Mathematics and Statistics Research Competition Question 1

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Contents

Let $\mathcal{N}_{10,2}$ be the set of positive integers whose digits in base-10 comprise only 0s and 1s. Examples of elements in $\mathcal{N}_{10,2}$ are: 1001, 110, and 11. Examples of elements not in $\mathcal{N}_{10,2}$ are: 4201, 690, and 12.

Consider a positive integer N. It can be constructed as the sum of elements in $\mathcal{N}_{10,2}$. For example, one construction of 1337 with 8 summands which are elements in $\mathcal{N}_{10,2}$ is as follows

$$1337 = 1000 + 111 + 111 + 111 + 1 + 1 + 1 + 1.$$

1 Problem 1

Problem. What are all of the constructions of 1337 using elements of $\mathcal{N}_{10,2}$?

We interpret the question as asking for how many unique ways there are to obtain 1337 as a sum of elements from $\mathcal{N}_{10,2}$. To do this, we look at the general case which seeks to find the number of unique ways to obtain a positive integer n as a sum of elements from $\mathcal{N}_{10,2}$.

Definition 1.1. For sake of convenience, we define a function $C(n)$ that counts the number of unique ways of constructing n as a sum of elements in $\mathcal{N}_{10,2}$. That is, $C(n)$ is the number of unique constructions such that

$$n = \sum_{a_i \in \mathcal{N}_{10,2}} a_i.$$

We start with an elementary example. Consider $n = 15$, and suppose we wish to find $C(15)$. Clearly, the only elements of $\mathcal{N}_{10,2}$ that are relevant here are 1, 10 and 11. With so few elements, we can easily calculate $C(15)$ manually. We find that there are three unique constructions of 15, which are

$$\begin{aligned}
15 &= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \\
15 &= 1 + 1 + 1 + 1 + 1 + 10 \\
15 &= 1 + 1 + 1 + 1 + 11.
\end{aligned}$$

Hence $C(15) = 3$. However, this method quickly breaks down for large n, where the number of relevant elements of $\mathcal{N}_{10,2}$ increases with the length of the number. For example, there are 15 relevant elements of $\mathcal{N}_{10,2}$ for $n = 1337$, which are $1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111$. As such, we need a better way to count $C(n)$.