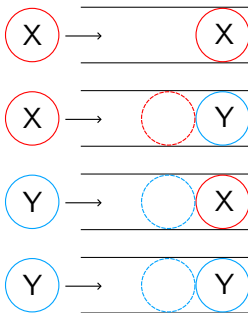
▶ Will the proportion of X particles in the tube stabilise as the number of shots goes to infinity?

▶ If so, is there a formula for this number?

We have:

$$\text{Proportion} = \frac{\#\text{X particles}}{\#\text{Particles}} = \frac{\#\text{Particles - Y particles}}{\#\text{Particles}}$$

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The number of X particles stays the same if an X particle hits an X particle.

This happens with probability p^2.

Otherwise, the number of particles increases by 1 with probability $1 - p^2$. So,

$$T_{n+1} = T_n + 1 - p^2.$$

Solving, we get

$$T_n = (1 - p^2)n + p^2.$$

Setting $p = \frac{1}{2}$ yields the solution for Problem 2: $T_n = \frac{3}{4}n + \frac{1}{4}$.

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After n shots, clearly the number of Y particles is

$$(1 - p)n.$$

Hence, the proportion is

$$\text{Proportion} = \frac{(1 - p^2)n + p^2 - (1 - p)n}{(1 - p^2)n + p^2}.$$

As n approaches infinity, the proportion approaches

$$\frac{p}{1 + p}.$$

Generalisation

▶ What happens if m consecutive X particles collapsed into n particles?

▶ Everything else remains the same, i.e. Y particles don't collapse, probability of X particle is $p \in (0, 1)$

▶ What is the average number of particles after k shots?

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Recursion

If an X particles hits $m - 1$ consecutive X particles, then they collapse into n X particles, decreasing the number of particles by $m - 1 - n$.

Otherwise, the number of particles increases by 1.

Let the probability of having $m - 1$ consecutive X particles be ϑ.

$$
\begin{aligned}
T_{k+1} &= T_k + 1 - p\vartheta + p\vartheta(n - m + 1) \\
&= T_k + 1 - p\vartheta(m - n)
\end{aligned}
$$

Calculating ϑ

ϑ is the probability that there are $m - 1$ X particles in a row.

The string of X particles must start with either a Y unless it is the beginning of the sequence.

$$\underbrace{X \ldots X}_{m-1} Y \tag{A}$$

$$\underbrace{X \ldots X}_{m-1} \tag{B}$$

Configuration A

$$\underbrace{X\ldots X}_{m-1}Y \tag{A}$$

If the sequence has a Y particle at the end, then the probability is simply $(1-p)p^{m-1}$.

But that doesn't account for previous collapses.

$$\underbrace{X\ldots X}_{m+(m-n)-1}\ Y \quad \longrightarrow \quad \underbrace{X\ldots X}_{n+(m-n)-1}\ Y \quad = \quad \underbrace{X\ldots X}_{m-1}Y$$

Configuration A

Adding $m - n$ particles to $m - 1$ reverts it back to $m - 1$, which can happen $\left\lfloor \dfrac{k - m}{m - n} \right\rfloor$ times. Summing this up gives

$$\sum_{a=0}^{\left\lfloor \frac{k-m}{m-n} \right\rfloor} (1 - p)p^{a(m-n)+m-1}.$$

Configuration B

$$\underbrace{X\ldots X}_{m-1} \tag{B}$$

Accounting for previous collapses, this configuration can obviously happen with a probability of p^k, if $k = a(m - n) + m - 1$ for some integer a. Hence, the probability is

$$\varepsilon = \begin{cases} p^k, & k - m + 1 \equiv 0 \bmod m - n \\ 0, & k - m + 1 \not\equiv 0 \bmod m - n. \end{cases}$$

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Adding these two probabilities gives us

$$\vartheta = \sum_{a=0}^{\lfloor \frac{k-m}{m-n} \rfloor} (1-p)p^{a(m-n)+m-1} + \varepsilon,$$

where

$$\varepsilon = \begin{cases} p^k, & k-m+1 \equiv 0 \bmod m-n \\ 0, & k-m+1 \not\equiv 0 \bmod m-n. \end{cases}$$

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Putting it all together

Recall that we obtained a recursion for T_k earlier.

$$T_{k+1} = T_k + 1 - p\vartheta(m - n)$$

We simply sum the left hand side from $m - 1$ to $k - 1$ to get a closed expression:

$$T_k = m - 1 + \sum_{b=m-1}^{k-1} \left(1 - p\vartheta(m - n)\right).$$

The formula works for $k \geq m$ since $T_k = k$ for $k < m$.

We can even expand it, getting

$$T_k = m-1+ \sum_{b=m-1}^{k-1} \left(1 - p \left(\sum_{a=0}^{\left\lfloor \frac{b-m}{m-n} \right\rfloor} (1-p)p^{a(m-n)+m-1} + \varepsilon \right) (m - n) \right)$$

However, there is a notable case where this nasty formula simplifies dramatically.

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Let us consider what happens when $n = m - 1 \iff m - n = 1$.

$$T_k = m - 1 + \sum_{b=m-1}^{k-1} \left(1 - p \left(\sum_{a=0}^{\left\lfloor \frac{b-m}{1} \right\rfloor} (1-p) p^{a(1)+m-1} + p^b \right) (1) \right)$$

Even ε becomes simplified into p^k since $z \equiv 0 \mod 1$ for any integer z. Amazingly, we can simplify ϑ into

$$\vartheta = \sum_{a=0}^{k-m} (1-p) p^{a+m-1} + p^k = p^{m-1}$$

since it is a geometric series. Now ϑ is constant as it no longer depends on k.

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Putting it back in, we get

$$T_k = m - 1 + \sum_{b=m-1}^{k-1} (1 - p(p^{m-1}))$$

$$= m - 1 + (k - m + 1)(1 - p^m)$$

Setting $m = 2, n = 1, p = \frac{1}{2}$, the parameters of Problem 3, we arrive at the exact same formula:

$$T_k = (1 - p^2)k + p^2.$$

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Use of computer simulation

Computer models allowed us to quickly compute average number of particles after any number of shots.

These helped us determine that a formula existed when finding the pattern.

They also helped us validate that our formula worked correctly.