LECTURE NOTES—FOURIER SERIES

EDWARD WANG

December 2022

Contents

1. Introduction

Fourier series have countless applications throughout applied mathematics and physics. Fourier series were first formally introduced by Joseph Fourier in his 1822 work *Théorie analytique de la chaleur (Analytical theory of heat)* in order to solve the Heat Equation. Whilst this is not a history lesson, it is still important to know that Fourier series are rooted in physics and are commonly used to solve differential equations.

2. Fourier series

But what *are* Fourier series? Despite sounding very fancy, the concept is very simple. The general concept of a Fourier series is to break a function down into a sum of sines and cosines. This is motivated by several reasons—they are easily differentiable, integrable, and manipulable.

For example, given some 2π periodic function $f(x)$, we can express it as some sum of sinusoidal waves like so:

$$f(x) \approx \frac{a_0}{2} + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx).$$

If, as the number of terms n approaches ∞, the approximation *converges* to f, then the resulting sum is called a *Fourier series*:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx).$$

You will have noticed that there are some coefficients: a_0, a_m, b_m. These are the *Fourier coefficients*, and determine how much of each sine wave of each frequency is included. Before we determine these coefficients, we first discuss the *orthogonality relations* of the sine and cosine functions.

3. Orthogonality of sine and cosine

There are three important formulae concerning sines and cosines:

Theorem 3.1 (Orthogonality of sine and cosine)
For integers m, n, the following identities hold:

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases} \tag{3.1}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases} \tag{3.2}$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \tag{3.3}$$

These identities may be proved using the product to sum trigonometric identities:

$$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)] \tag{3.4}$$

$$\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)] \tag{3.5}$$

$$\sin a \cos b = \frac{1}{2}[\sin(a-b) + \sin(a+b)] \tag{3.6}$$

Proof. We use 3.4 to prove the first formula:

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(x(m-n)) - \cos(x(m+n)) \, dx \\ &= \frac{1}{2} \left[\frac{\sin(x(m-n))}{m-n} - \frac{\sin(x(m+n))}{m+n} \right]_{-\pi}^{\pi}. \end{aligned}$$

Since $\sin k\pi = 0$ for any integer k, we have

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad m \neq n.$$

However, notice that the method only makes sense for $m \neq n$. If $m = n$, we may use the identity $\sin^2 a = \frac{1}{2}[1 - \cos(2a)]$ to obtain

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^2(mx)\, dx &= \frac{1}{2}\int_{-\pi}^{\pi} 1 - \cos(2mx)\, dx \\ &= \frac{1}{2}\left[x - \frac{\sin(2mx)}{2m}\right]_{-\pi}^{\pi} \\ &= \pi\end{aligned}$$

The other two identities are left as an exercise to the reader. □

4. Fourier Coefficients

Recall the Fourier series of $f(x)$, a 2π periodic function on $[-\pi, \pi]$:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx). \tag{4.1}$$

Let us assume that the series converges to f. We wish to calculate the coefficients a_0, a_m, b_m.

First, consider multiplying $f(x)$ by $\cos nx$ and integrating from $-\pi$ to π:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \cos nx\, dx &+ \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx \cos nx\, dx \\ &+ \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx \cos nx\, dx.\end{aligned}$$

Using the orthogonal relationships from earlier, we may simplify this to

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos nx\, dx &= 0 + a_n\pi + 0 \\ \boldsymbol{a_n} &\boldsymbol{= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx}.\end{aligned}$$

Now we multiply $f(x)$ by $\sin nx$ and integrate from $-\pi$ to π:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \sin nx\, dx &+ \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \sin nx \cos mx\, dx \\ &+ \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx \sin nx\, dx.\end{aligned}$$

Again, we simplify using the orthogonal relations:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin nx\, dx &= 0 + 0 + b_n\pi \\ \boldsymbol{b_n} &\boldsymbol{= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx}.\end{aligned}$$

Finally, to determine a_0, we simply integrate from $-\pi$ to π:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx\,dx \\ + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx\,dx\end{aligned}$$

This time, we simply notice that $\int_{-\pi}^{\pi} \cos kx\,dx$ and $\int_{-\pi}^{\pi} \sin kx\,dx$ both equal zero. Hence, we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0\pi + 0 + 0$$

$$\boldsymbol{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

Definition (Fourier series)

The *Fourier series* of a function $f(x)$ that is 2π periodic on the closed interval $[-\pi, \pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx),$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx \\ b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx. \end{aligned}$$

5. Fourier series and the Basel Problem

We now look at some applications of Fourier series. Let's go through an example of computing the Fourier series of a function.

Consider the function $f(x) = x$, which has been made periodic on the interval $[-\pi, \pi]$ so that it is 2π periodic. This just means that it repeats after $x = \pi$. We will calculate its Fourier coefficients.

6. Wave Equation in One Dimension

We now investigate the problem of a vibrating string.

Consider a string of length L lying on the x-axis, fixed at 2 ends: the origin $(0, 0)$ and $(L, 0)$. We will make some reasonable assumptions:

- The particles in the string move only up and down (there is no horizontal movement)
- The string is perfectly elastic

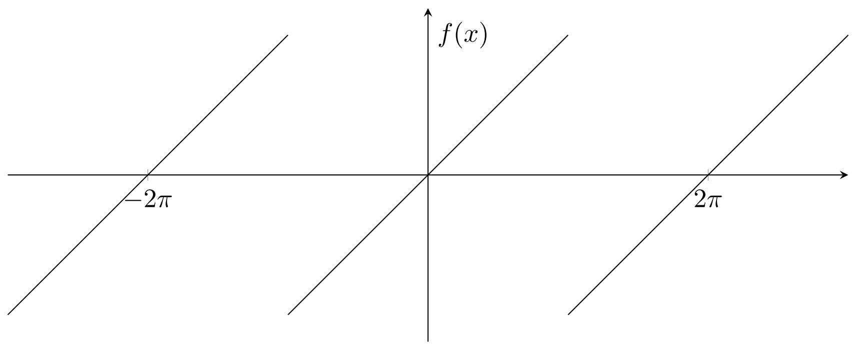


Figure 5.1. Graph of $f(x)$

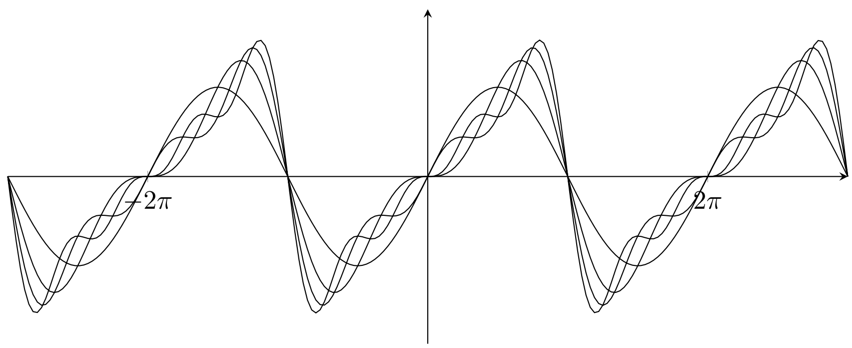


Figure 5.2. Graphs of the first 4 partial sums

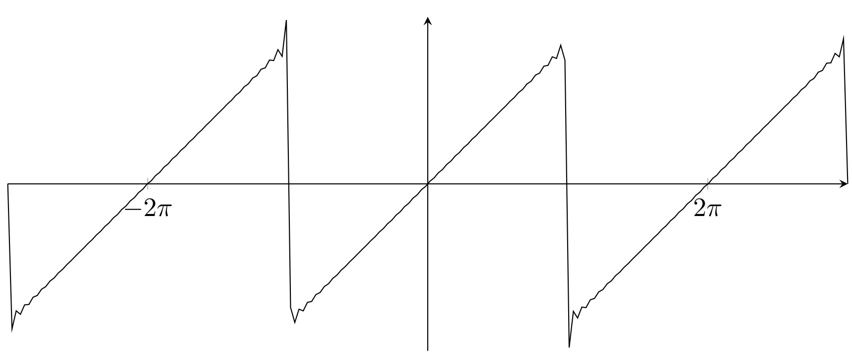


Figure 5.3. Graph of the 100th partial sum

Let the function $u(x,t)$ describe the displacement of the string at time t.

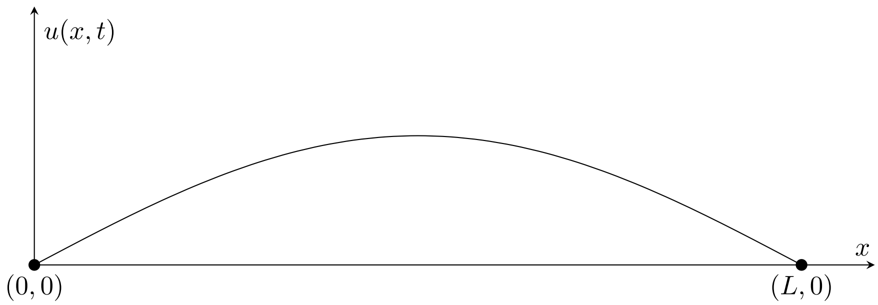
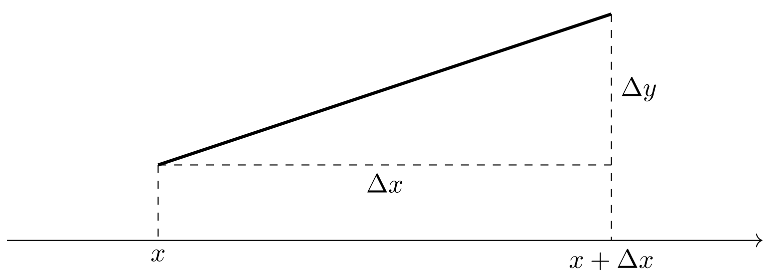


Figure 6.1. A possible shape of the string at some point in time

First, we find the mass of the string in some interval $[a, b]$. Let $\rho(x, t)$ describe the density of the string at x at time t. Focus on a tiny segment of the string at x.



Since this piece of string is so small, it approximates a straight line. Hence its length is clearly $\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$. Now the mass of this piece of string is simply $\rho(x,t)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$. Imagine summing this N times for all the tiny segments in the interval $[a, b]$. Thus an approximation of the mass of the string in the interval $[a, b]$ is

$$\sum_{i=1}^{N} \rho(x,t)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\Delta x_i.$$

If these segments are equally spaced, then $\Delta x_i = \Delta x$. Our approximation becomes the true mass of the string when Δx approaches 0 and N approaches ∞. Thus the mass of the string is

$$\lim_{\substack{N\to\infty \\ \Delta x \to 0}} \sum_{i=1}^{N} \rho(x,t)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\Delta x = \int_a^b \rho(x,t)\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\,dx.$$

For sake of brevity, we abbreviate $\frac{\partial u}{\partial x}$ to u_x, and similarly, $\frac{\partial u}{\partial t}$ to u_t.

Let us now concentrate on another small segment of the string in the interval $[x, x + \Delta x]$. This piece of string has mass

$$\int_x^{x+\Delta x} \rho(x,t)\sqrt{1 + u_x^2}\,dx.$$

Let $m = \rho(x,t)\sqrt{1+u_x^2}$ such that the mass is $\int_x^{x+\Delta x} m\,dx$.

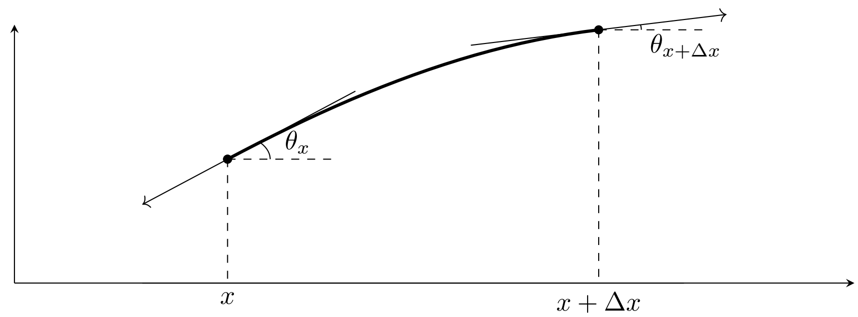


Figure 6.2.

Consider the tension forces T_x and $T_{x+\Delta x}$ at x and $x+\Delta x$ respectively. Since the net horizontal movement is 0, the sum of the horizontal components of the tension must be 0. This means that

$$T_x \cos\theta_x = T_{x+\Delta x}\cos\theta_{x+\Delta x}$$
$$T_{x+\Delta x}\cos\theta_{x+\Delta} - T_x\cos\theta_x = 0.$$

If we divide by Δx and let $\Delta x \to 0$, by the definition of the derivative, we get

$$\lim_{\Delta x\to 0}\frac{T_{x+\Delta x}\cos\theta_{x+\Delta} - T_x\cos\theta_x}{\Delta x} = 0$$
$$\frac{\partial}{\partial x}T_x\cos\theta_x = 0.$$

Hence $T_x\cos\theta_x$ must be a constant, which we will call τ.

Next, we consider the vertical tensional forces. We have

$$\begin{aligned}\sum T = T_{x+\Delta x}\sin\theta_{x+\Delta x} - T_x\sin\theta_x &= \tau\Big(\frac{\sin\theta_{x+\Delta x}}{\cos\theta_{x+\Delta x}} - \frac{\sin\theta_x}{\cos\theta_x}\Big)\\ &= \tau(\tan\theta_{x+\Delta x} - \tan\theta_x)\end{aligned}$$

Since $\tan\theta$ is simply the slope, we may replace it with u_x, giving us the net force

$$F = \tau(u_x(x+\Delta x,t) - u_x(x)).$$

We now have the force and the mass. We simply need the acceleration to apply Newton's second law. Thankfully, the acceleration is simply the second time derivative, u_{tt}. Applying Newton's second law, which states that $\sum F = ma$, we get:

$$\tau(u_x(x+\Delta x,t) - u_x(x)) = \int_x^{x+\Delta x} mu_{tt}\,dx. \tag{6.1}$$

The Intermediate Value Theorem for integrals states that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

for some $c\in[a,b]$. We use this to simplify 6.1:

$$\tau(u_x(x+\Delta x,t) - u_x(x)) = mu_{tt}\Delta x.$$

We then divide by Δx and let $\Delta \to 0$:

$$\begin{aligned} \tau \frac{u_x(x+\Delta x, t) - u_x(x,t)}{\Delta x} &= m u_{tt} \\ \tau u_{xx} &= m u_{tt}. \end{aligned}$$

If we define a constant $c = \sqrt{\frac{\tau}{m}}$, then we may rewrite the equation as

$$u_{tt} = c^2 u_{xx}.$$

This is the wave equation, in one spatial dimension.

Definition

The one-dimensional *wave equation* is

$$u_{tt} = c^2 u_{xx},$$

such that $u(x,t)$ describes the motion of a plucked string over time.

7. Solving the wave equation

Some bollocks