Mathematics

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4. Probability

4.1. Basics

Definition 4.1.1. The *sample space*, denoted Ω is the set of all possible outcomes of an experiment.

Each element $\omega \in \Omega$ is called an *outcome*. An *event* E is a subset of Ω, and is hence a set of possible outcomes. Thus Ω itself is an event, the *certain event*, and so is the empty set $\varnothing$, the *impossible event*.

Definition 4.1.2. If Ω is countable, then we say that it is a *discrete* sample space. Otherwise, if Ω is uncountable, we say that it is a *continuous* sample space.

Since events are just sets, we can use the usual properties of sets.

Proposition 4.1.1 (Set theory properties)**.** *Let A, B, C be subsets of Ω.*

1. *We have $A \cup \varnothing = A$ and $A \cap \varnothing = \varnothing$.*
2. *We have $A \cup \Omega = \Omega$ and $A \cap \Omega = A$.*
3. *We have $A \cup (\Omega \setminus A) = \Omega$ and $A \cap (\Omega \setminus A) = \varnothing$. In other words, $A \cup A^c = \Omega$ and $A \cap A^c = \varnothing$.*
4. *(Commutativity) We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$.*
5. *(Associativity) We have $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$.*
6. *(Distributivity) We have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.*
7. *(De Morgan's laws) We have $\Omega \setminus (A \cup B) = (\Omega \setminus A) \cap (\Omega \setminus B)$ and $\Omega \setminus (A \cap B) = (\Omega \setminus A) \cup (\Omega \setminus B)$. In other words, $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.*

Definition 4.1.3. Two events A and B are *mutually exclusive* or *disjoint* if $A \cap B = \varnothing$. A set of events $\{E_i\}_{i \in I}$ is mutually exclusive if for all $i \neq j$, $E_i \cap E_j = \varnothing$.

Definition 4.1.4. Let Ω be a sample space and let 2^Ω denote the power set of Ω. Then $\mathbb{P}\colon 2^\Omega \to [0,1]$ is a probability function that assigns to each event a probability. This function $\mathbb{P}$ must satisfy the following axioms formulated by Kolmogorov:

1. For all events E, we have $\mathbb{P}(E) \geq 0$.
2. The probability of the sample space is 1. That is, $\mathbb{P}(\Omega) = 1$.
3. (Countable additivity) Let $E_1, E_2, \ldots$ be a countable sequence of mutually exclusive (disjoint) events. Then,
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i).$$

Proposition 4.1.2. *Properties of the probability function:*

1. *We have* $\mathbb{P}(\varnothing) = 0$.
2. *We have* $\mathbb{P}(A^c) = \mathbb{P}(\Omega \setminus A) = 1 - \mathbb{P}(A)$.
3. *If* $A \subseteq B$, *then* $\mathbb{P}(A) \leq \mathbb{P}(B)$.
4. *(Finite additivity) If* $E_1, E_2, \ldots, E_n$ *are mutually exclusive events, then*
$$\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbb{P}(E_i)$$
5. *If* $A_1 \subseteq A_2 \subseteq \ldots$ *and* $B = \bigcup_{i=1}^{\infty} A_i$, *then* $\mathbb{P}(B) = \lim_{n\to\infty} \mathbb{P}(A_n)$.
6. *If* $A_1 \supseteq A_2 \supseteq \ldots$ *and* $B = \bigcap_{i=1}^{\infty} A_i$, *then* $\mathbb{P}(B) = \lim_{n\to\infty} \mathbb{P}(A_n)$.

Definition 4.1.5. We say that a collection of events $\{E_i\}_{i \in I}$ is *exhaustive* if
$$\bigcup_{i \in I} E_i = \Omega.$$

Definition 4.1.6. We define the *conditional probability* of event A *given* event B as
$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)},$$
given that $\mathbb{P}(B) > 0$.