

LECTURE NOTES—FOURIER SERIES

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1. INTRODUCTION

Fourier series have countless applications throughout applied mathematics and physics. Fourier series were first formally introduced by Joseph Fourier in his 1822 work *Théorie analytique de la chaleur* (*Analytical theory of heat*) in order to solve the Heat Equation. Whilst this is not a history lesson, it is still important to know that Fourier series are rooted in physics and are commonly used to solve differential equations.

2. FOURIER SERIES

But what *are* Fourier series? Despite sounding very fancy, the concept is very simple. The general concept of a Fourier series is to break a function down into a sum of sines and cosines. This is motivated by several reasons—they are easily differentiable, integrable, and manipulable.

For example, given some 2π periodic function $f(x)$, we can express it as some sum of sinusoidal waves like so:

$$f(x) \approx \frac{a_0}{2} + \sum_{m=1}^n (a_m \cos mx + b_m \sin mx).$$

If, as the number of terms n approaches ∞ , the approximation *converges* to f , then the resulting sum is called a *Fourier series*:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx).$$

You will have noticed that there are some coefficients: a_0, a_m, b_m . These are the *Fourier coefficients*, and determine how much of each sine wave of each frequency is included. Before we determine these coefficients, we first discuss the *orthogonality relations* of the sine and cosine functions.

3. ORTHOGONALITY OF SINE AND COSINE

There are three important formulae concerning sines and cosines:

THEOREM 3.1 (Orthogonality of sine and cosine)

For integers m, n , the following identities hold:

$$(3.1) \quad \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases}$$

$$(3.2) \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases}$$

$$(3.3) \quad \int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

These identities may be proved using the product to sum trigonometric identities:

$$(3.4) \quad \sin a \sin b = \frac{1}{2} [\cos(a - b) - \cos(a + b)]$$

$$(3.5) \quad \cos a \cos b = \frac{1}{2} [\cos(a - b) + \cos(a + b)]$$

$$(3.6) \quad \sin a \cos b = \frac{1}{2} [\sin(a - b) + \sin(a + b)]$$

Proof. We use 3.4 to prove the first formula:

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(x(m - n)) - \cos(x(m + n)) \, dx \\ &= \frac{1}{2} \left[\frac{\sin(x(m - n))}{m - n} - \frac{\sin(x(m + n))}{m + n} \right]_{-\pi}^{\pi}. \end{aligned}$$

Since $\sin k\pi = 0$ for any integer k , we have

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad m \neq n.$$

However, notice that the method only makes sense for $m \neq n$. If $m = n$, we may use the identity $\sin^2 a = \frac{1}{2}[1 - \cos(2a)]$ to obtain

$$\begin{aligned} \int_{-\pi}^{\pi} \sin^2(mx) dx &= \frac{1}{2} \int_{-\pi}^{\pi} 1 - \cos(2mx) dx \\ &= \frac{1}{2} \left[x - \frac{\sin(2mx)}{2m} \right]_{-\pi}^{\pi} \\ &= \pi \end{aligned}$$

The other two identities are left as an exercise to the reader. \square

4. FOURIER COEFFICIENTS

Recall the Fourier series of $f(x)$, a 2π periodic function on $[-\pi, \pi]$:

$$(4.1) \quad f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx).$$

Let us assume that the series converges to f . We wish to calculate the coefficients a_0, a_m, b_m .

First, consider multiplying $f(x)$ by $\cos nx$ and integrating from $-\pi$ to π :

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos nx dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos nx dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx \cos nx dx \\ &\quad + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx \cos nx dx. \end{aligned}$$

Using the orthogonal relationships from earlier, we may simplify this to

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos nx dx &= 0 + a_n \pi + 0 \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx. \end{aligned}$$

Now we multiply $f(x)$ by $\sin nx$ and integrate from $-\pi$ to π :

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin nx dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} \sin nx dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \sin nx \cos mx dx \\ &\quad + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx \sin nx dx. \end{aligned}$$

Again, we simplify using the orthogonal relations:

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin nx dx &= 0 + 0 + b_n \pi \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx. \end{aligned}$$

Finally, to determine a_0 , we simply integrate from $-\pi$ to π :

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx dx \\ &\quad + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx dx\end{aligned}$$

This time, we simply notice that $\int_{-\pi}^{\pi} \cos kx dx$ and $\int_{-\pi}^{\pi} \sin kx dx$ both equal zero. Hence, we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) dx &= a_0\pi + 0 + 0 \\ a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx\end{aligned}$$

DEFINITION (Fourier series)

The *Fourier series* of a function $f(x)$ that is 2π periodic on the closed interval $[-\pi, \pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx),$$

where

$$\begin{aligned}a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx \\ b_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx.\end{aligned}$$

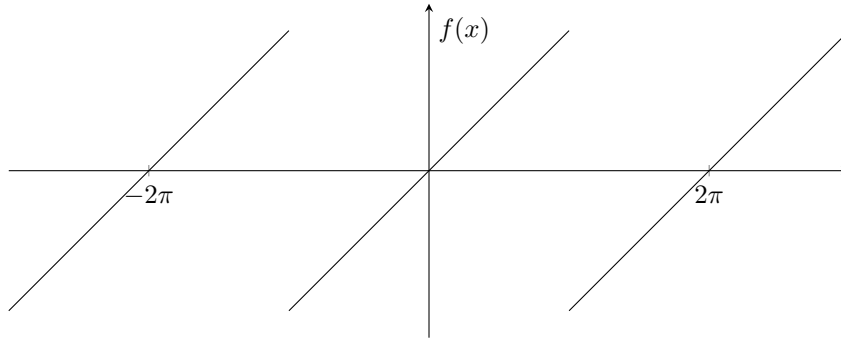
5. USING FOURIER SERIES

5.1. An example. We now look at some applications of Fourier series. Let's go through an example of computing the Fourier series of a function.

Consider the function $f(x) = x$, which has been made periodic on the interval $[-\pi, \pi]$ so that it is 2π periodic. This just means that it repeats after $x = \pi$. We will calculate its Fourier coefficients.

We know that $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$. Since $f(x) = x$, we have

$$\begin{aligned}a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x dx \\ &= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_{-\pi}^{\pi} \\ &= 0.\end{aligned}$$

FIGURE 5.1. Graph of $f(x)$

Next, we calculate a_m :

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos mx \, dx.$$

Notice $f(x) = x$ is an odd function, and $\cos mx$ is an even function. Therefore $x \cos mx$ is an odd function. When integrating an odd function with symmetric bounds, the result is always 0. Hence, $a_m = 0$.

Finally, we calculate b_m .

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin mx \, dx.$$

This is a simple integration by parts exercise:

$$\begin{aligned} b_m &= \frac{1}{\pi} \left[-\frac{x \cos mx}{m} + \frac{1}{m} \int \cos mx \, dx \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[\frac{\sin mx - mx \cos mx}{m^2} \right]_{-\pi}^{\pi}. \end{aligned}$$

Notice that $\sin m\pi = 0$ for any integer m , and $\cos mx = (-1)^m$. Hence we have

$$\begin{aligned} b_m &= \frac{1}{\pi} \left[\frac{-2\pi m(-1)^m}{m^2} \right] \\ &= -\frac{2(-1)^m}{m}. \end{aligned}$$

This gives us the Fourier series

$$f(x) = -\sum_{m=1}^{\infty} \frac{2(-1)^m}{m} \sin mx.$$

The graphs of some partial sums are shown in Figures 5.2 and 5.3.

5.2. The Basel Problem. You may have heard of the Basel problem, an astonishing identity where π seemingly randomly pops up. I will present a method of solving the problem using Fourier series.

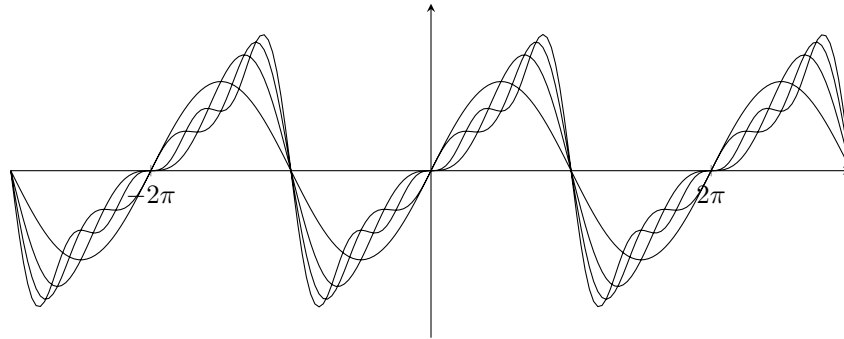


FIGURE 5.2. Graphs of the first 4 partial sums

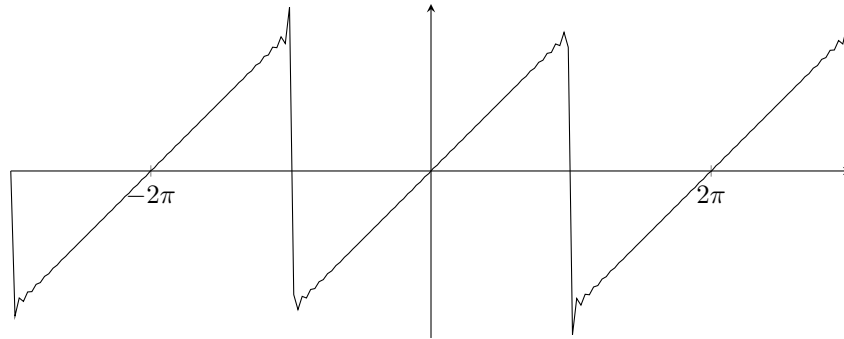


FIGURE 5.3. Graph of the 100th partial sum

THEOREM 5.1 (Basel problem)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof. Consider the function $f(x) = x^2$ which has been made periodic on the interval $[-\pi, \pi]$. We now find its Fourier series.

First, we calculate a_0 :

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx \\ &= \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} \\ &= \frac{2\pi^2}{3}. \end{aligned}$$

Next, we calculate a_m , using integration by parts:

$$\begin{aligned} a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos mx \, dx \\ &= \frac{1}{\pi} \left[\frac{x^2 \sin mx}{m} - \frac{2}{m} \int x \sin mx \, dx \right]_{-\pi}^{\pi}. \end{aligned}$$

We can use our previous result that $\int_{-\pi}^{\pi} x \sin mx \, dx = -\frac{2\pi(-1)^m}{m}$ to get

$$\begin{aligned} a_m &= \frac{1}{\pi} \left[\frac{x^2 \sin mx}{m} \right]_{-\pi}^{\pi} + \frac{2}{\pi m} \left[\frac{2\pi(-1)^m}{m} \right] \\ &= \frac{4(-1)^m}{m^2}. \end{aligned}$$

To calculate b_m we may notice that $x^2 \sin mx$ is odd, meaning that the symmetric integral will be 0. Hence, $b_m = 0$.

This means that our Fourier series for $f(x) = x^2$ on $[-\pi, \pi]$ is

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{m=1}^{\infty} \frac{4(-1)^m}{m^2} \cos mx.$$

□

6. WAVE EQUATION IN ONE DIMENSION

We now investigate the problem of a vibrating string.

Consider a string of length L lying on the x -axis, fixed at 2 ends: the origin $(0, 0)$ and $(L, 0)$. We will make some reasonable assumptions:

- The particles in the string move only up and down (there is no horizontal movement)
- The string is perfectly elastic

Let the function $u(x, t)$ describe the displacement of the string at time t .

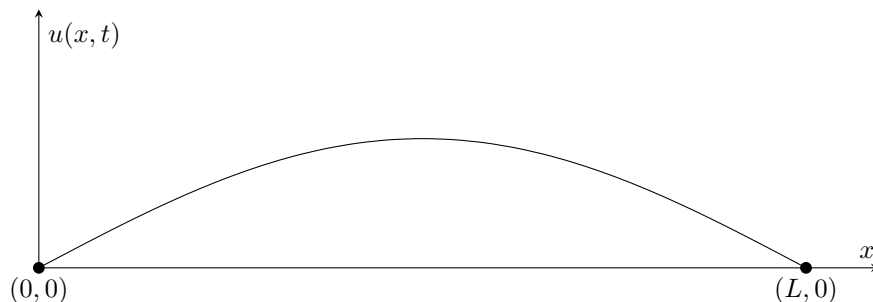
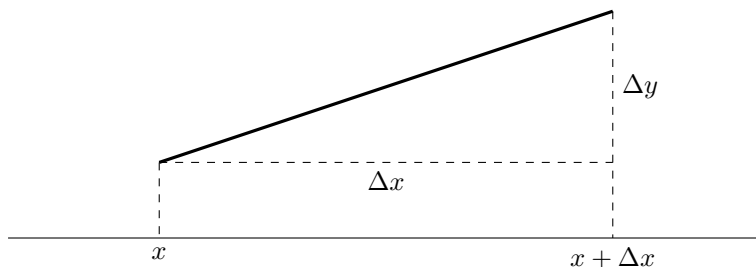


FIGURE 6.1. A possible shape of the string at some point in time

First, we find the mass of the string in some interval $[a, b]$. Let $\rho(x, t)$ describe the density of the string at x at time t . Focus on a tiny segment of the string at x .



Since this piece of string is so small, it approximates a straight line. Hence its length is clearly $\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x$. Now the mass of this piece of string is simply $\rho(x, t) \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x$. Imagine summing this N times for all the tiny segments in the interval $[a, b]$. Thus an approximation of the mass of the string in the interval $[a, b]$ is

$$\sum_{i=1}^N \rho(x_i, t) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i.$$

If these segments are equally spaced, then $\Delta x_i = \Delta x$. Our approximation becomes the true mass of the string when Δx approaches 0 and N approaches ∞ . Thus the mass of the string is

$$\lim_{\substack{N \rightarrow \infty \\ \Delta x \rightarrow 0}} \sum_{i=1}^N \rho(x, t) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \Delta x = \int_a^b \rho(x, t) \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} dx.$$

For sake of brevity, we abbreviate $\frac{\partial u}{\partial x}$ to u_x , and similarly, $\frac{\partial u}{\partial t}$ to u_t .

Let us now concentrate on another small segment of the string in the interval $[x, x + \Delta x]$. This piece of string has mass

$$\int_x^{x+\Delta x} \rho(x, t) \sqrt{1 + u_x^2} dx.$$

Let $m = \rho(x, t) \sqrt{1 + u_x^2}$ such that the mass is $\int_x^{x+\Delta x} m dx$.

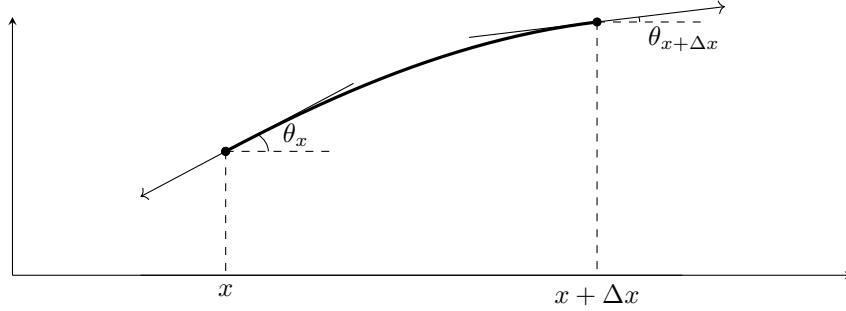


FIGURE 6.2.

Consider the tension forces T_x and $T_{x+\Delta x}$ at x and $x + \Delta x$ respectively. Since the net horizontal movement is 0, the sum of the horizontal components of the tension must be 0. This means that

$$\begin{aligned} T_x \cos \theta_x &= T_{x+\Delta x} \cos \theta_{x+\Delta x} \\ T_{x+\Delta x} \cos \theta_{x+\Delta x} - T_x \cos \theta_x &= 0. \end{aligned}$$

If we divide by Δx and let $\Delta x \rightarrow 0$, by the definition of the derivative, we get

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{T_{x+\Delta x} \cos \theta_{x+\Delta x} - T_x \cos \theta_x}{\Delta x} &= 0 \\ \frac{\partial}{\partial x} T_x \cos \theta_x &= 0. \end{aligned}$$

Hence $T_x \cos \theta_x$ must be a constant, which we will call τ .

Next, we consider the vertical tensional forces. We have

$$\begin{aligned} \sum T &= T_{x+\Delta x} \sin \theta_{x+\Delta x} - T_x \sin \theta_x = \tau \left(\frac{\sin \theta_{x+\Delta x}}{\cos \theta_{x+\Delta x}} - \frac{\sin \theta_x}{\cos \theta_x} \right) \\ &= \tau (\tan \theta_{x+\Delta x} - \tan \theta_x) \end{aligned}$$

Since $\tan \theta$ is simply the slope, we may replace it with u_x , giving us the net force

$$F = \tau(u_x(x + \Delta x, t) - u_x(x)).$$

We now have the force and the mass. We simply need the acceleration to apply Newton's second law. Thankfully, the acceleration is simply the second time derivative, u_{tt} . Applying Newton's second law, which states that $\sum F = ma$, we get:

$$(6.1) \quad \tau(u_x(x + \Delta x, t) - u_x(x)) = \int_x^{x+\Delta x} m u_{tt} dx.$$

The Intermediate Value Theorem for integrals states that

$$\int_a^b f(x) dx = f(c)(b - a)$$

for some $c \in [a, b]$. We use this to simplify 6.1:

$$\tau(u_x(x + \Delta x, t) - u_x(x)) = m u_{tt} \Delta x.$$

We then divide by Δx and let $\Delta \rightarrow 0$:

$$\tau \frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x} = m u_{tt}$$

$$\tau u_{xx} = m u_{tt}.$$

If we define a constant $c = \sqrt{\frac{\tau}{m}}$, then we may rewrite the equation as

$$u_{tt} = c^2 u_{xx}.$$

This is the wave equation, in one spatial dimension.

DEFINITION

The one-dimensional *wave equation* is

$$u_{tt} = c^2 u_{xx},$$

such that $u(x, t)$ describes the motion of a plucked string over time.

7. SOLVING THE WAVE EQUATION

Some bollocks