

LECTURE NOTES—FOURIER SERIES

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1. INTRODUCTION

Fourier series have countless applications throughout applied mathematics and physics. Fourier series were first formally introduced by Joseph Fourier in his 1822 work *Théorie analytique de la chaleur* (*Analytical theory of heat*) in order to solve the Heat Equation. Whilst this is not a history lesson, it is still important to know that Fourier series are rooted in physics and are commonly used to solve differential equations.

2. FOURIER SERIES

But what *are* Fourier series? Despite sounding very fancy, the concept is very simple. The general concept of a Fourier series is to break a function down into a sum of sines and cosines. This is motivated by several reasons—they are easily differentiable, integrable, and manipulable.

For example, given some 2π periodic function $f(x)$, we can express it as some sum of sinusoidal waves like so:

$$f(x) \approx \frac{a_0}{2} + \sum_{m=1}^n (a_m \cos mx + b_m \sin mx).$$

If, as the number of terms n approaches ∞ , the approximation *converges* to f , then the resulting sum is called a *Fourier series*:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx).$$

You will have noticed that there are some coefficients: a_0, a_m, b_m . These are the *Fourier coefficients*, and determine how much of each sine wave of each frequency is

included. Before we determine these coefficients, we first discuss the *orthogonality relations* of the sine and cosine functions.

3. ORTHOGONALITY OF SINE AND COSINE

There are three important formulae concerning sines and cosines:

THEOREM 3.1 (Orthogonality of sine and cosine)

For integers m, n , the following identities hold:

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases} \\ \int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases} \\ \int_{-\pi}^{\pi} \sin mx \cos nx \, dx &= 0\end{aligned}$$

These identities may be proved using the product to sum trigonometric identities:

$$(3.1) \quad \sin a \sin b = \frac{1}{2} [\cos(a - b) - \cos(a + b)]$$

$$(3.2) \quad \cos a \cos b = \frac{1}{2} [\cos(a - b) + \cos(a + b)]$$

$$(3.3) \quad \sin a \cos b = \frac{1}{2} [\sin(a - b) + \sin(a + b)]$$

Proof. We use 3.1 to prove the first formula:

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(x(m - n)) - \cos(x(m + n)) \, dx \\ &= \frac{1}{2} \left[\frac{\sin(x(m - n))}{m - n} - \frac{\sin(x(m + n))}{m + n} \right]_{-\pi}^{\pi}.\end{aligned}$$

Since $\sin k\pi = 0$ for any integer k , we have

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad m \neq n.$$

However, notice that the method only makes sense for $m \neq n$. If $m = n$, we may use the identity $\sin^2 a = \frac{1}{2}[1 - \cos(2a)]$ to obtain

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^2(mx) \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} 1 - \cos(2mx) \, dx \\ &= \frac{1}{2} \left[x - \frac{\sin(2mx)}{2m} \right]_{-\pi}^{\pi} \\ &= \pi\end{aligned}$$

The other two identities are left as an exercise to the reader. \square

4. FOURIER COEFFICIENTS

Recall the Fourier series of a 2π periodic function on $[-\pi, \pi]$, $f(x)$:

$$(4.1) \quad f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx).$$

Let us assume that the series converges to f . We wish to calculate the coefficients a_0, a_m, b_m .

5. WAVE EQUATION IN ONE DIMENSION

We now look at some applications of Fourier series. First, we investigate the problem of a vibrating string.

Consider a string of length L lying on the x -axis, fixed at 2 ends: the origin $(0, 0)$ and $(L, 0)$. We will make some reasonable assumptions:

- The particles in the string move only up and down (there is no horizontal movement)
- The string is perfectly elastic

Let the function $u(x, t)$ describe the displacement of the string at time t .

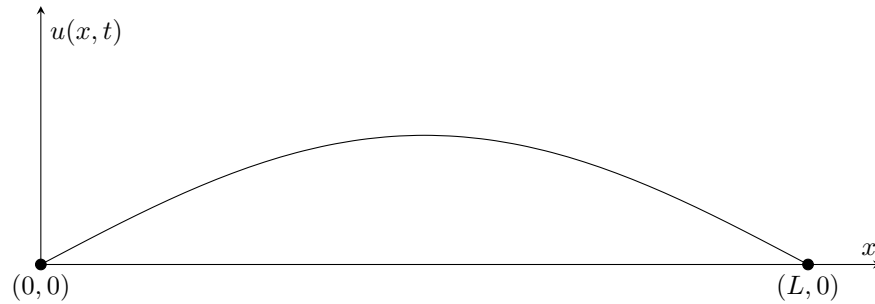
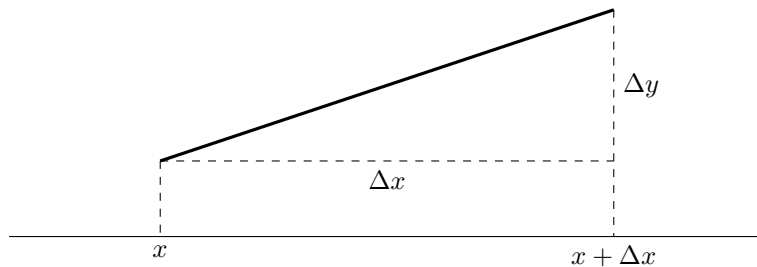


FIGURE 5.1. A possible shape of the string at some point in time

First, we find the mass of the string in some interval $[a, b]$. Let $\rho(x, t)$ describe the density of the string at x at time t . Focus on a tiny segment of the string at x .



Since this piece of string is so small, it approximates a straight line. Hence its length is clearly $\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x$. Now the mass of this piece of string is simply $\rho(x, t) \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x$. Imagine summing this N times for all the tiny segments in the interval $[a, b]$. Thus an approximation of the mass of the string in the interval $[a, b]$ is

$$\sum_{i=1}^N \rho(x, t) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i.$$

If these segments are equally spaced, then $\Delta x_i = \Delta x$. Our approximation becomes the true mass of the string when Δx approaches 0 and N approaches ∞ . Thus the mass of the string is

$$\lim_{N \rightarrow \infty} \lim_{\Delta x \rightarrow 0} \sum_{i=1}^N \rho(x, t) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \Delta x = \int_a^b \rho(x, t) \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} dx.$$

For sake of brevity, we abbreviate $\frac{\partial u}{\partial x}$ to u_x , and similarly, $\frac{\partial u}{\partial t}$ to u_t .

Let us now concentrate on another small segment of the string in the interval $[x, x + \Delta x]$. This piece of string has mass

$$\int_x^{x+\Delta x} \rho(x, t) \sqrt{1 + u_x^2} dx$$

which we will call m .

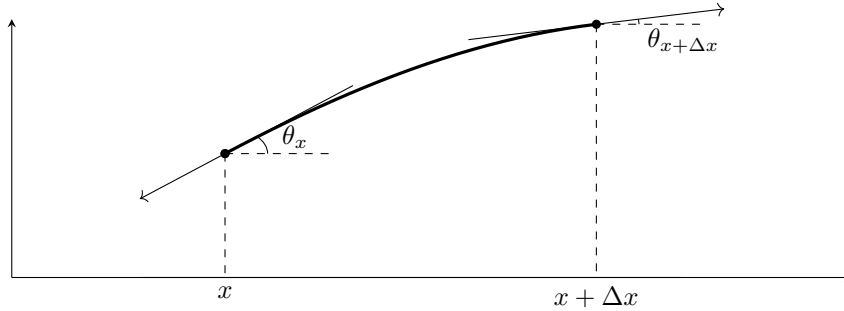


FIGURE 5.2.