Mathematics and Statistics Research Competition Question 1

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July 23, 2024

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Let $\mathcal{N}_{10,2}$ be the set of positive integers whose digits in base-10 comprise only 0s and 1s. Examples of elements in $\mathcal{N}_{10,2}$ are: 1001, 110, and 11. Examples of elements not in $\mathcal{N}_{10,2}$ are: 4201, 690, and 12.

Consider a positive integer N. It can be constructed as the sum of elements in $\mathcal{N}_{10,2}$. For example, one construction of 1337 with 8 summands which are elements in $\mathcal{N}_{10,2}$ is as follows

$$1337 = 1000 + 111 + 111 + 111 + 1 + 1 + 1 + 1.$$

1 Problem 1

Problem. What are all of the constructions of 1337 using elements of $\mathcal{N}_{10,2}$?

We interpret the question as asking for how many unique ways there are to obtain 1337 as a sum of elements from $\mathcal{N}_{10,2}$. To do this, we look at the general case which seeks to find the number of unique ways to obtain a positive integer n as a sum of elements from $\mathcal{N}_{10,2}$.

Definition 1.1. For sake of convenience, we define a function $C(n)$ that counts the number of unique ways of constructing n as a sum of elements in $\mathcal{N}_{10,2}$. That is, $C(n)$ is the number of unique constructions such that

$$n = \sum_{a_i \in \mathcal{N}_{10,2}} a_i.$$

We start with an elementary example. Consider $n = 15$, and suppose we wish to find $C(15)$. Clearly, the only elements of $\mathcal{N}_{10,2}$ that are relevant here are 1, 10 and 11. With so few elements, we can easily calculate $C(15)$ manually. We find that there are three unique constructions of 15, which are

$$\begin{aligned}
15 &= 1+1+1+1+1+1+1+1+1+1+1+1+1+1+1 \\
15 &= 1+1+1+1+1+10 \\
15 &= 1+1+1+1+11.
\end{aligned}$$

Hence $C(15) = 3$. However, this method quickly breaks down for large n, where the number of relevant elements of $\mathcal{N}_{10,2}$ increases with the length of the number. For example, there are 15 relevant elements of $\mathcal{N}_{10,2}$ for $n = 1337$, which are $1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111$. As such, we need a better way to count $C(n)$.

Let us consider a simpler case. Consider a fictional country Numberland whose currency system consists of $\mathcal{N}_{10,2}$. Suppose Dave wants to count the number of ways to make \$15 in this currency system. This is equivalent to asking what $C(15)$ is.

Suppose the number of ways to make $\$p$ using only \$1 coins is m, and the number of ways to make $\$q$ using only \$10 coins is n. Now, consider the terms mx^p and nx^q. Due to the multiplication principle, the number of ways to make $\$p + q$ using coins of either \$1 or \$10 is obviously mn. This is equivalent to looking at the coefficient of $p + q$ after multiplying mx^p and nx^q. Now consider a *generating function* that is defined as the polynomial $f(x) = \sum_{i=1}^{\infty} a_i x^i$. Let the coefficients a_i describe the number of ways to construct $\$i$ using coins of value $\$p$. Now let another generating function be defined as $g(x) = \sum_{i=1}^{\infty} b_i x^i$, where the coefficients b_i represent the number of ways to construct $\$i$ using coins of value $\$q$. If we multiply these functions together, we get another polynomial

$$h(x) = f(x)g(x) = (a_0 + a_1 x + a_2 x^2 + \dots)(b_0 + b_1 x + b_2 x^2 + \dots) = \sum_{i=1}^{\infty} c_i x^i.$$

Now, as before, each coefficient c_i represents the number of ways to make i using coins of value \$$p$ or \$$q$. Now, let us consider the generating functions of the different coin values of Numberland.

Consider the generating function g_1 of coins of value \$1. Clearly, we can make values of integer value \$$k$ in exactly 1 way, that is, $k = 1 + \cdots + 1$. Thus the generating function is simply

$$g_1(x) = 1 + x + x^2 + \ldots.$$

This is clearly a geometric series, and therefore we may simplify g_1 to

$$g_1(x) = \frac{1}{1-x}.$$

Next, consider the generating function g_{10} of coins of value \$10. Note that we can only make amounts that have values of multiples of 10. Moreover, we can only make those amounts in exactly 1 way, which is $10k = 10 + \cdots + 10$ for positive integers k. Thus we have

$$g_{10}(x) = \mathbf{1} + 0x + 0x^2 + \cdots + 0x^9 + \boldsymbol{x^{10}} + 0x^{11} + \cdots + 0x^{19} + \boldsymbol{x^{20}} + \cdots = 1 + x^{10} + x^{20} + \ldots.$$

Again, we can rewrite this as geometric series in the form

$$g_{10}(x) = \frac{1}{1-x^{10}}.$$

One may now notice a pattern that we now prove more formally.

Theorem 1.1. *The generating function for coins of value a is*

$$g_a(x) = \frac{1}{1-x^a}.$$

Proof. i really can't be fucked rn □

Theorem 1.2. *For positive integers n, $C(n)$ is equal to the coefficient of x^n in*

$$\prod_{a_i \in \mathcal{N}_{10,2}} \frac{1}{(1-x^{a_i})}.$$

Proof. yeah well we kinda proved it above didn't we □

2 Problem 2

Definition 2.1. Let $\mathcal{L}_{10,2}(N)$ be the minimum number of summands needed to construct N using elements in $\mathcal{N}_{10,2}$. For example, $\mathcal{L}_{10,2}(13) = 3$.

Problem. Determine $\mathcal{L}_{10,2}(N)$ for the following values of N:

- 1337
- 12345
- 190274876

Definition 2.2. Let d_N be the set of digits in the base 10 expansion of N.

For example, $d_{1337} = \{1, 3, 7\}$.

Theorem 2.1. *For a positive integer N, $\mathcal{L}_{10,2}(N) = \max\{d_N\}$.*

Theorem 2.2. *For a positive integer N, we have*

$$\mathcal{L}_{10,k}(N) = \left\lceil \frac{\max\{d_N\}}{k-1} \right\rceil.$$