LECTURE NOTES—FOURIER SERIES

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We now look at some applications of Fourier series. First, we investigate the problem of a vibrating string.

1. Wave Equation in One Dimension

Consider a string of length L lying on the x-axis, fixed at 2 ends: the origin $(0, 0)$ and $(L, 0)$. We will make some reasonable assumptions:

- The particles in the string move only up and down (there is no horizontal movement)
- The string is perfectly elastic

Let the function $u(x, t)$ describe the displacement of the string at time t.

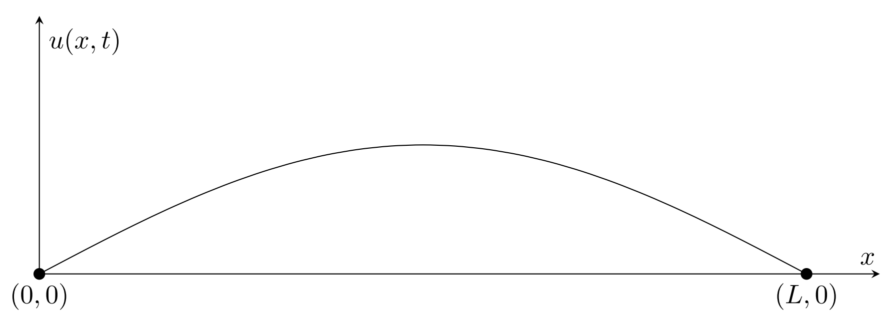
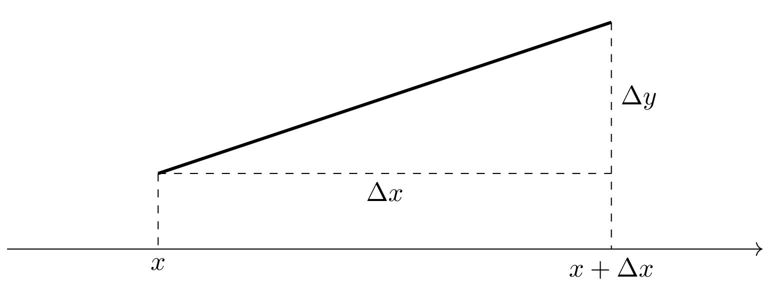


Figure 1.1. A possible shape of the string at some point in time

First, we find the mass of the string in some interval $[a, b]$. Let $\rho(x, t)$ describe the density of the string at x at time t. Focus on a tiny segment of the string at x.



Since this piece of string is so small, it approximates a straight line. Hence its length is clearly $\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$. Now the mass of this piece of string is simply $\rho(x,t)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$. Imagine summing this N times for all the tiny segments in the interval $[a, b]$. Thus an approximation of the mass of the string in the interval $[a, b]$ is

$$\sum_{i=1}^{N} \rho(x,t)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\Delta x_i.$$

If these segments are equally spaced, then $\Delta x_i = \Delta x$. Our approximation becomes the true mass of the string when Δx approaches 0 and N approaches ∞. Thus the mass of the string is

$$\lim_{N\to\infty} \lim_{\Delta x \to 0} \sum_{i=1}^{N} \rho(x,t)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\Delta x = \int_a^b \rho(x,t)\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\, dx.$$

For sake of brevity, we abbreviate $\frac{\partial u}{\partial x}$ to u_x, and similarly, $\frac{\partial u}{\partial t}$ to u_t.

Let us now concentrate on another small segment of the string in the interval $[x, x + \Delta x]$. This piece of string has mass

$$\int_x^{x+\Delta x} \rho(x,t)\sqrt{1 + u_x^2}\, dx$$

which we will call m.

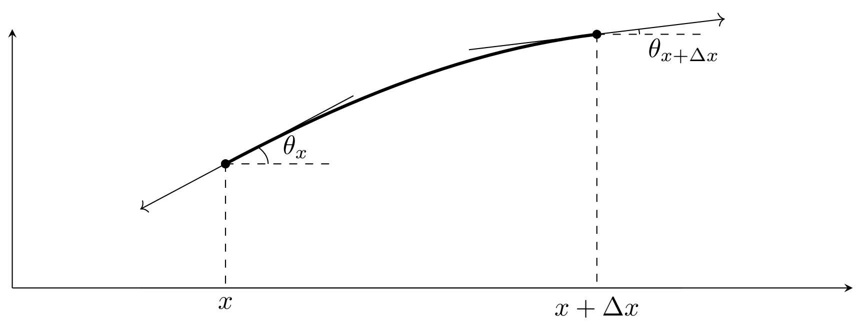


Figure 1.2.