

# LECTURE NOTES—FOURIER SERIES

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DECEMBER 2022

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## 1. INTRODUCTION

Fourier series have countless applications throughout applied mathematics and physics. Fourier series were first formally introduced by Joseph Fourier in his 1822 work *Théorie analytique de la chaleur* (*Analytical theory of heat*) in order to solve the Heat Equation. Whilst this is not a history lesson, it is still important to know the origin of the Fourier series, which are rooted in physics and are commonly used to solve differential equations.

## 2. FOURIER SERIES

But what *are* Fourier series? Despite sounding very fancy, the concept is very simple. The general concept of a Fourier series is to break a function down into a sum of sines and cosines. This is motivated by several reasons—they are easily differentiable, integrable, and manipulable.

For example, given some  $2\pi$  periodic function  $f(x)$ , we can express it as some sum of sinusoidal waves like so:

$$f(x) \approx \frac{a_0}{2} + \sum_{m=1}^N (a_m \cos mx + b_m \sin mx).$$

If, as the number of terms  $N$  approaches  $\infty$ , the approximation *converges* to  $f$ , then the resulting sum is called a *Fourier series*:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx).$$

You will have noticed that there are some coefficients:  $a_0, a_m, b_m$ . These are the *Fourier coefficients*, and determine how much of each sine wave of each frequency is included. Before we determine these coefficients, we first discuss the *orthogonality relations* of the sine and cosine functions.

### 3. ORTHOGONALITY OF SINE AND COSINE

There are three important formulae concerning sines and cosines:

**THEOREM 3.1** (Orthogonality of sine and cosine)

For integers  $m, n$ , the following identities hold:

$$(3.1) \quad \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases}$$

$$(3.2) \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases}$$

$$(3.3) \quad \int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$$

These identities may be proved using the product to sum trigonometric identities:

$$(3.4) \quad \sin a \sin b = \frac{1}{2} [\cos(a - b) - \cos(a + b)]$$

$$(3.5) \quad \cos a \cos b = \frac{1}{2} [\cos(a - b) + \cos(a + b)]$$

$$(3.6) \quad \sin a \cos b = \frac{1}{2} [\sin(a - b) + \sin(a + b)]$$

*Proof.* We use 3.4 to prove the first formula:

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(x(m - n)) - \cos(x(m + n)) \, dx \\ &= \frac{1}{2} \left[ \frac{\sin(x(m - n))}{m - n} - \frac{\sin(x(m + n))}{m + n} \right]_{-\pi}^{\pi}. \end{aligned}$$

Since  $\sin k\pi = 0$  for any integer  $k$ , we have

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad m \neq n.$$

However, notice that the method only makes sense for  $m \neq n$ . If  $m = n$ , we may use the identity  $\sin^2 a = \frac{1}{2}[1 - \cos(2a)]$  to obtain

$$\begin{aligned} \int_{-\pi}^{\pi} \sin^2(mx) dx &= \frac{1}{2} \int_{-\pi}^{\pi} 1 - \cos(2mx) dx \\ &= \frac{1}{2} \left[ x - \frac{\sin(2mx)}{2m} \right]_{-\pi}^{\pi} \\ &= \pi. \end{aligned}$$

The other two identities are left as an exercise to the reader.  $\square$

#### 4. FOURIER COEFFICIENTS

Recall the Fourier series of  $f(x)$ , a  $2\pi$  periodic function on  $[-\pi, \pi]$ :

$$(4.1) \quad f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx).$$

Let us assume that the series converges to  $f$ . We wish to calculate the coefficients  $a_0, a_m, b_m$ .

First, consider multiplying  $f(x)$  by  $\cos nx$  and integrating from  $-\pi$  to  $\pi$ :

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos nx dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos nx dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx \cos nx dx \\ &\quad + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx \cos nx dx. \end{aligned}$$

Using the orthogonal relationships from earlier, we may simplify this to

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos nx dx &= 0 + a_n \pi + 0 \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx. \end{aligned}$$

Now we multiply  $f(x)$  by  $\sin nx$  and integrate from  $-\pi$  to  $\pi$ :

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin nx dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} \sin nx dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \sin nx \cos mx dx \\ &\quad + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx \sin nx dx. \end{aligned}$$

Again, we simplify using the orthogonal relations:

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin nx dx &= 0 + 0 + b_n \pi \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx. \end{aligned}$$

Finally, to determine  $a_0$ , we simply integrate from  $-\pi$  to  $\pi$ :

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx dx \\ &\quad + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx dx.\end{aligned}$$

This time, we simply notice that  $\int_{-\pi}^{\pi} \cos kx dx$  and  $\int_{-\pi}^{\pi} \sin kx dx$  both equal zero. Hence, we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) dx &= a_0\pi + 0 + 0 \\ a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.\end{aligned}$$

**DEFINITION (Fourier series)**

The *Fourier series* of a function  $f(x)$  that is  $2\pi$  periodic on the closed interval  $[-\pi, \pi]$  is given by

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx),$$

where

$$\begin{aligned}a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx \\ b_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx.\end{aligned}$$

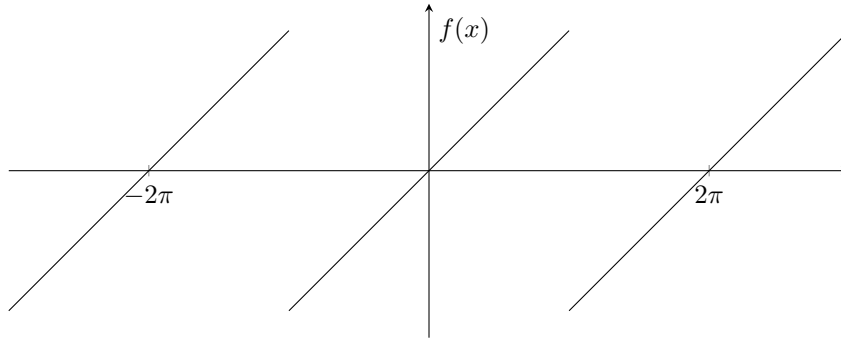
## 5. USING FOURIER SERIES

**5.1. An example.** We now look at some applications of Fourier series. Let's go through an example of computing the Fourier series of a function.

Consider the function  $f(x) = x$ , which has been made periodic on the interval  $[-\pi, \pi]$  so that it is  $2\pi$  periodic. This just means that it repeats after  $x = \pi$ . We will calculate its Fourier coefficients.

We know that  $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$ . Since  $f(x) = x$ , we have

$$\begin{aligned}a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x dx \\ &= \frac{1}{\pi} \left[ \frac{x^2}{2} \right]_{-\pi}^{\pi} \\ &= 0.\end{aligned}$$

FIGURE 5.1. Graph of  $f(x)$ 

Next, we calculate  $a_m$ :

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos mx \, dx.$$

Notice  $f(x) = x$  is an odd function, and  $\cos mx$  is an even function. Therefore  $x \cos mx$  is an odd function. When integrating an odd function with symmetric bounds, the result is always 0. Hence,  $a_m = 0$ .

Finally, we calculate  $b_m$ .

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin mx \, dx.$$

This is a simple integration by parts exercise:

$$\begin{aligned} b_m &= \frac{1}{\pi} \left[ -\frac{x \cos mx}{m} + \frac{1}{m} \int \cos mx \, dx \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[ \frac{\sin mx - mx \cos mx}{m^2} \right]_{-\pi}^{\pi}. \end{aligned}$$

Notice that  $\sin m\pi = 0$  for any integer  $m$ , and  $\cos m\pi = (-1)^m$ . Hence we have

$$\begin{aligned} b_m &= \frac{1}{\pi} \left[ \frac{-2\pi m(-1)^m}{m^2} \right] \\ &= -\frac{2(-1)^m}{m}. \end{aligned}$$

This gives us the Fourier series

$$f(x) = -\sum_{m=1}^{\infty} \frac{2(-1)^m}{m} \sin mx.$$

The graphs of some partial sums are shown in Figures 5.2 and 5.3.

**5.2. The Basel Problem.** You may have heard of the Basel problem, an astonishing identity where  $\pi$  seemingly randomly pops up. I will present a method of solving the problem using Fourier series.

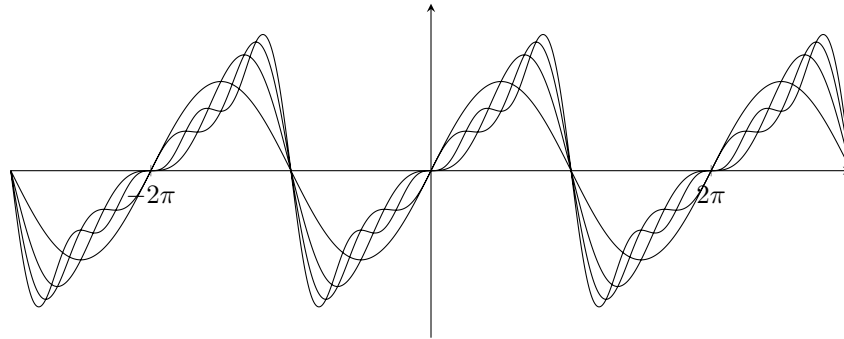


FIGURE 5.2. Graphs of the first 4 partial sums

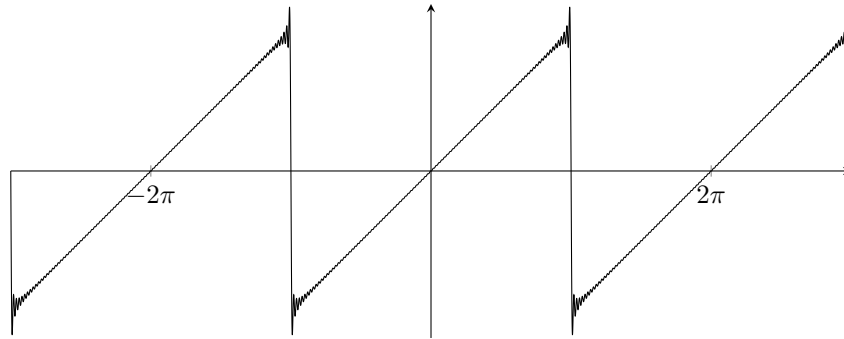


FIGURE 5.3. Graph of the 100th partial sum

**THEOREM 5.1** (Basel problem)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

*Proof.* Consider the function  $f(x) = x^2$  which has been made periodic on the interval  $[-\pi, \pi]$ . We now find its Fourier series.

First, we calculate  $a_0$ :

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx \\ &= \frac{2}{\pi} \left[ \frac{x^3}{3} \right]_0^{\pi} \\ &= \frac{2\pi^2}{3}. \end{aligned}$$

Next, we calculate  $a_m$ , using integration by parts:

$$\begin{aligned} a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos mx \, dx \\ &= \frac{1}{\pi} \left[ \frac{x^2 \sin mx}{m} - \frac{2}{m} \int x \sin mx \, dx \right]_{-\pi}^{\pi}. \end{aligned}$$

We can use our previous result that  $\int_{-\pi}^{\pi} x \sin mx \, dx = -\frac{2\pi(-1)^m}{m}$  to get

$$\begin{aligned} a_m &= \frac{1}{\pi} \left[ \frac{x^2 \sin mx}{m} \right]_{-\pi}^{\pi} + \frac{2}{\pi m} \left[ \frac{2\pi(-1)^m}{m} \right] \\ &= \frac{4(-1)^m}{m^2}. \end{aligned}$$

To calculate  $b_m$  we may notice that  $x^2 \sin mx$  is odd, meaning that the symmetric integral will be 0. Hence,  $b_m = 0$ .

This means that our Fourier series for  $f(x) = x^2$  on  $[-\pi, \pi]$  is

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{m=1}^{\infty} \frac{4(-1)^m}{m^2} \cos mx.$$

Let's now observe what happens when we plug in  $x = \pi$ . We get

$$\begin{aligned} f(\pi) = \pi^2 &= \frac{\pi^2}{3} + \sum_{m=1}^{\infty} \frac{4(-1)^m}{m^2} \cos(m\pi) \\ \frac{2\pi^2}{3} &= \sum_{m=1}^{\infty} \frac{4}{m^2}. \end{aligned}$$

Dividing both sides by 4 yields the result that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}. \quad \square$$

## 6. WAVE EQUATION IN ONE DIMENSION

We now investigate the problem of a vibrating string.

Consider a string of length  $L$  lying on the  $x$ -axis, fixed at 2 ends: the origin  $(0, 0)$  and  $(L, 0)$ . We will make some reasonable assumptions:

- The particles in the string move only up and down (there is no horizontal movement)
- The string is perfectly elastic

Let the function  $u(x, t)$  describe the displacement of the string at time  $t$ .

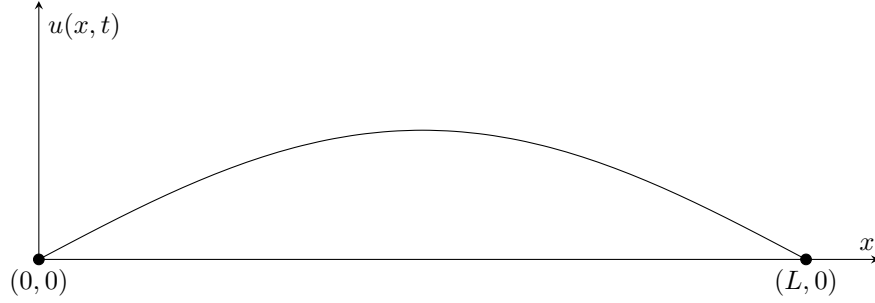
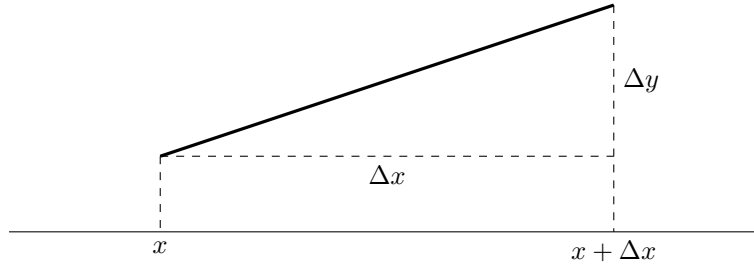


FIGURE 6.1. A possible shape of the string at some point in time

First, we find the mass of the string in some interval  $[a, b]$ . Let  $\rho(x, t)$  describe the density of the string at  $x$  at time  $t$ . Focus on a tiny segment of the string at  $x$ .



Since this piece of string is so small, it approximates a straight line. Hence its length is clearly  $\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x$ . Now the mass of this piece of string is simply  $\rho(x, t) \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x$ . Imagine summing this  $N$  times for all the tiny segments in the interval  $[a, b]$ . Thus an approximation of the mass of the string in the interval  $[a, b]$  is

$$\sum_{i=1}^N \rho(x_i, t) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i.$$

If these segments are equally spaced, then  $\Delta x_i = \Delta x$ . Our approximation becomes the true mass of the string when  $\Delta x$  approaches 0 and  $N$  approaches  $\infty$ . Thus the mass of the string is

$$\lim_{\substack{N \rightarrow \infty \\ \Delta x \rightarrow 0}} \sum_{i=1}^N \rho(x_i, t) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x = \int_a^b \rho(x, t) \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} dx.$$

For sake of brevity, we abbreviate  $\frac{\partial u}{\partial x}$  to  $u_x$ , and similarly,  $\frac{\partial u}{\partial t}$  to  $u_t$ .

Let us now concentrate on another small segment of the string in the interval  $[x, x + \Delta x]$ . This piece of string has mass

$$\int_x^{x+\Delta x} \rho(x, t) \sqrt{1 + u_x^2} dx.$$



Let  $m = \rho(x, t)\sqrt{1 + u_x^2}$  such that the mass is  $\int_x^{x+\Delta x} m dx$ .

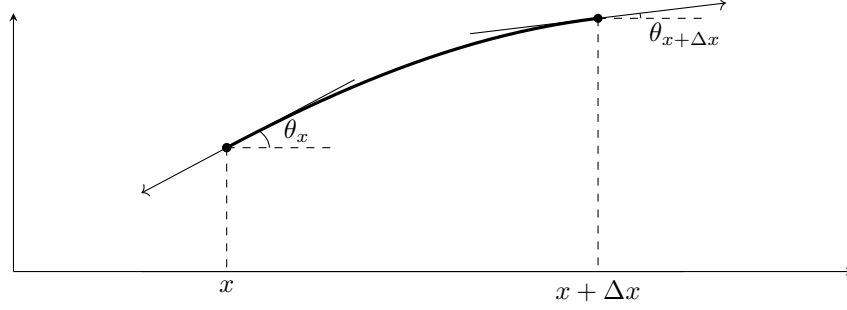


FIGURE 6.2.

Consider the tension forces  $T_x$  and  $T_{x+\Delta x}$  at  $x$  and  $x + \Delta x$  respectively. Since the net horizontal movement is 0, the sum of the horizontal components of the tension must be 0. This means that

$$\begin{aligned} T_x \cos \theta_x &= T_{x+\Delta x} \cos \theta_{x+\Delta x} \\ T_{x+\Delta x} \cos \theta_{x+\Delta x} - T_x \cos \theta_x &= 0. \end{aligned}$$

If we divide by  $\Delta x$  and let  $\Delta x \rightarrow 0$ , by the definition of the derivative, we get

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{T_{x+\Delta x} \cos \theta_{x+\Delta x} - T_x \cos \theta_x}{\Delta x} &= 0 \\ \frac{\partial}{\partial x} T_x \cos \theta_x &= 0. \end{aligned}$$

Hence  $T_x \cos \theta_x$  must be a constant, which we will call  $\tau$ .

Next, we consider the vertical tensional forces. We have

$$\begin{aligned} \sum T &= T_{x+\Delta x} \sin \theta_{x+\Delta x} - T_x \sin \theta_x = \tau \left( \frac{\sin \theta_{x+\Delta x}}{\cos \theta_{x+\Delta x}} - \frac{\sin \theta_x}{\cos \theta_x} \right) \\ &= \tau (\tan \theta_{x+\Delta x} - \tan \theta_x). \end{aligned}$$

Since  $\tan \theta$  is simply the slope, we may replace it with  $u_x$ , giving us the net force

$$F = \tau(u_x(x + \Delta x, t) - u_x(x)).$$

We now have the force and the mass. We simply need the acceleration to apply Newton's second law. Thankfully, the acceleration is simply the second time derivative,  $u_{tt}$ . Applying Newton's second law, which states that  $\sum F = ma$ , we get:

$$(6.1) \quad \tau(u_x(x + \Delta x, t) - u_x(x)) = \int_x^{x+\Delta x} m u_{tt} dx.$$

The Intermediate Value Theorem for integrals states that

$$\int_a^b f(x) dx = f(c)(b - a)$$

for some  $c \in [a, b]$ . We use this to simplify 6.1:

$$\tau(u_x(x + \Delta x, t) - u_x(x)) = m u_{tt} \Delta x.$$

We then divide by  $\Delta x$  and let  $\Delta \rightarrow 0$ :

$$\tau \frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x} = m u_{tt}$$
$$\tau u_{xx} = m u_{tt}.$$

If we define a constant  $c = \sqrt{\frac{\tau}{m}}$ , then we may rewrite the equation as

$$u_{tt} = c^2 u_{xx}.$$

This is the wave equation, in one spatial dimension.

**DEFINITION**

The one-dimensional *wave equation* is

$$u_{tt} = c^2 u_{xx},$$

such that  $u(x, t)$  describes the motion of a plucked string over time.

## 7. SOLVING THE WAVE EQUATION

Some bollocks