LECTURE NOTES—FOURIER SERIES

EDWARD WANG

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1. Introduction

Fourier series have countless applications throughout applied mathematics and physics. Fourier series were first formally introduced by Joseph Fourier in his 1822 work *Théorie analytique de la chaleur (Analytical theory of heat)* in order to solve the Heat Equation. Whilst this is not a history lesson, it is still important to know that Fourier series are rooted in physics and are commonly used to solve differential equations.

2. Fourier series

But what *are* Fourier series? Despite sounding very fancy, the concept is very simple. The general concept of a Fourier series is to break a function down into a sum of sines and cosines. This is motivated by several reasons—they are easily differentiable, integrable, and manipulable.

For example, given some 2π periodic function $f(x)$, we can express it as some sum of sinusoidal waves like so:

$$f(x) \approx \frac{a_0}{2} + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx).$$

If, as the number of terms n approaches ∞, the approximation *converges* to f, then the resulting sum is called a *Fourier series*:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx).$$

You will have noticed that there are some coefficients: a_0, a_m, b_m. These are the *Fourier coefficients*, and determine how much of each sine wave of each frequency is included. Before we determine these coefficients, we first discuss the *orthogonality relations* of the sine and cosine functions.

3. Orthogonality of sine and cosine

There are three important formulae concerning sines and cosines:

Theorem 3.1 (Orthogonality of sine and cosine)
For integers m, n, the following identities hold:

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases} \tag{3.1}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases} \tag{3.2}$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \tag{3.3}$$

These identities may be proved using the product to sum trigonometric identities:

$$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)] \tag{3.4}$$

$$\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)] \tag{3.5}$$

$$\sin a \cos b = \frac{1}{2}[\sin(a-b) + \sin(a+b)] \tag{3.6}$$

Proof. We use 3.4 to prove the first formula:

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(x(m-n)) - \cos(x(m+n)) \, dx \\ &= \frac{1}{2} \left[\frac{\sin(x(m-n))}{m-n} - \frac{\sin(x(m+n))}{m+n} \right]_{-\pi}^{\pi}. \end{aligned}$$

Since $\sin k\pi = 0$ for any integer k, we have

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad m \neq n.$$

However, notice that the method only makes sense for $m \neq n$. If $m = n$, we may use the identity $\sin^2 a = \frac{1}{2}[1 - \cos(2a)]$ to obtain

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^2(mx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} 1 - \cos(2mx)\,dx \\ &= \frac{1}{2}\left[x - \frac{\sin(2mx)}{2m}\right]_{-\pi}^{\pi} \\ &= \pi\end{aligned}$$

The other two identities are left as an exercise to the reader. □

4. Fourier Coefficients

Recall the Fourier series of a 2π periodic function on $[-\pi, \pi]$, $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx). \tag{4.1}$$

Let us assume that the series converges to f. We wish to calculate the coefficients a_0, a_m, b_m.

First, consider multiplying $f(x)$ by $\cos nx$ and integrating from $-\pi$ to π:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos nx\,dx &+ \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi}\cos mx\cos nx\,dx \\ &+ \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi}\sin mx\cos nx\,dx.\end{aligned}$$

Using the orthogonal relationships from earlier, we may simplify this to

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos nx\,dx &= 0 + a_n\pi + 0 \\ \boldsymbol{a_n} &\boldsymbol{= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}.\end{aligned}$$

Now we multiply $f(x)$ by $\sin nx$ and integrate from $-\pi$ to π:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\sin nx\,dx &+ \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi}\sin nx\cos mx\,dx \\ &+ \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi}\sin mx\sin nx\,dx.\end{aligned}$$

Again, we simplify using the orthogonal relations:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin nx\,dx &= 0 + 0 + b_n\pi \\ \boldsymbol{b_n} &\boldsymbol{= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx}.\end{aligned}$$

Finally, to determine a_0, we simply integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx\,dx \\ + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx\,dx$$

This time, we simply notice that $\int_{-\pi}^{\pi} \cos kx\,dx$ and $\int_{-\pi}^{\pi} \sin kx\,dx$ both equal zero. Hence, we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0\pi + 0 + 0$$

$$\boldsymbol{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

Definition (Fourier series)

The *Fourier series* of a function $f(x)$ that is 2π periodic on the closed interval $[-\pi, \pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx),$$

where

$$a_0 = \int_{-\pi}^{\pi} f(x)\,dx$$
$$a_m = \int_{-\pi}^{\pi} f(x)\cos mx\,dx$$
$$b_m = \int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

5. Wave Equation in One Dimension

We now look at some applications of Fourier series. First, we investigate the problem of a vibrating string.

Consider a string of length L lying on the x-axis, fixed at 2 ends: the origin $(0, 0)$ and $(L, 0)$. We will make some reasonable assumptions:

- The particles in the string move only up and down (there is no horizontal movement)
- The string is perfectly elastic

Let the function $u(x, t)$ describe the displacement of the string at time t.

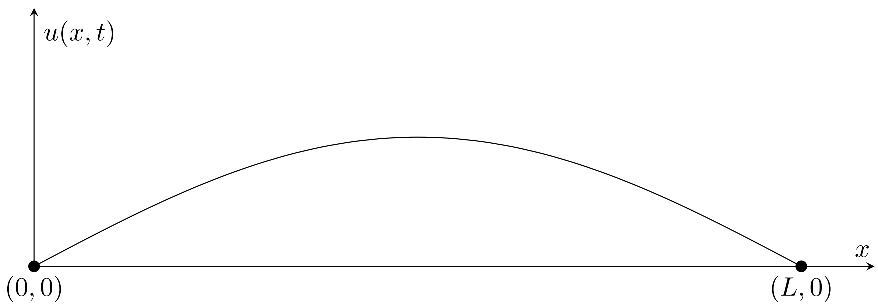
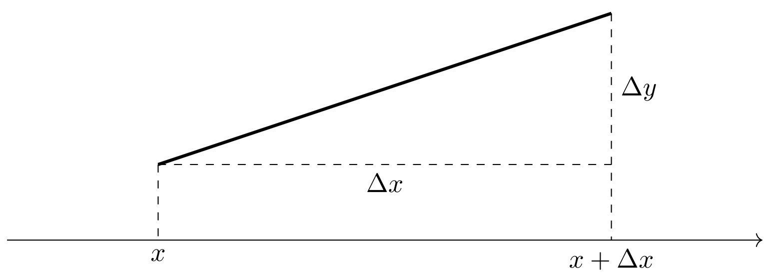


Figure 5.1. A possible shape of the string at some point in time

First, we find the mass of the string in some interval $[a, b]$. Let $\rho(x, t)$ describe the density of the string at x at time t. Focus on a tiny segment of the string at x.



Since this piece of string is so small, it approximates a straight line. Hence its length is clearly $\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$. Now the mass of this piece of string is simply $\rho(x, t)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$. Imagine summing this N times for all the tiny segments in the interval $[a, b]$. Thus an approximation of the mass of the string in the interval $[a, b]$ is

$$\sum_{i=1}^{N} \rho(x, t)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\Delta x_i.$$

If these segments are equally spaced, then $\Delta x_i = \Delta x$. Our approximation becomes the true mass of the string when Δx approaches 0 and N approaches ∞. Thus the mass of the string is

$$\lim_{\substack{N\to\infty \\ \Delta x\to 0}} \sum_{i=1}^{N} \rho(x, t)\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\Delta x = \int_a^b \rho(x, t)\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\, dx.$$

For sake of brevity, we abbreviate $\frac{\partial u}{\partial x}$ to u_x, and similarly, $\frac{\partial u}{\partial t}$ to u_t.

Let us now concentrate on another small segment of the string in the interval $[x, x + \Delta x]$. This piece of string has mass

$$\int_x^{x+\Delta x} \rho(x, t)\sqrt{1 + u_x^2}\, dx.$$

Let $m = \rho(x,t)\sqrt{1+u_x^2}$ such that the mass is $\int_x^{x+\Delta x} m\,dx$.

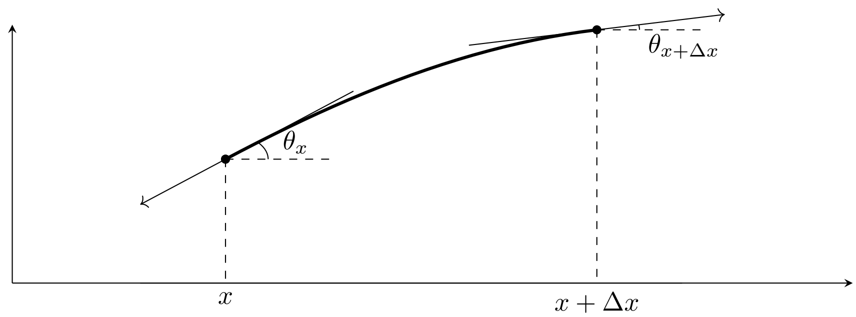


Figure 5.2.

Consider the tension forces T_x and $T_{x+\Delta x}$ at x and $x+\Delta x$ respectively. Since the net horizontal movement is 0, the sum of the horizontal components of the tension must be 0. This means that

$$T_x \cos\theta_x = T_{x+\Delta x}\cos\theta_{x+\Delta x}$$
$$T_{x+\Delta x}\cos\theta_{x+\Delta} - T_x\cos\theta_x = 0.$$

If we divide by Δx and let $\Delta x \to 0$, by the definition of the derivative, we get

$$\lim_{\Delta x\to 0}\frac{T_{x+\Delta x}\cos\theta_{x+\Delta} - T_x\cos\theta_x}{\Delta x} = 0$$
$$\frac{\partial}{\partial x}T_x\cos\theta_x = 0.$$

Hence $T_x\cos\theta_x$ must be a constant, which we will call τ.

Next, we consider the vertical tensional forces. We have

$$\begin{aligned}\sum T = T_{x+\Delta x}\sin\theta_{x+\Delta x} - T_x\sin\theta_x &= \tau\Big(\frac{\sin\theta_{x+\Delta x}}{\cos\theta_{x+\Delta x}} - \frac{\sin\theta_x}{\cos\theta_x}\Big)\\ &= \tau(\tan\theta_{x+\Delta x} - \tan\theta_x)\end{aligned}$$

Since $\tan\theta$ is simply the slope, we may replace it with u_x, giving us the net force

$$F = \tau(u_x(x+\Delta x,t) - u_x(x)).$$

We now have the force and the mass. We simply need the acceleration to apply Newton's second law. Thankfully, the acceleration is simply the second time derivative, u_{tt}. Applying Newton's second law, which states that $\sum F = ma$, we get:

$$\tau(u_x(x+\Delta x,t) - u_x(x)) = \int_x^{x+\Delta x} mu_{tt}\,dx. \tag{5.1}$$

The Intermediate Value Theorem for integrals states that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

for some $c\in[a,b]$. We use this to simplify 5.1:

$$\tau(u_x(x+\Delta x,t) - u_x(x)) = mu_{tt}\Delta x.$$

We then divide by Δx and let $\Delta \to 0$:

$$\begin{aligned}\tau \frac{u_x(x+\Delta x, t) - u_x(x,t)}{\Delta x} &= m u_{tt} \\ \tau u_{xx} &= m u_{tt}.\end{aligned}$$

If we define a constant $c = \sqrt{\frac{\tau}{m}}$, then we may rewrite the equation as

$$u_{tt} = c^2 u_{xx}.$$

This is the wave equation, in one spatial dimension.

DEFINITION

The one-dimensional *wave equation* is

$$u_{tt} = c^2 u_{xx},$$

such that $u(x,t)$ describes the motion of a plucked string over time.

6. SOLVING THE WAVE EQUATION

Some bollocks